Internet of Things
Homework: exercise n. 3

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Exercise text

A RFID system based on Dynamic Frame ALOHA is composed of $N = 4$ tags

1. Find the overall collision resolution efficiency η in the different cases in which the initial frame size is set to $r_1 = 1, 2, 3, 4, 5, 6$.
   - Assume that after the first frame, the frame size is correctly set to the current backlog size.
   - Assume as given the duration of the arbitration period with $N = 2, 3$ tags when $r = N$ ($L_2 = 4$, $L_3 = \frac{51}{8}$).
2. After computing the values of the efficiency with the different frame sizes, produce a plot with values of η over r_1.
3. For what values of r_1 we have the maximum value for η? Comment.

1 Collision resolution efficiency

Given, as computed during the exercise lectures,

$L_2 = 4$

$L_3 = \frac{51}{8}$

And adding, trivially,

$L_1 = 1$

We are now aiming at computing the missing L_4, which will be useful in the successive steps of the computation, which will in fact make use of the backlog size.

1.0 The missing piece: L_4

Being the frame size (r) equal to the number of tags (N), $r = N$, the generic recursive formula applies:

$$L_4 = 4 + \sum_{i=0}^{3} L_{4-i} \cdot P(S = i)$$

With S be the number of resolved tags.

The overall resolutive strategies then outlines the classic probabilistic characteristics of the problem by exploiting $\frac{Matching\ cases}{Possible\ cases}$, combining and permuting them in the classic slot allocation fashion.

The term accounting for possible cases, $\left(\frac{1}{r}\right)^N$, will, in fact, always be present.

$P(S=0) = \left(\frac{1}{4}\right)^4 \cdot \left(4 + \binom{4}{2} \cdot \frac{4!}{2 \cdot 2}\right) = \frac{10}{64}$, which accounts for the cases in which all the tags collide or the cases for which they collide in pairs (groups of 2), being the only possible one leading to no resolved tag.

$P(S=1) = \left(\frac{1}{4}\right)^4 \cdot 4 \cdot 4 \cdot 3 = \frac{3}{16}$, which accounts for the number of tags to solve, the ways of solving them and the ways of positioning the remaining ones.

$P(S=2) = \left(\frac{1}{4}\right)^4 \cdot \binom{4}{2} \cdot 4! = \frac{9}{16}$, which again considers the case in which pairs of tag collide.

$P(S=3) = 0$, being impossible solving 3 tags over 4: the remaining tag would be either collided with another one or solved itself.

$P(S=4) = \left(\frac{1}{4}\right)^4 \cdot 4! = \frac{6}{64}$, which accounts for the possible permutations of the 4 solved tags.

Which ultimately sums up to 1, as expected by enumerating all possible cases.

The overall result will therefore be:

$L_4 = 4 + \frac{10}{64} \cdot L_4 + \frac{3}{16} \cdot L_3 + \frac{9}{16} \cdot L_2 = 4 + \frac{10}{64} L_4 + \frac{3}{16} \cdot \frac{51}{8} + \frac{9}{16} \cdot 4 = \frac{953}{108} = 8{,}824.$

1.1 Case $r_1 = 1$

All tags will trivially collide in the only available slot, which translates to:

$P(S=0) = 1$

$P(S=1) = P(S=2) = P(S=3) = P(S=4) = 0$

Which ultimately leads to:

$L^* = r_1 + P(S=0) \cdot L_4 = 1 + 1 \cdot 8{,}824 = 9{,}824$

$\eta = \frac{N}{L^*} = \frac{4}{9{,}824} = 0{,}407$

1.2 Case $r_1 = 2$

The maximum number of tags that can be solved in 2 slots is expected to be, intuitively, equal to 1: as similarly commented earlier, the remaining tags would either collide or not be present at all.

$P(S=0) = \left(\frac{1}{2}\right)^4 \cdot \left(2 + \binom{4}{2}\right) = \frac{1}{2}$, which accounts for the cases in which all the tags collide or the cases for which they collide in pairs (groups of 2), being the only possible one leading to no resolved tag.

$P(S=1) = \left(\frac{1}{2}\right)^4 \cdot 4 \cdot 2 \cdot 1 = \frac{1}{2}$, which accounts for the number of tags to solve, the ways of solving them and the ways of positioning the remaining ones.

$P(S=2) = P(S=3) = P(S=4) = 0$, as expected.

Which ultimately sums up to 1, as expected by enumerating all possible cases.

The overall result will therefore be:

$L^* = r_1 + P(S=0) \cdot L_4 + P(S=1) \cdot L_3 = 2 + \frac{1}{2} \cdot 8{,}824 + \frac{1}{2} \cdot \frac{51}{8} = 9{,}600$

$\eta = \frac{N}{L^*} = \frac{4}{9{,}600} = 0{,}417$

1.3 Case $r_1 = 3$

The maximum number of tags that can be solved in 3 slots is expected to be, intuitively, equal to 2: as similarly commented earlier, the remaining tags would either collide or not be present at all.

$P(S=0) = \left(\frac{1}{3}\right)^4 \cdot \left(3 + \binom{4}{2} \cdot \frac{3!}{2}\right) = \frac{7}{27}$, which accounts for the cases in which all the tags collide or the cases for which they collide in pairs (groups of 2), being the only possible one leading to no resolved tag: in such a case, simmetry is also accounted for.

$P(S=1) = \left(\frac{1}{3}\right)^4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{8}{27}$, which accounts for the number of tags to solve, the ways of solving them and the ways of positioning the remaining ones.

$P(S=2) = \left(\frac{1}{3}\right)^4 \cdot \binom{4}{2} \cdot 3 \cdot 2 \cdot 1 = \frac{4}{9}$, which again considers the case in which pairs of tag collide.

$P(S=3) = P(S=4) = 0$, as expected.

Which ultimately sums up to 1, as expected by enumerating all possible cases.

The overall result will therefore be:

$L^* = r_1 + P(S=0) \cdot L_4 + P(S=1) \cdot L_3 + P(S=2) \cdot L_2 = 3 + \frac{7}{27} \cdot 8{,}824 + \frac{8}{27} \cdot \frac{51}{8} + \frac{4}{9} \cdot 4 = 8{,}954$

$\eta = \frac{N}{L^*} = \frac{4}{8{,}954} = 0{,}447$

1.4 Case $r_1 = 4$

Being $r = N$, as mentioned earlier, the recursive formula described at the beginning applies.
Therefore, $L^* = L_4 = 8{,}824$

$\eta = \frac{N}{L^*} = \frac{4}{8{,}824} = 0{,}453$

1.5 Case $r_1 = 5$

This time, we will need to consider all cases up to solving all tags, being all theoretically valid.

$P(S=0) = \left(\frac{1}{5}\right)^4 \cdot \left(5 + \binom{4}{2} \cdot \frac{5 \cdot 4}{2}\right) = \frac{13}{125}$, which accounts for the cases in which all the tags collide or the cases for which they collide in pairs (groups of 2), being the only possible one leading to no resolved tag: in such a case, simmetry is also accounted for.

$P(S=1) = \left(\frac{1}{5}\right)^4 \cdot 5 \cdot 4 \cdot 4 = \frac{16}{125}$, which accounts for the number of tags to solve, the ways of solving them and the ways of positioning the remaining ones.

$P(S=2) = \left(\frac{1}{5}\right)^4 \cdot \binom{4}{2} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2} = \frac{72}{125}$, which again considers the case in which pairs of tag collide and accounts for simmetry.

$P(S=3) = 0$, being impossible solving 3 tags over 4: the remaining tag would be either collided with another one or solved itself.

$P(S=4) = \left(\frac{1}{5}\right)^4 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{24}{125}$, which accounts for the possible permutations of the 4 solved tags.

Which ultimately sums up to 1, as expected by enumerating all possible cases.

The overall result will therefore be:

$L^* = r_1 + P(S=0) \cdot L_4 + P(S=1) \cdot L_3 + P(S=2) \cdot L_2 = 5 + \frac{13}{125} \cdot 8{,}824 + \frac{16}{125} \cdot \frac{51}{8} + \frac{72}{125} \cdot 4 = 9{,}038$

$\eta = \frac{N}{L^*} = \frac{4}{9{,}038} = 0{,}443$

1.6 Case $r_1 = 6$

This last time, again, we will need to consider all cases up to solving all tags, being all theoretically valid.

$P(S=0) = \left(\frac{1}{6}\right)^4 \cdot \left(6 + \binom{4}{2} \cdot \frac{6 \cdot 5}{2}\right) = \frac{2}{27}$, which accounts for the cases in which all the tags collide or the cases for which they collide in pairs (groups of 2), being the only possible one leading to no resolved tag: in such a case, simmetry is also accounted for.

$P(S=1) = \left(\frac{1}{6}\right)^4 \cdot 6 \cdot 5 \cdot 4 = \frac{5}{54}$, which accounts for the number of tags to solve, the ways of solving them and the ways of positioning the remaining ones.

$P(S=2) = \left(\frac{1}{6}\right)^4 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 4 \cdot 3}{2} = \frac{5}{9}$, which again considers the case in which pairs of tag collide and accounts for simmetry.

$P(S=3) = 0$, being impossible solving 3 tags over 4: the remaining tag would be either collided with another one or solved itself.

$P(S=4) = \left(\frac{1}{6}\right)^4 \cdot \binom{6}{2} \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{5}{18}$, which accounts for the possible permutations of the 4 solved tags and combinations of the remaining pairs of frame slots.

Which ultimately sums up to 1, as expected by enumerating all possible cases.

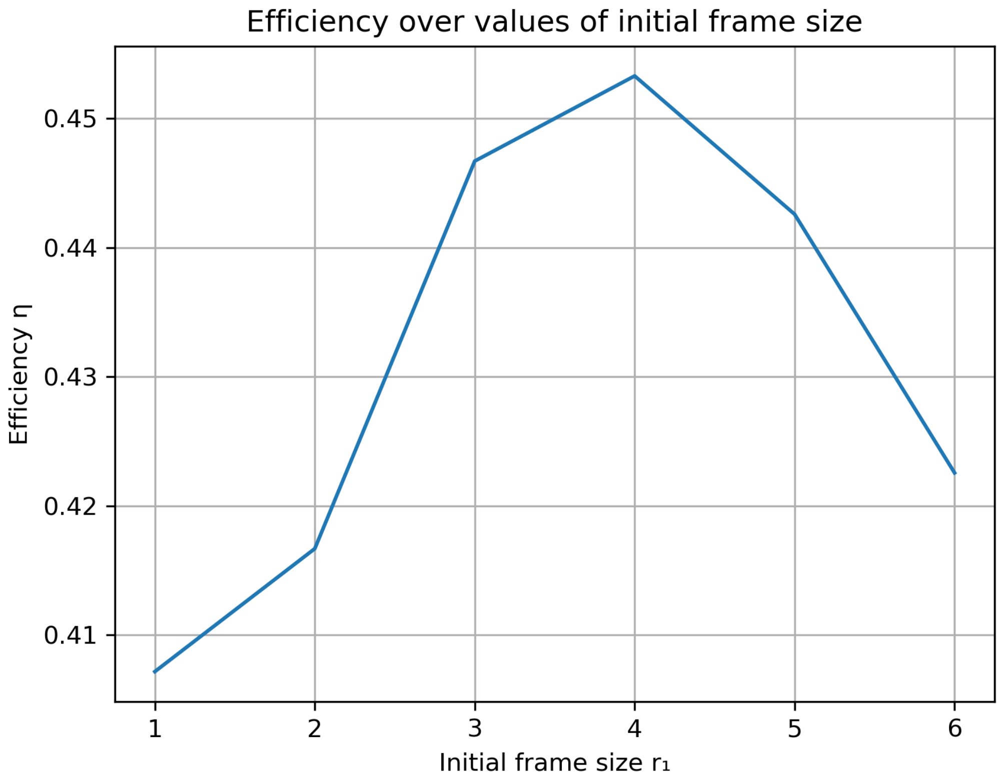
The overall result will therefore be:

$L^* = r_1 + P(S=0) \cdot L_4 + P(S=1) \cdot L_3 + P(S=2) \cdot L_2 = 6 + \frac{2}{27} \cdot 8{,}824 + \frac{5}{54} \cdot \frac{51}{8} + \frac{5}{9} \cdot 4 = 9{,}466$

$\eta = \frac{N}{L^*} = \frac{4}{9{,}466} = 0{,}423$

2 Efficiency plot

The collective efficiency plot is then produced using standard PYTHON libraries, with the result being shown in the following.



3 Comment on expected efficiency behaviour of the system

As anticipated by theory lectures, the case in which the DYNAMIC FRAME ALOHA's algorithm maximizes efficiency is the one in which the frame size (r) is dynamically updated to the current backlog (tags to solve at the next step, n).

Also, a direct consequence of the algotihm's behaviour will confirm the minimum being in $r = 1$.

$\Rightarrow max(\eta)$ is in $r = N$.

$\Rightarrow min(\eta)$ is in $r = 1$.

3.1 Demonstration: efficiency maximum and minimum points in $r = 1$ and $r = N$

Let $\eta = \frac{N}{r} \cdot \left(1 - \frac{1}{r}\right)^{N-1}$ be defined as the efficiency of the algorithm.

As known from Mathematical Analisys, to highlight maximum points we will just need to null the derivative $\frac{d\eta}{dr}$.

Note that, for the existance of the derivative, $r \neq 0$, which is trivially proved by the layout of slots and $N > 1$ being constant.

$$\frac{d\eta}{dr} = \frac{N \cdot \left(\frac{r-1}{r}\right)^{N+1} \cdot (r - N)}{(r-1)^3} = 0$$

Which, simplified, becomes:

$$\frac{N}{r^3} \cdot \left(\frac{r-1}{r}\right)^{N-2} \cdot (-r + N) = 0$$

And, when used to distinguish maximum points from minimum ones, translates to:

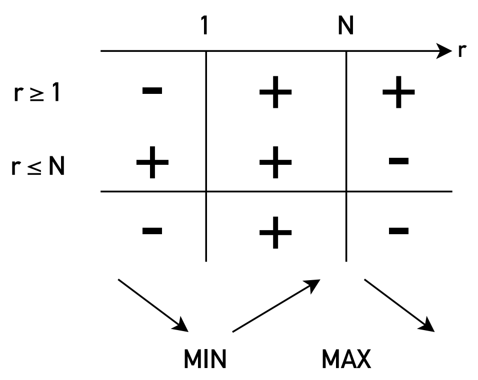
$\frac{N}{r^3} \cdot \left(\frac{r-1}{r}\right)^{N-2} \cdot (-r + N) \geq 0$ (with $r \neq 0$ from existance conditions).

Underlining the two cases:

$$\begin{cases} r - 1 \geq 0 \\ -r + N \geq 0 \end{cases} \Rightarrow \begin{cases} r \geq 1 \\ r \leq N \end{cases}$$

In which N is again trivially > 1.

This will ultimately yield to the final result:



Which confirms our expectations about the computed plot, underlining the minimum point being in $r = 1$ and the maximum one being in $r = N$.