Asset Allocation project - Exam B

Computational Finance - A.A. 2024-2025

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Introduction

The purpose of this project is to analyze and compare the performance of different Portfolio allocation strategies within the S&P 500 investment universe.
The assest at our disposal are indices constructed by the S&P Global Inc. which are commonly used by asset managers and investors to obtain a specific sector or factor exposition, allowing to profit from a precise view on the markets. In our case we have access to 11 sector indices and 5 factor indices, for which we have historical prices and market capitalization figures.
Our report focuses on constructing Portfolios and evaluating their effectiveness throughout quantitative techniques, providing insights into the dynamics of asset allocation.

The **S&P 500** is the most widely recognized stock market index, introduced in 1957 by Standard & Poor's as a means to track the performance of the five hundred largest publicly traded companies in the United States. The S&P 500 is one of the first indexes to use a **market-capitalization-weighted** approach, providing a more representative and dynamic view of the U.S. economy. In the last years this weighting criteria overexposed the index to tech companies, due to the appreciation they achieved, so an allocation across specific sector or factor will also allow us to better handle the Portfolio diversification.

The S&P 500 has shown strong performance over the last two years, delivering a total return exceeding 52%. Much of this growth can be attributed to the ever-expanding tech sector and the recent AI frenzies. The largest constituents of the S&P 500 are Apple, Microsoft, and Nvidia have been key drivers of this performance. Nvidia, in particular, has been a standout, with its value increasing nearly tenfold from early 2023 to October 2024. Apple and Microsoft also posted impressive gains during this period, with returns of 89% and 92%, respectively. We plot below a graph of the S&P500 to visualize its impressive performance.

Figure 1: S&P 500 From 03/01/2023 to 25/10/2024

Initial Remark

We would like to hereby remark that in our subsequent calculations we have generally assumed a risk-free rate of 0%, directly affecting the Sharpe Ratio value. However, we acknowledge that this assumption is an oversimplification, especially considering the significant fluctuations in interest rates observed in recent periods. During both the initial period (Part A) and the out-of-sample period (Part B), we witnessed a phase of notably high interest rates, with the risk-free rate reaching levels around 4% to 5%, as shown in the chart below. For the purposes of this project, however, this factor has not been taken into account.

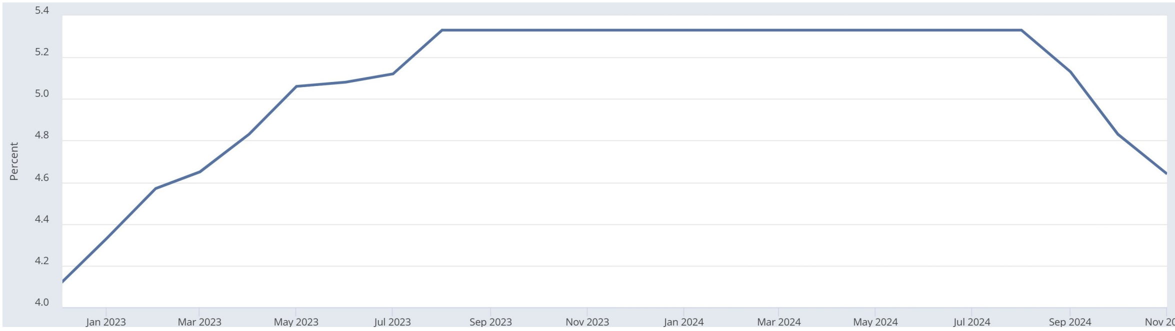


Figure 2: FED Funds Rate

Part A

Point 1

In this section, we computed the efficient frontier of Portfolios using Harry Markowitz's model. To ensure the analysis adhered to realistic investment scenarios, we applied the model's standard constraints:

$$\sum_{i=1}^{N} w_i = 1, \quad 0 \leq w_i \leq 1, \quad \forall i \in [1, \ldots, N]$$

where N represents the total number of assets.
These constraints require that the entire capital is fully allocated and prohibit short selling, ensuring that all Portfolio weights are non-negative. This approach allows for an optimized risk-return trade-off within the feasible investment space. We leveraged the existing Matlab Portfolio object from the Financial Toolbox to compute the efficient frontier.

This allowed us to visualize how Portfolios are distributed along the efficient frontier, demonstrating the successful optimization of the trade-off between risk and return. Overall, the Portfolios exhibit moderate returns with relatively low levels of volatility, reflecting a well-balanced risk-return profile.

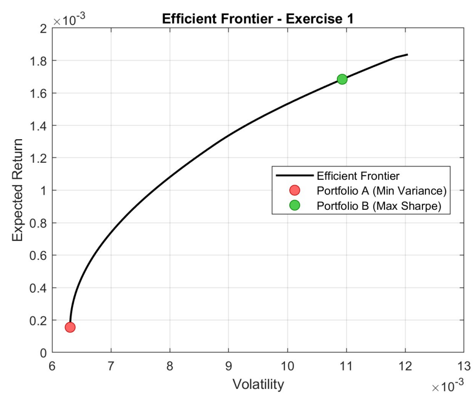


Figure 3: Efficient Frontier Visualization

We then computed the Minimum Variance Portfolio (**Portfolio A**) and the Maximum Sharpe ratio (**Portfolio B**), given the formula:

$$\text{Sharpe Ratio} = \frac{\mathbb{E}[R_p] - r_f}{\sigma_p}$$

Where:

- $\mathbb{E}[R_p]$ represents the Portfolio expected return
- r_f represents the risk free rate
- σ_p represents the volatility (standard deviation) of the Portfolio

Portfolio A	
Consumer Staples	37.91 %
Health Care	26.66 %
Low Volatility	17.47 %
Information Technology	9.47 %
Energy	6.57 %
Momentum	1.10 %
Other	0.82 %

Portfolio B	
Information Technology	75.91 %
Communication Services	13.50 %
Momentum	10.52 %
Other	0.07 %

Portfolio A: Expected log return: 1.5528e-04, volatility: 0.0063
Portfolio B: Expected log return: 0.0017, volatility: 0.0110

(We classify assets with a weight below 0.5% under 'Other', if applicable)

As we can observe, minimizing the variance leads to a Portfolio with a more diversified set of assets, reducing exposure to any single asset's fluctuations. Portfolio A is defensive, with significant allocations to Consumer Staples and Health Care, reflecting a focus on stability in uncertain market conditions. On the other hand, the Maximum Sharpe Ratio Portfolio will serve as our benchmark, as it represents the optimal trade-off between risk and return according to Markowitz's theory. It is the point on the efficient frontier where the slope, representing the ratio of excess return to risk, is steepest. Portfolio B, in contrast with A, is growth-oriented, with an heavy emphasis on Information Technology and Momentum. The low level of allocation to Consumer Staples in Portfolio B suggests a preference for higher-growth, higher-risk sectors, driven by optimism in tech and communications in 2023.

Point 2

Then, we incorporated additional constraints beyond the standard ones to reflect various potential investment choices.

$$\begin{cases} \sum_{i=1}^{N} w_i = 1, \;\; 0 \leq w_i \leq 1, \quad \forall i \in [1, \ldots, N] \\ \sum_{i \in \text{sensible sectors}} w_i > 0.10, \\ \sum_{i \in \text{cyclical sectors}} w_i < 0.30, \\ w_{\text{Consumer Staples}} = 0, \\ w_{\text{Low Volatility}} = 0, \\ \sum_{i=1}^{11} w_i < 0.80, \quad \forall i \in \text{sectors}. \end{cases}$$

The first constraint requires the exposure to sensitive sectors, which tend to react to economic or policy changes, to exceed 10%, while the exposure to cyclical sectors, which are more volatile and closely tied to economic cycles, was capped at 30%. Furthermore, investments in the Consumer Staples sector and the Low Volatility factor were completely excluded. Lastly, a maximum allocation limit of 80% was set for the global exposure on sectors to prevent excessive concentration and ensure adequate diversification. These constraints reflect a balanced approach to managing risk and targeting specific investment goals.

To optimize the efficient frontier, we once again utilized Matlab's built-in functions, prioritizing performance and efficiency. We also computed the Minimum Variance and Maximum Sharpe Ratio Portfolios.

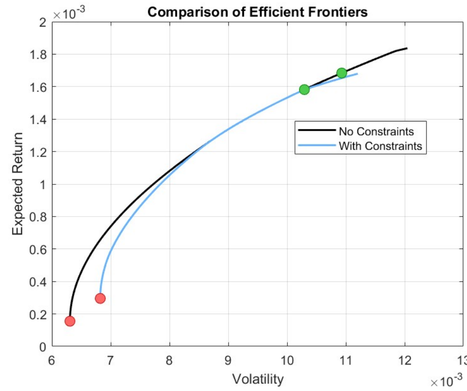


Figure 4: Comparison of Frontier with and without additional constraints

Portfolio C: Expected log return:2.9548e-04, volatility: 0.0068
Portfolio D: Expected log return: 0.0016, volatility: 0.0103

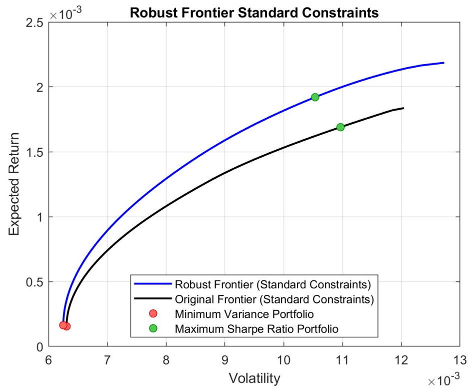
Portfolio C	
Health Care	56.03 %
Quality	23.63 %
Utilities	6.86 %
Energy	5.10 %
Information Technology	4.90 %
Momentum	3.48 %
Other	0 %

Portfolio D	
Information Technology	68.54 %
Communication Services	11.46 %
Momentum	20.00 %
Other	0 %

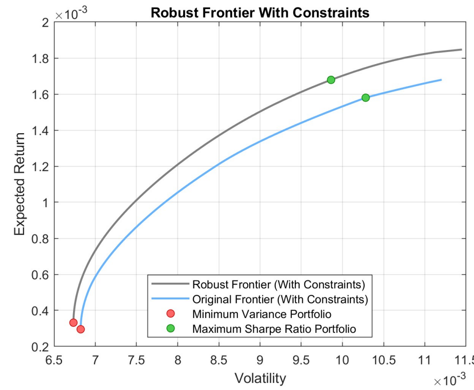
(We classify assets with a weight below 0.5% under 'Other,' if applicable)

Point 3

In this section, we enhanced the frontiers computed in Point 1) and 2) by applying a re-sampling method to develop robust frontiers under the given constraints. This approach addresses key limitations of traditional Mean-Variance Analysis, particularly in dealing with non-normally distributed data and capturing tail risks and extreme market events. Frontier Resampling also enables a dynamic perspective by accounting for shifting market conditions over time, making it a valuable tool for constructing Portfolios that can adapt to evolving economic environments. This adaptability ensures more resilient Portfolio optimization and provides a more comprehensive framework for risk and return analysis.



Efficient and Robust frontiers with standard constraints



Efficient and Robust frontiers standard and additional constraints

Through 150 simulations, we sampled returns as multivariate normal variables with mean and variance obtained from the market and we use them to compute the new expected returns and covariance matrix. This number of simulation allowed us to obtain good balance between precision and computational effort. The Portfolios on the frontier are computed as the average of the Portfolios of the simulated frontiers.
As can be seen by the plot above and as expected from the theoretical framework, the robust frontiers deliver superior returns while stabilizing volatility, offering enhanced diversification in the process.
Once again, we computed the Minimum Variance and Maximum Sharpe Ratio Portfolios, first with standard constraints (**Portfolio E** and **Portfolio G**), then incorporating the additional constraints from Point 2) (**Portfolio F** and **Portfolio H**).

Portfolio E: Expected log return: 1.7153e-04, volatility: 0.0063
Portfolio G: Expected log return: 0.0019, volatility: 0.0097
Portfolio F: Expected log return: 3.3203e-04, volatility: 0.0067
Portfolio H: Expected log return: 0.0017, volatility: 0.0099

Portfolio E	
Consumer Staples	37.49 %
Health Care	25.23 %
Low Volatility	16.54 %
Information Technology	7.05 %
Energy	5.02 %
Momentum	4.94 %
Communication Services	1.75 %
Quality	1.60 %
Other	0.38 %

Portfolio G	
Information Technology	45.22%
Communication Services	23.00%
Momentum	12.28%
Consumer Discretionary	7.38 %
Energy	4.89%
Industrials	1.86%
Consumer Staples	1.25%
Other	4.12 %

Portfolio F	
Health Care	54.44 %
Quality	18.70 %
Utilities	7.02 %
Momentum	5.30 %
Energy	4.89 %
Information Technology	4.43 %
Growth	2.28 %
Industrials	1.41 %
Communication Services	1.03 %
Other	0.5 %

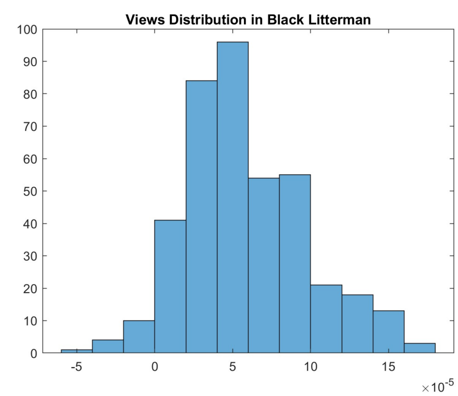
Portfolio H	
Information Technology	47.37%
Communication Services	20.90 %
Momentum	12.94 %
Consumer Discretionary	6.89 %
Energy	4.80 %
Industrials	1.68 %
Real Estate	1.04 %
Utilities	1.02 %
Other	3.36 %

(We classify assets with a weight below 0.5% under 'Other,' if applicable)

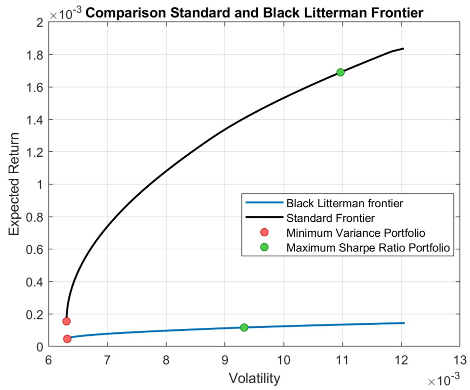
Point 4

We then computed the efficient frontier under standard constraints exploiting Black-Litterman model, a framework which leverages Bayesian statistics to build allocation strategies. This model improves the Markowitz one by incorporating investor views into the market equilibrium, resulting in more stable and customized Portfolio allocations. While Markowitz solely relies on historical data and expected returns, Black-Litterman adjusts these inputs, blending them with subjective market opinions to avoid extreme outcomes and enhance Portfolio diversification.

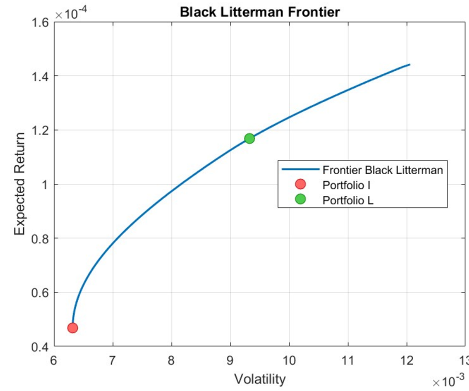
We integrated specific constraints to the Portfolio, including a view that the Technology sector would outperform the Financial sector by 2% annually, reflecting the growing importance of technology. Additionally, it was incorporated the assumption that, in a bull market, the Momentum factor would outperform the Low Volatility factor by 1% annually.



In order to compute the efficient frontier we considered the standard constraints and a moderate risk profile, keeping the aversion coefficient set to 1.2. By plotting the frontiers, we can see that the Black Litterman one, while having the same shape of the standard frontier, has a smaller scale. We noticed that is partially due to the choice of the aversion coefficient λ, indeed by increasing this parameter we obtain frontiers of similar scale and shape with respect to the standard one.



Comparison of Frontiers



Black Litterman Frontier

We finally computed the Minimum Variance (**Portfolio I**) and Maximum Sharpe Ratio (**Portfolio L**) Portfolios, showing that, as expected from the theoretical framework, these present a greater exposition to Technology sector and Momentum with respect to the base case. These findings show a practical model to adjust allocations in line with investor views in a quantitative framework.

Portfolio I: Expected log return: 4.6820e-05, volatility: 0.0063
Portfolio L: Expected log return: 1.1682e-04, volatility: 0.0093

Portfolio I	
Consumer Staples	37.91%
Health Care	26.66%
Low Volatility	17.48%
Information Technology	9.45%
Energy	6.58%
Momentum	1.11%
Communication Services	0.81%
Other	0%

Portfolio L	
Information Technology	58.00%
Momentum	35.33%
Communication Services	6.62%
Energy	0.05%
Other	0%

(We classify assets with a weight below 0.5% under 'Other,' if applicable)

Point 5

In this part, other than the standard constraints, we considered other two conditions: the cumulative allocation to assets classified as "cyclicals" must exceed 20% of the Portfolio's total weight and the total absolute difference between the weights of the constructed Portfolio and those of a capitalization-weighted benchmark Portfolio must be at least 20%. The Portfolios described here disregard any assumptions about expected returns and instead prioritize diversification and risk mitigation. Two distinct measures are used to quantify diversification. **Portfolio M** is constructed to maximize the diversification ratio, which assesses how effectively the Portfolio spreads its investments across assets.

$$DR = \frac{\sum_{i=1}^{N} w_i \sigma_i}{\sigma_P} \qquad \sigma_P = \text{volatility of the Portfolio.}$$

In contrast, **Portfolio N** is designed to maximize entropy, a metric derived from the volatilities of the assets. Originally a concept from physics, entropy in asset allocation represents diversification by encouraging a more evenly distributed weight among assets, thereby reducing concentration risk and enhancing portfolio stability.

$$H = -\sum_{i=1}^{N} \bar{\sigma}_i \cdot ln(\bar{\sigma}_i)$$

where $\bar{\sigma}_i$ is the volatility contribution of asset i to the portfolio: $\bar{\sigma}_i = \frac{w_i^2 \sigma_i^2}{\sum_{j=1}^{N} w_j^2 \sigma_j^2}$

The specific allocations for Portfolio M and Portfolio N are detailed below:

$$\begin{cases} \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0, \quad \forall i \\ \sum_{i \in \text{cyclicals}} w_i \geq 0.2 \\ \sum_{i=1}^{n} \left| w_i^{opt} - w_i^{\text{benchmark}} \right| \geq 0.2 \end{cases} \tag{1}$$

Portfolio M: Expected log return: 4.8939e-04, volatility: 0.0080
Portfolio N: Expected log return: 5.2429e-04, volatility: 0.0074

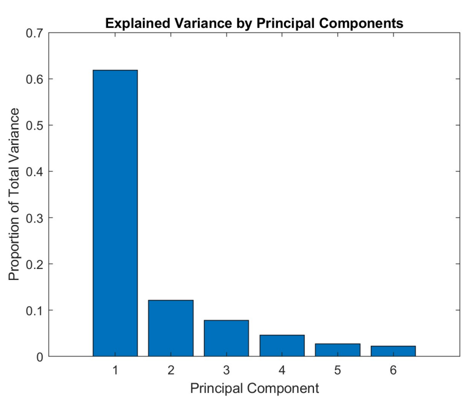
Portfolio M	
Information Technology	9.20%
Consumer Discretionary	14.94%
Communication Services	12.00%
Consumer Staples	13.43%
Energy	25.88%
Utilities	19.48%
Real Estate	5.06%
Other	0.01%

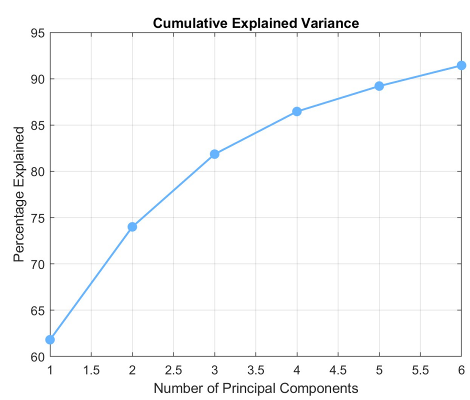
Portfolio N	
Information Technology	4.94%
Financials	5.79%
Health Care	8.16%
Consumer Discretionary	4.69%
Communication Services	4.37%
Industrials	6.40%
Consumer Staples	8.40%
Energy	4.13%
Utilities	5.29%
Real Estate	4.70%
Materials	5.62%
Momentum	7.24%
Value	7.05 %
Growth	7.03%
Quality	7.56%
Low Volatility	8.65%
Other	0%

(We classify assets with a weight below 0.5% under 'Other,' if applicable)

Point 6

In this exercise, we employed the Principal Component Analysis (PCA) to reduce the dimensionality of our problem to some key factors. Specifically, we determined the minimum number of factors needed to explain more than 90% of the cumulative variance, resulting in six factors. These six factors represent the most influential drivers of market behavior.





As we can observe, the first factor alone is able to explain approximately 62% of the cumulative variance while the fifth combined with the sixth explain less than 5%.

Next, we maximized the expected return, subject to the standard constraints and with a target volatility $\sigma_{tgt} = 0.7$ through the optimization Matlab function *fmincon*.

Portfolio P: Expected log return: 0.0011 , volatility: 0.7000

Portfolio P	
Communication Services	40.26%
Energy	25.56%
Information Technology	22.35%
Consumer Staples	7.96%
Utilities	3.86%
Other	0.01%

(We classify assets with a weight below 0.5% under 'Other,' if applicable)

Point 7

In this section, we determined the optimal asset allocation weights that maximize the VaR-modified Sharpe Ratio. This enhanced Sharpe Ratio serves as a risk-adjusted performance measure, where risk is represented by the Value at Risk (VaR) instead of volatility. The formula is as follows:

$$\text{VaR-modified Sharpe Ratio} = \frac{\mathbb{E}[R_p] - r_f}{VaR_{p,\alpha}}$$

To achieve this optimization, we utilized Matlab's *fmincon* function to minimize the negative VaR-modified Sharpe Ratio, effectively maximizing it. The standard Portfolio constraints, already outlined in point 2, were applied.

To compute the VaR-modified Sharpe Ratio as a function of Portfolio weights, we developed a MATLAB function. This function calculates the VaR using the Variance-Covariance method under the assumption of normally distributed returns. It takes as inputs the Portfolio weights, historical returns of the asset classes, and a specified confidence level.

Before proceeding with the computations of the weights, we performed some analysis on our historical returns in order to verify the normality assumption. At the beginning we visualized the distribution of returns plotted as an histograms and comparing it with the Gaussian CDF with same mean and standard deviation.

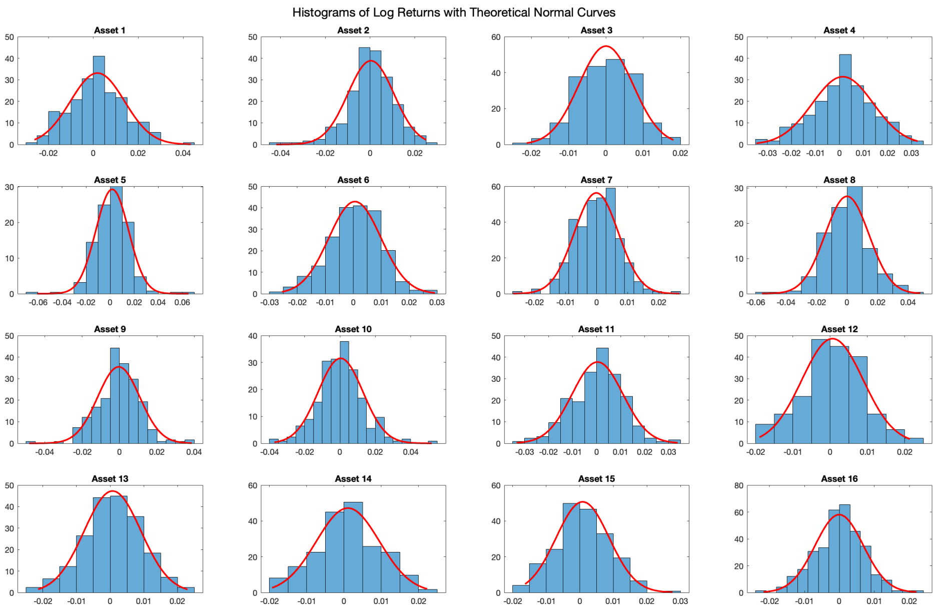


Figure 8: Histograms of returns distribution and Normal CDF

We employed the classical qq-plot which allows us to compare the distribution of the returns of each sectors with respect to the Gaussian distribution (see Figure 9). We also employed the Anderson-Darling test, which compares the empirical distribution function with the theoretical one. This test revealed us that the data follows a Gaussian distribution at 5% of significance level.

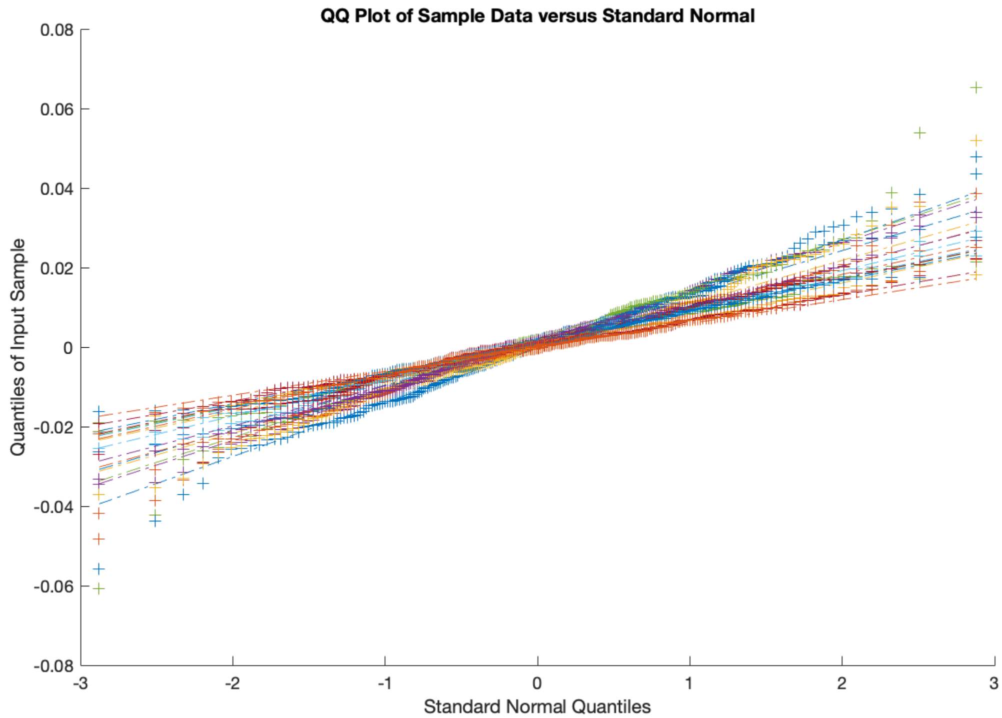


Figure 9: qq-plot of returns

Since the normality assumption is verified, we calculate the VaR in the formula for the modified Sharpe Ratio as:

$$VaR_{p,\alpha} = \mu_p + \sigma_p \cdot (\Phi^{-1}(1-\alpha))$$

where Φ is the cumulative distribution function of a standard normal random variable.

Our findings indicate that the confidence level used to compute the VaR (e.g., α equal to 0.05 or 0.01) does not significantly impact the final asset allocation, here we provide the results with α=0.05.

Portfolio Q: Expected log return: 0.0017, volatility: 0.0110

Portfolio Q	
Information Technology	75.90%
Communication Services	13.50%
Momentum	10.58%
Other	0.02%

(We classify assets with a weight below 0.5% under 'Other,' if applicable)

Point 8

The evaluation of the characteristics of the Portfolios in terms of performance, risk and diversification required different metrics to get to a complete and accurate view. This will be even more evident in the following section. First, we display a visual overview of the Portfolios.

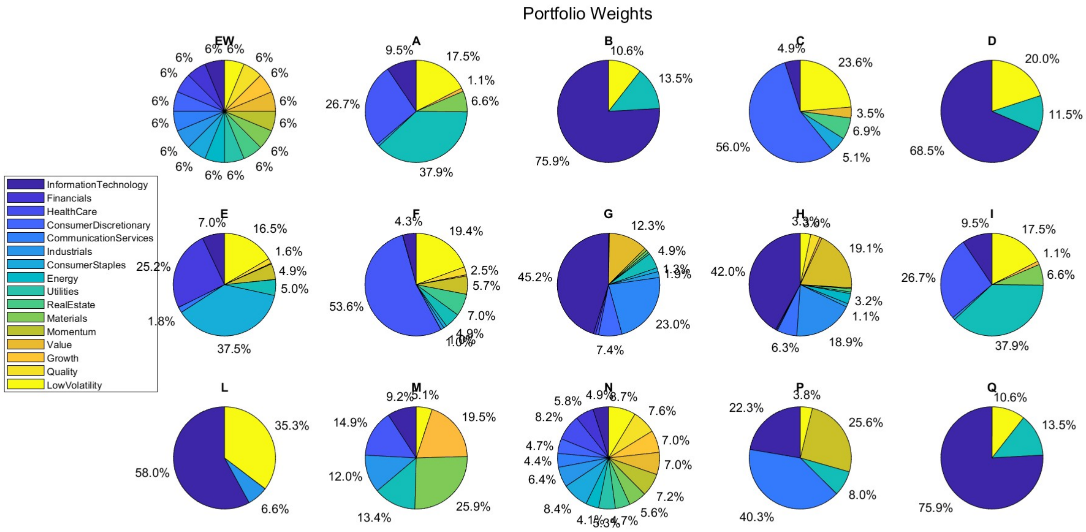


Figure 10: Portfolio's weights

In the matrix below are reported for each Portfolio the following metrics: **Annualized Return**, **Annualized Volatility**, **Sharpe Ratio**, **Maximum Drawdown** and **Calmar Ratio** assess performances and risks, while the **Diversification Ratio** measure the balance of the Portfolios. The use of the equally weighted Portfolio allowed us to obtain useful insights and features from the table.

By observing the performances, most portfolios can be grouped into two classes:

1. **first class**: A, C, E, F and I
2. **second class**: B, D, G, H, L and Q

The portfolios in the first class were computed minimizing the variance, showing an annualized return that is considerably lower than the equally weighted, around one half or even one third. The decrease of the volatility costed a lot in terms of gains.

On the opposite the portfolios in the second class are computed by maximizing the sharpe ratio, which leads to very high returns, also more than three times the ones of the equally weighted, but also to a relevant increase of the volatilities. These characteristics are driven by a greater concentration on few sectors, especially Information technology, which results to higher risk. As an example we could mention the outstanding return of around 52% of Portfolios B and Q, which are indeed very similar in composition.

This analysis is confirmed also by the Calmar ratio. Its values have different magnitude among the two allocation classes, always under -1% for the first and around -5% for the second. Between this numbers lies the Calmar ratio of the equally weighted Portfolio (-1.5514%), which also in this case acts as a intermediate point between the two classes. However the maximum drawdown seems to be less informative in this context, since it shows values that span a narrower range.

In terms of risk the Diversification ratio tells us the value of 1.315 is outperformed only by the Portfolio P and of course by M, computed via principal component analysis and maximizing this index, respectively. It is interesting to notice that maximizing the entropy did not improve this metric. As we notice above, pursuing high returns results in a cost in terms of risk and diversification since the investor need a greater exposure on high-performing sectors.

Portfolio	AnnRet	AnnVol	Sharpe	MaxDD	Calmar	DR
EW	0.15227	0.12108	1.2576	-0.09815	-1.5514	1.3150
A	0.04019	0.09989	0.4024	-0.09650	-0.4165	1.2874
B	0.52481	0.17423	3.0121	-0.09359	-5.6073	1.0809
C	0.07620	0.10806	0.7052	-0.08750	-0.8709	1.2205
D	0.48555	0.16331	2.9732	-0.08497	-5.7143	1.1148
E	0.04785	0.10002	0.4784	-0.09371	-0.5106	1.2847
F	0.08099	0.10828	0.7479	-0.08571	-0.9448	1.2295
G	0.43881	0.15948	2.7515	-0.08861	-4.9524	1.2007
H	0.42566	0.15365	2.7703	-0.08353	-5.0958	1.1829
I	0.04014	0.09989	0.4018	-0.09649	-0.4160	1.2874
L	0.42347	0.14790	2.8633	-0.07719	-5.4863	1.1596
M	0.13466	0.12664	1.0633	-0.09295	-1.4488	1.5193
N	0.13694	0.11688	1.1717	-0.09736	-1.4065	1.2880
P	0.30806	0.14612	2.1082	-0.08640	-3.5656	1.3964
Q	0.52475	0.17423	3.0119	-0.09360	-5.6061	1.0810

Table 1: Performance Metrics for Portfolios (Set 2)

Part B

Portfolio	AnnRet	AnnVol	Sharpe	MaxDD	Calmar	DR
EW	0.25886	0.10238	2.5284	-0.05686	-4.5527	1.4307
A	0.17604	0.08088	2.1764	-0.04455	-3.9515	1.4344
B	0.47764	0.20974	2.2774	-0.15200	-3.1424	1.0457
C	0.18584	0.09416	1.9737	-0.05291	-3.5127	1.3200
D	0.48643	0.20686	2.3515	-0.14965	-3.2504	1.0436
E	0.18961	0.08175	2.3193	-0.04404	-4.3051	1.4253
F	0.19737	0.09533	2.0702	-0.05258	-3.7535	1.3365
G	0.40513	0.17191	2.3566	-0.12602	-3.2148	1.1727
H	0.42504	0.17627	2.4113	-0.12838	-3.3107	1.1338
I	0.17601	0.08088	2.1762	-0.04455	-3.9511	1.4343
L	0.50217	0.20379	2.4642	-0.14705	-3.4151	1.0344
M	0.22948	0.10287	2.2307	-0.05195	-4.4173	1.6294
N	0.25665	0.09948	2.5800	-0.05356	-4.7914	1.4058
P	0.29796	0.13232	2.2518	-0.08834	-3.3729	1.4116
Q	0.47756	0.20970	2.2773	-0.15197	-3.1424	1.0458

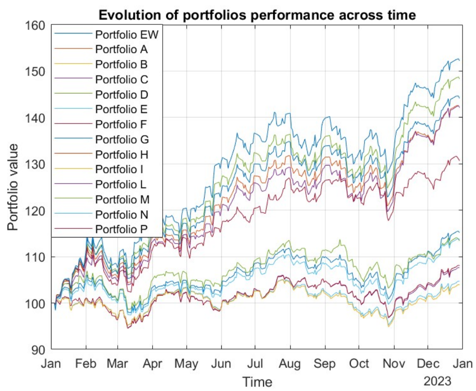
Table 2: Performance Metrics for Portfolios

During the first ten months of 2024, all Portfolios experienced significant appreciation, driven by the strong performance of the S&P 500, which gained over 22%. Notably, Portfolios that maximize the Sharpe ratio achieved considerably higher returns compared to those that minimize Portfolio variance.

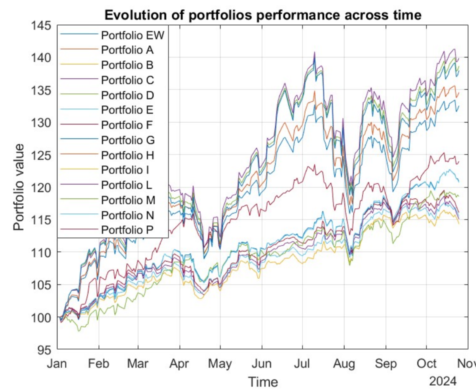
The best-performing Portfolios include B, D, L, and Q, all of which have a very similar asset allocation, with B and Q sharing identical weights. These Portfolios are heavily exposed to the Information Technology index, with allocations exceeding 58% and reaching up to 76%. They also have significant exposure to Communication Services and Momentum. Notably, the S&P 500 index disclosures reveal that Information Technology and Momentum indices share many constituents, such as Nvidia, Microsoft, Broadcom, and others (though no disclosure is available for Communication Services constituents). This creates a significant concentration risk, as the performance of these Portfolios is heavily dependent on a few companies, often within the same sector.

We can to some extent appreciate this concentration looking at the diversification ratio and the very small values it reaches on these Portfolios.

In the out-of-sample period, all Portfolios exhibit similar Sharpe ratios but significantly different maximum drawdowns, contrasting with the in-sample period. As expected, Portfolios that maximize the Sharpe ratio experience drawdowns two to three times larger than those of the minimum variance Portfolio. This could be a critical consideration for highly risk-averse investors when deciding which Portfolio to invest in.
Comparing Portfolios I and L, the out-of-sample period shows a smaller difference in returns but a larger difference in volatility than the in-sample period. This suggests that during the out-of-sample period, we can achieve reduced volatility without sacrificing a significant portion of returns.



In-sample Portfolios evolution



Out-of-sample Portfolios evolution

Figure 11: Portfolios Performance Over time

Finally, by plotting the share prices of our portfolios both in-sample and out-of-sample, we can highlight the key characteristics of our quantitative analysis. In both cases, we are able to distinguish portfolios in the two main classes found before: first minimizing the variance with lower returns and the second maximizing the sharpe ratio with higher returns. Moreover, it is possible to observe the different level of maximum drawdowns between in and out sample, in particular in August 2024.

In conclusion, it is worth noting that the last two years have been characterized by strong market performance, especially for the S&P 500. More concentrated portfolios benefited significantly from these favorable market conditions, though while incurring higher levels of risk.