MA Math Camp Exam 2019 Solutions

Instructions:

- The exam is from 11:40 - 12:55 pm.
- You may use any results covered in the class directly without proofs.
- All answers must be justified
- The exam is closed book. Calculators are not allowed.

1. Consider the 2×2 matrix.

$$A = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{1}{3} & 1 \end{bmatrix}$$

   (a) **(5 points)** Find the eigen values of this matrix A

   Solution : The eigen values are $\lambda = \frac{1}{3}, \frac{2}{3}$

   (b) **(3 points)** Find a 2×2 invertible matrix P and a 2×2 diagonal matrix Λ s.t. $A = P\Lambda P^{-1}$

   Solution : $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$

   (c) **(2 points)** Consider the quadratic form $Q = x'Ax$ where A is the matrix given above. Determine whether Q is positive definite, negative definite or indefinite.

   Solution : Positive definite since the eigen values are both positive.

2. **(5 points)** Define the kernel of a matrix A as :

$$ker(A) \equiv \{x \in \mathbb{R}^n | Ax = 0\}$$

   Prove that the $ker(A)$ is a subspace of $\mathbb{R}^n$ i.e. it contains the 0 vector and is closed under addition and scalar multiplication.

   Solution : **(i)** It contains $\mathbf{0}$ since $A\mathbf{0} = \mathbf{0}$. **(ii)** Closed under addition - Take $x, y \in ker(A)$. Then $x + y \in ker(A)$ since $A\,(x + y) = Ax + Ay = 0$. **(iii)** Take $x \in ker(A)$, then $\alpha x \in ker(A)$ since $A(\alpha x) = a\,(Ax) = 0$

3. **(10 points)** Solve the following system of equations :

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 2x_3 &= -1 \end{aligned}$$

Solution : $(x_1, x_2, x_3) = (4, -3, -1)$

4. State whether each of the following statements is true or false. If it is false, provide a counter-example. If true, give a sketch of a proof :

   (a) **(5 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly concave function. If f is twice-differentiable at $x_0 \in \mathbb{R}$, then we must have $f''(x_0) < 0$

   Solution : *False.* Take $f(x) = -x^4$

   (b) **(5 points)** If A and B are compact sets in the metric space (X, d), then $A \cup B$ is a compact set.

   Solution : *True.* To see this, take any sequence (x_n) in $A \cup B$. To show this is compact, we need to find a convergent subsequence of it. For the sequence (x_n), either infinite terms lie in A or in B. WLOG, suppose it is in A. We can construct a s subsequence (x_{n_k}) of (x_n) using only terms in A. Since A is compact, this implies there exists a convergent subsequence of (x_{n_k}), say $\left(x_{n_{k_j}}\right)$which converges to a point in A. But $\left(x_{n_{k_j}}\right)$ is also a subsequence of (x_n) and hence we have found a convergent subsequence of (x_n)

5. **(10 points)** Suppose $\mathcal{D}$ is a convex open set of $\mathbb{R}^n$ and let $f\colon \mathcal{D} \to \mathbb{R}$ be a concave function. Then show that the set of maximizers is a convex set i.e. show that :

$$M = \left\{x \in \mathcal{D} | f(x) \geq f(x') \ \forall x' \in \mathcal{D}\right\}$$

is a convex set.

Solution : Take $x, y \in M$ which implies $f(x) = f(y)$. We need to show that $\lambda x + (1-\lambda)y \in M$. Since f is concave, this implies :

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda) f(y) \\ &= f(x) \end{aligned}$$

which implies $\lambda x + (1-\lambda)y \in M$

6. Define $f : [1,\infty) \to [1,\infty)$ as $f(x) = x + \frac{1}{x}$.

   (a) **(5 points)** Show that $|f(x) - f(y)| < |x - y| \; \forall x, y$ such that $x \neq y$.

   Solution : For every $x, y \geq 1$ with $x \neq y$, we have :

$$\begin{aligned} |f(x) - f(y)| &= \left| x - y + \frac{y - x}{xy} \right| = \left| 1 - \frac{1}{xy} \right| |x - y| \\ &< |x - y| \end{aligned}$$

   (b) **(5 points)** Show that $f(\cdot)$ has no fixed point i.e. there does not exist x^* such that $f(x^*) = x^*$. Argue why this does not contradict Banach's Contraction Mapping Theorem.

   Solution : $f(x) > x \; \forall x$ which implies there is no fixed point. It does not contradict the Contraction Mapping Theorem because $\nexists \lambda \in [0,1)$ such that $|f(x) - f(y)| \leq \lambda |x - y|$. This is because $\left| 1 - \frac{1}{xy} \right| \to 1$ as $xy \to \infty$

7. Consider a consumer with a utility function $u(x_1, x_2) = x_1^{\frac{2}{3}} x_2^{\frac{1}{3}}$. Each good x_i costs p_i per unit and the consumer has a budget of M dollars. Find the consumer's optimal mix of consumption i.e. solve the program :

$$\begin{aligned} &\max_{x_1, x_2} \quad x_1^{\frac{2}{3}} x_2^{\frac{1}{3}} \\ &\text{subject to} \quad p_1 x_1 + p_2 x_2 = M \end{aligned}$$

   (a) **(10 points)** What is the optimal choice of consumption for the consumer i.e. what is (x_1^*, x_2^*)?

   Solution : $(x_1^*, x_2^*) = \left(\frac{2M}{3p_1} . \frac{M}{3p_2} \right)$

   (b) **(10 points)** Write down the value function $V(p_1, p_2, M)$. Verify that :

$$-\frac{\frac{\partial V}{\partial p_1}}{\frac{\partial V}{\partial M}} = x_1^*$$

   (This is known as Roy's Identity)

Solution : $V(\cdot) = M\left(\frac{2}{3p_1}\right)^{\frac{2}{3}}\left(\frac{1}{3p_2}\right)^{\frac{1}{3}}$

$$-\frac{\frac{\partial V}{\partial p_1}}{\frac{\partial V}{\partial M}} = \frac{\frac{2}{3}p_1^{-\frac{5}{3}}M\left(\frac{2}{3}\right)^{\frac{2}{3}}\left(\frac{1}{3p_2}\right)^{\frac{1}{3}}}{\left(\frac{2}{3p_1}\right)^{\frac{2}{3}}\left(\frac{1}{3p_2}\right)^{\frac{1}{3}}}$$
$$= \frac{2M}{3p_1}$$