Math Camp: Problem Set 1

Due August 25th, 2020

1 Set Theory

1. Prove $A \cap B = A$ if and only if $A \subseteq B$

2. Prove the intersection operator is associative: $(A \cap B) \cap C = A \cap (B \cap C)$ (hint, show set containment both ways)

3. Show the second of DeMorgan's Laws:

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

4. Let X and Y be two sets and $f : X \to Y$. Find an example in which $f(S_1 \cap S_2) \subsetneq f(S_1) \cap f(S_2)$.

5. Let X and Y be two sets and $f : X \to Y$. Prove that:

   - $f\left(f^{-1}(T)\right) = T$ for all $T \subseteq Y$ if and only if f is surjective.
   - $f^{-1}(f(S)) = S$ for all $S \subseteq X$ if and only if f is injective

   Note: $f^{-1}(T)$ represents the inverse *image* of f, not necessarily an inverse *function*.

6. Let R be a complete, transitive relation over a set X, and define the relation $\sim$ as follows: $a \sim b$ if and only if aRb and bRa. Let $I(x)$ be the collection $I(x) = \{y | y \sim x\}$. Show that for all x and y, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.

2 Analysis

1. Let $x, y \in \mathbb{R}^2$ and define $d(x, y)$ to be the maximum distance between their components: $d(x, y) = \max_i |x_i - y_i|$. Show that d satisfies the three properties of the Euclidean distance that we discussed in class (positive definiteness, symmetry, and the triangle inequality). Sketch the set of points $x \in \mathbb{R}^2$ such that $d(x, 0) = 1$.

2. A sequence $(x^k) = (x_1^k, ..., x_n^k)$ of $\mathbb{R}^n$ converges to a limit x iff each component converges to the corresponding component of x in $\mathbb{R}$.

3. Let x_n and y_n be sequences of $\mathbb{R}$ with $x_n \to x$ and $y_n \to y$. Prove that the sequence $z_n = x_n + y_n$ converges to $x + y$.

4. Is the sequence $a_n = \sum_{k=1}^{n} \frac{1}{2^k}$ Cauchy? (it might be useful to remember the geometric series formula)

5. Prove that the interior of a set is open; that is, $int(int(S)) = int(S)$.

6. Is any union of compact sets compact? Is a finite union of compact sets compact?

7. Show that the image of an open set by a continuous function is not necessarily an open set. Show that the image of an closed set by a continuous function is not necessarily a closed set.

8. Consider the sequence defined recursively by $x_1 = 2$ and $x_{n+1} = x_n/2 + 1/x_n$. You may assume $x_n \to x^*$. What is x^*? (Hint: $f(x) = x/2 + 1/x$ is a continuous function for $x > 0$)

9. Let $D \subseteq \mathbb{R}^n$. Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ with with $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say that:

   - f_n converges to f **pointwise** if for all $x \in \mathbb{R}$, $(f_n(x))$ converges to $f(x)$.
   - f_n converges to f **uniformly** if for any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that for all $n > N(\epsilon)$ and for all $x \in X$, $|f_n(x) - f(x)| < \epsilon$. (Note that N is not allowed to depend on x; it can only depend on ϵ).

   (a) Consider the sequence of functions $\{g_n\}$ defined by $g_n(x) = x^n$ defined on the closed interval $D = [0, 1]$. Does this sequence converge pointwise? If so, give its limit.

   (b) Does $\{g_n\}$ converge uniformly?

   (c) If a sequence of continuous functions converges pointwise, must its limit be continuous?