MA Math Camp Exam 2020 Solutions

Instructions:

- This is a 48 hour take home exam.
- You may use any results covered in the class directly without proofs.
- All answers must be justified.
- You may only consult the slides and lecture material for this exam. **If any indication of cheating is suspected, 10 points will be deducted for every suspected answer copied from the internet or from your classmates.** No chance of explanation will be given.
- Please write your answers clearly. Points will be deducted for bad handwriting.

1. (**5 points**) Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose f is increasing i.e. $x \geq y$ implies $f(x) \geq f(y)$. Prove that f is both quasiconvex and quasiconcave.

   Solution : Take any $x, y \in \mathbb{R}$. WLOG suppose $x \geq y$. Moreover $x \geq \lambda x + (1-\lambda)y \geq y \implies f(x) \geq f(\lambda x + (1-\lambda)y) \geq f(y)$ since f is increasing. Hence we have that :

   $$\max\{f(x), f(y)\} = f(x) \geq f(\lambda x + (1-\lambda)y) \geq f(y) = \min\{f(x), f(y)\}$$

   Hence f is quasiconcave and quasiconvex.

2. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$.

   (a) (**3 points**) Find the inverse of the matrix $\boldsymbol{A}$.

   Solution : The inverse is $A^{-1} = \begin{pmatrix} 5/12 & -1/12 \\ 1/6 & 1/6 \end{pmatrix}$

   (b) (**3 points**) Find the eigen values of this matrix

   Solution : The eigen values are $\lambda = 3, 4$

(c) (**3 points**) Find a set of linearly independent eigenvectors such that the Euclidean norm of each eigen-vector equals 1.

Solution : The set of linearly independent eigen vectors is :

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

(d) (**3 points**) Find a matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$

Solution : $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

(e) (**3 points**) Consider a sequence of matrices $\boldsymbol{A^1}$,$\boldsymbol{A^2}, \ldots, \boldsymbol{A^n}, \ldots$ where the n^{th} element of the sequence is the n^{th} power of $\boldsymbol{A}$. Does there exist a matrix $\boldsymbol{B}$ such that each of the elements of $\boldsymbol{A^n}$ converges to the corresponding element of $\boldsymbol{B}$ as $n \to \infty$

Solution : No. For any $n \in \mathbb{N}$ we have :

$$A^n = PD^nP^{-1} = P \begin{pmatrix} 3^n & 0 \\ 0 & 4^n \end{pmatrix} P^{-1}$$

which goes to infinity as $n \to \infty$

3. Are the following statements true or false? If you think it is true, provide a sketch of a proof. If you think it is false, provide a counterexample.

(a) (**5 points**) The intersection of 2 compact sets is a compact set.

Solution : True. For instance, using the sequential definition of compactness: Let K_1, K_2 be two compact sets of a metric space (X, d). Consider a sequence (x_n) of $K_1 \cap K_2$. Since (x_n) is a sequence of K_1 and K_1 is compact, it has a subsequence (x_{n_k}) that converges to $l \in K_1$. Since (x_{n_k}) is a sequence ofK_2 and K_2 is closed (because it is compact), $l \in K_2$. Hence (x_{n_k}) is a subsequence of (x_n) that converges in $K_1 \cap K_2$

(b) (**5 points**) The inverse image of a compact set by a continuous function is a compact set.

Solution : False. Consider $f(x) = 0 \ \forall x \in \mathbb{R}$. The set $\{0\}$ is compact, but $f^{-1}(\{0\}) = \mathbb{R}$ which is not compact.

(c) (**5 points**) Suppose $f : X \to Y$ is discontinuous at $x \in X$. Suppose $g : Y \to Z$ is a continuous function. Then $g \circ f : X \to Z$ is discontinous at x.

Solution : False. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ and $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = 1$. Thus $g \circ f(x) = 1 \ \forall x \in \mathbb{R}$ which is continuous.

4. On a non-empty set X, define the discrete metric d, as :

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

(a) (**5 points**) Verify that d is indeed a valid metric

Solution : Positive definiteness and symmetry are trivial. For triangle inequality, take $x, y, z \in X$. If $x = y$, then we have that $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then we must have that $d(x, y) = 1 \leq d(x, z) + d(z, y)$ since either $d(x, z) = 1$ or $d(z, y) = 1$.

(b) (**5 points**) Show that any subset of X is both open and closed

(i) Take any $S \subset X$. Consider any $x \in S$, then if we can show that it is an interior point, we are done. Now, by definition $B_1(x) = \{x\} \subset S$. Hence S is open.
(ii) To show that S is closed, we need to show that $S \supset S'$. We will show here that the set of limit points, $S' = \emptyset$. Take any $x \in X$. As before since, $B_1(x) = \{x\}$, we have that $B_1(x)/\{x\} \cap S = \emptyset$. Hence no point $x \in X$ is a limit point of S. Since $\emptyset \subset S$, we are done.

(c) (**5 points**) Show that a set S in X is compact if and only if it is finite.

Solution : $\implies$ Suppose S is compact. We know this implies that every open cover has a finite subcover. In particular consider the open cover $\{B_1(x)\}_{x \in S}$. Since S is compact, $\exists$ a finite set $\hat{S} \subset S$ s.t. $\{B_1(x)\}_{x \in \hat{S}}$ is also an open cover of S. But we know that :

$$\bigcup_{x \in \hat{S}} B_1(x) = \bigcup_{x \in \hat{S}} \{x\} = \hat{S} \supset S$$

This implies $\hat{S} = S$ which implies S is a finite set.
$\impliedby$ Suppose $S \subset X$ is finite. We need to show that it is compact. Take any open cover $\{E_\alpha\}_{\alpha \in A}$ of S. For each $x \in S$, $\exists \alpha_x \in A$ s.t. $x \in E_{\alpha_x}$. Then $\bigcup_{\alpha_x} E_{\alpha_x} \supset S$ by construction. Since S is finite, $\{E_{\alpha_x}\}_{x \in S}$ is a finite subcover.

5. (**10 points**) Consider the following maximization problem :

$$\max_{x \in \mathbb{R}^{\ltimes}} u(x)$$
$$s.t. \quad p \cdot x \leq w$$

where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $w \in \mathbb{R}$. (Note that $p \cdot x = \sum_{i=1}^{n} p_i x_i$). Suppose that the maximization problem has a solution for any (p, w) in some convex set $S \subset \mathbb{R}^n \times \mathbb{R}$. Show that the value function $v : S \to \mathbb{R}$ defined as $v(p, w) := \max_{x \in \mathbb{R}^n} \{u(x) \text{ s.t. } p \cdot x \leq w\}$ is quasiconvex.
(Hint : Kuhn Tucker is not required! Work with the definition of quasiconvexity)

Solution : For any (p, w) and $(p', w') \in S$, we want to show $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq \max\{v(p, w), v(p', w')\}$.

Notice that $(\lambda p + (1-\lambda)p') \cdot x \leq \lambda w + (1-\lambda)w'$ implies either $p \cdot x \leq w$ or $p' \cdot x \leq w'$. Then any feasible point under $(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w')$ must be feasible under either (p, w) or (p', w') and therefore either $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq v(p, w)$ or $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq v(p', w')$ and hence :

$$v\left(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w'\right) \leq \max\left\{v(p, w), v\left(p', w'\right)\right\}$$

6. Consider the function $f(x_1, x_2) = x_1^\alpha + x_2^\alpha$ defined for $x_1, x_2 \geq 0$ and $\alpha \in (0, 1]$. **Do notice that $\alpha = 1$ is included.** For a fixed α, consider the problem:

$$\max_{x_1, x_2} x_1^\alpha + x_2^\alpha$$
$$s.t. \; x_1 + x_2 = 1$$

(a) (**5 points**) Show that f is concave in (x_1, x_2) for a fixed α. (Hint : Use the result that a finite sum of concave functions is concave)

Solution : x^α is concave since $\alpha \in (0, 1]$. Hence f is concave since it is a finite sum of concave functions.

(b) (**5 points**) Does a solution exist for this maximization problem for all values of $\alpha \in (0, 1]$

Solution : The objective function is continuous and the constraint set $S = \{(x_1, x_2) \,|x_1 + x_2 = 1\}$ is closed and bounded and therefore compact. Hence a solution exists by Weierstrass' Theorem.

(c) (**15 points**) For a fixed α, find the solution to this maximization problem. (Hint : Consider separate cases for $\alpha < 1$ and $\alpha = 1$).

Solution : Consider the case $\alpha < 1$. Solving the FOCs of the Lagrangean, we get : $x_1 = x_2 = \frac{1}{2}$ is the only solution.

In the case $\alpha = 1$, there are infinite solutions since any x_1, x_2 such that $x_1 + x_2 = 1$ is a solution.

(d) (**5 points**) Do you need to check for the Second Order Conditions in this problem? Why or why not?

Solution : No. The objective function f is concave and the constraint is linear. Hence the Lagrangean is concave as well which is a sufficient condition for global maxima.

(e) (**5 points**) Find the value function $V^*(\alpha)$

Solution : $V^*(\alpha) = 2\left(\frac{1}{2}\right)^{\alpha}$

(f) (**5 points**) Verify that the Envelope theorem holds in this problem.

The Envelope theorem tells us that $V'(\alpha) = \mathcal{L}_{\alpha}\left(x_1^*(\alpha), x_2^*(\alpha), \alpha\right)$. Note that $\mathcal{L}_{\alpha}\left(x_1^*(\alpha), x_2^*(\alpha), \alpha\right)$ is the derivative of the Lagrangean with respect to α evaluated at $(x_1^*(\alpha), x_2^*(\alpha))$.

Solution : $V'(\alpha) = 2\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\alpha}$
$L_2'(x, \alpha) = \ln(x_1)\, x_1^{\alpha} + \ln(x_2)\, x_2^{\alpha}$.

This implies $L_2'(x^*(\alpha), \alpha) = 2\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\alpha}$