MA Math Camp Exam: Thur. 8 Sep.

Instructions: 1. The exam is from 1:10 - 2:25 pm.

2. You may use any results covered in the class directly without proofs.

3. The exam is closed book. No calculators are allowed.

1 Matrix Diagonalization (10 Points)

$$\mathbf{A} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

Can you find a square matrix $\mathbf{P}$ such that $\mathbf{P^{-1}AP} = \mathbf{D}$, where $\mathbf{D}$ is diagonal? If yes, find such a $\mathbf{P}$; if not, explain why.

Can you find an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q'AQ} = \mathbf{D}$, where $\mathbf{D}$ is diagonal? If yes, find such a $\mathbf{Q}$; if not, explain why.

Solution: Yes. Since $\mathbf{A}$ is a symmetric matrix, so there exists an orthogonal matrix $\mathbf{Q}$ to diagonalize $\mathbf{A}$.

The eigenvalues of A are 2 and 6. And the eigenvectors associated are respectively $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})'$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})'$.

Define $\mathbf{Q} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$; $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$,
we have $\mathbf{Q'AQ} = \mathbf{D}$.

And since an orthogonal matrix $\mathbf{Q}$ is found, a square matrix exists as well, just let $\mathbf{P} = \mathbf{Q}$.

2 Log-linearization (10 Points)

Log-linearize the following equation (it is a budget constraint in dynamic models) around $(d^*, c^*, i^*, k^*, h^*)$. (r, A, α, ϕ) are parameters.

$d_t = (1+r)d_{t-1} - Ak_t^\alpha h_t^{1-\alpha} + c_t + i_t + \frac{\phi}{2}(k_{t+1} - k_t)^2.$

(Hint: t here stands for time. Hence d_t and d_{t-1} have the same equilibrium value d^*, k_t and k_{t+1} have the same equilibrium value k^*).

Solution: Take the total derivative on both sides:

$dd_t = (1+r)dd_{t-1} - Ad(k_t^\alpha h_t^{1-\alpha}) + dc_t + di_t + \frac{\phi}{2}d[(k_{t+1} - k_t)^2]$

Replace $dx_t = \hat{x}_t x^*$:

$\hat{d}_t d^* = (1+r)\hat{d}_{t-1}d^* - A(k^{*\alpha}h^{*1-\alpha})[\alpha\hat{k}_t + (1-\alpha)\hat{h}_t] + \hat{c}_t c^* + \hat{i}_t i^* + \frac{\phi}{2}d[(k^* - k^*)^2][(\widehat{k_{t+1} - k_t})^2]$

Remove the last term because it is zero:

$\hat{d}_t d^* = (1+r)\hat{d}_{t-1}d^* - Ak^{*\alpha}h^{*1-\alpha}[\alpha\hat{k}_t + (1-\alpha)\hat{h}_t] + \hat{c}_t c^* + \hat{i}_t i^*$

3 Optimization (20 Points)

A firm produces two goods: A and B. Good A is the numeraire, price of which is normalized to 1. The price of good B is $p > 0$. The production process requires a single input C.

To produce x units of A, it needs $2x^2$ units of C. To produce y units of B, it needs $3y^2$ units of C.

Now the firm has only $z > 0$ units of C. The problem the firm faces is how to allocate the production resource between two goods to maximize the monetary value of its production.

(a) State the mathematical problem the firm faces. Specify choice variables and parameters.

(b) Does this problem have a solution, why?

(c) State all the candidates that could possibly be a solution.

(d) If your answer to (b) is yes, then give the solution, and calculate the monetary value of the production. If your answer to (b) is no, modify one constraint the firm faces. Make sure that the problem after your modification has a solution.

Solution:

$$\max_{x,y} \quad x + py$$
$$\text{s.t.} \quad 2x^2 + 3y^2 \leq z$$
$$x \geq 0, y \geq 0$$

In this model, p and z are parameters, x and y are choice variables.

So the constraint set is bounded and closed, so according to Weierstrass theorem, there must be a solution.

To maximize this problem, the inequality constraint $2x^2+3y^2 \leq z$ must bind. So it becomes a equality constraint: $2x^2 + 3y^2 = z$.

Let's divide the problem into three cases (hence three candidates):

1. $x = 0$, $y \neq 0$, according to the equality constraint, $y = \sqrt{\frac{z}{3}}$.

In this case: the candidate for maximum point is $(0, \sqrt{\frac{z}{3}})$, and monetary value of production is $f_1^* = p\sqrt{\frac{z}{3}}$.

2. $y = 0$, $x \neq 0$, according to the equality constraint, $x = \sqrt{\frac{z}{2}}$.

In this case: the candidate for maximum point is $(\sqrt{\frac{z}{2}}, 0)$, and monetary value of production is $f_2^* = \sqrt{\frac{z}{2}}$.

3. $x \neq 0, y \neq 0$, then the Lagrangian is $\mathcal{L} = x + py + \lambda(z - 2x^2 - 3y^2)$.

The first-order conditions are:

$1 - 4\lambda x = 0 \implies x = \frac{1}{4\lambda}$

$p - 6\lambda y = 0 \implies y = \frac{p}{6\lambda}$

Plug this result into the equality constraint, we can solve for $\lambda = \frac{\sqrt{3+2p^2}}{\sqrt{24z}}$.

In this case: the candidate for maximum point is $(\frac{\sqrt{6z}}{2\sqrt{3+2p^2}}, p\frac{\sqrt{6z}}{3\sqrt{3+2p^2}})$, and monetary value of production is $f_0^* = \frac{\sqrt{3+2p^2}}{\sqrt{6}}\sqrt{z}$.

Now let's compare three monetary values of production:

If $f_0^* \leq f_1^*$, then we will get $\frac{\sqrt{3+2p^2}}{\sqrt{6}} \leq \frac{p}{\sqrt{3}} \implies 3 \leq 0$, contradiction.

If $f_0^* \leq f_2^*$, then we will get $\frac{\sqrt{3+2p^2}}{\sqrt{6}} \leq \frac{1}{\sqrt{2}} \implies p \leq 0$, contradiction.

So the candidates in case 1 and case 2 cannot be maximum points. The maximum point is $(\frac{\sqrt{6z}}{2\sqrt{3+2p^2}}, p\frac{\sqrt{6z}}{3\sqrt{3+2p^2}})$, and monetary value of production is $f_0^* = \frac{\sqrt{3+2p^2}}{\sqrt{6}}\sqrt{z}$.

4 Convexity (15 Points)

Consider a 3-variable real valued function

$f(x, y, z) = -3x^3 - 6xy - y^3 + z^3$

Is there a subset of $A \subseteq \mathbb{R}^3$ such that f is **strictly convex** on A? If yes, give such a subset. If no, explain why.

Solution: Hessian matrix of f is $H_f(x) = \begin{bmatrix} -18x & -6 & 0 \\ -6 & -6y & 0 \\ 0 & 0 & 6z \end{bmatrix}$.

For f to be strictly convex, we need all the leading principal minors to be strictly positive.

$k = 1$, $-18x > 0 \implies x < 0$;

$k = 2$, $18 \times 6xy - 36 > 0 \implies xy > \frac{1}{3}$, then $y < 0$;

$k = 3$, $6z * (18 \times 6xy - 36) > 0 \implies z > 0$;

So A does exist: $A = \{(x, y, z) : x < 0, y < 0, z > 0, xy > \frac{1}{3}\}$, such that f is **strictly convex** on A.

5 Homogeneous Function (15 Points)

Let f be a function, $f : \mathbb{R}^n \to \mathbb{R}$. f is homogeneous of degree $k \in \mathbb{N}$, and $k > 1$. f is also in $\mathcal{C}^2$ (meaning f is twice differentiable). Show that

(a) $f(\mathbf{0}) = 0$;

(b) $f'(\mathbf{0}) = \mathbf{0}$ (here $\mathbf{0}$ is a n vector, and 0 is a scalar).

(Hint: Euler's Homogeneous Function Theorem)

Proof: Since f is homogeneous of degree k and f is differentiable, so according to Euler Theorem: $kf(\mathbf{x}) = f'(\mathbf{x}) \cdot \mathbf{x}$,

Let $\mathbf{x} = \mathbf{0}$, then $kf(\mathbf{x}) = 0$. Since $k \neq 0$, we have $f(\mathbf{x}) = 0$.

The first derivative f' is homogeneous of degree $k - 1$ and also differentiable. The difference is that f' is no longer a real valued function but a vector of function, of which every entry is an homogeneous function of degree $(k - 1)$. So by using the Euler theorem again on any entry f'_i, we get $(k - 1)f'_i(\mathbf{x}) = f''_i(\mathbf{x}) \cdot \mathbf{x}$.

So for all $i = 1, ..., n$, $f'_i(\mathbf{x}) = 0$.

Therefore $f'(\mathbf{0}) = \mathbf{0}$.

6 Integration (10 Points)

Solve the following integrations:

(a) $\int_1^2 x^2 \ln(x) dx$

(b) $\int_0^1 x\sqrt{x+1} dx$

(Hint: $(x^a)' = ax^{a-1}$, $(\ln(x))' = \frac{1}{x}$. Apply the strategy of integration by parts)

Solution:

(a)

$$\int_1^2 x^2 \ln(x) dx$$

$$= \frac{1}{3} \int_1^2 \ln(x) dx^3$$

$$= \frac{1}{3} x^3 \ln(x)|_1^2 - \frac{1}{3} \int_1^2 x^3 \times \frac{1}{x} dx$$

$$= \frac{8}{3} \ln(2) - \frac{1}{3} \int_1^2 x^2 dx$$

$$= \frac{8}{3} \ln(2) - \frac{1}{9} x^3|_1^2$$

$$= \frac{8}{3} \ln(2) - \frac{7}{9}$$

(b)

$$\int_0^1 x\sqrt{x+1}dx$$

$$= \int_0^1 x\sqrt{x+1}d(x+1)$$

$$= \frac{2}{3}\int_0^1 xd(x+1)^{3/2}$$

$$= \frac{2}{3}x(x+1)^{3/2}|_0^1 - \frac{2}{3}\int_0^1 (x+1)^{3/2}dx$$

$$= \frac{4}{3}\sqrt{2} - \frac{2}{3}\int_0^1 (x+1)^{3/2}d(x+1)$$

$$= \frac{4}{3}\sqrt{2} - \frac{2}{3} \times \frac{2}{5}(x+1)^{5/2}|_0^1$$

$$= \frac{4}{3}\sqrt{2} - \frac{4}{15}(4\sqrt{2}-1)$$

$$= \frac{4}{15}\sqrt{2} + \frac{4}{15}$$