MA Math Camp Exam: Tuesday, September 5

Instructions:

- The exam is from 1:10 - 2:25 pm.
- You may use any results covered in the class directly without proofs.
- The exam is closed book. No calculators are allowed.

1 Linear Algebra

1. Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

   (a) Do the columns of A form a basis for $\mathbb{R}^3$?

   (b) Is A invertible?

   (c) Let $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$. How many solutions does the equation $Ax = b$ have? (Note, you do not need to give the solution(s), if any).

2. Now consider the matrix:

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

   (a) What are the eigenvalues of A?

   (b) Is A invertible?

   (c) Diagonalize A, if possible.

2 Log-Linearization

Log-linearize the following equation around the steady state π^*, r^* (ϕ, α, and ϵ are fixed parameters):

$$1 + \pi_t = \left(\phi + \alpha(1 + r_t)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}$$

3 Calculus

1. Consider the function given by:

$$f(x) = \int_{x}^{x^2} g(t)dt$$

   What is $f'(x)$?

2. Suppose f, g are twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If f and g are increasing and convex, is the composite function $(f \circ g)(x) \equiv f(g(x))$ increasing and convex? If so demonstrate why; if not provide a counterexample.

4 Optimization

A consumer is trying to maximize her utility from work and consumption. She can work any amount L hours at a wage $w > 0$. She can spend her wages on two goods; let c_1 and c_2 represent her consumption of goods 1 and 2, respectively. Let $p_1 > 0$ and $p_2 > 0$ be the prices of goods 1 and 2, respectively.

The consumer has no other form of wealth, so total expenditures must be less than or equal to labor income. Her utility from consumption and labor is:

$$u(c_1, c_2, L) = \alpha \ln(c_1) + c_2 - \frac{\gamma}{2}L^2,$$

where $\alpha > 0$ and $\gamma > 0$ are parameters describing her like for good 1 and her dislike of work. Assume $\alpha < w^2/(\gamma p_2^2)$.

(a) State the consumer's maximization problem and budget constraint. Be clear about what variables the consumer is free to choose.

(b) Write down the Lagrangian. (You will only need one Lagrange multiplier for the budget constraint; do not worry about any positivity constraints)

(c) Are there any points where the constraint qualification fails?

(d) Solve the program (that is, solve for c_1, c_2, and L in terms of w, p_1, p_2, α, and γ).

5 Analysis

Indicate whether the following statements are true or false. If true, provide a brief proof. If not, provide a counterexample:

(a) If d_1 and d_2 are metrics on a metric space X, then the metric d defined by $d(x, y) \equiv d_1(x, y) + d_2(x, y)$ is a metric as well

(b) If f is a continuous function and S is a closed set, then $f(S)$ is closed