Math Camp: Problem Set 1 Suggested Solutions

1 Set Theory

1. Prove $A \cap B = A$ if and only if $A \subseteq B$

   Solution: ($\Rightarrow$) Suppose $A \cap B = A$, and let $a \in A$. Since $A = A \cap B$, $a \in A \cap B$, so by definition, $a \in B$. Therefore $A \subseteq B$.

   ($\Leftarrow$) Suppose $A \subseteq B$. Since $A \cap B \subseteq A$ always, we need only show that $A \subseteq A \cap B$. Let $a \in A$. Since $A \subseteq B$, $a \in B$, so $a \in A \cap B$, meaning $A \subseteq A \cap B$. Thus we've shown that $A \cap B = A$.

2. Prove the intersection operator is associative: $(A \cap B) \cap C = A \cap (B \cap C)$ (hint, show set containment both ways)

   Solution: We show the two sets are equivalent:

$$\begin{aligned} a \in (A \cap B) \cap C &\Leftrightarrow a \in A \cap B \text{ and } a \in C \\ &\Leftrightarrow a \in A \text{ and } a \in B \text{ and } a \in C \\ &\Leftrightarrow a \in A \text{ and } a \in B \cap C \\ &\Leftrightarrow a \in A \cap (B \cap C) \end{aligned}$$

3. Show the second of DeMorgan's Laws:

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

   Solution: Let $B_i = A_i^c$. According to the first law:

$$(B_1 \cup B_2)^c = B_1^c \cap B_2^c$$

   Complementing both sides of this equation gives

$$B_1 \cup B_2 = (B_1^c \cap B_2^c)^c$$

   Finally, using the fact that $B_i^c = (A_i^c)^c = A_i$, we see:

$$B_1 \cup B_2 = (A_1 \cap A_2)^c$$

   which is the second law.

4. Let X and Y be two sets and $f : X \to Y$. Find an example in which $f(S_1 \cap S_2) \subsetneq f(S_1) \cap f(S_2)$.

   Solution: Let $X = \{x_1, x_2\}$ and let $Y = \{y_1\}$. Define $S_1 = \{x_1\}$ and $S_2 = \{x_2\}$, and suppose $f(x_1) = f(x_2) = y_1$.
   With this setup, $f(S_1 \cap S_2) = f(\emptyset) = \emptyset$, while

   $$f(S_1) \cap f(S_2) = \{y_1\} \cap \{y_1\} = \{y_1\}$$

5. Let X and Y be two sets and $f : X \to Y$. Prove that:

   - $f\left(f^{-1}(T)\right) = T$ for all $T \subseteq Y$ if and only if f is surjective.

     Solution: ($\Rightarrow$) Suppose $f(f^{-1}(T)) = T$ for all $T \subseteq Y$. We show f is surjective by contradiction. Let $y \in Y$, and suppose there is no x such that $f(x) = y$. Then $f^{-1}(\{y\}) = \emptyset$, so $f(f^{-1}(y)) = \emptyset$. However, this contradicts the fact that $f(f^{-1}(\{y\})) = \{y\}$, so f must be onto.

     ($\Leftarrow$) Now suppose f is surjective, and let T be a subset of Y. We show subset containment both ways:

     – Let $y \in T$. Since f is surjective, there exists an x such that $f(x) = y$. Thus $f(x) \in T$, so $x \in f^{-1}(T)$, meaning $f(x) \in f(f^{-1}(T))$. Since $y = f(x)$, we have $y \in f(f^{-1}(T))$. We've shown $T \subseteq f(f^{-1}(T))$.
     – Let $y \in f(f^{-1}(T))$. By definition $y = f(x)$ for some $x \in f^{-1}(T)$. Since $x \in f^{-1}(T)$, we have $f(x) \in T$. Since $y = f(x)$, $y \in T$. Thus we've shown $f(f^{-1}(T)) = T$ (note we didn't need surjectivity for this part of the proof), so we're done.

   - $f^{-1}(f(S)) = S$ for all $S \subseteq X$ if and only if f is injective

     Solution: ($\Rightarrow$) Suppose $f^{-1}(f(S)) = S$ for all $S \subseteq X$. We show f is injective. Suppose $f(x_1) = f(x_2)$. Then $f^{-1}(f(\{x_1\})) = f^{-1}(f(\{x_2\}))$. By assumption, $f^{-1}(f(S)) = S$, so we find $\{x_1\} = \{x_2\}$, meaning f is 1-1.

     ($\Leftarrow$) Now suppose f is injective, and let S be a subset of X. We show subset containment both ways:

     – Let $x \in S$. Then $f(x) \in f(S)$ by definition, and $x \in f^{-1}(f(S))$ by definitions. Thus $S \subseteq f^{-1}(f(S))$ (note we did not need injectivity here)
     – Let $x \in f^{-1}(f(S))$. By definition, $f(x) = y \in f(S)$. Since $y \in f(S)$, $y = f(z)$ for some $z \in S$. However, since f is 1-1, $f(x) = y = f(z)$ implies $x = z$, so $x \in S$.

6. Let R be a complete, transitive relation over a set X, and define the relation $\sim$ as follows: $a \sim b$ if and only if aRb and bRa. Let $I(x)$ be the collection $I(x) = \{y | y \sim x\}$.

   Show that for all x and y, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.

Solution: We first note that $\sim$ is transitive: suppose $x \sim y$ and $y \sim z$. By definition, xRy and yRz, so by the transitivity of R we have xRz. Additionally, because zRy and yRx, we conclude zRx. Thus by definition, $x \sim z$ (also, by definition, $z \sim x$, i.e., $\sim$ is symmetric)

Let $x, y \in X$. We have two cases:

- Suppose $x \sim y$. We show $I(x) = I(y)$. Let $z \in I(x)$. By definition, $z \sim x$, and by assumption $x \sim y$, so $z \sim y$, meaning $z \in I(y)$. By the same logic, we can show $I(y) \subseteq I(x)$, meaning $I(y) = I(x)$.
- Suppose $x \not\sim y$. We show $I(x) \cap I(y) = \emptyset$. Suppose $z \in I(x) \cap I(y)$. Then $z \sim x$ and $z \sim y$, which, by transitivity, implies $x \sim y$, contradicting our assumption. Thus $z \notin I(x) \cap I(y)$, so $I(x) \cap I(y) = \emptyset$.

2 Analysis

1. Let $x, y \in \mathbb{R}^2$ and define $d(x, y)$ to be the maximum distance between their components: $d(x, y) = \max_i |x_i - y_i|$. Show that d satisfies the three properties of the Euclidean distance that we discussed in class (positive definiteness, symmetry, and the triangle inequality). Sketch the set of points $x \in \mathbb{R}^2$ such that $d(x, 0) = 1$.

   Solution:

   - Positive definiteness follows from the fact that the absolute value is positive definite
   - Symmetry follows from the fact that $|x_i - y_i| = |y_i - x_i|$
   - Triangle Inequality:

   $$\begin{aligned} d(x,z) &= \max(|x_1 - z_1|, |x_2 - z_2|) \\ &= \max(|x_1 - y_1 + y_1 - z_1|, |x_2 - y_2 + y_2 - z_2|) \\ &\leq \max(|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|) \\ &\leq \max(|x_1 - y_1|, |x_2 - y_2|) + \max(|y_1 - z_1|, |y_2 - z_2|) \\ &= d(x,y) + d(y,z) \end{aligned}$$

     In the second line we have cleverly added zero. The third line uses the triangle inequality for the absolute value function. The fourth line uses the fact that $\max(a+b, c+d) \leq \max(a, c) + \max(b, d)$.

     The set of points such that $d(x, 0) = 1$ is a square with edges $(\pm 1, \pm 1)$.

2. Let (x_n) be a sequence in $(\mathbb{R}^k, d)$. The sequence (x_n) converges to $x \in \mathbb{R}^k$ iff the sequence (x_n^i) converges to x^i in $(\mathbb{R}, d)$ for any $i \in 1, 2, \ldots, k$ (Note that we use superscript to index coordinates of vectors, since we used subscript to index terms of sequences)

   Solution: **$(\Longrightarrow)$:** Take any $i \in \{1, 2, \ldots, k\}$. WTS : $x_n^i \to x^i$ in $(\mathbb{R}, d)$. Take any $\varepsilon > 0$, we want to find N^i st $d\left(x_n^i, x^i\right) < \varepsilon$ for any $n > N^i$. Because $x_n \to x$, there exists N s.t.

$d(x_n, x) < \varepsilon$ for any $n > N$. Let $N^i := N$ and I claim that this is the N we need. This is because for any $n > N^i = N$, we have :

$$\begin{aligned} d\left(x_n^i, x^i\right) &= \left|x_n^i - x^i\right| = \sqrt{\left(x_n^i - x^i\right)^2} \\ &\leq \sqrt{\sum_{j=1}^{k}\left(x_n^j - x^j\right)^2} = d(x_n, x) < \varepsilon \end{aligned}$$

($\Longleftarrow$) : Take any $\varepsilon > 0$, we want to find N s.t. $d(x_n, x) < \varepsilon$ for any $n > N$. Because $x_n^i \to x^i$, there exists N^i st $d\left(x_n^i, x^i\right) < \varepsilon/\sqrt{k}$ for any $n > N^i$. Let $N := \max\{N_1, \ldots, N_k\}$. I claim that this is the N we need since for any $n > N$, we have $n > N^i$ and thus $d\left(x_n^i, x^i\right) < \varepsilon/\sqrt{k}$ for any i and therefore :

$$d(x_n, x) = \sqrt{\sum_{j=1}^{k}\left(x_n^j - x^j\right)^2} < \sqrt{k\left(\varepsilon/\sqrt{k}\right)^2} = \varepsilon$$

3. Let x_n and y_n be sequences of $\mathbb{R}$ with $x_n \to x$ and $y_n \to y$. Prove that the sequence $z_n = x_n + y_n$ converges to $x + y$.

   Solution: Fix $\epsilon > 0$. Since $x_n \to x$, there exists N_1 such that for all $n \geq N_1$, $d(x_n, x) < \epsilon/2$. Similarly, there exists N_2 such that for all $n \geq N_2$, $d(y_n, y) < \epsilon/2$.

   Define $N = \max(N_1, N_2)$. Then for all $n \geq N$:

$$\begin{aligned} d(x_n + y_n, x + y) &= |x_n + y_n - (x + y)| \\ &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

4. Is the sequence $a_n = \sum_{k=1}^{n} \frac{1}{2^k}$ Cauchy? (it might be useful to remember the geometric series formula)

   Solution: Fix $\epsilon > 0$ and let N be such that $1/2^N < \epsilon$. Let $n, m \geq N$ with $n < m$. Then:

$$\begin{aligned} d(a_n, a_m) &= \sum_{k=n+1}^{m} \frac{1}{2^k} \\ &= \frac{1}{2^n}\sum_{k=1}^{m-n} \frac{1}{2^k} \\ &< \frac{1}{2^n}\sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= \frac{1}{2^n} \\ &< \epsilon \end{aligned}$$

Therefore a_n is Cauchy. In the fourth line I've used the fact that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

5. Prove that the interior of a set is open; that is, $int(int(S)) = int(S)$.

   Solution: Since the interior of a set is always a subset of that set, we need only show $int(S) \subseteq int(int(S))$. Let $x \in int(S)$. By definition, there exists a radius $r > 0$ such that $B(x,r) \subseteq S$. However, since $B(x,r)$ is open, for any $x' \in B(x,r)$ we can find a radius r' such that $B(x',r') \subseteq B(x,r) \subseteq S$. Therefore $x' \in int(S)$, so $B(x,r) \subseteq int(S)$, so $x \in int(int(S))$.

6. Is any union of compact sets compact? Is a finite union of compact sets compact?

   Solution: An arbitrary union of compact sets need not be compact. For instance, in $\mathbb{R}$, $\cup_{i=1}^{\infty}[i, i+1] = [1, \infty)$, which is not bounded and therefore not compact.

   A finite union of compact sets is compact. Let $S = \cup_{i=1}^{n} S_i$, where S_i are compact subsets of a $\mathbb{R}^n$. Consider a sequence (x_n) of S; we want to show it has a convergent subsequence.

   The sequence (x_n) must have an infinite number of terms in a least one of the sets S_i (if not, it has a finite number of terms in a finite number of sets, and is therefore finite). Let (x_{n_k}) be the subsequence of (x_n) with terms only in S_i. Since S_i is compact, (x_{n_k}) has a convergent subsequence $x_{n_{k_l}}$, which means x_n has a convergent subsequence, so S is compact.

7. Show that the image of an open set by a continuous function is not necessarily an open set. Show that the image of an closed set by a continuous function is not necessarily a closed set.

   Solution: For the first, take $f(x) = x^2$ over $(-1, 1)$. For the second, take $f(x) = 1/(1 + x)$ over $[0, \infty)$.

8. Consider the sequence defined recursively by $x_1 = 2$ and $x_{n+1} = x_n/2 + 1/x_n$. You may assume $x_n \to x^*$. What is x^*? (Hint: $f(x) = x/2 + 1/x$ is a continuous function for $x > 0$)

   Solution: We first note that x_n is monotonic. Define $f(x) = x/2 + 1/x$. This function has derivative $f'(x) = 1/2 - 1/x^2$, so for $x \geq \sqrt{2}$, the function is increasing. Thus for $x_n \geq \sqrt{2}$, $f(x_n) \geq f(\sqrt{2}) = \sqrt{2}$ So, by induction, $x_n \geq \sqrt{2}$ for all n.

   To find the limit, note $(x_n) \to x^*$. Since f is continuous, $f(x_n) \to f(x^*)$. However, $(f(x_n)) = (x_{n+1})$, so we must have $f(x^*) = x^*$, or $l = \sqrt{2}$. (we can't have $l = -\sqrt{2}$ since $x_n \geq \sqrt{2}$ for all n).

9. Let $D \subseteq \mathbb{R}^n$. Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ with with $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$, we say that:

   - f_n converges to f **pointwise** if for all $x \in \mathbb{R}$, $(f_n(x))$ converges to $f(x)$.
   - f_n converges to f **uniformly** if for any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that for all $n > N(\epsilon)$ and for all $x \in X$, $|f_n(x) - f(x)| < \epsilon$. (Note that N is not allowed to depend on x; it can only depend on ϵ).

(a) Consider the sequence of functions $\{g_n\}$ defined by $g_n(x) = x^n$ defined on the closed interval $X = [0,1]$. Does this sequence converge pointwise? If so, give its limit.

Solution: This sequence does converge pointwise, to the function:

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

To show this, fix $x \in [0,1)$ and fix $\epsilon > 0$. Let $N > \log(\epsilon)/\log(x)$. Then for $n \geq N$, $g_n(x) = x^n < x^{\frac{\log(\epsilon)}{\log(x)}} = \epsilon$. If $x = 1$, then $(g_n(x)) = (1,1,1,...)$, which clearly coverges to 1.

(b) Does $\{g_n\}$ converge uniformly?

Solution: No. Fix ϵ and fix N. For any $n \geq N$, we can find $x \neq 1$ such that $g_n(x) > \epsilon$ by taking x sufficiently close to 1.

(c) If a sequence of continuous functions converges pointwise, must its limit be continuous?

Solution: No. For example, in the previous problem a pointwise limit converged to a discontinuous function.