Final Exam Solutions
Columbia MA Math Camp 2021

September 13th, 2021

Instructions :

- This is a 1h15 exam (+10 minutes for scanning and uploading if the exam is taken remotely).
- Students taking the exam remotely should upload their answers on Gradescope : be careful that the 1h25 limit will be enforced automatically after you open this exam.
- The exam is closed book, it is not allowed to use any material either from the class or external. Calculators are not allowed. Students are not allowed to cooperate. Any suspicion of cheating will be taken very seriously.
- All answers must be justified. Keep in mind that answers are supposed to be short.
- You may use any result that was seen in the class without proof.
- Please write clearly and concisely.
- The exam has a total of 100 points.

1. **(25 points)** *True or False :* For each of the following statements, state whether they are true or false. Justify true statements with a short argument and provide a counterexample for false statements.

   (a) Any convergent sequence in $\mathbb{R}$ is bounded.

   Solution : True. Let (x_n) a sequence converging to x in $\mathbb{R}$. Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x - \epsilon \leq x_n \leq x + \epsilon$. Define $\overline{x}_N := \max\{x_k, k \leq N\}$ and $\underline{x}_N := \min\{x_k, k \leq N\}$, which both exist since the set is finite. By construction for any $n \geq 0$:

   $$\min\{x - \epsilon, \underline{x}_N\} \leq x_n \leq \max\{x + \epsilon, \overline{x}_N\}$$

   (b) An arbitrary intersection of open sets is open.

   Solution : False. Consider the sets indexed by $n \in \mathbb{N}$ $O_n := (-1/n, 1/n)$. Clearly they are all open but $\bigcap_{n\in\mathbb{N}} O_n = \{0\}$ which is not open.

(c) The image of a closed set by a continuous function is closed.

Solution : False. Consider for example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$. Observe that $f(\mathbb{R}) = \mathbb{R}_{++}$: clearly $\mathbb{R}$ is closed, $\mathbb{R}_{++}$ isn't, and f is continuous.

(d) For any $n \times n$ matrices A and B, $Tr(AB) = Tr(BA)$

Solution : True. Let $A = (a_{ij})$ and $B = (b_{ij})$. Observe that :

$$Tr(AB) = \sum_{k=1}^{n}(AB)_{kk} = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{kj}b_{jk}\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} b_{jk}a_{kj}\right) = \sum_{j=1}^{n}(BA)_{jj} = Tr(BA)$$

(e) The interval $[0,1]$ is open in the metric space $([0,1], d_2)$.

Solution : True. By construction the whole set is always open in any metric space.

2. **(15 points)** Let E, F, G three sets. Consider two functions $f : E \to F$, $g : F \to G$. Show that if $g \circ f$ is injective, then f is injective.

Solution : Let $x, x' \in E$ such that $f(x) = f(x')$. Taking the image by g this means that $g(f(x)) = g(f(x'))$, i.e $g \circ f(x) = g \circ f(x')$. By injectivity of $g \circ f$, this implies $x = x'$, hence f is injective.

3. **(15 points)** Consider the following matrix :

$$M := \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

(i) Is it invertible ?

Solution : Compute $det(M) = \frac{2}{3} \times \frac{2}{3} - \frac{1}{3} \times \frac{1}{3} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$. Observing that $det(M) > 0$, we conclude that M is invertible.

(ii) Is it diagonalizable in $\mathbb{R}$? If so, find a diagonal matrix Λ and an invertible matrix P such that $M = P\Lambda P^{-1}$.

Solution : We start by computing the eigenvalues of M. For $\lambda \in \mathbb{R}$, we have :

$$\begin{aligned} det(M - \lambda I_2) &= det\begin{pmatrix} \frac{2}{3} - \lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda \end{pmatrix} \\ &= \lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} \end{aligned}$$

Hence $det(M - \lambda I_2) = 0$ if and only if $\lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = 0$. The discriminant of this polynomial is given by $(4/3)^2 - 4/3 = 4/9$ hence the two (real) roots of the polynomial are given by :

$$\lambda = \frac{1}{2}\left(\frac{4}{3} \pm \sqrt{\frac{4}{9}}\right)$$

i.e M has two distincts real eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{3}$. Therefore, M is diagonalizable.

To find the change of basis matrix P, we solve for corresponding eigenvectors.

$$MX = X \Leftrightarrow \begin{cases} \frac{2}{3}x + \frac{1}{3}y = x \\ \frac{1}{3}x + \frac{2}{3}y = y \end{cases} \Leftrightarrow x = y$$

Therefore $(1,1)$ is an eigenvector for the eivenvalue 1. Similarly solving for $MX = \frac{1}{3}X$ yields that $(1,-1)$ is an eigenvector for the eigenvalue $\frac{1}{3}$. Hence defining the matrices :

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \; P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we have that $M = P\Lambda P^{-1}$.

(iii) Write a simple expression for M^k for $k \in \mathbb{N}$.

Solution : For any $k \in \mathbb{N}$, we have :

$$M^k = (P\Lambda P^{-1})^k = P\Lambda(P^{-1}P)\Lambda ... (P^{-1}P)\Lambda P^{-1} = P\Lambda^k P^{-1}$$

(You can show this result formally by induction but stating it directly is acceptable). Therefore we can write :

$$M^k = P \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{3})^k \end{pmatrix} P^{-1}$$

4. **(15 points)** Consider the following function defined on $(0,1)$:

$$\begin{aligned} f : (0,1) &\to \mathbb{R} \\ x &\mapsto -x\ln(x) \end{aligned}$$

(i) Verify that f is concave on $(0,1)$

Solution : f is twice differentiable on $(0,1)$, and we have for all $x \in (0,1)$:

$$\begin{aligned} f'(x) &= -\ln(x) - 1 \\ f''(x) &= -\frac{1}{x} \end{aligned}$$

Hence $f''(x) < 0$ for all $x \in (0,1)$ so f is (strictly) concave.

(ii) Does f have a maximum ? If so, say where it is attained and its value.

Solution : Since f is concave and defined on an open set, it is sufficient to find a critical point to find a maximum. We solve :

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow -\ln(x) - 1 = 0 \\ &\Leftrightarrow x = \frac{1}{e} \end{aligned}$$

Therefore, f attains a (unique) maximum at $x = 1/e$, whose value is $f(1/e) = (1/e)(-\ln(1/e)) = (1/e)ln(e) = 1/e$.

(iii) Show that f does not have a minimum. What is its infimum ?

Solution : Observe that $\lim_{x\to 1} f(x) = 0$ because the function $-x\ln(x)$ defined on $\mathbb{R}_{++}$ is actually continuous at 1. But on $(0,1)$, we have $f(x) > 0$, therefore f does not have an minimum and $\inf_{x\in(0,1)} f(x) = 0$.

5. **(30 Points)** Consider the following constrained optimization problem :

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} x_1^\alpha x_2^\beta$$

$$\text{subject to : } p_1x_1 + p_2x_2 \leq m$$

where $\alpha, \beta \in (0,1)$, $p_1, p_2, m \in \mathbb{R}_{++}$, are parameters.

(a) Does a solution exist ?

Solution : The objective function is continuous and the constraint set is compact – the latter can be shown directly by proving that it is closed and bounded, see the proof of that exact case in the lecture notes. Therefore, the maximization problem has a solution by Weierstrass Theorem.

(b) Argue that neither of the non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ can bind at a maximum.

Solution : Consider (x_1, x_2) a feasible point such that either $x_1 = 0$ or $x_2 = 0$. Then, the value of the objective function is $x_1^\alpha x_2^\beta = 0$. Consider instead the point $(\overline{x}_1, \overline{x}_2) = (\frac{m}{2p_1}, \frac{m}{2p_2})$. Clearly this point is feasible. Furthermore it yields strictly positive outcome in the objective function :

$$\overline{x}_1^\alpha \overline{x}_2^\beta > 0 = x_1^\alpha x_2^\beta$$

Therefore, no point where either of the non-negativity constraint bind can ever be optimal.

(c) Write the Lagrangian of the problem and the first order conditions (observe that we now only need one constraint).

Solution : Since the non-negativity constraint cannot bind, we can equivalently rewrite the problem as :

$$\max_{(x_1,x_2)\in\mathbb{R}^2_{++}} x_1^\alpha x_2^\beta$$

$$\text{subject to : } p_1x_1 + p_2x_2 \leq m$$

The Lagrangian of this problem is given by :

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^\alpha x_2^\beta + \lambda(m - p_1x_1 - p_2x_2)$$

First order Kuhn-Tucker conditions are given by :

$$\begin{aligned}\alpha x_1^{\alpha-1}x_2^{\beta} - p_1\lambda &= 0\\ \beta x_1^{\alpha}x_2^{\beta-1} - p_2\lambda &= 0\\ \lambda \geq 0,\, p_1x_1 + p_2x_2 &\leq m\\ \lambda(m - p_1x_1 - p_2x_2) &= 0\end{aligned}$$

(d) Argue that the budget constraint $p_1x_1 + p_2x_2 \leq m$ must bind at a maximum.

Solution : If the budget constraint does not bind, then we must have $\lambda = 0$. Plugging this into the necessary conditions for a maximum and observing that the constraint qualification always holds, we obtain a contradiction since e.g $0 = p_1\lambda = \alpha x_1^{\alpha-1}x_2^{\beta} > 0$. **Note :** this argument is only complete if you verify that the CQ holds everywhere (otherwise, you could have points where the necessary conditions do not hold).

Another valid approach consists of assuming that the optimum is such that $p_1x_1 + p_2x_2 < m$ and finding an improvement by constructing $\tilde{x}_1 := x_1 + (m - p_1x_1 + p_2x_2)/p_1$ and observing that the point $(\tilde{x}_1, x_2)$ is a strict improvement

(e) Solve the problem.

Solution : Observing (and directly verifying) that the CQ holds everywhere, we know that solving in the first order condition will give us a candidate maximizer. Furthermore, we know that the budget constraint binds and $\lambda \neq 0$ (otherwise there is a contradiction). Therefore :

$$\lambda = \frac{\alpha}{p_1}x_1^{\alpha-1}x_2^{\beta} = \frac{\beta}{p_2}x_1^{\alpha}x_2^{\beta-1}$$

Hence :

$$\frac{x_2}{x_1} = \frac{\beta}{\alpha}\frac{p_1}{p_2} \Leftrightarrow p_2x_2 = \frac{\beta}{\alpha}p_1x_1$$

Plugging this in the budget constraint yields :

$$p_1x_1 + \frac{\beta}{\alpha}p_1x_1 = m$$

Rearranging finally yields :

$$x_1 = \frac{\alpha}{\alpha+\beta}\frac{m}{p_1},\; x_2 = \frac{\beta}{\alpha+\beta}\frac{m}{p_2}$$

This is the unique candidate, therefore it has to be the maximum.