MA Math Camp Exam: Tuesday, September 4

Instructions:

- The exam is from 4:10 - 5:25 pm.
- You may use any results covered in the class directly without proofs.
- All answers must be justified
- The exam is closed book. No calculators are allowed.

1 Linear Algebra

1. (10 points) Let $x_1, ..., x_n$ be a collection of n vectors that span $\mathbb{R}^n$. Are $x_1, ..., x_n$ linearly independent? Prove or give a counterexample.

   Solution: Yes. Suppose $x_1, ..., x_n$ were linearly dependent. Then by removing linearly dependent vectors, we could construct a basis for $\mathbb{R}^n$ that had less than n elements, a contradiction.

2. (25 points) Consider the following model:

$$x_{t+2} = \alpha x_{t+1} + x_t \tag{1}$$

   (a) Define $z_{t+2} = \begin{pmatrix} x_{t+2} \\ x_{t+1} \end{pmatrix}$. Write equation (1) in the matrix form $z_{t+2} = Az_{t+1}$

   Solution: We can write

$$\underbrace{\begin{pmatrix} x_{t+2} \\ x_{t+1} \end{pmatrix}}_{z_{t+2}} = \underbrace{\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix}}_{z_{t+1}}$$

   (b) Use part (a) to write an expression relating z_{t+2}, A and z_1.

   Solution: By repeatedly substituting we find

$$z_{t+2} = Az_{t+1} = A^2 z_t = \ldots = A^{t+1} z_1$$

(c) Calculate the eigenvalues of A as a function of α

Solution: The eigenvalues are the solution of the characteristic polynomial $det(A - \lambda I) = 0$, or:

$$\lambda^2 - \alpha\lambda - 1 = 0$$

Applying the quadratic formula gives

$$\lambda = \frac{\alpha \pm \sqrt{4 + \alpha^2}}{2}$$

(d) Is A diagonalizable? (note, you don't need to diagonalize it)

Solution: Yes. Since A has two distinct eigenvalues, its eigenvectors are linearly independent, so it is diagonalizable. (Alternatively, since A is symmetric we know it is orthogonally diagonalizable.)

(e) Is there any value of α such that $\lim_{t\to\infty} A^t$ exists?

Solution: There is no such value of α

- If $\alpha > 0$, one of the eigenvalues is $\frac{\alpha+\sqrt{\alpha^2+4}}{2} > 1$, so A^t diverges
- If $\alpha < 0$, one of the eigenvalues if $\frac{\alpha-\sqrt{\alpha^2+4}}{2} < -1$, so again A^t diverges
- If $\alpha = 0$, then $A^t = A$ for t odd, and $A^t = I$ for t even, so A^t oscillates and does not converge

2 Calculus and Optimization

1. (10 points) A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree k if for any $\lambda \in \mathbb{R}$,

$$f(\lambda x) = \lambda^k f(x)$$

Show that any differentiable, homogeneous function of degree k satisfies the following relationship:

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = kf(x)$$

Solution: Differentiate the expression $f(\lambda x) = \lambda^k f(x)$ with respect to λ and apply the chain rule:

$$f'(\lambda x)x = k\lambda^{k-1} f(x)$$

Setting $\lambda = 1$ and expanding the left-hand side gives the result:

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = kf(x)$$

2. (25 points) Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be given by $f(x, y) = x^\alpha + y^\alpha$ where $\alpha > 1$. Define the value function

$$V(\alpha, C) = \min f(x, y) \text{ s.t. } x + y = C$$

where $C > 0$.

(a) Show that the constraint set is closed and bounded

Solution: The constraint set is $g^{-1}(\{0\})$ where $g(x, y) = C - x - y$. This is the inverse image of a closed set by a continuous function, so the constraint set is closed.

Every point in the constraint set is contained in a ball of radius 1 about (0,0), so the constraint set is bounded (remember the domain is $\mathbb{R}^2_+$).

(b) Does a solution to this problem exist?

Solution: Yes. The objective function is continuous and the constraint set is closed and bounded, so a minimum exists.

(c) Show that f is convex.

Solution: We can use the Hessian of f:

$$H(x, y) = \begin{pmatrix} \alpha(\alpha - 1)x^{\alpha-2} & 0 \\ 0 & \alpha(\alpha - 1)y^{\alpha-2} \end{pmatrix}$$

Both diagonal elements and the determinant are weakly positive, so f is convex[1]

(d) Find $V(\alpha, C)$ using the method of Lagrange. Does the constraint qualification ever fail?

Solution: The derivative of the constraint function is $\begin{pmatrix} -1 & -1 \end{pmatrix}$, which is always rank 1, so the constraint qualification never fails.
The Lagrangian is

$$\mathcal{L}(x, y, \lambda) = x^\alpha + y^\alpha + \lambda(C - x - y)$$

The FOC with respect to x and y are:

$$\begin{aligned} \alpha x^{\alpha-1} &= \lambda \\ \alpha y^{\alpha-1} &= \lambda \end{aligned}$$

[1] In fact, f is strictly convex, but we'd have to work a little harder to show that because the Hessian is 0 at the origin, and hence not positive definite there

Since $\alpha > 1$, this holds iff $x = y$. From the constraint we have $x = y = \frac{C}{2}$. Since f is convex and the constraints are linear, this critical point is a minimum, not a maximum.

The value function is therefore $V(\alpha, C) = 2\left(\frac{C}{2}\right)^{\alpha} = 2^{1-\alpha}C^{\alpha}$.

(e) Verify that $\frac{\partial V}{\partial C}$ equals the Lagrange multiplier found in part (d).

Solution: The Lagrange multiplier satisfies $\lambda = \alpha x^{\alpha-1}$, or $\lambda = \alpha\left(\frac{C}{2}\right)^{\alpha-1}$. Direct differentiation of the value function wrt C gives

$$\frac{\partial V}{\partial C} = 2^{1-\alpha}\alpha C^{\alpha-1} = \alpha\left(\frac{C}{2}\right)^{\alpha-1} = \lambda$$

3 Log-Linearization

1. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$. Let $p_i \geq 0$ for $i = 1, ..., n$ and $\sum_{i=1}^{n} p_i = 1$. Define the expectation of f as the weighted average

$$\mathbb{E}\left[f(x)\right] \equiv \sum_{i=1}^{n} p_i f(x_i)$$

Log-linearize $\mathbb{E}\left[f(x)\right]$ around $(x_1, x_2, \ldots, x_n) = (x^*, x^*, \ldots, x^*)$

Solution: Using the relationship $\hat{x} = \frac{dx}{x^*}$ and the properties of the d operator we see:

$$\begin{aligned}
\widehat{\sum_{i=1}^{n} p_i f(x_i)} &= \frac{d\left(\sum_{i=1}^{n} p_i f(x_i)\right)}{\sum_{i=1}^{n} p_i f(x_i^*)} \\
&= \frac{\sum_{i=1}^{n} p_i df(x_i)}{\sum_{i=1}^{n} p_i f(x_i^*)} \\
&= \frac{\sum_{i=1}^{n} p_i f'(x_i^*) dx_i}{\sum_{i=1}^{n} p_i f(x_i^*)} \\
&= \frac{\sum_{i=1}^{n} p_i f'(x_i^*) x_i^* \hat{x}_i}{\sum_{i=1}^{n} p_i f(x_i^*)}
\end{aligned}$$

Using the fact that $x_i^* = x^*$ for all i, this simplifies to:

$$\begin{aligned}
\widehat{\mathbb{E}\left[f(x)\right]} &= \frac{f'(x^*)x^*}{f(x^*)} \sum_{i=1}^{n} p_i \hat{x}_i \\
&= \epsilon_f(x^*)\mathbb{E}\left[\hat{x}\right],
\end{aligned}$$

where $\epsilon_f(x^*)$ is the elasticity of f at x^*.

4 Analysis

1. (15 points) Are the following statements true or false? If true, give a brief proof why; if false, provide a counterexample.

   (a) An arbitrary intersection of compact subsets of $\mathbb{R}^n$ is compact

   Solution: True. Each set is closed and bounded. An arbitrary intersection of closed sets is closed, and an arbitrary intersection of bounded sets is bounded, so the intersection is closed and bounded, therefore compact.

   (b) The inverse image of a compact set by a continuous function is compact

   Solution: False. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$. Then $\{0\}$ is a compact set, but $f^{-1}(\{0\}) = \mathbb{R}$, which is not compact.

   (c) If $S \subseteq \mathbb{R}^n$ is bounded, then any sequence of S converges.

   Solution: False. Consider the set $[0, 1]$. The sequence (0, 1, 0, 1, 0, 1,...) does not converge.