Problem Set 2
MA Math Camp 2021
Due Date : Monday August 30th, 2021

Answers should be typed and submitted in PDF format on Gradescope (see the course website for details). Be sure to answer every question thoroughly, and try to write complete and rigorous yet concise proofs. You can contact me if you have *specific* questions about the problem set, or if you think you have spotted a typo or mistake.

1. Let A a non-empty bounded subset of $\mathbb{R}$. Show that $\inf A$ and $\sup A$ belong to the closure of A.
2. Prove the following theorem. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces. Let set S be a subset of X, and $f : S \to Y$ be a continuous function. Let T be a set s.t. $f(S) \subset T \subset Y$, and $g : T \to Z$ be a continuous function. Then $g \circ f : S \to Z$ is a continuous function.
3. Prove that the Euclidean space $\left(\mathbb{R}^k, d_2\right)$ is a complete metric space. (Hint: First prove a Cauchy sequence in a metric space is bounded.)
4. Check whether the following sets are subspaces of the n-dimensional real vector space $\mathbb{R}^n$, equipped with its usual addition and scalar product.
   a) $\{\mathbf{0}\}$
   b) $\{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} = \alpha\mathbf{z}, \text{ for some } \alpha \in \mathbb{R}\}$,where $\mathbf{z} \in \mathbb{R}^n$.
   c) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 = 0\}$
   d) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 \neq 0\}$
   e) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 + x_2 = 0\}$
   f) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 = 0 \text{ or } x_2 = 0\}$
   g) (When $n = 1$) the set of integers $\mathbb{Z}$.
   h) (When $n = 3$) $S := \{(t - 2s, -s, t) : t, s \in \mathbb{R}\}$.
   i) $KerA := \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$, where A is an $n \times n$ real matrix.
5. Let E a vector space and F and G two vector subspaces of E. Show that $F \cup G$ is a vector space if and only if $F \subset G$ or $G \subset F$. Show that E cannot be written as the union of two vector subspaces different from E itself.
6. Consider the following collection of vectors in $\mathbb{R}^4$:

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

   is it an independent family ?
7. Show that the following $\|\cdot\|$ are valid norms in $\mathbb{R}^n$.

a) $\|\mathbf{x}\| := \max_{i=1}^{n} |x_i|$.

b) $\|\mathbf{x}\| := \sum_{i=1}^{n} |x_i|$.

8. Find non-zero 2×2 matrices, A, B such that $AB = 0$.

9. Show that for any two $n \times n$ matrices A and B, $Tr(AB) = Tr(BA)$.

10. Let A an $n \times n$ matrix and denote by I_n the identity matrix of size n. Show that : there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$ if and only if for any matrix B of size n, $AB = BA$.

11. Determine the rank of the following matrices:

    a) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$

    b) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix}$

    c) $\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$

12. Is it possible that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2 + \mathbf{v}_3$ are linearly independent?

13. State whether each of the following statements is true or false. Justify it accordingly with a short proof or a counterexample.

    a) No system of linear equations can have exactly k solutions for any $k \geq 2$.

    b) If $A\mathbf{x} = \mathbf{0}$ has a solution, then $A\mathbf{x} = \mathbf{b}$ has a solution.

    c) If an $n \times n$ matrix A is full rank, then $A\mathbf{x} = \mathbf{b}$ has a solution.

    d) If an $n \times n$ matrix A has rank less than n, then $A\mathbf{x} = \mathbf{b}$ has no solution.

    e) If an $n \times n$ matrix A is full rank, all its eigenvalues are distinct.

    f) Every diagonal real matrix has real eigenvalues.

    g) An $n \times n$ matrix A has a zero eigenvalue if and only if it has rank less than n.

14. Let A be an $n \times n$ positive definite real matrix.

    a) Verify that $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

    $$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T A \mathbf{y}$$

    is a valid inner product.

    b) Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\left(\mathbf{x}^T A \mathbf{x}\right)\left(\mathbf{y}^T A \mathbf{y}\right) \geq \left(\mathbf{x}^T A \mathbf{y}\right)^2$.

15. Let A be an idempotent matrix (i.e $A^2 = A$). Show that the eigenvalues of A must be either 0 or 1.

16. Let A a symmetric invertible $n \times n$ matrix. Show that A^{-1} is symmetric.