MA Math Camp Exam: Wednesday, Sept 4^{th}, 2019

Instructions:

- The exam is from 11:40 - 12:55 pm.
- You may use any results covered in the class directly without proofs.
- All answers must be justified
- The exam is closed book. Calculators are not allowed.

1. Consider the 2×2 matrix.
$$A = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{1}{3} & 1 \end{bmatrix}$$
   (a) **(5 points)** Find the eigen values of this matrix A.
   (b) **(3 points)** Find a 2×2 invertible matrix P and a 2×2 diagonal matrix Λ s.t. $A = P\Lambda P^{-1}$
   (c) **(2 points)** Consider the quadratic form $Q = x'Ax$ where A is the matrix given above. Determine whether Q is positive definite, negative definite or indefinite.

2. **(5 points)** Define the kernel of a matrix A as :
$$ker(A) \equiv \{x \in \mathbb{R}^n | Ax = 0\}$$
   Prove that the $ker(A)$ is a subspace of $\mathbb{R}^n$ i.e. it contains the 0 vector and is closed under addition and scalar multiplication.

3. **(10 points)** Solve the following system of equations :
$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 2x_3 &= -1 \end{aligned}$$

4. State whether each of the following statements is True or False. If it is False, provide a counter-example. If True, give a sketch of a proof :

   (a) **(5 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly concave function. If f is twice-differentiable at $x_0 \in \mathbb{R}$, then we must have $f''(x_0) < 0$.

   (b) **(5 points)** If A and B are compact sets in the metric space (X, d), then $A \cup B$ is a compact set.

5. **(10 points)** Suppose $\mathcal{D}$ is a convex open set of $\mathbb{R}^n$ and let $f\colon \mathcal{D} \to \mathbb{R}$ be a concave function. Then show that the set of maximizers is a convex set i.e. show that :

$$M = \left\{x \in \mathcal{D} | f(x) \geq f(x') \ \ \forall x' \in \mathcal{D}\right\}$$

   is a convex set.

6. Define $f : [1, \infty) \to [1, \infty)$ as $f(x) = x + \frac{1}{x}$.

   (a) **(5 points)** Show that $|f(x) - f(y)| < |x - y| \ \forall x, y$ such that $x \neq y$.

   (b) **(5 points)** Show that $f(\cdot)$ has no fixed point i.e. there does not exist x^* such that $f(x^*) = x^*$. Argue why this does not contradict Banach's Contraction Mapping Theorem.

7. Consider a consumer with a utility function $u(x_1, x_2) = x_1^{\frac{2}{3}} x_2^{\frac{1}{3}}$. Each good x_i costs p_i per unit and the consumer has a budget of M dollars. Find the consumer's optimal mix of consumption i.e. solve the program :

$$\max_{x_1, x_2} \quad x_1^{\frac{2}{3}} x_2^{\frac{1}{3}}$$
$$\text{subject to} \quad p_1 x_1 + p_2 x_2 = M$$

   (a) **(10 points)** What is the optimal choice of consumption for the consumer i.e. what is (x_1^*, x_2^*)?

   (b) **(10 points)** Write down the value function $V(p_1, p_2, M)$. Verify that :

$$-\frac{\frac{\partial V}{\partial p_1}}{\frac{\partial V}{\partial M}} = x_1^*$$

   (This is known as **Roy's Identity**)