MA Math Camp Exam: Tuesday, September 5

Instructions:

- The exam is from 1:10 - 2:25 pm.
- You may use any results covered in the class directly without proofs.
- The exam is closed book. No calculators are allowed.

1 Linear Algebra

1. Consider the matrix:
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

   (a) Do the columns of A form a basis for $\mathbb{R}^3$?

   Solution: Yes. One way to see this is by performing Gaussian elimination, which confirms that this matrix is rank 3, so the dimension of the column space is 3. Thus the columns of A are linearly independent, and any collection of 3 linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

   (b) Is A invertible?

   Solution: Yes. Recall a square matrix is invertible if and only if its columns are linearly independent.

   (c) Let $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$. How many solutions does the equation $Ax = b$ have? (Note, you do not need to give the solution(s), if any).

   Solution: This equation has exactly one solution, namely $x = A^{-1}b$.

2. Now consider the matrix:
$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

   (a) What are the eigenvalues of A?

   Solution: The characteristic equation is $(2-\lambda)^2 - 4 = 0$, which has solutions $\lambda = 0$ and $\lambda = 4$.

(b) Is A invertible?

Solution: No. Since A has a 0 eigenvalue, its determinant is 0, so it is not invertible. Another way to see this is that the columns of A are multiples of one another, so they are linearly dependent.

(c) Diagonalize A, if possible.

Solution: Since A has two distinct eigenvalues, we can diagonalize it. The eigenvector associated with $\lambda = 4$ is $\begin{pmatrix} 2 & 1 \end{pmatrix}'$, while the eigenvalue associated with $\lambda = 0$ is $\begin{pmatrix} -2 & 1 \end{pmatrix}'$. Thus:

$$A = PDP^{-1} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix}$$

2 Log-Linearization

Log-linearize the following equation around the steady state π^*, r^* (ϕ, α, and ϵ are fixed parameters):

$$1 + \pi_t = \left(\phi + \alpha(1 + r_t)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}$$

Solution: We see

$$\begin{aligned}
\widehat{1 + \pi_t} &= \widehat{\left(\left(\phi + \alpha(1 + r_t)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}\right)} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{1}{1-\epsilon}\widehat{\left(\phi + \alpha(1 + r_t)^{1-\epsilon}\right)} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{1}{1-\epsilon}\frac{d\left(\phi + \alpha(1 + r_t)^{1-\epsilon}\right)}{\phi + \alpha(1 + r^*)^{1-\epsilon}} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{1}{1-\epsilon}\frac{\alpha d\left((1 + r_t)^{1-\epsilon}\right)}{\phi + \alpha(1 + r^*)^{1-\epsilon}} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{1}{1-\epsilon}\frac{\alpha(1-\epsilon)(1 + r^*)^{-\epsilon}dr_t}{\phi + \alpha(1 + r^*)^{1-\epsilon}} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{\alpha(1 + r^*)^{-\epsilon}dr_t}{\phi + \alpha(1 + r^*)^{1-\epsilon}} \\
\frac{\pi^*}{1 + \pi^*}\widehat{\pi}_t &= \frac{\alpha(1 + r^*)^{-\epsilon}r^*}{\phi + \alpha(1 + r^*)^{1-\epsilon}}\widehat{r}_t
\end{aligned}$$

3 Calculus

1. Consider the function given by:

$$f(x) = \int_x^{x^2} g(t)dt$$

What is $f'(x)$?

Solution: By the Fundamental Theorem of Calculus:

$$f'(x) = 2xg(x^2) - g(x)$$

2. Suppose f, g are twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If f and g are increasing and convex, is the composite function $(f \circ g)(x) \equiv f(g(x))$ increasing and convex? If so demonstrate why; if not provide a counterexample.

   Solution: The composite function is convex and increasing. To see this write $h(x) = f(g(x))$. Then:

   $$h'(x) = f'(g(x))g'(x) \geq 0,$$

   since $f', g' \geq 0$. The second derivative is

   $$h''(x) = f''(g(x))g'(x)^2 + f'(g(x))g''(x) \geq 0,$$

   since each term in the sum is weakly positive. Since $h''(x) \geq 0$ everywhere, h is convex.

4 Optimization

A consumer is trying to maximize her utility from work and consumption. She can work any amount L hours at a wage $w > 0$. She can spend her wages on two goods; let c_1 and c_2 represent her consumption of goods 1 and 2, respectively. Let $p_1 > 0$ and $p_2 > 0$ be the prices of goods 1 and 2, respectively.

The consumer has no other form of wealth, so total expenditures must be less than labor income. Her utility from consumption and labor is:

$$u(c_1, c_2, L) = \alpha \ln(c_1) + c_2 - \frac{\gamma}{2}L^2,$$

where $\alpha > 0$ and $\gamma > 0$ are parameters describing her like for good 1 and her dislike of work.

(a) State the consumer's maximization problem and budget constraint. Be clear about what variables the consumer is free to choose.

Solution: The consumer's problem is:

$$\max_{c_1, c_2, L} \alpha \ln(c_1) + c_2 - \frac{\gamma}{2}L^2 \text{ subject to } p_1 c_1 + p_2 c_2 = wL$$

(b) Write down the Lagrangian. (You will only need one Lagrange multiplier for the budget constraint; do not worry about any positivity constraints)

Solution: The Lagrangian is:

$$L(c_1, c_2, L, \lambda) = \alpha \ln(c_1) + c_2 - \frac{\gamma}{2}L^2 - \lambda(p_1 c_1 + p_2 c_2 - wL)$$

(c) Are there any points where the constraint qualification fails?

Solution: The constraint qualification is that $g'(c_1, c_2, L)$ must be rank 1, since there is one constraint. We see:

$$g'(c_1, c_2, L) = \begin{pmatrix} p_1 & p_2 & w \end{pmatrix}$$

Since this is not the zero vector, the constraint derivative is rank 1 everywhere, so the constraint qualification never fails.

(d) Solve the program (that is, solve for c_1, c_2, and L in terms of w, p_1, p_2, α, and γ).

Solution: Since the objective function is concave and the constraints are linear, a critical point of the Lagrangian will be a maximum of the constrained problem. The system of first-order conditions is:

$$\begin{aligned}
\frac{\alpha}{c_1} &= \lambda p_1 \\
1 &= \lambda p_2 \\
\gamma L &= \lambda w \\
p_1 c_1 + p_2 c_2 &= wL
\end{aligned}$$

From the second equation we see $\lambda = 1/p_2$. This implies $c_1 = \alpha p_2/p_1$ and $L = w/(\gamma p_2)$ (interestingly, L does not depend on the price of good 1 - the reason for this is that utility is linear in good 2, so good 2 establishes the value of money). Finally, using the budget constraint we see $c_2 = (wL - p_1 c_1)/p_2 = \frac{w^2}{\gamma p_2^2} - \alpha$. Thus there is a unique solution given by:

$$\begin{aligned}
c_1 &= \frac{\alpha p_2}{p_1} \\
c_2 &= \frac{w^2}{\gamma p_2^2} - \alpha \\
L &= \frac{w}{\gamma p_2} \\
\lambda &= \frac{1}{p_2}
\end{aligned}$$

The comparative statics all make sense here (I didn't ask you to think about these, but it's a good habit to get into, and can be a useful way to check whether you answer is right!):

- Higher α means the consumer likes good 1 more, so consumption of 1 increases and consumption of 2 decreases.
- If w increases, labor is more valuable, so more labor is provided. Interestingly, though, the additional wages all go towards consumption of good 2. This is because good 1 suffers from decreasing marginal utility, so once the marginal utilities are equalized the consumer prefers to put all additional consumption towards good 2.
- If γ goes up, the consumer dislikes work more, and works less. The decreased consumption all comes from good 2 (same intuition as above)
- If the price of good 1 or 2 increases, then consumption of that good goes down.

5 Analysis

Indicate whether the following statements are true or false. If true, provide a brief proof. If not, provide a counterexample:

(a) If d_1 and d_2 are metrics on a metric space X, then the metric d defined by $d(x,y) = d_1(x,y) + d_2(x,y)$ is a metric as well

Solution: This is true. To see this we need to show three things:

- Note $d(x,y) = d_1(x,y) + d_2(x,y) \geq 0$. Also, if $d(x,y) = 0$ then we must have $d_1(x,y) = d_2(x,y) = 0$, which means $x = y$.
- The metric is symmetric: $d(x,y) = d_1(x,y) + d_2(x,y) = d_1(y,x) + d_2(y,x) = d(y,x)$.
- The metric satisfies the triangle inequality: for any $x, y, z \in X$ we have

$$d(x,y) = d_1(x,y) + d_2(x,y) \leq d_1(x,z) + d_1(z,y) + d_2(x,z) + d_2(z,y) = d(x,z) + d(z,y)$$

(b) If f is a continuous function and S is a closed set, then $f(S)$ is closed

Solution: This is false. Consider $f(x) = e^{-x}$ and $S = [0,\infty)$. Then $f(S) = (0,1]$, which is not closed.