Math Camp: Problem Set 2 Solutions

Due Monday, August 26th

1. Show that the norm satisfies the triangle inequality: for any $x, y \in \mathbb{R}^n$:

$$\|x+y\| \leq \|x\| + \|y\|$$

Solution: This follows from Cauchy Schwarz:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y \rangle \\
&= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\
&= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

2. The Euclidean distance between two points $x, y \in \mathbb{R}^n$ is defined as $d(x, y) = \|x - y\|$. Show that the above result implies the following triangle inequality: for any $x, y, z \in \mathbb{R}^n$:

$$d(x, y) \leq d(x, z) + d(z, y)$$

Solution: We can use our standard trick of adding 0:

$$\begin{aligned}
d(x, y) &= \|x - y\| \\
&= \|x - z + z - y\| \\
&\leq \|x - z\| + \|z - y\| \\
&= d(x, z) + d(z, y)
\end{aligned}$$

3. Give an example of two matrices A and B such that A, B are non-zero, but $AB = 0$.

Solution: (Many possible answers). One example is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Let A be an $m \times n$ matrix and B a $n \times m$ matrix. Show that $tr(AB) = tr(BA)$.

Solution: For this we need to use the definition of matrix multiplication:

$$\begin{aligned} tr(AB) &= \sum_{i=1}^{m} (AB)_{ii} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} \\ &= \sum_{j=1}^{n} (BA)_{jj} \\ &= tr(BA) \end{aligned}$$

5. Find the rank of the following matrices

   - $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$

     Solution: Subtracting twice the first row from the second gives

     $$\begin{pmatrix} 1 & 3 & 4 \\ 0 & -6 & -7 \end{pmatrix}$$

     This is in row-echelon form, so the rank is the number of nonzero rows; namely, 2.

   - $\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix}$

     Solution: Adding twice the second row to the first row and three times the second row to the third row gives:

     $$\begin{pmatrix} 0 & 9 & 9 & 9 \\ -1 & 4 & 3 & 1 \\ 0 & 14 & 14 & 14 \end{pmatrix}$$

     The first and third rows are linearly dependent; subtracting 14/9 times the first row from the third gives

     $$\begin{pmatrix} 0 & 9 & 9 & 9 \\ -1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

     By flipping the first and second rows we put the matrix in row-echelon form, so its rank is 2.

6. Let A and B be $n \times n$ matrices with $AB = I$. Show that $BA = I$. (Hint: what can we conclude about the rank of B?)

   Solution: We claim B is rank n. Note $Bx = 0 \Rightarrow ABx = 0 \Rightarrow x = 0$, so the columns of B are linearly independent, and therefore form a basis for $\mathbb{R}^n$.

   Let $x \in \mathbb{R}^n$. Since the columns of B are a basis for $\mathbb{R}^n$, there exists an y such that $By = x$. We therefore see

   $$BAx = B\underbrace{AB}_{I}y = By = x$$

Thus $BAx = x$ for every vector x, so $BA = I$ (convince yourself that this last step is true).

7. Let X be an $m \times n$ matrix with $m > n$. Show that $rank(X'X) = n$ iff $rank(X) = n$.

   Solution: Suppose X is rank n. Since $m > n$, that implies the columns of X are linearly independent.

   Suppose $X'X\beta = 0$. Multiplying on the left by β' we see:

$$\begin{aligned} \beta' X' X \beta &= 0 \\ (\Leftrightarrow)(X\beta)' X\beta &= 0 \\ (\Leftrightarrow)\|X\beta\| &= 0 \\ (\Leftrightarrow)X\beta &= 0, \end{aligned}$$

   which happens iff $\beta = 0$ since X is full rank. Therefore $X'X\beta = 0$ only has the trivial solution, so $X'X$ is full rank.

   Now suppose $X'X$ is rank n. Then $X'X\beta = 0$ only has the trivial solution. Suppose $X\beta = 0$. Multiplying on the left by X' gives $X'X\beta = 0$, so $\beta = 0$. Therefore the columns of X are linearly independent, so X is full rank.

8. An $n \times n$ matrix A is said to be idempotent if $A^2 = A$. Let X be an $m \times n$ matrix such that $X'X$ is invertible. Show that $M = I_m - X(X'X)^{-1}X'$ is idempotent.

   Solution: This is just a direct computation:

$$\begin{aligned} M^2 &= (I_m - X(X'X)^{-1}X')(I_m - X(X'X)^{-1}X') \\ &= I_m - 2X(X'X)^{-1}X' + X\underbrace{(X'X)^{-1}X'X}_{I_n}(X'X)^{-1}X' \\ &= I_m - X(X'X)^{-1}X' \\ &= M \end{aligned}$$

9. Let A be an idempotent matrix. Show that the eigenvalues of A must be 0 or 1.

   Solution: Suppose λ is an eigenvalue of an idempotent matrix A. Then $Ax = \lambda x$ for some vector x. We calculate A^2x two ways:

$$\begin{aligned} A^2x &= Ax = \lambda x \\ A^2x &= A(Ax) = A(\lambda x) = \lambda Ax = \lambda^2 x \end{aligned}$$

   The first line follows from idempotetency, while the second follows from the vector that x is an eigenvector of A. Thus $\lambda = \lambda^2$, so $\lambda = 0$ or 1.

10. Let V_1 and V_2 be vector subspaces of $\mathbb{R}^n$.

    - Is $V_1 \cap V_2$ a vector subspace of $\mathbb{R}^n$?
      Solution: Yes. First note that $0 \in V_1$ and $0 \in V_2$, so $0 \in V_1 \cap V_2$.
      Now we show closure under vector addition. Suppose $v, w \in V_1 \cap V_2$. Since $v, w \in V_1$ and V_1 is closed under vector addition, $v + w \in V_1$. By the same logic $v + w \in V_2$, so $v + w \in V_1 \cap V_2$.
      Finally, we show closure under scalar multiplication. Let $v \in V_1 \cap V_2$ and $\alpha \in \mathbb{R}$. Then $\alpha v \in V_1$ because V_1 is closed under scalar multiplication. Similarly, $\alpha v \in V_2$, so $\alpha v \in V_1 \cap V_2$.

- Is $V_1 \cup V_2$ a vector subspace of $\mathbb{R}^n$?
  Solution: Not necessarily. Let V_1 be the x-axis in $\mathbb{R}^2$ and V_2 be the y-axis. Then $V_1 \cup V_2$ is not closed under vector addition, since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in $V_1 \cup V_2$.

- Define $V_1 + V_2 = \{v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\}$. Is $V_1 + V_2$ a vector subspace of $\mathbb{R}^n$?
  Solution: Yes. Since $0 \in V_1$ and V_2, $0 = 0 + 0$ is in $V_1 + V_2$.

  To show closure under vector addition, let $v, w \in V_1 + V_2$. By definition, $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$. Similarly $w = w_1 + w_2$. Then $v + w = (v_1 + w_1) + (v_2 + w_2)$. Since V_1 and V_2 are closed under vector addition, $v_1 + w_1 \in V_1$ and $v_2 + w_2 \in V_2$, meaning $v + w \in V_1 + V_2$.
  Finally, we show closure under scalar multiplication. Let $v \in V_1 + V_2$. Again, we can write $v = v_1 + v_2$. Thus $\alpha v = \alpha v_1 + \alpha v_2$, which - since V_1 and V_2 are closed under scalar multiplication - is in $V_1 + V_2$.

11. Let A be an $m \times n$ matrix. Show that $ker(A)$ is a vector subspace of $\mathbb{R}^n$.

    Solution: We first note that $A0 = 0$, so $0 \in ker(A)$. Now we show closure under vector addition. Let $x, y \in ker(A)$. Then $A(x + y) = Ax + Ay = 0$, so $x + y \in ker(A)$. Finally, we show closure under scalar multiplication. Let $\alpha \in \mathbb{R}$ and $x \in ker(A)$. Then $A(\alpha x) = \alpha Ax = \alpha 0 = 0$, so $\alpha x \in ker(A)$.

12. Find the eigenvalues and associated eigenvectors of the following matrices. Diagonalize these matrices.

    - $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$
      Solution: The characteristics polynomial of the matrix is:
      $$(1 - \lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3)$$
      Therefore the eigenvalues of this matrix are -2 and 3. Since A has two distinct eigenvalues, it is diagonalizable.
      The eigenvector associated with -2 satisfies $3x_1 + 3x_2 = 0$, or $x_1 = -x_2$, so an the eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or any scalar multiple of this).
      The eigenvector associated with 3 satisfies $-2x_1 + 3x_2 = 0$, or $x_1 = \frac{3}{2}x_2$, so an eigenvector is $\begin{pmatrix} 1.5 \\ 1 \end{pmatrix}$.
      To diagonalize A, we form P by stacking the eigenvectors of A and D by forming the diagonal matrix with the eigenvalues of A. Thus:
      $$A = PDP^{-1} = \frac{2}{5} \begin{pmatrix} 1.5 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1.5 \end{pmatrix}$$

    - $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Solution: First note this matrix is symmetric, so it is orthogonally diagonalizable. The characteristic polynomial of this matrix is:

$$P(\lambda) = (1-\lambda)^3 - (1-\lambda)$$

We can factor this as $P(\lambda) = \lambda(1-\lambda)(\lambda-2)$, so the eigenvalues of A are 0, 1, 2.

The eigenvector associated with 0 satisfies $x_1 + x_3 = 0$ and $x_2 = 0$, so any multiple of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ will do.

The eigenvector associated with 1 satisfies $x_1+x_3 = x_1$ and $x_1+x_3 = x_3$, so $x_1 = x_2 = 0$, so it is a multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The eigenvector associated with 2 satisfies $x_1 + x_3 = 2x_1$, so $x_1 = x_3$. It also satisfies $2x_2 = x_2$, so $x_2 = 0$, so it is a multiple of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Define P as the 3×3 matrix where these vectors have been normalized to length 1. Since A is orthogonally diagonalizable,

$$A = P'DP$$

where $D = diag(0, 1, 2)$.

13.
- Let A be an $n \times n$ square matrix. Show that A' has the same eigenvalues as A.

  Solution: Note $det(A-\lambda I) = det((A-\lambda I)') = det(A'-\lambda I)$, so λ is an eigenvalue of A if and only if λ is an eigenvalue of A'.

- Show that if $\lambda \neq 0$ is an eigenvalue of an invertible matrix A, λ^{-1} is an eigenvalue of A^{-1}

  Solution: Let $\lambda \neq 0$ be an eigenvalue of A. Then $Ax = \lambda x$ for some nonzero vector x. Multiplying on the left by A^{-1} we see $x = \lambda A^{-1}x$, so $A^{-1}x = \lambda^{-1}x$, so λ^{-1} is an eigenvalue of A^{-1} (with the same eigenvector associated to λ for A).

14. Let A be a symmetric, invertible $n \times n$ matrix. Show that A^{-1} is symmetric.

    Solution: Let A be symmetric, and suppose $AB = I$. Transposing, we see $B'A' = I$. Since A is symmetric, $B'A = I$. From question 5, we know that B and B' are inverses for A, and by the uniqueness of the inverse, $B = B'$, so B is symmetric. That is, A^{-1} is symmetric.