Problem Set 1
MA Math Camp 2021
Due Date : August 23rd, 2021

Answers should be typed and submitted in PDF format on Gradescope (see the course website for details). Be sure to answer every question thoroughly, and try to write complete and rigorous yet concise proofs. You can contact me if you have *specific* questions about the problem set, or if you think you have spotted a typo or mistake.

1. Let P, Q two statements.
   a. Show that the statements $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent, using a truth table.
   b. Show the contrapositive principle ($P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$).
2. Write the negation of each of the following statement. Interpret each statement and state (if possible) which is true or false.
   a. $\forall x \in \mathbb{R}, x^2 \geq 0$
   b. $\forall x \in \mathbb{R}, x^2 > 0$
   c. $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, x \leq M$
   d. $\forall x \in \mathbb{R}, \exists M \in \mathbb{R}, x \leq M$
   e. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \theta \in \mathbb{R}, |x-y|^2 \leq \theta |x-y|$
   f. $\exists \theta \in \mathbb{R}, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x-y|^2 \leq \theta |x-y|$
   g. $\forall (x,y) \in \mathbb{R}^2, x+y=0 \Rightarrow (x=0 \text{ and } y=0)$
   h. $\forall (x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2=0 \Rightarrow (x=0 \text{ and } y=0 \text{ and } z=0)$
3. Write explicitly :
   a. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$
   b. $\mathcal{P}(\mathcal{P}(\{a,b\}))$
4. Let E and F two sets.
   a. Show that $E \subseteq F \Leftrightarrow \mathcal{P}(E) \subseteq \mathcal{P}(F)$.
   b. Compare $\mathcal{P}(E \cup F)$ and $\mathcal{P}(E) \cup \mathcal{P}(F)$ (is one included in the other ?).
5. Let E a non-empty set and let A, B, C subsets of E. Show that :
   a. $A = B \Leftrightarrow A \cap B = A \cup B$
   b. $A \cap B^c = A \cap C^c \Leftrightarrow A \cap B = A \cap C$
   c. $\begin{cases} A \cap B \subseteq A \cap C \\ A \cup B \subseteq A \cup C \end{cases} \Rightarrow B \subseteq C$

d. Define the symmetrical difference, denoted Δ, of A and B as :

$$\begin{aligned} A\Delta B :&= \{(x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\} \\ &= (A \cup B) \setminus (A \cap B) \\ &= (A \setminus B) \cup (B \setminus A) \end{aligned}$$

The three definitions above are equivalent. Show that :

$$A^c \Delta B^c = A \Delta B$$

6. Let R be a complete, transitive relation over a set X. Define the relation $\sim$ as follows : $a \sim b$ iff aRb and bRa. For any $x \in X$, define the set $I(x)$ as :

$$I(x) := \{y \in X | y \sim x\}$$

Show that for any $x, y \in X$, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$

7. Let I an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a function defined over I taking values in $\mathbb{R}$. Write mathematical statements (using quantifiers) to express the following statements :

   a. f takes the value zero
   b. f is the zero function (takes the value zero everywhere)
   c. f is not a constant function
   d. f never takes the same value twice

8. Let X, Y two sets and $f \in Y^X$.

   a. Show that for any $A, B \in \mathcal{P}(E)$, $f(A \cap B) \subseteq f(A) \cap f(B)$
   b. Show that f is injective if and only if for any $A, B \in \mathcal{P}(E)$: $f(A \cap B) = f(A) \cap f(B)$
   c. Find an example of a function f such that there exists $A, B \in \mathcal{P}(E)$ for which $f(A \cap B) \subsetneq f(A) \cap f(B)$

9. Let X, Y two sets and $f \in Y^X$.

   a. Show that for all $A \in \mathcal{P}(X)$, $A \subseteq f^{-1}(f(A))$ and this holds with equality for all A if and only if f is injective.
   b. Show that for all $B \in \mathcal{P}(Y)$, $f(f^{-1}(Y)) \subseteq Y$ and this holds with equality for all B if and only if f is surjective.

10. Show that ther does not exist any surjective function from E into $\mathcal{P}(E)$ (this is a famous result due to Cantor). Hint : consider $\phi : E \to \mathcal{P}(E)$ and assume by contradiction that it is surjective, then consider the set $A := \{x \in E, x \notin \phi(x)\}$.

11. Show the two following results by induction :

   a. For any $n \in \mathbb{N}$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

b. For any $n \in \mathbb{N}$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

12. Verify that the d_∞ norm on $\mathbb{R}^k$, defined as $d_\infty(x,y) := \max_{1\leq i\leq k} |x_i - y_i|$ is indeed a distance. In $\mathbb{R}^2$, draw the set of points x such that $d(x,0) = 1$.

13. Consider (E,d) a metric space. Prove that for an arbitrary set $S \subseteq E$, the interior of S is open.

14. Consider the metric space $(\mathbb{R}, d_2)$ and two convergent sequences $x_n \to x$, $y_n \to y$. Prove the following results :

    a. If for all $n \in \mathbb{N}$, $x_n \leq y_n$ then $x \leq y$

    b. $x_n + y_n \to x + y$

    c. If $x \neq 0$, $\frac{1}{x_n} \to \frac{1}{x}$

15. Prove that the two definitions we gave for closed sets are equivalent. In other words, let (E,d) a metric space and $S \subseteq E$, prove that the two following statements are equivalent :

    - S^c is an open set
    - S contains all of its limit points

16. Prove that if a sequence (x_n) converges in $(\mathbb{R}, d_2)$, then so does $(|x_n|)$. Is the converse true ? If not, find a counterexample.

17. Show that the image of an open set by a continuous function is not necessarily an open set. Show that the image of a closed set by a continuous function is not necessarily a closed set.