Math Camp: Problem Set 2

Due August 29th, 2020

1. Show that the norm satisfies the triangle inequality: for any $x, y \in \mathbb{R}^n$:

$$\|x + y\| \leq \|x\| + \|y\|$$

   (Hint : Use the Cauchy-Schwarz Inequality)

2. The Euclidean distance between two points $x, y \in \mathbb{R}^n$ is defined as $d(x, y) = \|x - y\|$. Show that the above result implies the following triangle inequality: for any $x, y, z \in \mathbb{R}^n$:

$$d(x, y) \leq d(x, z) + d(z, y)$$

   (Hint : Use the result proved in the previous question)

3. Give an example of two matrices A and B such that A, B are non-zero, but $AB = 0$.

4. Show that $tr(AB) = tr(BA)$.

5. Find the rank of the following matrix
   - $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$
   - $\begin{pmatrix} 2 & 1 & 3 & 7 \\ -1 & 4 & 3 & 1 \\ 3 & 2 & 5 & 11 \end{pmatrix}$

6. Let A and B be $n \times n$ matrices with $AB = I$. Show that $BA = I$. (Hint: what can we conclude about the rank of B?)

7. Let X be an $m \times n$ matrix with $m > n$. Show that $rank(X'X) = n$ iff $rank(X) = n$.

8. An $n \times n$ matrix A is said to be idempotent if $A^2 = A$. Let X be an $m \times n$ matrix such that $X'X$ is invertible. Show that $M = I_m - X(X'X)^{-1}X'$ is idempotent.

9. Let A be an idempotent matrix. Show that the eigen values of A must either be 0 or 1.

10. Let V_1 and V_2 be vector subspaces of $\mathbb{R}^n$.
    - Is $V_1 \cap V_2$ a vector subspace of $\mathbb{R}^n$?
    - Is $V_1 \cup V_2$ a vector subspace of $\mathbb{R}^n$?
    - Define $V_1 + V_2 = \{v_1 + v_2 | v_1 \in V_1, v_2 \in V_2\}$. Is $V_1 + V_2$ a vector subspace of RN?

11. Let A be an $m \times n$ matrix. Show that $ker(A)$ is a vector subspace of $\mathbb{R}^n$.

12. Find the eigenvalues and associated eigenvectors of the following matrices. Diagonalize these matrices.
    - $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$
    - $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

13. - Let A be an $n \times n$ square matrix. Show that A' has the same eigenvalues as A.
    - Show that if $\lambda \neq 0$ is an eigenvalue of an invertible matrix A, λ^{-1} is an eigenvalue of A^{-1}

14. Let A be a symmetric, invertible $n \times n$ matrix. Show that A^{-1} is symmetric.

15. Calculate the inverse of the following matrix

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$