Final Exam
Columbia MA Math Camp 2021

September 13th, 2021

Instructions :

- This is a 1h15 exam (+10 minutes for scanning and uploading if the exam is taken remotely).
- Students taking the exam remotely should upload their answers on Gradescope : be careful that the 1h25 limit will be enforced automatically after you open this exam.
- The exam is closed book, it is not allowed to use any material either from the class or external. Calculators are not allowed. Students are not allowed to cooperate. Any suspicion of cheating will be taken very seriously.
- All answers must be justified. Keep in mind that answers are supposed to be short.
- You may use any result that was seen in the class without proof.
- Please write clearly and concisely.
- The exam has a total of 100 points.

1. **(25 points)** *True or False :* For each of the following statements, state whether they are true or false. Justify true statements with a short argument and provide a counterexample for false statements.
   (a) Any convergent sequence in $\mathbb{R}$ is bounded.
   (b) An arbitrary intersection of open sets is open.
   (c) The image of a closed set by a continuous function is closed.
   (d) For any $n \times n$ matrices A and B, $Tr(AB) = Tr(BA)$
   (e) The interval $[0, 1]$ is open in the metric space $([0, 1], d_2)$.

2. **(15 points)** Let E, F, G three sets. Consider two functions $f : E \to F$, $g : F \to G$. Show that if $g \circ f$ is injective, then f is injective.

3. **(15 points)** Consider the following matrix :

$$M := \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

(i) Is it invertible ?

(ii) Is it diagonalizable in $\mathbb{R}$? If so, find a diagonal matrix Λ and an invertible matrix P such that $M = P\Lambda P^{-1}$.

(iii) Write a simple expression for M^k for $k \in \mathbb{N}$.

4. **(15 points)** Consider the following function defined on $(0, 1)$:

$$\begin{aligned} f : (0,1) &\to \mathbb{R} \\ x &\mapsto -x\ln(x) \end{aligned}$$

(i) Verify that f is concave on $(0, 1)$

(ii) Does f have a maximum ? If so, say where it is attained and its value.

(iii) Show that f does not have a minimum. What is its infimum ?

5. **(30 Points)** Consider the following constrained optimization problem :

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} x_1^\alpha x_2^\beta$$
$$\text{subject to : } p_1x_1 + p_2x_2 \leq m$$

where $\alpha, \beta \in (0, 1)$, $p_1, p_2, m \in \mathbb{R}_{++}$, are parameters.

(a) Does a solution exist ?

(b) Argue that neither of the non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ can bind at a maximum.

(c) Write the Lagrangian of the problem and the first order conditions (observe that we now only need one constraint).

(d) Argue that the budget constraint $p_1x_1 + p_2x_2 \leq m$ must bind at a maximum.

(e) Solve the problem.