MA Math Camp Exam:
9 a.m. Sept 8^{th} to 9 a.m. Sept 10^{th}, 2020

Instructions:

- This is a 48 hour take home exam.
- You may use any results covered in the class directly without proofs.
- All answers must be justified.
- You may only consult the slides and lecture material for this exam. **If any indication of cheating is suspected, 10 points will be deducted for every suspected answer copied from the internet or from your classmates.** No chance of explanation will be given.
- Please write your answers clearly. Points will be deducted for bad handwriting.

1. (**5 points**) Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose f is increasing i.e. $x \geq y$ implies $f(x) \geq f(y)$. Prove that f is both quasiconvex and quasiconcave.

2. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$.

   (a) (**3 points**) Find the inverse of the matrix $\boldsymbol{A}$.

   (b) (**3 points**) Find the eigen values of this matrix

   (c) (**3 points**) Find a set of linearly independent eigenvectors such that the Euclidean norm of each eigen-vector equals 1.

   (d) (**3 points**) Find a matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$

   (e) (**3 points**) Consider a sequence of matrices $\boldsymbol{A^1}$,$\boldsymbol{A^2}, \ldots, \boldsymbol{A^n}, \ldots$ where the n^{th} element of the sequence is the n^{th} power of $\boldsymbol{A}$. Does there exist a matrix $\boldsymbol{B}$ such that each of the elements of $\boldsymbol{A^n}$ converges to the corresponding element of $\boldsymbol{B}$ as $n \to \infty$

3. Are the following statements true or false? If you think it is true, provide a sketch of a proof. If you think it is false, provide a counterexample.

   (a) (**5 points**) The intersection of 2 compact sets is a compact set.

   (b) (**5 points**) The inverse image of a compact set by a continuous function is a compact set.

   (c) (**5 points**) Suppose $f : X \to Y$ is discontinuous at $x \in X$. Suppose $g : Y \to Z$ is a continuous function. Then $g \circ f : X \to Z$ is discontinous at x.

4. On a non-empty set X, define the discrete metric d, as :

$$d\left(x, y\right) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

   (a) (**5 points**) Verify that d is indeed a valid metric

   (b) (**5 points**) Show that any subset of X is both open and closed

   (c) (**5 points**) Show that a set S in X is compact if and only if it is finite.

5. (**10 points**) Consider the following maximization problem :

$$\max_{x \in \mathbb{R}^{\ltimes}} u(x)$$
$$s.t. \quad p \cdot x \leq w$$

   where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $w \in \mathbb{R}$. (Note that $p \cdot x = \sum_{i=1}^{n} p_i x_i$). Suppose that the maximization problem has a solution for any (p, w) in some convex set $S \subset \mathbb{R}^n \times \mathbb{R}$. Show that the value function $v : S \to \mathbb{R}$ defined as $v\left(p, w\right) := \max_{x \in \mathbb{R}^n} \{u(x) \text{ s.t. } p \cdot x \leq w\}$ is quasiconvex.
   (Hint : Kuhn Tucker is not required! Work with the definition of quasiconvexity)

6. Consider the function $f\left(x_1, x_2\right) = x_1^{\alpha} + x_2^{\alpha}$ defined for $x_1, x_2 \geq 0$ and $\alpha \in (0, 1]$. **Do notice that $\alpha = 1$ is included.** For a fixed α, consider the problem:

$$\max_{x_1, x_2} x_1^{\alpha} + x_2^{\alpha}$$
$$s.t. \; x_1 + x_2 = 1$$

   (a) (**5 points**) Show that f is concave in (x_1, x_2) for a fixed α. (Hint : Use the result that a finite sum of concave functions is concave)

   (b) (**5 points**) Does a solution exist for this maximization problem for all values of $\alpha \in (0, 1]$

(c) (**15 points**) For a fixed α, find the solution to this maximization problem. (Hint : Consider separate cases for $\alpha < 1$ and $\alpha = 1$).

(d) (**5 points**) Do you need to check for the Second Order Conditions in this problem? Why or why not?

(e) (**5 points**) Find the value function $V^*(\alpha)$

(f) (**5 points**) Verify that the Envelope theorem holds in this problem.

The Envelope theorem tells us that $V'(\alpha) = \mathcal{L}_\alpha(x_1^*(\alpha), x_2^*(\alpha), \alpha)$. Note that $\mathcal{L}_\alpha(x_1^*(\alpha), x_2^*(\alpha), \alpha)$ is the derivative of the Lagrangean with respect to α evaluated at $(x_1^*(\alpha), x_2^*(\alpha))$.