MA Math Camp Exam 2022

Columbia Unversity

September 7th 2022

Instructions:

- This is a 1h15 exam.
- The exam has a total of 100 points.
- The exam is closed book, it is not allowed to use any material either from the class or external. Calculators are not allowed. Students are not allowed to cooperate. Any suspicion of cheating will be taken very seriously.
- All answers must be justified.
- You may use any result that was seen in class without proof.
- Please write clearly and concisely.
- Good luck!

1. **(25 points)** *True or False:* For each of the following statements, state whether they are true or false. Justify true statements with a short argument and provide a counterexample for false statements.

   (a) Let X be a set endowed with the discrete metric*. Every subset of the resulting metric space (X, d) is both open and closed.

   Solution: *True.* Take any set $S \subseteq X$. Consider an arbitrary $x \in S$. By definition, $B_{\frac{1}{2}}(x) = \{x\} \subseteq S$. Therefore, x is an interior point of S and thus S is open. Now, because every subset of X is open, if we take a set S we know that its complement S^c is open. Therefore S is closed.

*Recall that the discrete metric is defined for any $x, y \in X$ as

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

(b) The image of a continuous function over an open set is open.

Solution: *False.* Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ for all x. Take the open set $(0, 1)$ and $f((0, 1)) = \{1\}$ which is not open.

(c) Let A be an $n \times n$ matrix. If $det(A) = 0$ then at least one eigenvalue must be 0.

Solution: *True.* Recall that for a $n \times n$ matrix A we have $det(A) = \lambda_1 \lambda_2 \dots \lambda_n$. Therefore, $det(A) = 0 \iff \lambda_i = 0$ for some $i = 1, \dots, n$

(d) In a metric space, any intersection of open sets is open.

Solution: *False.* Consider, in the metric space $\mathbb{R}, d_2$ the family of open sets $E_n = (-\frac{1}{n}, \frac{1}{n})$. We have $\bigcap_{n=1}^{+\infty} E_n = \{0\}$, which is not an open set.

(e) Any collection of vectors that include the zero vector cannot be linearly independent.

Solution: *True.* A set including the zero vector is always linearly dependent. To see this, let $\mathbf{v_1}, \dots \mathbf{v_n}$ be n vectors of a vector space V. Suppose it contains the zero vector $\mathbf{0}$, wlog say $\mathbf{v_1} = \mathbf{0}$. Then, set $\lambda_i = 0$ for all $i \neq 1$ and $\lambda_1 = 1$. We thus have a collection of scalars, not all zeros, such that:

$$\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \dots \lambda_n \mathbf{v_n} = 1\mathbf{0} + \mathbf{0} = \mathbf{0}$$

2. **(20 points)** Consider two concave functions $f, \phi : \mathbb{R} \to \mathbb{R}$.

(a) Prove that the function $\phi \circ f$ defined by $\phi(f(x))$ for all $x \in \mathbb{R}$, is concave, if ϕ is weakly increasing [†].

Solution: Let $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. We want to show that:

$$\phi(f(\lambda x + (1 - \lambda)y)) \geq \lambda \phi(f(x)) + (1 - \lambda)\phi(f(y))$$

Since f is concave, we have that

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$

Since ϕ is weakly increasing and concave, we have

$$\phi(f(\lambda x + (1 - \lambda)y)) \geq \phi(\lambda f(x) + (1 - \lambda)f(y)) \geq \lambda \phi(f(x)) + (1 - \lambda)\phi(f(y))$$

(b) Would $\phi \circ f$ still be concave if the requirement of monotonicity (ϕ being weakly increasing) were to be dropped? Motivate your answer!

Solution: If we dropped monotonicity, the composite function $\phi \circ f$ may not be concave anymore. For instance, consider $\phi(x) = -x$ and $f(x) = -x^2$. Now ϕ is not weakly increasing but it is still concave. The function f, on the other hand, is concave. The composite function is $(\phi \circ) f(x) = \phi(f(x)) = -f(x) = x^2$ which is a strictly convex function (not concave).

[†] Recall that ϕ is weakly increasing if and only if

$$s \geq t \implies \phi(s) \geq \phi(t).$$

3. **(15 points)** Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Is the matrix diagonalizable? If yes, diagonalize it, i.e., find a 2×2 matrix such that $A = P\Lambda P^{-1}$, where Λ is a diagonal matrix.

Solution: The characteristic polynomial of the matrix A $\det(A - \lambda I_n) = 0 \iff (1-\lambda)^2 - 4 = 0$. Given this, we can find the two roots: $\lambda = 3$ and $\lambda = -1$. These are the two eigenvalues of the matrix. Since the matrix has two distinct eigenvalues it is diagonalizable. Two of the associated eigenvectors (given $Ax = \lambda x$) are: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. We can thus define:

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

so that

$$A = P\Lambda P^{-1}$$

4. **(25 points)** A consumer's utility maximization problem is:

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \; & \alpha \ln x_1 + \beta \ln x_2 \\ \text{s.t. } & p_1x_1 + p_2x_2 \leq m \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

where, $\alpha > 0$, $\beta > 0$ $p_1 > 0$, $p_2 > 0$, $m > 0$ are parameters.

(a) Does a solution exist? If yes, why?

Solution: The objective function is the sum of continuous functions and it is therefore continuous. Furthermore, the constraint set:

$$D := \{(x_1, x_2) \in \mathbb{R}^2 : p_1x_1 + p_2x_2 \leq m, x_1 \geq 0, x_2 \geq 0\}$$

is a nonempty compact set. Therefore, by Weiestrass theorem, the problem must have a solution.

(b) Argue that the budget constraint binds at any possible maximum.

Solution: Suppose the budget constraint does not bind at the optimum and $m - p_1x_1 - p_2x_2 = c > 0$. Then one could increase x_2 by c/x_2, which still satisfies all the constraints and obtain a strictly higher value of the objective function.

(c) Argue that the constraints $x_1 \geq 0$ and $x_2 \geq 0$ are not binding at the maximum.

Solution: It is enough to notice that the objective function is not defined for $x_1 = 0$ or $x_2 = 0$, because $\ln x$ is not defined for $x = 0$. Therefore, the contraints $x_1 \geq 0$ and $x_2 \geq 0$ are not binding at the maximum.

(d) Are there any potential maximizers where the constraint qualification would fail?

Solution: Given the previous points, the only binding constraint is the budget constraint $g(x_1, x_2) = m - p_1 x_1 - p_2 x_2$. We have that:

$$\nabla g(x_1, x_2) = \begin{pmatrix} -p_1 \\ -p_2 \end{pmatrix}$$

The set whose only element is this vector is linearly independent since $p_1 > 0$ and $p_2 > 0$ and thus $\nabla g(x_1, x_2) \neq (0, 0)$. Hence, there are no points at which the constraint qualification fails.

(e) Set up the Lagrange and solve the problem.

Solution: Given the previous points, we can write a new problem that has the same set of maximizers as the original problem, solving this instead:

$$\max_{(x_1, x_2) \in \mathbb{R}^2_{++}} \alpha \ln x_1 + \beta \ln x_2$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 = m$$

The Lagrangian of this problem is:

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha \ln x_1 + \beta \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

whose related FOCs are:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0 \iff \lambda = \frac{\alpha}{p_1 x_1} \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\beta}{x_2} - \lambda p_2 = 0 \iff \lambda = \frac{\beta}{p_2 x_2} \end{cases}$$

So that $p_1 x_1 = \frac{\alpha}{\beta} p_2 x_2$, which plugged in the budget constraint $m - p_1 x_1 - p_2 x_2 = 0$ leads to the solution:

$$(x_1, x_2) = \left(\frac{\alpha}{\alpha + \beta} \frac{m}{p_1}, \frac{\beta}{\alpha + \beta} \frac{m}{p_2} \right)$$

which is the unique maximizer of the problem.

5. **(15 points)** Let $V = \mathbb{R}^3$. Consider the sets

$$W_1 = \{x \in \mathbb{R}^3 \quad x_1 = 0\}$$
$$W_2 = \{x \in \mathbb{R}^3 \quad x_2 = 0\}.$$

(a) Show that W_1 and W_2 are vector subspaces of V.

Solution: Since the zero vector belong to W_1, we just need to check that:

$$\alpha x + \beta y \in W_1 \quad \forall \alpha, \beta \in \mathbb{R} \quad \forall x, y \in W_1$$

As $x, y \in W_1$ we have that their first component is 0:

$$x_1 = y_1 = 0$$

and thus the first component of the vector $\alpha x + \beta y$ is 0 as well

$$\alpha x_1 + \beta y_1 = 0$$

implying that $\alpha x + \beta y \in W_1$. The same reasoning applies to W_2

(b) Show that $W_1 \cup W_2$ **is not** a vector subspace of V.

Solution: Take two vectors:

$$x = (0, 1, 1), \quad y = (1, 0, 1)$$

We have that $x \in W_1$ and $y \in W_2$ so that they both belong to $W_1 \cup W_2$. On the other hand, their sum

$$x + y = (1, 1, 2)$$

does not. Therefore, $W_1 \cup W_2$ is not a vector subspace of V.

(c) What about $W_1 \cap W_2$?

Solution: $W_1 \cap W_2$ is the intersection of two vector subspaces and thus it is a a vector subspace itself.