MA Math Camp Exam 2022

Columbia Unversity

September 7th 2022

Instructions:

- This is a 1h15 exam.
- The exam has a total of 100 points.
- The exam is closed book, it is not allowed to use any material either from the class or external. Calculators are not allowed. Students are not allowed to cooperate. Any suspicion of cheating will be taken very seriously.
- All answers must be justified.
- You may use any result that was seen in class without proof.
- Please write clearly and concisely.
- Good luck!

1. **(25 points)** *True or False:* For each of the following statements, state whether they are true or false. Justify true statements with a short argument and provide a counterexample for false statements.

   (a) Let X be a set endowed with the discrete metric*. Every subset of the resulting metric space (X, d) is both open and closed.

   (b) The image of a continuous function over an open set is open.

   (c) Let A be an $n \times n$ matrix. If $det(A) = 0$ then at least one eigenvalue must be 0.

   (d) In a metric space, any intersection of open sets is open.

   (e) Any collection of vectors that include the zero vector cannot be linearly independent.

*Recall that the discrete metric is defined for any $x, y \in X$ as

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

2. **(20 points)** Consider two concave functions $f, \phi : \mathbb{R} \to \mathbb{R}$.

   (a) Prove that the function $\phi \circ f$ defined by $\phi(f(x))$ for all $x \in \mathbb{R}$, is concave, if ϕ is weakly increasing [†].

   (b) Would $\phi \circ f$ still be concave if the requirement of monotonicity (ϕ being weakly increasing) were to be dropped? Motivate your answer!

3. **(15 points)** Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

   Is the matrix diagonalizable? If yes, diagonalize it, i.e., find a 2×2 matrix such that $A = P\Lambda P^{-1}$, where Λ is a diagonal matrix.

4. **(25 points)** A consumer's utility maximization problem is:

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \ & \alpha \ln x_1 + \beta \ln x_2 \\ \text{s.t. } & p_1 x_1 + p_2 x_2 \leq m \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

   where, $\alpha > 0$, $\beta > 0$ $p_1 > 0$, $p_2 > 0$, $m > 0$ are parameters.

   (a) Does a solution exist? If yes, why?

   (b) Argue that the budget constraint binds at any possible maximum.

   (c) Argue that the constraints $x_1 \geq 0$ and $x_2 \geq 0$ are not binding at the maximum.

   (d) Are there any potential maximizers where the constraint qualification would fail?

   (e) Set up the Lagrange and solve the problem.

5. **(15 points)** Let $V = \mathbb{R}^3$. Consider the sets

$$\begin{aligned} W_1 &= \{x \in \mathbb{R}^3 \quad x_1 = 0\} \\ W_2 &= \{x \in \mathbb{R}^3 \quad x_2 = 0\}. \end{aligned}$$

   (a) Show that W_1 and W_2 are vector subspaces of V.

   (b) Show that $W_1 \cup W_2$ **is not** a vector subspace of V.

   (c) What about $W_1 \cap W_2$?

[†] Recall that ϕ is weakly increasing if and only if

$$s \geq t \implies \phi(s) \geq \phi(t).$$