

Problem Set 2

MA Math Camp 2022

Due Date : Monday August 29th, 2022

Answers should be typed and submitted in PDF format on Gradescope (see the course website for details). Be sure to answer every question thoroughly, and try to write complete and rigorous yet concise proofs. You can contact me if you have *specific* questions about the problem set, or if you think you have spotted a typo or mistake.

1. Show that the image of an open set by a continuous function is not necessarily an open set. Show that the image of a closed set by a continuous function is not necessarily a closed set.
2. Prove the following theorem. Let (X, d_X) , (Y, d_Y) , (Z, d_Z) be metric spaces. Let set S be a subset of X , and $f : S \rightarrow Y$ be a continuous function. Let T be a set s.t. $f(S) \subset T \subset Y$, and $g : T \rightarrow Z$ be a continuous function. Then $g \circ f : S \rightarrow Z$ is a continuous function.
3. Check whether the following sets are subspaces of the n -dimensional real vector space \mathbb{R}^n , equipped with its usual addition and scalar product.
 - a) $\{\mathbf{0}\}$
 - b) $\{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} = \alpha \mathbf{z}, \text{ for some } \alpha \in \mathbb{R}, \text{ where } \mathbf{z} \in \mathbb{R}^n\}$.
 - c) $\{(x_1, \dots, x_n) \in \mathbb{R}^n, x_1 = 0\}$
 - d) $\{(x_1, \dots, x_n) \in \mathbb{R}^n, x_1 \neq 0\}$
 - e) $\{(x_1, \dots, x_n) \in \mathbb{R}^n, x_1 + x_2 = 0\}$
 - f) $\{(x_1, \dots, x_n) \in \mathbb{R}^n, x_1 = 0 \text{ or } x_2 = 0\}$
 - g) (When $n = 1$) the set of integers \mathbb{Z} .
 - h) (When $n = 3$) $S := \{(t - 2s, -s, t) : t, s \in \mathbb{R}\}$.
 - i) $\text{Ker } A := \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$, where A is an $n \times n$ real matrix.
4. Let E a vector space and F and G two vector subspaces of E . Show that $F \cup G$ is a vector subspace if and only if $F \subset G$ or $G \subset F$. Show that E cannot be written as the union of two vector subspaces different from E itself.
5. Consider the following collection of vectors in \mathbb{R}^4 :

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

is it an independent family ?

6. Show that the following $\|\cdot\|$ are valid norms in \mathbb{R}^n .
 - a) $\|\mathbf{x}\| := \max_{i=1}^n |x_i|$.
 - b) $\|\mathbf{x}\| := \sum_{i=1}^n |x_i|$.

7. Find non-zero 2×2 matrices, A, B such that $AB = 0$.
8. Show that for any two $n \times n$ matrices A and B , $\text{Tr}(AB) = \text{Tr}(BA)$.
9. Let A an $n \times n$ matrix and denote by I_n the identity matrix of size n . Show that : there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$ if and only if for any matrix B of size n , $AB = BA$.
10. Determine the rank of the following matrices:
 - a) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$
 - b) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix}$
 - c) $\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$
11. Is it possible that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent?
12. State whether each of the following statements is true or false. Justify it accordingly with a short proof or a counterexample.
 - a) No system of linear equations can have exactly k solutions for any $k \geq 2$.
 - b) If $A\mathbf{x} = \mathbf{0}$ has a solution, then $A\mathbf{x} = \mathbf{b}$ has a solution.
 - c) If an $n \times n$ matrix A is full rank, then $A\mathbf{x} = \mathbf{b}$ has a solution.
 - d) If an $n \times n$ matrix A has rank less than n , then $A\mathbf{x} = \mathbf{b}$ has no solution.
 - e) If an $n \times n$ matrix A is full rank, all its eigenvalues are distinct.
 - f) Every diagonal real matrix has real eigenvalues.
 - g) An $n \times n$ matrix A has a zero eigenvalue if and only if it has rank less than n .
13. Let A be an $n \times n$ positive definite real matrix.
 - a) Verify that $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T A \mathbf{y}$$
 is a valid inner product.
 - b) Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $(\mathbf{x}^T A \mathbf{x})(\mathbf{y}^T A \mathbf{y}) \geq (\mathbf{x}^T A \mathbf{y})^2$.
14. Let A be an idempotent matrix (i.e $A^2 = A$). Show that the eigenvalues of A must be either 0 or 1.
15. Let A a symmetric invertible $n \times n$ matrix. Show that A^{-1} is symmetric.

16. Find the eigenvalues and associated eigenvectors of the following matrices. Diagonalize these matrices

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$