

**Solution for Problem Set 3**  
**MA Math Camp 2023**

1. State if and where the following function are differentiable, and compute their derivative :

- a)  $f : x \mapsto \frac{1}{1+x^2}$  defined over  $\mathbb{R}$
- b)  $f : x \mapsto \sqrt{x^2 - 1}$  defined over  $[1, \infty)$
- c)  $f : x \mapsto a^x$  defined over  $\mathbb{R}$ , with  $a > 0$
- d)  $f : (x, y) \mapsto \cos(x) \sin(y)$  over  $\mathbb{R}^2$

**Solution :**

- a)  $f'(x) = \frac{-2x}{(1+x^2)^2}$  differentiable over  $\mathbb{R}$
- b)  $f'(x) = \frac{x}{\sqrt{x^2-1}}$  differentiable over  $(1, \infty)$
- c) Observe that  $f(x) = \exp(x \ln(a))$ , hence  $f'(x) = \ln(a) \exp(x \ln(a)) = \ln(a)a^x$  for  $x \in \mathbb{R}$
- d)  $\nabla f(x) = (-\sin(x) \sin(y), \cos(x) \cos(y))$  for  $x, y \in \mathbb{R}^2$

2. a) Verify that Schwarz theorem (symmetry of the second order derivatives) holds for the following  $C^2$  functions :

- i.  $f(x, y) := x \exp(xy)$
- ii.  $f(x, y) := \ln(x^2 + y^2 + 1)$

**Solution :**

- i. Let  $f(x, y) := x \exp(xy)$ , we have :

$$\nabla f(x, y) = \exp(xy) \begin{pmatrix} xy + 1 \\ x^2 \end{pmatrix}$$

and :

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \exp(xy)(x + x(xy + 1)) = \exp(xy)(2x + yx^2) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

- ii. Let  $f(x, y) := \ln(x^2 + y^2 + 1)$ , we have :

$$\nabla f(x, y) = \frac{2}{1 + x^2 + y^2} \begin{pmatrix} x \\ y \end{pmatrix}$$

and :

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{-2xy}{(1 + x^2 + y^2)^2} = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

b) The function is clearly  $C^2$  over  $\mathbb{R}^2 \setminus \{(0,0)\}$ , furthermore its gradient then is :

$$\nabla f(x, y) = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^3(x^2 + y^2) - 2x^2y^3 \\ 3xy^2(x^2 + y^2) - 2xy^4 \end{pmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^5 - x^2y^3 \\ 3xy^2 + xy^4 \end{pmatrix}$$

We can verify easily that  $f$  is  $C^1$  over  $\mathbb{R}^2$  and  $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ . Let's show that the cross partial derivatives exist at  $(0,0)$  but do not coincide (hence if  $f$  was  $C^2$  over  $\mathbb{R}^2$ , Schwarz theorem would be violated) :

$$\begin{aligned} \frac{1}{h} \left( \frac{\partial f}{\partial x}(0, h) - \underbrace{\frac{\partial f}{\partial x}(0, 0)}_{=0} \right) &= \frac{1}{h} \frac{1}{h^4} (h^3(h^2 + 0^2) - 2h^3 \cdot 0^2) = 1 \xrightarrow{h \rightarrow 0} 1 = \frac{\partial^2 f}{\partial y \partial x}(0, 0) \\ \frac{1}{h} \left( \frac{\partial f}{\partial y}(h, 0) - \underbrace{\frac{\partial f}{\partial y}(0, 0)}_{=0} \right) &= \frac{1}{h} \frac{1}{h^4} (3h^3 \cdot 0^2 + h \cdot 0^4) = 0 \xrightarrow{h \rightarrow 0} 0 = \frac{\partial^2 f}{\partial x \partial y}(0, 0) \end{aligned}$$

3. Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  a  $C^1$  function. Show that the following function is continuous on  $\mathbb{R}^2$  :

$$f(x, y) := \begin{cases} \frac{F(x) - F(y)}{x - y} & \text{if } x \neq y \\ F'(x) & \text{if } x = y \end{cases}$$

**Solution :** Since  $F$  is continuous, the function  $f$  is continuous at every point  $(x, y)$  with  $x \neq y$ . To show that it is continuous at any point  $(a, a) \in \mathbb{R}^2$ , we first restate the mean value theorem : for any  $x, y \in \mathbb{R}$ , there exists  $c$  between  $x$  and  $y$  such that :

$$F(x) - F(y) = F'(c)(x - y)$$

Let  $\epsilon > 0$ . Since  $F'$  is continuous at  $a$ , there exists  $\delta > 0$  such that if  $|t - a| < \delta$ ,  $|F'(t) - F'(a)| < \epsilon$ . Now consider  $(x, y)$  close enough to  $(a, a)$  in the sense that  $|x - a| < \delta$  and  $|y - a| < \delta$  (NB : this is equivalent to taking  $(x, y)$  in the ball of radius  $\delta$  around  $(a, a)$  according to the sup norm). Take  $c$  between  $x$  and  $y$  as in the statement of the mean value theorem. Since  $c$  is between  $x$  and  $y$ , directly  $|c - a| < \delta$  (using the triangular inequality and that  $c = \lambda x + (1 - \lambda)y$  for some  $\lambda \in [0, 1]$ ). Now consider two cases :

- If  $x = y$ , then  $|f(x, y) - f(a, a)| = |F'(x) - F'(a)| < \epsilon$  since  $|x - a| < \delta$
- If  $x \neq y$ , then  $|f(x, y) - f(a, a)| = |F'(c) - F'(a)| < \epsilon$  since  $|c - a| < \delta$

This shows that  $f$  is continuous at  $(a, a)$ . **Remark :** A proof that does not use the continuity of  $F'$  cannot be complete since if  $F'$  is not continuous,  $f$  is not continuous along the diagonal  $\{x = y\}$  hence it cannot be continuous on  $\mathbb{R}^2$ .

4. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  a differentiable function. Differentiate the functions :  $u(x) = f(x, -x)$ ,  $g(x, y) = f(y, x)$ .

**Solution :** We have :

$$\begin{aligned}u'(x) &= \frac{\partial f}{\partial x}(x, -x) - \frac{\partial f}{\partial y}(x, -x) \\ \frac{\partial g}{\partial x}(x, y) &= \frac{\partial f}{\partial y}(y, x) \\ \frac{\partial g}{\partial y}(x, y) &= \frac{\partial f}{\partial x}(y, x)\end{aligned}$$

5. For the following functions from a given interval  $I$  to  $\mathbb{R}$ , compute  $\sup_{x \in I} f(x)$ ,  $\inf_{x \in I} f(x)$ , state if these are attained and at which point(s) :

a)  $f(x) = x(1 - x)$  on  $I = [0, 1]$

**Solution :**  $f$  is continuous and  $I$  is compact, hence  $f$  attains its supremum and its infimum. Observe that  $f(x) \geq 0$  for all  $x \in [0, 1]$  and  $f(0) = f(1) = 0$ , hence  $\min_I f = 0$ . This implies that the maximum must be attained on  $(0, 1)$ , hence the interior condition  $f'(x) = 0$  must be verified at that point. We can directly compute that the derivative only cancels out at a single point  $x = 1/2$ , therefore this has to be the maximum and we have  $\max_I f = f(1/2) = 1/4$ .

b)  $f(x) = 1 - e^{-x}$  on  $I = \mathbb{R}^+$

**Solution :**  $f$  is continuous but  $I$  is not bounded so we cannot a priori conclude on the existence of extrema. However, we can observe that  $1 \geq e^{-x} > 0$  hence  $f$  is bounded. Since  $f(0) = 0$ , we have  $\min_I f = 0$ . Observing that  $\lim_{+\infty} f = 1$  ensures that  $\sup_I f = 1$  – but the supremum is not attained.

c)  $f(x) = 3x^4 - 4x^3 + 6x^2 - 12x + 1$  on  $I = \mathbb{R}$

**Solution :** Again, since the domain is not bounded we cannot a priori conclude about the existence of extrema even though  $f$  is continuous. Considering the limit as  $x \rightarrow \infty$ , we see that  $\sup_I f = \infty$ . Computing the derivative of  $f$  yields  $f'(x) = 12(x - 1)(x^2 + 1)$ . It cancels out at a single point,  $x_0 = 1$ . Furthermore  $f''(x) = 12(3x^2 - 2x + 1) > 0$  for any  $x \in \mathbb{R}$ , hence  $f$  is convex and  $f$  attains a local minimum at  $x_0$ . Since  $f'$  is increasing and zero at  $x_0 = 1$ ,  $f$  is decreasing on  $(-\infty, 1]$  and increasing on  $[1, +\infty)$ . Therefore  $f$  is bounded below and attains its infimum at  $x_0$  :  $\min_I f = f(1) = -6$ .

d)  $f(x) = \frac{1}{\sqrt{x^2 - x + 1}}$  on  $I = [0, 1]$

**Solution :** We can verify that the denominator is strictly positive over  $I$  (e.g by observing that  $x^2 - x + 1 = (x^2 - 1) + x$ ), hence  $f$  is continuous over  $I$ . Over  $x \in I$ ,  $3/4 \leq x^2 - x + 1 \leq 1$ , hence since the function  $x \mapsto 1/\sqrt{x}$  is decreasing, we have  $\min_I f = f(1) = 1$ ,  $\max_I f = f(1/2) = \sqrt{4/3}$ .

6. Find the maximum and minimum of  $f(x, y) = x^2 - y^2$  on the unit circle  $x^2 + y^2 = 1$  using the Kuhn-Tucker method. Using the substitution  $y^2 = 1 - x^2$  solve the same problem as a single variable unconstrained problem. Do you get the same results? Why or why not?

**Solution :** As the object  $f$  is continuous and the unit circle is compact, by Weierstrass this program has global minimum and maximum.

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}^2} \quad & x^2 - y^2 \\ \text{s.t.} \quad & x^2 + y^2 = 1 \end{aligned}$$

Lagrangian:  $\mathcal{L}(x, y, \lambda) = x^2 - y^2 + \lambda(1 - x^2 - y^2)$

$$\text{FOC: } \begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 2(1 - \lambda)x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = -2(\lambda + 1)y = 0 \end{cases}$$

Solve the two equations along with the constraint, we have either  $x = 0, y = \pm 1$  or  $x = \pm 1, y = 0$ , which are candidates for optimizers. Notice that  $-1 = f(0, 1) = f(0, -1) < f(1, 0) = f(-1, 0) = 1$  and that  $x^2 - y^2 \leq x^2 \leq x^2 + y^2 = 1$  and  $x^2 - y^2 \geq -y^2 \geq -(x^2 + y^2) = -1$ , we know the maximum is 1 and the minimum is -1.

If we substitute  $y^2 = 1 - x^2$  into the objective function, we get the unconstrained problem

$$\max_{x \in \mathbb{R}} 2x^2 - 1$$

which has no solution. The reason for the difference is that we have not imposed the constraint that  $1 - x^2 \geq 0$ , but this is necessary since  $1 - x^2$  must equal  $y^2$  for some real number  $y$ .

7. A consumer's utility maximization problem is

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}_{++} \times \mathbb{R}_+} \quad & \alpha \ln x + y \\ \text{s.t.} \quad & px + qy \leq m \\ & y \geq 0 \end{aligned}$$

where,  $\alpha > 0, p > 0, q > 0, m > 0$  are parameters.

a) Argue that the budget constraint must hold with equality.

**Solution :** Suppose not, and  $m - px - qy = c > 0$  at an optimum. Then we could increase  $y$  by  $c/q$ , which still satisfies all the constraints and obtain a strictly higher value of the objective function, a contradiction.

b) Write the Lagrangian. State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

**Solution :** The Lagrangian is given by :

$$\mathcal{L}(x, y, \lambda, \mu) = \alpha \ln(x) + y + \lambda(m - px - qy) + \mu y$$

Assuming the budget constraint holds with equality, the necessary conditions are :

$$\begin{aligned} \frac{\alpha}{x} - \lambda &= 0 \\ 1 + \mu - \lambda q &= 0 \\ y \geq 0, \mu &\geq 0, y\mu = 0 \end{aligned}$$

$$px + qy = m$$

Since the objective function is concave and the constraints are linear, these conditions are also sufficient for a maximum.

- c) Are there any admissible points where the constraint qualification fails?

**Solution :** Denote  $h(x, y) := m - px - qy$  and  $g(x, y) = y$ . We have :

$$(\nabla g(x, y), \nabla h(x, y)) = \left( \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Those vectors are linearly independent since  $p > 0$ , hence there are no points at which the constraint qualification fails – observe that this holds whether or not the positivity constraint on  $y$  is active : if it isn't, then the family composed of only the vector  $(p, q)$  is independent.

- d) Solve for the maximizer  $(x^*, y^*)$ .

**Solution :** First consider the case  $y = 0$ . Then we must have  $x = m/p$ , which implies  $\alpha/m = \lambda$ . Plugging this into the second FOC yields :

$$1 + \mu = \frac{\alpha q}{m} \Leftrightarrow \mu = \frac{\alpha q}{m} - 1$$

For this to be consistent (i.e for  $(m/p, 0)$  to be a candidate point), we need  $\frac{\alpha q}{m} - 1 \geq 0$ .

Next consider the case  $y > 0$ , then  $\lambda q = 1$  and  $\alpha/x = p/q$ , hence  $x = \frac{\alpha q}{p}$ . Plugging this into the budget constraint, we get :

$$\begin{aligned} px + qy = m &\Leftrightarrow \alpha q + qy = m \\ &\Leftrightarrow y = \frac{m}{q} - \alpha \end{aligned}$$

This must be nonnegative for this to be a consistent solution, so this is only possible if  $\frac{m}{q} - \alpha \geq 0$ .

Summing up, the solution is given by :

$$\begin{cases} x = \frac{m}{p}, y = 0 & \text{if } \alpha q \geq m \\ x = \frac{\alpha q}{p}, y = \frac{m}{q} - \alpha & \text{if } \alpha q \leq m \end{cases}$$

- e) Find the value function  $v(p, q, m)$ . What does the Envelope Theorem tell you about the derivative of  $v(p, q, m)$  with respect to  $q$ ?

**Solution :** Substituting the solution into the objective function gives :

$$v(p, q, m) = \begin{cases} \alpha \ln \left( \frac{m}{p} \right) & \text{if } \alpha q \geq m \\ \alpha \ln \left( \frac{\alpha q}{p} \right) + \frac{m}{p} - \alpha & \text{if } \alpha q \leq m \end{cases}$$

The envelope theorem tells us that the derivative of the value function with respect to  $q$  is equal to the derivative of the Lagrangian with respect to  $q$ , evaluated at the optimum point. Verify

it by first differentiating  $v(p, q, m)$  in  $q$  directly :

$$\frac{\partial v}{\partial q}(p, q, m) = \begin{cases} 0 & \text{if } \alpha q \geq m \\ \frac{\alpha}{q} - \frac{m}{q^2} & \text{if } \alpha q \leq m \end{cases}$$

Now observe that the derivative of the Lagrangian with respect to  $q$  is :

$$\frac{\partial \mathcal{L}}{\partial q}(x, y, \lambda, \mu | p, q, m) = -\lambda y$$

At the optimum we have, when  $\alpha q \geq m$ ,  $y = 0$ , which is consistent with the previous derivation.

When  $\alpha q \leq m$ ,  $\lambda = 1/q$  and  $y = m/q - \alpha$ , which also gives the same expression as before.

8. A firm produces two outputs,  $x$  and  $y$ , using a single input  $z$ . The price of  $x$  has been normalized to 1; the price of  $y$  is  $p$ . The firm's program is

$$\begin{aligned} \max_{(x,y) \in \mathbb{R}^2} \quad & x + py \\ \text{s.t.} \quad & x^2 + y^2 \leq z \\ & x \geq 0, y \geq 0 \end{aligned}$$

$p > 0$  and  $z > 0$  are parameters.

- a) Write the Lagrangian.

**Solution :** The Lagrangian is given by :

$$\mathcal{L}(x, y, \lambda, \mu, \nu) = x + py + \lambda(z - y^2 - x^2) + \mu y + \nu x$$

- b) State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

**Solution :** First order necessary conditions are given by :

$$\begin{aligned} 1 - 2\lambda x + \nu &= 0 \\ p - 2\lambda y + \mu &= 0 \\ y^2 + x^2 &\leq z \\ x &\geq 0 \\ y &\geq 0 \\ \lambda(z - y^2 - x^2) &= 0 \\ \nu x &= 0 \\ \mu y &= 0 \\ \lambda, \mu, \nu &\geq 0 \end{aligned}$$

Since the objective function is linear (therefore concave), and the constraints are convex, these conditions are sufficient for a maximum.

- c) Are there any admissible points where the constraint qualification fails? Can any of these points

be a solution to the program?

**Solution :** First consider points at which only the first constraint binds, i.e  $x^2 + y^2 = z$ , but  $x, y > 0$ . The CQ will fail only if the gradient  $\begin{pmatrix} 2x \\ 2y \end{pmatrix}$  is equal to zero, which cannot be the case since  $x, y > 0$ . Consider next the case where the first constraint binds,  $y = 0$ , but  $x > 0$ . The gradient associated to the second constraint is  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  which is linearly independent from the gradient of the first constraint  $\begin{pmatrix} 2x \\ 2y \end{pmatrix}$ , hence the CQ holds. The case  $x = 0$  but  $y > 0$  is symmetric. Lastly, it can never be the case that all three constraints bind since  $z > 0$  and if the first constraint does not bind the CQ is directly always verified : the gradients associated to the other two constraints  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  are always linearly independent. We conclude that there are no points at which the CQ fails.

d) Solve for the maximizer  $(x^*, y^*)$ .

**Solution :** First suppose that only the first constraint binds. Then  $\nu = \mu = 0$ . Dividing the second FOC by the first yields  $px = y$  hence  $p^2 x^2 = y^2$ . We substitute in the constraint to get  $x^2 + p^2 x^2 = z$ , which then gives :

$$x = \sqrt{\frac{z}{1+p^2}}$$

$$y = p\sqrt{\frac{z}{1+p^2}}$$

This is our first solution candidate. Since this solution has  $x$  and  $y$  strictly positive, we can never have a maximum at which both of the nonnegativity constraints bind :  $x = y = 0$  yields a strictly lower value than the previous point. Now suppose  $\nu > 0$  but  $\mu = 0$ . Then  $x = 0$  so from the first constraint  $1 + \nu = 0$ , which cannot be satisfied for any  $\nu > 0$ . The same goes for the symmetric case  $\nu = 0$  and  $\mu > 0$ . Hence the only candidate is the one we previously found. This must be the maximum since the Kuhn-Tucker conditions are sufficient here.

e) Find the value function,  $f^*(p, z)$ .

**Solution :** Substituting the optimal  $(x^*, y^*)$  into the objective function yields :

$$f^*(p, z) = p^2 \sqrt{\frac{z}{1+p^2}} + \sqrt{\frac{z}{1+p^2}} = \sqrt{z(1+p^2)}$$

f) What does the Envelope Theorem tell you about the derivative of  $f(p, z)$  with respect to  $z$ ?

**Solution :** Applying the Envelope Theorem yields :

$$\frac{\partial f^*}{\partial z}(p, z) = \left[ \frac{\partial \mathcal{L}}{\partial z}(x, y, p, z) \right]_{x=x^*(p,z), y=y^*(p,z)}$$

$$= \lambda^*(p, z)$$

We can verify by plugging in the value of the multiplier that this coincides with directly differentiating the expression from the previous question in  $z$ .