Multivariate Calculus 1 - MA Math Camp 2022

Andrea Ciccarone

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Derivatives in one dimension

- The fundamental concept in calculus is the **derivative**. The main idea underlying derivatives is a rate of change.
- Consider a function $f : \mathbb{R} \to \mathbb{R}$ and take:

$$\frac{f(x) - f(x_0)}{x - x_0}$$

  This is giving us the average rate of change of the function f between x and x_0.
- The idea of the derivative is to let x go to x_0 and study what happens at this rate of change. In this sense, we can talk about the **instantaneous rate of change**:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Definition of Derivative

Definition 1.1

- Let $A \subseteq \mathbb{R}$, and $x_0 \in A \cap A'$. A function $f : A \to \mathbb{R}$ is said to be **differentiable** at x_0 iff the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

  exists. In that case, In that case, define the **derivative of** f **at** x_0 as the limit above, denoted as $f'(x_0)$.
- A function $f : A \to \mathbb{R}$ is said to be **differentiable** iff $A \subseteq A'$ and f is differentiable at any $x_0 \in A$.
- Let $\hat{A}$ be the set of points in $A \cap A'$ at which f is differentiable. Then the function $f' : \hat{A} \to \mathbb{R}$ is called the **derivative (function)** of f.

An Example

- Notice that by just applying the change of variable $x = x_0 + h$ we can rewrite the limit as:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- For example, if we want to show that the function $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \sqrt{x}$ is differentiable (and compute the associated derivative) we observe:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{x + h - x} = \frac{1}{\sqrt{x+h} - \sqrt{x}}$$

so that:

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x}}$$

Differentiability implies Continuity

- If a function is differentiable at x_0 it is continuous at x_0. The contrary is not always true (Example?)
- Remember that a function is continuous at x_0 iff the limit of the function at x_0 is equal to the value of the function at x_0
- We can thus prove that a differentiable function is continuous by writing:

$$
\begin{aligned}
\left[\lim_{x\to x_0} f(x)\right] - f(x_0) &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right] \\
&= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\cdot\lim_{x\to x_0}(x-x_0) \\
&= f'(x)\cdot 0 = 0
\end{aligned}
$$

Common Derivatives and Properties

- **Common Derivatives:**
  - $(x^\alpha)' = \alpha x^{\alpha-1}$
  - $(\ln x)' = \frac{1}{x}$
  - $(e^x)' = e^x$
  - $(\sin x)' = \cos x$
- **Properties:**
  - $(f+g)' = f' + g'$
  - $(\lambda f)' = \lambda f'$
  - $(fg)' = f'g + fg'$
  - $(f/g)' = \frac{f'g - fg'}{g^2}$

Derivatives as Affine Transformations

- An other interpretation of a derivative is that it is the best linear approximation of a function at x_0.
- In order to appreciate this interpretation, let us introduce first order expansions

Definition 1.4

Let $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. We say that f admits a first order expansion around x_0 if there exists $a, b \in \mathbb{R}$ and a function $\epsilon : A \to \mathbb{R}$ such that:

$$\forall x \in A, f(x) = a + b(x - x_0) + (x - x_0)\varepsilon(x)$$
$$\text{and } \lim_{x \to x_0} \varepsilon(x) = 0$$

Derivatives as Affine Transformations (ctd.)

- If a function is differentiable at x_0 it will always have a first order expansion at x_0 (set $\epsilon(x) = \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$, $a = f(x_0)$, $b = f'(x_0)$)
- If a function has a first order expansion, it will be differentiable and we can show $a = f(x_0)$, $b = f'(x_0)$:
  - If $\forall x \in A, f(x) = a + b(x - x_0) + (x - x_0)\varepsilon(x)$ then $f(x_0) = a$
  - For $x \in A \setminus \{0\}$ we have:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x) - a}{x - x_0} = b + \epsilon(x) \xrightarrow[x \to x_0]{} b$$

  - Thus $f'(x_0) = b$
- All this combined implies the next result.

Derivatives as Affine Transformations (ctd.)

Theorem 1.5

Let $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. The following are equivalent :

1. f is differentiable at x_0
2. f has a first order expansion at x_0

Furthermore the coefficients of the first order expansion when they exist are $a = f(x_0)$, $b = f'(x_0)$.

- The function $f(x_0) + (x - x_0)f'(x_0)$ is the affine transformation of f at x_0. Geometrically speaking, it is line that is tangent to f at x_0.

Mean Value Theorem

- The following theorem is an important result with many useful implications:

Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$, differentiable on (a, b), and continuous on $[a, b]$. Then there exists $x \in (a, b)$ s.t.

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

- In words, MVT states that we can find some x in the interior of the interval $[a, b]$ such that the average rate of change is equal to the instantaneous rate of change at that point
- We are not going to prove it, but the proof is on the lecture notes (you are also going to see it in Math Methods)

MVT and IVT

- An important implication of MVT is that if $f' > 0$ then f is stringly increasing.
- Take any x_1, x_2 in (a, b) with $x_1 < x_2$. By MVT there exists some $x \in (x_1, x_2)$ s.t.
$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
- Thus $f(x_2) - f(x_1) = f'(x) \cdot (x_2 - x1) > 0 \implies f$ is strictly increasing.

Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ continuous and u is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ s.t. $u = f(c)$.

L'Hospital Rule

- Using MVT, it is also possible to obtain a result which is very useful in computing some limits.
- In order to state this result we need to define the extended real line $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ by extending the order $\leq$ s.t. $+\infty > a$ and $-\infty < a$ for any $a \in \mathbb{R}$.

L'Hospital Rule

Let $-\infty \leq a < b \leq +\infty$, and $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R} \setminus \{0\}$ are differentiable in (a, b). If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are both 0 or $\pm\infty$, and $\lim_{x \to a} f'(x)/g'(x)$ has a finite value or is $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The statement is also true for $x \to b$.

L'Hospital Rule - Example

- Suppose we want to find the limit of $\frac{\ln(x)}{\sqrt{x}}$ when $x \to +\infty$.
- Notice that both the nominator and the denominator diverge to $+\infty$, so we can apply L'Hospital
- In particular, we have:

$$\frac{(\ln x)'}{(\sqrt{x})'} = \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{2}{\sqrt{x}} \to 0$$

- We can thus conclude that our function converges to 0 as x diverges to $+\infty$.

Total Derivatives - Introduction

- As economists, you are going to work a lot with functions from $\mathbb{R}^n$ to $\mathbb{R}$ (eg. utility functions).
- We thus want to extend the concept of derivatives to multivariate functions.
- We mentioned that the derivative at x_0 is slope of the "best" linear approximation we can find for $f(x)$ around x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- "Best" means that the relative error term $\epsilon(x)$ goes to 0. In other words, $f'(x_0)$ is the value of m such that:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - m(x - x_0)}{x - x_0} = 0$$

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Total Derivatives

- If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ then we can write a linear approximation (around x_0) of f as:

$$f(x) \approx f(x_0) + C(x - x_0)$$

  where A is an $m \times n$ matrix.

- Thus, we are going to define the **total derivative** of f at x_0 as the matrix C such that:

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - C(x - x_0)\|}{\|x - x_0\|} = 0$$

Total Derivative - Definition

Definition 2.1

- Let $A \subset \mathbb{R}^n$ and $x \in int\,(A)$. A function $f : A \to \mathbb{R}^m$ is said to be **differentiable at** x iff $\exists$ an $m \times n$ real matrix C s.t.

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ch\|}{\|h\|} = 0$$

  In this case, define the **(total) derivative of** f **at** x as the matrix C, denoted as $f'(x)$, or $Df(x)$.
- A function $f : A \to \mathbb{R}$ is said to be **differentiable** iff A is open and f is differentiable at any $x \in A$.
- Let $A_1 \subset int\,(A)$ be the set of points at which f is differentiable. Then the function $f' : A_1 \to \mathbb{R}^{mn}$ is called the **derivative (function)** of f.