Solutions for Problem Set 1
MA Math Camp 2023

1. Let P, Q two statements.
   a. Show that the statements $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent, using a truth table.
   b. Show the contrapositive principle $(P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P)$.

   Solution : One way to write the truth tables (if this is not clear enough, also include explicitly $\neg P$ and $\neg Q$) :

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

   For the contrapositive, recall that $P \Rightarrow Q$ is the statement $\neg P \vee Q$, and $\neg Q \Rightarrow P$ is the statement $\neg(\neg Q) \vee \neg P$, which is logically equivalent to $Q \vee \neg P$, which is itself logically equivalent to $\neg P \vee Q$. We can also write the truth table directly to verify it.

2. Write the negation of each of the following statement. Interpret each statement and state (if possible) which is true or false.
   a. $\forall x \in \mathbb{R}, x^2 \geq 0$
   b. $\forall x \in \mathbb{R}, x^2 > 0$
   c. $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, x \leq M$
   d. $\forall x \in \mathbb{R}, \exists M \in \mathbb{R}, x \leq M$
   e. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists \theta \in \mathbb{R}, |x-y|^2 \leq \theta|x-y|$
   f. $\exists \theta \in \mathbb{R}, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, |x-y|^2 \leq \theta|x-y|$
   g. $\forall (x, y) \in \mathbb{R}^2, x + y = 0 \Rightarrow (x = 0 \text{ and } y = 0)$
   h. $\forall (x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 0 \Rightarrow (x = 0 \text{ and } y = 0 \text{ and } z = 0)$

   Solution :
   a. Negation : $\exists x \in \mathbb{R}, x^2 < 0$; the statement is true.
   b. Negation : $\exists x \in \mathbb{R}, x^2 \leq 0$; the statement is false ($0^2 = 0$).
   c. Negation : $\forall M \in \mathbb{R}, \exists x \in \mathbb{R}, x < M$; the statement is false.
   d. Negation : $\exists x \in \mathbb{R}, \forall M \in \mathbb{R}, x > M$; the statement is true.
   e. Negation : $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall \theta \in \mathbb{R}, |x-y|^2 > \theta|x-y|$; the statement is true.
   f. Negation : $\forall \theta \in \mathbb{R}, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, |x-y|^2 > \theta|x-y|$; the statement is false.

g. $\exists (x,y) \in \mathbb{R}^2, x+y=0$ and ($x \neq 0$ or $y \neq 0$) ; the statement is false ($1+(-1)=0$).

h. $\exists (x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2=0$ and ($x \neq 0$ or $y \neq 0$ and $z \neq 0$) ; the statement is true.

3. Write explicitly :

   a. $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$

   b. $\mathcal{P}(\mathcal{P}(\{a,b\}))$

Solution :

$$\begin{aligned}
\mathcal{P}(\emptyset) &= \{\emptyset\} \\
\mathcal{P}\big(\mathcal{P}(\emptyset)\big) &= \{\emptyset, \{\emptyset\}\} \\
\mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\emptyset)\big)\Big) &= \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(\{a,b\}) = &\big\{\emptyset, \{a\}, \{b\}, \{a,b\}\big\} \\
\mathcal{P}\big(\mathcal{P}(\{a,b\})\big) = &\big\{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \\
&\{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a,b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\} \\
&\{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a,b\}\}, \{\emptyset, \{b\}, \{a,b\}\}, \{\{a\}, \{b\}, \{a,b\}\} \\
&\{\emptyset, \{a\}, \{b\}, \{a,b\}\}\big\}
\end{aligned}$$

4. Let E and F two sets.

   a. Show that $E \subseteq F \Leftrightarrow \mathcal{P}(E) \subseteq \mathcal{P}(F)$.

   b. Compare $\mathcal{P}(E \cup F)$ and $\mathcal{P}(E) \cup \mathcal{P}(F)$ (is one included in the other ?).

Solution :

a. We prove the result by double implication :

($\Rightarrow$) First assume that $E \subseteq F$. Consider any $A \in \mathcal{P}(E)$; by definition for all $x \in A$, $x \in E$ hence $x \in F$ since $E \subseteq F$. This means $A \subseteq F$, i.e $A \in \mathcal{P}(F)$. Since A was arbitrary, this means $\mathcal{P}(E) \subseteq \mathcal{P}(F)$. T

($\Leftarrow$) Now assume that $\mathcal{P}(E) \subseteq \mathcal{P}(F)$, i.e for all $A \in \mathcal{P}(E)$, $A \in \mathcal{P}(F)$. In particular, $E \in \mathcal{P}(E)$ hence $E \in \mathcal{P}(F)$: in other words $E \subseteq F$.

This completes the proof.

b. We have $\mathcal{P}(E) \cup \mathcal{P}(F) \subseteq \mathcal{P}(E \cup F)$. Indeed, let $A \in \mathcal{P}(E) \cup \mathcal{P}(F)$: either $A \subseteq E \subseteq E \cup F$ or $A \subseteq F \subseteq E \cup F$. In both cases, $A \subseteq E \cup F$ i.e $A \in \mathcal{P}(E \cup F)$. The reverse inclusion does not hold in general ; to show it, consider a simple example : $E = \{a\}, F = \{b\}$ with $a \neq b$. Then $\{a,b\} \in \mathcal{P}(E \cup F)$ but $\{a,b\} \notin \mathcal{P}(E) \cup \mathcal{P}(F)$.

5. Let E a non-empty set and let A, B, C subsets of E. Show that :

   a. $A = B \Leftrightarrow A \cap B = A \cup B$

Solution : That $A = B$ implies $A \cap B = A \cup B$ is trivial since in that case $A \cap B = A = B = A \cup B$. We show the converse. Assume $A \cap B = A \cup B$. We have $A \subseteq A \cup B = A \cap B \subseteq B$, hence $A \subseteq B$. Similarly $B \subseteq A \cup B = A \cap B \subseteq A$, hence $B \subseteq A$. We conclude that $A = B$ and this completes the proof.

b. $A \cap B^c = A \cap C^c \Leftrightarrow A \cap B = A \cap C$

Solution :

($\Rightarrow$) Assume $A \cap B^c = A \cap C^c$. Let $x \in A \cap B$, i.e $x \in A$ and $x \in B$. Hence $x \notin B^c$ so $x \notin A \cap B^c$. Since $A \cap B^c = A \cap C^c$, $x \notin A \cap C^c$, so $x \notin C^c$, hence $x \in A$ and $x \in C$, showing $x \in A \cap C$. This shows $A \cap B \subseteq A \cap C$. The proof is symmetrical to get $A \cap C \subseteq A \cap B$. Another method : observe that $A \cap B = A \setminus (A \cap B^c) = A \setminus (A \cap C^c) = A \cap C$.

($\Leftarrow$) The proof is exactly symmetrical for the other direction (recall that the complement of the complement is the set itself, so this is the same proof up to a relabeling).

c. $\begin{cases} A \cap B \subseteq A \cap C \\ A \cup B \subseteq A \cup C \end{cases} \Rightarrow B \subseteq C$

Solution : Let $A \cap B \subseteq A \cap C$ and $A \cup B \subseteq A \cup C$. Take $x \in B$ and consider two cases :

- Either $x \in A$, in which case $x \in A \cap B \subseteq A \cap C$, hence $x \in C$
- Or $x \in A^c$, but then we have $x \in B \subseteq A \cup C \subseteq A \cup C$, hence $x \in A \cup C$ but $x \notin A$, so it must be that $x \in C$

In both cases, $x \in C$ so we conclude $B \subseteq C$.

d. Define the symmetrical difference, denoted Δ, of A and B as :

$$\begin{aligned} A \Delta B &:= \{(x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\} \\ &= (A \cup B) \setminus (A \cap B) \\ &= (A \setminus B) \cup (B \setminus A) \end{aligned}$$

The three definitions above are equivalent. Show that :

$$A^c \Delta B^c = A \Delta B$$

Solution : We can write directly :

$$\begin{aligned} A^c \Delta B^c &= (A^c \cup B^c) \setminus (A^c \cap B^c) \\ &= (A^c \cup B^c) \cap (A^c \cap B^c)^c \\ &= (A^c \cup B^c) \cap (A \cup B) \\ &= (A \cap B)^c \cap (A \cup B) \\ &= (A \cup B) \setminus (A \cap B) \\ &= A \Delta B \end{aligned}$$

6. Let R be a complete, transitive relation over a set X. Define the relation $\sim$ as follows : $a \sim b$ iff aRb and bRa. For any $x \in X$, define the set $I(x)$ as :

$$I(x) := \{y \in X | y \sim x\}$$

Show that for any $x, y \in X$, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$

Solution : Let $x, y \in X$. First observe that $\sim$ is transitive : this is a direct consequence of the transitivity of R. Consider two cases :

- If $x \sim y$, then by transitivity of $\sim$, for any $z \in I(y)$ $y \sim z$ implies $x \sim z$, hence $z \in I(x)$. Symmetrically any $z \in I(x)$ is in $I(y)$ hence in this case $I(x) = I(y)$
- If not (i.e $\neg(x \sim y)$), assume by contradiction that there exists $z \in I(x) \cap I(y)$. Then $x \sim z$ and $z \sim y$, so transitivity implies $x \sim y$, a contradiction. We conclude that $I(x) \cap I(z) = \emptyset$ in this case.

7. Let I an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a function defined over I taking values in $\mathbb{R}$. Write mathematical statements (using quantifiers) to express the following statements :

   a. f takes the value zero

   Solution : $\exists x \in I, f(x) = 0$

   b. f is the zero function (takes the value zero everywhere)

   Solution : $\forall x \in I, f(x) = 0$

   c. f is not a constant function

   Solution : $\exists x, y \in I, f(x) \neq f(y)$

   d. f never takes the same value twice

   Solution : $\forall x, y \in I, f(x) = f(y) \Rightarrow x = y$

8. Let X, Y two sets and $f \in Y^X$.

   a. Show that for any $A, B \in \mathcal{P}(E)$, $f(A \cap B) \subseteq f(A) \cap f(B)$

   Solution : Let $y \in f(A \cap B)$; this means that there exists $x \in A \cap B$ such that $f(x) = y$. Since $x \in A$, $y \in f(A)$ by definition, and similarly $y \in f(B)$ since $x \in B$. Hence $y \in f(A) \cap f(B)$ and this proves the result.

   b. Show that f is injective if and only if for any $A, B \in \mathcal{P}(E)$: $f(A \cap B) = f(A) \cap f(B)$

Solution : We already know from the previous question that one inclusion holds for an arbitrary function f, so it is sufficient to show that :

$$f \text{ injective } \Leftrightarrow \forall A, B \in \mathcal{P}(E), f(A) \cap f(B) \subseteq f(A \cap B)$$

We first show the "if" direction by contraposition. Assume that there exists A, B such that $f(A \cap B) \subsetneq f(A) \cap f(B)$, i.e there exists $y \in f(A) \cap f(B)$ such that $y \notin f(A \cap B)$. In other words, there exists $x \in (A \cup B) \setminus (A \cap B)$ such that $f(x) = y$. But since $y \in f(A \cap B)$ there also exists $x' \in A \cap B$ such that $f(x') = y$. Hence we have $x' \neq x$ by construction but $f(x) = f(x')$, hence f is not injective. This shows (by contraposition) that f injective $\Rightarrow \forall A, B \in \mathcal{P}(E), f(A) \cap f(B) \subseteq f(A \cap B)$. To show the "only if" direction, assume $\forall A, B \in \mathcal{P}(E), f(A) \cap f(B) \subseteq f(A \cap B)$. Let $x, x' \in X$ such that $f(x) = f(x') =: y$. By assumption, we have $f(\{x\} \cap \{x'\}) = f(\{x\}) \cap f(\{x'\}) = \{y\}$. If $x' \neq x$, $\{x\} \cap \{x'\} = \emptyset$, and the previous equality becomes $f(\emptyset) = \emptyset = \{y\}$ which is a clear contradiction. Therefore $x = x'$, and we conclude that f is injective. This completes the proof.

c. Find an example of a function f such that there exists $A, B \in \mathcal{P}(E)$ for which $f(A \cap B) \subsetneq f(A) \cap f(B)$

Solution : Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and let $A := [-1, 0]$, $B = [0, 1]$. We have

$$f(A \cap B) = f(\{0\}) = \{0\}$$

and

$$f(A) = f(B) = [0, 1]$$

hence $f(A) \cap f(B) = [0, 1]$, so clearly $f(A \cap B) \subsetneq f(A) \cap f(B)$. Observe that non-injectivity is key to constructing this example, as expected.

9. Let X, Y two sets and $f \in Y^X$.

a. Show that for all $A \in \mathcal{P}(X)$, $A \subseteq f^{-1}(f(A))$ and this holds with equality for all A if and only if f is injective.

Solution : The inclusion is direct by definition. Let $x \in A$, then $f(x) \in f(A)$ by definition of the image, hence $x \in f^{-1}(f(A))$ by definition of the inverse image.

If $A \subsetneq f^{-1}(f(A))$, there exists $y \in f(A)$ such that there exists $x \notin A$ with $f(x) = y$ (otherwise the reverse inclusion would hold); but since $y \in f(A)$, there also exists $x' \in A$ such that $f(x') = y$. Since $x' \neq x$ but $f(x) = f(x')$, f is not injective. By contraposition, if f is injective, $A = f^{-1}(f(A))$ for all A.

Now assume conversely that $A = f^{-1}(f(A))$ for all A and let $x, x' \in X$ such that $f(x) = f(x')$. $f(\{x\}) = f(\{x'\})$, therefore $f^{-1}(f(\{x\})) = f^{-1}(f(\{x'\}))$, but we know that $f^{-1}(f(\{x\})) = \{x\}$ and $f^{-1}(f(\{x'\})) = \{x'\}$, hence $x = x'$ and f is injective. This completes the proof.

b. Show that for all $B \in \mathcal{P}(Y)$, $f(f^{-1}(B)) \subseteq B$ and this holds with equality for all B if and only if f is surjective.

Solution : The inclusion is direct by definition : let $y \in f(f^{-1}(B))$, this means that there exists $x \in f^{-1}(B)$, with $f(x) = y$; by definition of the inverse image $x \in f^{-1}(B)$ means $f(x) \in B$, hence $y \in B$.

If $f(f^{-1}(B)) \subsetneq B$, there exists $y \in B$ such that $y \notin f(f^{-1}(B))$; if there exists $x \in X$ such that $f(x) = y$, then $xf^{-1}(B)$ which implies $y \in f(f^{-1}(B))$ and that would be a contradiction. Therefore, there exists no $x \in X$ such that $f(x) = y$ and this shows that f is not surjective. By contraposition, we have just shown that f surjective implies $f(f^{-1}(B)) = B$ (since the reverse inclusion always holds).

Now assume that for every $B \in \mathcal{P}(Y)$, $f(f^{-1}(B)) = B$. For an arbitrary $y \in Y$, consider the set $\{y\}$. Assume by contradiction that there exists no $x \in X$ such that $f(x) = y$, then $f^{-1}(\{y\}) = \emptyset$, and $f(f^{-1}(\{y\})) = \emptyset$ but we know $f(f^{-1}(\{y\})) = \{y\}$ hence this is a contradiction. This means that for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$, i.e f is surjective.

10. Show that ther does not exist any surjective function from E into $\mathcal{P}(E)$ (this is a famous result due to Cantor). Hint : consider $\phi : E \to \mathcal{P}(E)$ and assume by contradiction that it is surjective, then consider the set $A := \{x \in E, x \notin \phi(x)\}$.

Solution : Let E and arbitrary set, and consider a function $\phi : E \to \mathcal{P}(E)$. Assume by contradiction that it is surjective. Now consider the set $A := \{x \in E, x \notin \phi(x)\}$. Since ϕ is surjective, there must exist $x \in E$ such that $\phi(x) = A$. Now consider two cases :

- If $x \in A$, then $x \notin \phi(x)$, but $\phi(x) = A$ hence we have $x \in A$ and $x \notin A$, which is a clear contradiction.
- If $x \notin A$, then it must mean that $x \in \phi(x)$ by definition of A, hence $x \in A$ since $A = \phi(x)$. Again we find $x \in A$ and $x \notin A$, a contradiction.

We conclude that there exists no surjection from a set to its power set.

11. Show the two following results by induction :

a. For any $n \in \mathbb{N}$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Solution : Proceeding by induction :

- (Initialization, $n = 0$) It is direct to verify $\sum_{k=0}^{0} k = 0 = 0(0+1)/2$
- (Induction) Let $n \in \mathbb{N}$, assume $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$. Then, we can rewrite :

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + (n+1)$$

Using our induction hypothesis, we have :

$$\sum_{k=0}^{n+1} k = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Hence by induction this holds for all n.

b. For any $n \in \mathbb{N}$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Similarly $\sum_{k=0}^{0} k = 0 = 0(0+1)(2 \cdot 0+1)/6$. Now assume for some $n \in \mathbb{N}$, $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Write :

$$\begin{aligned}\sum_{k=0}^{n+1} k^2 &= \sum_{k=0}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6}\end{aligned}$$

hence the result holds for all n.

12. Verify that the d_∞ norm on $\mathbb{R}^k$, defined as $d_\infty(x,y) := \max_{1\le i\le k} |x_i - y_i|$ is indeed a distance. In $\mathbb{R}^2$, draw the set of points x such that $d(x,0) = 1$.

Solution : The first two properties of a distance are direct to verify. To prove the triangle inequality, observe that for any x, y, z :

$$\begin{aligned}\max_{1\le i\le n} |x_i - z_i| + \max_{1\le i\le n} |z_i - y_i| &\ge \max_{1\le i\le n} (|x_i - z_i| + |z_i - y_i|) \\ &\ge \max_{1\le i\le n} (|x_i - y_i|)\end{aligned}$$

Where we used that the sum of the maxes is always weakly higher than the max of the sum and then applied the triangle inequality on $\mathbb{R}$ to each coordinate.

13. Consider (E,d) a metric space. Prove that for an arbitrary set $S \subseteq E$, the interior of S is open.

Solution : Let $x \in int(S)$. By definition of the interior there exists $r > 0$ such that $B(x,r) \subset S$. We want to show that $B(x,r) \subset int(S)$. Assume not, then there exists $x' \in B(x,r)$ such that $x' \notin int(S)$. This means that for all $\epsilon > 0$, there exists $x'' \in B(x', \epsilon) \setminus S$. For ϵ small enough, $B(x', \epsilon) \subset B(x,r)$, therefore there exists $x'' \in B(x,r) \setminus S$, which contradicts $B(x,r) \subset S$. We conclude that $B(x,r) \subset int(S)$; since x was arbitrary, S is open.

14. Consider the metric space $(\mathbb{R}, d_2)$ and two convergent sequences $x_n \to x$, $y_n \to y$. Prove the following results :

a. If for all $n \in \mathbb{N}$, $x_n \le y_n$ then $x \le y$

Solution : Suppose by contradiction $x > y$. Denote $\varepsilon := \frac{x-y}{2} > 0$. $x_n \to x \Rightarrow \exists N_x \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$, for $\forall n \geq N_x$, $y_n \to y \Rightarrow \exists N_y \in \mathbb{N}$ s.t. $|y_n - y| < \varepsilon$, for $\forall n \geq N_y$. Let $N := \max\{N_x, N_y\}$, then for $\forall n \geq N$, we have $x_n > x - \varepsilon = y + \varepsilon > y_n$, which contradicts $x_n \leq y_n$.

b. $x_n + y_n \to x + y$

Solution : Take any $\varepsilon > 0$. We want to find$N \in \mathbb{N}$ s.t. $|\,(x_n + y_n) - (x + y)\,| < \varepsilon$ for any $n \geq N$. $x_n \to x \Rightarrow \exists N_x \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon/2$, for $\forall n \geq N_x$. $y_n \to y \Rightarrow \exists N_y \in \mathbb{N}$ s.t. $|y_n - y| < \varepsilon/2$, for $\forall n \geq N_y$. Let $N := \max\{N_x, N_y\}$, then for $\forall n \geq N$, we have: $|\,(x_n + y_n) - (x + y)\,| = |\,(x_n - x) + (y_n - y)\,| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

c. If $x \neq 0$, $\frac{1}{x_n} \to \frac{1}{x}$

Solution : Take any $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ s.t. $|\frac{1}{x_n} - \frac{1}{x}| < \varepsilon$ for any $n \geq N$. $x_n \to x \Rightarrow \exists N \in \mathbb{N}$ s.t. $|x_n - x| < \min\left\{\frac{|x|}{2}, \frac{\varepsilon|x|^2}{2}\right\}$, for $\forall n \geq N$. The condition $|x_n - x| < |x|/2$ implies that $||x_n| - |x|| \leq |x_n - x| < |x|/2$ and therefore $|x|/2 < |x_n| < 3|x|/2$. Then for $\forall n \geq N$, we have $|\frac{1}{x_n} - \frac{1}{x}| = \frac{|x_n - x|}{|x||x_n|} < \frac{\varepsilon|x|^2/2}{|x||x|/2} = \varepsilon$.

15. Prove that the two definitions we gave for closed sets are equivalent. In other words, let (E, d) a metric space and $S \subseteq E$, prove that the two following statements are equivalent :

- S^c is an open set
- S contains all of its limit points

Solution : Assume that S^c is an open set. Let $x \in E$ a limit point of S, assume by contradiction that $x \notin S$, i.e $x \in S^c$. By definition of S^c being open, there exists $r > 0$ such that $B(x, r) \subset S^c$, but by definition of x being a limit point of S there must exist $x' \in B(x, r)$ such that $x' \in S$, which contradicts $B(x, r) \subset S^c$. We conclude that $x \in S$, therefore S contains all of its limit points.

Conversely assume that x contains all of its limit points. Assume by contradiction that S^c is not open, i.e there exists $x \in S^c$ such that for all $r > 0$, there exists $x' \in B(x, r)$ such that $x' \notin S^c$. Since $x' \notin S^c$ is equivalent to $x' \in S$, this equivalently means that x is a limit point of S, which by assumption entails $x \in S$, contradicting $x \in S^c$. Therefore S^c is open.

16. Prove that if a sequence (x_n) converges in $(\mathbb{R}, d_2)$, then $(|x_n|)$. Is the converse true ? If not, find a counterexample.

Solution : Suppose $x_n \to x \in \mathbb{R}$. Take any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$, for $\forall n \geq N$. We have $||x_n| - |x|| \leq |x_n - x| < \varepsilon$. Therefore $|x_n| \to |x|$.

The converse is not true. Take sequence (x_n) s.t.

$$x_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ -1, & \text{if } n \text{ is even.} \end{cases}$$

Clearly (x_n) does not converge. Note that $|x_n| = 1, \forall n \in \mathbb{N}$. Therefore $|x_n| \to 1$.

17. Show that the image of an open set by a continuous function is not necessarily an open set. Show that the image of a closed set by a continuous function is not necessarily a closed set.

Solution : Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ for all x. Take the open set $(0, 1)$, we have $f((0, 1)) = \{1\}$, which is not open. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = e^x$. Take the closed set $\mathbb{R}$ (the whole set, which is closed) : $g(\mathbb{R}) = (0, \infty)$ which is not closed.