Solution for Problem Set 3
MA Math Camp 2023
Due Date : Monday August 28th, 2023

1. Let A a non-empty bounded subset of $\mathbb{R}$. Show that $\inf A$ and $\sup A$ belong to the closure of A.

   Solution : Assume by contradiction that $\sup(A) \notin Cl(A)$. This implies that there exists $r > 0$ such that $B(\sup(A), r) \subset A^c$ (otherwise $\sup(A)$ would be a limit point of A). Consider any $x \in \mathbb{R}$ such that $\sup(A) - r < x < \sup(A)$. By construction, $x \geq y$ for any $y \in A$, but $x < \sup(A)$. This contradicts the definition of the supremum as the least upper bound, therefore $\sup(A) \in Cl(A)$. The proof is symmetrical for the infimum.

2. Prove the following theorem. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces. Let set S be a subset of X, and $f : S \to Y$ be a continuous function. Let T be a set s.t. $f(S) \subset T \subset Y$, and $g : T \to Z$ be a continuous function. Then $g \circ f : S \to Z$ is a continuous function.

   Solution : Take any $x_0 \in S$. WTS: $g \circ f$ is continuous at x_0.

   Take any $\varepsilon > 0$, as g is continuous at $f(x_0)$, there exists $\tau > 0$ s.t. $d_Z(g(y), g(f(x_0))) < \varepsilon$ for any $y \in T$ with $d_Y(y, f(x_0)) < \tau$.

   As f is continuous, there exists $\delta > 0$ s.t. $d_Y(f(x), f(x_0)) < \tau$ for any $x \in S$ with $d_X(x, x_0) < \delta$.

   Then for any $x \in S$ with $d_X(x, x_0) < \delta$, we have $d_Y(f(x), f(x_0)) < \tau$, and therefore $d_Z(g(y), g(f(x_0))) < \varepsilon$. QED.

   Alternative proof:

   Assume f and g are continuous . Let V be an open set on $(g(f(S)), d_Z)$. Since g is continuous $g^{-1}(V)$ is an open set in $(f(S), d_Y)$. Since f is continuous, $f^{-1}\left(g^{-1}(V)\right)$ is an open set in S. As $f^{-1}\left(g^{-1}(V)\right) = (g \circ f)^{-1}(V)$, we have just shown the result. QED.

3. Prove that the Euclidean space $\left(\mathbb{R}^k, d_2\right)$ is a complete metric space. (Hint: First prove a Cauchy sequence in a metric space is bounded.)

   Solution : First prove the following result:

   In metric space (X, d), let (x_n) be a Cauchy sequence, then (x_n) is a bounded sequence.

   Because (x_n) is Cauchy, there exists $N \in \mathbb{N}$ s.t. $d(x_m, x_n) < 1$ for $\forall m, n \geq N$. Therefore $x_n \in B_1(x_N)$ for $\forall n \geq N$. Let

   $r := \max\{d(x_1, x_N), d(x_2, x_N), ..., d(x_{N-1}, x_N)\} + 1$. Then we have $x_n \in B_r(x_N)$, for $\forall n \in \mathbb{N}$. Therefore the sequence (x_n) is bounded. QED.

   Now let's prove our main result.

   Take any Cauchy sequence (x_n) in $\left(\mathbb{R}^k, d_2\right)$, we want to find an $x \in \mathbb{R}^k$ s.t. $x_n \to x$.

   From the result above we know that (x_n) is bounded. By Bolzano-Weierstrass theorem, we can find a subsequence $(x_{n_k})_k \subset (x_n)$ s.t. $x_{n_k} \to x \in \mathbb{R}^k$. Now we only need to show $x_n \to x$.

   Take any $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ for $\forall n \geq N$.

   (x_n) is Cauchy $\Rightarrow \exists N \in \mathbb{N}$ s.t. $d(x_m, x_n) < \varepsilon/2$ for $\forall m, n \geq N$.

$(x_{n_k}) \to x \Rightarrow \exists K \in \mathbb{N}$ s.t. $d(x_{n_K}, x) < \varepsilon/2$ and $n_K \geq N$.

Then for $\forall n \geq N$, we have $d(x_n, x) \leq d(x_n, x_{n_K})) + d(x_{n_K}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. QED.

4. Check whether the following sets are subspaces of the n-dimensional real vector space $\mathbb{R}^n$, equipped with its usual addition and scalar product.

   a) $\{\mathbf{0}\}$
   Solution : This is a vector subspace of $\mathbb{R}^n$, the proof is direct.

   b) $\{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} = \alpha\mathbf{z},$ for some $\alpha \in \mathbb{R}\}$,where $\mathbf{z} \in \mathbb{R}^n$.
   Solution : This is a vector subspace of $\mathbb{R}^n$.

   c) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 = 0\}$
   Solution : This is a vector subspace of $\mathbb{R}^n$. Let $V = \{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 = 0\}$. Clearly $\mathbf{0} \in V$. Consider any $\lambda \in \mathbb{R}$, $(0, x_2, .., x_n), (0, x'_2, .., x'_n) \in V$, $\lambda(0, x_2, .., x_n) + (0, x'_2, .., x'_n) = (0, \lambda x_2 + x'_2, .., x_n + x'_n) \in V$

   d) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 \neq 0\}$
   Solution : This is not a vector subspace, since $\mathbf{0} \notin \{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 \neq 0\}$.

   e) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 + x_2 = 0\}$
   Solution : This is a vector subspace of $\mathbb{R}^n$. It clearly contains $\mathbf{0}$, furthermore if $x_1 + x_2 = 0$, and $x'_1 + x'_2 = 0$, then for any λ, $\lambda x_1 + x'_1 + \lambda x_2 + x'_2 = \lambda(x_1 + x_2) + x'_1 + x'_2 = 0$.

   f) $\{(x_1, ..., x_n) \in \mathbb{R}^n, x_1 = 0$ or $x_2 = 0\}$
   Solution : This is not a vector subspace of $\mathbb{R}^n$. Just consider vectors $(1, 0, ...)$ and $(0, 1, , ...)$, they both belong to this set but their sum does not.

   g) (When $n = 1$) the set of integers $\mathbb{Z}$.
   Solution : This is not a vector subspace of $\mathbb{R}^n$: $.5 \times 1 \notin \mathbb{Z}$.

   h) (When $n = 3$) $S := \{(t - 2s, -s, t) : t, s \in \mathbb{R}\}$.
   Solution : This is a vector subspace of $\mathbb{R}^n$. Take $s = t = 0$, we can see that $(0, 0, 0)^T \in S$. Take any $x_1 = (t_1 - 2s_1, -s_1, t_1), x_2 = (t_2 - 2s_2, -s_2, t_2) \in S$, we have $x_1 + x_2 = ((t_1 + t_2) - 2(s_1 + s_2), -(s_1 + s_2), t_1 + t_2) \in S$. $\forall \lambda \in \mathbb{R}$, $\lambda x_1 = ((\lambda t_1) - 2(\lambda s_1), -(\lambda s_1), (\lambda t_1)) \in S$.

   i) $KerA := \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$,where A is an $n \times n$ real matrix.
   Solution : Let $v, w \in Ker(A)$ and $\lambda \in \mathbb{R}$. $A(\lambda v + w) = \lambda Av + Aw = 0$, hence $\lambda v + w \in Ker(A)$. Clearly $A\mathbf{0} = 0$ hence $\mathbf{0} \in Ker(A)$. Hence $Ker(A)$ is a vector space.

5. Let E a vector space and F and G two vector subspaces of E. Show that $F \cup G$ is a vector space if and only if $F \subset G$ or $G \subset F$. Show that E cannot be written as the union of two vector subspaces different from E itself.

   Solution : Clearly if $F \subset G$ or $G \subset F$, $F \cup G$ is a vector subspace of E. Conversely, assume $F \cup G$ is a vector subspace of E. Assume that $F \not\subseteq G$ and $G \not\subseteq F$ i.e there exist $x \in G \setminus F$, $y \in F \setminus G$. Since $x, y \in F \cup G$, for every $\lambda, \mu \in \mathbb{R}$, $x + y \in F \cup G$. If $x + y \in F$, then $x + y - x = y \in F$, which is a contradiction. If $x + y \in G$, then $x + y - y = x \in G$, which is also a contradiction.

   Assume by contradiction that $E = F \cup G$ where $F \subsetneq E$ and $G \subsetneq E$ are vector subspaces. Then by the previous result $F \cup G = F$ i.e $E = F$ or $F \cup G = G$ i.e $E = G$, which proves the claim.

6. Consider the following collection of vectors in $\mathbb{R}^4$:

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

is it an independent family ?

Solution : No it is not. This can be proven directly by writing out the definition and solving for scalars such that the linear combination is zero (and observing that we can find a non-zero solution).

7. Show that the following $\|\cdot\|$ are valid norms in $\mathbb{R}^n$.

a) $\|\mathbf{x}\| := \max_{i=1}^n |x_i|$.

b) $\|\mathbf{x}\| := \sum_{i=1}^n |x_i|$.

Solution :

a) $|x_i|$ is nonnegative for all i, so the maximum of these numbers must be nonnegative; the maximum of these numbers can only be zero if they are all zero, so the first property of a norm is satisfied.

To prove the second property, note that if $|x_i| \geq |x_j|$ for all $j \neq i$ then $|\alpha x_i| \geq |\alpha x_j|$ for all $j \neq i$, so $|\alpha| \max_i |x_i| = \max_i |\alpha x_i|$.

To prove the last property, note that $||\mathbf{x}+\mathbf{y}|| = \max_i |x_i+y_i| \leq \max_i(|x_i|+|y_i|) \leq \max_j |x_j| + \max_k |y_k| = ||\mathbf{x}|| + ||\mathbf{y}||$.

b) $|x_i|$ is nonnegative for all i, so the sum of these numbers must be nonnegative; the sum of these numbers can only be zero if they are all zero, so the first property of a norm is satisfied.

The second property is satisfied since $||\alpha\mathbf{x}|| = \sum_i |\alpha x_i| = |\alpha| \sum_i |x_i| = |\alpha| ||\mathbf{x}||$.

The third property is satisfied since $||\mathbf{x}+\mathbf{y}|| = \sum_i |x_i+y_i| \leq \sum_i (|x_i|+|y_i|) = ||\mathbf{x}|| + ||\mathbf{y}||$.

8. Find non-zero 2×2 matrices, A, B such that $AB = 0$.

Solution : For example, $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

9. Show that for any two $n\times n$ matrices A and B, $Tr(AB) = Tr(BA)$.

Solution : Let $A = (a_{ij})_{1\leq i,j\leq n}$, $B = (b_{ij})_{1\leq i,j\leq n}$. Write :

$$\begin{aligned} Tr(AB) &= \sum_{i=1}^n (AB)_{ii} \\ &= \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik} b_{ki} \right) \\ &= \sum_{k=1}^n \sum_{i=1}^n b_{ki} a_{ik} \\ &= \sum_{k=1}^n (BA)_{kk} = Tr(BA) \end{aligned}$$

10. Let A an $n \times n$ matrix and denote by I_n the identity matrix of size n. Show that : there exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_n$ if and only if for any matrix B of size n, $AB = BA$.

    Solution : The only if direction is immediate. To prove the if direction, assume for any matrix B of size n, $AB = BA$. Define $B^{lk} = (b_{ij}^{lk})$ as the matrix such that :

$$b_{ij}^{lk} = \begin{cases} 1 & \text{if } (l,k) = (i,j) \\ 0 & \text{otherwise} \end{cases}$$

    Observe that :

$$(AB^{lk})_{ij} = \sum_{m=1}^{n} a_{im} b_{mj}^{lk} = \begin{cases} a_{ik} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$
$$(B^{lk}A)_{ij} = \sum_{m=1}^{n} b_{im}^{lk} a_{mj} = \begin{cases} a_{lj} & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}$$

    Since $(AB^{lk})_{ij} = (B^{lk}A)_{ij}$ and this holds for all l, k, this directly implies $a_{ij} = 0$ if $i \neq j$ and $a_{11} = a_{ii}$ for all i, hence $A = a_{11} I_n$, which proves the result.

11. Determine the rank of the following matrices:

    a) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{pmatrix}$

    b) $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{pmatrix}$

    c) $\begin{pmatrix} 1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 2 \\ -1 & 1 & -1 & -3 \\ -2 & -5 & -2 & 0 \end{pmatrix}$

    Solution : a) 2; b) 3; c) 3.

12. Is it possible that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2 + \mathbf{v}_3$ are linearly independent?

    Solution : No. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$. Let $V = \left(\mathbf{v}_1^T, \mathbf{v}_2^T, \mathbf{v}_3^T\right)^T$, $W = \left((\mathbf{v}_1 + \mathbf{v}_2)^T, (\mathbf{v}_1 + \mathbf{v}_3)^T, (\mathbf{v}_2 + \mathbf{v}_3)^T\right)^T$, then $V, W \in \mathcal{M}_{(3,n)}$ and we have $W = AV$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

    As $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent, we have $rank\,(W) = 3$.

    $W = AV \Rightarrow 3 = Rank\,(W) \leq \min\{Rank\,(A), Rank\,(V)\} \Rightarrow Rank\,(V) \geq 3 \Rightarrow Rank\,(V) = 3$.

    This implies $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Contradiction!

    Remark: As $\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3$ can be represented by linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the former always has an equal or smaller rank than the latter.

13. State whether each of the following statements is true or false. Justify it accordingly with a short proof or a counterexample.

a) No system of linear equations can have exactly k solutions for any $k \geq 2$.

b) If $A\mathbf{x} = \mathbf{0}$ has a solution, then $A\mathbf{x} = \mathbf{b}$ has a solution.

c) If an $n \times n$ matrix A is full rank, then $A\mathbf{x} = \mathbf{b}$ has a solution.

d) If an $n \times n$ matrix A has rank less than n, then $A\mathbf{x} = \mathbf{b}$ has no solution.

e) If an $n \times n$ matrix A is full rank, all its eigenvalues are distinct.

f) Every diagonal real matrix has real eigenvalues.

g) An $n \times n$ matrix A has a zero eigenvalue if and only if it has rank less than n.

Solution :

a) TRUE. From the lecture notes we know the number of solutions of a system of linear equations can only be zero, one or infinite.

b) FALSE. For example $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

c) TRUE. We can show that $Rank\left([A|\mathbf{b}]\right) = Rank\left(A\right) = n$.

d) FALSE. Take $\mathbf{b} = \mathbf{0}$.

e) FALSE. Consider I_n.

f) TRUE. For a diagonal matrix, eigenvalues are just diagonal terms.

g) TRUE. A square matrix has less than full rank iff its determinant is zero; since the determinant is the product of the eigenvalues and can equal zero only if at least one eigenvalue equals zero, a matrix has less than full rank iff it has a zero eigenvalue.

14. Let A be an $n \times n$ positive definite real matrix.

a) Verify that $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T A \mathbf{y}$$

is a valid inner product.

b) Show that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\left(\mathbf{x}^T A \mathbf{x}\right)\left(\mathbf{y}^T A \mathbf{y}\right) \geq \left(\mathbf{x}^T A \mathbf{y}\right)^2$.

Solution :

a) Commutativity: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} = \left(\mathbf{x}^T A \mathbf{y}\right)^T = \mathbf{y}^T A^T \mathbf{x} = \mathbf{y}^T A \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle$.

Linearity: $\langle \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2, \mathbf{y} \rangle = \left(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2\right)^T A\mathbf{y} = \lambda_1 \mathbf{x}_1^T A \mathbf{y} + \lambda_2 \mathbf{x}_2^T A \mathbf{y} = \lambda_1 \langle \mathbf{x}_1, \mathbf{y} \rangle + \lambda_2 \langle \mathbf{x}_2, \mathbf{y} \rangle$.

Positive definiteness: As A is positive definite, we have for any $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T A \mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$, and equality holds iff $\mathbf{x} = \mathbf{0}$.

Or from Choleski Decomposition A positive $\Leftrightarrow \exists$ real lower triangle matrix P with all positive entries on its diagonal s.t. $A = PP^T$, and so $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T P P^T \mathbf{y} = \left(P^T \mathbf{x}\right)^T \left(P^T \mathbf{y}\right)$ is the dot product of $P^T \mathbf{x}$ and $P^T \mathbf{y}$. Therefore it should be a valid inner product.

b) As $\langle \cdot, \cdot \rangle$ is a valid inner product, by Cauchy-Schwarz inequality for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $||\mathbf{x}||\ ||\mathbf{y}|| \geq |\langle \mathbf{x}, \mathbf{y} \rangle| \Leftrightarrow ||\mathbf{x}||^2\ ||\mathbf{y}||^2 \geq |\langle \mathbf{x}, \mathbf{y} \rangle|^2$ i.e. $\left(\mathbf{x}^T A \mathbf{x}\right)\left(\mathbf{y}^T A \mathbf{y}\right) \geq \left(\mathbf{x}^T A \mathbf{y}\right)^2$.

15. Let A be an idempotent matrix (i.e $A^2 = A$). Show that the eigenvalues of A must be either 0 or 1

Solution : Let λ an eigenvalue of A and x an associated eigenvector. Then we have :

$$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda^2 x$$
$$A^2x = Ax = \lambda x$$

Therefore $\lambda x = \lambda^2 x$, hence since $x \neq 0$, $\lambda = \lambda^2$, i.e $\lambda \in \{0, 1\}$.

16. Let A a symmetric invertible $n \times n$ matrix. Show that A^{-1} is symmetric.

Solution : Let B such that $AB = I_n$. Taking the transpose yields $B^T A^T = I_n$, but since $A^T = A$, $B^T A = I_n$. Since the inverse of A must be unique and B and B^T are inverses for A, we conclude $B = B^T$ – i.e A's inverse is symmetric.