Polynomial regression

June 17, 2019

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt

        import pandas as pd
        import seaborn as sns

        %matplotlib inline

        from sklearn.datasets import load_boston
        from sklearn.linear_model import LinearRegression
        from sklearn.metrics import mean_squared_error

        boston_dataset = load_boston()

In [2]: print(boston_dataset.keys())
        print('******************************************************************************')
        print(boston_dataset.DESCR)
```

```
dict_keys(['filename', 'feature_names', 'DESCR', 'target', 'data'])
******************************************************************************
.. _boston_dataset:

Boston house prices dataset
---------------------------

**Data Set Characteristics:**

    :Number of Instances: 506

    :Number of Attributes: 13 numeric/categorical predictive. Median Value (attribute 14) is usu

    :Attribute Information (in order):
        - CRIM     per capita crime rate by town
        - ZN       proportion of residential land zoned for lots over 25,000 sq.ft.
        - INDUS    proportion of non-retail business acres per town
        - CHAS     Charles River dummy variable (= 1 if tract bounds river; 0 otherwise)
        - NOX      nitric oxides concentration (parts per 10 million)
```

```
        - RM       average number of rooms per dwelling
        - AGE      proportion of owner-occupied units built prior to 1940
        - DIS      weighted distances to five Boston employment centres
        - RAD      index of accessibility to radial highways
        - TAX      full-value property-tax rate per $10,000
        - PTRATIO  pupil-teacher ratio by town
        - B        1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town
        - LSTAT    % lower status of the population
        - MEDV     Median value of owner-occupied homes in $1000's

    :Missing Attribute Values: None

    :Creator: Harrison, D. and Rubinfeld, D.L.

This is a copy of UCI ML housing dataset.
https://archive.ics.uci.edu/ml/machine-learning-databases/housing/


This dataset was taken from the StatLib library which is maintained at Carnegie Mellon Universit

The Boston house-price data of Harrison, D. and Rubinfeld, D.L. 'Hedonic
prices and the demand for clean air', J. Environ. Economics & Management,
vol.5, 81-102, 1978.   Used in Belsley, Kuh & Welsch, 'Regression diagnostics
...', Wiley, 1980.   N.B. Various transformations are used in the table on
pages 244-261 of the latter.

The Boston house-price data has been used in many machine learning papers that address regressio
problems.

.. topic:: References

   - Belsley, Kuh & Welsch, 'Regression diagnostics: Identifying Influential Data and Sources of
   - Quinlan,R. (1993). Combining Instance-Based and Model-Based Learning. In Proceedings on the
```

In [3]:
```python
boston = pd.DataFrame(boston_dataset.data, columns=boston_dataset.feature_names)
boston.head()
```

Out[3]:
```
      CRIM    ZN  INDUS  CHAS    NOX     RM   AGE     DIS  RAD    TAX  \
0  0.00632  18.0   2.31   0.0  0.538  6.575  65.2  4.0900  1.0  296.0
1  0.02731   0.0   7.07   0.0  0.469  6.421  78.9  4.9671  2.0  242.0
2  0.02729   0.0   7.07   0.0  0.469  7.185  61.1  4.9671  2.0  242.0
3  0.03237   0.0   2.18   0.0  0.458  6.998  45.8  6.0622  3.0  222.0
4  0.06905   0.0   2.18   0.0  0.458  7.147  54.2  6.0622  3.0  222.0

   PTRATIO       B  LSTAT
0     15.3  396.90   4.98
```

```
1      17.8  396.90   9.14
2      17.8  392.83   4.03
3      18.7  394.63   2.94
4      18.7  396.90   5.33
```

In [4]:
```
boston['MEDV'] = boston_dataset.target
boston.head()
```

Out[4]:
```
      CRIM    ZN  INDUS  CHAS    NOX     RM   AGE     DIS  RAD    TAX  \
0  0.00632  18.0   2.31   0.0  0.538  6.575  65.2  4.0900  1.0  296.0
1  0.02731   0.0   7.07   0.0  0.469  6.421  78.9  4.9671  2.0  242.0
2  0.02729   0.0   7.07   0.0  0.469  7.185  61.1  4.9671  2.0  242.0
3  0.03237   0.0   2.18   0.0  0.458  6.998  45.8  6.0622  3.0  222.0
4  0.06905   0.0   2.18   0.0  0.458  7.147  54.2  6.0622  3.0  222.0

   PTRATIO       B  LSTAT  MEDV
0     15.3  396.90   4.98  24.0
1     17.8  396.90   9.14  21.6
2     17.8  392.83   4.03  34.7
3     18.7  394.63   2.94  33.4
4     18.7  396.90   5.33  36.2
```

0.1 Let's verify the absence of null or missing values in the training data

In [5]:
```
boston.isnull().sum()
```

Out[5]:
```
CRIM       0
ZN         0
INDUS      0
CHAS       0
NOX        0
RM         0
AGE        0
DIS        0
RAD        0
TAX        0
PTRATIO    0
B          0
LSTAT      0
MEDV       0
dtype: int64
```

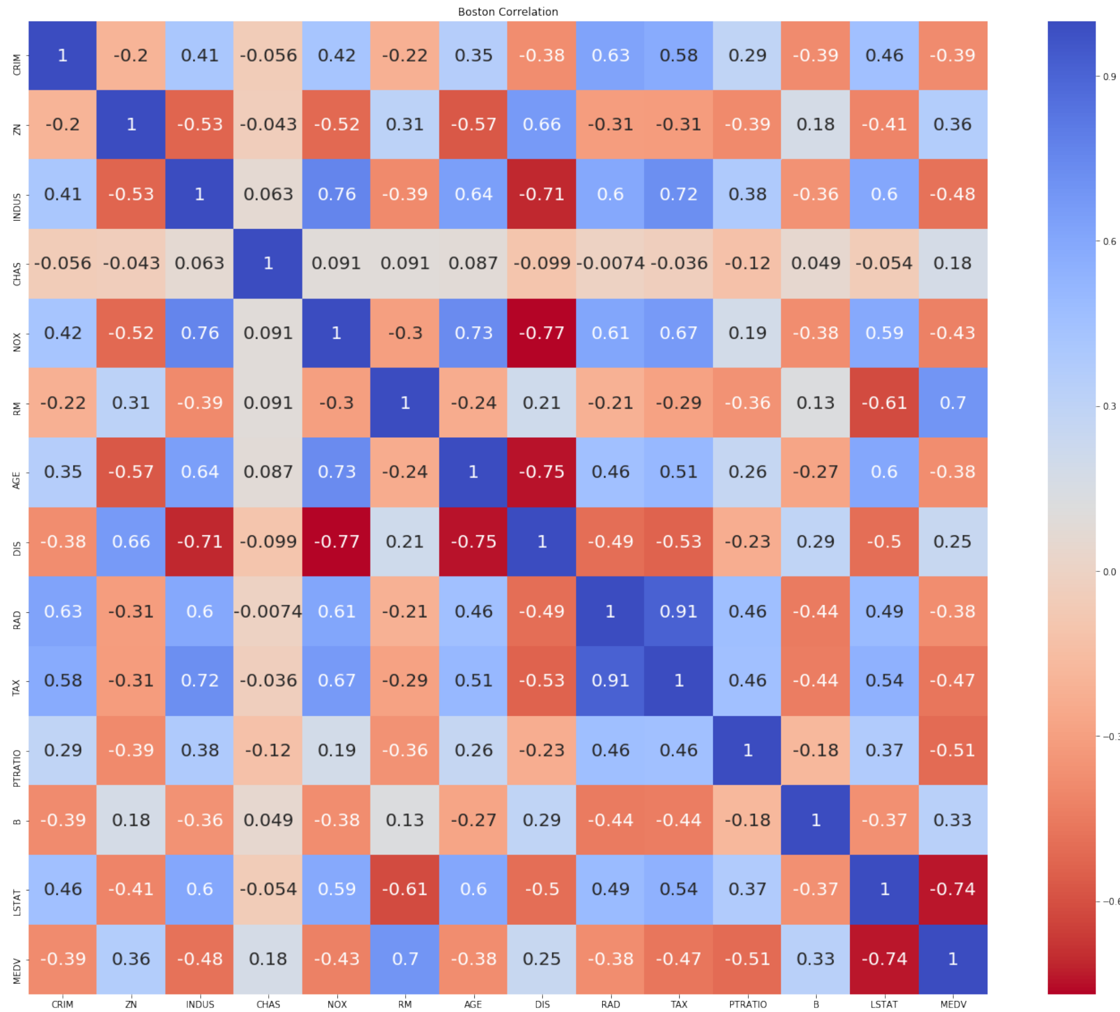
0.2 Let's analyze the (Pearson) correlation between the features. This is an important operation when we deal with a great number of features, in fact this dataset has 13 features which cannot be drawn. This problem is called curse of dimensionality. Then we evaluate the correlation in order to reduce manually the features by extracting 2 or 3 features with the higher correlation value

In [6]:
```
boston.corr(method='pearson')
```

Out[6]:
```
             CRIM        ZN     INDUS      CHAS       NOX        RM       AGE  \
CRIM     1.000000 -0.200469  0.406583 -0.055892  0.420972 -0.219247  0.352734
ZN      -0.200469  1.000000 -0.533828 -0.042697 -0.516604  0.311991 -0.569537
INDUS    0.406583 -0.533828  1.000000  0.062938  0.763651 -0.391676  0.644779
CHAS    -0.055892 -0.042697  0.062938  1.000000  0.091203  0.091251  0.086518
NOX      0.420972 -0.516604  0.763651  0.091203  1.000000 -0.302188  0.731470
RM      -0.219247  0.311991 -0.391676  0.091251 -0.302188  1.000000 -0.240265
AGE      0.352734 -0.569537  0.644779  0.086518  0.731470 -0.240265  1.000000
DIS     -0.379670  0.664408 -0.708027 -0.099176 -0.769230  0.205246 -0.747881
RAD      0.625505 -0.311948  0.595129 -0.007368  0.611441 -0.209847  0.456022
TAX      0.582764 -0.314563  0.720760 -0.035587  0.668023 -0.292048  0.506456
PTRATIO  0.289946 -0.391679  0.383248 -0.121515  0.188933 -0.355501  0.261515
B       -0.385064  0.175520 -0.356977  0.048788 -0.380051  0.128069 -0.273534
LSTAT    0.455621 -0.412995  0.603800 -0.053929  0.590879 -0.613808  0.602339
MEDV    -0.388305  0.360445 -0.483725  0.175260 -0.427321  0.695360 -0.376955

              DIS       RAD       TAX   PTRATIO         B     LSTAT      MEDV
CRIM    -0.379670  0.625505  0.582764  0.289946 -0.385064  0.455621 -0.388305
ZN       0.664408 -0.311948 -0.314563 -0.391679  0.175520 -0.412995  0.360445
INDUS   -0.708027  0.595129  0.720760  0.383248 -0.356977  0.603800 -0.483725
CHAS    -0.099176 -0.007368 -0.035587 -0.121515  0.048788 -0.053929  0.175260
NOX     -0.769230  0.611441  0.668023  0.188933 -0.380051  0.590879 -0.427321
RM       0.205246 -0.209847 -0.292048 -0.355501  0.128069 -0.613808  0.695360
AGE     -0.747881  0.456022  0.506456  0.261515 -0.273534  0.602339 -0.376955
DIS      1.000000 -0.494588 -0.534432 -0.232471  0.291512 -0.496996  0.249929
RAD     -0.494588  1.000000  0.910228  0.464741 -0.444413  0.488676 -0.381626
TAX     -0.534432  0.910228  1.000000  0.460853 -0.441808  0.543993 -0.468536
PTRATIO -0.232471  0.464741  0.460853  1.000000 -0.177383  0.374044 -0.507787
B        0.291512 -0.444413 -0.441808 -0.177383  1.000000 -0.366087  0.333461
LSTAT   -0.496996  0.488676  0.543993  0.374044 -0.366087  1.000000 -0.737663
MEDV     0.249929 -0.381626 -0.468536 -0.507787  0.333461 -0.737663  1.000000
```

In [7]:
```python
f, (ax1) = plt.subplots(1, 1, figsize=(24, 20))
correlation_matrix = boston.corr(method='pearson')
sns.heatmap(correlation_matrix, cmap='coolwarm_r', annot = True, annot_kws={'size':20},
ax1.set_title('Boston Correlation')
```

Out[7]:
```
Text(0.5, 1.0, 'Boston Correlation')
```



0.3 Analyzing the correlation, we point out that:

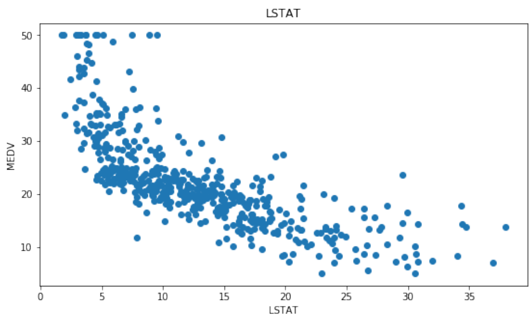
0.3.1 1. MEDV has a strong positive correlation with RM (0.7)

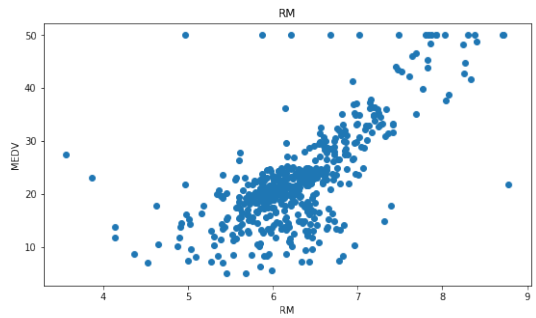
0.3.2 2. MEDV has a strong negative correlation with LSTAT (-0.74)

0.3.3 3. RAD and TAX have a correlation of 0.91; this features pair is strongly correlated to each other then we should not select both together for training the model because we could not understand what is the feature that influence more the model

In [8]:
```
plt.figure(figsize=(20, 5))

features = ['LSTAT', 'RM']
target = boston['MEDV']

for i, col in enumerate(features):
    plt.subplot(1, len(features) , i+1)
```

```
    x = boston[col]
    y = target
    plt.scatter(x, y, marker='o')
    plt.title(col)
    plt.xlabel(col)
    plt.ylabel('MEDV')
```





1 Linear Regression

```
In [22]: VARIABLE = 'LSTAT' #RM
         X = pd.DataFrame(np.c_[boston[VARIABLE]], columns = [VARIABLE])
         y = boston['MEDV'].values.reshape((y.shape[0], 1))

         X = np.concatenate([np.ones((X.shape[0], 1)), X], axis=1)
         m = X.shape[0]
         n = X.shape[1]
         print('Training examples: ', m)
         print('Features: ', n)
```

```
Training examples:  506
Features:  2
```

```
In [23]: def computeCostVectorized(X, y, theta):
             m=y.size
             h=X.dot(theta)
             J= ((h-y).T).dot(h-y)
             return J/(2*m)

         def gradientDescentVectorized(X, y,theta=[[0],[0]],alpha=0.1, n_iter=1500):
             m=y.size
             J = np.zeros((n_iter,1))
             for i in range(n_iter):
                 h = X.dot(theta)
                 theta = theta - (alpha/m) * np.dot((X.T),(h-y))
```

```
            J[i] = computeCostVectorized(X,y,theta)
        return theta, J
```

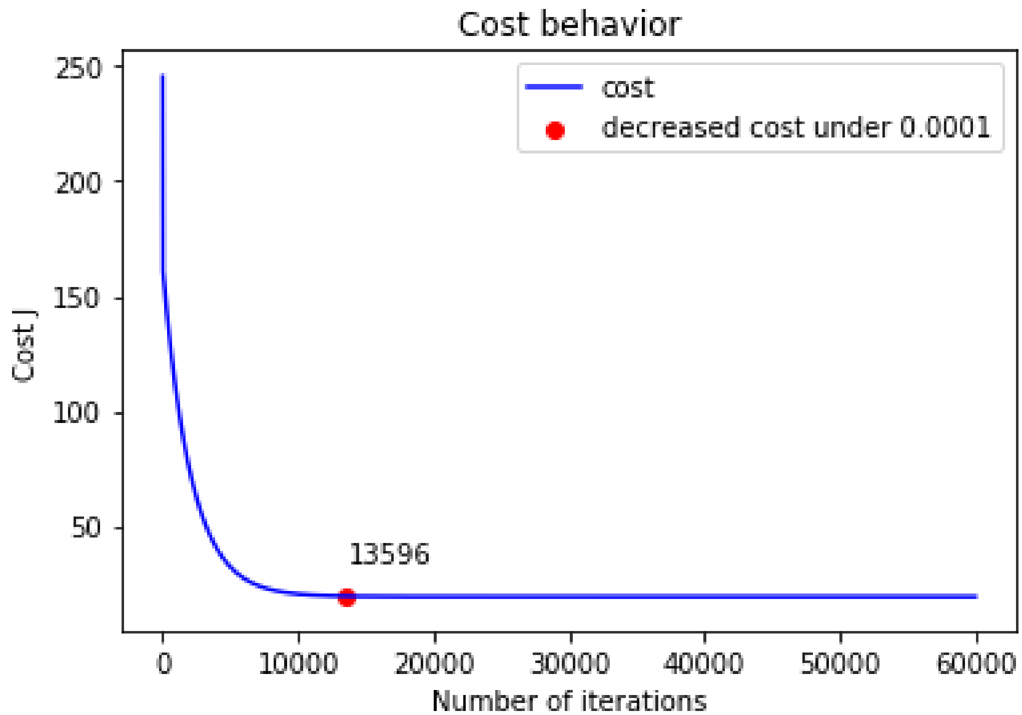
In [24]:
```
num_iters = 60000
alpha = 0.001 #0.1
theta = np.zeros((2, 1))
theta, J_history = gradientDescentVectorized(X,y, theta, alpha, num_iters)
print(theta)
```

```
[[34.55382208]
 [-0.95004823]]
```

1.0.1 Let's define a function to show when the Gradient Descent has reached convergence (the cost function curve is almost flat, which means the cost value change under a threshold value)

In [25]:
```
def find_flat(thr = 0.001):
    for i in range(num_iters):
        if abs(J_history[i]-J_history[i-1]) < thr:
            return i
        if i == num_iters-1:
            return 0
```

In [26]:
```
thr = 0.0001
k = find_flat(thr)
# Plot the convergence graph
plt.plot([i for i in range(num_iters)], J_history, '-b', label = 'cost')
plt.scatter(k, J_history[k], c='r', label = 'decreased cost under {}'.format(thr))
plt.text(k, J_history[k]+15, k)
plt.xlabel('Number of iterations') # Set the xaxis label
plt.ylabel('Cost J') # Set the yaxis label
plt.title('Cost behavior')
plt.legend()
plt.show()
```



1.0.2 Let's check if we are gone on well with the gradient descent by using the normal equation

```
In [27]: def normalEquation(X, y):
             pseudo_inv = np.linalg.pinv(X.T.dot(X))
             product = pseudo_inv.dot(X.T)
             return product.dot(y)
         theta_ne = normalEquation(X,y)
         print(theta_ne)
```

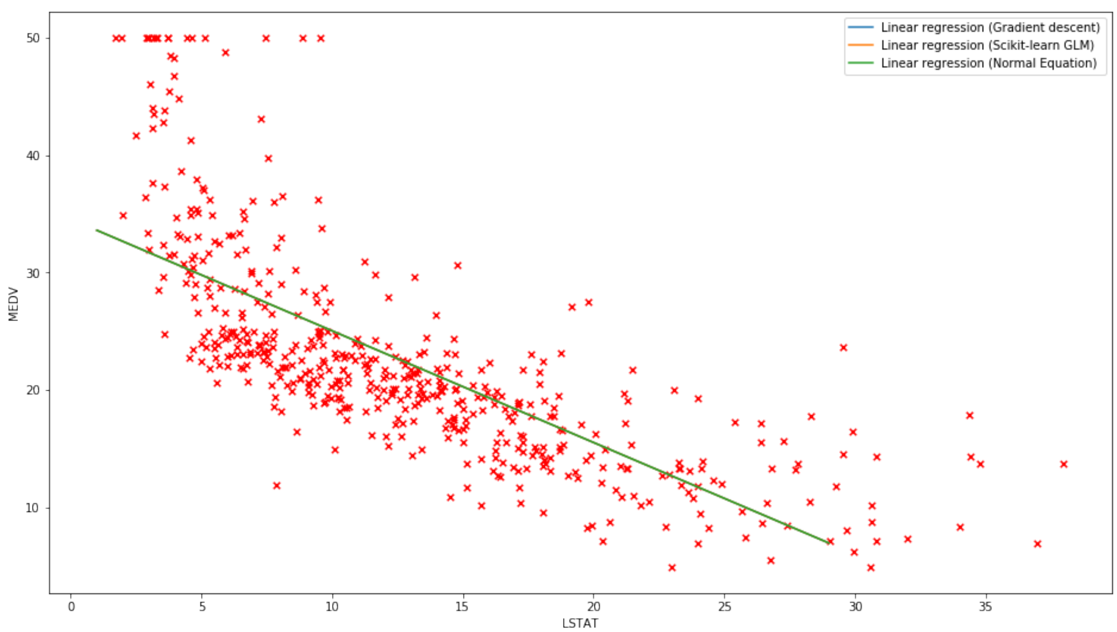
```
[[34.55384088]
 [-0.95004935]]
```

```
In [28]: xx = np.arange(1,30)
         yy = theta[0] + theta[1] * xx

         # Plot gradient descent
         plt.figure(figsize=(16,9))
         plt.scatter(X[:,1], y, s=30, c='r', marker='x', linewidths=1)
         plt.plot(xx,yy, label='Linear regression (Gradient descent)')

         # Compare with Scikit-learn Linear regression
         regr = LinearRegression()
```

```
regr.fit(X[:,1].reshape(-1,1), y.ravel())
plt.plot(xx, regr.intercept_ + regr.coef_ * xx, label='Linear regression (Scikit-learn

print(regr.intercept_)
print(regr.coef_)

# Compare with Normal Equations
plt.plot(xx, theta_ne[0] + theta_ne[1] * xx, label='Linear regression (Normal Equation)

#plt.xlim(-2,10)
plt.xlabel(VARIABLE)
plt.ylabel('MEDV')
plt.legend(loc=1)
```

```
34.5538408793831
[-0.95004935]
```

Out[28]: <matplotlib.legend.Legend at 0x7fe5c68d1518>



1.1 This plot shows a model that is not perfectly fitting the data. This problem is caused by some values, called outliers; We need to make a preprocess of the data by removing this values in order to solve this problem.

2 Remove Outliers

2.1 An Outlier is a data point that differs significantly from other observations. The outliers cause serious problems in our analysis then we need to reject these values.

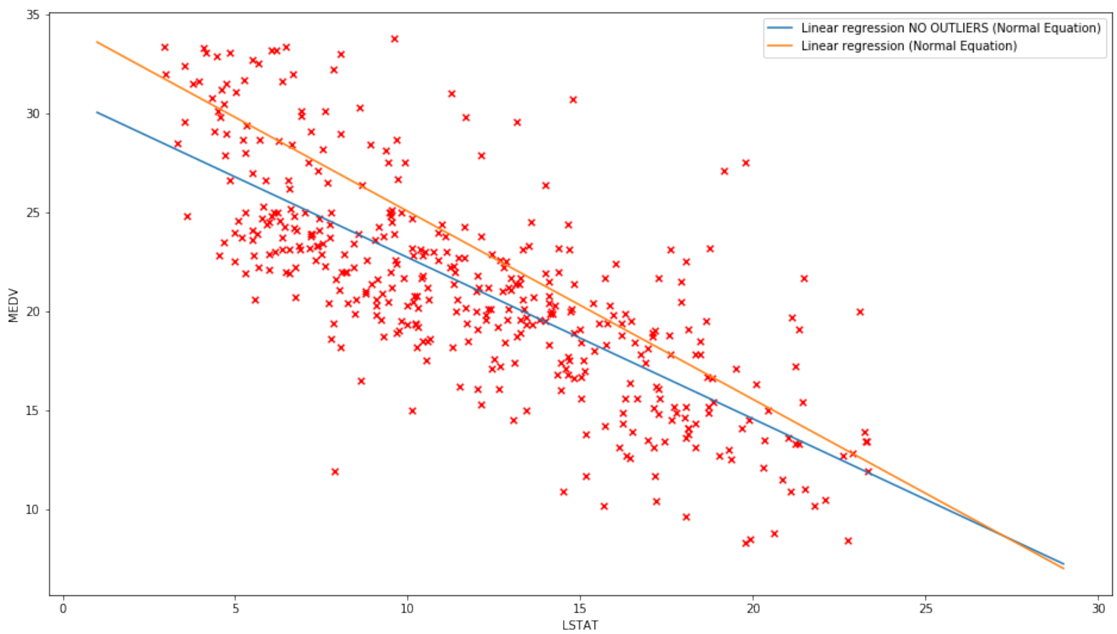
2.2 We use the Z-score method, which describes a data point in terms of its relationship to the mean and standard deviation of the dataset; we choose to discard all the data point with a Z-score greater than 2.

In [36]:
```
VARIABLE = 'LSTAT' #RM
X_o = pd.DataFrame(np.c_[boston[VARIABLE]], columns = [VARIABLE])
y_o = boston['MEDV'].values.reshape((y.shape[0], 1))

X_o['temp'] = pd.Series(y_o.reshape(y_o.shape[0]), index = X_o.index)
X_o = X_o.mask((X_o - X_o.mean()).abs() > 2 * X_o.std())
X_o = X_o.mask((X_o - X_o.mean()).abs() > 2 * X_o.std()).dropna()
y_o = X_o.as_matrix(columns=X_o.columns[1:])
temp = X_o.as_matrix(columns=X_o.columns[:1])

X_o = np.concatenate([np.ones((temp.shape[0], 1)), temp], axis=1)
m_o = X_o.shape[0]
n_o = X_o.shape[1]
print('Training examples: ', m_o)
print('Features: ', n_o)

theta_ne_o = normalEquation(X_o,y_o)
print(theta_ne_o)

xx = np.arange(1,30)

# Plot gradient descent
plt.figure(figsize=(16,9))
plt.scatter(X_o[:,1], y_o, s=30, c='r', marker='x', linewidths=1)
plt.plot(xx, theta_ne_o[0] + theta_ne_o[1] * xx, label='Linear regression NO OUTLIERS (
plt.plot(xx, theta_ne[0] + theta_ne[1] * xx, label='Linear regression (Normal Equation)

#plt.xlim(-2,10)
plt.xlabel(VARIABLE)
plt.ylabel('MEDV')
plt.legend(loc=1)
```

```
Training examples:  405
Features:  2
[[30.86534272]
 [-0.81489421]]
```

```
/home/andrea/.local/lib/python3.5/site-packages/ipykernel_launcher.py:8: FutureWarning: Method .

/home/andrea/.local/lib/python3.5/site-packages/ipykernel_launcher.py:9: FutureWarning: Method .
  if __name__ == '__main__':
```

Out[36]: <matplotlib.legend.Legend at 0x7fe5c4188c88>



2.3 The plot shows how the new model (the blue line) fits better than the previous one

2.4 Let's evaluate the behaviour of our model by using the Root Mean Squared Error (RMSE)

$$RMSE = \sqrt{\frac{\sum_{i=1}^{m}(\hat{y_{(i)}} - y_{(i)})}{m}}$$

In [37]:
```python
y_pred = np.zeros((X.shape[0], 1))
y_pred = X.dot(theta)

rmse = np.sqrt(mean_squared_error(y_pred, y))
print('RMSE with outliers: ', rmse)

y_pred_o = np.zeros((X_o.shape[0], 1))
y_pred_o = X_o.dot(theta_ne_o)
```

```
rmse = np.sqrt(mean_squared_error(y_pred_o, y_o))
print('RMSE without outliers: ', rmse)
```

```
RMSE with outliers:  6.203464131433289
RMSE without outliers:  3.5171166550418955
```

2.5 The idea, that the blue model is better than the orange, is confirmed by the RMSE metric

3 Polynomial Regression

3.1 Sometimes we can study different models by using some techniques:

3.1.1 1. To get a feature and create a new one by raising it to power

3.1.2 2. To create a new features by multiplying some features

3.2 This approach is used to get a better model!

```
In [38]: def polynomial_features(X, degree):
             for i in range(1, degree):
                 colname = dataframe.columns[0]+'^'+str(i+1)
                 X[colname] = np.power(X[VARIABLE], i+1)
             return X
```

3.2.1 Let's take our feature (VARIABLE) and raises it to power and let's define the following models:

Model 1:

$$h_\theta(x) = \theta_0 + \theta_1(q)$$

Model 2:

$$h_\theta(x) = \theta_0 + \theta_1(q) + \theta_2(q^2)$$

Model 3:

$$h_\theta(x) = \theta_0 + \theta_1(q) + \theta_2(q^2) + \theta_3(q^3)$$

Model 4:

$$h_\theta(x) = \theta_0 + \theta_1(q) + \theta_2(q^2) + \theta_3(q^3) + \theta_4(q^4)$$

Model 5:

$$h_\theta(x) = \theta_0 + \theta_1(q) + \theta_2(q^2) + \theta_3(q^3) + \theta_4(q^4) + \theta_5(q^5)$$

where q substitute VARIABLE in the models.

```
In [39]: dataframe = pd.DataFrame(X[:, 1], columns = [VARIABLE])

         X_1 = np.concatenate([np.ones((X.shape[0],1)), polynomial_features(dataframe, 1).values
         theta_ne_1 = normalEquation(X_1,y)
         print(theta_ne_1)
```

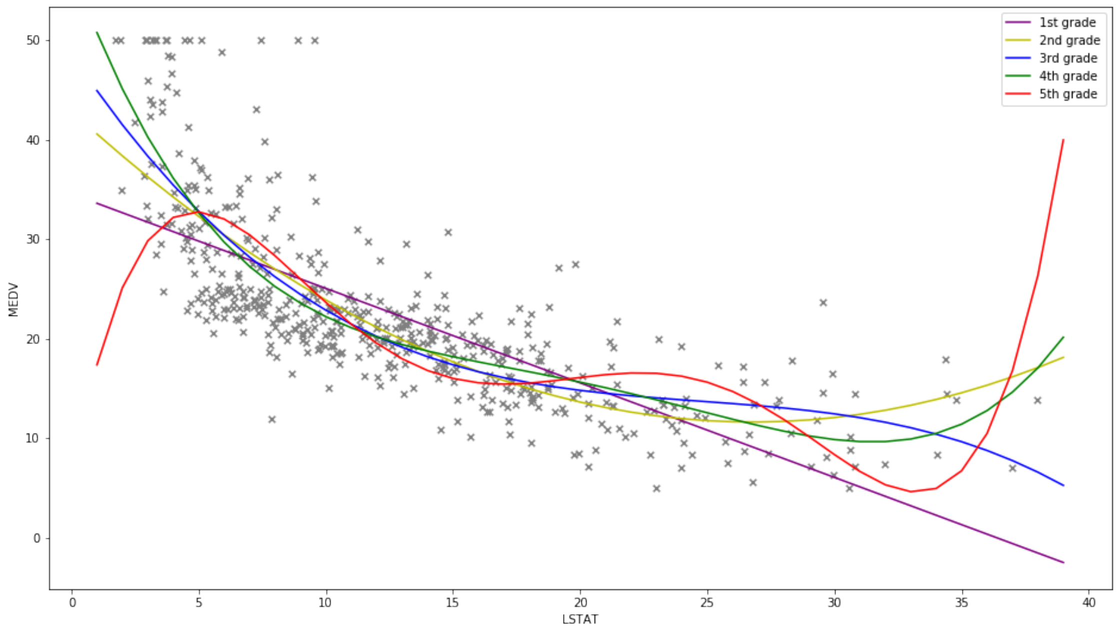
```
X_2 = np.concatenate([np.ones((X.shape[0],1)), polynomial_features(dataframe, 2).values
theta_ne_2 = normalEquation(X_2,y)
print(theta_ne_2)

X_3 = np.concatenate([np.ones((X.shape[0],1)), polynomial_features(dataframe, 3).values
theta_ne_3 = normalEquation(X_3,y)
print(theta_ne_3)

X_4 = np.concatenate([np.ones((X.shape[0],1)), polynomial_features(dataframe, 4).values
theta_ne_4 = normalEquation(X_4,y)
print(theta_ne_4)

X_5 = np.concatenate([np.ones((X.shape[0],1)), polynomial_features(dataframe, 5).values
theta_ne_5 = normalEquation(X_5,y)
print(theta_ne_5)
```

```
[[34.55384088]
 [-0.95004935]]
[[42.86200733]
 [-2.3328211 ]
 [ 0.04354689]]
[[ 4.86496253e+01]
 [-3.86559278e+00]
 [ 1.48738477e-01]
 [-2.00386767e-03]]
[[ 5.73099582e+01]
 [-7.02846102e+00]
 [ 4.95481226e-01]
 [-1.63101737e-02]
 [ 1.94867818e-04]]
[[ 5.83047794e+00]
 [ 1.37100067e+01]
 [-2.31938916e+00]
 [ 1.51199510e-01]
 [-4.29819581e-03]
 [ 4.43546502e-05]]
```

4 Plot with Normal Equation

In [40]:
```
xx = np.arange(1,40)
yy1 = theta_ne_1[0] + theta_ne_1[1] * xx
yy2 = theta_ne_2[0] + theta_ne_2[1] * xx + theta_ne_2[2] * (xx**2)
yy3 = theta_ne_3[0] + theta_ne_3[1] * xx + theta_ne_3[2] * (xx**2) + theta_ne_3[3]*(xx*
yy4 = theta_ne_4[0] + theta_ne_4[1] * xx + theta_ne_4[2] * (xx**2) + theta_ne_4[3]*(xx*
yy5 = theta_ne_5[0] + theta_ne_5[1] * xx + theta_ne_5[2] * (xx**2) + theta_ne_5[3]*(xx*
```

```
# Plot Normal Equation
plt.figure(figsize=(16,9))
plt.scatter(X[:,1], y, s=30, c='grey', marker='x', linewidths=1)
plt.plot(xx,yy1,c='purple', label='1st grade')
plt.plot(xx,yy2,c='y', label='2nd grade')
plt.plot(xx,yy3,c='b', label='3rd grade')
plt.plot(xx,yy4,c='g', label='4th grade')
plt.plot(xx,yy5,c='r', label='5th grade')


plt.xlabel(VARIABLE)
plt.ylabel('MEDV')
plt.legend(loc=1);
plt.show()
```



5 Plot using Gradient Descent

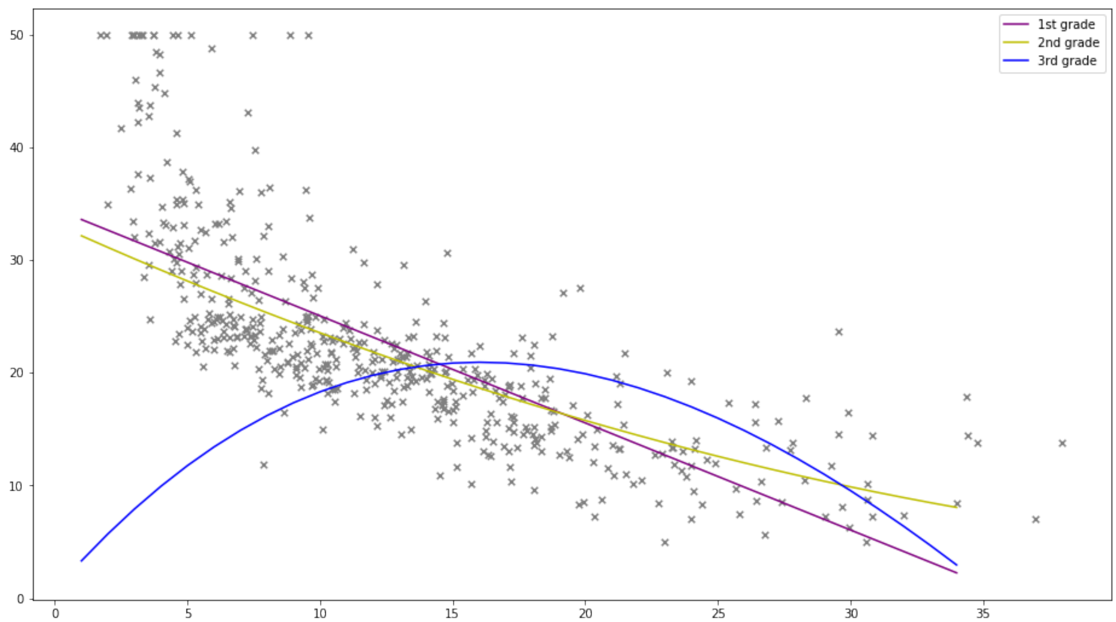
5.1 Let's try to plot the result of the gradient descent in order to compare the two approaches

In [41]:
```
alpha1 = 0.001
num_iters1 = 60000
theta_1 = np.zeros((2,1))
theta_1, J_history_1 = gradientDescentVectorized(X_1,y, theta_1, alpha1, num_iters1)
```

```
alpha2 = 0.000019
num_iters2 = 1000000
theta_2 = np.zeros((3,1))
theta_2, J_history_2 = gradientDescentVectorized(X_2,y, theta_2, alpha2, num_iters2)


alpha3 = 0.000000019
num_iters3 = 5000000
theta_3 = np.zeros((4,1))
theta_3, J_history_3 = gradientDescentVectorized(X_3,y, theta_3, alpha3, num_iters3)
print(theta_3, theta_ne_3)
```

```
[[ 7.16395671e-01]
 [ 2.69471781e+00]
 [-1.00187305e-01]
 [ 6.72516345e-04]] [[ 4.86496253e+01]
 [-3.86559278e+00]
 [ 1.48738477e-01]
 [-2.00386767e-03]]
```

```
In [42]: xx = np.arange(1,35)
         yy1 = theta_1[0] + theta_1[1] * xx
         yy2 = theta_2[0] + theta_2[1] * xx + theta_2[2] * (xx**2)
         yy3 = theta_3[0] + theta_3[1] * xx + theta_3[2] * (xx**2) + theta_3[3]*(xx**3)

         plt.figure(figsize=(16,9))
         plt.scatter(X[:,1], y, s=30, c='grey', marker='x', linewidths=1)
         plt.plot(xx,yy1,c='purple', label='1st grade')
         plt.plot(xx,yy2,c='y', label='2nd grade')
         plt.plot(xx,yy3,c='b',label='3rd grade')

         plt.legend(loc=1)
         plt.show()
```



5.2 Observation

5.3 We can see that gradient descent convergence is too slow (we need millions of iterations to reach a discrete result) above all when the polynomial degree becomes greater than 2

6 Polynomial regression with LSTAT and RM (3D plot)

6.1 We try to use both LSTAT and RM features in order to show how it is difficult to evaluate the model when the dimension increases (the curse of dimensionality)

In [43]:
```
X_trid = pd.DataFrame(np.c_[boston['RM'],boston['LSTAT']], columns = ['RM','LSTAT'])
X_trid.head()
```

Out[43]:

	RM	LSTAT
0	6.575	4.98
1	6.421	9.14
2	7.185	4.03
3	6.998	2.94
4	7.147	5.33

In [44]:
```
y_trid = boston['MEDV'].values.reshape((y.shape[0],1))
X_trid = np.concatenate([np.ones((X_trid.shape[0],1)), X_trid], axis=1)

m_trid = X_trid.shape[0]
n_trid = X_trid.shape[1]
```

```
        print('#Training examples: ', m_trid)
        print('#Features: ',n_trid)
```

```
#Training examples:  506
#Features:  3
```

```
In [45]: alpha_trid = 0.001
         num_iters_trid = 300000
         theta_trid = np.zeros((3,1))
         theta_trid, J_history_trid = gradientDescentVectorized(X_trid, y_trid, theta_trid, alph
         print(theta_trid)
```

```
[[-1.01575607]
 [ 5.04738744]
 [-0.64572484]]
```

```
In [46]: theta_ne_trid = normalEquation(X_trid,y_trid)
         print(theta_ne_trid)
```

```
[[-1.35827281]
 [ 5.09478798]
 [-0.64235833]]
```

```
In [47]: dataframe = pd.DataFrame((X_trid[:,1:]), columns = ['RM','LSTAT'])


         def polynomial_features(X, degree):
             for i in range(1, degree):
                 colname = 'RM_%d'%(i+1)
                 colname2 = 'LSTAT_%d'%(i+1)
                 X[colname] = X['RM']**(i+1)
                 X[colname2] = X['LSTAT']**(i+1)
             return X

         dataframe.head()
```

Out[47]:

	RM	LSTAT
0	6.575	4.98
1	6.421	9.14
2	7.185	4.03
3	6.998	2.94
4	7.147	5.33

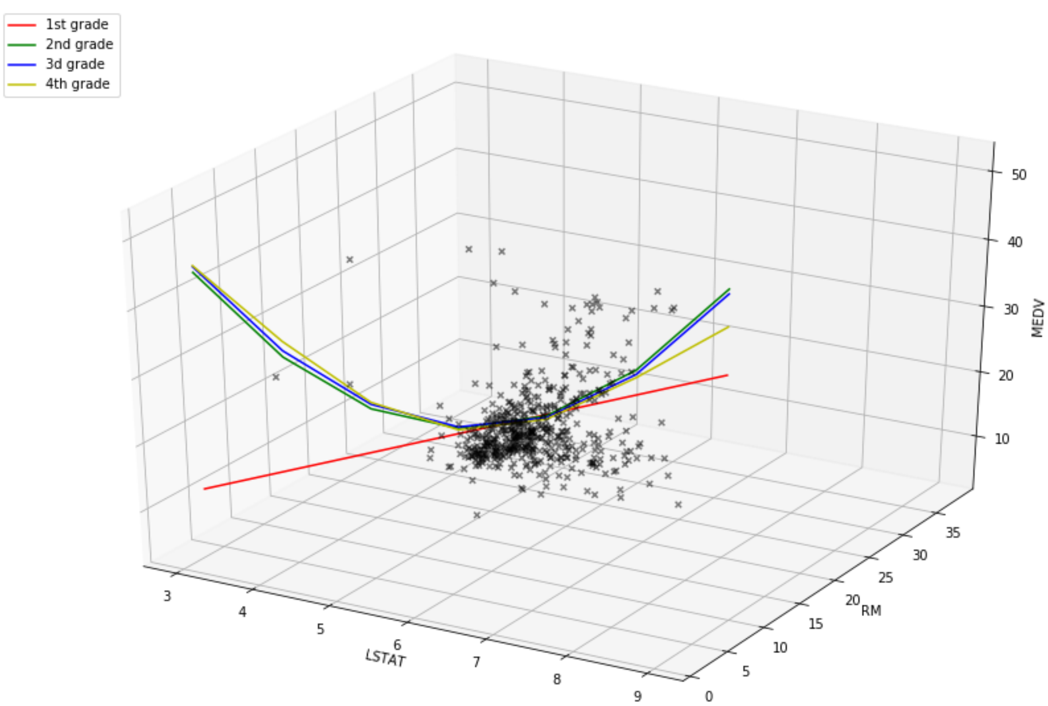
```
In [48]: X_trid1 = np.concatenate([np.ones((X_trid.shape[0],1)), polynomial_features(dataframe,
         theta_ne_trid1 = normalEquation(X_trid1,y_trid)
         print(theta_ne_trid1)
```

```
X_trid2 = np.concatenate([np.ones((X_trid.shape[0],1)), polynomial_features(dataframe,
theta_ne_trid2 = normalEquation(X_trid2,y_trid)
print(theta_ne_trid2)

X_trid3 = np.concatenate([np.ones((X_trid.shape[0],1)), polynomial_features(dataframe,
theta_ne_trid3 = normalEquation(X_trid3,y_trid)
print(theta_ne_trid3)

X_trid4 = np.concatenate([np.ones((X_trid.shape[0],1)), polynomial_features(dataframe,
theta_ne_trid4 = normalEquation(X_trid4,y_trid)
print(theta_ne_trid4)
```

```
[[-1.35827281]
 [ 5.09478798]
 [-0.64235833]]
[[ 1.05084032e+02]
 [-2.60093622e+01]
 [-1.41622862e+00]
 [ 2.35606882e+00]
 [ 2.18496064e-02]]
[[ 1.01800688e+02]
 [-2.28433641e+01]
 [-1.93480744e+00]
 [ 1.81384692e+00]
 [ 5.55531810e-02]
 [ 2.80246479e-02]
 [-6.30200944e-04]]
[[ 2.78159601e+01]
 [ 3.73999759e+01]
 [-3.45122253e+00]
 [-1.48047332e+01]
 [ 2.22683073e-01]
 [ 1.99209718e+00]
 [-7.52358061e-03]
 [-8.44400275e-02]
 [ 9.33456558e-05]]
```

In [49]:
```
from mpl_toolkits import mplot3d
# Create the figure
fig = plt.figure(figsize=(15, 10))
ax = fig.add_subplot(111, projection='3d')

# Plot the values
ax.scatter(X_trid[:,1], X_trid[:,2], y, c = 'black', marker='x', alpha=0.5)
ax.set_xlabel('LSTAT')
ax.set_ylabel('RM')
```

```
ax.set_zlabel('MEDV ')

xx1 = np.arange(3,10)
xx2 = np.arange(3,10)
zz1 = theta_ne_trid1[0] + theta_ne_trid1[1] * xx1 + theta_ne_trid1[2] * xx2
zz2 = theta_ne_trid2[0] + theta_ne_trid2[1] * xx1 + theta_ne_trid2[2] * xx2 + theta_ne_
        theta_ne_trid2[4]*(xx2**2)
zz3 = theta_ne_trid3[0] + theta_ne_trid3[1] * xx1 + theta_ne_trid3[2] * xx2 + theta_ne_
        theta_ne_trid3[4]*(xx2**2) + theta_ne_trid3[5]*(xx1**3) + theta_ne_trid3[6]*(xx
zz4 = theta_ne_trid4[0] + theta_ne_trid4[1] * xx1 + theta_ne_trid4[2] * xx2 + theta_ne_
        theta_ne_trid4[4]*(xx2**2) + theta_ne_trid4[5]*(xx1**3) + theta_ne_trid4[6]*(xx
        theta_ne_trid4[7] * (xx1**4) + theta_ne_trid4[8]*(xx2**4)


ax.plot(xx1,xx2,zz1, c='r', label='1st grade')
ax.plot(xx1,xx2,zz2,c='g' , label='2nd grade')
ax.plot(xx1,xx2,zz3,c='b',  label='3d grade')
ax.plot(xx1,xx2,zz4, c='y', label='4th grade')
plt.legend(loc=2);

plt.draw()
```



6.2 Observation

6.3 With this plot we can see that it is already difficult to make evaluation about our models

7 Exercise

7.1 The Dataset is divided into Training, Validation and Test

7.2 1. Let's define the dataframe with the two features and the target variable

```
In [50]: dset = pd.DataFrame(np.c_[boston['RM'], boston['LSTAT'], boston['MEDV']], columns = ['R
         dset = dset.sample(frac=1).reset_index(drop=True)
         rows = dset.shape[0]
         dset.head()
```

Out[50]:

	RM	LSTAT	MEDV
0	5.344	23.09	20.0
1	6.302	6.72	24.8
2	6.565	9.51	24.8
3	6.185	18.13	14.1
4	5.961	9.88	21.7

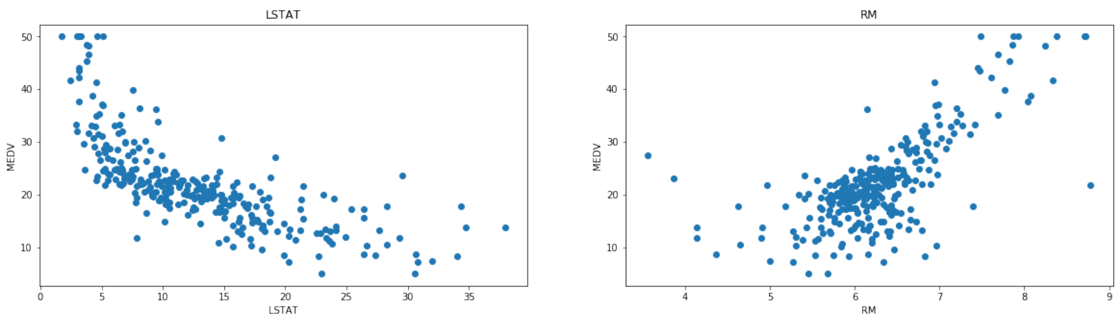
7.3 2. Let's split the dataset in 3 batches: Training (60%), Validation and Test (20%)

```
In [51]: print("Total examples: ", rows)
         train_size = round(rows*.6)
         print("Training examples: ", train_size)
         sec_size = round(rows*.2)
         print("Validation/Test examples: ", sec_size)
         print('\nSplitting result:')
         train = dset.iloc[:train_size]
         print("Training: ", train.shape)
         valid = dset.iloc[train_size:train_size+sec_size]
         print("Validation: ", valid.shape)
         test = dset.iloc[train_size+sec_size:]
         print("Test: ", test.shape)
```

```
Total examples:  506
Training examples:  304
Validation/Test examples:  101

Splitting result:
Training:  (304, 3)
Validation:  (101, 3)
Test:  (101, 3)
```

```
In [52]: plt.figure(figsize=(20, 5))
```

```
features = ['LSTAT', 'RM']
target = train['MEDV']

for i, col in enumerate(features):
    plt.subplot(1, len(features) , i+1)
    xxxx = train[col]
    yyyy = target
    plt.scatter(xxxx, yyyy, marker='o')
    plt.title(col)
    plt.xlabel(col)
    plt.ylabel('MEDV')
```




```
In [53]: X = train.loc[:, ['RM', 'LSTAT']]
         X_new = pd.DataFrame(np.c_[X['RM']*X['LSTAT']], columns = ['RM*LSTAT'])
         y = train.loc[:, ['MEDV']]

         print(X.shape)
```

```
(304, 2)
```

7.4 3. Let's define some models to carry out the training phase

7.4.1 We define some models by combining and by raising to power some features. This is a good approach if we use features with a meaning to be fused but in this case RM and LSTAT have not.

Modello 1:

$$h_\theta(x) = \theta_0 + \theta_1(RM * LSTAT)$$

Modello 2:

$$h_\theta(x) = \theta_0 + \theta_1(RM * LSTAT) + \theta_2(RM^2 * LSTAT)$$

Modello 3:

$$h_\theta(x) = \theta_0 + \theta_1(RM * LSTAT) + \theta_2(RM * LSTAT^2)$$

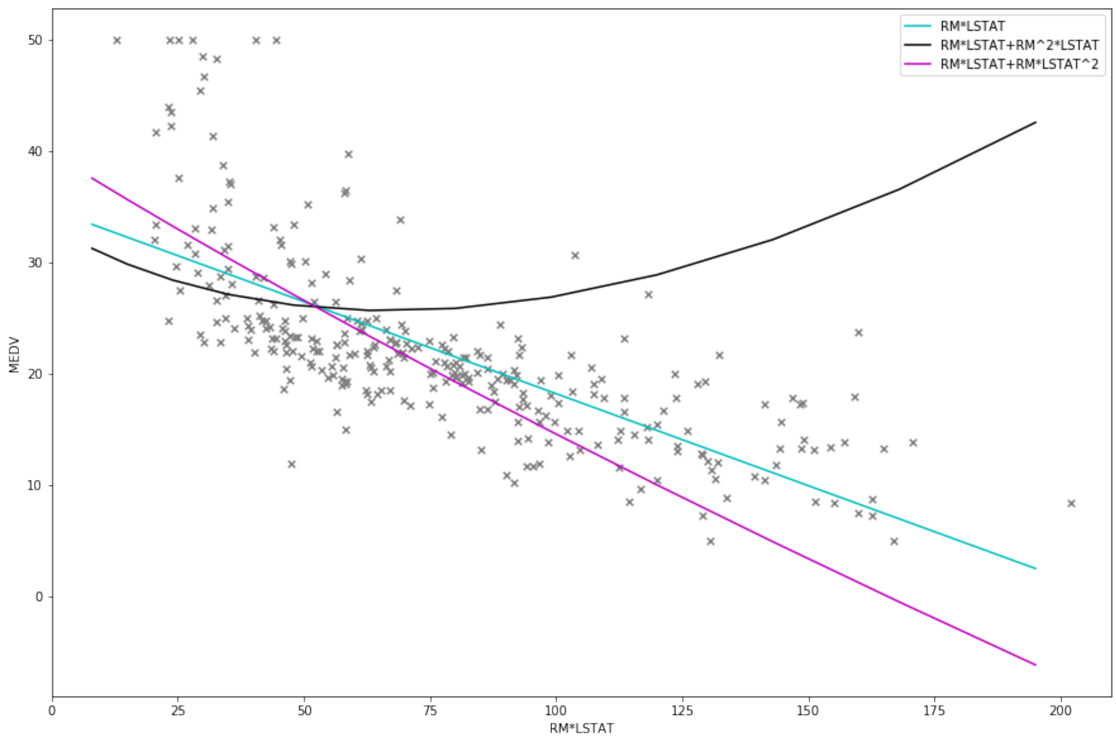
7.5 4. Let's apply the normal equation to find the theta parameters of our models

In [54]:
```
def new_polynomial(X, feat, degree):
    colname = dataframe.columns[0]+'^'+str(degree)
    X[colname] = np.power(X[feat], degree)
    return X[colname]
```

In [55]:
```
rm = pd.DataFrame(np.c_[X['RM']], columns = ['RM'])
lstat = pd.DataFrame(np.c_[X['LSTAT']], columns = ['LSTAT'])

X_1 = np.concatenate([np.ones((X.shape[0],1)), X_new.values], axis=1)
theta_ne_1 = normalEquation(X_1,y)
print(theta_ne_1)

X_2 = np.concatenate([np.ones((X.shape[0],1)), X_new.values, np.c_[new_polynomial(rm, '
theta_ne_2 = normalEquation(X_2,y)
print(theta_ne_2)

X_3 = np.concatenate([np.ones((X.shape[0],1)), X_new.values, np.c_[new_polynomial(lstat
theta_ne_3 = normalEquation(X_3,y)
print(theta_ne_3)
```

```
[[34.72424377]
 [-0.16546041]]
[[ 3.33438816e+01]
 [-3.19328315e-01]
 [ 2.82006249e-02]]
[[ 3.98070998e+01]
 [-2.99299328e-01]
 [ 4.22262190e-03]]
```

In [56]:
```
xx1 = np.arange(2,14)
xx2 = np.arange(4,16)
yy1 = theta_ne_1[0] + theta_ne_1[1] * xx1 * xx2
yy2 = theta_ne_2[0] + theta_ne_2[1] * xx1 * xx2 + theta_ne_2[2] * (xx1**2) * xx2
yy3 = theta_ne_3[0] + theta_ne_3[1] * xx1 * xx2 + theta_ne_3[2] * (xx2**2) * xx1

plt.figure(figsize=(15, 10))
plt.scatter(X_1[:,1], y, s=30, c='grey', marker='x', linewidths=1)
plt.plot(xx1 * xx2,yy1,c='c', label='RM*LSTAT')
plt.plot(xx1 * xx2,yy2,c='black', label='RM*LSTAT+RM^2*LSTAT')
plt.plot(xx1 * xx2,yy3,c='m', label='RM*LSTAT+RM*LSTAT^2')

plt.xlabel('RM*LSTAT')
plt.ylabel('MEDV')
plt.xlim(0,210)
plt.legend(loc=0);
plt.show()
```



7.6 RMSE on training set

```
In [57]: X = train.loc[:, ['RM', 'LSTAT']]
         X_train = pd.DataFrame(np.c_[X['RM']*X['LSTAT']], columns = ['RM*LSTAT'])
         y_train = train.loc[:, ['MEDV']]

         rm_train = pd.DataFrame(np.c_[X['RM']], columns = ['RM'])
         lstat_train = pd.DataFrame(np.c_[X['LSTAT']], columns = ['LSTAT'])

In [58]: y_pred = np.zeros((sec_size, 1))
         X_train1 = np.concatenate([np.ones((X_train.shape[0],1)), X_train], axis=1)
         y_pred = X_train1.dot(theta_ne_1)

         rmse = np.sqrt(mean_squared_error(y_train, y_pred))
         print('RM*LSTAT: ', rmse)

         y_pred = np.zeros((sec_size, 1))
         X_train2 = np.concatenate([np.ones((X_train.shape[0],1)), X_train, np.c_[new_polynomial
         y_pred = X_train2.dot(theta_ne_2)

         rmse = np.sqrt(mean_squared_error(y_train, y_pred))
         print('RM*LSTAT+RM^2*LSTAT: ', rmse)
```

```
y_pred = np.zeros((sec_size, 1))
X_train3 = np.concatenate([np.ones((X_train.shape[0],1)), X_train, np.c_[new_polynomial
y_pred = X_train3.dot(theta_ne_3)

rmse = np.sqrt(mean_squared_error(y_train, y_pred))
print('RM*LSTAT+RM*LSTAT^2: ', rmse)
```

```
RM*LSTAT:  5.827539871205015
RM*LSTAT+RM^2*LSTAT:  5.665232932749886
RM*LSTAT+RM*LSTAT^2:  5.609669764756434
```

7.7 5. Validation phase, let's evaluate the RMSE metric for the validation set

```
In [59]: X = valid.loc[:, ['RM', 'LSTAT']]
         X_valid = pd.DataFrame(np.c_[X['RM']*X['LSTAT']], columns = ['RM*LSTAT'])
         y_valid = valid.loc[:, ['MEDV']]

         rm_valid = pd.DataFrame(np.c_[X['RM']], columns = ['RM'])
         lstat_valid = pd.DataFrame(np.c_[X['LSTAT']], columns = ['LSTAT'])
```

```
In [60]: y_pred = np.zeros((sec_size, 1))
         X_valid1 = np.concatenate([np.ones((X_valid.shape[0],1)), X_valid], axis=1)
         y_pred = X_valid1.dot(theta_ne_1)

         rmse = np.sqrt(mean_squared_error(y_valid, y_pred))
         print('RM*LSTAT: ', rmse)

         y_pred = np.zeros((sec_size, 1))
         X_valid2 = np.concatenate([np.ones((X_valid.shape[0],1)), X_valid, np.c_[new_polynomial
         y_pred = X_valid2.dot(theta_ne_2)

         rmse = np.sqrt(mean_squared_error(y_valid, y_pred))
         print('RM*LSTAT+RM^2*LSTAT: ', rmse)

         y_pred = np.zeros((sec_size, 1))
         X_valid3 = np.concatenate([np.ones((X_valid.shape[0],1)), X_valid, np.c_[new_polynomial
         y_pred = X_valid3.dot(theta_ne_3)

         rmse = np.sqrt(mean_squared_error(y_valid, y_pred))
         print('RM*LSTAT+RM*LSTAT^2: ', rmse)
```

```
RM*LSTAT:  7.180927065920508
RM*LSTAT+RM^2*LSTAT:  6.948146433593475
RM*LSTAT+RM*LSTAT^2:  6.939091354892269
```

7.8 6. Testing phase

```
In [61]: X = test.loc[:, ['RM', 'LSTAT']]
         X_test = pd.DataFrame(np.c_[X['RM']*X['LSTAT']], columns = ['RM*LSTAT'])
         y_test = test.loc[:, ['MEDV']]

         rm_test = pd.DataFrame(np.c_[X['RM']], columns = ['RM'])
         lstat_test = pd.DataFrame(np.c_[X['LSTAT']], columns = ['LSTAT'])
```

```
In [62]: y_pred = np.zeros((sec_size, 1))
         X_test1 = np.concatenate([np.ones((X_test.shape[0],1)), X_test], axis=1)
         y_pred = X_test1.dot(theta_ne_1)

         rmse = np.sqrt(mean_squared_error(y_test, y_pred))
         print('RM*LSTAT: ', rmse)

         y_pred = np.zeros((sec_size, 1))
         X_test2 = np.concatenate([np.ones((X_test.shape[0],1)), X_test, np.c_[new_polynomial(rm
         y_pred = X_test2.dot(theta_ne_2)

         rmse = np.sqrt(mean_squared_error(y_test, y_pred))
         print('RM*LSTAT+RM^2*LSTAT: ', rmse)

         y_pred = np.zeros((sec_size, 1))
         X_test3 = np.concatenate([np.ones((X_test.shape[0],1)), X_test, np.c_[new_polynomial(ls
         y_pred = X_test3.dot(theta_ne_3)

         rmse = np.sqrt(mean_squared_error(y_test, y_pred))
         print('RM*LSTAT+RM*LSTAT^2: ', rmse)
```

```
RM*LSTAT:  6.758619856001923
RM*LSTAT+RM^2*LSTAT:  6.5088802488127895
RM*LSTAT+RM*LSTAT^2:  6.655358551807103
```

7.9 Observation

7.9.1 This approach can be used to find a model that fits well the data but actually this has no physical sense because the two features are meaningless when they are mixed.