COMPUTING INTEGRALS OF EXPM

For computing $\int_0^t e^{\tau A} d\tau$ I can simply compute e^C, where

$$C = \begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}$$

To do so we need to compute powers of C:

$$C^2 = \begin{bmatrix} A & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2 & A \\ 0 & 0 \end{bmatrix}$$

$$C^3 = \begin{bmatrix} A^2 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A^3 & A^2 \\ 0 & 0 \end{bmatrix}$$

$$C^K = \begin{bmatrix} A^K & A^{K-1} \\ 0 & 0 \end{bmatrix} \qquad e^C \simeq \left(\sum_{K=0} a_K C^K \right)^{-1} \left(\sum_{K=0} b_K C^K \right)$$

Good news: we can compute powers of C by only computing powers of A which is half its size, so it should be 8 times faster.
After computing the powers of C we'll need to invert a matrix that has this structure:

$$M = \begin{bmatrix} M_1 & M_2 \\ 0 & \alpha I \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \qquad \alpha = a_0 \in \mathbb{R}$$

$$M M^{-1} = \begin{bmatrix} M_1 & M_2 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} = \begin{bmatrix} M_1 N_1 + M_2 N_3 & M_1 N_2 + M_2 N_4 \\ \alpha N_3 & \alpha N_4 \end{bmatrix}$$

So we know that $N_3 = 0$, $N_4 = \alpha^{-1} I$ and:

$$\begin{cases} M_1 N_1 = I \\ M_1 N_2 + \alpha^{-1} M_2 = 0 \end{cases} \qquad \begin{cases} N_1 = M_1^{-1} \\ N_2 = -\alpha^{-1} M_1^{-1} M_2 \end{cases}$$

We only need to invert M_1 to compute the inverse of M

May be the same trick can be applied to compute

$$x_{int}(t) = \int_0^t x(\tau)\,d\tau \qquad x_{int2}(t) = \int_0^t \int_0^{\tau} x(\tau_1)\,d\tau_1\,d\tau$$

$$x(t) = e^{tA}x(0) + \int_0^t e^{\tau A}\,d\tau\, b \qquad \dot{x} = Ax + b$$

We compute the first two terms as:

$$x_{int}(t) = e^{tA_1}\begin{bmatrix}0_{n+1}\\ 1\end{bmatrix} \qquad A_1 \triangleq \begin{bmatrix}A_0 & x_0(0)\\ 0 & 0\end{bmatrix} \qquad A_0 = \begin{bmatrix}A & b\\ 0 & 0\end{bmatrix}$$

$$x_{int2}(t) = e^{tA_2}\begin{bmatrix}0_{n+2}\\ 1\end{bmatrix} \qquad A_2 \triangleq \begin{bmatrix}A_0 & x_0(0) & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \qquad x_0(0) = \begin{bmatrix}x(0)\\ 1\end{bmatrix}$$

Actually computations could be faster by having zero initial conditions

$$\dot{x}(t) = A\,x(t) + b \qquad \bar{x}(t) \triangleq x(t) - x(0) \Rightarrow \bar{x}(0) = 0$$

$$\dot{\bar{x}}(t) = \dot{x}(t) = A\left(x(t) - x(0)\right) + b = A\,x(t) + \underbrace{b - A\,x(0)}_{\bar{b}}$$

$$\bar{x}(t) = \int_0^t e^{\tau A}\,d\tau\,\bar{b} \Rightarrow x(t) = x(0) + \bar{x}(t) = x(0) + \int_0^t e^{\tau A}\,d\tau\,\left(b - A\,x(0)\right)$$

Using this expression I can move $x(0)$ inside $\bar{b}$ so A_1, A_2 become

$$A_1 \triangleq \begin{bmatrix}A & \bar{b} & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$A_2 \triangleq \begin{bmatrix}A & \bar{b} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So to compute expm of A_1 and A_2 we need powers of A_1, A_2:

$$A_1^2 = \begin{bmatrix} A & \bar{b} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & \bar{b} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2 & A\bar{b} & \bar{b} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_1^3 = \begin{bmatrix} A^2 & A\bar{b} & \bar{b} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & \bar{b} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^3 & A^2\bar{b} & A\bar{b} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_1^k = \begin{bmatrix} A^k & A^{k-1}\bar{b} & A^{k-2}\bar{b} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ $\Leftarrow$ We only need powers of A!

$$A_2^2 = \begin{bmatrix} A & \bar{b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & \bar{b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2 & A\bar{b} & \bar{b} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_2^3 = \begin{bmatrix} A^2 & A\bar{b} & \bar{b} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & \bar{b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^3 & A^2\bar{b} & A\bar{b} & \bar{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1^3 & \begin{smallmatrix}\bar{b} \\ 0 \\ 0\end{smallmatrix} \\ 0 & 0 \end{bmatrix}$$

$$A_2^4 = \begin{bmatrix} A^3 & A^2\bar{b} & A\bar{b} & \bar{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & \bar{b} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^4 & A^3\bar{b} & A^2\bar{b} & A\bar{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2^k = \begin{bmatrix} A^k & A^{k-1}\bar{b} & A^{k-2}\bar{b} & A^{k-3}\bar{b} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $\Leftarrow$ We only need powers of A

Basically, once we have computed powers of A, then computing powers of A_1 and A_2 is very cheap! The only remaining operation in expm is the inverse of a polynomial of A_1 / A_2. I am hopeful that even these two inverses could be easily computed together saving computation time. The derivation of their expressions should follow the same approach used above for deriving the inverse of M.