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A simple DFT algorithm

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Taking as an example a set of 16 index samples:

$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15$$

we divide into even and odd positions. We obtain:

$$0\ 2\ 4\ 6\ 8\ 10\ 12\ 14 \qquad 1\ 3\ 5\ 7\ 9\ 11\ 13\ 15$$

which correspond respectively to the terms $2n$ e $2n+1$.

We repeat the same operation in the two subsets and find the 4 subsets:

$$0\ 4\ 8\ 12 \quad 2\ 6\ 10\ 14 \quad 1\ 5\ 9\ 13 \quad 3\ 7\ 11\ 15$$

that correspond to the terms $4n, 4n+2, 4n+1, 4n+3$.

We can therefore express the DFT as:

$$X(k) = \sum_{n}^{\frac{N}{4}-1} x\,(4n) W_N^{-4kn} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+2) W_N^{-k(4n+2)} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+1) W_N^{-k(4n+1)} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+3) W_N^{-k(4n+3)}$$

where $W_N^{-kn} = e^{-j2\pi\frac{kn}{N}}$.

It is shown [1] that $W_N^{-4kn} = W_{N/4}^{-kn}$ from which we can therefore rewrite:

$$X(k) = \sum_{n}^{\frac{N}{4}-1} x\,(4n) W_{N/4}^{-kn} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+2) W_{N/4}^{-k(n+1/2)} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+1) W_{N/4}^{-k(n+1/4)} + \sum_{n}^{\frac{N}{4}-1} x\,(4n+3) W_{N/4}^{-k(n+3/4)}$$

which we rewrite as:

$$X(k) = \sum_{n}^{\frac{N}{4}-1} x\,(4n)W_{N/4}^{-k(n)} + W_{N/4}^{-k/2} \sum_{n}^{\frac{N}{4}-1} x\,(4n+2)W_{N/4}^{-k(n)}$$
$$+ W_{N/4}^{-k/4} \sum_{n}^{\frac{N}{4}-1} x\,(4n+1)W_{N/4}^{-k(n)} + W_{N/4}^{-k(3/4)} \sum_{n}^{\frac{N}{4}-1} x\,(4n+3)W_{N/4}^{-k(n)}$$

Focusing on the terms $W_{N/4}^{-k/2}$, $W_{N/4}^{-k/4}$ e $W_{N/4}^{-k(3/4)}$, based on (1) we can rewrite them as W_N^{-2k}, W_N^{-k} e W_N^{-3k} which can be shown (2) to be periodic of period $\frac{N}{4}$, $\frac{N}{2}$ e $\frac{N}{6}$ respectively.

If the number of samples N ***is divisible by 2,4 and*** 6 and knowing that $W_{N/4}^{-k}$ is periodic of $\frac{N}{4}$ the total number of operations is:

- Sums: $4 \cdot \frac{N}{4} \cdot \frac{N}{4}$ operations knowing that $k \in [0, N-1]$
- $\frac{N}{4} + \frac{N}{2} + \frac{N}{6}$ multiplications for $W_{N/4}^{-k/2}$, $W_{N/4}^{-k/4}$ e $W_{N/4}^{-k(3/4)}$

for a total of $\left(\frac{N}{2}\right)^2 + \frac{11}{12}N$

Demonstrations

(1) $W_N^{-4kn} = e^{j2\pi\frac{-4kn}{N}} = e^{j2\pi\frac{-kn}{N/4}} = W_{N/4}^{-kn}$

(2) W_N^{-2k} is periodic of $\frac{N}{4}$:

$$e^{-j4\pi\frac{(k+N/4)}{N}} = e^{-j4\pi\frac{k}{N}} \cdot e^{-j\pi} = -\,e^{-j2\pi\frac{2kn}{N}} = -\,W_N^{-2k}$$

W_N^{-k} is periodic of $\frac{N}{2}$:

$$e^{-j2\pi\frac{(k+N/2)}{N}} = e^{-j2\pi\frac{k}{N}} \cdot e^{-j\pi} = -\,e^{-j2\pi\frac{kn}{N}} = -\,W_N^{-k}$$

W_N^{-3k} is periodic of $\frac{N}{6}$:

$$e^{-j6\pi\frac{(k+N/6)}{N}} = e^{-j6\pi\frac{k}{N}} \cdot e^{-j\pi} = -\,e^{-j2\pi\frac{kn}{N}} = -\,W_N^{-3k}$$