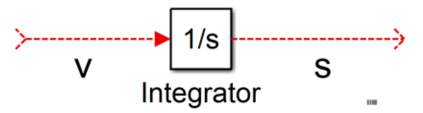
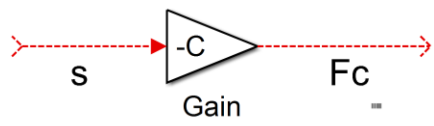
From ODE to Simulink and Vice Versa Introduction

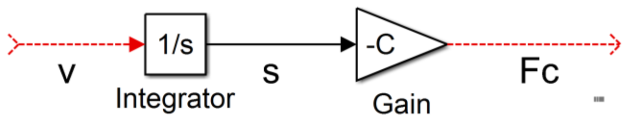
**So far…**

$$s = \int v\,dt \quad \Longrightarrow$$




$$Fc = -C \cdot s \quad \Longrightarrow$$




Prof. Dr.-Ing. Patrick Metzler Modelling and Simulation using Simulink© 1

So far, we generated our model by concatenating the graphical representations of simple laws like

$$s = \int v\,dt$$

or

$$Fc = -C \cdot s$$

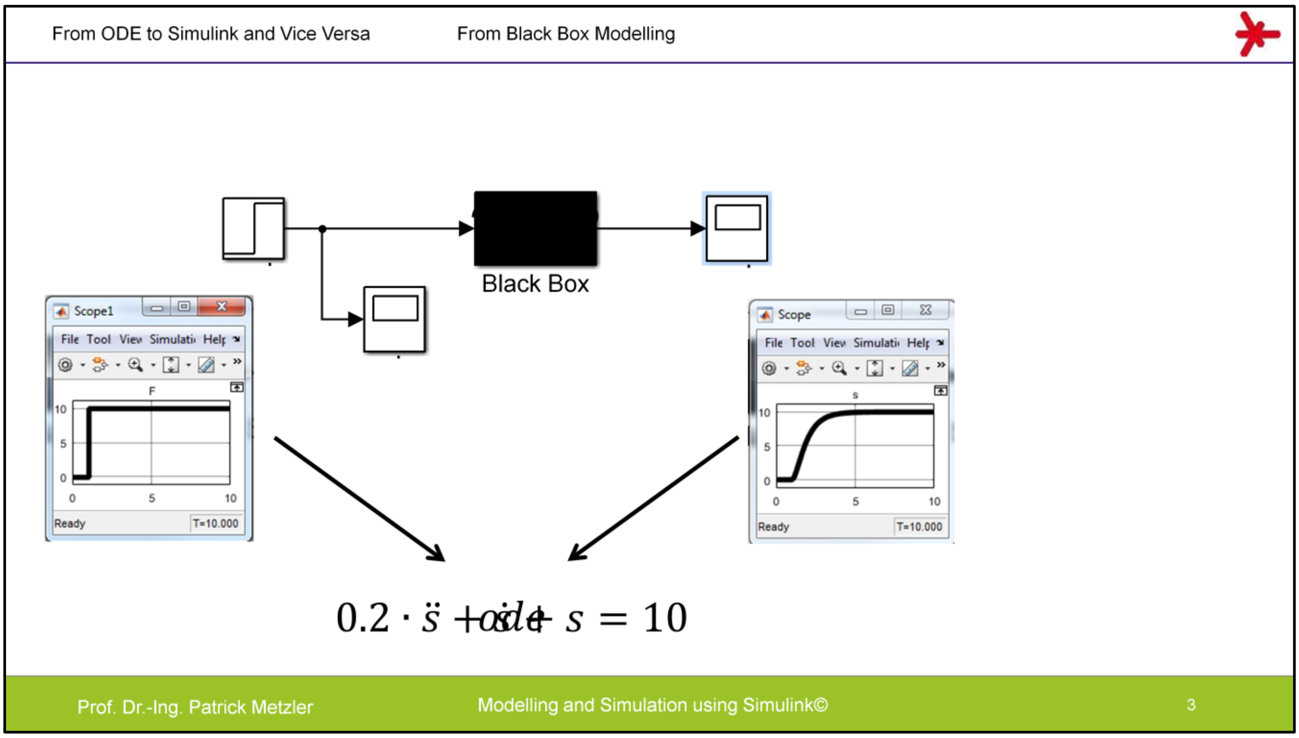
By this way we build a complete Simulink model.

…now **Ordinary Differential Equation (ode)**

$$\boxed{m\cdot\ddot{s} + D\cdot\dot{s} + C\cdot s = F}$$



But sometimes you want to enter an ordinary differential equation (TLA: ode) directly in Simulink. Either someone has already combined the simple laws to a more complex ode on a sheet of paper.




Or sometimes there are no laws to be known or the laws are too complicated and by black box modelling you just came up with a differential equation. This approximately describes the behavior of your system, but black box modelling on its own does not tell you which parameter is representing what.

For example in our msd-system (<u>m</u>ass <u>s</u>pring <u>d</u>amper), black box modelling would give you the equation

$$0.2 \cdot \ddot{s} + \dot{s} + s = 10$$

without telling you what the 0.2 is all about.

…ode to simulink, ode does not feature derivatives of the input signal

$$a3 \cdot \dddot{v} + a2 \cdot \ddot{v} + a1 \cdot \dot{v} + a0 \cdot v = b0 \cdot u$$



Generating a Simulink Modell from an ode is very simple, if there are no derivatives of the input signal in the ode. Let us denote u as the input of a system and v as its output. (Unlike a Simulink model, an ode does not intrinsically tell you what is input and what is output. But it's good practice to place the output related terms on the left side of the equation and the input related terms on the right side of the equation.

To develop the Simulink model of an equation like

$$a3 \cdot \dddot{v} + a2 \cdot \ddot{v} + a1 \cdot \dot{v} + a0 \cdot v = b0 \cdot u$$

you can follow these steps

Step 1
First solve the equation for the summand with the highest order derivation of v.

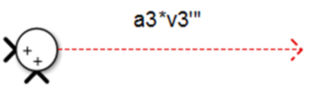
$$a3 \cdot \dddot{v} + a2 \cdot \ddot{v} + a1 \cdot \dot{v} + a0 \cdot v = b0 \cdot u$$

$$\Downarrow$$

$$a3 \cdot \dddot{v} = b0 \cdot u - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$

Step 2
Secondly, start your model with the summand with the highest order derivation of v as the output of a sum block.

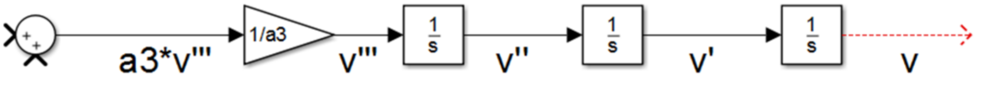
$$a3 \cdot \dddot{v} = b0 \cdot u - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$




Please note, within a Simulink model I use the ' to indicate a derivation.

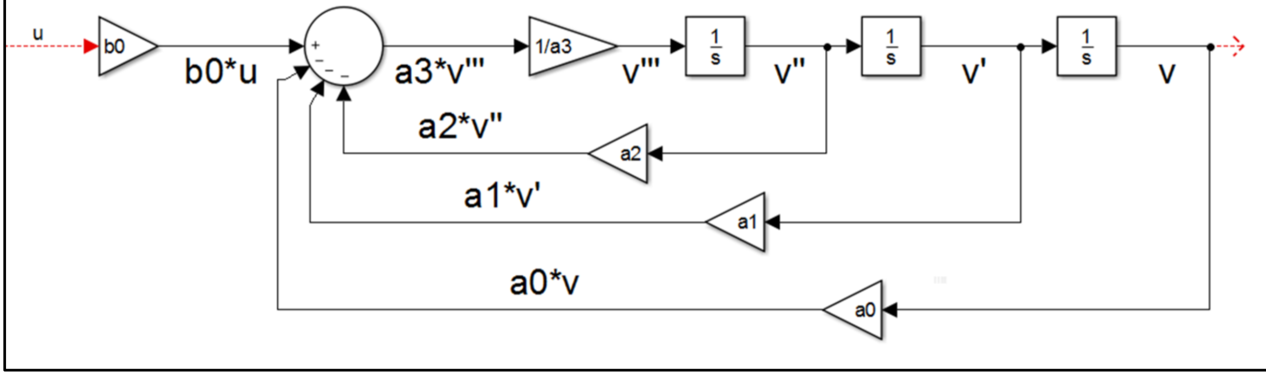
Step 3
What we are interested in is the output v. This we get by dividing by a3 and integrating three times.
Or more generally: divide by the factor included in the summand with the highest grade of derivation.
Integrate until the output is v.

$$a3 \cdot \dddot{v} = b0 \cdot u - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$

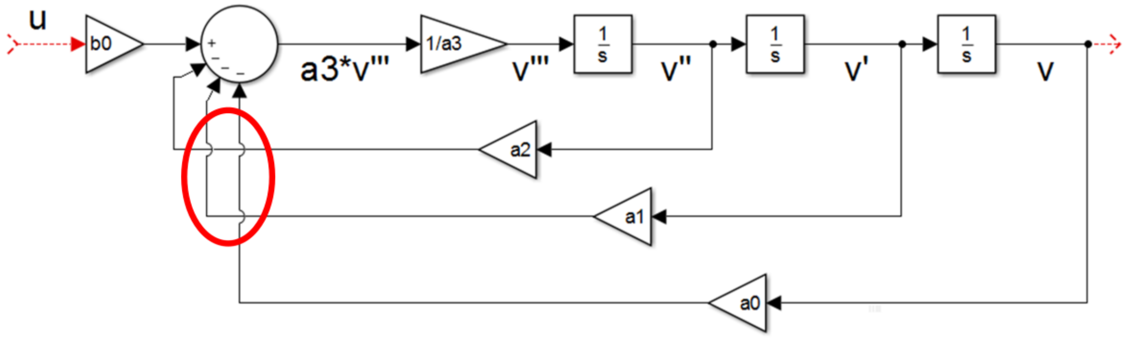


Step 4
Add the right side terms with the right signs to the summand block.

$$a3 \cdot \dddot{v} = b0 \cdot u - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$



Try to avoid unnecessary crossings of signal lines:



## …ode to simulink, ode features derivatives of the input signal

$$a3 \cdot \dddot{v} + a2 \cdot \ddot{v} + a1 \cdot \dot{v} + a0 \cdot v = b0 \cdot u + b1 \cdot \dot{u}$$



Things are more complicated if your ode features derivatives of the input signal.
The easiest example would be this equation.

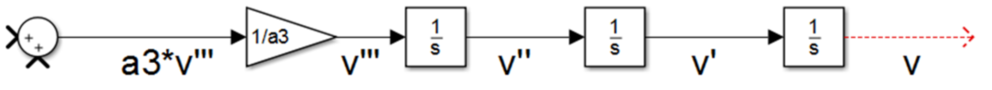
$$a3 \cdot \dddot{v} + a2 \cdot \ddot{v} + a1 \cdot \dot{v} + a0 \cdot v = b0 \cdot u + b1 \cdot \dot{u}$$

$$a3 \cdot \dddot{v} = b0 \cdot u + b1 \cdot \dot{u} - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$



Now Step 1 looks
like this.

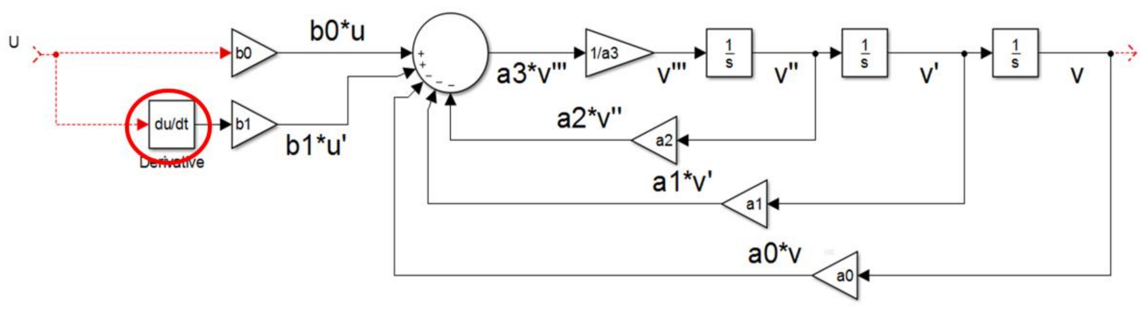
$$a3 \cdot \dddot{v} = b0 \cdot u + b1 \cdot \dot{u} - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$






Step 2 and Step 3 are as in the previous example,

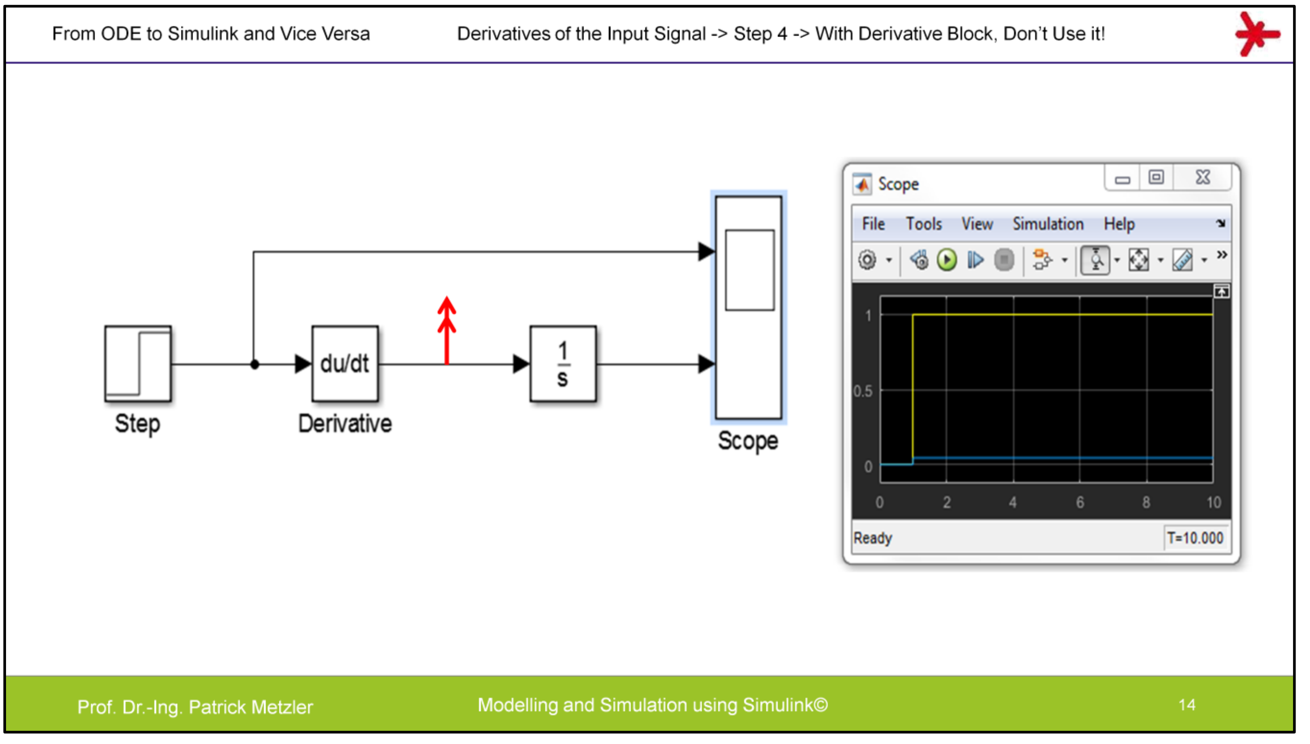
$$a3 \cdot \dddot{v} = b0 \cdot u + b1 \cdot \dot{u} - a2 \cdot \ddot{v} - a1 \cdot \dot{v} - a0 \cdot v$$





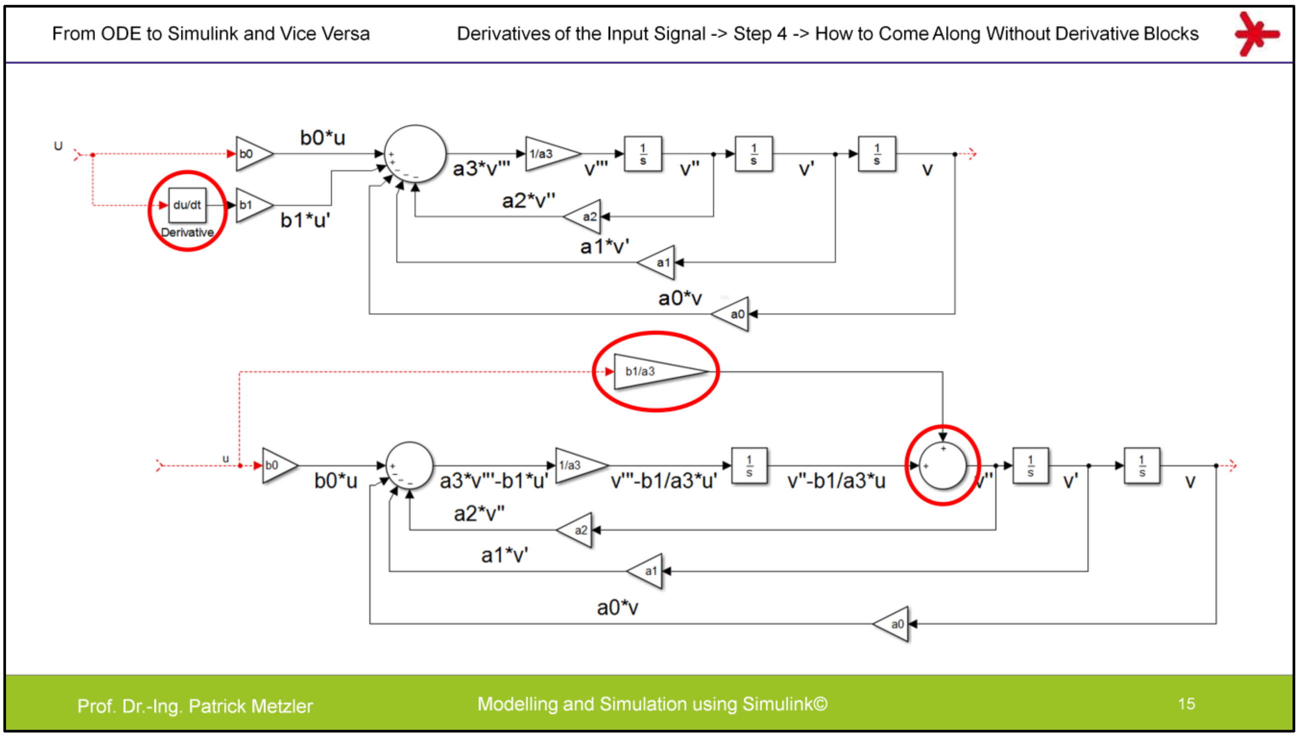

And Step 4 doesn’t look like a big deal if you use the derivative block.

But a very important rule of thumb for Simulink is: Don’t use the Derivative Block. Of course there are exceptions to this rule but first let’s look at why this rule is important:




Depending on the solver configuration, derivative blocks can lead to wrong numeric results, as the blue output of this system should be the same as the yellow input. (Good news, good old Euler handles this example correctly.)

Furthermore, if our system is fed with a step signal, we have unreasonably high in-between signals. In the way this Simulink model is laid out, you could never build it e.g. with analog electronics. Therefore it's good practice to avoid derivative blocks.

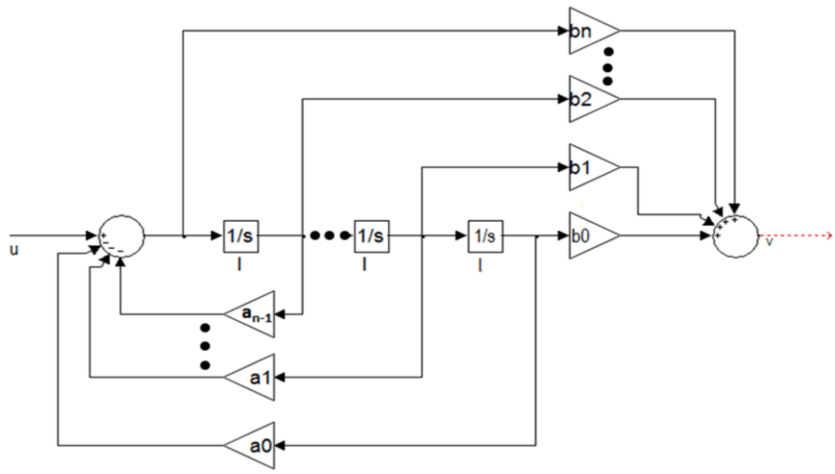



The idea now is to combine a derivative block and an integrator block to compensate each other. In order to do so, the entry point is moved downstream the signal lines.

While in this small example it is easy to do this by hand, for more complex systems this gets complicated. So I'll just give you a recipe for how to do it.

…ode to simulink, general recipe

$$1\cdot v^{(n)}(t)+\ldots+a_1\cdot v^{(1)}(t)+a_0\cdot v(t)=b_0\cdot u(t)+b_1\cdot u^{(1)}(t)+\ldots b_n\cdot u^{(n)}(t)$$



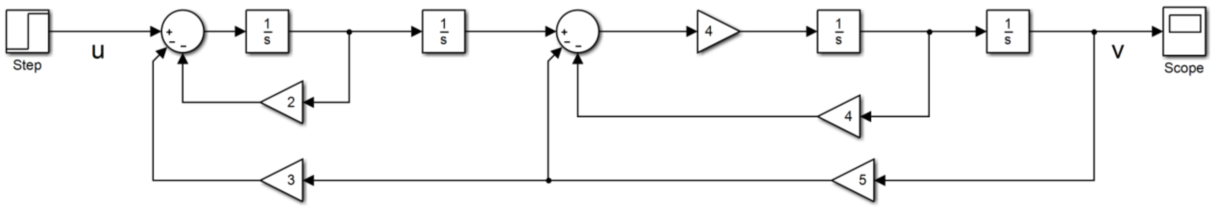



So here is the first main result:

A system with an ode like this where a *number in high parenthesis describes the order of the derivative.*

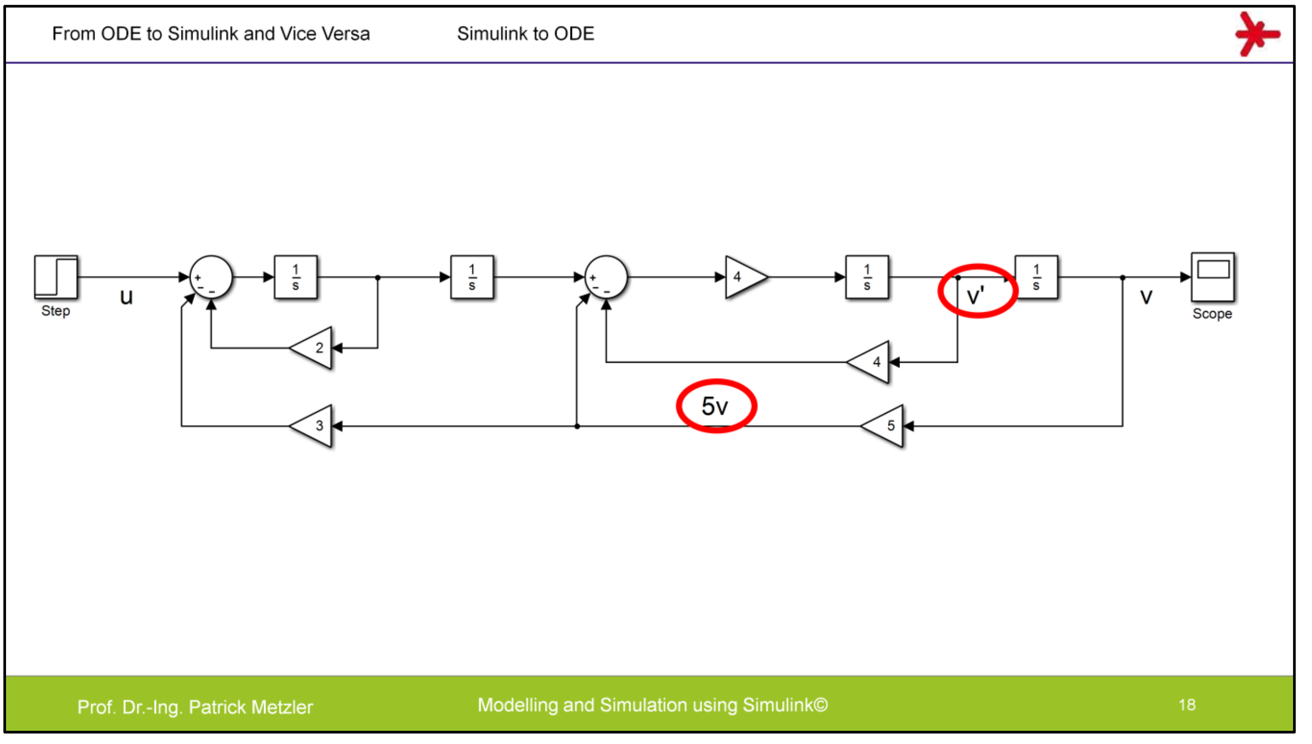
can be modelled in Simulink like this:

## From simulink to ode





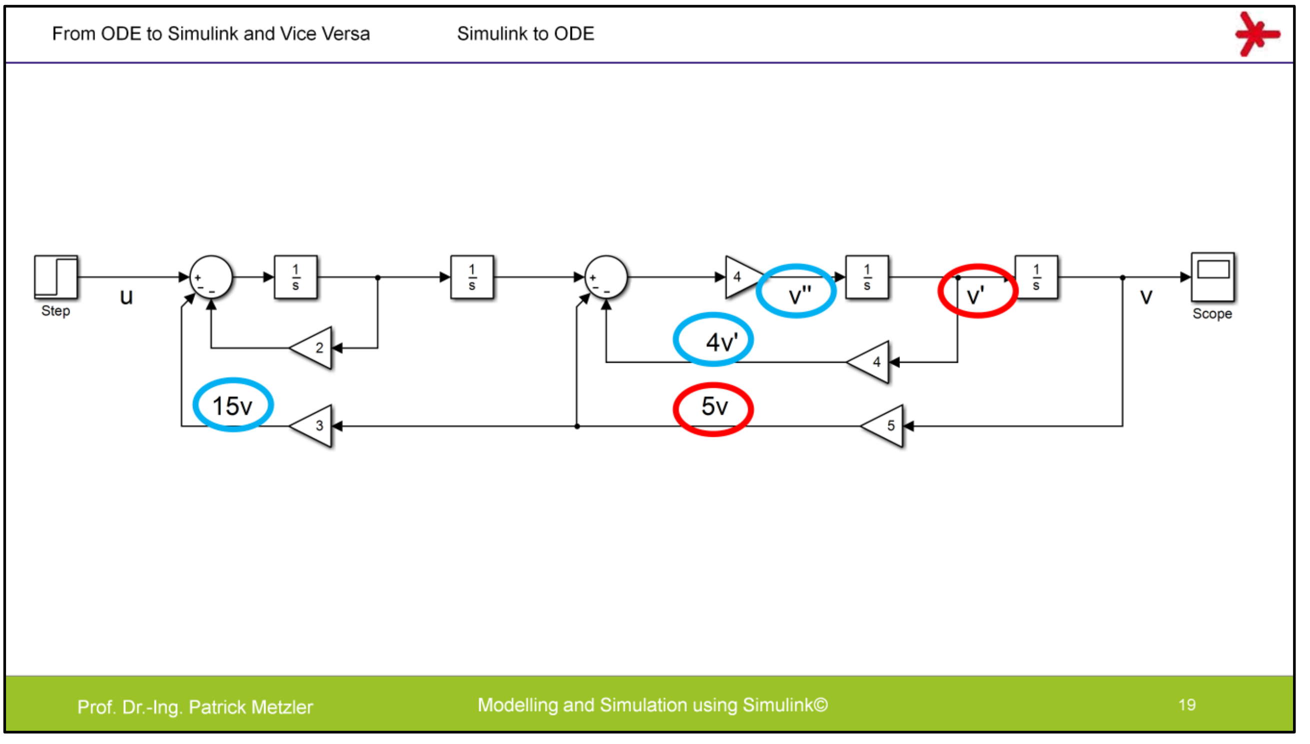

Here is a simple example of how to go the other way round: How do we get from the model to the ode?



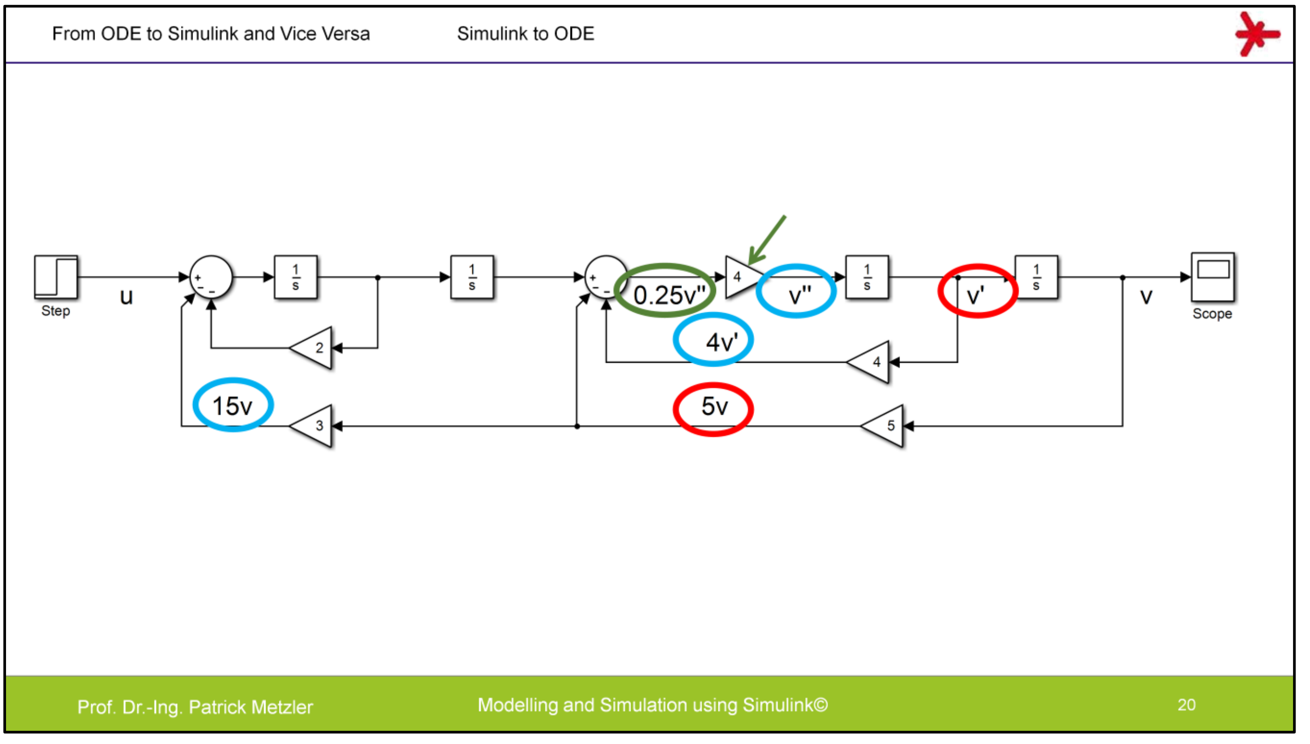

We walk from the outside (inputs and outputs), taking a step towards the inner part of the model.

We follow the gain block from its input to its output. With the input of the gain block being the output v of the model, the output of the gain block is 5 times v.

Going from the output of the integrator back to its input, we have to reverse the integrating process i. e. we have to calculate the derivative of the integrator output.



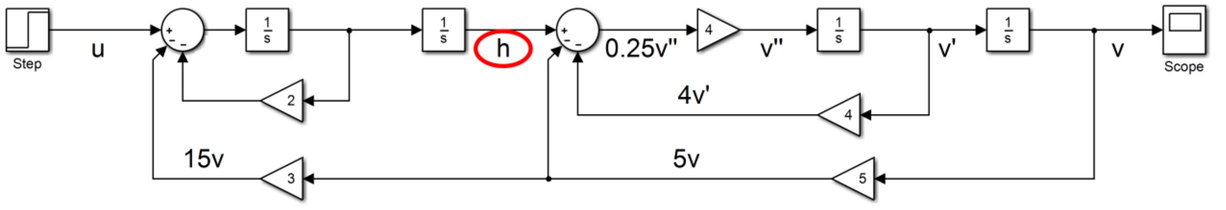

And from these results, step another block forward or back respectively




And from these results, step another block forward or back respectively ..as long as we can do that..

Note also the gain block with gain 4. Which input multiplied by 4 makes an output of v’’?. Right - 0.25 v’’.

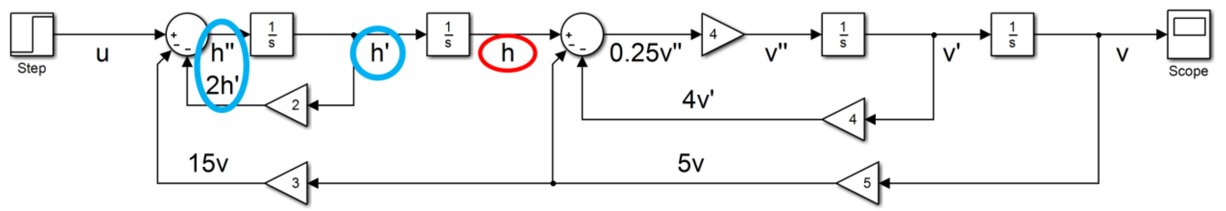
It is not possible to step any further without going backwards through a sum block. Some folks can do that using mental arithmetic. But to have a clear systematic approach let us add some help variables.






Having n sum blocks we may need up to n-1 help variables. In our example there are 2 sum blocks, thus we need one help variable.

Choose an unknown input signal as a help variable. And then go inside again.




Left sum block:

$$u - 15 \cdot v - 2 \cdot \dot{h} = \ddot{h}$$

Right sum block:

$$h - 5 \cdot v - 4 \cdot \dot{v} = 0.25 \cdot \ddot{v}$$



Now we can write down the equations of the two sum blocks.

Left sum block:

$$u - 15 \cdot v - 2 \cdot \dot{h} = \ddot{h}$$

Right sum block:

$$h - 5 \cdot v - 4 \cdot \dot{v} = 0.25 \cdot \ddot{v}$$

$$h = 0.25 \cdot \ddot{v} + 5 \cdot v + 4 \cdot \dot{v}$$

$$\dot{h} = 0.25 \cdot \dddot{v} + 5 \cdot \dot{v} + 4 \cdot \ddot{v}$$

$$\ddot{h} = 0.25 \cdot v^{(4)} + 5 \cdot \ddot{v} + 4 \cdot \dddot{v}$$

$$u - 15 \cdot v - 2 \cdot (0.25 \cdot \dddot{v} + 5 \cdot \dot{v} + 4 \cdot \ddot{v}) = 0.25 \cdot v^{(4)} + 5 \cdot \ddot{v} + 4 \cdot \dddot{v}$$

$$v^{(4)} + 18 \cdot \dddot{v} + 52 \cdot \ddot{v} + 40 \cdot \dot{v} + 60 \cdot v = 4 \cdot u$$



Now solving for the unknown signal h,

Calculating the derivatives of h

and inserting all these results e.g. in the equation of the left sum block we get

a first version of the ode describing the system.

And after some housekeeping this one.

Left sum block:

$$u - 15 \cdot v - 2 \cdot \dot{h} = \ddot{h}$$

Right sum block:

$$h - 5 \cdot v - 4 \cdot \dot{v} = 0.25 \cdot \ddot{v}$$

$$h = 0.25 \cdot \ddot{v} + 5 \cdot v + 4 \cdot \dot{v}$$

$$\dot{h} = 0.25 \cdot \dddot{v} + 5 \cdot \dot{v} + 4 \cdot \ddot{v}$$

$$\ddot{h} = 0.25 \cdot v^{(4)} + 5 \cdot \ddot{v} + 4 \cdot \dddot{v}$$

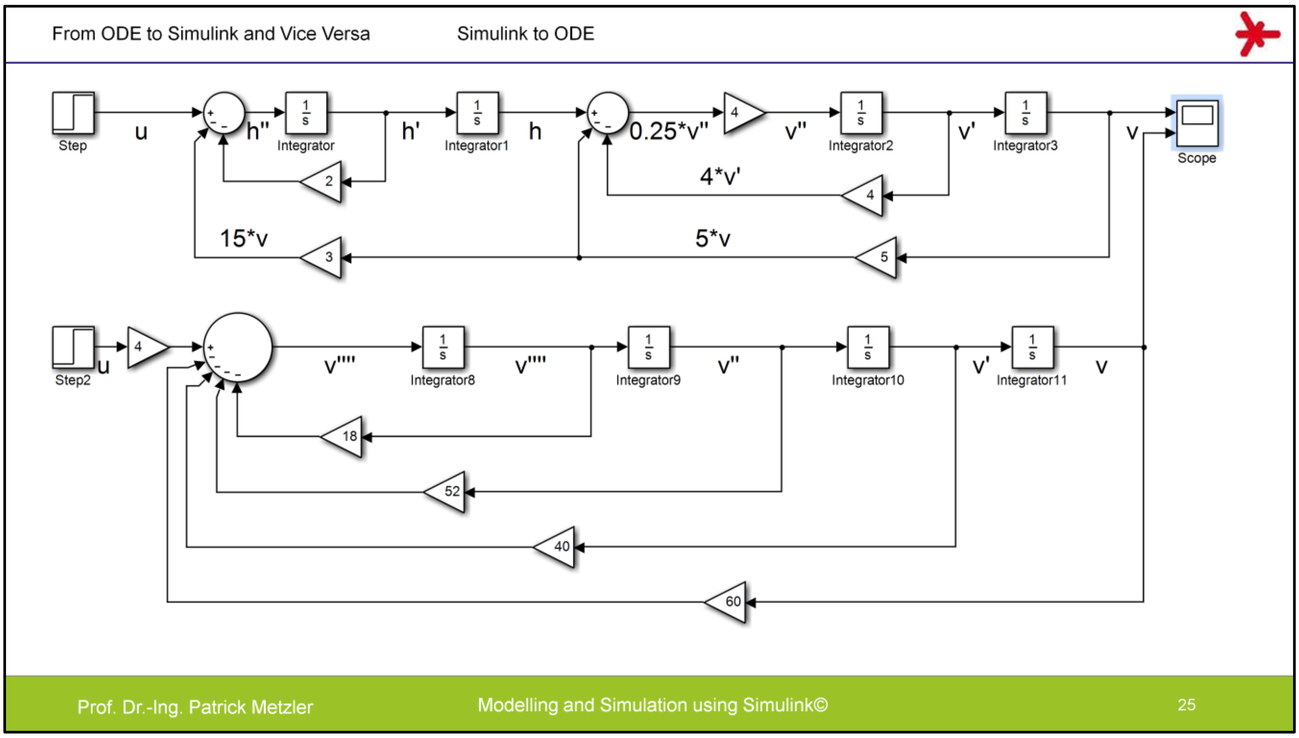
$$u - 15 \cdot v - 2 \cdot (0.25 \cdot \dddot{v} + 5 \cdot \dot{v} + 4 \cdot \ddot{v}) = 0.25 \cdot v^{(4)} + 5 \cdot \ddot{v} + 4 \cdot \dddot{v}$$

$$v^{(4)} + 18 \cdot \dddot{v} + 52 \cdot \ddot{v} + 40 \cdot \dot{v} + 60 \cdot v = 4 \cdot u$$

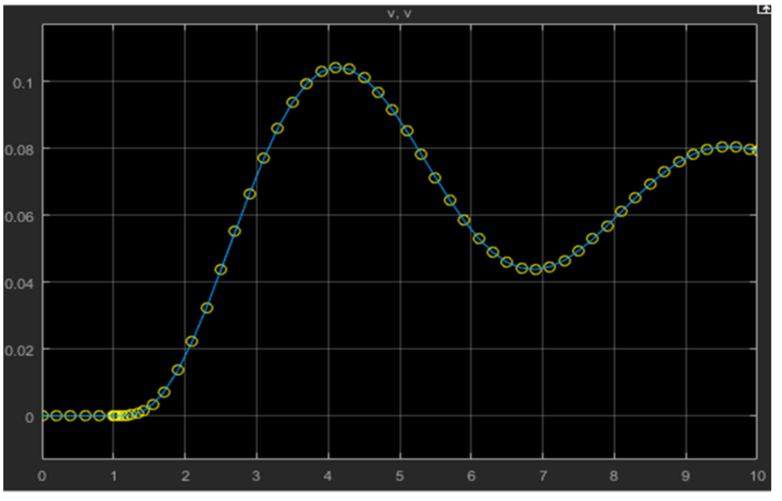
# Try to develop the Simulink Model!



To check the ode we just found, let us transform it into a model again using our general recipe to do so.
Pause this presentation and try it for your own!



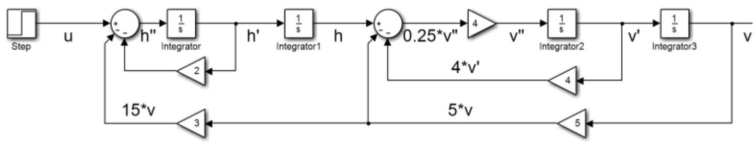

If we followed our recipe correctly, both models must lead to the same curves.






For better comparison, I show one signal as solid cyan line and the other signal as yellow circles. And indeed both signals are identical.

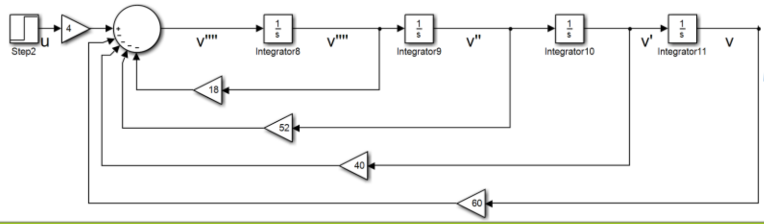
## Summary




Simulink to ode

$$v^{(4)} + 18 \cdot \dddot{v} + 52 \cdot \ddot{v} + 40 \cdot \dot{v} + 60 \cdot v = 4 \cdot u$$

ode to Simulink






So we just did a round trip. First from simulink to ode and then from ode to simulink again.

And that's exactly what this unit was about: how can you go from a simulink model to an ode and from an ode to simulink.

For the sake of simplicity I did not cover nonlinear differential equations or nonlinear simulink models respectively. But the principles are the same.

## Physical Units

$$m \cdot \ddot{s} + D \cdot \dot{s} + C \cdot s = Fg$$

$$\frac{m}{kg} \cdot \frac{\ddot{s}}{m/s^2} + \frac{D}{\frac{N}{m/s}} \cdot \frac{\dot{s}}{m/s} + \frac{C}{N/m} \cdot \frac{s}{m} = \frac{Fg}{N}$$

$$\frac{m}{\color{blue}{kg}} \cdot \frac{\ddot{s}}{\color{blue}{m/s^2}} + \frac{D}{\color{blue}{\frac{N}{m/s}}} \cdot \frac{\dot{s}}{\color{blue}{m/s}} + \frac{C}{\color{blue}{N/m}} \cdot \frac{s}{\color{blue}{m}} = \frac{Fg}{\color{blue}{N}}$$



Simulink itself can only handle numbers. So you need a process to reduce the physical quantities comprising units and numbers to just numbers. An easy way to do this is:

Start with the equation and divide each quantity by its coherent si-unit. A coherent si-unit can be written as a combination of si-base units as factors in numerator or denominator. The force unit 1N can be written as kg/m/s/s and is thus a coherent si-unit. A bar cannot be written as N/m/m but has to be written as $10^5$ N/m/m and is thus not a coherent si-unit.

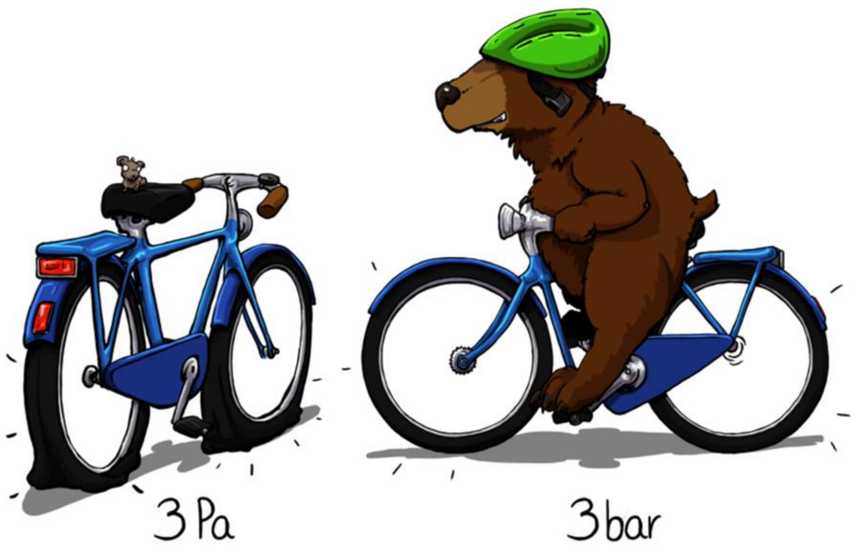
As all the summands had the same unit in the first place, all summands are now divided by the same unit. Which means we did an

equivalent transformation. It's a little tricky now that you have two sets of symbols, one set for the physical quantities and another set for the units. The letter 'm', for example, can now mean a mass or the unit meter, 's' may mean a length or the unit second. To distinguish the two sets, let us keep the quantities black and mark the units blue.

Instead of the quantity mass we now use the dimensionless number m/kg, instead of acceleration a we use a/m/s$^2$ and so on. To make things very clear, we should use e.g. a variable name m_in_kg for the term $\frac{m}{kg}$. But for simplicity people just stick with the variable name m. And that’s what we’ve been using all through this Mooc so far, when we said m=0.2, D=1 and so on.

And there are engineers and scientists that spend their whole carrier without ever doing something more advanced than this. But on the other hand sometimes, coherent si base units don’t come in handy.

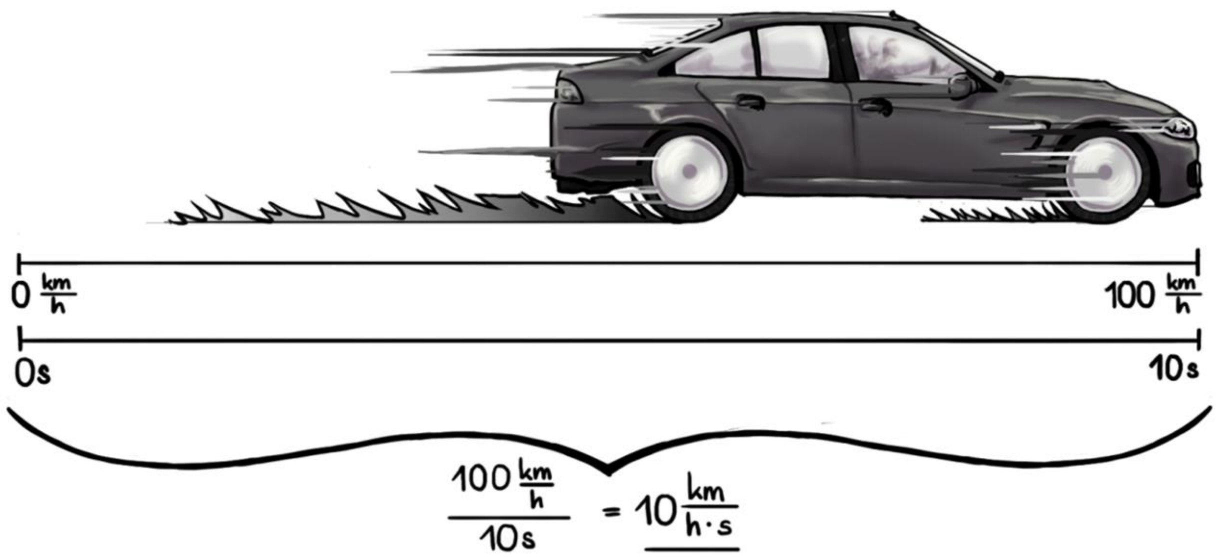
## Coherent SI-Units Unhandy 1






For example there is no hydraulic or pneumatic engineer out there calculating pressures in $N/m^2$.

They either take psi or bar depending on the region they are working in.






The most famous unit for an acceleration is not m/s² but km/h*s because that's what you read in car data sheets all the time: for example it takes 10 s for a certain car to accelerate from 0 to 100 km/h. So the acceleration is 100km/h/10s=10km/h*s.

$$\frac{m}{kg}\cdot\frac{\ddot{s}}{m/s^2}+\frac{D}{\frac{N}{m/s}}\cdot\frac{\dot{s}}{m/s}+\frac{C}{N/m}\cdot\frac{s}{m}=\frac{Fg}{N}$$

$$\frac{m}{kg}\cdot\frac{t}{t}\cdot\frac{\ddot{s}}{m/s^2}\cdot\frac{m/h^2}{m/h^2}+\frac{D}{\frac{N}{m/s}}\cdot\frac{\frac{kN}{mm}\cdot ms}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{m/s}\cdot\frac{mm/s}{mm/s}+\frac{C}{N/m}\cdot\frac{N/m}{N/m}\cdot\frac{s}{m}\cdot\frac{mm}{mm}=\frac{Fg}{N}\cdot\frac{kN}{kN}$$

$$\frac{m}{kg}\cdot\frac{t}{t}\cdot\frac{\ddot{s}}{m/s^2}\cdot\frac{m/h^2}{m/h^2}+\frac{D}{\frac{N}{m/s}}\cdot\frac{\frac{kN}{mm}\cdot ms}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{m/s}\cdot\frac{mm/s}{mm/s}+\frac{C}{N/m}\cdot\frac{N/m}{N/m}\cdot\frac{s}{m}\cdot\frac{mm}{mm}=\frac{Fg}{N}\cdot\frac{kN}{kN}$$



Let us go back to our msd-example and take the mass m in tons, the Damping constant in kN/(mm/ms), the spring constant in N/m and the gravity force in kN. The acceleration should be in m/h$^2$, the speed in mm/s and the position in mm. As we will see, these choices are not very wise, but I just picked a real mess in order to show the recipe.

Now each part of our dimensionless equation is multiplied with a pure number 1 having the desired units both in its numerator and denominator

$$\frac{m}{kg}\cdot\frac{t}{t}\cdot\frac{\ddot{s}}{m/s^2}\cdot\frac{m/h^2}{m/h^2}+\frac{D}{\frac{N}{m/s}}\cdot\frac{\frac{kN}{mm}\cdot ms}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{m/s}\cdot\frac{mm/s}{mm/s}+\frac{C}{N/m}\cdot\frac{N/m}{N/m}\cdot\frac{s}{m}\cdot\frac{mm}{mm}=\frac{Fg}{N}\cdot\frac{kN}{kN}$$

$$\frac{m}{t}\cdot\frac{t}{kg}\cdot\frac{\ddot{s}}{m/h^2}\cdot\frac{m/h^2}{m/s^2}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{\frac{kN}{mm}\cdot ms}{\frac{N}{m/s}}\cdot\frac{\dot{s}}{mm/s}\cdot\frac{mm/s}{m/s}+\frac{C}{N/m}\cdot\frac{N/m}{N/m}\cdot\frac{s}{mm}\cdot\frac{mm}{m}=\frac{Fg}{kN}\cdot\frac{kN}{N}$$



Now the denominators are exchanged.

There are fractions including physical quantities (framed red)

and fractions not including physical quantities (framed green).

None of the fractions have net units. The fractions without physical quantities are reduced to a simple number.

$$\frac{m}{t}\cdot\frac{1000kg}{kg}\cdot\frac{\ddot{s}}{m/h^2}\cdot\frac{m/(60\cdot 60s)^2}{m/s^2}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{\frac{1000N}{0.001m}\cdot 0.001s}{\frac{N}{m/s}}\cdot\frac{\dot{s}}{mm/s}\cdot\frac{0.001m/s}{m/s}+\frac{C}{N/m}\cdot\frac{N/m}{N/m}\cdot\frac{s}{mm}\cdot\frac{0.001m}{m}=\frac{Fg}{kN}\cdot\frac{1000N}{N}$$

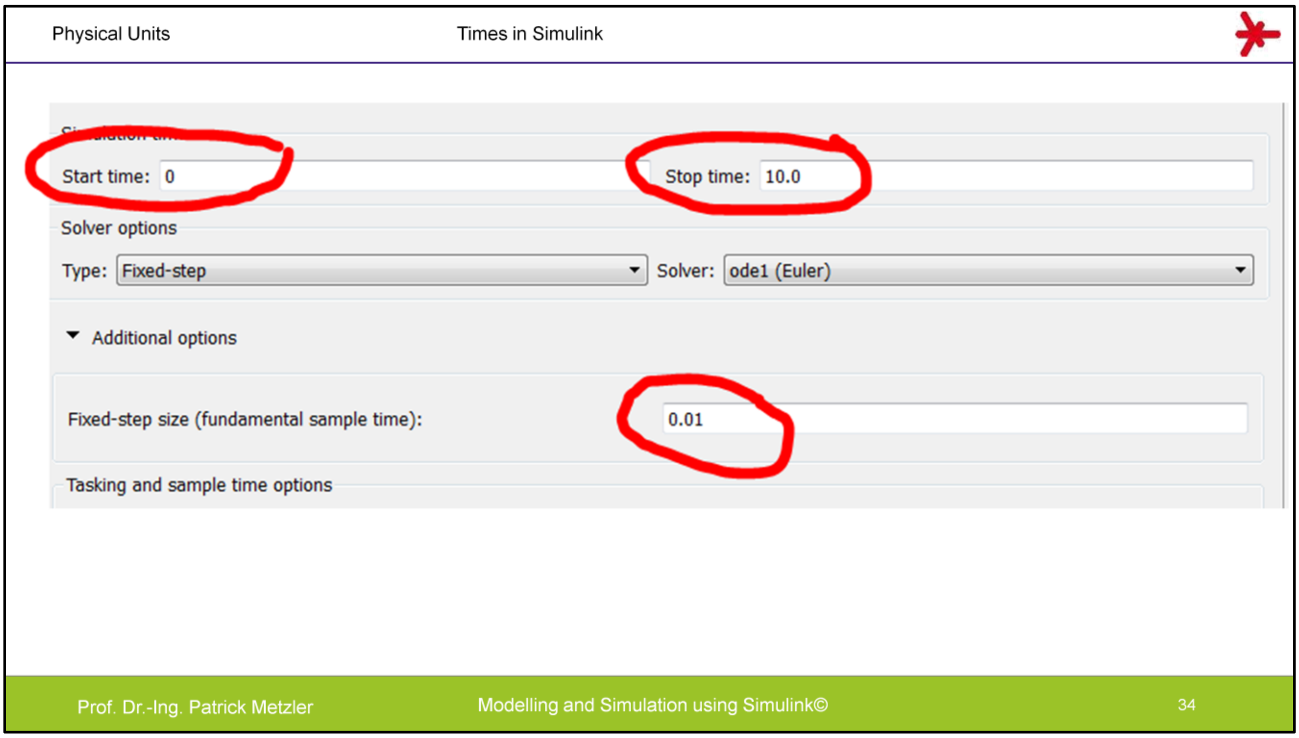
$$\frac{m}{t}\cdot\frac{1000}{1}\cdot\frac{\ddot{s}}{m/h^2}\cdot\frac{1/(60\cdot 60)^2}{1}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{1000}{1}\cdot\frac{\dot{s}}{mm/s}\cdot\frac{0.001}{1}+\frac{C}{N/m}\cdot\frac{s}{mm}\cdot\frac{0.001}{1}=\frac{Fg}{kN}\cdot\frac{1000}{1}$$

$$\frac{m}{t}\cdot 1000\cdot\frac{\ddot{s}}{\frac{m}{h^2}}\cdot\frac{1}{3600\cdot 3600}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot 1000\cdot\frac{\dot{s}}{\frac{mm}{s}}\cdot 0.001+\frac{C}{\frac{N}{m}}\cdot\frac{s}{mm}\cdot 0.001=\frac{Fg}{kN}\cdot 1000$$

$$\frac{m}{t}\cdot\frac{\ddot{s}}{\frac{m}{h^2}}\cdot\frac{1}{36\cdot 360}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{\frac{mm}{s}}+\frac{C}{\frac{N}{m}}\cdot\frac{s}{mm}\cdot 0.001=\frac{Fg}{kN}\cdot 1000$$



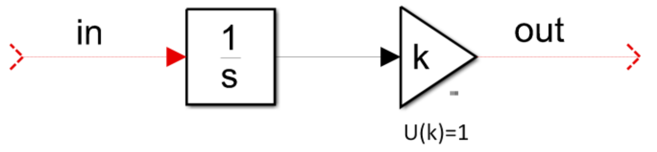
After some calculation shown in a smaller a font, we finally have a result, where each physical quantity is related to the unit we have chosen for this quantity. We call such an equation a normalized dimensionless equation.




Working with Simulink there are three additional quantities: the Start time, the Stop time and the fundamental sample time. In a pure simulation mode, the unit of those free times can be chosen as you like. The only restriction is, that the unit has to be the same for all three times.

Working with hardware and support packages, the unit is automatically set to 1 s.

Physical Units | Integral Block With Units




First integration step:

$$in \cdot dt = out$$

$$N(in) \cdot U(in) \cdot N(dt) \cdot U(dt) = N(out) \cdot U(out)$$

$$\frac{N(in) \cdot U(in) \cdot N(dt) \cdot U(dt)}{U(out)} = N(out)$$

Prof. Dr.-Ing. Patrick Metzler | Modelling and Simulation using Simulink© | 35

When you enter such a normalized dimensionless equation in Simulink, things are a little bit tricky with the integration blocks. You have to be careful about what is the unit of your fundamental simulation step, what is the unit you scaled the incoming signal with and what is the unit you scaled the outgoing signal with. To ease the following calculation, let us assume the initial condition of our integrator is zero and we just want to calculate the first simulation step.

But our argumentation is not limited to this situation. Let the function U(s) return the unit of a physical signal s. The function N(s) returns the according number of a physical signal s.

We will soon see that an integrator block alone in general is not sufficient to deliver the desired result:

We need an additional gain block with amplification k. k is only a number, this means:

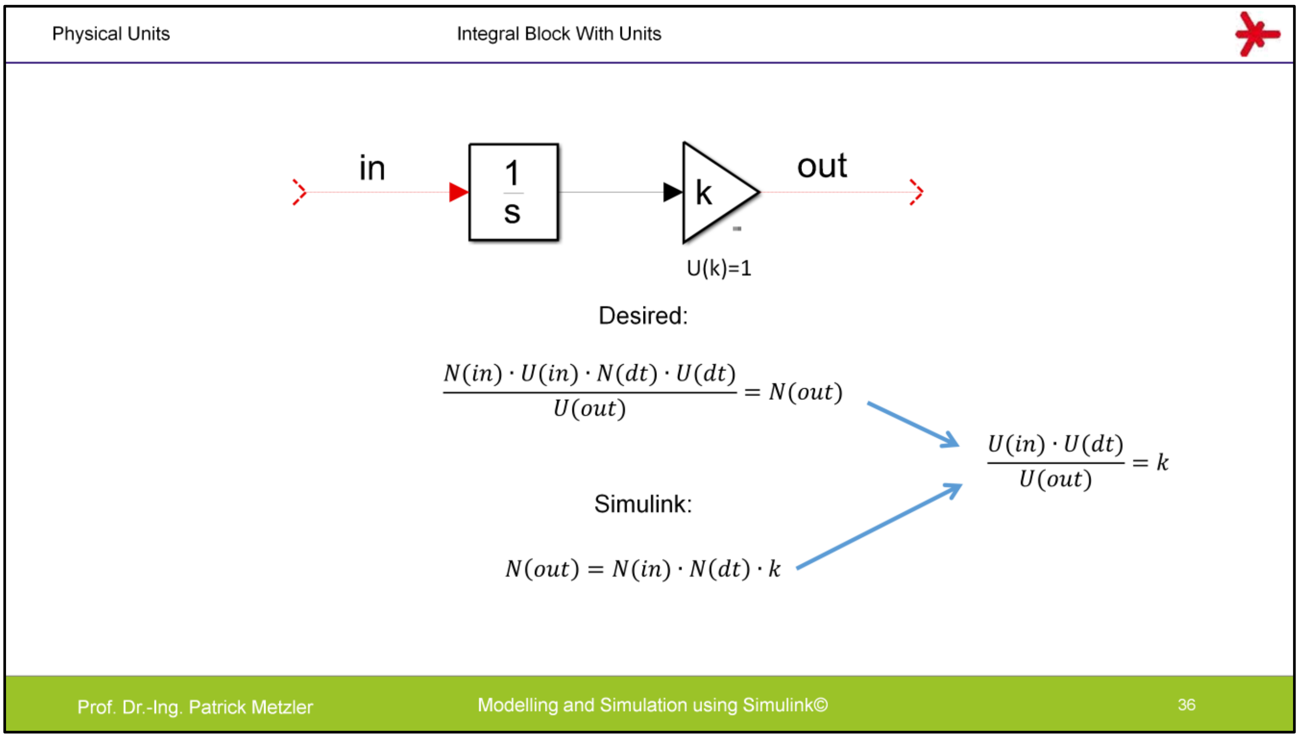
U(k)=1

Writing our desired behavior

using N and U we get:

$$N(s_{in}) \cdot U(s_{in}) \cdot N(dt) \cdot U(dt) = N(s_{out}) \cdot U(s_{out})$$

and solving for N(out):




We already know how Simulink calculates $N(s_{out})N(s_{out}) =_{Simulink} N(s_{in}) \cdot N(dt) \cdot k$

And thus we can calculate the necessary k

$$\frac{U(s_{in}) \cdot U(dt)}{U(s_{out})} = k$$


$$\frac{U(in) \cdot U(dt)}{U(out)} = k$$

$$U(in) \cdot U(dt) = U(out) \quad \Rightarrow \quad k = 1$$

Good practice:

$$\frac{a}{m/h^2} \quad \frac{v}{m/h} \quad \frac{s}{m} \quad dt = 1h \qquad or \qquad \frac{a}{m/s^2} \quad \frac{v}{m/s} \quad \frac{s}{m} \quad dt = 1s$$

Bad practice:

$$\frac{a}{m/h^2} \quad \frac{v}{m/s} \quad \frac{s}{m} \quad dt = 1h \qquad or \qquad \frac{a}{m/s^2} \quad \frac{v}{m/h} \quad \frac{s}{m} \quad dt = 1s$$



You can skip the gain block, if

U(in) $\cdot U(dt) = U(out)$

And this is best practice. Feel free to choose a fundamental time step matching your problem like a year, a month, a week, a day, an hour, a minute, a second, a millisecond, a microsecond…

But then try to pick the remaining units in a way that for every integrator, k has the value 1 which means you can skip the gain block.

$$\frac{m}{t}\cdot\frac{\ddot{s}}{\frac{m}{h^2}}\cdot\frac{1}{36\cdot 360}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{\frac{mm}{s}}+\frac{C}{\frac{N}{m}}\cdot\frac{s}{mm}\cdot 0.001=\frac{Fg}{kN}\cdot 1000$$

| Values in original units | Values in new Units |
|---|---|
| m=0.2 kg | m=0.0002 t |
| D=1 N/m/s | D=0.001 KN/mm/ms |
| C=1 N/m | C=1 N/m |
| g=10 m/s^2 | g=10 m/s^2 |

| Name | Value |
|---|---|
| C | 1 |
| D | 0.001 |
| g | 10 |
| m | 0.0002 |



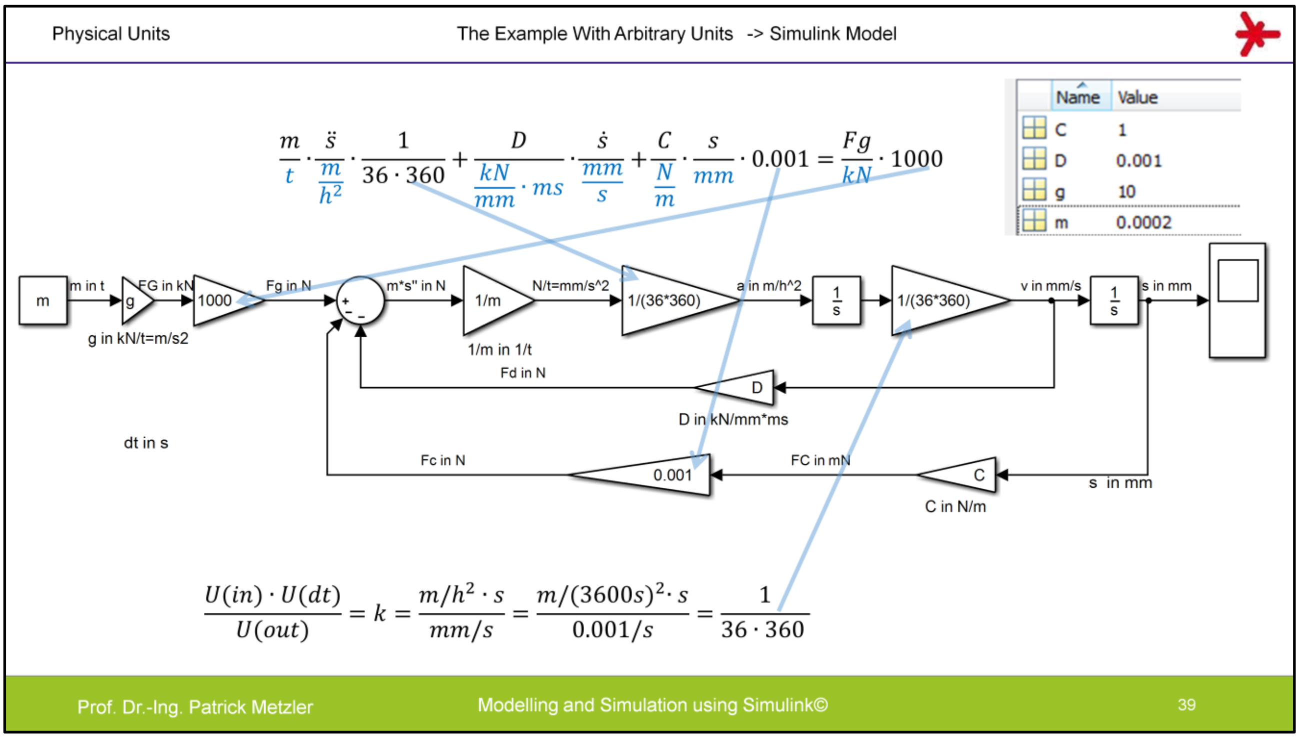
Back to our msd example in the normalized dimensionless form.

Our example is a bad practice one in terms of U(in) * U(dt) not being U(out) for all the integrators. But in this way our example is most general. So let us try to enter this example in simulink.

First we have to enter our parameters in the corresponding units that we picked.

In the left column of this table you see the original values we used so far. In the right column you see the values according to our dimensionless normalized equation.

These values are then entered into the model explorer of our simulink model.




Here is finally the simulink model in our arbitrary units.

Three gain blocks are caused by number-factors in our dimensionless normalized equation.

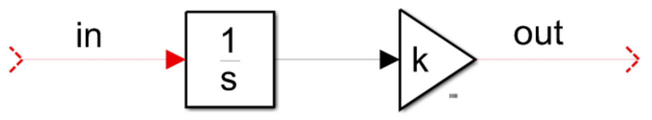
One Factor is caused by the correction factor k for integration.

The second integrator does not need a factor k, as this factor would be 1.

## Summary

$$\frac{m}{kg}\cdot\frac{\ddot{s}}{m/s^2}+\frac{D}{\frac{N}{m/s}}\cdot\frac{\dot{s}}{m/s}+\frac{C}{N/m}\cdot\frac{s}{m}=\frac{Fg}{N}$$

$$\frac{m}{t}\cdot\frac{\ddot{s}}{\frac{m}{h^2}}\cdot\frac{1}{36\cdot 360}+\frac{D}{\frac{kN}{mm}\cdot ms}\cdot\frac{\dot{s}}{\frac{mm}{s}}+\frac{C}{\frac{N}{m}}\cdot\frac{s}{mm}\cdot 0.001=\frac{Fg}{kN}\cdot 1000$$




$$\frac{U(in)\cdot U(dt)}{U(out)}=k \qquad U(in)\cdot U(dt)=U(out) \;\Rightarrow\; k=1$$



Summary:

Life is easy, if you use coherent Si-units.

As sometime this is no good match, you can use any units using dimensionless normalized equations.

Be aware of the integrators needing a correction gain k, if you did not chose $U(in)\cdot U(dt)=U(out)$