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Machine Learning Dualities in Statistical Physics

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Talk based on an ICML25 paper with P. Gupta and N. Iqbal
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Here is a link to the ICML25 paper:



ICML25

There's also a longer but older arxiv preprint (2411.04838), with complementary results I won't talk about today.

Duality is a key idea of modern theoretical physics: the same physical system may have two (or more!) surprisingly different descriptions. Biased set of examples:

- Bosonisation, Sine-Gordon/Thirring [Coleman, Mandelstam, ...]
- Particle/Vortex duality in 2+1 dimensions (3d mirror symmetry)
[Dasgupta/Halperin, Peskin, Song/Tarch/Tong, Intriligator/Seiberg,...]
- ...[Very long list]...
- AdS/CFT (string theory/conformal field theory) [Maldacena, Witten,...]:
- 2d Ising Model Kramers-Wannier [Kramers/Wannier]

These different descriptions, or *frames*, are often related at different regimes: strong coupling/weak coupling, high temperature/low temperature (hence “surprising”).

Motivation – Dualities

In all cases we have a “dictionary” between *observable quantities* in the two frames.

Slogan

Whatever can in principle be computed in one frame, may be computed in the other frame.

But the strong/weak relation implies that computational efficiency may be different in the two frames.

Motivation – Dualities

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Algorithm analogy

Embracing an extended Church-Turing-Deutsch-like thesis, frames can be thought of as algorithms that compute observables (outputs) based on different initial conditions (inputs). Dualities are algorithms taking different inputs, but producing the same outputs.

Can we make this precise in some settings?

Let's try to do so in statistical physics!

Proto-task

Given a distribution of states \mathbf{s}_α , $\alpha \in C$, guess an *energy function* (Hamiltonian) determining Boltzmann weights

$$\{\mathbf{s}_\alpha\}_{\alpha \in C} \mapsto \{\mathbf{s}_\alpha \mapsto e^{-H(\mathbf{s}_\alpha)}\}$$

such that the observed data is “close” to the respective probability distribution

$$p(\mathbf{s}_\alpha) = \frac{e^{-H(\mathbf{s}_\alpha)}}{Z}, \quad Z = \sum_{\beta} e^{-H(\mathbf{s}_\beta)} .$$

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Machines can perform some of these tasks very well, and fast.

Motivation – Statistical physics tasks

Example: Fully Visible Boltzmann Machines (FVBM). Let $\mathbf{s} = \{s_i\}_{i \in L}$, $s_i \in \{\pm 1\}$. Let

$$H(\mathbf{s}) = -\frac{1}{2} \sum_{i \neq j} w_{ij} s_i s_j .$$

The proto-task corresponds to finding w_{ij} such that the negative log-likelihood

$$\mathcal{L}^{FVBM} = \log(Z) + \frac{1}{|C|} \sum_{\alpha \in C} H(\mathbf{s}_\alpha)$$

is minimised. Gradients:

$$\frac{\partial \mathcal{L}^{FVBM}}{\partial w_{ij}} = \langle s_i s_j \rangle_H - \langle s_i s_j \rangle_{data} .$$

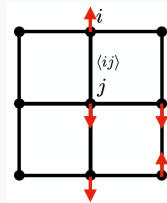
Note:

- It is very expensive to compute $\langle s_i s_j \rangle_H$, because it is very expensive to compute Z .
- Standard algorithms involve a Monte Carlo estimation of $\langle s_i s_j \rangle_H$
- See e.g. Contrastive Divergence [\[Hinton\]](#)

Motivation – Statistical physics tasks

Special case: The *2d Ising model* on a square lattice is an example. Let $i \in L$ label the sites of the lattice, and

$$w_{ij} = \begin{cases} \beta = J/k_B T & i, j \text{ neighbouring, } \langle ij \rangle \\ 0 & \text{otherwise,} \end{cases}$$



that is

$$H(\mathbf{s}) = -\beta \sum_{\langle ij \rangle} s_i s_j .$$

In this case, the problem of finding the parameters is called *Inverse Ising Problem*.

There is a huge literature (e.g. review [Nguyen/Zecchina/Berg]) that we will not use today until the very end (see arxiv paper 2411.04838 [Ferrari et al.]).

We will be concerned with a different, more general problem. Consider a more sophisticated statistical physicist that is *agnostic* about:

- Parametrisation of the underlying configuration of states. Spins on a lattice? Which lattice?
- Precise form of the Hamiltonian/energy
- The actual *observable* that is actually being measured.

Guiding principle

The less is assumed the more a potential description may be surprising/useful! Still, one needs to find well-defined training tasks...

Look at dualities in statistical physics.

Today we will consider one of the most elementary examples of a *duality*, Kramers-Wannier duality in the 2d Ising model [Kramers-Wannier, ...]. Recall $s_i \in \{\pm 1\}$, and

$$H(\mathbf{s}) = -\beta \sum_{\langle ij \rangle} s_i s_j .$$

Note that there is a \mathbb{Z}_2 symmetry

$$s_i \mapsto -s_i ,$$

The energy is lowest (“ground state”) when spins are “aligned”, $s_i = +1$ or $s_i = -1$.

Kramers-Wannier duality is in this case a *self*-duality relating different β regimes.¹

¹Physically, the duality is related to *gauging* the \mathbb{Z}_2 symmetry.

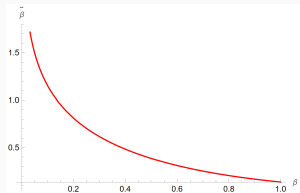
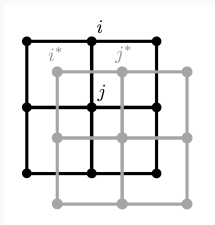
2d Ising and Kramers-Wannier

More in details, the *self*-duality relates models

$$H(\mathbf{s}) = -\beta \sum_{\langle ij \rangle} s_i s_j \leftrightarrow \tilde{H}(\mathbf{s}^*) = -\tilde{\beta} \sum_{\langle i^* j^* \rangle} s_{i^*} s_{j^*}$$

where \tilde{H} is defined on the *dual lattice*² L^\vee and

$$\sinh(2\beta) \sinh(2\tilde{\beta}) = 1 .$$



The partition functions agree up to an overall constant.

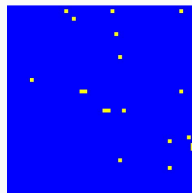
²In general, $L^\vee := \{y \in \mathbb{R}^n \mid \forall x \in L, (y, x) \in \mathbb{Z}\}$.

2d Ising and Kramers-Wannier

Remark: this is a large-temperature/low-temperature duality.

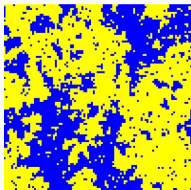


$$\beta \ll 0.44$$



$$\beta \gg 0.44$$

with a self-dual point at $\beta \simeq 0.44$



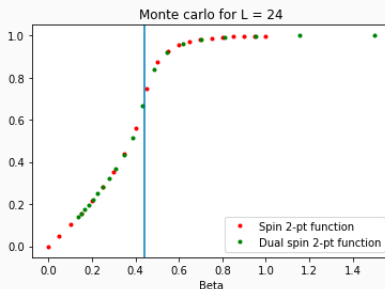
Important: the map between spin configurations is *not* 1:1!

2d Ising and Kramers-Wannier

Correlation functions, however, need to match

$$\langle s_i s_j \dots \rangle_H = \langle e^{-2\tilde{\beta} \tilde{s}_{i^*} \tilde{s}_{j^*} \dots} \rangle_{\tilde{H}}$$

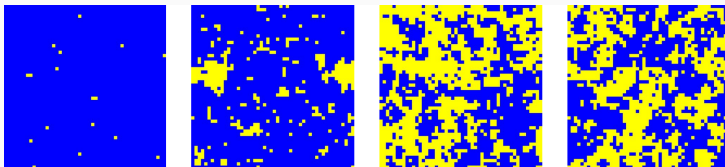
Example: Implement Monte Carlo in *e.g. Mathematica* to compute the correlations on a 24×24 lattice:



Can this matching of data be used to *learn* the duality?

Duality as an optimisation problem – Data generation

Yes we can! Start from the theory at a given β , and generate lattice spin configurations $C = \{L_\alpha\}_\alpha$ using Markov Chain Monte Carlo (Metropolis)



We can then compute

$$\langle \mathcal{O}_{ij}(\mathbf{s}) \mathcal{O}_{kl}(\mathbf{s}) \dots \mathcal{O}_{mn}(\mathbf{s}) \rangle_{data} := \frac{1}{|C|} \sum_{\alpha \in C} \mathcal{O}_{ij}(\mathbf{s}_\alpha) \mathcal{O}_{kl}(\mathbf{s}_\alpha) \dots \mathcal{O}_{mn}(\mathbf{s}_\alpha) .$$

which is an estimate for

$$\langle \mathcal{O}_{ij}(\mathbf{s}) \mathcal{O}_{kl}(\mathbf{s}) \dots \mathcal{O}_{mn}(\mathbf{s}) \rangle_H .$$

Duality as an optimisation problem – Duality learning

Once the data has thus been prepared, we can define the following problem:

Kramers-Wannier Duality Learning Task - Rough sketch

Given *some* lattice $L^?$ with spins $s_{i'} \in \{\pm 1\}$, as well as

- A reasonably general Hamiltonian

$$H^t(\mathbf{s}') = -t_0 \sum_{\langle i'j' \rangle} s_{i'} s_{j'} - t_1 \sum_{(i'j'k'l')} s_{i'} s_{j'} s_{k'} s_{l'} - \dots$$

- A (sufficiently local) observable $\mathcal{O}_{ij}^\theta : \mathbf{s}' \rightarrow \mathbb{R}$ parametrised by θ

find $(t = t^*, \theta = \theta^*)$ such that all distances between all correlation functions

$$|\langle \mathcal{O}_{ij}(\mathbf{s}) \mathcal{O}_{kl}(\mathbf{s}) \dots \mathcal{O}_{mn}(\mathbf{s}) \rangle_H - \langle \mathcal{O}_{ij}^\theta(\mathbf{s}') \mathcal{O}_{kl}^\theta(\mathbf{s}') \dots \mathcal{O}_{mn}^\theta(\mathbf{s}') \rangle_{H^t}|^2$$

are minimised.

Note: The above strategy may be thought of as a decoding/encoding process.

See also [\[Betzler/Krippendorf\]](#)

Duality as an optimisation problem – Expected solutions

Provided the problem is set up correctly, one should recover two solutions:

The original model:

- $L^? = L$
- $H^{t*} = H(\beta)$
- $\mathcal{O}_{ij}^{\theta*} = \mathcal{O}_{ij} = s_i s_j$.

The dual model:

- $L^? = L^\vee$
- $H^{t*} = H(\tilde{\beta})$
- $\mathcal{O}_{ij}^{\theta*} = e^{-\tilde{\beta} s_{i*} s_{j*}}$.

Duality as an optimisation problem – Expected solutions

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- $H^{t*} = H(\tilde{\beta})$
- $\mathcal{O}_{ij}^{\theta*} = e^{-\tilde{\beta} s_{i*} s_{j*}}$.

To achieve this result, one needs to:

- identify a suitable parametrisation of the Hamiltonian H^t (as before) and observable \mathcal{O}_{ij}^θ
- define an appropriate *loss function*
- develop an *optimisation scheme*.

Parametrisation of the observables

We assume that the lattice $L^?$ is also square and so $\mathbf{s}^* \cong \mathbf{s}$.

In principle, one could parametrise \mathcal{O}_{ij}^θ as any NN with a large input (the spins) and one output (the value of the observable).

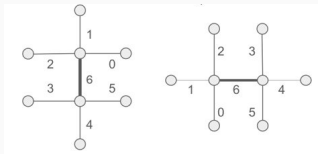
Observable Intuition

Focus attention on some neighbouring links.

Reasonable example:

$$\mathcal{O}^\theta(\mathbf{s}) = \theta_2 \text{Gumbel-Softmax}(\mathbf{s})(\theta_1)^T \delta^{ij}(\mathbf{s}) + \theta_3$$

where $\delta^{ij}(\mathbf{s})$ is a vector with the values of the *neighbouring links* of $\langle ij \rangle$ (=6 below).



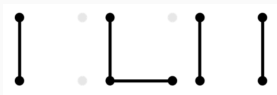
Recall desired matches (with $\mathbf{s} = \mathbf{s}^*$):

$$|\langle \mathcal{O}_{ij}(\mathbf{s}) \mathcal{O}_{kl}(\mathbf{s}) \dots \mathcal{O}_{mn}(\mathbf{s}) \rangle_H - \langle \mathcal{O}_{ij}^\theta(\mathbf{s}) \mathcal{O}_{kl}^\theta(\mathbf{s}) \dots \mathcal{O}_{mn}^\theta(\mathbf{s}) \rangle_{H^t}|^2$$

Standard approaches involve *kernel methods* [\[Li/Zwersky/Zemel\]](#)³, however:

- Care is needed because $s_i s_i = 1$. The issue can be solved by modifying kernels.
- Correlation functions involving far-away spins carry little information, noise-to-signal problem

Solution: just focus on a handful of these correlators! For instance:



³Cf. *Maximum Mean Discrepancy*.

Loss function

More formally, let

- $f_{ij} : \mathbf{s} \rightarrow \mathbb{R}$ a function labelled by a link $\langle ij \rangle$ (such as the observables);
- A be a set of subsets of links.

and for $a \in A$ define the “product of the function values over links”

$$\psi^a(f) = \prod_{\langle ij \rangle \in a} f_{ij} .$$

Loss function

We define the loss \mathcal{L} as the Mean-Squared-Error

$$\mathcal{L} = \sum_{a \in A} l_a^2 .$$

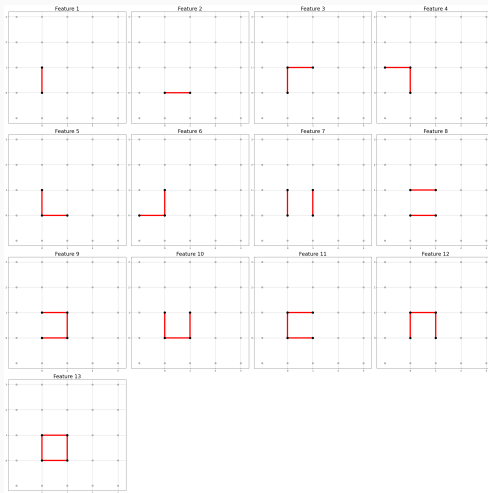
with

$$l_a = \langle \psi^a(\mathcal{O}_{ij}) \rangle_H - \left\langle \psi^a \left(\mathcal{O}_{ij}^\theta \right) \right\rangle_{H^t}$$

for a suitable subset A (see next slide).

Loss function

Here are the selected features (subset A) in a little more details:



To compute the loss, we need to compute gradients with respect to the parameters t and θ entering the pair $(H^t, \mathcal{O}_{ij}^\theta)$. This can be done as follows:

- $\partial_\theta \mathcal{L}$ is easy to compute: simply use backpropagation `loss.backward()`
- $\partial_t \mathcal{L}$ is computationally hard, as t directly affects the probability measure of the sampling.
- As far as I can tell, classic methods such as CD cannot be used in this case.

However, a simple computation shows that the $\partial_t \mathcal{L}$ gradients can be expressed in terms of expectation values:⁴

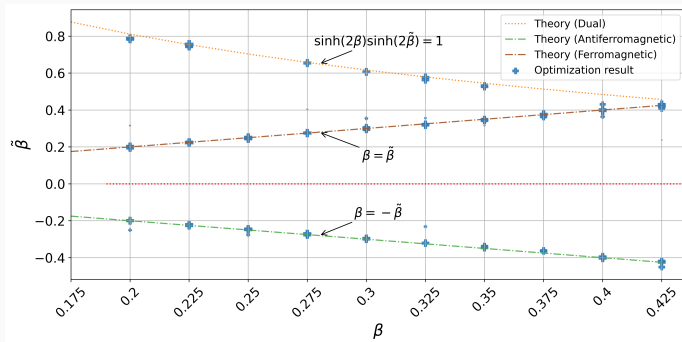
$$\partial_t \mathcal{L} = 2 \left\langle \sum_{a \in A} I_a \left(\sum_{\langle ij \rangle} \langle s_i s_j \rangle_{H^t} - \sum_{\langle ij \rangle} s_i s_j \right) \psi^a \right\rangle_{H^t}$$

Thus, they can be estimated using MCMC! Slow but feasible...

⁴Similar to REINFORCE

First results – KW rediscovery

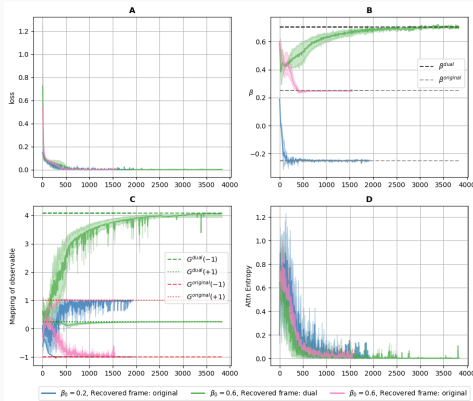
Run the optimisation problem on a 8×8 lattice at different temperatures, first with H^t truncated to the Ising Hamiltonian (that is $t = (t_0)$, in the graph $\tilde{\beta}$).



Rejoice, the duality is rediscovered! As well as the antiferromagnetic image of the original model $\beta \mapsto -\beta$.

First results – KW rediscovery

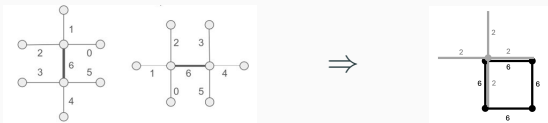
Let's have a look at the training dynamics as well as the discovered observables



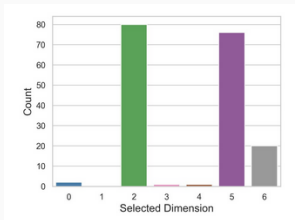
All looks pretty good. Low value of attention entropy indicates one link is mostly picked. Expressions $\mathcal{O}_{ij}^{\theta^*} = s_i s_j$, $\mathcal{O}_{ij}^{\theta^*} = e^{-2\tilde{\beta} s_i^* s_j^*}$ are found!

First results – KW rediscovery

It is interesting to see whether it is *really* the original links $\langle ij \rangle$ or the dual links $\langle j^* i^* \rangle$ that are discovered. Recall:



That is, link 6 recovers the original lattice, 2 or 5 recover the dual. Let us look at the links selected by the optimisation problem:



Some details and problems:

- Computing gradients takes time, as MCMC chains need to be equilibrated
- The MCMC-based gradients are also noisy. It is advisable to use variance reduction techniques such as *control variates*.
- The training rarely converges when the original model is in the ordered phase. This is similar to observations in the Inverse Ising problem –deep?
- 8×8 is very small, still it takes hours to do these runs.
- Adding more couplings complicates things...

In any case, given the success with KW in the standard 2d Ising model, one can consider generalisations. Obvious candidates:

- Next-to-nearest neighbour 2d Ising
- Plaquette (see below).

The former (studied in e.g. [Zandlivet, Lee/Song/Kim/Kim]) has interesting dualities to brick-wall or honeycomb lattices (see e.g. [Strycharski/Koza]). The latter has not been studied in great details, at least as far as I know.

Plaquette generalisation task

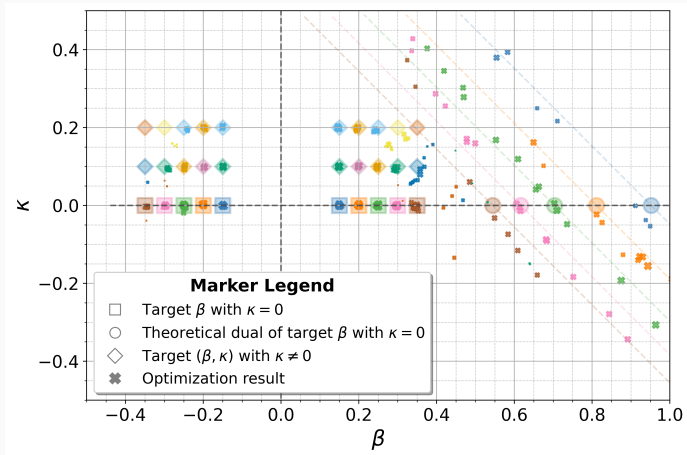
What does our set-up outputs for the plaquette model

$$H(\mathbf{s}) = -\beta \sum_{\langle ij \rangle} s_i s_j - \kappa \sum_{(ijkl)} s_i s_j s_k s_l ,$$

where $(ijkl)$ label *plaquettes*?

Generalisation – Plaquette results

Some results for the plaquette model, square lattice:



Some observations on the emergence of anti-diagonal “clusters”:

- When models are deep in the ordered phase (almost all spins aligned), an excellent approximation for the model is given in terms of ‘rare, isolated spin flips
- These happen with probability $p = p(\tilde{\beta} + \tilde{\kappa})$
- The ML model learned that the model becomes effectively a one-parameter model in this regime
- The loss is very low, 0 up to noise!

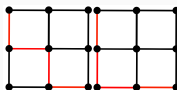
Approximate duals.

Even on a square lattice, one can find “approximate duals” deep in the ordered phase.

Symmetry (re)-discovery and inverse Ising

A slightly less clueless physicist may notice that $s_i s_j$ is a *chunk* of a topological line, for purely kinematical reasons. For a subset of links a_{mn} connecting sites m, n :

$$\psi^{a_{mn}}(s_i s_j) = \prod_{\langle ij \rangle \in a_{mn}} s_i s_j = s_m s_n .$$



Similarly, the dual observable *is* a chunk of a topological line –the one that implements the (dual) \mathbb{Z}_2 -symmetry:

$$(s_{i^*} s_{j^*} \mapsto -s_{i^*} s_{j^*}) \Rightarrow \left(e^{-H(\tilde{\beta})} \mapsto e^{-H(\tilde{\beta})} e^{-2\tilde{\beta} s_{i^*} s_{j^*}} \right)$$

Automated symmetry discovery [Ferrari et al., in progress]

Minimise a suitable loss functions to find topological lines, schematically:

$$\mathcal{L} \sim \sum_{m,n} \sum_{a,b \in \text{Paths}(m,n)} \left(\langle \psi^a(\mathcal{O}_{ij}^\theta) - \psi^b(\mathcal{O}_{ij}^\theta) \rangle_H \right)^2 .$$

Once the functional form of topological lines is known, one can try to match those across models. Learning couplings t is then an *inverse Ising* problem.

Line-based Kramers-Wannier Duality Learning - Rough sketch

Given a lattice L^* with spins $s_{i'} \in \{\pm 1\}$, as well as

- A reasonably general Hamiltonian

$$H^t(\mathbf{s}') = -t_0 \sum_{\langle i'j' \rangle} s_{i'} s_{j'} - t_1 \sum_{(i'j'k'l')} s_{i'} s_{j'} s_{k'} s_{l'} - \dots$$

- A chunk of a topological line $\mathcal{O}_{ij}^* : \mathbf{s}' \rightarrow \mathbb{R}$

find $(t = t^*)$ such that all distances between all correlation functions

$$|\langle \mathcal{O}_{ij}(\mathbf{s}) \mathcal{O}_{kl}(\mathbf{s}) \dots \mathcal{O}_{mn}(\mathbf{s}) \rangle_H - \langle \mathcal{O}_{ij}^*(\mathbf{s}') \mathcal{O}_{kl}^*(\mathbf{s}') \dots \mathcal{O}_{mn}^*(\mathbf{s}') \rangle_{H^t}|^2$$

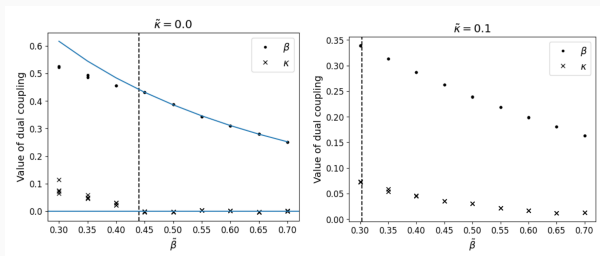
are minimised.

Symmetry discovery and inverse Ising

The problem can in general be approached exploiting consistency conditions on correlators [Albert/Swensen]⁵. In 2411.04838 [Ferrari et al.] we tested this in the Next-to-Nearest Neighbour Ising model

$$H(\mathbf{s}) = -\beta \sum_{\langle ij \rangle} s_i s_j - \kappa \sum_{[ij]} s_i s_j .$$

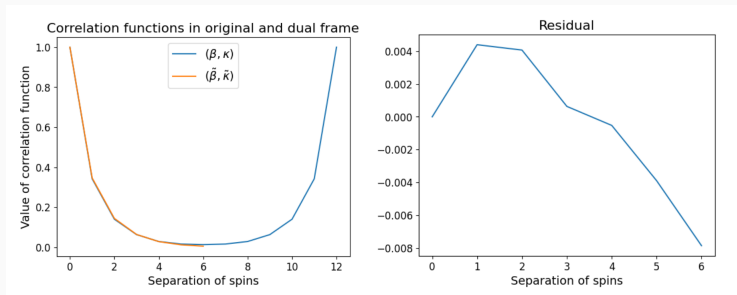
Strictly speaking this is no longer defined on the square lattice, but we view it for small κ as a deformation thereof:



⁵Schematically, the idea is to exploit relations $\langle s_j s_{j+\delta} \rangle = \left\langle \tanh \left(\sum_{\delta'} t_{j,j+\delta'} s_{j+\delta'} \right) s_{j+\delta} \right\rangle$.

Symmetry discovery and inverse Ising

Some preliminary matching of correlators on a 12×12 lattice,
 $(\beta, \kappa) = (0.261, 0.0194)$, , $(\tilde{\beta}, \tilde{\kappa}) = (0.45, 0.05)$:



As spin separation approaches half the lattice size, correlators start diverging.

To conclude:

- The standard problem of fitting parameters in a Hamiltonian of a statistical model/energy of an energy-based model has interesting generalisations
- It is possible to define duality discovery in statistical physics as an instance of these more general problems
- Solutions so far lead to the rediscovery of known dualities as well as hints towards interesting physics

Several future directions:

- Study the ML problem better! (New ways to optimise etc.)
- Learn topological defects!
- Generalise duality discovery! New dualities?
- More modestly, apply our framework to more statistical physics models!

The End