At what altitude does the horizon cease to be visible?

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At altitudes greater than a few kilometers the horizon is indistinct because of contrast reduction, even on extraordinarily clear days. An airplane passenger flying over an ocean cannot point to the apparent boundary between earth and sky and confidently proclaim its distance. To determine the distance to the horizon to within 10% requires knowing its angular position to within a few minutes of arc. This is unattainable in a realistic atmosphere, even one that is very clean. Scattering by the air molecules themselves is not quite sufficient to blur the horizon, but it does not take many particles before the horizon becomes indistinct.

I. INTRODUCTION

In a recent article in this Journal[1] French calculated the distance to what he called the "visible horizon," taking atmospheric refraction into account. According to various dictionaries and glossaries we have consulted, the visible horizon is where the earth and sky appear to meet. It is "the circle which bounds that part of the earth's surface visible from a given point."[2] For a circle on the earth's surface to qualify as the visible horizon therefore requires that it be visible, a seeming tautology until one realizes that (incoherent) scattering in the atmosphere places an upper limit on the distance over which objects can be seen because of contrast reduction. French has shown that the distance to the horizon is increased (by not more than about 10%) when atmospheric refraction is taken into account. But, if the atmosphere scatters coherently (i.e., refracts), it also necessarily scatters incoherently, which reduces contrast. In the following we shall show that above about 2-km altitude the horizon defined by the paths of tangential light rays cannot be seen because of contrast reduction even when the sea-level horizontal visual range is over 200 km.

II. ON A CLEAR DAY YOU CANNOT SEE FOREVER

When you look at any object, especially a distant one, you unavoidably receive not only light from that object, but light scattered by all the molecules and particles along your line of sight as well. This *airlight* reduces contrast between an object and the horizon sky. At sufficiently large distances, the brightness of airlight is indistinguishable from that of the horizon sky. Thus there is not enough contrast to see even dark objects at such distances. A good example of this is provided by a series of parallel ridges, one behind the other, covered with the same dark green vegetation. Ridges nearby, several kilometers or less, will show their natural color, but those somewhat farther will have a milky bluish cast, even on a fairly clear day. Each successive ridge will be less distinct than its predecessor. Indeed, this is how we judge distances to faraway objects of unknown size. The farthest ridges fade into the horizon sky. Thus there is a range beyond which we cannot see the ridges because of insufficient contrast, even though our line of sight may not be restricted by the earth's curvature.

Scattering by particles suspended in the atmosphere (aerosols) contributes to airlight, thereby reducing contrast, but such particles are not the only contributors: Air molecules themselves scatter light. What determines the brightness of airlight is *optical* distances, not physical distances, to which both particles and molecules contribute.

The *optical thickness* τ (see, e.g., Ref. 3) along a path connecting two points in an incoherently scattering medium is defined as

$$\tau = \int_1^2 \beta \, ds\,, \tag{1}$$

where the scattering coefficient β is the number density of scatterers times the scattering cross section per scatterer. At visible wavelengths absorption can be neglected, particularly in fairly clean atmospheres. Optical thicknesses are dimensionless; they are distances measured in units of scattering mean-free paths. Since scattering is wavelength dependent, in general, so are optical thicknesses.

Consider a line of sight of length d between an observer and a dark object (Fig. 1). Sunlight is scattered by the molecules and particles lying along this line of sight. At any distance x the amount of light scattered toward the observer is proportional to $L_0 \beta$, where L_0 is the radiance (power per unit area per steradian) of the incident sunlight and β is the scattering coefficient. The probability that a photon will travel a distance x without being scattered is $\exp(-\beta x)$. Thus the radiance L of the airlight at the ob-

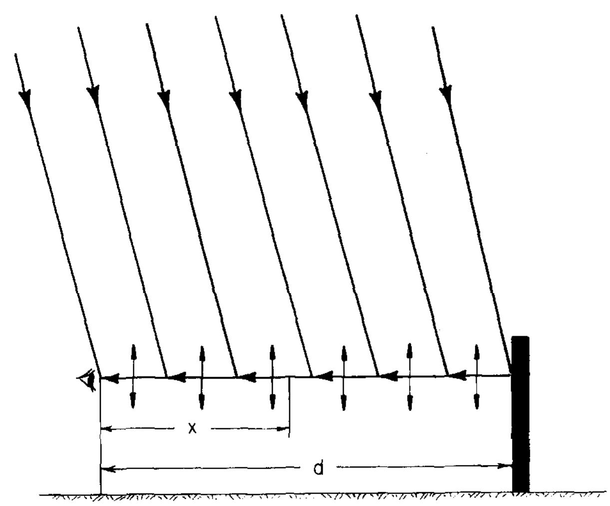

Fig. 1. An observer at a distance d from a dark object sees light scattered by everything—molecules and particles—lying along the line of sight.

server is given by the integral

$$L = GL_0 \int_0^d \beta \exp(-\beta x)\, dx = GL_0(1 - e^{-\tau}), \quad (2)$$

where $\tau = \beta d$ is the optical thickness of the line of sight. G accounts for geometrical reduction of radiance by scattering: Incident sunlight is confined to a small range of direction, whereas there is a finite probability that it will be scattered in any direction. We will not need G, but introducing it helps to explain why the horizon sky is so much less bright than the sun. For isotropic scatterers (molecules are almost isotropic scatterers) G is about 10^{-5} (i.e., the solid angle subtended by the sun times $1/4\pi$).

Although we implicitly assumed that β is uniform, Eq. (2) holds even if β varies, in which instance τ is given by Eq. (1).

The radiance of the horizon sky is very nearly equal to the limit of L in Eq. (2) as τ becomes indefinitely large: $L_\infty = GL_0$. That is, optical thicknesses of near-horizontal paths through the entire atmosphere are sufficiently large that $\exp(-\tau)$ is small compared with unity. We shall justify this assertion later.

Radiance is a radiometric quantity, that is, it describes the power received without taking into account the portion of it that actually stimulates the human eye or by what relative amount it does so. Brightness is the corresponding photometric quantity that takes into account the response function of the eye. Because we are interested in what an observer sees we must convert radiances to brightnesses (or luminances). The brightness B is related to the radiance L by an integral over all wavelengths λ:

$$B = \int K(\lambda) L(\lambda) d\lambda, \quad (3)$$

where the luminous efficiency of the human eye K peaks at about 555 nm and vanishes outside the spectral region 385–760 nm.

The contrast C between an object and the horizon sky is

$$C = (B - B_\infty)/B_\infty, \quad (4)$$

where B is the brightness of the object and B_∞ is that of the horizon sky. If an object at a distance d from an observer is black (which gives the greatest contrast), its apparent brightness is solely the result of airlight, in which instance its contrast follows from Eqs. (2)–(4):

$$C = -\int KL_0 \exp(-\beta d)\, d\lambda \Big/ \int KL_0\, d\lambda, \quad (5)$$

where we have assumed that the line of sight is uniform in its scattering properties.

To determine the greatest distance at which a dark object can be seen against the horizon sky, we need to know the threshold of brightness contrast ϵ: the smallest contrast between an object and its surroundings such that it can be detected. As evidenced by many observations (see Ref. 4, pp. 86 *et seq.*), ϵ does not have an invariable value: It depends on the observer, the angular size of the object, the presence of nearby objects, and the magnitude of the brightness. For purposes of analysis here we have chosen ϵ to be 0.02, partly because it defines what is called the meteorological range (Ref. 4, p. 103) and partly because this value appears to err on the side of overestimating the greatest distance at which objects can be detected. The precise value of ϵ does not affect our conclusions; they would be much the same if we were to choose $\epsilon = 0.01$ or even 0.005 because contrast is an exponential function of distance [see Eq. (5)].

For an atmosphere completely free of particles, β is the molecular scattering coefficient β_0. Values of β_0 at standard temperature and pressure have been tabulated by Penndorf[5] for a range of wavelengths. At visible wavelengths the sea-level molecular scattering coefficient is given to good approximation by

$$\beta_0 = (1.16 \times 10^9/\lambda^4)\ \mathrm{km}^{-1}, \quad (6)$$

where the wavelength λ is in nanometers.

Using Eq. (6) and values of KL_0 for noon sunlight tabulated in Ref. 6, we evaluated Eq. (5) numerically to determine the distance d for which $C = -0.02$. On a clear day you *cannot* see forever, even on a flat earth: 330 km is the greatest distance at which an object can be seen against the horizon sky on the clearest possible day at sea level, assuming a contrast threshold of 2%.

If β does not vary with wavelength, Eq. (5) reduces to

$$-\ln|C| = \beta d = \tau. \quad (7)$$

For $d = 330$ km it follows from Eqs. (6) and (7) that $\lambda = 560$ nm. This effective wavelength is shifted slightly longward of the peak of the eye's sensitivity because the molecular scattering coefficient decreases with increasing wavelength. From now on we shall do all calculations of visual ranges by determining optical thicknesses at 560 nm and by using Koschmieder's relation (Ref. 4, pp. 104 *et seq.*):

$$\tau = -\ln(0.02) = 3.9. \quad (8)$$

That is, we shall assume that dark objects cannot be seen against the horizon sky for optical thicknesses at 560 nm greater than 3.9.

One sometimes encounters the assertion that if the density gradient of the atmosphere or the radius of the earth were different, light rays could travel around the world. An especially fanciful discussion of this is given in one of Daedalus's[7] "plausible schemes":

> If the Earth had a radius of only 13 km smaller, a ray at the surface would follow its curvature exactly, and it would appear flat...people would not have realized that the Earth was round until they discovered that, with a good telescope, you could see the back of your head.

But, the same molecules that refract (coherently scatter) light must necessarily incoherently scatter it, thereby reducing its brightness. The circumference of the earth is about 40 000 km. At sea level this corresponds to an optical thickness (at 560 nm) in a pure molecular atmosphere of about 450. In traveling around the world the brightness of a ray will therefore be reduced by a factor of e^{450} of its initial value. This huge number ought to lay to rest notions about light traveling around the world or, indeed, merely across oceans and continents.

Up to this point we have considered only horizontal paths at sea level. In Sec. III we determine optical thicknesses along slant paths.

III. OPTICAL THICKNESSES OF SLANT PATHS

The paths of interest here are those that begin at some height h above the earth's surface and end on this surface; that is, they are paths along lines of sight for an observer at an altitude h. We shall take these paths to be straight lines. Ignoring refractive curving errs on the side of overestimat-

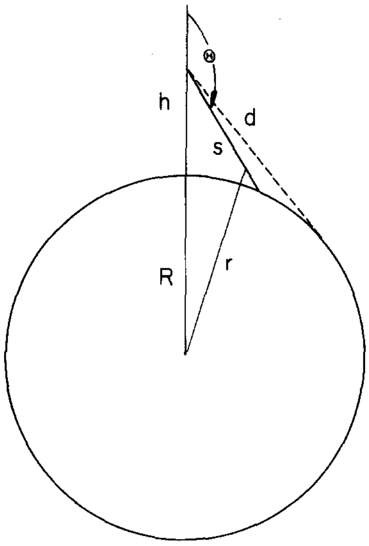

Fig. 2. The path length from an observer at altitude h to a point at a distance r from the center of the earth is s. The total path length to the earth's surface is d. A tangential line of sight (dashed line) defines the horizon (ignoring refraction).

ing the greatest altitude at which the horizon is visible. Moreover, except within about 100 m or less of the earth's surface, the refractive correction is small.

The number density of molecules in the earth's atmosphere, particularly in the troposphere where most of them are concentrated, varies approximately exponentially with height. If r is the distance from the earth's center and R is its radius, the number density N at any height is

$$N = N_0 \exp[-(r-R)/H], \tag{9}$$

where N_0 is the number density at mean sea level and the *scale height* H is about 8.4 km. If Eqs. (1) and (9) are combined we obtain

$$\tau = \beta_0 \int_1^2 \exp\left(\frac{-(r-R)}{H}\right) ds, \tag{10}$$

where β_0 is the molecular scattering coefficient at sea level.

Consider an observer whose line of sight makes an angle Θ with the local vertical. The arc length s along this path and the radial distance r are related by (see Fig. 2)

$$r = [s^2 + 2s(h+R)\cos\Theta + (h+R)^2]^{1/2}. \tag{11}$$

The distance d to the surface is obtained from Eq. (11) by setting $r = R$ and $s = d$.

For the moment we restrict ourselves to lines of sight that are tangent to the earth; that is, $\Theta = \Theta_g$, where

$$\cos\Theta_g = -[(h^2+2Rh)/(h^2+2Rh+R^2)]^{1/2}. \tag{12}$$

For such tangential paths the optical thickness is

$$\tau = \beta_0 \int_0^d \exp\left(\frac{-R\{[1+f(s)]^{1/2}-1\}}{H}\right) ds,$$

$$f(s) = \frac{(s-d)^2}{R^2}, \tag{13}$$

where $d = (h^2+2Rh)^{1/2}$. The values of the function $f(s)$ are small compared with unity for all s, provided that $h \ll R$. To good approximation, therefore, the optical thickness is

$$\tau = \beta_0 (2Rh)^{1/2} \left(\frac{H}{h}\right)^{1/2} \int_0^{(h/H)^{1/2}} \exp(-x^2) dx, \tag{14}$$

where we have transformed the variable of integration and also neglected h^2 in comparison with $2Rh$. The quantity $\beta_0(2Rh)^{1/2}$ is just the optical thickness of a path in an atmosphere with uniform density equal to the sea-level value. The quantity

$$\left(\frac{H}{h}\right)^{1/2} \int_0^{(h/H)^{1/2}} \exp(-x^2) dx \tag{15}$$

is therefore the correction factor that accounts for the variation of molecular density with altitude. Alternatively, we may look upon Eq. (14) as defining an effective distance d_e to the horizon, which is the geometrical distance $(2Rh)^{1/2}$ times Eq. (15).

Note that for $h \ll H$, Eq. (15) is nearly unity. For altitudes much smaller than the scale height the atmosphere may be taken to be uniform for purposes of calculating optical thicknesses. This is a general result; it is not restricted to small altitudes (compared with H) nor even to tangential paths. Consider, for example, a radial path. The optical thickness of such a path is obtained from Eqs. (10) and (11) with $\Theta = \pi$:

$$\tau = \beta_0 H[1 - \exp(-h/H)]. \tag{16}$$

In the limit $h/H \to \infty$ the optical thickness of a radial path through the *entire* atmosphere is

$$\tau_n = \beta_0 H, \tag{17}$$

which is often called the *normal optical thickness*. This is just the optical thickness of a uniform atmosphere confined to altitudes below $h = H$. If we calculate optical thicknesses of radial paths by assuming that the atmosphere has a finite thickness H and is uniform in density equal to the sea-level value, we overestimate τ by *at most* (for $h = H$) about 37%. For both small and large (compared with unity) values of h/H, the error is negligible.

The optical thickness of a tangential path in a finite, uniform atmosphere is

$$\tau = \begin{cases} \beta_0(2Rh)^{1/2}, & h < H, \\ \beta_0(2RH)^{1/2}, & h > H, \end{cases} \tag{18}$$

provided that $h \ll R$. For small values of h/H, the approximate result Eq. (18) is quite close to the exact result Eq. (14). As with radial paths, the optical thickness along a tangential path is overestimated; the maximum error is about 26%.

To determine whether or not the horizon is visible in a pure molecular atmosphere we need to know if the optical thickness along a slant path to the horizon can exceed 3.9 [see Eq. (8)]. It follows from Eqs. (6) and (14) that τ at 560 nm is less than 2.42 (2.78 if we ignore the variation of density with height) at all altitudes. This corresponds to a contrast of -0.09. In a molecular atmosphere, therefore, the boundary between the bright horizon sky and dark ground below it is visible at all altitudes.

In addition to molecular scattering there is also scattering by atmospheric aerosols, which increases optical thicknesses. In very clean environments the total normal optical thickness may be dominated by the molecular contribution (see Fig. 3 in Ref. 8 for curves of aerosol normal optical thickness in a wide variety of locations, from the South Pole to Central Alaska). Measurements of the turbidity ($2.3\,\tau_{\text{aerosol}}$ at 500 nm) over the United States during 1961–1966[9] show minimum aerosol normal optical thicknesses of about 0.06 (the molecular normal optical thickness at 500 nm is about 0.15). The mean annual aerosol optical thickness varies from about 0.1 to 0.5, with the higher values in urban areas.

To properly account for aerosol scattering requires a bit of care. The scale height H_a for the decrease of aerosol concentration with height is considerably less than the molecular scale height H. Typically, H_a is about 1 or 2 km (see, e.g., Ref. 10). The total optical thickness of a line of sight to the horizon is the sum of molecular and aerosol contributions, each given by expressions of the form Eq. (14) with appropriate values of the surface scattering coefficient and the scale height. It therefore follows that to the same degree of approximation as Eqs. (18), the optical thickness including aerosol scattering is given by

$$\tau = \begin{cases} 39(h/H)^{1/2}[\tau_{n,m} + \tau_{n,a}(H/H_a)], & h < H_a, \\ 39[\tau_{n,m}(h/H)^{1/2} + \tau_{n,a}(H/H_a)^{1/2}], & h > H_a, \end{cases} \quad (19)$$

for $R = 6370$ km and $H = 8.4$ km; $\tau_{n,m}$ and $\tau_{n,a}$ are the molecular and aerosol normal optical thicknesses, respectively. Note that because the aerosol is concentrated more toward the surface, scattering by it contributes disproportionately to the total optical thickness.

The molecular normal optical thickness at 560 nm is 0.1. Let us take the aerosol normal optical thickness to be $\frac{1}{2}$ this value, which is somewhat less than the minimum observed value reported in Ref. 9. Such a total normal optical thickness corresponds to a meteorological range (i.e., a surface visual range assuming a contrast threshold of -0.02) of 220 km. According to the International Visibility Code reproduced on p. 43 of Ref. 11, a meteorological range of greater than 50 km is classified as "exceptionally clear." For an aerosol scale height of 2 km, it follows from Eqs. (8) and (19) that the contrast between horizon sky and underlying dark ground is -0.02 at an altitude of less than 1 km. The contrast is -0.005 at less than 2 km.

From the preceding analysis it is difficult to avoid the conclusion that above at most a kilometer or two there is insufficient contrast to see the horizon. Yet passengers in an airplane flying at 10 km above an ocean on a clear day would see a dark region below them and a bright one near the horizon. What is at issue, therefore, is not the existence of dark and bright regions in the passengers' field of view, but rather the distinctness of the boundary between these two regions. This we address in Sec. IV.

IV. BRIGHTNESS VARIATIONS AT DIFFERENT ALTITUDES

We want to answer as accurately as possible the question: At different altitudes and for different aerosol loadings, how does the brightness near the horizon vary vertically with angle?

We define the normalized brightness B_n to be the actual brightness divided by that of the horizon:

$$B_n = 1 - \exp(-\tau), \quad (20)$$

where τ is the optical thickness at 560 nm of an observer's line of sight at altitude h. Path lengths of lines of sight below the horizon are finite; those of lines of sight above are infinite. Although the corresponding optical thicknesses are finite, they tend to be much greater slightly above the horizon, which justifies taking the brightness there to be unity regardless of altitude or aerosol loading.

If we assume that the concentration of scatterers decreases exponentially with height, where H is the scale height, then by following steps similar to those that led to Eq. (14) we obtain the following for the optical thickness as a function of angle θ from the horizon:

Below horizon ($\theta < 0$):

$$\tau = \tau_n \left(\frac{\pi R}{2H}\right)^{1/2} \exp\left(\frac{\delta^2 R}{2H}\right) \exp\left(\frac{-h}{H}\right) \times \operatorname{erf}\left(\frac{\delta}{(2H/R)^{1/2}}\right) - \operatorname{erf}\left[\left(\frac{\delta^2 - 2h/R}{2H/R}\right)^{1/2}\right].$$

Above horizon ($\theta > 0$):

$$\tau = \tau_n \left(\frac{\pi R}{2H}\right)^{1/2} \exp\left(\frac{\delta^2 R}{2H}\right) \exp\left(\frac{-h}{H}\right) \operatorname{erfc}\left(\frac{-\delta}{(2H/R)^{1/2}}\right), \quad (21)$$

where $\delta = -\sin\theta + (2h/R)^{1/2}\cos\theta$ and τ_n is the normal optical thickness; erf and erfc denote the error function and the complementary error function. The limit of Eq. (21) as θ approaches 0 from negative values is Eq. (14). To obtain the total optical thickness we merely calculate the molecular and aerosol contributions separately and add them.

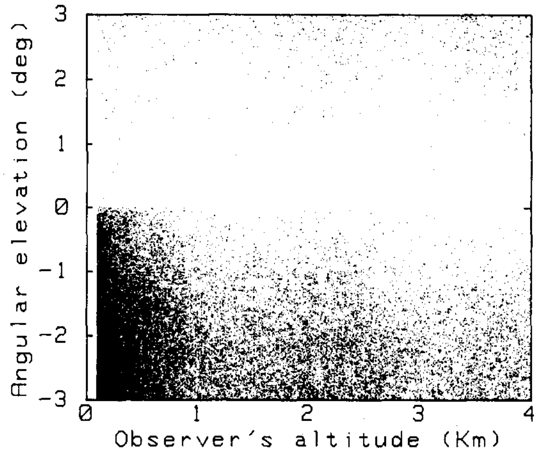

Fig. 3. The density of dots is distributed according to $\exp(-\tau)$, where τ is the total optical thickness of the slant paths above and below the horizon (0°). The molecular normal optical thickness is 0.1 and the aerosol normal optical thickness is 0.05; the aerosol scale height is $\frac{1}{4}$ the molecular scale height.

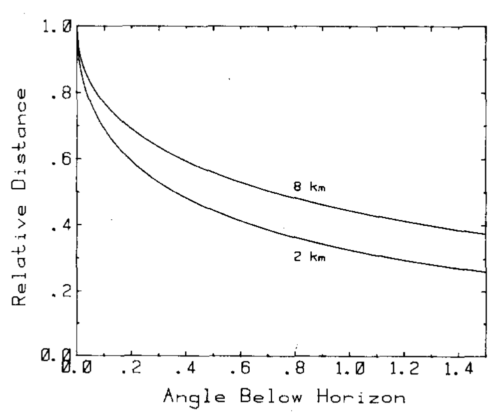

Fig. 4. Distance to a point on the earth relative to the distance to the horizon as a function of angle below the horizon for observers at altitudes of 2 and 8 km.

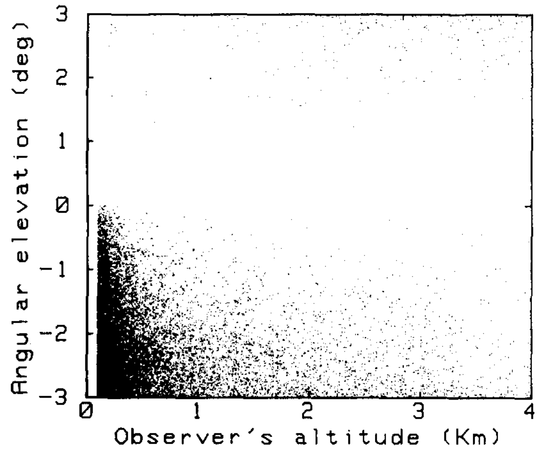

Fig. 5. Same as Fig. 3 except the aerosol normal optical thickness is 0.15.

Figure 3 shows a plot of dots distributed in density according to $\exp(-\tau)$ as a function of angular elevation at different altitudes. After much experimentation we adopted this method as the best way of conveying the horizon's indistinctness. Figure 3 is for a molecular optical thickness of 0.10, an aerosol normal optical thickness of 0.05, and an aerosol scale height $\frac{1}{3}$ the molecular scale height. As we pointed out in Sec. III this corresponds to an extraordinarily clear day, although still within the bounds of reality. Yet even on such a day the horizon is not well defined above about 1 km. To avoid clutter we did not include contours of equal brightness; a contour corresponding to 0.98 is nearly coincident with the horizon at low altitudes, but drops to two-tenths of a degree below it at an altitude of 4 km. Thus the angular ambiguity above about 1 km is a tenth of a degree or more. This may not seem to be much, but its magnitude becomes apparent from Fig. 4, which shows distance to the earth as a function of angle below the horizon. Note that if there is an error in angular position of the horizon of only two-tenths of a degree, the error in the distance to the horizon is almost 50% at an altitude of 2 km. Small angular errors lead to large distance errors because at the horizon the derivative of the distance to the earth with respect to angle below the horizon is infinite.

Figure 5 is the same as Fig. 3 except the aerosol optical thickness has been increased to the more realistic value of 0.15. This does not, however, correspond to a heavily polluted atmosphere: The meteorological range is over 100 km. The indistinctness of the horizon is clearly evident: It lies somewhere in a bright band, but where precisely our eyes cannot tell us.

V. A FEW CAVEATS

To keep our analysis as simple as possible we have made many assumptions, explicitly and implicitly. But wherever there was a choice we tried to err on the side of undermining our assertions about the indistinctness of the horizon. For example, we took the earth to be perfectly dark, which is at least approximated by the oceans. A reflecting ground tends to reduce contrast for two reasons: The ground is brighter, and reflected light is an additional source of light that can be scattered to contribute to airlight. We have also ignored multiple scattering. That is, we have assumed that a photon scattered out of the line of sight (see Fig. 1) will not be scattered back into it. This tends to underestimate airlight and hence overestimate contrast.

We have assumed that the aerosol concentration varies only radially. More important, we have assumed uniform illumination of the line of sight. This is plausible on a clear day, especially if the sun is nearly overhead. But on days when lines of sight are only partially illuminated, contrast can be markedly enhanced. This is frequently observed. Suppose for example, that the line of sight in Fig. 1 is partially obscured by clouds. This means that airlight is decreased and hence contrast is increased. All else being equal (i.e., equal aerosol concentrations) contrast tends to be better on overcast days, particularly if the clouds are stratiform. Of course, it is difficult to assert without measurements that on two consecutive days, one clear and one cloudy, the aerosol loading is the same: Whatever weather system brought the clouds might have brought particles with them. Fortunately, the consequences of nonuniform illumination are frequently observable at the same place and at the same time: Just observe a dark ridge on a day with broken clouds. Sunlight penetrating gaps in the clouds will illuminate the molecules and particles in its path, whereas regions below thick clouds will be dark. This gives rise to alternating dark and bright bands, called crepuscular rays, which appear to diverge from the sun because of perspective. The contrast of parts of the ridge behind bright crepuscular rays will be noticeably less than that of parts behind dark ones. An illustration of this can be seen in a full-page advertisement on the inside cover of the 31 December 1984 issue of *Time* magazine.

Our primary interest has been in what an observer at high altitudes sees. Although clouds may enhance contrast for the sea-level observer, they are more likely to merely obscure the horizon completely for airplane passengers.

We have also ignored absorption. It is conceivable that absorption could enhance contrast by diminishing airlight. Rather than trying to incorporate absorption into our analysis, we merely state that our conclusions are strictly valid only for weakly absorbing aerosols.

VI. CONCLUSIONS

No one, we think, would deny that the horizon cannot be seen on a very hazy or cloudy day. What we have tried to show is that even on the clearest days the horizon is indistinct. It is indistinct to the extent that an airplane passenger flying at 10 km above an ocean on a clear day cannot point to the apparent boundary between earth and sky and confidently proclaim that the distance to that boundary is (ignoring refraction) 357 km. To determine the distance to the horizon to within 10% requires knowing its angular position to within a few minutes of arc. This is unattainable in a realistic atmosphere, even one that is very clean. The molecules of the atmosphere are themselves a kind of pollution in that they scatter light. And so do small, naturally occurring particles that inevitably inhabit an atmosphere overlying a planet studded with deserts, oceans, and forests. The molecules by themselves are not quite sufficient to blur the horizon, but it does not take many particles before the horizon above a few kilometers becomes indistinct.

ACKNOWLEDGMENTS

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visibility of the horizon than we had originally intended. We also thank Christopher Fairall for reviewing succeeding drafts.

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Energy analysis of the conical-spring oscillator

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Traditionally, the spring-hanging mass oscillator has been modeled as a textbook harmonic oscillator with an augmented hanging mass to account for the finite mass of the spring. This traditional model distorts the partition of energy between the spring and the hanging mass while the present analysis reveals the true partition of energy. Changes in effective spring mass and spring constant are tracked across the range of hanging mass in terms of kinetic and potential energy distributions on the coil spring. Results for the case farthest from the textbook, that of the spring along, reveal how, as hanging mass is removed, the spring effectively shortens, stiffens, and uses it own free-end as a hanging mass.

I. INTRODUCTION

The massless spring is a creature of the textbook, not of the laboratory, and communication of this awful truth is the task of the lab instructor. Between textbook and lab the massless spring transforms into some hundred closely spaced turns of brass wire wound in a slight taper. A typical spring has a mass of some 200 g and a stiffness of about 10 N/m. To reduce the spring mass of such a soft spring to 10% of the hanging mass requires the addition of a 2-kg load that produces 2 m of stretch. Inescapably, lab experiments are performed with the spring mass as an appreciable fraction of the hanging mass.

Customarily, the finite spring mass is accounted for by a "correction" to the hanging mass. Through the literature (see MacDonald's fine list of references[1]) runs the notion that a fraction of the spring mass can be added to the hanging mass so as to make the customary expression for a massless spring produce the observed frequencies. Indeed, simply adding or subtracting a fraction of the spring mass can account, numerically, for any change in frequency, however induced. The price paid for this simplicity is ignorance of the physical roots of the frequency change.

Historically, adjustment of only the mass is justified because the low-frequency case of finite, but low, spring mass is nearest to the case of the massless spring and corresponds nearly to static stretch of the spring. However, as hanging mass is removed and frequency rises, the spring form becomes rather different from its static shape. At zero hanging mass, and corresponding high frequency, what started as a mass correction becomes the entire mass and includes a hidden contribution from a spring constant rather different from the static value. By broadening one's perspective to include not just frequency changes, but also the associated energy distributions on the coil, one can form a comprehensive physical picture that assimilates even the case of the spring alone to that of the classical oscillator.

The energy analysis which follows sharply separates inertia effects from spring constant effects. This splitting leads to replacement of the traditional mass fraction f with two dimensionless fractions: one proportional strictly to effective spring mass and one proportional strictly to spring constant. Of course, in the low-frequency, low-spring-mass limit the new mass fraction assumes the traditional value and the spring-constant fraction is unity, which corresponds to static stretching. For springs of arbitrary taper at low frequencies, energy analysis yields the exact expression for the mass correction.

Energy analysis looks beyond just the frequency of oscillation into the elements that compose it: a generalized spring constant and a generalized mass. These elements are broken down in turn into the distributions of energy that define them: potential energy for spring constant and kinetic energy for effective spring mass. For all frequencies, the energy distributions on the coil plus the new dimensionless fractions, taken together, show the physical connection between the real-world conical-spring oscillator and the idealized oscillator of the textbook.