Chapter 1

Hurwitz Numbers

1.1 Maps of Riemann surfaces and Hurwitz numbers

In the following we will assume the Riemann surfaces that we will consider to be compact and connected. Let $f\colon X \to Y$ be a non-constant holomorphic map of Riemann surfaces (hence surjective[1])

Check where conne
is needed.

Definition 1.1.
- The *ramification index* of f at $x \in X$ is the integer $k_x \in \mathbb{Z}_{\geq 1}$ s.t. f is locally of the form z_x^k, where z is a local coordinate centered in x.
- The *differential length* of f at $x \in X$ is $\nu_x = k_x - 1$.
- A point $x \in X$ s.t. $\nu_x > 0$ is called a *ramification point*. The *ramification locus* of f is the set of its ramification points. If $\nu_x = 0$ we say that f is *unramified* at x. If $\nu_x = 1$ we say that f has *simple ramification* at x.
- If $x \in X$ is a ramification point, then $f(x) \in Y$ is called a *branch point*. The *branch locus* B_f of f is the set of its branch points.

The ramification locus and branch locus are finite sets of X and Y respectively.[2] We assume f non-constant. For any $y, y' \in Y \setminus B_f$ we have $|f^{-1}(y)| = |f^{-1}(y')|$.[3] We call *degree* of f the integer $\deg f := |f^{-1}(y)|$ for any $y \in Y \setminus B_f$. Then:

Theorem 1.2 (Riemann-Hurwitz [**CM**])**.** *Let g_X and g_Y denote the genus of X and Y respectively. Then*

$$\underbrace{2g_X - 2}_{\chi(X)} = \big(\underbrace{2g_Y - 2}_{\chi(Y)}\big) \deg f + \sum_{x \in X} \nu_x \tag{1.3}$$

We also have that for any $y \in Y$

$$\deg f = \sum_i k_{x_i} \tag{1.4}$$

where i labels the elements of $f^{-1}(y) = \{x_1, \dots, x_n\}$. Denote $d := \deg f$.

Definition 1.5. Let $y \in Y$, $f^{-1} = \{x_1, \dots, x_n\}$. We call *ramification profile* of f at y the partition of d given by $(k_{x_1}, \dots, k_{x_n})$. We say that f is *unramified* / *simply ramified* / *fully ramified* if its ramification profile is $(1, \dots, 1)$ / $(2, 1, \dots, 1)$ / (d) respectively.

[1] [**CM**]
[2] [**CM**]
[3] [**CM**]

Definition 1.6. Two holomorphic maps of Riemann surfaces $f\colon X \to Y$ and $\tilde{X} \to Y$ are called isomorphic if there is an *isomorphism* of Riemann surfaces $\phi\colon X \to \tilde{X}$ s.t. $f = g \circ \phi$.

An *automorphism* of $f\colon X \to Y$ is an isomorphism $\psi\colon X \to X$ s.t. $f = f \circ \psi$. The group of automorphisms of f is denoted $\mathrm{Aut}(f)$.

Definition 1.7. Let Y be a connected compact Riemann surface of genus g, $B = \{b_1, \ldots, b_n\} \subset Y$ finite subset, $d \in \mathbb{Z}_{\geq 1}$, $\eta_1, \ldots, \eta_n \vdash d$. We define the *(degree d) connected Hurwitz number* to be

$$H^{\circ}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \sum_{[f]} \frac{1}{|\,\mathrm{Aut}(f)|} \tag{1.8}$$

and the *(degree d) Hurwitz number* to be

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \sum_{[f]} \frac{1}{|\,\mathrm{Aut}(f)|} \tag{1.9}$$

where both sums runs over isomorphism classes of holomorphic maps $f\colon X \to Y$ s.t.

- X is a compact Riemann surface of genus h[4]
- the branch locus of f is B
- the ramification profile of f at b_i is η_i

In the case of connected Hurwitz numbers we further require X to be connected. Numbers (1.9) are also called *disconnected Hurwitz number* since X is allowed to be disconnected.

Note that for any $f\colon X \to Y$ as in the definition above we have

$$\sum_{x \in X} \nu_x = \sum_{i=1}^{n} (d - \ell(\eta_i)) = nd - \sum_{i=1}^{n} \ell(\eta_i) \tag{1.10}$$

where $\ell(\eta_i)$ denotes the length of the partition η_i (i.e. the number of cycles in η_i), so due to Riemann-Hurwitz theorem we have

$$2h - 2 = (2g - 2)d + nd - \sum_{i=1}^{n} \ell(\eta_i) \tag{1.11}$$

In particular the ramification profile of $f : X \to Y$ and the genus of Y uniquely determines the genus of X.

1.2 Maps of Riemann surfaces as ramified covers

Maps of Riemann surfaces as above can be regarded as ramified covers:

Definition 1.12. A *ramified cover* is a continuous function between compact topological surfaces $f\colon X \to Y$ s.t. there is a finite set $B \subset Y$ and

- $f^{-1}(B)$ is finite,
- $p\colon X \setminus f^{-1}(B) \to Y \setminus B$ is a covering.

[4]For X disconnected, its genus is defined by $\chi = 2g - 2$, where the Euler characteristic χ is naturally additive under disjoint unions. The Riemann-Hurwitz formula then applies identically in also for X disconnected.

Vice-versa, we have

Theorem 1.13 (Riemann's Existence Thm. [**CM**]). *Let Y compact Riemann surface, X^0 topological surface, $\{b_1, \ldots, b_n\} \subset Y$ finite subset, $f^0 \colon X^0 \to Y \setminus \{b_1, \ldots, b_n\}$ covering of finite degree. Then there exists a unique (up to isomorphisms) compact Riemann surface X s.t.*

- *X^0 is a dense subset of X,*
- *f^0 extends to a holomorphic map of Riemann surfaces $f \colon X \to Y$.*

Definition 1.14. Let $y_0 \in Y \setminus B_f$ and fix a bijection $f^{-1}(y_0) \cong \{1, \ldots, d\}$. The ramified cover $f \colon X \to Y$ determines a group homomorphism

$$\Phi \colon \pi_1(Y \setminus B_f, y_0) \to S_d \ , \qquad \gamma \mapsto \sigma_\gamma \tag{1.15}$$

called *monodromy representation.*

Now consider $b \in B_f$ and $\gamma \in \pi_1(Y \setminus B_f, y_0)$ simple loop winding once around b (and with zero winding number around the other branch points). If the ramification profile of f at b is $\eta = (k_1, \ldots, k_l)$, then σ_γ has cycle type η (to see this recall that the local expression of f around ramification points is $z^k$and consider a circle around y_0 of unit radius in the chart).

Definition 1.16. Let Y be a connected Riemann surface of genus g, $y_0, b_1, \ldots, b_n \in Y$ points, $d \in \mathbb{Z}_{\geq 1}$ and $\eta_1, \ldots, \eta_n \vdash d$. A *monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$ is a group homomorphism $\Phi \colon \pi_1(Y \setminus \{b_1, \ldots, b_n\}, y_0) \to S_d$ s.t. if γ_k is a small loop around b_k then $\Phi(\gamma_k)$ has cycle type η_k.

If moreover the subgroup $\operatorname{im} \Phi \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$ we say that Φ is a *connected monodromy representation.*

We obtained that a degree d map $f \colon X \to Y$ between compact connected Riemann surfaces s.t. the ramification profile over each branch point is η_i gives rise to a connected monodromy representation Φ of type $(g_Y, d, \eta_1, \ldots, \eta_n)$. If we let X to be non connected, then the monodromy representation may not be connected anymore, more precisely: the monodromy representation is connected if and only if X is connected.[5] We also have that isomorphic maps give the same monodromy representation.

Conversely:

Theorem 1.17 ([**CM**]). *Let Y be a Riemann surface of genus g, Φ a monodromy representation of type $(g, d, \eta_1, \ldots, \eta_n)$, $B = \{b_1, \ldots, b_n\} \subset Y$ a finite subset. Then exists a holomorphic map of Riemann surface covering Y with branch locus B whose associated monodromy is F. Such map is unique up to isomorphisms.*

Our discussion leads to a bijection between isomorphisms classes of holomorphic maps with a given ramification profile and monodromy representations. More precisely:

Theorem 1.18 ([**CM**]). *Let M° (resp $M^\bullet$) be the set of connected monodromy representations (resp. monodromy representations) of type $(g, d, \eta_1, \ldots, \eta_n)$. Then*

$$H^\circ_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\circ|}{d!} \tag{1.19}$$

[5]For further details: [**CM**].

and

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \frac{|M^{\bullet}|}{d!} \tag{1.20}$$

where h is determined by (1.11)*.*

Although the information carried by connected Hurwitz numbers is usually more interesting for geometrical purposes, it turns out that it is easier to compute the (possibly disconnected) Hurwitz numbers. We will see later how it is possible to recover the connected Hurwitz numbers from the disconnected ones.

We mention that using some "degeneration formulas" (which heuristically correspond to shrink the Riemann surface Y producing nodal curves) all disconnected degree d Hurwitz numbers are determined in therms of Hurwitz numbers of the form $H^{\bullet}_{h \xrightarrow{d} 0}(\eta_1, \eta_2, \eta_3)$.[6] For this (and other) reason we later restrict our discussion to the case $g = 0$.

1.3 Interlude: representation theory of S_d

Using theorem 1.18 the problem of computing degree d Hurwitz numbers can be translated into a problem in representation theory of the symmetric group S_d. In order to show this we need some facts about representation theory (see **[FH]** and **[CM]** for more details).

Definition 1.21. The *group algebra* of the symmetric group S_d is the complex algebra generated by the elements of S_d, that is

$$\mathbb{C}[S_d] := \left\{ \sum_{\sigma \in S_d} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{C} \right\} \tag{1.22}$$

with operations

$$\sum_{\sigma \in S_d} a_\sigma \sigma + \sum_{\sigma \in S_d} b_\sigma \sigma = \sum_{\sigma \in S_d} (a_\sigma + b_\sigma)\sigma \tag{1.23}$$

$$\left(\sum_{\sigma \in S_d} a_\sigma \sigma \right) \cdot \left(\sum_{\sigma \in S_d} b_\sigma \sigma \right) = \sum_{\sigma \in S_d} \sum_{\sigma' \in S_d} a_\sigma b_{\sigma'} (\sigma \cdot \sigma') \tag{1.24}$$

$$t \cdot \left(\sum_{\sigma \in S_d} a_\sigma \sigma \right) = \sum_{\sigma \in S_d} (t a_\sigma)\sigma \tag{1.25}$$

where $t \in \mathbb{C}$. Expression in the r.h.s. of (1.24) (before multiplying σ and σ') is called formal expansion of the product.

We define *class algebra* of S_d the center of the group algebra

$$\mathcal{Z}\mathbb{C}[S_d] = \{ x \in \mathbb{C}[S_d] \mid yx = xy \quad \text{for all} \quad y \in \mathbb{C}[S_d] \} \tag{1.26}$$

The following functions are very important for our discussion:

[6] **[CM]**

Definition 1.27. A *class function* on S_d is a map $\alpha\colon S_d \to \mathbb{C}$ which is constant on conjugacy classes, i.e. $\forall h \in S_d$ we have $\alpha(h^{-1}gh) = \alpha(h)$. Let $\mathbb{C}_{\text{class}}$ denote the vector space of class functions on S_d. We define on $\mathbb{C}_{\text{class}}$ the following Hermitian inner product

$$(\alpha, \beta) := \frac{1}{d!} \sum_{\sigma \in S_d} \alpha(\sigma)\overline{\beta(\sigma)} = \frac{1}{d!} \sum_{C \subset S_d} |C|\alpha(C)\overline{\beta(C)} \tag{1.28}$$

for any $\alpha, \beta \in \mathbb{C}_{\text{class}}$, where $\sum_C$ runs over the conjugacy classes of S_d.

We have the following

Lemma 1.29 ([**FH**]).

$$\mathcal{Z}\mathbb{C}[S_d] = \left\{ \sum_{\sigma \in S_d} \alpha(\sigma)\sigma \;\middle|\; \alpha \in \mathbb{C}_{class} \right\} \tag{1.30}$$

For $\eta \vdash d$ denote by $C_\eta \subset S_d$ the conjugacy class corresponding to all elements of cycle type η. Let $\alpha_\eta\colon S_d \to \mathbb{C}$ the class function which takes value 1 on elements of C_η and 0 otherwise. It is clear that the set $\{\alpha_\eta \mid \eta \vdash d\}$ gives a basis of $\mathbb{C}_{\text{class}}$. Let $c_\eta \in \mathcal{Z}\mathbb{C}[S_d]$ denote the corresponding element in the center of the group algebra, that is

$$c_\eta = \sum_{s \in S_d} \alpha_\eta(\sigma)\sigma = \sum_{\sigma \in C_\eta} \sigma \tag{1.31}$$

We get that $\{c_\eta \mid \eta \vdash d\}$ form a basis of $\mathcal{Z}\mathbb{C}[S_d]$ as a complex vector space, called *conjugacy class basis*

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\eta \vdash d} \langle c_\eta \rangle_{\mathbb{C}} \tag{1.32}$$

Note that the identity element c_e in $\mathcal{Z}\mathbb{C}[S_d]$ corresponds to the partition $e = (1, \ldots, 1)$.

A (complex) representation of S_d is a homomorphism $\rho\colon S_d \to \operatorname{Aut}(V_\rho)$ making the finite dimensional complex vector space V_ρ into a S_d-module. We define the dimension of the representation to be $\dim \rho := \dim_{\mathbb{C}} V_\rho$. Given any representation ρ, notice that this extends to a homomorphism $\mathbb{C}[S_d] \to \operatorname{End}(V_\rho)$, making V_ρ into a $\mathbb{C}[S_d]$-module.

Representations of S_d are described by their characters, which are defined as follows:

Definition 1.33. Let ρ be a representation of S_d. The *character* of ρ is the class function $\chi_\rho \in \mathbb{C}_{\text{class}}$ defined by

$$\chi_\rho(\sigma) := \operatorname{tr}(\rho(\sigma)) \tag{1.34}$$

The fact that χ_ρ is a class function follows from the cyclicity of the trace. From the definition it follows that χ_ρ does not depend on the choice of basis for V_ρ (due to the corresponding property of the trace) and that $\chi_\rho(e) = \dim \rho$ (since $\rho(e) = \text{id}$). It is also easy to see that $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

A representation ρ is *irreducible* if V_ρ does not contain any nontrivial S_d-submodules. For complex representation we have the following fundamental result

Theorem 1.35 ([**FH**]). *In terms of the inner product* (1.28) *the characters of the irreducible representations of S_d are orthonormal:*

$$(\chi_{\rho_1}, \chi_{\rho_2}) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \ncong \rho_2 \end{cases} \tag{1.36}$$

where ρ_1, ρ_2 are irreducible representations.

Any complex representation decomposes uniquely into the direct sum of irreducible representations. More precisely, we have

$$R = \bigoplus_{\rho} V_\rho^{\oplus \dim V_\rho} \tag{1.37}$$

where R is the obvious representation of S_d on $\mathbb{C}[S_d]$, called *regular representation*, while the (big) direct sum runs over all the irreducible representations of S_d. This also implies the following isomorphism of algebras

$$\mathbb{C}[S_d] \cong \bigoplus_{\rho} \mathrm{End}(V_\rho) \tag{1.38}$$

which is defined extending $S_d \to \bigoplus_\rho \mathrm{End}(V_\rho)$ by linearity.

Theorem 1.39 ([**FH**])**.** *To each partition $\lambda \vdash d$ corresponds a unique irreducible representation V_λ of S_d. The corresponding character, denoted χ^λ, is given by Frobenius formula [**FH**]. In particular χ^λ is a real class function.*

By dimensional arguments, we get that characters of irreducible representations form another basis of $\mathbb{C}_{\text{class}}$. Moreover, denoting

$$e_\lambda := \sum_{\sigma \in S_d} \chi^\lambda(\sigma)\sigma \tag{1.40}$$

we get that $\{e_\lambda \mid \lambda \vdash d\}$ gives another basis of $\mathcal{Z}\mathbb{C}[S_d]$

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\lambda \vdash d} \langle e_\lambda \rangle_{\mathbb{C}} \tag{1.41}$$

which will be called *character basis* or *idempotent basis*. Indeed from characters orthogonality (and the fact that $\overline{\chi^\lambda} = \chi^\lambda$) we get

$$e_{\lambda_i} \cdot e_{\lambda_j} = \begin{cases} e_{\lambda_i} & \text{if } e_{\lambda_i} = e_{\lambda_j} \\ 0 & \text{otherwise} \end{cases} \tag{1.42}$$

Since $\mathcal{Z}\mathbb{C}[S_d]$ admits a basis of idempotent elements, we say that it is a *semisimple algebra*. The formulas for the change of basis are given by the characters themselves

$$e_\lambda = \frac{\dim \lambda}{d!} \sum_{\eta \vdash d} \chi^\lambda(C_\eta) c_\eta \qquad \text{and} \qquad c_\eta = |C_\eta| \sum_{\lambda \vdash d} \frac{\chi^\lambda(C_\eta)}{\dim \lambda} e_\lambda \tag{1.43}$$

where $\dim \lambda := \dim V_\lambda$. In order to simplify the notation, set also $\chi^\lambda_\eta := \chi^\lambda(C_\eta)$.

1.4 Burnside's formula

Now we are ready to use the notions that we introduced in order to rewrite the expression of Hurwitz numbers in terms of characters of the irreducible representations of S_d. Existence of a idempotent basis for $\mathcal{Z}\mathbb{C}[S_d]$ and formulas (1.43) will be crucial.

Recall that

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \frac{|M^{\bullet}|}{d!} \tag{1.44}$$

where $M^{\bullet}$ is the set of monodromy representations of type $(g, d, \eta_1, \dots, \eta_r)$ where $\eta_1, \dots, \eta_n$ are partitions of d.

In order to account for the case $g \neq 0$ one needs

Definition 1.45. Let $d \in \mathbb{Z}_{\geq 1}$, $\eta \vdash d$. Denote by $\xi(\eta)$ the centralizer of C_η. We define the *kommutator* to be the element

$$\mathfrak{K} := \sum_{\eta \vdash d} |\xi(\eta)| c_\eta^2 \in \mathcal{Z}\mathbb{C}[S_d] \tag{1.46}$$

Then we have the following

Proposition 1.47 ([**CM**])**.**

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \frac{1}{d!}[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1} \tag{1.48}$$

where $[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$ *denotes the coefficient of* c_e *after writing the product* $\mathfrak{K}^g c_{\eta_n} \dots c_{\eta_2} c_{\eta_1}$ *as a linear combination of the basis elements* $c_\eta \in \mathcal{Z}\mathbb{C}[S_d]$*. As usual* h *is determined by Riemann-Hurwitz formula.*

Maybe add proof fo $g = 0$.

By changing basis from the conjugacy basis to the idempotent basis we get

Theorem 1.49 (Burnside's Character Formula [**CM**])**.**

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g} \prod_{i=1}^{n} f_{C_j}(\lambda) \tag{1.50}$$

where

$$f_{C_i}(\lambda) := |C_i| \frac{\chi^\lambda(C_i)}{\dim \lambda} \tag{1.51}$$

and $C_i := C_{\eta_i}$.

Recall that from the change of basis formula,

$$c_\eta = \sum_{\lambda \vdash d} f_{C_\eta}(\lambda) e_\lambda \tag{1.52}$$

From Bournside's formula we see that such coefficients for the change of basis correspond to the contribution of the ramification profile η to the disconnected Hurwitz numbers.

1.5 The generating function

In the following we will restrict ourself to the case of$g = 0$. Recall that there are some degeneration formulas which allows to express all the Hurwitz numbers in terms of those for $g = 0$. For a similar discussion for arbitrary g see [**CM**]. We will follow instead [**O1**].

For $g=0$ the Riemann-Hurwitz formula implies

$$2h-2=-2d+nd-\sum_{i=1}^{n}\ell(\eta_i) \tag{1.53}$$

and since this fixes h in terms of $(d,\eta_1,\dots,\eta_n)$, we simply denote

$$H_d^\bullet(\eta_1,\dots,\eta_n):=H^\bullet_{h\xrightarrow{d}0}(\eta_1,\dots,\eta_n)=\sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2\prod_{i=1}^{n}f_{C_j}(\lambda) \tag{1.54}$$

Let b be the number of branch points which have simple ramification, i.e. $\eta=(2)$. We denote

$$H_{d,b}^\bullet(\eta_1,\dots,\eta_{n-b}):=H_d^\bullet(\eta_1,\dots,\eta_{n-b},\underbrace{(2),\dots,(2)}_{b\text{ times}})=\sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2 f_2(\lambda)^b\prod_{i=1}^{n-b}f_{C_j}(\lambda) \tag{1.55}$$

where $f_2:=f_{C_{(2)}}$. Analogous definitions hold for connected Hurwitz numbers, replacing $H^\bullet$ with H°.

Rather than considering the different Hurwitz numbers separately, it worth to collect them together into generating functions. Fix $m\in\mathbb{Z}_{\geq 0}$ to be the number of branch points with non-simple ramification profile. Then

Definition 1.56. Let $\{p_{i,j}\}$ and q be some variables, $i\in\{1,\dots,m\}$, $j\in\mathbb{Z}_{\geq 0}$. Then we define the *Hurwitz potential* to be

$$\mathfrak{H}^\bullet(p_{i,j},q,\beta):=\sum_{d,b=0}^{\infty}q^d\frac{\beta^b}{b!}\sum_{\eta_1\vdash d}\cdots\sum_{\eta_m\vdash d}p_1^{\eta_1}\cdots p_m^{\eta_m}H_{d,b}^\bullet(\eta_1,\dots,\eta_m) \tag{1.57}$$

where for $\eta=(l_1,\dots,l_k)\vdash d$ we have

$$p_i^\eta:=(p_{i,1}^{l_1}+\cdots+p_{i,d}^{l_1})\cdots(p_{i,1}^{l_k}+\cdots+p_{i,d}^{l_k}) \tag{1.58}$$

An analogous definition holds for the *connected Hurwitz potential* $\mathfrak{H}^\circ$.

For fixed i, the polynomials of the form p^η for all partitions $\eta\vdash d$ form a basis for the space of all homogeneous polynomials of degree d in d variables with rational coefficients. They are called *power sum polynomials*. Therefore, given $\mathfrak{H}$, it can be expanded uniquely as in (1.57) giving all the Hurwitz numbers. For more details on power sum polynomials see [**FH**] or [**M**].

We see that in the expansion of the Hurwitz potential

- q keeps track of the degree d
- β keeps track of the number of simple ramification points (we divided by $b!$ in order to not distinguish between them)
- i indicizes the non-simple ramification points
- j gives the ramification profiles

The first advantage of considering generating function in place of the single Hurwitz numbers is

Theorem 1.59 (**[CM]**).

$$1 + \mathfrak{H}^\bullet = e^{\mathfrak{H}^\circ} \tag{1.60}$$

Moreover, for any $\lambda \vdash d$, we have the following relation

$$s_\lambda(p_1, \dots, p_d) = \frac{1}{d!} \sum_{\eta \vdash d} \chi_\eta^\lambda |C_\eta| p^\eta \tag{1.61}$$

where s_λ is the *Schur polynomial* associated to λ, and is homogeneous of degree d, see **[FH]** or **[M]** (note the similarity with the base change $c_\eta \to e_\lambda$). Putting formulas together we get

$$\begin{aligned}
\mathfrak{H}^\bullet(p_{i,j}, q, \beta) &= \sum_{d,b=0}^{\infty} q^d \frac{\beta^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^2 f_2(\lambda)^b \prod_{i=1}^m f_{C_i}(\lambda) \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{\beta f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \prod_{i=1}^m d! |C_i| \chi^\lambda(C_i) \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{\beta f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \prod_{i=1}^m s_\lambda(p_{i,1}, \dots, p_{i,d})
\end{aligned} \tag{1.62}$$

The case $m = 2$ correspond to the so called *double Hurwitz numbers*, which are the ones we are interested in. For a reason that will be more clear later we denote the corresponding Hurwitz potential by τ

$$\begin{aligned}
\tau(p_j, p'_j, q, \beta) &= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{\beta f_2(\lambda)} s_\lambda(p_1, \dots, p_d) s_\lambda(p'_1, \dots, p'_d) \\
&= \sum_{\lambda} q^{|\lambda|} e^{\beta f_2(\lambda)} s_\lambda(P) s_\lambda(P')
\end{aligned} \tag{1.63}$$

where $\sum_\lambda$ runs over the paritions of arbitrary integers and $|\lambda|$ is the size of the partition λ, $\lambda \vdash |\lambda|$. We used P and P' to denote the set of variables $\{p_1, p_2, \dots\}$ and $\{p'_1, p'_2, \dots\}$ respectively (of course s_λ depends only on the first $|\lambda|$ variables).

Before moving on, notice that for double Hurwitz numbers the Riemann-Hurwitz formula gives

$$2h = b + 2 - \ell(\eta) - \ell(\eta') \tag{1.64}$$

1.6 Frobenius formula for $C_{(2)}$

Our generating function for double Hurwitz numbers takes a very simple form, it only remains to rewrite $f_2(\lambda)$ in a nicer way. To do this we will use the Frobenius formula combined with the *half-infinite wedge formalism.* Regarding the first, note that we did not wrote the general Frobenius formula, since it is quite complicated. However it has a nicer expression in the special case of $\chi^\lambda(C_{(2)})$, that is the case needed to compute f_2.

For a given partition λ, define its rank r to be the length of the diagonal of its Young diagram, and let a_i and b_i be the number of boxes to the right and below of the i-th box of the diagonal, reading from the upper left to the lower right. Then $\begin{pmatrix} a_1 a_2 \dots a_n \\ b_1 b_2 \dots b_n \end{pmatrix}$ is called

Frobenius notation of the partition. Let also define the *modified Frobenius notation* to be Add picture, FH pag 51
$\begin{pmatrix} a'_1 a'_2 \dots a'_n \\ b'_1 b'_2 \dots b'_n \end{pmatrix}$ where $a'_i = a_i + 1/2$ and $b'_i = b_i + 1/2$. Then

Lemma 1.65 (**[FH]**).

$$\chi^\lambda(C_{(2)}) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} (a_i(a_i+1) - b_i(b_i+1)) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} ((a'_i)^2 - (b'_i)^2) \tag{1.66}$$

From this and $|C_{(2)}| = \binom{d}{2}$ it follows that

$$f_2(\lambda) = \frac{|C_{(2)}|}{\dim \lambda} \chi^\lambda(C_{(2)}) = \frac{1}{2} \sum_{i=1}^{r} ((a'_i)^2 - (b'_i)^2) \tag{1.67}$$

Draw the Young diagram of λ over the real line with opposite orientation as in the picture and consider the black and white stones as showed. Suppose that λ is made of $k = \ell(\lambda)$ cycles of lengths $\{\lambda_1, \dots, \lambda_k\}$ where $\lambda_1 \geq \lambda_2 \geq \dots$. Now consider the ordered set $\{\tilde{\lambda}\}_{i \in \mathbb{Z}_{\geq 0}} = \{\lambda_1, \dots, \lambda_k, 0, 0, \dots\}$ ending with infinitely many zeros. Then it is clear that black stones are placed in correspondence to elements of

$$\mathfrak{S}_\bullet(\lambda) := \{\lambda_i - i + 1/2\} \subset \mathbb{Z} + \frac{1}{2} \tag{1.68}$$

and white stones in correspondence to elements of $\mathfrak{S}_\circ(\lambda) := (\mathbb{Z} + \frac{1}{2}) \setminus \mathfrak{S}_\bullet(\lambda)$. Moreover, the coefficients in the modified Frobenius notation are given by

$$\begin{aligned} \{a'_i\} &= \mathfrak{S}^+_\bullet(\lambda) := \mathfrak{S}_\bullet \cap (\mathbb{Z}_{\geq 0} + 1/2) \\ \{b'_i\} &= \mathfrak{S}^-_\circ(\lambda) := \mathfrak{S}_\circ \cap (\mathbb{Z}_{\leq 0} - 1/2) = (\mathbb{Z}_{\leq 0} - 1/2) \setminus \mathfrak{S}_\bullet(\lambda) \end{aligned} \tag{1.69}$$

Hence we obtained

$$f_2(\lambda) = \sum_{k \in \mathfrak{S}^+_\bullet} \frac{k^2}{2} - \sum_{k \in \mathfrak{S}^-_\circ} \frac{k^2}{2} \tag{1.70}$$

The aim of the following section is to construct a system described by a vector space whose basis elements correspond to partitions (a.k.a. Young diagrams) and which admits an operator $\mathcal{F}_2$ which takes value $f_2(\lambda)$ in correspondence to the configuration λ. Actually we will be able to do more, that is, to recover the whole Hurwitz potential

$$\tau(p_j, p'_j, q, \beta) = \sum_\lambda q^{|\lambda|} e^{\beta f_2(\lambda)} s_\lambda(P) s_\lambda(P') \tag{1.71}$$

as a correlator of some other operator, that is $\tau = (v, Av)$ for a certain vector v, a certain operator A acting on it, and some inner product $(-,-)$ defined on the vector space. Such a system will be described by configurations of black and white stones as before.

1.7 Half infinite wedge formalism

Now we introduce the *half infinite wedge formalism*, also known as *infinite wedge formalism*. As a references see [**MJD**; **O2**; **J**].

Let V be a vector space with basis $\{\underline{k}\}$, $k \in \mathbb{Z}+\frac{1}{2}$. We define the vector space $\bigwedge^{\frac{\infty}{2}} V$ to be spanned by vectors

$$v_S := \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots \tag{1.72}$$

where $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z}+\frac{1}{2}$ is a subset s.t. both

$$S^+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \quad \text{and} \quad S^- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S \tag{1.73}$$

are finite. We equip $\bigwedge^{\frac{\infty}{2}} V$ with the inner product $(-,-)$ in which the basis $\{v_S\}$ (for all possible choices of S) is orthonormal.

For our purposes S is identified with $\mathfrak{S}_\bullet(\lambda)$ defined before, we denote by v_λ the vector in $\bigwedge^{\frac{\infty}{2}} V$ corresponding to $S = \mathfrak{S}_\bullet(\lambda)$. In this case $S^+ = \mathfrak{S}_\bullet^+$ and $S^- = \mathfrak{S}_\circ^-$, and finiteness condition simply corresponds to the fact that we have finitely many black stones on the left of the origin and finitely many white stones on the right or the origin, which is automatic for $\mathfrak{S}_\bullet(\lambda)$.

Define the operators ψ_k and ψ_k^* on $\bigwedge^{\frac{\infty}{2}} V$ as follows. For $v \in \bigwedge^{\frac{\infty}{2}} V$ let

$$\psi_k(v) := \underline{k} \wedge v \tag{1.74}$$

and let ψ_k^* to be the adjoint operator, that is s.t.

$$(v', \psi_k^* v) = (\psi_k v', v) \tag{1.75}$$

From the definition of these operators and of the inner product, note that they satisfy

$$\begin{aligned} \psi_j \psi_k^* + \psi_k^* \psi_j &= \delta_{jk} \\ \psi_j \psi_k + \psi_k \psi_j &= 0 \\ \psi_j^* \psi_k^* + \psi_k^* \psi_j^* &= 0 \end{aligned} \tag{1.76}$$

which are known as *fermionic commutation relations.*

These operators are related to the usual creation and annihilation operators for the *Fermi sea*, by identifying black stones with electrons and white stones with empty energy levels (but note that in usual physical literature, and in [**MJD**], one should exchange $\psi_k \leftrightarrow \psi_k^*$). Finiteness condition amouts to requiring that the considered state is a finite energy excitation of the vacuum in the Fermi sea. We identify the vacuum of the Fermi sea with the vector $v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \underline{-\frac{5}{2}} \wedge \dots$, where all stones are black (resp. white) on the right (resp. left) of the origin.

To see this, note that if $\underline{k}$ is not present in v_S (“the energy level k is empty”) then ψ_k add that vector to v_S (”creates the electron”) whereas ψ_k^* maps v_S to 0 (“the state is annihilated”). Instead, if $\underline{k}$ is present in v_S (“the energy level k is occupied”) then ψ_k maps v_S to 0 (“Pauli principle”) whereas ψ_k^* removes that vector from v_S (“the electron at the energy level k is annihilated”).

Introduce the normal ordered product

$$: \psi_k \psi_k^* : = \begin{cases} \psi_k \psi_k^* & k > 0 \\ -\psi_k^* \psi_k & k < 0 \end{cases} \tag{1.77}$$

Hence for $k > 0$ (resp. $k < 0$), $: \psi_k \psi_k^* :$ gives 1 if we have a black (resp. white) stone in k and zero otherwise. Finiteness condition ensures that any operator of the form $\sum_k a_k : \psi_k \psi_k^* :$ is finite for any choice of the coefficients a_k.

Introduce the following operators, called *energy operator* and *charge operators* respectively

$$H = \sum_{k \in \mathbb{Z}+1/2} k : \psi_k \psi_k^* : \qquad C = \sum_{k \in \mathbb{Z}+1/2} : \psi_k \psi_k^* : \tag{1.78}$$