Hurwitz numbers in half infinite wedge space

From geometry to operators

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Let $f\colon X \to Y$ be a (non-constant holomorphic) map of (compact) and connected Riemann surfaces (hence surjective).

Definition

- The *ramification index* of f at $x \in X$ is the integer $k_x \in \mathbb{Z}_{\geq 1}$ s.t. f is locally of the form z^k, where z is a local coordinate centered in x.
- The *differential length* of f at $x \in X$ is $\nu_x = k_x - 1$.
- A point $x \in X$ s.t. $\nu_x > 0$ is called a *ramification point*. The *ramification locus* of f is the set of its ramification points. If $\nu_x = 0$ we say that f is *unramified* at x. If $\nu_x = 1$ we say that f has *simple ramification* at x.
- If $x \in X$ is a ramification point, then $f(x) \in Y$ is called a *branch point*. The *branch locus* B_f of f is the set of its branch points.

For any $y, y' \in Y \setminus B_f$ we have $|f^{-1}(y)| = |f^{-1}(y')|$.

Definition

$$\deg f := |f^{-1}(y)| \ , \qquad y \in Y \setminus B_f \ .$$

For any $y \in Y$, $f^{-1}(y) = \{x_1, \dots, x_n\}$, we have $d := \deg f = \sum_i k_{x_i}$.

Definition

We call *ramification profile* of f at y the partition of d given by $(k_{x_1}, \dots, k_{x_n})$. We say that f is *unramified* / *simply ramified* / *fully ramified* if its ramification profile is $(1, \dots, 1)$ / $(2, 1, \dots, 1)$ / (d) respectively.

Theorem (Riemann-Hurwitz)

Let g_X and g_Y denote the genus of X and Y respectively. Then

$$\underbrace{2g_X - 2}_{\chi(X)} = \underbrace{(2g_Y - 2)}_{\chi(Y)} \deg f + \sum_{x \in X} \nu_x$$

Definition

$f\colon X \to Y$ and $g\colon \tilde{X} \to Y$ are *isomorphic* if there is $\phi\colon X \xrightarrow{\sim} \tilde{X}$ s.t. $f = g \circ \phi$.
An *automorphism* of $f\colon X \to Y$ is $\psi\colon X \xrightarrow{\sim} X$ s.t. $f = f \circ \psi$.
The group of automorphisms of f is denoted $\mathrm{Aut}(f)$.

Definition

Let Y RS of genus g, $B = \{b_1, \dots, b_n\} \subset Y$, $d \geq 0$, $\eta_1, \dots, \eta_n \vdash d$.
We define the *(degree d) connected Hurwitz number* to be

$$H^{\circ}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|}$$

and the *(disconnected) (degree d) Hurwitz number* to be

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|}$$

where both sums runs over isomorphism classes of degree d holomorphic maps $f\colon X \to Y$ s.t.

- X is a compact RS of genus h
- the branch locus of f is B
- the ramification profile of f at b_i is η_i

In the case of connected Hurwitz numbers we further require X to be connected.

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Maps of Riemann surfaces are ramified covers:

Definition

A *ramified cover* is a continuous function between compact topological surfaces $f\colon X \to Y$ with a finite set $B \subset Y$ s.t.

- $f^{-1}(B)$ is finite,
- $f\colon X \setminus f^{-1}(B) \to Y \setminus B$ is a covering.

Vice-versa, we have

Theorem (Riemann's Existence Thm.)

Let Y compact Riemann surface, X^0 topological surface, $\{b_1, \dots, b_n\} \subset Y$ finite subset, $f^0\colon X^0 \to Y \setminus \{b_1, \dots, b_n\}$ covering of finite degree. Then there exists a unique (up to isomorphisms) compact Riemann surface X s.t.

- *X^0 is a dense subset of X,*
- *f^0 extends to a holomorphic map of Riemann surfaces $f\colon X \to Y$.*

This will be useful to reconstruct X and f given Y and some additional data.

Definition

The map $f\colon X \to Y$ is said y_0-*labelled* if $y_0 \in Y \setminus B_f$ and is chosen an isomorphism $L\colon f^{-1}(y_0) \xrightarrow{\sim} \{1,\dots,d\}$. We also say that L is a *labelling*. An *isomorphism of y_0-labelled maps* (f, L) and (f', L') is an isomorphism of Riemann surfaces $\phi\colon X \to X'$ s.t.

$$f' \circ \phi = f \quad \text{and} \quad L' \circ \phi = L$$

Definition

A y_0-labelled map $f\colon X \to Y$ determines a group homomorphism

$$\Phi\colon \pi_1(Y \setminus B_f, y_0) \to S_d \ , \qquad \gamma \mapsto \sigma_\gamma$$

called *monodromy representation*.

We have that $(f, L) \cong (f', L')$ imply $\Phi = \Phi'$.

Now consider $b \in B_f$ and $\gamma \in \pi_1(Y \setminus B_f, y_0)$ simple loop winding once around b (and with zero winding number around the other branch points). If the ramification profile of f at b is $\eta = (k_1, \ldots, k_l)$, then σ_γ has cycle type η (to see this recall that the local expression of f around ramification points is z^k and consider a circle around y_0 of unit radius in the chart).

Definition

Let Y be a connected Riemann surface of genus g, $y_0, b_1, \ldots, b_n \in Y$ points, $d \in \mathbb{Z}_{\geq 1}$ and $\eta_1, \ldots, \eta_n \vdash d$. A *monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$ is a group homomorphism $\Phi \colon \pi_1(Y \setminus \{b_1, \ldots, b_n\}, y_0) \to S_d$ s.t. if γ_k is a small loop around b_k then $\Phi(\gamma_k)$ has cycle type η_k.
If moreover the subgroup $\operatorname{im}\Phi \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$ we say that Φ is a *connected monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$.

Note that if two labellings L, L' of $f \colon X \to Y$ are given, then $L' = \sigma \cdot L$ for some $\sigma \in S_d$ and $\Phi'(\gamma) = \sigma \cdot \Phi(\gamma) \cdot \sigma^{-1}$. In particular the type of the monodromy representation does note depend on the chosen labelling.

We obtained that a degree d map $f\colon X \to Y$ between compact connected Riemann surfaces s.t. the ramification profile over each branch point is η_i gives rise to a connected monodromy representation Φ of type $(g_Y, d, \eta_1, \ldots, \eta_n)$. If we let X to be non connected, then the monodromy representation may not be connected anymore, more precisely: the monodromy representation is connected if and only if X is connected. We also have that isomorphic maps give the same monodromy representation.
Conversely:

Theorem

Let Y be a Riemann surface of genus g, Φ a monodromy representation of type $(g, d, \eta_1, \ldots, \eta_n)$, $B = \{b_1, \ldots, b_n\} \subset Y$ a finite subset. Then exists a y_0-labelled holomorphic map of Riemann surface covering Y with branch locus B whose associated monodromy is F. Such map is unique up to isomorphisms of y_0-labelled maps.

Sketch of proof In the proof of this result the Riemann's Existence theorem is fundamental. We construct explicitly the topological space X_0 and the covering $f^0 \colon X^0 \to Y \setminus B$ in such a way that the map $f \colon X \to Y$ given by the Riemann's Existence theorem has the desired monodomy representation.
Take cycles $\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g$ in Y generating $\pi_1(Y, y_0)$ all containing a point $p \in Y$ in such a way that $Y \setminus \{\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g\}$ is the fundamental polygon describing Y. Denote by γ_i a loop containing y_0, winding once around b_i, never around the other elements of B, and fully contained in $Y \setminus \{\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g\}$.
PICTURE

PICTURE

Consider segments s_j connecting p with the points b_j. Open the previous polygon in correspondence of these segments, so that we get a new polygon

$$P := s_1\bar{s}_1 \cdots s_n\bar{s}_n\alpha_1\beta_1\bar{\alpha}_1\bar{\beta}_1 \cdots \alpha_g\beta_g\bar{\alpha}_g\bar{\beta}_g$$

Then it suffices to take d copies of P and glue their boundaries appropriately to produce X^0 in such a way that the natural projection to $Y \setminus B$ has the desired monodromy representation:

$$s_{j,k} \sim \bar{s}_{j,\Phi(\gamma_j)(k)}$$
$$\bar{\alpha}_{i,k} \sim \alpha_{i,\Phi(\beta_i)(k)}$$
$$\beta_{i,k} \sim \bar{\beta}_{i,\Phi(\alpha_i)(k)}$$

■

The previous theorem ensures that we have a bijection between isomorphism classes of y_0-labeled Hurwitz covers and connected monodromy representations of type $(g, d, \eta_1, \dots, \eta_n)$. Then we can prove

Theorem

Let M° (resp $M^\bullet$) be the set of connected monodromy representations (resp. monodromy representations) of type $(g, d, \eta_1, \dots, \eta_n)$. Then

$$H^\circ_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \frac{|M^\circ|}{d!}$$

and

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) = \frac{|M^\bullet|}{d!}$$

where h is determined by Riemann-Hurwitz.

Sketch of proof We give the proof in the connected case, the other case is analogous.

Take $f\colon X \to Y$ Hurwitz cover. Clearly there are $d!$ possible choices of a y_0-labelling $L\colon f^{-1}(y_0) \to \{1, \dots, d\}$. An automorphism $\phi \in \operatorname{Aut}(f)$ is an isomorphism $\phi\colon X \xrightarrow{\sim} X$ satisfying $f = f \circ \phi$. In particular ϕ gives an isomorphism $(f, L) \cong (f, L')$ where $L' = \phi \cdot L := L \circ \phi^{-1}$. Here $\phi \cdot L$ denotes the left action of $\operatorname{Aut}(f)$ on the possible y_0-labellings of f. Such action is free (i.e. $\phi \cdot L = L$ imply $\phi = \mathrm{id}_X$) so the number of isomorphism classes of y_0-labelings of f is $d!/|\operatorname{Aut}(f)|$.
From the previous theorem isomorphism classes of y_0-labelled maps for the given f are in bijection with the distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. Therefore

$$m_f = \frac{d!}{|\operatorname{Aut}(f)|}$$

where m_f the number of distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. So we have

$$H^{\circ}_{h \xrightarrow{d} g}(\eta_1, \dots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|} = \sum_{[f]} \frac{m_f}{d!} = \frac{|M^{\circ}|}{d!} \qquad \blacksquare$$

Although the information carried by connected Hurwitz numbers is usually more interesting for geometrical purposes, it turns out that it is easier to compute the (possibly disconnected) Hurwitz numbers. We will see later how it is possible to recover the connected Hurwitz numbers from the disconnected ones.
We mention that using some "degeneration formulas" (which heuristically correspond to shrink the Riemann surface Y producing nodal curves) all disconnected degree d Hurwitz numbers are determined in therms of Hurwitz numbers of the form $H^{\bullet}_{h \xrightarrow{d} 0}(\eta_1, \eta_2, \eta_3)$. For this reason (and others) we later restrict our discussion to the case $g = 0$.
Using the last theorem the problem of computing degree d Hurwitz numbers can be translated into a problem in representation theory of the symmetric group S_d. In order to show this we need some facts about representation theory.

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Definition

The *group algebra* of the symmetric group S_d is the complex algebra

$$\mathbb{C}[S_d] := \left\{ \sum_{\sigma \in S_d} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{C} \right\}$$

We define *class algebra* of S_d the center of the group algebra

$$\mathcal{Z}\mathbb{C}[S_d] = \{x \in \mathbb{C}[S_d] \mid yx = xy \quad \text{for all} \quad y \in \mathbb{C}[S_d]\}$$

Definition

A *class function* on S_d is a map $\alpha \colon S_d \to \mathbb{C}$ s.t. $\alpha(h^{-1}gh) = \alpha(h)\ \forall h \in S_d$. Let $\mathbb{C}_{\text{class}}$ denote the vector space of class functions on S_d, together with the following Hermitian inner product

$$(\alpha, \beta) := \frac{1}{d!} \sum_{\sigma \in S_d} \alpha(\sigma)\overline{\beta(\sigma)} = \frac{1}{d!} \sum_{C \subset S_d} |C| \alpha(C) \overline{\beta(C)}$$

where $\alpha, \beta \in \mathbb{C}_{\text{class}}$ and $\sum_C$ runs over the conjugacy classes of S_d.

Lemma

$$\mathcal{Z}\mathbb{C}[S_d] = \left\{ \sum_{\sigma \in S_d} \alpha(\sigma)\sigma \;\middle|\; \alpha \in \mathbb{C}_{class} \right\}$$

For $\eta \vdash d$, denote C_η the conjugacy class in S_d of elements of cycle type η. It follows from the lemma that we can find a basis of $\mathcal{Z}\mathbb{C}[S_d]$ by considering functions α_η taking value 1 on elements of C_η and zero otherwise

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\eta \vdash d} \langle c_\eta \rangle_{\mathbb{C}} \quad \text{where} \quad c_\eta := \sum_{\sigma \in S_d} \alpha_\eta(s)\sigma = \sum_{\sigma \in C_\eta} \sigma$$

Another important basis of $\mathcal{Z}\mathbb{C}[S_d]$ is obtained taking as basis of $\mathbb{C}_{\text{class}}$ the characters of irreducible representations.

Representations of S_d are described by their characters

Definition

Let ρ be a representation of S_d. The *character* of ρ is the class function $\chi_\rho \in \mathbb{C}_{\text{class}}$ defined by

$$\chi_\rho(\sigma) := \operatorname{tr}(\rho(\sigma))$$

For complex representation we have the following fundamental result

Theorem

In terms of the inner product defined in $\mathbb{C}_{class}$ the characters of the irreducible representations of S_d are orthonormal:

$$(\chi_{\rho_1}, \chi_{\rho_2}) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \ncong \rho_2 \end{cases}$$

where ρ_1, ρ_2 are irreducible representations.

Theorem

To each partition $\lambda \vdash d$ corresponds a unique irreducible representation V_λ of S_d. The corresponding character, denoted χ^λ, is given by Frobenius formula.

By dimensional arguments, characters of irreducible representations form another basis of $\mathbb{C}_{\text{class}}$. Moreover,

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\lambda \vdash d} \langle e_\lambda \rangle_{\mathbb{C}} \quad \text{where} \quad e_\lambda := \sum_{\sigma \in S_d} \chi^\lambda(\sigma)\sigma$$

From characters orthogonality (and the fact that $\overline{\chi^\lambda} = \chi^\lambda$) we get

$$e_{\lambda_i} \cdot e_{\lambda_j} = \begin{cases} e_{\lambda_i} & \text{if } e_{\lambda_i} = e_{\lambda_j} \\ 0 & \text{otherwise} \end{cases}$$

for this reason $\{e_\lambda\}$ is called *idempotent basis* of $\mathcal{Z}\mathbb{C}[S_d]$.
The formulas for the change of basis are given by the characters

$$e_\lambda = \frac{\dim \lambda}{d!} \sum_{\eta \vdash d} \chi^\lambda(C_\eta) c_\eta \qquad \text{and} \qquad c_\eta = |C_\eta| \sum_{\lambda \vdash d} \frac{\chi^\lambda(C_\eta)}{\dim \lambda} e_\lambda$$

where $\dim \lambda := \dim V_\lambda$.

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Recall that

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^{\bullet}|}{d!}$$

where $M^{\bullet}$ is the set of monodromy representations of type $(g, d, \eta_1, \ldots, \eta_r)$.

Definition

Let $d \in \mathbb{Z}_{\geq 1}$, $\eta \vdash d$. Denote by $\xi(\eta)$ the centralizer of C_η. We define the *kommutator* to be the element

$$\mathfrak{K} := \sum_{\eta \vdash d} |\xi(\eta)| c_\eta^2 \in \mathcal{Z}\mathbb{C}[S_d]$$

Theorem

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{1}{d!}[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$$

where $[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$ *denotes the coefficient of* c_e *in* $\mathfrak{K}^g c_{\eta_n} \ldots c_{\eta_2} c_{\eta_1}$.

By changing basis from the conjugacy basis to the idempotent basis we get

Theorem (Burnside's Character Formula)

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g} \prod_{i=1}^{n} f_{C_j}(\lambda)$$

where

$$f_{C_i}(\lambda) := |C_i| \frac{\chi^{\lambda}(C_i)}{\dim \lambda}$$

and $C_i := C_{\eta_i}$.

Recall that from the change of basis formula,

$$c_\eta = \sum_{\lambda \vdash d} f_{C_\eta}(\lambda) e_\lambda$$

From Bournside's formula we see that such coefficients $f_{C_\eta}(\lambda)$ for the change of basis correspond to the contribution of the ramification profile η to the disconnected Hurwitz numbers.

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In the following we will restrict ourselves to the case of $g = 0$. Recall that there are some degeneration formulas which allows to express all the Hurwitz numbers in terms of those for $g = 0$.
Since Riemann-Hurwitz formula fixes h in terms of $(d, \eta_1, \dots, \eta_n)$, we denote

$$H_d^\bullet(\eta_1, \dots, \eta_n) := H^\bullet_{h \xrightarrow{d} 0}(\eta_1, \dots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{n} f_{C_j}(\lambda)$$

Let b be the number of branch points which have simple ramification, i.e. $\eta = (2)$. We denote

$$H_{d,b}^\bullet(\eta_1, \dots, \eta_{n-b}) := H_d^\bullet(\eta_1, \dots, \eta_{n-b}, \overbrace{(2), \dots, (2)}^{b \text{ times}})$$
$$= \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 f_2(\lambda)^b \prod_{i=1}^{n-b} f_{C_j}(\lambda)$$

where $f_2 := f_{C_{(2)}}$. Analogous definitions hold for connected Hurwitz numbers, replacing $H^\bullet$ with H°.

Rather than considering the different Hurwitz numbers separately, it worth to collect them together into generating functions. Fix $m \in \mathbb{Z}_{\geq 0}$ to be the number of branch points with non-simple ramification profile.

Definition

Let $\{p_{i,j}\}$ and q be some variables, $i \in \{1, \dots, m\}$, $j \in \mathbb{Z}_{\geq 0}$. We define the *Hurwitz potential* to be

$$\mathfrak{H}^\bullet(p_{i,j}, q, z) := \sum_{d,b=0}^{\infty} q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} H^\bullet_{d,b}(\eta_1, \dots, \eta_m) \tag{1}$$

where for $\eta = (l_1, \dots, l_k) \vdash d$ we defined

$$p_i^\eta := (p_{i,1}^{l_1} + \cdots + p_{i,d}^{l_1}) \cdots (p_{i,1}^{l_k} + \cdots + p_{i,d}^{l_k})$$

We also introduce the *modified Hurwitz potential* to be

$$\mathfrak{h}^\bullet_d(\eta_q, \dots, \eta_m) := \sum_{b=0}^{\infty} \frac{z^b}{d!} H^\bullet_{d,b}(\eta_1, \dots, \eta_m)$$

Analogous definitions hold for the *(modified) connected Hurwitz potential* $\mathfrak{H}^\circ$ and $\mathfrak{h}^\circ$.

For fixed i, the polynomials of the form p^η for all partitions $\eta \vdash d$ form a basis for the space of all homogeneous polynomials of degree d in d variables with rational coefficients. They are called *power sum polynomials*. Therefore, given $\mathfrak{H}^\bullet$, it can be expanded uniquely as in (1) giving all the Hurwitz numbers.
We see that in the expansion of the Hurwitz potential

- q keeps track of the degree d
- z keeps track of the number of simple ramification points (we divided by $b!$ in order to not distinguish between them)
- i indicizes the non-simple ramification points
- j gives the ramification profiles

The first advantage of considering generating function in place of the single Hurwitz numbers is

Theorem

$$1 + \mathfrak{H}^\bullet = e^{\mathfrak{H}^\circ}$$

In order to simplify the notation, in the following we denote $\chi^\lambda_\eta := \chi^\lambda(C_\eta)$. For any $\lambda \vdash d$, we have the following relation, which can be regarded as corollary of Frobenius formula

$$s_\lambda(p_1, \dots, p_d) = \frac{1}{d!} \sum_{\eta \vdash d} \chi^\lambda_\eta |C_\eta| p^\eta$$

where s_λ is the *Schur polynomial* associated to λ, it is homogeneous polynomial of degree d. Putting formulas together we get

$$\begin{aligned}
&\mathfrak{H}^\bullet(p_{i,j}, q, z) = \\
&= \sum_{d,b=0}^{\infty} q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 f_2(\lambda)^b \prod_{i=1}^m f_{C_i}(\lambda) \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!} \right)^{2-m} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \prod_{i=1}^m \frac{1}{d!} |C_{\eta_i}| \chi^\lambda_{\eta_i} \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!} \right)^{2-m} \prod_{i=1}^m s_\lambda(p_{i,1}, \dots, p_{i,d})
\end{aligned}$$

The case $m = 2$ corresponds to the so called *double Hurwitz numbers*, which are the ones we are interested in

$$\begin{aligned}\mathfrak{H}^{\bullet}(\{p_j, p'_j\}, q, z) &= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} s_\lambda(p_1, \ldots, p_d) s_\lambda(p'_1, \ldots, p'_d) \\ &= \sum_{\lambda} q^{|\lambda|} e^{z f_2(\lambda)} s_\lambda(P) s_\lambda(P')\end{aligned}$$

We denoted $P := \{p_1, p_2, \ldots\}$ and $P' := \{p'_1, p'_2, \ldots\}$ (of course s_λ depends only on the first $|\lambda|$ variables of each set). For $m = 2$ we also have

$$H^{\bullet}_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}$$

$$\mathfrak{h}^{\bullet}_d(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta e^{z f_2(\lambda)} \chi^\lambda_{\eta'}$$

where, for $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ given by a_i cycles of length i, $\sum_i i a_i = d$,

$$\mathfrak{z}(\eta) := \frac{d!}{|C_\eta|}$$

We recall the well known fact that for $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ given by a_i cycles of length i, $\sum_i ia_i = d$,

$$|C_\eta| = \frac{d!}{\prod_{i=1}^{\ell(\eta)} a_i!\, i^{a_i}}$$

so

$$\mathfrak{z}(\eta) = \prod_{i=1}^{\ell(\eta)} a_i!\, i^{a_i}$$

Before moving on, notice that for double Hurwitz numbers the Riemann - Hurwitz formula gives

$$2h = b + 2 - \ell(\eta) - \ell(\eta')$$

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Given a partition λ, define its rank r to be the length of the diagonal of its Young diagram. Let a_i and b_i be the number of boxes to the right and below of the i-th box of the diagonal, reading from the upper left to the lower right.
We call $\begin{pmatrix} a_1 a_2 \ldots a_n \\ b_1 b_2 \ldots b_n \end{pmatrix}$ and $\begin{pmatrix} a'_1 a'_2 \ldots a'_n \\ b'_1 b'_2 \ldots b'_n \end{pmatrix}$ the *Frobenius notation* and the *modified Frobenius notation* of the partition respectively, where $a'_i = a_i + 1/2$ and $b'_i = b_i + 1/2$.
PICTURE

Lemma (Frobenius formula for $C_{(2)}$)

$$\chi^\lambda(C_{(2)}) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} (a_i(a_i+1) - b_i(b_i+1)) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} ((a'_i)^2 - (b'_i)^2)$$

From Frobenius formula and $|C_{(2)}| = \binom{d}{2}$ it follows that

$$f_2(\lambda) = \frac{|C_{(2)}|}{\dim \lambda}\chi^\lambda(C_{(2)}) = \frac{1}{2}\sum_{i=1}^{r}((a_i')^2 - (b_i')^2)$$

Draw the Young diagram of λ over the real line with opposite orientation as in the picture.
PICTURE

Suppose that λ is made of $k = \ell(\lambda)$ cycles of lengths $\{\lambda_1, \ldots, \lambda_k\}$ where $\lambda_1 \geq \lambda_2 \geq \ldots$. Now consider the ordered set $\{\tilde{\lambda}_i\}_{i \in \mathbb{Z}_{\geq 0}} = \{\lambda_1, \ldots, \lambda_k, 0, 0, \ldots\}$ ending with infinitely many zeros. Then it is clear that black stones are placed in correspondence to elements of

$$\mathfrak{S}_\bullet(\lambda) := \{\tilde{\lambda}_i - i + 1/2\} \subset \mathbb{Z} + \frac{1}{2}$$

and white stones in correspondence to elements of $\mathfrak{S}_\circ(\lambda) := (\mathbb{Z} + \frac{1}{2}) \setminus \mathfrak{S}_\bullet(\lambda)$. Moreover, the coefficients in the modified Frobenius notation are given by

$$\{a_i'\} = \mathfrak{S}_\bullet^+(\lambda) := \mathfrak{S}_\bullet \cap (\mathbb{Z}_{\geq 0} + 1/2)$$
$$\{b_i'\} = \mathfrak{S}_\circ^-(\lambda) := \mathfrak{S}_\circ \cap (\mathbb{Z}_{\leq 0} - 1/2) = (\mathbb{Z}_{\leq 0} - 1/2) \setminus \mathfrak{S}_\bullet(\lambda)$$

Hence we obtained

$$f_2(\lambda) = \sum_{k \in \mathfrak{S}_\bullet^+} \frac{k^2}{2} - \sum_{k \in \mathfrak{S}_\circ^-} \frac{k^2}{2}$$

Notice also that

$$|\lambda| = \lambda_1 + \cdots + \lambda_k = a_1' + \cdots + a_k' + b_1' + \cdots + b_k' = \sum_{k \in \mathfrak{S}_\bullet^+} k - \sum_{k \in \mathfrak{S}_\circ^-} k$$

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Definition

Let V be a vector space with basis $\{\underline{k}\}$, $k \in \mathbb{Z} + \frac{1}{2}$. We define the vector space $\bigwedge^{\frac{\infty}{2}} V$ to be spanned by vectors

$$v_S := \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots$$

where $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ is a subset s.t. both

$$S^+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \quad \text{and} \quad S^- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$$

are finite. We equip $\bigwedge^{\frac{\infty}{2}} V$ with the inner product $(-,-)$ in which the basis $\{v_S\}$ (for all possible choices of S) is orthonormal.

For our purposes S is identified with $\mathfrak{S}_\bullet(\lambda)$ defined before, we denote by v_λ the vector in $\bigwedge^{\frac{\infty}{2}} V$ corresponding to $S = \mathfrak{S}_\bullet(\lambda)$. In this case $S^+ = \mathfrak{S}_\bullet^+$ and $S^- = \mathfrak{S}_\circ^-$, and finiteness condition simply corresponds to the fact that we have finitely many black stones on the left of the origin and finitely many white stones on the right or the origin, which is automatic for $\mathfrak{S}_\bullet(\lambda)$.

Definition

For all $k \in \mathbb{Z} + \frac{1}{2}$ define the operators ψ_k and ψ_k^* on $\bigwedge^{\frac{\infty}{2}} V$ by

$$\psi_k(v) := \underline{k} \wedge v \quad \text{and} \quad (v', \psi_k^* v) = (\psi_k v', v)$$

where $v, v' \in \bigwedge^{\frac{\infty}{2}} V$.

From the definitions we have

$$\psi_j \psi_k^* + \psi_k^* \psi_j = \delta_{jk} \quad \text{and} \quad \psi_j \psi_k + \psi_k \psi_j = \psi_j^* \psi_k^* + \psi_k^* \psi_j^* = 0$$

which are known as *fermionic commutation relations*.
These operators are related to the usual creation and annihilation operators for the *Fermi sea*, by identifying black stones with electrons and white stones with empty energy levels (but in physical literature ψ_k and ψ_k^* are exchanged).
Finiteness condition amounts to considering state which are finite energy excitation of the vacuum state. We identify the vacuum of the Fermi sea with the vector $v_\emptyset := \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \underline{-\frac{5}{2}} \wedge \ldots$, where all stones are black (resp. white) on the right (resp. left) of the origin.
All vectors in $\bigwedge^{\frac{\infty}{2}} V$ can be obtained from $v_\emptyset$ by applying ψ_k and ψ_k^* a finite number of times: $v_\emptyset$ is cyclic.

Definition

Introduce the *normal ordered product*

$$:\psi_j \psi_k^*: \; := \begin{cases} \psi_j \psi_k^* & k > 0 \\ -\psi_k^* \psi_j & k < 0 \end{cases}$$

For $k > 0$ (resp. $k < 0$), $:\psi_k \psi_k^*:$ gives 1 if we have a black (resp. white) stone in k and zero otherwise.

For $j \neq k$ the normal ordered product is the ordinary product due to the fermionic commutation relation $:\psi_j \psi_k^*: = \psi_j \psi_k^* = -\psi_k^* \psi_j$.

The effect of $:\psi_j \psi_k^*:$ for $j \neq k$ is to take the black stone in k and move it to the position j, unless there is no black stone in k or the position j is already occupied: in such cases it gives the zero vector.

Finiteness condition ensures that any operator of the form $\sum_{j,k} a_{jk} :\psi_j \psi_k^*:$ is well defined for any choice of the coefficients $a_{j,k}$.

The assignement $E_{jk} \mapsto :\psi_j \psi_k^*:$ can be understood as a projective representation of $\mathfrak{gl}(V)$ on $\bigwedge^{\frac{\infty}{2}} V$.

Definition

$$\mathcal{F}_2 = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} :\psi_k \psi_k^*:$$

It satisfies

$$\mathcal{F}_2 v_\lambda = f_2(\lambda) v_\lambda$$

Definition

We define the following operators, called *energy operator* and *charge operators* respectively

$$H := \sum_{k \in \mathbb{Z}+1/2} k :\psi_k \psi_k^*: \qquad C := \sum_{k \in \mathbb{Z}+1/2} :\psi_k \psi_k^*:$$

The physical interpretation of these operators is obvious when we regard a black stone at position $k > 0$ as an electron of energy k and charge $+1$ and a white stone at position $k < 0$ as a positron of energy $-k$ and charge -1.
We have

$$H v_\lambda = |\lambda| v_\lambda$$

For the charge operator we have

$$Cv_S = (|S^+| - |S^-|)v_s$$

Suppose $Cv_S = c\,v_S$, then $|S^+| = |S^-| + c$. Pick $k \in S$, $k < \min(S^-)$, this means that the stone at k and all those on its right are black. Then $|S^+| = |S^-| + c$ implies that $k = s_{-(k-1/2)+c}$. Hence

$$c = \lim_{i \to +\infty} \left(s_i + i - \frac{1}{2}\right)$$

In particular, all vectors corresponding to Young diagrams ($S = \mathfrak{S}_{\bullet}(\lambda)$) have zero charge. Conversely, any vector of zero charge can be obtained from a Young diagram by taking the partition $\lambda_i = s_i + i - \frac{1}{2}$ (omitting the infinitely many λ_i which vanish). We denote by $\bigwedge_0^{\frac{\infty}{2}} V$ the subspace of $\bigwedge^{\frac{\infty}{2}} V$ of zero charge. From this we get that

$$\{v_\lambda \,|\, \lambda \vdash d\}$$

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ of energy d. Considering all positive values of d we get a basis of the whole charge zero subspace $\bigwedge_0^{\frac{\infty}{2}} V$.

Definition

For $n \in \mathbb{Z}_{\neq 0}$ define
$$\alpha_n := \sum_{k \in \mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*:$$

It satisfy $\alpha_n \underline{k} = \underline{k-n}$ and the *Heisenberg commutation relations*
$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}$$
From the definition it follows that α_n and α_{-n} are adjoint for every $n \in \mathbb{Z}_{\neq 0}$. The effect of this operator in a certain configuration of stones (or electrons) is the following. Pick one and move it to the right by n positions. If the new position is a white stone, make it black and replace the initial black stone by a white one. If the new position is occupied by a black stone, then the operator gives the zero vector (Pauli principle). Then repeat this for all black stones and sum all the resulting vectors to give the final state. Finiteness condition ensures that this operation is well defined.

Definition

For any partition $\eta = \{\eta_1, \dots, \eta_{\ell(\eta)}\} \vdash |\eta|$ define

$$\alpha_\eta := \prod_{i=1}^{\ell(\eta)} \alpha_{\eta_i} \quad \text{and} \quad \alpha_{-\eta} := \prod_{i=1}^{\ell(\eta)} \alpha_{-\eta_i}$$

The Heisenberg commutation relation ensures that the ordering of the multiplied operators is not relevant and also that α_η and $\alpha_{-\eta}$ are adjoint. Note also that α_η (resp. $\alpha_{-\eta}$) decreases (resp. increases) the energy of states by $|\eta|$. Moreover, it is possible to prove that

$$\{\alpha_{-\eta} v_\emptyset \,\big|\, \eta \vdash d\}$$

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ of energy d, orthogonal with respect to the inner product $(-,-)$.

The two basis of $\bigwedge_0^{\frac{\infty}{2}} V$ can be related as follows. Let λ be any partition and $n \in \mathbb{Z}_{>0}$. Then graphically we can see that

$$\alpha_n v_\lambda = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \text{ skew hook} \\ |\lambda/\lambda'| = n}} (-1)^{r(\lambda'/\lambda)-1} v_{\lambda'}$$

where $r(\lambda'/\lambda)$ is the number of rows of λ' touched by λ'/λ. Compare to

Lemma (*Murnaghan-Nakayama rule*)

$$\chi^\lambda_{\{(n),\eta\}} = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \text{ skew hook} \\ |\lambda/\lambda'| = n}} (-1)^{r(\lambda'/\lambda)-1} \chi^{\lambda'}_\eta$$

Using these identities we get that for any two partitions λ and η s.t. $|\lambda| = |\eta|$

$$\alpha_\eta v_\lambda = \chi^\lambda_\eta v_\emptyset$$

and taking the adjoint

$$\alpha_{-\eta} v_\emptyset = \sum_{\lambda \vdash |\eta|} \chi^\lambda_\eta v_\lambda$$

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For $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ and $\eta' = \{\eta'_1, \ldots, \eta'_{\ell(\eta')}\} \vdash d$ we obtain

$$\begin{aligned}(v_\emptyset, \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} v_\emptyset) &= \sum_{\lambda \vdash d} \chi^\lambda_{\eta'} (v_\emptyset, \alpha_\eta \mathcal{F}_2^b v_\lambda) = \sum_{\lambda \vdash d} f_2(\lambda)^b \chi^\lambda_{\eta'} (v_\emptyset, \alpha_\eta v_\lambda) \\ &= \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} (v_\emptyset, v_\emptyset) = \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}\end{aligned}$$

and then

$$H^\bullet_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \langle \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} \rangle$$

where the *correlator* of the operator A is defined by $\langle A \rangle := (v_\emptyset, A v_\emptyset)$. Similarly

$$\mathfrak{h}^\bullet_d(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta e^{z f_2(\lambda)} \chi^\lambda_{\eta'} = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \langle \alpha_\eta e^{z \mathcal{F}_2} \alpha_{-\eta'} \rangle$$

To rewrite the Hurwitz potential we should be able to recover the Schur polynomials from the half infinite wedge formalism.

Definition

Given a sequence $t = (t_1, t_2, \ldots)$ define

$$\Gamma_{\pm}(t) = \exp\left(\sum_{n=1}^{\infty} t_n \alpha_{\pm n}\right)$$

Note that $\Gamma_+(t)$ and $\Gamma_-(t)$ are adjoint.

Lemma

Consider a set of variables $P = (p_1, p_2, \ldots)$ and denote $p^{(k)} = p_1^k + p_2^k + \cdots$. Then

$$\Gamma_-\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) v_\emptyset = \sum_\lambda s_\lambda(P) v_\lambda$$

where $\sum_\lambda$ runs over all possible partitions (of any number).

Introduce the following abbreviations

$$\Gamma_+ := \Gamma_+\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) \quad \text{and} \quad \Gamma_- := \Gamma_-\left(p'^{(1)}, \frac{p'^{(2)}}{2}, \frac{p'^{(3)}}{3}, \cdots\right)$$

where $P = (p_1, p_2, \ldots)$, $P' = (p'_1, p'_2, \ldots)$. Using the lemma we have

$$\left(v_\emptyset, \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- v_\emptyset\right) = \sum_{\lambda,\lambda'} s_\lambda(P) s_{\lambda'}(P') \left(v_\lambda, q^H e^{z\mathcal{F}_2} v_{\lambda'}\right)$$

$$= \sum_{\lambda,\lambda'} q^{|\lambda'|} s_\lambda(P) e^{z f_2(\lambda')} s_{\lambda'}(P') \left(v_\lambda, v_{\lambda'}\right) = \sum_{\lambda} q^{|\lambda|} s_\lambda(P) e^{z f_2(\lambda)} s_\lambda(P')$$

so

$$\mathfrak{H}^\bullet(\{p_j, p'_j\}, q, z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- \right\rangle$$

Summarizing, we obtained the following formulas

$$H^{\bullet}_{d,b}(\eta,\eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\left\langle \alpha_{\eta}\mathcal{F}_2^b\alpha_{-\eta'}\right\rangle$$

$$\mathfrak{h}^{\bullet}_{d}(\eta,\eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\left\langle \alpha_{\eta}e^{z\mathcal{F}_2}\alpha_{-\eta'}\right\rangle$$

$$\mathfrak{H}^{\bullet}(\{p_j,p'_j\},q,z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2}\Gamma_-\right\rangle$$

From this expression it was shown that the Hurwitz potential $\mathfrak{H}^{\bullet}$ is the τ-function for the Toda lattice hierarchy of Ueno and Takasaki, implying (infinitely) many recursive relations on $\mathfrak{H}^{\bullet}$. More about this in a forthcoming seminar!

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Introducing other operators it is possible to rewrite the previous correlators in such a way that their explicit computation is much simplified. In order to simplify the notation, denote $E_{ij} = :\psi_j \psi_k^*:$.

Definition

For $r \in \mathbb{Z}$ define

$$\mathcal{E}_r(z) := \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-r/2)} E_{k-r,k} + \frac{\delta_{r,0}}{\varsigma(z)}$$

where $\varsigma(z) := e^{z/2} - e^{-z/2}$.

The exponent $r/2$ in the definition is used in order to have

$$\mathcal{E}_r(z)^* = \mathcal{E}_{-r}(z)$$

The operators $\mathcal{E}$ satisfy the following commutation relation

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(\det \left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right])\mathcal{E}_{a+b}(z+w)$$

The operators $\mathcal{E}$ specialize to the standard bosonic operators on $\bigwedge^{\frac{\infty}{2}} V$

$$\alpha_k = \mathcal{E}_k(0) \; , \qquad k \neq 0$$

Let's explain the meaning of the additional term arising when $r = 0$. We would like to have an operator which acts as

$$\underline{k} \mapsto e^{zk}\underline{k}$$

and the natural candidate is

$$\sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}\psi_k\psi_k^*$$

However applying it to the vacuum the second definition gives for $\Re z > 0$

$$\sum_{k\in\mathbb{Z}_{\leq 0}-\frac{1}{2}} e^{zk} = e^{-z/2}\sum_{k\in\mathbb{Z}_{\leq 0}} e^{zk} = \frac{e^{-z/2}}{1-e^{-z}} = \frac{1}{\varsigma(z)}$$

and diverges for $\Re z = 0$. So we regularize

$$\sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}\psi_k\psi_k^* = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}E_{k,k} + \sum_{k\in\mathbb{Z}_{\leq 0}-\frac{1}{2}} e^{zk} \quad \mapsto \quad \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk}E_{k,k} + \frac{1}{\varsigma(z)}$$

For $r \neq 0$ we don't have any problem since $E_{k-r,k} = \psi_{k-r}\psi_k^*$.

Definition

For $k \in \mathbb{Z}_{>0}$ define operators $\mathcal{F}_k$ and $\mathcal{P}_k$ by

$$\mathcal{F}_k = \frac{\mathcal{P}_k}{k!} = [z^k]\mathcal{E}_0(z)$$

where $[z^k]\mathcal{E}_0(z)$ stands for the coefficient of z^k in $\mathcal{E}_0(z)$.

Lemma

For λ partition

$$\mathcal{P}_k v_\lambda = \mathbf{p}_k(\lambda) v_\lambda$$

where $\lambda = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$ and

$$\begin{aligned}\mathbf{p}_k(\lambda) &= \sum_{i=1}^{\infty}\left[\left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i + \frac{1}{2}\right)^k\right] + (1-2^{-k})\zeta(-k)\\ &= \sum_{i \in \mathfrak{S}_\bullet^+} i^k - \sum_{i \in \mathfrak{S}_\circ^-} i^k + (1-2^{-k})\zeta(-k)\end{aligned}$$

where ζ is the Riemann zeta function.

The meaning of the term $(1-2^{-k})\zeta(-k)$ is as before, that is we would like to have

$$\sum_{i=1}^{\infty}\left(\lambda_i - i + \frac{1}{2}\right)^k$$

but if we consider the partition $\emptyset$ we get the a divergent contribution.
Since $\zeta(-2)=0$ we get that the new definition of $\mathcal{F}_2$ coincides with the previous one.
The last formula we need is

Lemma

For $n \in \mathbb{Z}_{\geq 1}$

$$e^{z\mathcal{F}_2}\alpha_{-n}e^{-z\mathcal{F}_2} = \mathcal{E}_{-n}(nz)$$

Using $\mathcal{F}_2 v_\emptyset = 0$ we have

$$\mathfrak{h}^\bullet_d(\eta,\eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \alpha_\eta e^{z\mathcal{F}_2}\alpha_{-\eta'}\rangle = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)}\alpha_{\eta_i}\prod_{j=1}^{\ell(\eta')}\left(e^{z\mathcal{F}_2}\alpha_{-\eta'_j}e^{-z\mathcal{F}_2}\right)\rangle$$

$$= \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)}\mathcal{E}_{\eta_i}(0)\prod_{j=1}^{\ell(\eta')}\mathcal{E}_{-\eta'_j}(z\eta'_j)\rangle$$

Using the commutation relation for the operators $\mathcal{E}$ it is possible to compute the correlator in the previous formula by moving the operators with negative energy on the right and those of positive energy on the left. Then only operators of zero energy survive, so we get a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\mathcal{E}_0(n_1 z)\mathcal{E}_0(n_2 z)\cdots$ for some positive integers $m_1, m_2, \ldots, n_1, n_2, \ldots$. Now recall

$$\mathcal{E}_0(nz) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{nzk}E_{k-r,k} + \frac{1}{\varsigma(nz)}$$

and $E_{k-r,k}v_\emptyset = 0$ for all k, r. Hence $\langle\mathcal{E}_0(nz)\rangle = \varsigma^{-1}(nz)$ and we get as final result a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\varsigma^{-1}(n_1 z)\varsigma^{-1}(n_2 z)\cdots$.

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