Hurwitz numbers in half infinite wedge space

From geometry to correlators

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Let $f\colon X \to Y$ be a (non-constant holomorphic) map of (compact) and connected Riemann surfaces (hence surjective).
For any $y \in Y$, $f^{-1}(y) = \{x_1, \ldots, x_n\}$, we have $\deg f = \sum_i k_{x_i}$, where k_x denotes the ramification index of f at x. We call *ramification profile* of f at y the partition of d given by $(k_{x_1}, \ldots, k_{x_n})$.

Theorem (Riemann-Hurwitz)

Let h and g denote the genus of X and Y respectively. Then

$$\underbrace{2h-2}_{\chi(X)} = \underbrace{(2g-2)}_{\chi(Y)} \deg f + \sum_{x \in X} \nu_x$$

where $\nu_x := k_x - 1$ is the differential length of f at x.

Definition

$f\colon X \to Y$ and $g\colon \tilde{X} \to Y$ are *isomorphic* if there is $\phi\colon X \xrightarrow{\sim} \tilde{X}$ s.t. $f = g \circ \phi$.
An *automorphism* of $f\colon X \to Y$ is $\psi\colon X \xrightarrow{\sim} X$ s.t. $f = f \circ \psi$.
The group of automorphisms of f is denoted $\mathrm{Aut}(f)$.

Definition

Let Y RS of genus g, $B = \{b_1, \ldots, b_n\} \subset Y$, $d \geq 0$, $\eta_1, \ldots, \eta_n \vdash d$.
We define the *(degree d) connected Hurwitz number* to be

$$H^{\circ}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|}$$

and the *(disconnected) (degree d) Hurwitz number* to be

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|}$$

where both sums runs over isomorphism classes of *Hurwitz covers*, i.e. degree d holomorphic maps $f\colon X \to Y$ s.t.

- X is a compact RS of genus h
- the branch locus of f is B
- the ramification profile of f at b_i is η_i

In the case of connected Hurwitz numbers we further require X to be connected.

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Recall that maps of Riemann surfaces can be regarded as ramified covers.

Definition

A *ramified cover* is a continuous function between compact topological surfaces $f: X \to Y$ with a finite set $B \subset Y$ s.t.

- $f^{-1}(B)$ is finite,
- $f: X \setminus f^{-1}(B) \to Y \setminus B$ is a covering.

Vice-versa, we have

Theorem (Riemann's Existence Thm.)

Let Y compact Riemann surface, X^0 topological surface, $B = \{b_1, \ldots, b_n\} \subset Y$ marked points, $f^0: X^0 \to Y \setminus B$ covering of finite degree. Then there exists a unique (up to isomorphisms) compact Riemann surface X s.t.

- *X^0 is a dense subset of X,*
- *f^0 extends to a holomorphic map of Riemann surfaces $f: X \to Y$.*

This will be useful to reconstruct X and f given Y and some additional data.

Definition

The map $f\colon X \to Y$ is said y_0-*labelled* if $y_0 \in Y \setminus B_f$ and is chosen an isomorphism $L\colon f^{-1}(y_0) \xrightarrow{\sim} \{1, \ldots, d\}$. We also say that L is a *labelling*. An *isomorphism of* y_0-*labelled maps* (f, L) and (f', L') is an isomorphism of Riemann surfaces $\phi\colon X \to X'$ s.t.

$$f' \circ \phi = f \quad \text{and} \quad L' \circ \phi = L$$

Definition

A y_0-labelled map $f\colon X \to Y$ determines a group homomorphism

$$\Phi\colon \pi_1(Y \setminus B_f, y_0) \to S_d \ , \qquad \gamma \mapsto \sigma_\gamma$$

called *monodromy representation*.

Definition

Let Y be a connected Riemann surface of genus g, $y_0, b_1, \ldots, b_n \in Y$ points, $d \in \mathbb{Z}_{\geq 1}$ and $\eta_1, \ldots, \eta_n \vdash d$. A *monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$ is a group homomorphism $\Phi\colon \pi_1(Y \setminus \{b_1, \ldots, b_n\}, y_0) \to S_d$ s.t. if γ_k is a small loop around b_k then $\Phi(\gamma_k)$ has cycle type η_k.
If moreover the subgroup $\operatorname{im}\Phi \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$ we say that Φ is a *connected monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$.

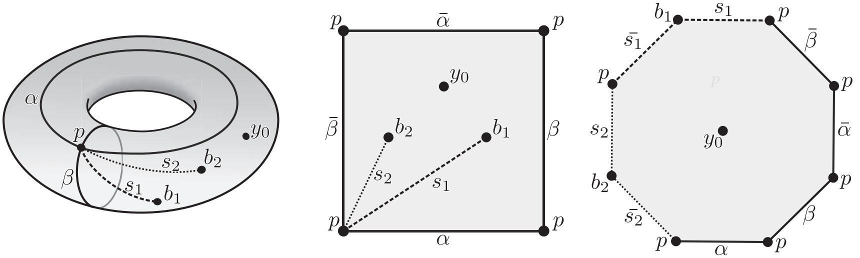
We have that $(f, L) \cong (f', L')$ imply $\Phi = \Phi'$.
Note also that if two labellings L, L' of $f\colon X \to Y$ are given, then $L' = \sigma \cdot L$ for some $\sigma \in S_d$ and $\Phi'(\gamma) = \sigma \cdot \Phi(\gamma) \cdot \sigma^{-1}$. So monodromy representations of (f, L) and (f, L') are of the same type.

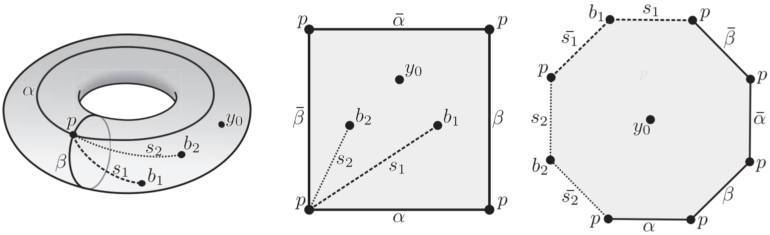
A degree d, y_0-labelled map (f, L) between compact Riemann surfaces s.t. the ramification profile over each branch point is η_i gives rise to a monodromy representation Φ of type $(g, d, \eta_1, \ldots, \eta_n)$. The monodromy representation will be connected if and only if X is connected.
Conversely:

Theorem

Let Y be a Riemann surface of genus g, Φ a monodromy representation of type $(g, d, \eta_1, \ldots, \eta_n)$, $B = \{b_1, \ldots, b_n\} \subset Y$ a finite subset. Then exists a y_0-labelled holomorphic map of Riemann surface covering Y with branch locus B whose associated monodromy is Φ. Such map is unique up to isomorphisms of y_0-labelled maps.

Sketch of proof In the proof of this result the Riemann's Existence theorem is fundamental. We construct explicitly the topological space X_0 and the covering $f^0\colon X^0 \to Y \setminus B$ in such a way that the map $f\colon X \to Y$ given by the Riemann's Existence theorem has the desired monodomy representation.
Take cycles $\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g$ in Y generating $\pi_1(Y, y_0)$ all containing a point $p \in Y$ in such a way that $Y \setminus \{\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g\}$ is the fundamental polygon describing Y. Denote by γ_i a loop containing y_0, winding once around b_i, never around the other elements of B, and fully contained in $Y \setminus \{\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g\}$.





Consider segments s_j connecting p with the points b_j. Open the previous polygon in correspondence of these segments, so that we get a new polygon

$$P := s_1\bar{s}_1 \cdots s_n\bar{s}_n\alpha_1\beta_1\bar{\alpha}_1\bar{\beta}_1 \cdots \alpha_g\beta_g\bar{\alpha}_g\bar{\beta}_g$$

Then it suffices to take d copies of P and glue their boundaries appropriately to produce X^0 in such a way that the natural projection to $Y \setminus B$ has the desired monodromy representation:

$$s_{j,k} \sim \bar{s}_{j,\Phi(\gamma_j)(k)}$$

$$\bar{\alpha}_{i,k} \sim \alpha_{i,\Phi(\beta_i)(k)}$$

$$\beta_{i,k} \sim \bar{\beta}_{i,\Phi(\alpha_i)(k)}$$

■

The previous theorem ensures that we have a bijection between isomorphism classes of y_0-labeled Hurwitz covers and monodromy representations of type $(g, d, \eta_1, \ldots, \eta_n)$. Then we can prove

Theorem

Let M° (resp $M^\bullet$) be the set of connected monodromy representations (resp. monodromy representations) of type $(g, d, \eta_1, \ldots, \eta_n)$. Then

$$H^\circ_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\circ|}{d!}$$

and

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\bullet|}{d!}$$

where h is determined by Riemann-Hurwitz.

Sketch of proof We give the proof in the connected case, the other case is analogous.

Take $f\colon X \to Y$ Hurwitz cover. Clearly there are $d!$ possible choices of a y_0-labelling $L\colon f^{-1}(y_0) \to \{1, \ldots, d\}$. An automorphism $\phi \in \mathrm{Aut}(f)$ is an isomorphism $\phi\colon X \xrightarrow{\sim} X$ satisfying $f = f \circ \phi$. In particular ϕ gives an isomorphism $(f, L) \cong (f, L')$ where $L' = \phi \cdot L := L \circ \phi^{-1}$. Here $\phi \cdot L$ denotes the left action of $\mathrm{Aut}(f)$ on the possible y_0-labellings of f. Such action is free (i.e. $\phi \cdot L = L$ imply $\phi = \mathrm{id}_X$) so the number of isomorphism classes of y_0-labelings of f is $d!/|\mathrm{Aut}(f)|$.
From the previous theorem isomorphism classes of y_0-labelled maps for the given f are in bijection with the distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. Therefore

$$m_f = \frac{d!}{|\mathrm{Aut}(f)|}$$

where m_f the number of distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. So we have

$$H^\circ_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|} = \sum_{[f]} \frac{m_f}{d!} = \frac{|M^\circ|}{d!}$$

where in the last step we used again previous theorem. ■

Although the information carried by connected Hurwitz numbers is usually more interesting for geometrical purposes, it turns out that it is easier to compute the (possibly disconnected) Hurwitz numbers. We will see later how it is possible to recover the connected Hurwitz numbers from the disconnected ones.

We mention that using some "degeneration formulas" (which heuristically correspond to shrink the Riemann surface Y producing nodal curves) all disconnected degree d Hurwitz numbers are determined in therms of Hurwitz numbers of the form $H^{\bullet}_{h\xrightarrow{d}0}(\eta_1, \eta_2, \eta_3)$. For this reason (and others) we later restrict our discussion to the case $g = 0$.

Using the last theorem the problem of computing degree d Hurwitz numbers can be translated into a problem in representation theory of the symmetric group S_d. In order to show this we need some facts about representation theory.

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Definition

The *group algebra* of the symmetric group S_d is the complex algebra

$$\mathbb{C}[S_d] := \left\{ \sum_{\sigma \in S_d} a_\sigma \sigma \;\middle|\; a_\sigma \in \mathbb{C} \right\}$$

We define *class algebra* of S_d the center of the group algebra

$$\mathcal{Z}\mathbb{C}[S_d] = \{ x \in \mathbb{C}[S_d] \;|\; yx = xy \quad \text{for all} \quad y \in \mathbb{C}[S_d] \}$$

Definition

A *class function* on S_d is a map $\alpha \colon S_d \to \mathbb{C}$ s.t. $\alpha(h^{-1}gh) = \alpha(h)\ \forall h \in S_d$. Let $\mathbb{C}_{\text{class}}$ denote the vector space of class functions on S_d, together with the following Hermitian inner product

$$(\alpha, \beta) := \frac{1}{d!} \sum_{\sigma \in S_d} \alpha(\sigma)\overline{\beta(\sigma)} = \frac{1}{d!} \sum_{C \subset S_d} |C|\alpha(C)\overline{\beta(C)}$$

where $\alpha, \beta \in \mathbb{C}_{\text{class}}$ and $\sum_C$ runs over the conjugacy classes of S_d.

Representations of S_d are described by their characters

Definition

Let ρ be a representation of S_d. The *character* of ρ is the class function $\chi_\rho \in \mathbb{C}_{\text{class}}$ defined by

$$\chi_\rho(\sigma) := \mathrm{tr}(\rho(\sigma))$$

Theorem

In terms of the inner product defined in $\mathbb{C}_{class}$ the characters of the complex irreducible representations of S_d are orthonormal:

$$(\chi_{\rho_1}, \chi_{\rho_2}) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \ncong \rho_2 \end{cases}$$

where ρ_1, ρ_2 are irreducible representations.

Theorem

To each partition $\lambda \vdash d$ corresponds a unique irreducible representation V_λ of S_d. The corresponding character, denoted χ^λ, is given by Frobenius formula.

For $\eta, \lambda \vdash d$, denote C_η the conjugacy class in S_d of elements of cycle type η, and define the following elements in $\mathcal{Z}\mathbb{C}[S_d]$:

$$c_\eta := \sum_{\sigma \in C_\eta} \sigma \quad \text{and} \quad e_\lambda := \sum_{\sigma \in S_d} \chi^\lambda(\sigma)\sigma$$

Then one can prove

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\eta \vdash d} \langle c_\eta \rangle_{\mathbb{C}} = \bigoplus_{\lambda \vdash d} \langle e_\lambda \rangle_{\mathbb{C}}$$

From characters orthogonality (and the fact that $\overline{\chi^\lambda} = \chi^\lambda$) we get

$$e_{\lambda_i} \cdot e_{\lambda_j} = \begin{cases} e_{\lambda_i} & \text{if } e_{\lambda_i} = e_{\lambda_j} \\ 0 & \text{otherwise} \end{cases}$$

for this reason $\{e_\lambda\}$ is called *idempotent basis* of $\mathcal{Z}\mathbb{C}[S_d]$.
The formulas for the change of basis are given by the characters

$$e_\lambda = \frac{\dim \lambda}{d!} \sum_{\eta \vdash d} \chi^\lambda(C_\eta) c_\eta \qquad \text{and} \qquad c_\eta = |C_\eta| \sum_{\lambda \vdash d} \frac{\chi^\lambda(C_\eta)}{\dim \lambda} e_\lambda$$

where $\dim \lambda := \dim V_\lambda$.

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Recall that

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\bullet|}{d!}$$

For $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ where $i \in \mathbb{Z}_{\geq 1}$ appears a_i times in the partition, $\sum_i i a_i = d$, the size of the centralizer of C_η is given by

$$\mathfrak{z}(\eta) = \frac{d!}{|C_\eta|} = \prod_{i=1}^{\ell(\eta)} a_i!\, i^{a_i}$$

Definition

Let $d \in \mathbb{Z}_{\geq 1}$, $\eta \vdash d$. We define the *kommutator* to be the element

$$\mathfrak{K} := \sum_{\eta \vdash d} \mathfrak{z}(\eta) c_\eta^2 \in \mathcal{Z}\mathbb{C}[S_d]$$

Theorem

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{1}{d!}[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$$

where $[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$ *denotes the coefficient of* c_e *in* $\mathfrak{K}^g c_{\eta_n} \ldots c_{\eta_2} c_{\eta_1}$.

Theorem

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{1}{d!}[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$$

where $[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$ *denotes the coefficient of* c_e *in* $\mathfrak{K}^g c_{\eta_n} \ldots c_{\eta_2} c_{\eta_1}$.

By changing basis from the conjugacy basis to the idempotent basis we get

Theorem (Burnside's Character Formula)

$$H^{\bullet}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^{2-2g} \prod_{i=1}^{n} f_{C_i}(\lambda)$$

where $C_i := C_{\eta_i}$ *and*

$$f_{C_\eta}(\lambda) := |C_\eta| \frac{\chi^\lambda(C_\eta)}{\dim \lambda}$$

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In the following we will restrict ourselves to the case of $g = 0$. Recall that there are some degeneration formulas which allows to express all the Hurwitz numbers in terms of those for $g = 0$.
Since Riemann-Hurwitz formula fixes h in terms of $(d, \eta_1, \ldots, \eta_n)$, we denote

$$H^\bullet_d(\eta_1, \ldots, \eta_n) := H^\bullet_{h \xrightarrow{d} 0}(\eta_1, \ldots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^n f_{C_i}(\lambda)$$

Let b be the number of branch points which have simple ramification, i.e. $\eta = (2)$. We denote

$$\begin{aligned} H^\bullet_{d,b}(\eta_1, \ldots, \eta_m) &:= H^\bullet_d(\eta_1, \ldots, \eta_m, \overbrace{(2), \ldots, (2)}^{b \text{ times}}) \\ &= \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 f_2(\lambda)^b \prod_{i=1}^m f_{C_i}(\lambda) \end{aligned}$$

where $f_2 := f_{C_{(2)}}$.Analogous definitions hold for connected Hurwitz numbers, replacing $H^\bullet$ with H°.

Rather than considering the different Hurwitz numbers separately, it worth to collect them together into generating functions.Fix $m \in \mathbb{Z}_{\geq 0}$ to be the number of branch points with non-simple ramification profile.

Definition

Let $\{p_{i,j}\}$, q and z be some variables, $i \in \{1, \dots, m\}$, $j \in \mathbb{Z}_{\geq 0}$. We define the *Hurwitz potential* to be

$$\mathfrak{H}^\bullet(p_{i,j}, q, z) := \sum_{d,b=0}^{\infty} q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} H^\bullet_{d,b}(\eta_1, \dots, \eta_m)$$

where for $\eta_i = (l_1, \dots, l_k) \vdash d$ we defined

$$p_i^{\eta_i} := (p_{i,1}^{l_1} + \cdots + p_{i,d}^{l_1}) \cdots (p_{i,1}^{l_k} + \cdots + p_{i,d}^{l_k})$$

We also introduce the *modified Hurwitz potential* to be

$$\mathfrak{h}^\bullet_d(\eta_q, \dots, \eta_m, z) := \sum_{b=0}^{\infty} \frac{z^b}{b!} H^\bullet_{d,b}(\eta_1, \dots, \eta_m)$$

Analogous definitions hold for the *(modified) connected Hurwitz potential* $\mathfrak{H}^\circ$ and $\mathfrak{h}^\circ$.

For fixed i, the polynomials of the form p^η for all partitions $\eta \vdash d$ form a basis for the space of all homogeneous polynomials of degree d in d variables with rational coefficients. They are called *power sum polynomials*. Therefore, given $\mathfrak{H}^\bullet$, it can be expanded uniquely giving all the Hurwitz numbers.

The first advantage of considering generating function in place of the single Hurwitz numbers is

Theorem

$$\mathfrak{H}^\bullet = e^{\mathfrak{H}^\circ}$$

In order to simplify the notation, in the following we denote $\chi^\lambda_\eta := \chi^\lambda(C_\eta)$. For any $\lambda \vdash d$, we have the following relation, which can be regarded as corollary of Frobenius formula

$$s_\lambda(p_1, \ldots, p_d) = \frac{1}{d!} \sum_{\eta \vdash d} \chi^\lambda_\eta |C_\eta| p^\eta$$

where s_λ is the *Schur polynomial* associated to λ, it is homogeneous polynomial of degree d. Putting formulas together we get

$$\begin{aligned}
&\mathfrak{H}^\bullet(p_{i,j}, q, z) = \\
&= \sum_{d,b=0}^{\infty} q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^2 f_2(\lambda)^b \prod_{i=1}^m f_{C_i}(\lambda) \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \prod_{i=1}^m \frac{1}{d!} |C_{\eta_i}| \chi^\lambda_{\eta_i} \\
&= \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \prod_{i=1}^m s_\lambda(p_{i,1}, \ldots, p_{i,d})
\end{aligned}$$

The case $m = 2$ corresponds to the so called *double Hurwitz numbers*, which are the ones we are interested in

$$\mathfrak{H}^{\bullet}(\{p_j, p'_j\}, q, z) = \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} s_\lambda(p_1, \dots, p_d) s_\lambda(p'_1, \dots, p'_d)$$
$$= \sum_{\lambda} q^{|\lambda|} e^{z f_2(\lambda)} s_\lambda(P) s_\lambda(P')$$

We denoted $P := \{p_1, p_2, \dots\}$ and $P' := \{p'_1, p'_2, \dots\}$ (of course s_λ depends only on the first $|\lambda|$ variables of each set). For $m = 2$ we also have

$$H^{\bullet}_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}$$
$$\mathfrak{h}^{\bullet}_{d}(\eta, \eta', z) = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta e^{z f_2(\lambda)} \chi^\lambda_{\eta'}$$

Before moving on, notice that for double Hurwitz numbers the Riemann - Hurwitz formula gives

$$2h = b + 2 - \ell(\eta) - \ell(\eta')$$

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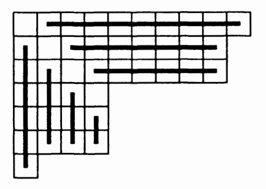
Given a partition λ, define its rank r to be the length of the diagonal of its Young diagram. Let a_i and b_i be the number of boxes to the right and below of the i-th box of the diagonal, reading from the upper right to the lower left.

We call $\begin{pmatrix} a_1 a_2 \dots a_n \\ b_1 b_2 \dots b_n \end{pmatrix}$ and $\begin{pmatrix} a'_1 a'_2 \dots a'_n \\ b'_1 b'_2 \dots b'_n \end{pmatrix}$ the *Frobenius notation* and the *modified Frobenius notation* of the partition respectively, where $a'_i = a_i + 1/2$ and $b'_i = b_i + 1/2$.



$\lambda = (10, 9, 9, 4, 4, 4, 1)$

Frobenius notation $\begin{pmatrix} 9 & 7 & 6 & 0 \\ 2 & 3 & 4 & 6 \end{pmatrix}$

modified Frobenius notation $\begin{pmatrix} 9.5 & 7.5 & 6.5 & 0.5 \\ 2.5 & 3.5 & 4.5 & 6.5 \end{pmatrix}$

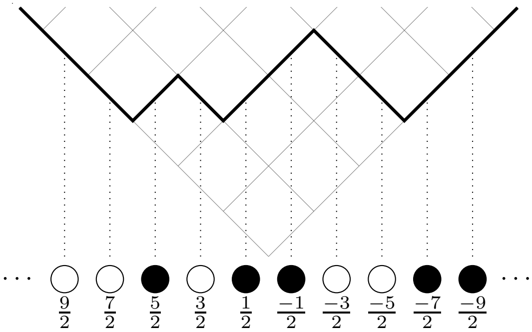
Lemma (Frobenius formula for $C_{(2)}$)

$$\chi^\lambda(C_{(2)}) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} (a_i(a_i+1) - b_i(b_i+1)) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} ((a'_i)^2 - (b'_i)^2)$$

From Frobenius formula and $|C_{(2)}| = \binom{d}{2}$ it follows that

$$f_2(\lambda) = \frac{|C_{(2)}|}{\dim \lambda}\chi^\lambda(C_{(2)}) = \frac{1}{2}\sum_{i=1}^{r}((a_i')^2 - (b_i')^2)$$

Draw the Young diagram of λ rotated by 135° over the real line with opposite orientation as in the picture.



Suppose that λ is made of $k = \ell(\lambda)$ cycles of lengths $\{\lambda_1, \dots, \lambda_k\}$ where $\lambda_1 \geq \lambda_2 \geq \dots$. Now consider the ordered set $\{\tilde{\lambda}_i\}_{i \in \mathbb{Z}_{\geq 0}} = \{\lambda_1, \dots, \lambda_k, 0, 0, \dots\}$ ending with infinitely many zeros. Then it is clear that black stones are placed in correspondence to elements of

$$\mathfrak{S}_\bullet(\lambda) := \{\tilde{\lambda}_i - i + 1/2\} \subset \mathbb{Z} + \frac{1}{2}$$

and white stones in correspondence to elements of $\mathfrak{S}_\circ(\lambda) := (\mathbb{Z} + \frac{1}{2}) \setminus \mathfrak{S}_\bullet(\lambda)$. Moreover, the coefficients in the modified Frobenius notation are given by

$$\{a'_i\} = \mathfrak{S}^+_\bullet(\lambda) := \mathfrak{S}_\bullet \cap (\mathbb{Z}_{\geq 0} + 1/2)$$
$$\{b'_i\} = \mathfrak{S}^-_\circ(\lambda) := \mathfrak{S}_\circ \cap (\mathbb{Z}_{\leq 0} - 1/2) = (\mathbb{Z}_{\leq 0} - 1/2) \setminus \mathfrak{S}_\bullet(\lambda)$$

Hence we obtained

$$f_2(\lambda) = \sum_{k \in \mathfrak{S}^+_\bullet} \frac{k^2}{2} - \sum_{k \in \mathfrak{S}^-_\circ} \frac{k^2}{2}$$

Notice also that

$$|\lambda| = \lambda_1 + \dots + \lambda_k = a'_1 + \dots + a'_k + b'_1 + \dots + b'_k = \sum_{k \in \mathfrak{S}^+_\bullet} k - \sum_{k \in \mathfrak{S}^-_\circ} k$$

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Definition

Let V be a vector space with basis $\{\underline{k}\}$, $k \in \mathbb{Z} + \frac{1}{2}$. We define the vector space $\bigwedge^{\frac{\infty}{2}} V$ to be spanned by vectors

$$v_S := \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots$$

where $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ is a subset s.t. both

$$S^+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \quad \text{and} \quad S^- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \setminus S$$

are finite. We equip $\bigwedge^{\frac{\infty}{2}} V$ with the inner product $(-,-)$ in which the basis $\{v_S\}$ (for all possible choices of S) is orthonormal.

For our purposes S is identified with $\mathfrak{S}_\bullet(\lambda)$ defined before, we denote by v_λ the vector in $\bigwedge^{\frac{\infty}{2}} V$ corresponding to $S = \mathfrak{S}_\bullet(\lambda)$. In this case $S^+ = \mathfrak{S}_\bullet^+$ and $S^- = \mathfrak{S}_\circ^-$, and finiteness condition simply corresponds to the fact that we have finitely many black stones on the left of the origin and finitely many white stones on the right or the origin, which is automatic for $\mathfrak{S}_\bullet(\lambda)$.

Definition

For all $k \in \mathbb{Z} + \frac{1}{2}$ define the operators ψ_k and ψ_k^* on $\bigwedge^{\frac{\infty}{2}} V$ by

$$\psi_k(v) := \underline{k} \wedge v \quad \text{and} \quad (v', \psi_k^* v) = (\psi_k v', v)$$

where $v, v' \in \bigwedge^{\frac{\infty}{2}} V$.

From the definitions we have

$$\psi_j \psi_k^* + \psi_k^* \psi_j = \delta_{jk} \quad \text{and} \quad \psi_j \psi_k + \psi_k \psi_j = \psi_j^* \psi_k^* + \psi_k^* \psi_j^* = 0$$

which are known as *fermionic commutation relations*.
These operators are related to the usual creation and annihilation operators for the *Fermi sea*, by identifying black stones with electrons and white stones with empty energy levels (but in physical literature ψ_k and ψ_k^* are exchanged).
Finiteness condition amounts to considering state which are finite energy excitation of the vacuum state.
We define *vacuum* the vector $v_\emptyset := \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \underline{-\frac{5}{2}} \wedge \ldots$, where all stones are black (resp. white) on the right (resp. left) of the origin. This is exactly the vacuum in the Fermi sea model.

Definition

Introduce the *normal ordered product*

$$:\psi_j\psi_k^*: \;:= \begin{cases} \psi_j\psi_k^* & k > 0 \\ -\psi_k^*\psi_j & k < 0 \end{cases}$$

For $k > 0$ (resp. $k < 0$), $:\psi_k\psi_k^*:$ gives 1 (resp. -1) if we have a black (resp. white) stone in k and zero otherwise.
For $j \neq k$ the normal ordered product is the ordinary product due to the fermionic commutation relation $:\psi_j\psi_k^*: = \psi_j\psi_k^* = -\psi_k^*\psi_j$.
The effect of $:\psi_j\psi_k^*:$ for $j \neq k$ is to take the black stone in k and move it to the position j, unless there is no black stone in k or the position j is already occupied: in such cases it gives the zero vector.
Finiteness condition ensures that any operator of the form $\sum_{j,k} a_{jk} :\psi_j\psi_k^*:$ is well defined for any choice of the coefficients $a_{j,k}$.

Definition

$$\mathcal{F}_2 = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} :\psi_k\psi_k^*:$$

It satisfies

$$\mathcal{F}_2 v_\lambda = f_2(\lambda) v_\lambda$$

Definition

We define the following operators, called *energy operator* and *charge operator* respectively

$$H := \sum_{k\in\mathbb{Z}+1/2} k :\psi_k\psi_k^*: \qquad C := \sum_{k\in\mathbb{Z}+1/2} :\psi_k\psi_k^*:$$

The physical interpretation of these operators is obvious when we regard a black stone at position $k>0$ as an electron of energy k and charge $+1$ and a white stone at position $k<0$ as a positron of energy $-k$ and charge -1.
We have

$$H v_\lambda = |\lambda| v_\lambda$$

For the charge operator we have

$$Cv_S = (|S^+| - |S^-|)v_s$$

Suppose $Cv_S = c\,v_S$, then $|S^+| = |S^-| + c$. Pick $k \in S$, $k < \min(S^-)$, this means that the stone at k and all those on its right are black. Then $|S^+| = |S^-| + c$ implies that $k = s_{-(k-1/2)+c}$. Hence

$$c = \lim_{i \to +\infty} \left(s_i + i - \frac{1}{2} \right)$$

In particular, all vectors corresponding to Young diagrams ($S = \mathfrak{S}_\bullet(\lambda)$) have zero charge. Conversely, any vector of zero charge can be obtained from a Young diagram by taking the partition $\lambda_i = s_i + i - \frac{1}{2}$ (omitting the infinitely many λ_i which vanish). We denote by $\bigwedge_0^{\frac{\infty}{2}} V$ the subspace of $\bigwedge^{\frac{\infty}{2}} V$ of zero charge. From this we get that

$$\{v_\lambda \,\big|\, \lambda \vdash d\}$$

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ of energy d. Considering all positive values of d we get a basis of the whole charge zero subspace $\bigwedge_0^{\frac{\infty}{2}} V$.

Definition

For $n \in \mathbb{Z}_{\neq 0}$ define

$$\alpha_n := \sum_{k \in \mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*:$$

They satisfy $\alpha_n \underline{k} = \underline{k-n}$ and the *Heisenberg commutation relations*

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}$$

From the definition it follows that α_n and α_{-n} are adjoint for every $n \in \mathbb{Z}_{\neq 0}$. The effect of this operator in a certain configuration of stones (or electrons) is the following. Pick one and move it to the right by n positions. If the new position is a white stone, make it black and replace the initial black stone by a white one. If the new position is occupied by a black stone, then the operator gives the zero vector (Pauli principle). Then repeat this for all black stones and sum all the resulting vectors to give the final state.

Definition

For any partition $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash |\eta|$ define

$$\alpha_\eta := \prod_{i=1}^{\ell(\eta)} \alpha_{\eta_i} \quad \text{and} \quad \alpha_{-\eta} := \prod_{i=1}^{\ell(\eta)} \alpha_{-\eta_i}$$

The Heisenberg commutation relation ensures that the ordering of the multiplied operators is not relevant and also that α_η and $\alpha_{-\eta}$ are adjoint. Note also that α_η (resp. $\alpha_{-\eta}$) decreases (resp. increases) the energy of states by $|\eta|$. Moreover, it is possible to prove that

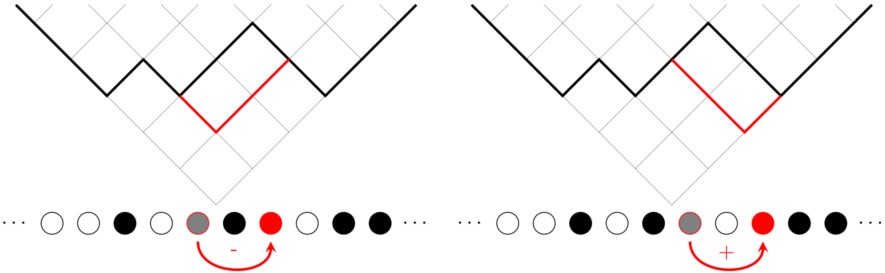
$$\{\alpha_{-\eta} v_\emptyset \,|\, \eta \vdash d\}$$

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ of energy d, orthogonal with respect to the inner product $(-,-)$.

The two bases of $\bigwedge_0^{\frac{\infty}{2}} V$ can be related as follows. Let λ be any partition and $n \in \mathbb{Z}_{>0}$. Then graphically we can see that

$$\alpha_n v_\lambda = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \text{ skew hook} \\ |\lambda/\lambda'| = n}} (-1)^{r(\lambda/\lambda')-1} v_{\lambda'}$$

where $r(\lambda/\lambda')$ is the number of rows of λ touched by λ/λ'. Indeed moving a black stone corresponds to remove a skew hook of length n from the partition (the strip of length 2 in the picture).



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where $r(\lambda/\lambda')$ is the number of rows of λ touched by λ/λ'. Compare to

Lemma (*Murnaghan-Nakayama rule*)

$$\chi^\lambda_{\{(n),\eta\}} = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \textit{ skew hook} \\ |\lambda/\lambda'| = n}} (-1)^{r(\lambda/\lambda')-1} \chi^{\lambda'}_\eta$$

Using these identities we get that for any two partitions λ and η s.t. $|\lambda| = |\eta|$

$$\alpha_\eta v_\lambda = \chi^\lambda_\eta v_\emptyset$$

and taking the adjoint

$$\alpha_{-\eta} v_\emptyset = \sum_{\lambda \vdash |\eta|} \chi^\lambda_\eta v_\lambda$$

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For $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ and $\eta' = \{\eta'_1, \ldots, \eta'_{\ell(\eta')}\} \vdash d$ we obtain

$$\begin{aligned}\left(v_\emptyset, \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} v_\emptyset\right) &= \sum_{\lambda \vdash d} \chi^\lambda_{\eta'} \left(v_\emptyset, \alpha_\eta \mathcal{F}_2^b v_\lambda\right) = \sum_{\lambda \vdash d} f_2(\lambda)^b \chi^\lambda_{\eta'} \left(v_\emptyset, \alpha_\eta v_\lambda\right) \\ &= \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} \left(v_\emptyset, v_\emptyset\right) = \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}\end{aligned}$$

Therefore

$$H^\bullet_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \left\langle \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} \right\rangle$$

where the *correlator* of the operator A is defined by $\langle A \rangle := (v_\emptyset, A v_\emptyset)$. Similarly

$$\mathfrak{h}^\bullet_d(\eta, \eta', z) = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta e^{z f_2(\lambda)} \chi^\lambda_{\eta'} = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \left\langle \alpha_\eta e^{z \mathcal{F}_2} \alpha_{-\eta'} \right\rangle$$

To rewrite the Hurwitz potential we should be able to recover the Schur polynomials from the half infinite wedge formalism.

Definition

Given a sequence $t = (t_1, t_2, \dots)$ define

$$\Gamma_\pm(t) = \exp\left(\sum_{n=1}^{\infty} t_n \alpha_{\pm n}\right)$$

Note that $\Gamma_+(t)$ and $\Gamma_-(t)$ are adjoint.

Lemma

Consider a set of variables $P = (p_1, p_2, \dots)$ and denote $p^{(k)} = p_1^k + p_2^k + \cdots$. Then

$$\Gamma_-\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) v_\emptyset = \sum_\lambda s_\lambda(P) v_\lambda$$

where $\sum_\lambda$ runs over all possible partitions (of any number).

Introduce the following abbreviations

$$\Gamma_+ := \Gamma_+\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) \quad \text{and} \quad \Gamma_- := \Gamma_-\left(p'^{(1)}, \frac{p'^{(2)}}{2}, \frac{p'^{(3)}}{3}, \cdots\right)$$

where $P = (p_1, p_2, \ldots)$, $P' = (p'_1, p'_2, \ldots)$. Using the lemma we have

$$\left(v_\emptyset, \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- v_\emptyset\right) = \sum_{\lambda,\lambda'} s_\lambda(P) s_{\lambda'}(P') \left(v_\lambda, q^H e^{z\mathcal{F}_2} v_{\lambda'}\right)$$

$$= \sum_{\lambda,\lambda'} q^{|\lambda'|} s_\lambda(P) e^{z f_2(\lambda')} s_{\lambda'}(P') \left(v_\lambda, v_{\lambda'}\right) = \sum_{\lambda} q^{|\lambda|} s_\lambda(P) e^{z f_2(\lambda)} s_\lambda(P')$$

Hence

$$\mathfrak{H}^\bullet(\{p_j, p'_j\}, q, z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- \right\rangle$$

Summarizing, we obtained the following formulas

$$H^{\bullet}_{d,b}(\eta,\eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\left\langle \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'}\right\rangle$$

$$\mathfrak{h}^{\bullet}_{d}(\eta,\eta',z) = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\left\langle \alpha_\eta e^{z\mathcal{F}_2} \alpha_{-\eta'}\right\rangle$$

$$\mathfrak{H}^{\bullet}(\{p_j,p'_j\},q,z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2}\Gamma_-\right\rangle$$

From this expression it was shown that the Hurwitz potential $\mathfrak{H}^{\bullet}$ is the τ-function for the Toda lattice hierarchy of Ueno and Takasaki, implying (infinitely) many recursive relations on $\mathfrak{H}^{\bullet}$. More about this in a forthcoming seminar!

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Definition

For $r \in \mathbb{Z}$ define

$$\mathcal{E}_r(z) := \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-r/2)} E_{k-r,k} + \frac{\delta_{r,0}}{\varsigma(z)}$$

where $E_{ij} = :\psi_j \psi_k^*:$ and $\varsigma(z) := e^{z/2} - e^{-z/2}$.

The exponent $r/2$ in the definition is used in order to have

$$\mathcal{E}_r(z)^* = \mathcal{E}_{-r}(z)$$

The operators $\mathcal{E}$ satisfy the following commutation relation

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(\det \left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]) \mathcal{E}_{a+b}(z+w)$$

The operators $\mathcal{E}$ specialize to the standard bosonic operators on $\bigwedge^{\frac{\infty}{2}} V$

$$\alpha_k = \mathcal{E}_k(0) \ , \qquad k \neq 0$$

Lemma

For $n \in \mathbb{Z}_{\geq 1}$

$$e^{z\mathcal{F}_2} \alpha_{-n} e^{-z\mathcal{F}_2} = \mathcal{E}_{-n}(nz)$$

Using $\mathcal{F}_2 v_\emptyset = 0$ we have

$$\mathfrak{h}^\bullet_d(\eta,\eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \alpha_\eta e^{z\mathcal{F}_2}\alpha_{-\eta'}\rangle = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)} \alpha_{\eta_i} \prod_{j=1}^{\ell(\eta')} \left(e^{z\mathcal{F}_2}\alpha_{-\eta'_j} e^{-z\mathcal{F}_2}\right)\rangle$$

$$= \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)} \mathcal{E}_{\eta_i}(0) \prod_{j=1}^{\ell(\eta')} \mathcal{E}_{-\eta'_j}(z\eta'_j)\rangle$$

Using the commutation relation for the operators $\mathcal{E}$ it is possible to compute the correlator in the previous formula by moving the operators with negative energy on the right and those of positive energy on the left. Then only operators of zero energy survive, so we get a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\mathcal{E}_0(n_1 z)\mathcal{E}_0(n_2 z)\cdots$ for some positive integers $m_1, m_2, \ldots, n_1, n_2, \ldots$. Now recall

$$\mathcal{E}_0(nz) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{nzk} E_{k-r,k} + \frac{1}{\varsigma(nz)}$$

and $E_{k-r,k} v_\emptyset = 0$ for all k, r. Hence $\langle \mathcal{E}_0(nz)\rangle = \varsigma^{-1}(nz)$ and we get as final result a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\varsigma^{-1}(n_1 z)\varsigma^{-1}(n_2 z)\cdots$.

Bibliography I

Miwa, T., M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*. 2000.

Okounkov, A. *Toda equations for Hurwitz numbers*. 2000.

– .*Infinite wedge and random partitions*. 2001.

Fulton, W. and J. Harris. *Representation Theory*. 2004.

Okounkov, A. and R. Pandharipande. *Gromov–Witten theory, Hurwitz theory, and completed cycles*. 2006.

– .*The equivariant Gromov–Witten theory of* P^1. 2006.

Johnson, P. *Double Hurwitz numbers via the infinite wedge*. 2015.

Macdonald, I.G. and A.V. Zelevinsky. *Symmetric Functions and Hall Polynomials*. 2015.

Cavalieri, R. and E. Miles. *Riemann Surfaces and Algebraic Curves: A First Course in Hurwitz Theory*. 2016.