Chapter 1

Hurwitz Numbers

1.1 Maps of Riemann surfaces and Hurwitz numbers

In the following we will assume the Riemann surfaces that we will consider to be compact and connected. Let $f\colon X \to Y$ be a non-constant holomorphic map of Riemann surfaces (hence surjective[1])

Definition 1.1.
- The *ramification index* of f at $x \in X$ is the integer $k_x \in \mathbb{Z}_{\geq 1}$ s.t. f is locally of the form z^k, where z is a local coordinate centered in x.
- The *differential length* of f at $x \in X$ is $\nu_x = k_x - 1$.
- A point $x \in X$ s.t. $\nu_x > 0$ is called a *ramification point*. The *ramification locus* of f is the set of its ramification points. If $\nu_x = 0$ we say that f is *unramified* at x. If $\nu_x = 1$ we say that f has *simple ramification* at x.
- If $x \in X$ is a ramification point, then $f(x) \in Y$ is called a *branch point*. The *branch locus B_f* of f is the set of its branch points.

The ramification locus and branch locus are finite sets of X and Y respectively.[2] We assume f non-constant. For any $y, y' \in Y \setminus B_f$ we have $|f^{-1}(y)| = |f^{-1}(y')|$.[3] We call *degree* of f the integer $\deg f := |f^{-1}(y)|$ for any $y \in Y \setminus B_f$. Then:

Theorem 1.2 (Riemann-Hurwitz [**CM**]). *Let g_X and g_Y denote the genus of X and Y respectively. Then*

$$\underbrace{2g_X - 2}_{\chi(X)} = \underbrace{(2g_Y - 2)}_{\chi(Y)} \deg f + \sum_{x \in X} \nu_x$$

We also have that for any $y \in Y$

$$\deg f = \sum_i k_{x_i}$$

where i labels the elements of $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Denote $d := \deg f$.

Definition 1.3. Let $y \in Y$, $f^{-1} = \{x_1, \ldots, x_n\}$. We call *ramification profile* of f at y the partition of d given by $(k_{x_1}, \ldots, k_{x_n})$. We say that f is *unramified / simply ramified / fully ramified* if its ramification profile is $(1, \ldots, 1)$ / $(2, 1, \ldots, 1)$ / (d) respectively.

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[1]**CM**
[2]**CM**
[3]**CM**

Definition 1.4. Two holomorphic maps of Riemann surfaces $f\colon X \to Y$ and $g\colon \tilde{X} \to Y$ are called *isomorphic* if there is an isomorphism of Riemann surfaces $\phi\colon X \to \tilde{X}$ s.t. $f = g \circ \phi$.

An *automorphism* of $f\colon X \to Y$ is an automorphism $\psi\colon X \to X$ s.t. $f = f \circ \psi$. The group of automorphisms of f is denoted $\mathrm{Aut}(f)$.

Definition 1.5. Let Y be a connected compact Riemann surface of genus g, $B = \{b_1, \ldots, b_n\} \subset Y$ finite subset, $d \in \mathbb{Z}_{\geq 1}$, $\eta_1, \ldots, \eta_n \vdash d$. We define the *(degree d) connected Hurwitz number* to be

$$H^{\circ}_{h\overset{d}{\to}g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|}$$

and the *(degree d) Hurwitz number* to be

$$H^{\bullet}_{h\overset{d}{\to}g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\operatorname{Aut}(f)|} \tag{1.6}$$

where both sums runs over isomorphism classes of holomorphic maps $f\colon X \to Y$ s.t.
  • X is a compact Riemann surface of genus h[4]
  • the branch locus of f is B
  • the ramification profile of f at b_i is η_i
In the case of connected Hurwitz numbers we further require X to be connected. Numbers (1.6) are also called *disconnected Hurwitz number* since X is allowed to be disconnected.

Note that for any $f\colon X \to Y$ as in the definition above we have

$$\sum_{x \in X} \nu_x = \sum_{i=1}^{n}(d - \ell(\eta_i)) = nd - \sum_{i=1}^{n} \ell(\eta_i)$$

where $\ell(\eta_i)$ denotes the length of the partition η_i (i.e. the number of cycles in η_i), so due to Riemann-Hurwitz theorem we have

$$2h - 2 = (2g - 2)d + nd - \sum_{i=1}^{n} \ell(\eta_i) \tag{1.7}$$

In particular the ramification profile of $f : X \to Y$ and the genus of Y uniquely determines the genus of X.

1.2 Maps of Riemann surfaces as ramified covers

Maps of Riemann surfaces are ramified covers:

Definition 1.8. A *ramified cover* is a continuous function between compact topological surfaces $f\colon X \to Y$ with a finite set $B \subset Y$ s.t.
  • $f^{-1}(B)$ is finite,
  • $f\colon X \setminus f^{-1}(B) \to Y \setminus B$ is a covering.

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[4]For X disconnected, its genus is defined by $\chi = 2g - 2$, where the Euler characteristic χ is naturally additive under disjoint unions. The Riemann-Hurwitz formula then applies identically in also for X disconnected.

Vice-versa, we have

Theorem 1.9 (Riemann's Existence Thm. [**CM**]). *Let Y compact Riemann surface, X^0 topological surface, $\{b_1, \ldots, b_n\} \subset Y$ finite subset, $f^0 \colon X^0 \to Y \setminus \{b_1, \ldots, b_n\}$ covering of finite degree. Then there exists a unique (up to isomorphisms) compact Riemann surface X s.t.*

- X^0 *is a dense subset of X,*
- f^0 *extends to a holomorphic map of Riemann surfaces $f \colon X \to Y$.*

Definition 1.10. The map $f \colon X \to Y$ is said y_0-*labelled* if $y_0 \in Y \setminus B_f$ and is chosen an isomorphism $L \colon f^{-1}(y_0) \to \{1, \ldots, d\}$. We also say that L is a *labelling*. An *isomorphism of y_0-labelled maps* (f, L) and (f', L') is an isomorphism of Riemann surfaces $\phi \colon X \to X'$ s.t.

$$f' \circ \phi = f \quad \text{and} \quad L' \circ \phi = L$$

Definition 1.11. A y_0-labelled map $f \colon X \to Y$ determines a group homomorphism

$$\Phi \colon \pi_1(Y \setminus B_f, y_0) \to S_d , \qquad \gamma \mapsto \sigma_\gamma$$

called *monodromy representation*.

We have that $(f, L) \cong (f', L')$ imply $\Phi = \Phi'$.

Now consider $b \in B_f$ and $\gamma \in \pi_1(Y \setminus B_f, y_0)$ simple loop winding once around b (and with zero winding number around the other branch points). If the ramification profile of f at b is $\eta = (k_1, \ldots, k_l)$, then σ_γ has cycle type η (to see this recall that the local expression of f around ramification points is z^k and consider a circle around y_0 of unit radius in the chart).

Definition 1.12. Let Y be a connected Riemann surface of genus g, $y_0, b_1, \ldots, b_n \in Y$ points, $d \in \mathbb{Z}_{\geq 1}$ and $\eta_1, \ldots, \eta_n \vdash d$. A *monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$ is a group homomorphism $\Phi \colon \pi_1(Y \setminus \{b_1, \ldots, b_n\}, y_0) \to S_d$ s.t. if γ_k is a small loop around b_k then $\Phi(\gamma_k)$ has cycle type η_k.

If moreover the subgroup $\operatorname{im} \Phi \subset S_d$ acts transitively on $\{1, 2, \ldots, d\}$ we say that Φ is a *connected monodromy representation of type* $(g, d, \eta_1, \ldots, \eta_n)$.

Note that if two labellings L, L' of $f \colon X \to Y$ are given, then $L' = \sigma \cdot L$ for some $\sigma \in S_d$ and $\Phi'(\gamma) = \sigma \cdot \Phi(\gamma) \cdot \sigma^{-1}$. In particular the type of the monodromy representation does note depend on the chosen labelling.

We obtained that a degree d map $f \colon X \to Y$ between compact connected Riemann surfaces s.t. the ramification profile over each branch point is η_i gives rise to a connected monodromy representation Φ of type $(g_Y, d, \eta_1, \ldots, \eta_n)$. If we let X to be non connected, then the monodromy representation may not be connected anymore, more precisely: the monodromy representation is connected if and only if X is connected.[5] We also have that isomorphic maps give the same monodromy representation.

Conversely:

Theorem 1.13 ([**CM**]). *Let Y be a Riemann surface of genus g, Φ a monodromy representation of type $(g, d, \eta_1, \ldots, \eta_n)$, $B = \{b_1, \ldots, b_n\} \subset Y$ a finite subset. Then exists a y_0-labelled holomorphic map of Riemann surface covering Y with branch locus B whose associated monodromy is F. Such map is unique up to isomorphisms of y_0-labelled maps.*

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[5]For further details: [**CM**].

Sketch of proof. In the proof of this result the Riemann's Existence theorem is fundamental. We construct explicitly the topological space X_0 and the covering $f^0 \colon X^0 \to Y \setminus B$ in such a way that the map $f \colon X \to Y$ given by the Riemann's Existence theorem has the desired monodomy representation.

Take cycles $\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g$ in Y generating $\pi_1(Y, y_0)$ all containing a point $p \in Y$ in such a way that $Y \setminus \{\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g\}$ is the fundamental polygon describing Y. Denote by γ_i a loop containing y_0, winding once around b_i, never around the other elements of B, and fully contained in $Y \setminus \{\alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g\}$.
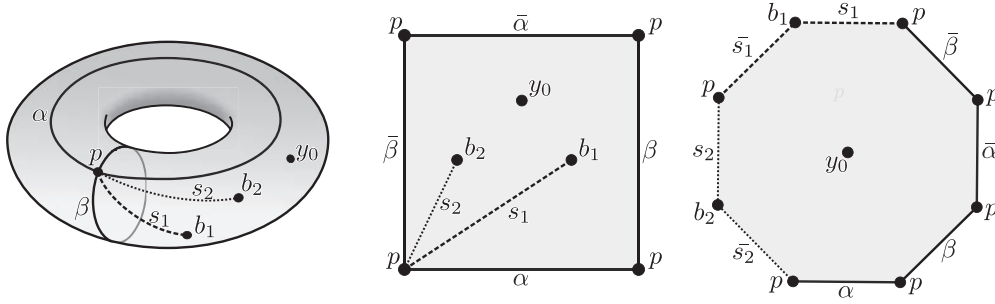
Consider segments s_j connecting p with the points b_j. Open the previous polygon in correspondence of these segments, so that we get a new polygon

$$P := s_1 \bar{s}_1 \cdots s_n \bar{s}_n \alpha_1 \beta_1 \bar{\alpha}_1 \bar{\beta}_1 \cdots \alpha_g \beta_g \bar{\alpha}_g \bar{\beta}_g$$

Then it suffices to take d copies of P and glue their boundaries appropriately to produce X^0 in such a way that the natural projection to $Y \setminus B$ has the desired monodromy representation:

$$s_{j,k} \sim \bar{s}_{j, \Phi(\gamma_j)(k)}$$
$$\bar{\alpha}_{i,k} \sim \alpha_{i, \Phi(\beta_i)(k)}$$
$$\beta_{i,k} \sim \bar{\beta}_{i, \Phi(\alpha_i)(k)}$$

$\square$

The previous theorem ensures that we have a bijection between isomorphism classes of y_0-labeled Hurwitz covers and connected monodromy representations of type $(g, d, \eta_1, \ldots, \eta_n)$. Then we can prove

Theorem 1.14 ([**CM**]). *Let M° (resp $M^\bullet$) be the set of connected monodromy representations (resp. monodromy representations) of type $(g, d, \eta_1, \ldots, \eta_n)$. Then*

$$H^\circ_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\circ|}{d!}$$

and

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\bullet|}{d!}$$

where h is determined by (1.7).

Sketch of proof. We give the proof in the connected case, the other case is analogous.

Take $f \colon X \to Y$ Hurwitz cover. Clearly there are $d!$ possible choices of a y_0-labelling $L \colon f^{-1}(y_0) \to \{1, \ldots, d\}$. An automorphism $\phi \in \mathrm{Aut}(f)$ is an isomorphism $\phi \colon X \xrightarrow{\sim} X$

satisfying $f = f \circ \phi$. In particular ϕ gives an isomorphism $(f, L) \cong (f, L')$ where $L' = \phi \cdot L := L \circ \phi^{-1}$. Here $\phi \cdot L$ denotes the left action of $\mathrm{Aut}(f)$ on the possible y_0-labellings of f. Such action is free (i.e. $\phi \cdot L = L$ imply $\phi = \mathrm{id}_X$) so the number of isomorphism classes of y_0-labelings of f is $d!/|\mathrm{Aut}(f)|$.

From the previous theorem isomorphism classes of y_0-labelled maps for the given f are in bijection with the distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. Therefore

$$m_f = \frac{d!}{|\mathrm{Aut}(f)|}$$

where m_f the number of distinct monodromy representations arising from f by different labelings of $f^{-1}(y_0)$. So we have

$$H^{\circ}_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) := \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|} = \sum_{[f]} \frac{m_f}{d!} = \frac{|M^{\circ}|}{d!}$$

$\square$

Although the information carried by connected Hurwitz numbers is usually more interesting for geometrical purposes, it turns out that it is easier to compute the (possibly disconnected) Hurwitz numbers. We will see later how it is possible to recover the connected Hurwitz numbers from the disconnected ones.

We mention that using some "degeneration formulas" (which heuristically correspond to shrink the Riemann surface Y producing nodal curves) all disconnected degree d Hurwitz numbers are determined in therms of Hurwitz numbers of the form $H^{\bullet}_{h \xrightarrow{d} 0}(\eta_1, \eta_2, \eta_3)$.[6] For this reason (and others) we later restrict our discussion to the case $g = 0$.

1.3 Interlude: representation theory of S_d

Using theorem 1.14 the problem of computing degree d Hurwitz numbers can be translated into a problem in representation theory of the symmetric group S_d. In order to show this we need some facts about representation theory (see [**FH**], [**M**] and [**CM**] for more details).

Definition 1.15. The *group algebra* of the symmetric group S_d is the complex algebra generated by the elements of S_d, that is

$$\mathbb{C}[S_d] := \left\{ \sum_{\sigma \in S_d} a_\sigma \sigma \mid a_\sigma \in \mathbb{C} \right\}$$

with operations

$$\sum_{\sigma \in S_d} a_\sigma \sigma + \sum_{\sigma \in S_d} b_\sigma \sigma = \sum_{\sigma \in S_d} (a_\sigma + b_\sigma) \sigma$$

$$\left(\sum_{\sigma \in S_d} a_\sigma \sigma \right) \cdot \left(\sum_{\sigma \in S_d} b_\sigma \sigma \right) = \sum_{\sigma \in S_d} \sum_{\sigma' \in S_d} a_\sigma b_{\sigma'} (\sigma \cdot \sigma') \tag{1.16}$$

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[6][**CM**]

$$t \cdot \left(\sum_{\sigma \in S_d} a_\sigma \sigma \right) = \sum_{\sigma \in S_d} (ta_\sigma)\sigma$$

where $t \in \mathbb{C}$. Expression in the r.h.s. of (1.16) (before multiplying σ and σ') is called *formal expansion* of the product.

We define *class algebra* of S_d the center of the group algebra

$$\mathcal{Z}\mathbb{C}[S_d] = \{x \in \mathbb{C}[S_d] \mid yx = xy \quad \text{for all} \quad y \in \mathbb{C}[S_d]\}$$

The following functions are very important for our discussion:

Definition 1.17. A *class function* on S_d is a map $\alpha \colon S_d \to \mathbb{C}$ which is constant on conjugacy classes, i.e. $\forall h \in S_d$ we have $\alpha(h^{-1}gh) = \alpha(h)$. Let $\mathbb{C}_{\text{class}}$ denote the vector space of class functions on S_d. We define on $\mathbb{C}_{\text{class}}$ the following Hermitian inner product

$$(\alpha, \beta) := \frac{1}{d!} \sum_{\sigma \in S_d} \alpha(\sigma)\overline{\beta(\sigma)} = \frac{1}{d!} \sum_{C \subset S_d} |C|\alpha(C)\overline{\beta(C)} \tag{1.18}$$

for any $\alpha, \beta \in \mathbb{C}_{\text{class}}$, where $\sum_C$ runs over the conjugacy classes of S_d.

We have the following

Lemma 1.19 ([FH]).

$$\mathcal{Z}\mathbb{C}[S_d] = \left\{ \sum_{\sigma \in S_d} \alpha(\sigma)\sigma \mid \alpha \in \mathbb{C}_{class} \right\}$$

For $\eta \vdash d$ denote by $C_\eta \subset S_d$ the conjugacy class corresponding to all elements of cycle type η. Let $\alpha_\eta \colon S_d \to \mathbb{C}$ the class function which takes value 1 on elements of C_η and 0 otherwise. It is clear that the set $\{\alpha_\eta \mid \eta \vdash d\}$ gives a basis of $\mathbb{C}_{\text{class}}$. Let $c_\eta \in \mathcal{Z}\mathbb{C}[S_d]$ denote the corresponding element in the center of the group algebra, that is

$$c_\eta = \sum_{s \in S_d} \alpha_\eta(\sigma)\sigma = \sum_{\sigma \in C_\eta} \sigma$$

We get that $\{c_\eta \mid \eta \vdash d\}$ form a basis of $\mathcal{Z}\mathbb{C}[S_d]$ as a complex vector space, called *conjugacy class basis*

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\eta \vdash d} \langle c_\eta \rangle_\mathbb{C}$$

Note that the identity element c_e in $\mathcal{Z}\mathbb{C}[S_d]$ corresponds to the partition $e = (1, \ldots, 1)$.

A (complex) representation of S_d is a homomorphism $\rho \colon S_d \to \text{Aut}(V_\rho)$ making the finite dimensional complex vector space V_ρ into a S_d-module. We define the dimension of the representation to be $\dim \rho := \dim_\mathbb{C} V_\rho$. Given any representation ρ, notice that this extends to a homomorphism $\mathbb{C}[S_d] \to \text{End}(V_\rho)$, making V_ρ into a $\mathbb{C}[S_d]$-module.

Representations of S_d are described by their characters, which are defined as follows:

Definition 1.20. Let ρ be a representation of S_d. The *character* of ρ is the class function $\chi_\rho \in \mathbb{C}_{\text{class}}$ defined by

$$\chi_\rho(\sigma) := \text{tr}(\rho(\sigma))$$

The fact that χ_ρ is a class function follows from the cyclicity of the trace. From the definition it follows that χ_ρ does not depend on the choice of basis for V_ρ (due to the corresponding property of the trace) and that $\chi_\rho(e) = \dim \rho$ (since $\rho(e) = \mathrm{id}$). It is also easy to see that $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

A representation ρ is *irreducible* if V_ρ does not contain any nontrivial S_d-submodules. For complex representation we have the following fundamental result

Theorem 1.21 ([**FH**]). *In terms of the inner product* (1.18) *the characters of the irreducible representations of S_d are orthonormal:*

$$(\chi_{\rho_1}, \chi_{\rho_2}) = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{if } \rho_1 \ncong \rho_2 \end{cases}$$

where ρ_1, ρ_2 are irreducible representations.

Any complex representation decomposes uniquely into the direct sum of irreducible representations. More precisely, we have

$$R = \bigoplus_\rho V_\rho^{\oplus \dim V_\rho}$$

where R is the obvious representation of S_d on $\mathbb{C}[S_d]$, called *regular representation*, while the (big) direct sum runs over all the irreducible representations of S_d. This also implies the following isomorphism of algebras

$$\mathbb{C}[S_d] \cong \bigoplus_\rho \mathrm{End}(V_\rho)$$

which is defined extending $S_d \to \bigoplus_\rho \mathrm{End}(V_\rho)$ by linearity.

Theorem 1.22 ([**FH**]). *To each partition $\lambda \vdash d$ corresponds a unique irreducible representation V_λ of S_d. The corresponding character, denoted χ^λ, is given by Frobenius formula [**FH**]. In particular χ^λ is a real class function.*

By dimensional arguments, we get that characters of irreducible representations form another basis of $\mathbb{C}_{\mathrm{class}}$. Moreover, denoting

$$e_\lambda := \sum_{\sigma \in S_d} \chi^\lambda(\sigma)\sigma$$

we get that $\{e_\lambda \mid \lambda \vdash d\}$ gives another basis of $\mathcal{Z}\mathbb{C}[S_d]$

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\lambda \vdash d} \langle e_\lambda \rangle_\mathbb{C}$$

which will be called *character basis* or *idempotent basis*. Indeed from characters orthogonality (and the fact that $\overline{\chi^\lambda} = \chi^\lambda$) we get

$$e_{\lambda_i} \cdot e_{\lambda_j} = \begin{cases} e_{\lambda_i} & \text{if } e_{\lambda_i} = e_{\lambda_j} \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{Z}\mathbb{C}[S_d]$ admits a basis of idempotent elements, then it is a *semisimple algebra*. The formulas for the change of basis are given by the characters

$$e_\lambda = \frac{\dim \lambda}{d!} \sum_{\eta \vdash d} \chi^\lambda(C_\eta) c_\eta \qquad \text{and} \qquad c_\eta = |C_\eta| \sum_{\lambda \vdash d} \frac{\chi^\lambda(C_\eta)}{\dim \lambda} e_\lambda \qquad (1.23)$$

where $\dim \lambda := \dim V_\lambda$.

1.4   Burnside's formula

Now we are ready to use the notions that we introduced in order to rewrite the expression of Hurwitz numbers in terms of characters of the irreducible representations of S_d. Existence of a idempotent basis for $\mathcal{Z}\mathbb{C}[S_d]$ and formulas (1.23) will be crucial.

Recall that

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{|M^\bullet|}{d!}$$

where $M^\bullet$ is the set of monodromy representations of type $(g, d, \eta_1, \ldots, \eta_r)$ with $\eta_1, \ldots, \eta_n$ partitions of d.

For $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ where $i \in \mathbb{Z}_{\geq 1}$ appears a_i times in the partition, $\sum_i i a_i = d$, the size of the centralizer of C_η is given by

$$\mathfrak{z}(\eta) = \frac{d!}{|C_\eta|} = \prod_{i=1}^{\ell(\eta)} a_i! \, i^{a_i}$$

In order to account for the case $g \neq 0$ one needs

Definition 1.24. Let $d \in \mathbb{Z}_{\geq 1}$, $\eta \vdash d$. We define the *kommutator* to be the element

$$\mathfrak{K} := \sum_{\eta \vdash d} \mathfrak{z}(\eta) c_\eta^2 \in \mathcal{Z}\mathbb{C}[S_d]$$

Then we have the following

Proposition 1.25 ([CM]).

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \frac{1}{d!}[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$$

where $[c_e]\mathfrak{K}^g c_{\eta_n} \cdots c_{\eta_2} c_{\eta_1}$ denotes the coefficient of c_e after writing the product $\mathfrak{K}^g c_{\eta_n} \ldots c_{\eta_2} c_{\eta_1}$ as a linear combination of the basis elements $c_\eta \in \mathcal{Z}\mathbb{C}[S_d]$. As usual h is determined by Riemann-Hurwitz formula.

By changing basis from the conjugacy basis to the idempotent basis we get

Theorem 1.26 (Burnside's Character Formula [CM]).

$$H^\bullet_{h \xrightarrow{d} g}(\eta_1, \ldots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^{2-2g} \prod_{i=1}^n f_{C_j}(\lambda)$$

where

$$f_{C_i}(\lambda) := |C_i| \frac{\chi^\lambda(C_i)}{\dim \lambda}$$

and $C_i := C_{\eta_i}$.

Recall that from the change of basis formula,

$$c_\eta = \sum_{\lambda \vdash d} f_{C_\eta}(\lambda) e_\lambda$$

From Burnside's formula we see that such coefficients $f_{C_\eta}(\lambda)$ for the change of basis correspond to the contribution of the ramification profile η to the disconnected Hurwitz numbers.

1.5 The generating function

In the following we will restrict ourselves to the case of $g = 0$. Recall that there are some degeneration formulas which allows to express all the Hurwitz numbers in terms of those for $g = 0$. For a similar discussion for arbitrary g see [**CM**]. We will follow instead [**O1**].

For $g = 0$ the Riemann-Hurwitz formula implies

$$2h - 2 = -2d + nd - \sum_{i=1}^{n} \ell(\eta_i)$$

and since this fixes h in terms of $(d, \eta_1, \ldots, \eta_n)$, we simply denote

$$H_d^\bullet(\eta_1, \ldots, \eta_n) := H_{h \xrightarrow{d} 0}^\bullet(\eta_1, \ldots, \eta_n) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{n} f_{C_j}(\lambda)$$

Let b be the number of branch points which have simple ramification, i.e. $\eta = (2)$. We denote

$$H_{d,b}^\bullet(\eta_1, \ldots, \eta_{n-b}) := H_d^\bullet(\eta_1, \ldots, \eta_{n-b}, \underbrace{(2), \ldots, (2)}_{b \text{ times}}) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 f_2(\lambda)^b \prod_{i=1}^{n-b} f_{C_j}(\lambda)$$

where $f_2 := f_{C_{(2)}}$. Analogous definitions hold for connected Hurwitz numbers, replacing $H^\bullet$ with H°.

Rather than considering the different Hurwitz numbers separately, it worth to collect them together into generating functions. Fix $m \in \mathbb{Z}_{\geq 0}$ to be the number of branch points with non-simple ramification profile. Then

Definition 1.27. Let $\{p_{i,j}\}$ and q be some variables, $i \in \{1, \ldots, m\}$, $j \in \mathbb{Z}_{\geq 0}$. Then we define the *Hurwitz potential* to be[7]

$$\mathfrak{H}^\bullet(p_{i,j}, q, z) := \sum_{d,b=0}^{\infty} q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} H_{d,b}^\bullet(\eta_1, \ldots, \eta_m) \qquad (1.28)$$

where for $\eta = (l_1, \ldots, l_k) \vdash d$ we have

$$p_i^\eta := (p_{i,1}^{l_1} + \cdots + p_{i,d}^{l_1}) \cdots (p_{i,1}^{l_k} + \cdots + p_{i,d}^{l_k})$$

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[7]For $d = 0$ we set $H_{d,b}^\bullet(\eta_1, \ldots, \eta_m) = 0$, this is consistent with the fact that there is no covering of degree zero.

We also introduce the *modified Hurwitz potential* to be

$$\mathfrak{h}_d^\bullet(\eta_q, \ldots, \eta_m) := \sum_{b=0}^\infty \frac{z^b}{b!} H_{d,b}^\bullet(\eta_1, \ldots, \eta_m)$$

Analogous definitions hold for the *(modified) connected Hurwitz potential* $\mathfrak{H}^\circ$ and $\mathfrak{h}^\circ$.

Note that from the definition we have

$$\mathfrak{H}^\bullet(\psi_{i,j}, q, z) = \sum_{d=0}^\infty q^d \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \mathfrak{h}_d^\bullet(\eta_q, \ldots, \eta_m)$$

For fixed i, the polynomials of the form p^η for all partitions $\eta \vdash d$ form a basis for the space of all homogeneous polynomials of degree d in d variables with rational coefficients. They are called *power sum polynomials*. Therefore, given $\mathfrak{H}^\bullet$, it can be expanded uniquely as in (1.28) giving all the Hurwitz numbers. For more details on power sum polynomials see [**FH**] or [**M**].

We see that in the expansion of the Hurwitz potential
- q keeps track of the degree d
- z keeps track of the number of simple ramification points (we divided by $b!$ in order to not distinguish between them)
- i indicizes the non-simple ramification points
- j gives the ramification profiles

The first advantage of considering generating function in place of the single Hurwitz numbers is

Theorem 1.29 ([CM]).

$$\mathfrak{H}^\bullet = e^{\mathfrak{H}^\circ}$$

In order to simplify the notation, in the following we denote $\chi_\eta^\lambda := \chi^\lambda(C_\eta)$.

For any $\lambda \vdash d$, we have the following relation[8] which can be regarded as corollary of Frobenius formula

$$s_\lambda(p_1, \ldots, p_d) = \frac{1}{d!} \sum_{\eta \vdash d} \chi_\eta^\lambda |C_\eta| p^\eta$$

where s_λ is the *Schur polynomial* associated to λ, it is homogeneous polynomial of degree d. Putting formulas together we get

$$\mathfrak{H}^\bullet(p_{i,j}, q, z) = \sum_{d,b=0}^\infty q^d \frac{z^b}{b!} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!}\right)^2 f_2(\lambda)^b \prod_{i=1}^m f_{C_i}(\lambda)$$

$$= \sum_{d=0}^\infty q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \sum_{\eta_1 \vdash d} \cdots \sum_{\eta_m \vdash d} p_1^{\eta_1} \cdots p_m^{\eta_m} \prod_{i=1}^m \frac{1}{d!} |C_{\eta_i}| \chi_{\eta_i}^\lambda$$

$$= \sum_{d=0}^\infty q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} \left(\frac{\dim \lambda}{d!}\right)^{2-m} \prod_{i=1}^m s_\lambda(p_{i,1}, \ldots, p_{i,d})$$

---

[8]See [**FH**] or [**M**]

The case $m = 2$ corresponds to the so called *double Hurwitz numbers*, which are the ones we are interested in

$$\mathfrak{H}^\bullet(\{p_j, p'_j\}, q, z) = \sum_{d=0}^{\infty} q^d \sum_{\lambda \vdash d} e^{z f_2(\lambda)} s_\lambda(p_1, \dots, p_d) s_\lambda(p'_1, \dots, p'_d)$$

$$= \sum_\lambda q^{|\lambda|} e^{z f_2(\lambda)} s_\lambda(P) s_\lambda(P')$$

where $\sum_\lambda$ runs over the paritions of arbitrary integers and $|\lambda|$ is the size of the partition λ, $\lambda \vdash |\lambda|$. We used P and P' to denote the set of variables $\{p_1, p_2, \dots\}$ and $\{p'_1, p'_2, \dots\}$ respectively (of course s_λ depends only on the first $|\lambda|$ variables).

For the case $m = 2$ we also have

$$H^\bullet_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}$$

and

$$\mathfrak{h}^\bullet_d(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta e^{z f_2(\lambda)} \chi^\lambda_{\eta'}$$

Before moving on, notice that for double Hurwitz numbers the Riemann-Hurwitz formula gives

$$2h = b + 2 - \ell(\eta) - \ell(\eta')$$

1.6 Frobenius formula for $C_{(2)}$

Here we want to rewrite $f_2(\lambda)$ in a different way using Frobenius formula. We did not wrote the general Frobenius formula, since it is quite complicated. However it has a nicer expression in the special case of $\chi^\lambda(C_{(2)})$, that is the case needed to compute $f_2(\lambda)$.

For a given partition λ, define its rank r to be the length of the diagonal of its Young diagram, and let a_i and b_i be the number of boxes to the right and below of the i-th box of the diagonal, reading from the upper right to the lower left. Then $\begin{pmatrix} a_1 a_2 \dots a_n \\ b_1 b_2 \dots b_n \end{pmatrix}$ is called *Frobenius notation* of the partition.
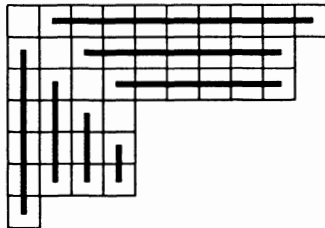
$\lambda = (10, 9, 9, 4, 4, 4, 1)$

Frobenius notation $\begin{pmatrix} 9 & 7 & 6 & 0 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

modified Frobenius notation $\begin{pmatrix} 9.5 & 7.5 & 6.5 & 0.5 \\ 2.5 & 3.5 & 4.5 & 5.5 \end{pmatrix}$

Let also define the *modified Frobenius notation* to be $\begin{pmatrix} a'_1 a'_2 \dots a'_n \\ b'_1 b'_2 \dots b'_n \end{pmatrix}$ where $a'_i = a_i + 1/2$ and $b'_i = b_i + 1/2$. Then
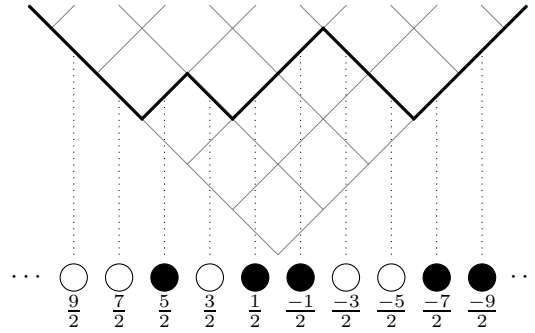
Lemma 1.30 (Frobenius formula [**FH**])**.**

$$\chi^\lambda(C_{(2)}) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} (a_i(a_i+1) - b_i(b_i+1)) = \frac{\dim \lambda}{d(d-1)} \sum_{i=1}^{r} ((a_i')^2 - (b_i')^2)$$

From this and $|C_{(2)}| = \binom{d}{2}$ it follows that

$$f_2(\lambda) = \frac{|C_{(2)}|}{\dim \lambda} \chi^\lambda(C_{(2)}) = \frac{1}{2} \sum_{i=1}^{r} ((a_i')^2 - (b_i')^2)$$

Draw the Young diagram of λ rotated by $135°$ over the real line with opposite orientation as in the picture and consider the black and white stones as showed.

Suppose that λ is made of $k = \ell(\lambda)$ cycles of lengths $\{\lambda_1, \ldots, \lambda_k\}$ where $\lambda_1 \geq \lambda_2 \geq \ldots$. Now consider the ordered set $\{\tilde{\lambda}_i\}_{i \in \mathbb{Z}_{\geq 0}} = \{\lambda_1, \ldots, \lambda_k, 0, 0, \ldots\}$ ending with infinitely many zeros. Then it is clear that black stones are placed in correspondence to elements of

$$\mathfrak{S}_\bullet(\lambda) := \{\tilde{\lambda}_i - i + 1/2\} \subset \mathbb{Z} + \frac{1}{2}$$

and white stones in correspondence to elements of $\mathfrak{S}_\circ(\lambda) := (\mathbb{Z} + \frac{1}{2}) \setminus \mathfrak{S}_\bullet(\lambda)$. Moreover, the coefficients in the modified Frobenius notation are given by

$$\{a_i'\} = \mathfrak{S}_\bullet^+(\lambda) := \mathfrak{S}_\bullet \cap (\mathbb{Z}_{\geq 0} + 1/2)$$
$$\{b_i'\} = \mathfrak{S}_\circ^-(\lambda) := \mathfrak{S}_\circ \cap (\mathbb{Z}_{\leq 0} - 1/2) = (\mathbb{Z}_{\leq 0} - 1/2) \setminus \mathfrak{S}_\bullet(\lambda)$$

Hence we obtained

$$f_2(\lambda) = \sum_{k \in \mathfrak{S}_\bullet^+} \frac{k^2}{2} - \sum_{k \in \mathfrak{S}_\circ^-} \frac{k^2}{2}$$

Notice also that

$$|\lambda| = \lambda_1 + \cdots + \lambda_k = a_1' + \cdots + a_k' + b_1' + \cdots + b_k' = \sum_{k \in \mathfrak{S}_\bullet^+} k - \sum_{k \in \mathfrak{S}_\circ^-} k$$

The aim of the following section is to write the formula of double Hurwitz numbers and those for the corresponding potentials in terms of matrix elements in the half infinite wedge space. This half infinite wedge space corresponds to the space of possible configurations of black and white stones as before.

1.7 Half infinite wedge formalism

Now we introduce the *half infinite wedge formalism*, also known as *infinite wedge formalism*. As a references see [**MJD**; **O2**; **J**].

Let V be a vector space with basis $\{\underline{k}\}$, $k \in \mathbb{Z} + \frac{1}{2}$. We define the vector space $\bigwedge^{\frac{\infty}{2}} V$ to be spanned by vectors
$$v_S := \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \dots$$
where $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ is a subset s.t. both
$$S^+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \frac{1}{2} \right) \quad \text{and} \quad S^- = \left(\mathbb{Z}_{\leq 0} - \frac{1}{2} \right) \setminus S$$

are finite. We equip $\bigwedge^{\frac{\infty}{2}} V$ with the inner product $(-,-)$ in which the basis $\{v_S\}$ (for all possible choices of S) is orthonormal.

For our purposes S is identified with $\mathfrak{S}_\bullet(\lambda)$ defined before, we denote by v_λ the vector in $\bigwedge^{\frac{\infty}{2}} V$ corresponding to $S = \mathfrak{S}_\bullet(\lambda)$. In this case $S^+ = \mathfrak{S}_\bullet^+$ and $S^- = \mathfrak{S}_\circ^-$, and finiteness condition simply corresponds to the fact that we have finitely many black stones on the left of the origin and finitely many white stones on the right or the origin, which is automatic for $\mathfrak{S}_\bullet(\lambda)$.

Define the operators ψ_k and ψ_k^* on $\bigwedge^{\frac{\infty}{2}} V$ as follows. For $v \in \bigwedge^{\frac{\infty}{2}} V$ let
$$\psi_k(v) := \underline{k} \wedge v$$

and let ψ_k^* to be the adjoint operator, that is s.t.
$$(v', \psi_k^* v) = (\psi_k v', v)$$

From the definition of these operators and of the inner product, note that they satisfy
$$\psi_j \psi_k^* + \psi_k^* \psi_j = \delta_{jk}$$
$$\psi_j \psi_k + \psi_k \psi_j = 0$$
$$\psi_j^* \psi_k^* + \psi_k^* \psi_j^* = 0$$

which are known as *fermionic commutation relations*.

These operators are related to the usual creation and annihilation operators for the *Fermi sea*, by identifying black stones with electrons and white stones with empty energy levels (but note that in usual physical literature, and in [**MJD**], one should exchange $\psi_k \leftrightarrow \psi_k^*$). Finiteness condition amounts to requiring that the considered state is a finite energy excitation of the vacuum in the Fermi sea. We identify the vacuum of the Fermi sea with the vector $v_\emptyset = -\frac{1}{2} \wedge -\frac{3}{2} \wedge -\frac{5}{2} \wedge \dots$, where all stones are black (resp. white) on the right (resp. left) of the origin.

To see this, note that if $\underline{k}$ is not present in v_S ("the energy level k is empty") then ψ_k add that vector to v_S ("creates the electron") whereas ψ_k^* maps v_S to 0 ("the state is annihilated"). Instead, if $\underline{k}$ is present in v_S ("the energy level k is occupied") then ψ_k maps v_S to 0 ("Pauli principle") whereas ψ_k^* removes that vector from v_S ("the electron at the energy level k is annihilated").

Note also that all the vectors in $\bigwedge^{\frac{\infty}{2}} V$ can be obtained from $v_\emptyset$ by applying creation and annihilation operators a finite number of times, that is $v_\emptyset$ is cyclic.

Introduce the *normal ordered product*

$$:\psi_j\psi_k^*: \ := \begin{cases} \psi_j\psi_k^* & k > 0 \\ -\psi_k^*\psi_j & k < 0 \end{cases}$$

For $k > 0$ (resp. $k < 0$), $:\psi_k\psi_k^*:$ gives 1 if we have a black (resp. white) stone in k and zero otherwise. For $j \neq k$ the normal ordered product is the ordinary product due to the fermionic commutation relation $:\psi_j\psi_k^*: = \psi_j\psi_k^* = -\psi_k^*\psi_j$. The effect of $:\psi_j\psi_k^*:$ for $j \neq k$ is to take the black stone in k and move it to the position j, unless there is no black stone in k or the position j is already occupied: in such cases it gives the zero vector. Finiteness condition ensures that any operator of the form $\sum_{j,k} a_{jk} :\psi_j\psi_k^*:$ is well defined for any choice of the coefficients $a_{j,k}$. Note that the assignement $E_{jk} \mapsto \ :\psi_j\psi_k^*:$ can be understood as a projective representation of $\mathfrak{gl}(V)$ on $\bigwedge^{\frac{\infty}{2}} V$.

Now we can define the operator $\mathcal{F}_2$

$$\mathcal{F}_2 = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} :\psi_k\psi_k^*:$$

which satisfies

$$\mathcal{F}_2 v_\lambda = f_2(\lambda) v_\lambda$$

Let's see how we can rewrite the whole quantities

$$H_{d,b}^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi_\eta^\lambda f_2(\lambda)^b \chi_{\eta'}^\lambda$$

$$\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi_\eta^\lambda e^{z f_2(\lambda)} \chi_{\eta'}^\lambda$$

$$\mathfrak{H}^\bullet(\{p_j, p_j'\}, q, z) = \sum_\lambda q^{|\lambda|} s_\lambda(P) e^{z f_2(\lambda)} s_\lambda(P')$$

in the half infinte wedge space. To do so we should be able to rewrite also quantities $|\lambda|$, χ_η^λ and $s_\lambda(P)$ in terms of matrix elements in the half infinite wedge space.

We introduce the following operators, called *energy operator* and *charge operator* respectively

$$H := \sum_{k \in \mathbb{Z}+1/2} k :\psi_k\psi_k^*: \qquad C := \sum_{k \in \mathbb{Z}+1/2} :\psi_k\psi_k^*:$$

The physical interpretation of these operators is obvious when we regard a black stone at position $k > 0$ as an electron of energy k and charge $+1$ and a white stone at position $k < 0$ as a positron of energy $-k$ and charge -1.

First notice that from our previous discussion we have

$$H v_\lambda = |\lambda| v_\lambda$$

and so

$$q^H e^{z\mathcal{F}_2} v_\lambda = q^{|\lambda|} e^{f_2(\lambda)} v_\lambda$$

We have

$$C v_S = (|S^+| - |S^-|) v_s$$

Suppose v_S is of charge c, i.e. $Cv_S = c\,v_S$, then $|S^+| = |S^-| + c$. Pick $k \in S$, $k < \min(S^-)$, this means that the stone at k and all those on its right are black. Then $|S^+| = |S^-| + c$ implies that $k = s_{-(k-1/2)+c}$. Hence the charge of the vector v_S is given by

$$c = \lim_{i \to +\infty} \left(s_i + i - \frac{1}{2} \right)$$

In particular, all vectors corresponding to Young diagrams ($S = \mathfrak{S}_\bullet(\lambda)$) have zero charge. Conversely, any vector of zero charge can be obtained from a Young diagram by taking the partition $\lambda_i = s_i + i - \frac{1}{2}$ (omitting the infinitely many λ_i which vanish). We denote by $\bigwedge_0^{\frac{\infty}{2}} V$ the subspace of $\bigwedge^{\frac{\infty}{2}} V$ of zero charge. From this we get that

$$\{ v_\lambda \mid \lambda \vdash d \}$$

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ given by vectors of energy d. Considering all positive values of d we get a basis of the whole charge zero subspace $\bigwedge_0^{\frac{\infty}{2}} V$.

Now define the following operators for all $n \in \mathbb{Z}_{\neq 0}$

$$\alpha_n := \sum_{k \in \mathbb{Z} + \frac{1}{2}} :\psi_{k-n} \psi_k^*:$$

satisfying $\alpha_n \underline{k} = \underline{k-n}$ and the *Heisenberg commutation relations*[9]

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}$$

From the definition it follows that α_n and α_{-n} are adjoint for every $n \in \mathbb{Z}_{\neq 0}$.

The effect of this operator in a certain configuration of stones (or electrons) is the following. Pick one and move it to the right by n positions. If the new position is a white stone, make it black and replace the initial black stone by a white one. If the new position is occupied by a black stone, then the operator gives the zero vector (Pauli principle). Then repeat this for all black stones and sum all the resulting vectors to give the final state. Finiteness condition ensures that this operation is well defined.

For any partition $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash |\eta|$ define

$$\alpha_\eta := \prod_{i=1}^{\ell(\eta)} \alpha_{\eta_i} \quad \text{and} \quad \alpha_{-\eta} := \prod_{i=1}^{\ell(\eta)} \alpha_{-\eta_i}$$

The Heisenberg commutation relation ensures that the ordering of the multiplied operators is not relevant and also that α_η and $\alpha_{-\eta}$ are adjoint. Note also that α_η (resp. $\alpha_{-\eta}$) decreases (resp. increases) the energy of states by $|\eta|$. Moreover, it is possible to prove[10] that

$$\{ \alpha_{-\eta} v_\emptyset \mid \eta \vdash d \}$$

---

[9]Note that we realized operators satisfying the *bosonic* algebra using operators satisfying the *fermionic* one. Indeed we have an isomorphism of vector spaces given by $v_\lambda \leftrightarrow s_\lambda$ called *Boson-Fermion correspondence*, and the operators α_n essentially realize some natural bosonic operators in the space of Schur polynomials as operators in the fermionic space $\bigwedge_0^{\frac{\infty}{2}} V$. The correspondence $v_\lambda \leftrightarrow s_\lambda$ is the essence of the strong relation between representation theory of S_d and the infinite wedge formalism. See [**MJD**] for details.

[10][**J**]

is a basis for the subspace of $\bigwedge_0^{\frac{\infty}{2}} V$ of energy d, orthogonal with respect to the inner product $(-,-)$. To see this it suffices to prove that $\alpha_{\eta'}\alpha_{-\eta}v_\emptyset \propto \delta_{\eta,\eta'}v_\emptyset$.

The two bases of $\bigwedge_0^{\frac{\infty}{2}} V$ can be related as follows. Let λ be any partition and $n \in \mathbb{Z}_{>0}$. From the following picture
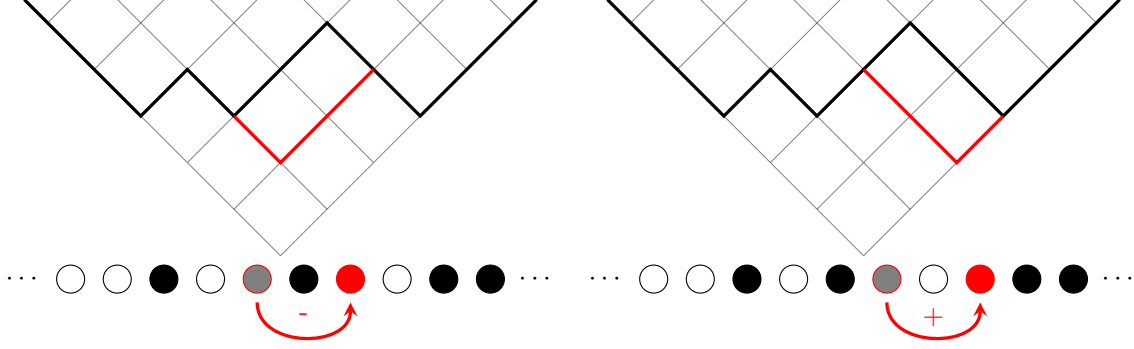
we can see that

$$\alpha_n v_\lambda = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \text{ skew hook} \\ |\lambda/\lambda'|=n}} (-1)^{r(\lambda'/\lambda)-1} v_{\lambda'}$$

where $\lambda' < \lambda$ means that $\lambda'_i < \lambda_i$ for all $i \le \ell(\lambda)$ and $r(\lambda'/\lambda)$ is the number of rows of λ' touched by λ'/λ. This is very similar to

Lemma 1.31 (*Murnaghan-Nakayama rule* [**FH**])**.**

$$\chi^\lambda_{\{(n),\eta\}} = \sum_{\substack{\lambda' < \lambda \\ \lambda/\lambda' \text{ skew hook} \\ |\lambda/\lambda'|=n}} (-1)^{r(\lambda'/\lambda)-1} \chi^{\lambda'}_\eta$$

Using these identities we get that for any two partitions λ and η such that $|\lambda| = |\eta|$

$$\alpha_\eta v_\lambda = \chi^\lambda_\eta v_\emptyset$$

and than taking the adjoint[11] we arrive at

$$\alpha_{-\eta} v_\emptyset = \sum_{\lambda \vdash |\eta|} \chi^\lambda_\eta v_\lambda$$

Using this result, for $\eta = \{\eta_1, \ldots, \eta_{\ell(\eta)}\} \vdash d$ and $\eta' = \{\eta'_1, \ldots, \eta'_{\ell(\eta')}\} \vdash d$ we obtain

$$\left(v_\emptyset, \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} v_\emptyset\right) = \sum_{\lambda \vdash d} \chi^\lambda_{\eta'} \left(v_\emptyset, \alpha_\eta \mathcal{F}_2^b v_\lambda\right) = \sum_{\lambda \vdash d} f_2(\lambda)^b \chi^\lambda_{\eta'} \left(v_\emptyset, \alpha_\eta v_\lambda\right)$$

$$= \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} \left(v_\emptyset, v_\emptyset\right) = \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'}$$

Finally we get

$$H^\bullet_{d,b}(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi^\lambda_\eta f_2(\lambda)^b \chi^\lambda_{\eta'} = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \left\langle \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} \right\rangle$$

---

[11]Write $\alpha_{-\eta} v_\emptyset = \sum_{\lambda \vdash |\eta|} (v_\lambda, \alpha_{-\eta} v_\emptyset) v_\lambda = \sum_{\lambda \vdash |\eta|} (\alpha_\eta v_\lambda, v_\emptyset) v_\lambda$.

where the *correlator* of the operator A is defined by $\langle A \rangle := (v_\emptyset, A v_\emptyset)$. Similarly

$$\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \sum_{\lambda \vdash d} \chi_\eta^\lambda e^{zf_2(\lambda)} \chi_{\eta'}^\lambda = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \langle \alpha_\eta e^{z\mathcal{F}_2} \alpha_{-\eta'} \rangle$$

Now it remains to rewrite

$$\mathfrak{H}^\bullet(\{p_j, p_j'\}, q, z) = \sum_\lambda q^{|\lambda|} s_\lambda(P) e^{zf_2(\lambda)} s_\lambda(P')$$

To do so we should be able to recover the Schur polynomials from the half infinite wedge formalism.

Given a sequence $t = (t_1, t_2, \ldots)$ define

$$\Gamma_\pm(t) = \exp\left(\sum_{n=1}^\infty t_n \alpha_{\pm n}\right)$$

Note that $\Gamma_+(t)$ and $\Gamma_-(t)$ are adjoint. The following result can also be related to some results in representation theory of S_d

Lemma 1.32 ([**O2**], [**MJD**]). *Consider a set of variables $P = (p_1, p_2, \ldots)$ and denote $p^{(k)} = p_1^k + p_2^k + \cdots$. Then*

$$\Gamma_-\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) v_\emptyset = \sum_\lambda s_\lambda(P) v_\lambda$$

where $\sum_\lambda$ runs over all possible partitions (of any number).

Introduce the following abbreviations

$$\Gamma_+ := \Gamma_+\left(p^{(1)}, \frac{p^{(2)}}{2}, \frac{p^{(3)}}{3}, \cdots\right) \quad \text{and} \quad \Gamma_- := \Gamma_-\left(p'^{(1)}, \frac{p'^{(2)}}{2}, \frac{p'^{(3)}}{3}, \cdots\right)$$

where $P = (p_1, p_2, \ldots)$, $P' = (p_1', p_2', \ldots)$. Using the lemma we have

$$\left(v_\emptyset, \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- v_\emptyset\right) = \left(\Gamma_- v_\emptyset, q^H e^{z\mathcal{F}_2} \Gamma_- v_\emptyset\right) = \sum_{\lambda, \lambda'} s_\lambda(P) s_{\lambda'}(P') \left(v_\lambda, q^H e^{z\mathcal{F}_2} v_{\lambda'}\right)$$

$$= \sum_{\lambda, \lambda'} s_\lambda(P) e^{zf_2(\lambda')} s_{\lambda'}(P') \left(v_\lambda, q^H v_{\lambda'}\right) = \sum_{\lambda, \lambda'} q^{|\lambda'|} s_\lambda(P) e^{zf_2(\lambda')} s_{\lambda'}(P') \left(v_\lambda, v_{\lambda'}\right)$$

$$= \sum_\lambda q^{|\lambda|} s_\lambda(P) e^{zf_2(\lambda)} s_\lambda(P')$$

so

$$\mathfrak{H}^\bullet(\{p_j, p_j'\}, q, z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- \right\rangle$$

Summarizing, we obtained the following formulas

$$H_{d,b}^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \left\langle \alpha_\eta \mathcal{F}_2^b \alpha_{-\eta'} \right\rangle$$

$$\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \left\langle \alpha_\eta e^{z\mathcal{F}_2} \alpha_{-\eta'} \right\rangle$$

$$\mathfrak{H}^\bullet(\{p_j, p_j'\}, q, z) = \left\langle \Gamma_+ q^H e^{z\mathcal{F}_2} \Gamma_- \right\rangle$$

The last one was first derived in [**O1**], and later the first two were derived in [**OP1**] (although in a quite different form). In [**O1**] it was shown also that the Hurwitz potential $\mathfrak{H}^\bullet$ is the τ-function for the Toda lattice hierarchy of Ueno and Takasaki, implying (infinitely) many recursive relations on $\mathfrak{H}^\bullet$. The technique applied here was later generalized to Gromov-Witten theory in [**OP1**; **OP2**].

Notice that, by considering the energies of the operators in the correlators, it is clear that we should have $|\eta| = |\eta'|$ in order to have nonzero values, as expected since we should have $|\eta| = d = |\eta'|$.

Introducing other operators it is possible to rewrite the previous correlators in such a way that their explicit computation is much simplified. See [**J**; **OP1**; **OP2**] for more details. In order to simplify the notation, denote

$$E_{ij} = :\psi_j \psi_k^*:$$

Definition 1.33. Let $r \in \mathbb{Z}$. We define

$$\mathcal{E}_r(z) := \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-r/2)} E_{k-r,k} + \frac{\delta_{r,0}}{\varsigma(z)}$$

where

$$\varsigma(z) := e^{z/2} - e^{-z/2}$$

Let's explain the meaning of the additional term arising when $r = 0$. We would like to have an operator which acts as

$$\underline{k} \mapsto e^{zk}\underline{k}$$

and the natural candidate is

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} \psi_k \psi_k^*$$

However applying it to the vacuum the second definition gives for $\Re z > 0$

$$\sum_{k \in \mathbb{Z}_{\leq 0}-\frac{1}{2}} e^{zk} = e^{-z/2} \sum_{k \in \mathbb{Z}_{\leq 0}} e^{zk} = \frac{e^{-z/2}}{1 - e^{-z}} = \frac{1}{\varsigma(z)}$$

and diverges for $\Re z = 0$. So we regularize

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} \psi_k \psi_k^* = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} E_{k,k} + \sum_{k \in \mathbb{Z}_{\leq 0}-\frac{1}{2}} e^{zk} \quad \mapsto \quad \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{zk} E_{k,k} + \frac{1}{\varsigma(z)}$$

For $r \neq 0$ we don't have any problem since $E_{k-r,k} = \psi_{k-r}\psi_k^*$.

The exponent $r/2$ in the definition is used in order to have

$$\mathcal{E}_r(z)^* = \mathcal{E}_{-r}(z)$$

The operators $\mathcal{E}$ satisfy the following commutation relation

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(\det \left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]) \mathcal{E}_{a+b}(z+w)$$

The operators $\mathcal{E}$ specialize to the standard bosonic operators on $\bigwedge^{\frac{\infty}{2}} V$

$$\alpha_k = \mathcal{E}_k(0) , \qquad k \neq 0$$

Definition 1.34. For $k \in \mathbb{Z}_{>0}$ define operators $\mathcal{F}_k$ and $\mathcal{P}_k$ by

$$\mathcal{F}_k = \frac{\mathcal{P}_k}{k!} = [z^k]\mathcal{E}_0(z)$$

where $[z^k]\mathcal{E}_0(z)$ stands for the coefficient of z^k in $\mathcal{E}_0(z)$.

Lemma 1.35 ([OP1]). *For $\lambda = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$ partition*

$$\mathcal{P}_k v_\lambda = \mathbf{p}_k(\lambda) v_\lambda$$

where

$$\mathbf{p}_k(\lambda) = \sum_{i=1}^\infty \left[\left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i + \frac{1}{2}\right)^k \right] + (1 - 2^{-k})\zeta(-k)$$

where ζ is the Riemann zeta function.

The meaning of the term $(1 - 2^{-k})\zeta(-k)$ is as before, that is we would like to have

$$\sum_{i=1}^\infty \left(\lambda_i - i + \frac{1}{2}\right)^k$$

but if we consider the partition $\emptyset$ we get the following divergent contribution

$$\sum_{i=1}^\infty \left(-i + \frac{1}{2}\right)^k$$

which can be regularized as before giving $(1 - 2^{-k})\zeta(-k)$.

Notice that the lemma imply that $\mathcal{F}_2$ really coincides with the operator defined previously. Indeed we can rewrite the formula above as

$$\mathbf{p}_k(\lambda) = \sum_{i \in \mathfrak{S}_\bullet^+} i^k - \sum_{i \in \mathfrak{S}_\circ^-} i^k + (1 - 2^{-k})\zeta(-k)$$

and $\zeta(-2) = 0$.

The last formula we need is

Lemma 1.36 ([OP2]).
$$e^{z\mathcal{F}_2}\alpha_{-n}e^{-z\mathcal{F}_2} = \mathcal{E}_{-n}(nz)$$

Now we can rewrite the modified Hurwitz potential in a better form which makes computations simpler. Using $\mathcal{F}_2 v_\emptyset = 0$ we have

$$\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \alpha_\eta e^{z\mathcal{F}_2}\alpha_{-\eta'}\rangle = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)} \alpha_{\eta_i} \prod_{j=1}^{\ell(\eta')} \left(e^{z\mathcal{F}_2}\alpha_{-\eta'_j}e^{-z\mathcal{F}_2}\right)\rangle$$

$$= \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\langle \prod_{i=1}^{\ell(\eta)} \mathcal{E}_{\eta_i}(0) \prod_{j=1}^{\ell(\eta')} \mathcal{E}_{-\eta'_j}(z\eta'_j)\rangle$$

Using the commutation relation for the operators $\mathcal{E}$ it is possible to compute the correlator in the previous formula by moving the operators with negative energy on the right and

those of positive energy on the left. Then only operators of zero energy survive, so we get a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\mathcal{E}_0(n_1 z)\mathcal{E}_0(n_2 z)\cdots$ for some positive integers $m_1, m_2, \ldots, n_1, n_2, \ldots$. Now recall

$$\mathcal{E}_0(nz) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} e^{nzk} E_{k-r,k} + \frac{1}{\varsigma(nz)}$$

and $E_{k-r,k} v_\emptyset = 0$ for all k, r. Hence $\langle \mathcal{E}_0(nz) \rangle = \varsigma^{-1}(nz)$ and we get as final result a sum of terms of the form $\varsigma(m_1 z)\varsigma(m_2 z)\cdots\varsigma^{-1}(n_1 z)\varsigma^{-1}(n_2 z)\cdots$.

Example 1.37. We illustrate the previous procedure by computing $\mathfrak{h}_d^\bullet(\eta, \eta')$ when $\eta = (\eta_1, \eta_2)$ and $\eta' = (\eta_1', \eta_2')$, satisfying $\eta_1 + \eta_2 = \eta_1' + \eta_2' = d$. We assume $\eta_1 > \eta_1' > \eta_2' > \eta_2$. Then

$$\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')} \langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle$$

Moving $\mathcal{E}_{\eta_2}(0)$ to the right we have

$$\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2}(0)\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle + \varsigma(z\eta_2\eta_1')\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle$$

Moving again $\mathcal{E}_{\eta_2}(0)$ to the right in the first term we get

$$\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2}(0)\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2')\mathcal{E}_{\eta_2}(0) \rangle + \varsigma(z\eta_2\eta_2')\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2-\eta_2'}(z\eta_2') \rangle$$

The first term vanish since $\mathcal{E}_{\eta_2}(0)$ has negative energy. Similarly

$$\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{\eta_2-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle + \varsigma(z\eta_1\eta_1')\langle \mathcal{E}_{\eta_1+\eta_2-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle$$

and

$$\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2-\eta_2'}(z\eta_2') \rangle + \varsigma(z\eta_1\eta_1')\langle \mathcal{E}_{\eta_1-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2-\eta_2'}(z\eta_2') \rangle$$

where the first terms vanish since $\mathcal{E}_{\eta_2-\eta_1'}(z\eta_1')$ and $\mathcal{E}_{-\eta_1'}(z\eta_1')$ have positive energy. It remains to consider

$$\langle \mathcal{E}_{\eta_1+\eta_2-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{-\eta_2'}(z\eta_2')\mathcal{E}_{\eta_1+\eta_2-\eta_1'}(z\eta_1') \rangle + \varsigma\big[z((\eta_1 + \eta_2 - \eta_1')\eta_2' - (-\eta_2')\eta_1')\big]\langle \mathcal{E}_{\eta_1+\eta_2-\eta_1'-\eta_2'}(z(\eta_1' + \eta_2')) \rangle =$$
$$= \varsigma(zd\eta_2')\langle \mathcal{E}_0(zd) \rangle$$

and

$$\langle \mathcal{E}_{\eta_1-\eta_1'}(z\eta_1')\mathcal{E}_{\eta_2-\eta_2'}(z\eta_2') \rangle =$$
$$= \langle \mathcal{E}_{\eta_2-\eta_2'}(z\eta_2')\mathcal{E}_{\eta_1-\eta_1'}(z\eta_1') \rangle + \varsigma\big[z((\eta_1 - \eta_1')\eta_2' - (\eta_2 - \eta_2')\eta_1')\big]\langle \mathcal{E}_{\eta_1-\eta_1'+\eta_2-\eta_2'}(z(\eta_1' + \eta_2')) \rangle$$
$$= \varsigma(z(\eta_1\eta_2' - \eta_2\eta_1'))\langle \mathcal{E}_0(zd) \rangle$$

Putting our results together

$$
\begin{aligned}
\langle \mathcal{E}_{\eta_1}(0)\mathcal{E}_{\eta_2}(0)\mathcal{E}_{-\eta_1'}(z\eta_1')\mathcal{E}_{-\eta_2'}(z\eta_2')\rangle &= \\
&= \varsigma(z\eta_2\eta_2')\varsigma(z\eta_1\eta_1')\varsigma(z(\eta_1\eta_2' - \eta_2\eta_1'))\langle\mathcal{E}_0(zd)\rangle + \varsigma(z\eta_2\eta_1')\varsigma(z\eta_1\eta_1')\varsigma(zd\eta_2')\langle\mathcal{E}_0(zd)\rangle \\
&= \frac{\varsigma(z\eta_2\eta_2')\varsigma(z\eta_1\eta_1')\varsigma(z(\eta_1\eta_2' - \eta_2\eta_1')) + \varsigma(z\eta_2\eta_1')\varsigma(z\eta_1\eta_1')\varsigma(zd\eta_2')}{\varsigma(zd)}
\end{aligned}
$$

and using the following identity for ς

$$
\varsigma(a - b)\varsigma(c) + \varsigma(b - c)\varsigma(a) + \varsigma(c - a)\varsigma(b) = 0
$$

we finally get

$$
\mathfrak{h}_d^\bullet(\eta, \eta') = \frac{1}{\mathfrak{z}(\eta)\mathfrak{z}(\eta')}\frac{\varsigma(z\eta_1\eta_1')\varsigma(z\eta_1\eta_2')\varsigma(z\eta_2 d)}{\varsigma(zd)}
$$