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Chapter 1

The LSZ Reduction Formula

1.1 A new approach to Quantum Field Theory

In this chapter we will make a first contact between operator formalism and path integral formalism of QFT. QFT has main objective of compute:

(i) scattering amplitudes;

(ii) cross-section, where well defined and separated states interacts, creating new final states.

By now, these objective are obtained using canonical quantization formalism. This tool is useful for several situations but, as we will see, there is something more in QFT.

Another point to stress is that the approach we used so far is a perturbative method, which is a really good and precise method, and gave very good results. Nevertheless this method gives an incomplete information. With the standard approach of computing cross section using operator formalism, we missed some informations about the process we are studiying. So is important to have a different tool that allows us to obtain this non perturbative information. This tool is called **Path Integral formalism**.

With this method we will obtain all results we obtained with the perturbative method, but we will be able to understand better our previous results and also something more about QFT, such as non-perturbative effects. This is important because QFT plays, beside its role in Fundamental Interactions, a fundamental role also in Statistical Mechanics, Cosmology, Topology, Geometry and String Theory, where perturbative approach is not so efficient (or it is useless at all).

First of all we need to understand which are limits of operatorial formalism, and then which is the relation between the two formalism.

Limits of the Operatorial Formalism

In the operator formalism approach the step one has to do are:

(i) Pick a time variable t;

(ii) Define an Hamiltonian $H = H(\phi, \pi)$ and the conjugated field

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)}$$

(iii) Promote ϕ and π these into operators

(iv) Impose that at equal time some (anti-)commutation rules between fields are satisfied

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

Pros:

(i) Clear physical content (states, operators), especially manifest unitarity (thanks to Hamiltonian formalism)

Cons:

(i) Covariance not manifest at intermediate stages

(ii) Derivation of Feynman rules can be complicated.
For example, consider a Lagrangian with a term that mixes interaction and kinetic terms

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda\phi^2\partial_\mu\phi\partial^\mu\phi$$

The conjugated field is

$$\pi = \partial_0\phi(1 + 2\lambda\phi^2)$$

and the Hamiltonian reads

$$H = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\frac{\pi^2}{1+2\lambda\phi^2} + \dots$$

So we don't have only π^2 but also an interaction term between π and ϕ. Assume $\lambda \ll 1$, then using perturbative approach

$$H = \frac{1}{2}\pi^2(1 - 2\lambda\phi^2 + 4\lambda^2\phi^4 + \dots)$$

thus we obtain an infinite number of vertex related to terms $\lambda^n\phi^{2n}\pi$. In this case deriving Feynman rules is a really hard task.

(iii) In order to obtain a nice analysis of gauge theories (in particular non-abelian ones) at levels up to the tree level the perturbative approach is not so good. To obtain proprieties of these theories we must go beyond using use path integral approach.

1.2 Correlators and the LSZ reduction formula

Important: *In this section (and also often in the following) we will use the metric*

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In this section we will obtain the connection between perturbative formalism and path integral formalism. Recall from previous courses that *correlators* or *Green functions* contains most of informations about QFT

$$G(x_1, x_2, \dots, x_n) = \langle 0|\, T[\phi(x_1), \dots, \varphi(x_n)]\, |0\rangle$$

Covariant convention for states

Srednicki [14] chap 3.

We start with a very simple theory, a free massive scalar field with lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

whose equation of motion is

$$\Box\phi = m^2\phi$$

and normalized solutions are in the form of plane waves[I]

$$\frac{1}{\sqrt{(2\pi)^3 2k^0}}e^{\pm ikx} \qquad \text{where} \quad k^2 = \mathbf{k}^2 - (k^0)^2 = -m^2$$

[I] $x^\mu = (t, \mathbf{x})$ and $k^\mu = (k^0, \mathbf{k})$.

I can write scalar field in terms of α and $\alpha^\dagger$ operators:[II]

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3 2k^0}} \left(\alpha(\mathbf{k})e^{ikx} + \alpha^\dagger(\mathbf{k})e^{-ikx}\right) \tag{1.1}$$

When we quantize the theory we introduce the commutation relations

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = 0 = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})]$$

Equivalently, these relation can be imposed saying that α, $\alpha^\dagger$ are annihilation and creation operators that satisfy

$$[\alpha(\mathbf{k}), \alpha(\mathbf{k}')] = 0 = [\alpha(\mathbf{k}), \alpha(\mathbf{k}')]$$

$$[\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}')$$

Let's introduce an alternative notation in oder to obtain a covariant algebra for ladder operators and a covariant normalization for states. We introduce the manifestly Lorentz invariant[III] measure (under $SO^+(3,1)$)

$$\mathrm{d}^4k\, \delta(k^2+m^2)\theta(k^0)$$

Notice that the theta function picks an arrow of time (i.e. particles or antiparticles). Using proprieties of delta function, we know that $\delta(k^2+m^2)$ can be splitted as follows

$$\delta(k^2+m^2) = \frac{1}{2k^0}\left[\delta\left(k^0 - \sqrt{m^2+\mathbf{k}^2}\right) + \delta\left(k^0 + \sqrt{m^2+\mathbf{k}^2}\right)\right]$$

Then we have ($\omega := \sqrt{m^2+\mathbf{k}^2}$)

$$\int \mathrm{d}k^0 \delta(k^2+m^2)\theta(k^0) = \frac{1}{2\omega}$$

thus we see that if we replace the measure in eq. (1.1) with a measure proportional to $\mathrm{d}^3\mathbf{k}/2\omega$ then this will be Lorentz invariant. In particular we choose the Lorentz invariant differential as follows, obtaining a new expansion of $\phi(x)$ in terms of new ladder operators

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}}{2k^0}\left(a(k)e^{ikx} + a^\dagger(k)e^{-ikx}\right)$$

As a consequence of commutation relations, new ladder operators obey the following covariant algebra

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0$$

$$[a(k), a^\dagger(k')] = (2\pi)^3 2k^0 \delta^3(\mathbf{k}-\mathbf{k}')$$

and we obtain a Fock space made of covariant normalized states

$$\langle 0|0\rangle = 1, \qquad |k\rangle = a^\dagger(k)\,|0\rangle$$

$$\langle k|k'\rangle = (2\pi)^3 2k^0 \delta^3(\mathbf{k}-\mathbf{k}')$$

In the free theory we can also write down a and $a^\dagger$ in function of the field, in particular for the free scalar theory we have

$$a^\dagger(k) = -i\int \mathrm{d}^3x\, (e^{ikx} \overleftrightarrow{\partial_0} \phi) \tag{1.2}$$

where $\overleftrightarrow{\partial_0} := \overrightarrow{\partial_0} - \overleftarrow{\partial_0}$.

[II] If I consider a real field $\phi(x) = \phi^\dagger(x)$

[III] We don't consider full Lorentz group but only transformations that preserve time, i.e. transformations with determinant equal to 1 (proper Lorentz transformations).

Scattering amplitude in interacting theory

Srednicki [14], chap 5.

When we do a scattering experiment (which depend on the momenta of particles) a and $a^\dagger$ will became time dependent operators. Consider a generic lagrangian in the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_0$ is the free field Lagrangian while $\mathcal{L}_{int}$ is an interaction term. Initial and final states takes the from[IV]

$$|i\rangle = \lim_{t\to-\infty} a^\dagger(k_1,t)\dots a^\dagger(k_n,t)\,|0\rangle$$
$$|f\rangle = \lim_{t\to+\infty} a^\dagger(k'_1,t)\dots a^\dagger(k'_n,t)\,|0\rangle$$

This states must be normalized, for instance we can take

$$\langle i|i\rangle = 1 = \langle f|f\rangle$$

Now we look for an explicit expression for the scattering amplitude

$$S_{fi} = \langle f|i\rangle$$

in the interaction formalism. In order to obtain an useful expression for that we better try to express latter operators in terms of fields of the theory. Let's start from the free field, we can write, using eq. (1.2)

$$\begin{aligned}
a^\dagger(k,+\infty) - a^\dagger(k,-\infty) &= \int_{-\infty}^{+\infty} \mathrm{d}t\,\partial_0 a^\dagger(k,t) \\
&= \int \mathrm{d}t\,\partial_0 \left(-i\int_{-\infty}^{+\infty} \mathrm{d}^3x (e^{ikx}\partial_0\phi - \partial_0 e^{ikx}\phi)\right) \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(\partial_0^2 + (k^0)^2)\phi \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(\partial_0^2 + m^2 + \mathbf{k}^2)\phi \\
&= -i\int \mathrm{d}^4x\, [e^{ikx}(\partial_0^2 + m^2)\phi - \phi\vec{\nabla}^2 e^{ikx}] \\
&= -i\int \mathrm{d}^4x\, [e^{ikx}(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi] + \text{surface term} \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(-\partial_\mu\partial^\mu + m^2)\phi
\end{aligned} \tag{1.3}$$

and vanishes in the free theory because $(\Box - m^2)\phi = 0$.

Let's assume that (1.2) holds also in the interaction case, then also the latter formula holds, but in this case the last term in general is not vanishing. Consider scattering amplitude

$$S_{fi} = \langle f|i\rangle = \langle 0|\, a(k'_1,+\infty)\dots a(k'_m,+\infty)a^\dagger(k_1,-\infty)\dots a^\dagger(k_n,-\infty)\,|0\rangle$$

since operators are time ordered I can write

$$S_{fi} = \langle 0|\, T\left[a(k'_1,+\infty)\dots a(k'_m,+\infty)a^\dagger(k_1,-\infty)\dots a^\dagger(k_n,-\infty)\right]|0\rangle$$

Using eq. (1.3) and its complex conjugated, i can rewrite[V]:

[IV] Notice that now a and $a^\dagger$ are ladder operators for the interacting theory, no more for the free theory.

[V] The time ordering product kills undesired ladder operators in eq. (1.3) since their action on the vacuum gives zero.

$$S_{fi} = i^{m+n} \int \prod_{i=1}^{m} \mathrm{d}^4 x_i' \, e^{-ik_i' x_i'} (-\Box_{x_i'} + m^2) \prod_{j=1}^{n} \mathrm{d}^4 x_j \, e^{ik_j x_j} (-\Box_{x_j} + m^2) \times \\ \times \underbrace{\langle 0 | \, T \left[\phi(x_1') \dots \phi(x_m') \phi(x_1) \dots \phi(x_n) \right] | 0 \rangle}_{G(x_1', \dots, x_n', x_1, \dots, x_n)} \tag{1.4}$$

Eq. (1.4) is the **Lehmann-Symanzik-Zimmermann (LSZ) reduction formula**, it express the scattering amplitude in terms of Green functions. Here is clear the importance of the Green functions, as they allows to compute any scattering amplitudes.

Let's recall a crucial assumption we made: formula (1.2) must holds in the interacting case. This is the case only if a couple of conditions are satisfied:

(i) $\langle 0 | \, \phi(x) \, | 0 \rangle = 0$

This means that whenever you start from the state $|0\rangle$ the operator $a^\dagger$ must creates a particle state, orthogonal to the vacuum. This is obvious for free theories, but does not holds in general for interacting theories. In some cases $\phi(x) \, |0\rangle$ is a linear combination of a one particle state and the vacuum state.

So, if[VI] $v = \langle 0 | \, \phi(x) \, | 0 \rangle = \langle 0 | \, e^{-i\hat{p}x} \phi(0) e^{i\hat{p}x} \, | 0 \rangle = \langle 0 | \, \phi(0) \, | 0 \rangle$ is a Lorentz invariant number different to zero, we will shift the field $\phi(x)$ by the constant v:

$$\phi(x) \to \phi(x) + v = \tilde{\phi}(x)$$

This is just a change in the name of the operator of interest, and does not affect the physics. However, the shifted $\tilde{\phi}(x)$ obeys, by costruction, $\langle 0 | \, \tilde{\phi}(x) \, | 0 \rangle = 0$.

(ii) $\langle k | \, \phi(x) \, | 0 \rangle = e^{-ikx}$

This is what it is in the free theory, and we know that in free theory, $a^\dagger(\pm\infty)$ creates a correctly normalized one-particle state. Thus, for $a^\dagger(\pm\infty)$ creates a correctly normalized one-particle state in the interacting theory, we must have $e^{-ikx} = \langle k | \, \phi(x) \, | 0 \rangle = \langle k | \, e^{-i\hat{p}x} \phi(0) e^{i\hat{p}x} \, | 0 \rangle = e^{-ikx} \langle k | \, \phi(0) \, | 0 \rangle$.

So, if $Z_\phi^{1/2} = \langle k | \, \phi(0) \, | 0 \rangle$ is a Lorentz invariant number different to one, we will rescale (or, *renormalize* $\phi(x)$ by a multiplicative constant:

$$\phi(x) \to Z_\phi^{1/2} \phi = \phi_{int}$$

This is just a change of the name of the operator of interest, and does not affect the physics. However, the rescaled $\phi(x)$ obeys, by costruction, $\langle k | \, \phi(0) \, | 0 \rangle = 1$.

Let us recap. The basic formula for scattering amplitude in terms of the fields of an interacting quantum field theory is the LSZ formula (1.4). The LSZ formula is valid *provided* that the fields obeys

$$\langle 0 | \, \phi(x) \, | 0 \rangle = 0 \quad \text{and} \quad \langle k | \, \phi(x) \, | 0 \rangle = e^{-ikx} \tag{1.5}$$

These normalization conditions may conflict with our original choice of field and parameter normalization in the lagrangian. Consider, for example, a lagrangian originally specified as

$$\mathcal{L} = \underbrace{-\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2}_{\mathcal{L}_0} + \underbrace{\frac{1}{6} g \phi^3}_{\mathcal{L}_{int}}$$

After shifting and rescaling (and renaming some parameters), we will have instead

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_{CT} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{6} Z_g g \phi^3 + Y \phi$$

where

$$\mathcal{L}_{CT} = -\frac{1}{2} (Z_\phi - 1) \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} (Z_m - 1) m^2 \phi^2 + \frac{1}{6} (Z_g - 1) g \phi^3 + Y \phi$$

[VI] We assume that $\hat{p} \, |0\rangle = 0$.

is a new term called *counter-Lagrangian* and terms $Z_\bullet$ and Y are called *counter-terms*. Here Z_ϕ, Z_m, Z_g and Y are yet unknown constant. They must be chosen to ensure the validity of eq.(1.5); this gives us two conditions in four unknown. We fix the parameter $Z_m m^2$ by requiring to be equal to the actual mass of the particle (equivalently, the energy of the first excited state relative to the ground state), and we fix the parameter $Z_g g$ by requiring some particular scattering cross section to depend on the coupling term in some particular way. So we have four conditions in four unknown, and it is possible to calculate Y and the three Zs order by order in powers of g.

Next, we must develop the tools needed to compute the correlation functions $G(x_1, \ldots, x_n)$ in an interacting quantum field theory.

Chapter 2

The Path integral in Quantum Mechanics

Path Integral will be the main tool we will use in this course in order to study QFT. We will start introducing this tool in special case of Quantum Mechanics.

2.1 Intuitive Introduction to Path Integrals

Rattazzi [10] sec. 1.1.1 - 1.1.3

One of the important experiments that show the fundamental difference between Quantum and Classical Mechanics is the double slit experiment. It is interesting with respect to the path integral formalism because it leads to a conceptual motivation for introducing it.

Consider a source S of approximatively mono-energetic particles, electrons for instance, placed at position (x_i, y_i). The flux of electrons is measured on a screen facing the source. Imagine now placing a third screen in between the ohers, with two slits on it, which can be opened or closed. When the first is open and the second closed we measure the flux F_1, when the first slit is closed and the second open we measure a flux F_2 and when both slits are open we measure the flux F.

One finds in general $F = F_1 + F_2 + F_{int}$, and the structure of F_{int} precisely corresponds to the interference between two waves passing respectively through 1 and 2:

$$F = |\phi_1 + \phi_2|^2 = \underbrace{|\phi_1|^2}_{F_1} + \underbrace{|\phi_2|^2}_{F_2} + \underbrace{\phi_1\phi_2^* + \phi_2\phi_1^*}_{F_{int}}$$

where ϕ_i is the probability amplitude for a point-like particle position and $|\phi_i|^2$ the corresponding probability density.

The idea behind the path integral approach to QM is to take the implications of the double slit experiment to its extreme consequences. One can immagine adding extra screens and drilling more and more holes through them, generalising the result of the double slit experiment by the superposition principle.

Let's denote as follows the superposition of N slits fluxes:

$$\Phi = \sum_{i=1}^{N} \phi(y_A^i)$$

where $\phi(y_A^i)$ denotes the flux of the particle whose trajectory goes through the i-th slit positioned in the coordinate y_A^i of the screen placed in x_A. Nothing stops us from taking the ideal limit where $N \to \infty$ and the holes fill all the surface. The sum $\sum_i$ becomes now an integral over y_A:

$$\Phi = \int \mathrm{d}y_A \phi(y_A)$$

We can go on and further refine our trajectories by adding more and more screens between the source and the final screen. In the limit in which the added screens become infinitesimally close, we have specified all possible paths $y(x)$:

$$\Phi = \int \mathrm{d}x \int \mathrm{d}y_x \phi(y(x))$$

where $\phi(y(x))$ is the flux corresponding to the path $y(x)$.

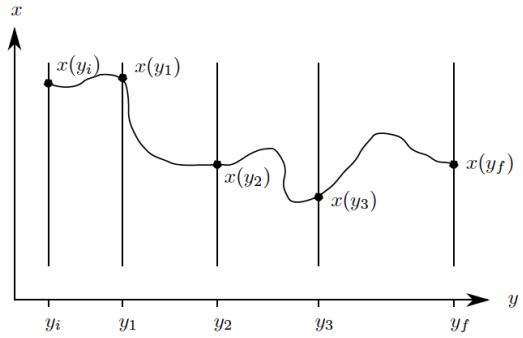


Figure 2.1: y_i denotes position of screens, while $x(y_i)$ are the positions where the particle is found in each screen. Notice that the role of x and y are inverted respect to our notation.

We then arrived at a formal representation of the probability amplitude as a sum over all possible trajectories:

$$\Phi = \sum_{\substack{\text{All trajectories} \\ \{x(t),y(t)\}}} \phi(\{x\}) \tag{2.1}$$

How do I make sense of this? What Φ is and how do I make sense of this sum? Moreover, I'd like that for $\hbar \to 0$ I should go back to Classical Mechanics.[I] This implies that ϕ must depends on $\hbar$ in someway. Since $\hbar$ has dimensionality [Energy]×[Time], one can guess that

$$\phi(\gamma) = e^{i\frac{S[\gamma]}{\hbar}} \tag{2.2}$$

where S is the action which describes the classical trajectories via the principle of least action. This means that we associate to each trajectory γ a phase related to the action $S[\gamma]$. Recall that the classical trajectories are given by the stationary points of $S[\gamma]$ ($\delta S[\gamma] = 0$).

Let's analyze our guess. The choice $\phi(\gamma) = f(S[\gamma]/\hbar)$ is natural, since in this way the argument is adimensional. The choice of the exponential function seems promising for two reasons:

(i) The requirement $\delta S[\gamma] = 0$ for $\hbar \to 0$ is heuristically seen to hold. In a macroscopic, classical, situation the gradient $\delta S/\delta\gamma$ is for most trajectories much greater than $\hbar$. Around such trajectories the phase $e^{iS/\hbar}$ oscillates extremely rapidly and the sum over neighbouring trajectories will tend to cancel.

On the other hand, on a classical trajectory γ_{cl} the action S is stationary. Therefore in the neighbourhood of γ_{cl}, S varies very little, so that all trajectories in a tube centred around γ_{cl} add up coherently in the sum over trajectories.

Indeed, this means that in the exact limit $\hbar \to 0$ these effects becomes dramatic and only the classical trajectory survives.

(ii) Eq. (2.2) leads to crucial composition property. Indeed the action for a path γ_{12} obtained by joining two subsequent paths γ_1 and γ_2 satisfies the simple additive relation

$$S[\gamma_{12}] = S[\gamma_1] + S[\gamma_2]$$

[I] See Rattazzi and MacKenzie chap. 3.

Thanks to eq.(2.2) the additivity of S translates into a factorization property for the amplitude:

$$\phi[\gamma_{12}] = \phi[\gamma_1]\phi[\gamma_2]$$

This feature is required for our theory, as it describes composition of paths and the corresponding probability.

2.2 From Schroedinger Equation to the Path Integral

Rattazzi [10], sec. 1.1.4; Sredniki [14] chap. 6; Peskin [7] chap. 9.1; MacKenzie [5] sec. 2.1, 2.2.1

In the previous discussion gave us the idea behind Path Integrals formulation in QM, now we will derive it more formally starting from Schrödinger equation.

The transition amplitude for a particle in QM reads

$$\langle x_f(t_f)|x_i(t_i)\rangle = \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t_i)}\,|x_i\rangle$$

In order to evaluate this quantity we split $t_f - t_i$ in N pieces with N large. Let $\delta t = (t_f - t_i)/N$. Recall that $\int \mathrm{d}x\,|x\rangle\langle x| = 1 = \int \mathrm{d}p\,|p\rangle\langle p|$. Then

$$\langle x_f(t_f)|x_i(t_i)\rangle = \int \left(\prod_{j=1}^{N-1}\mathrm{d}x_j\right) \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x_{N-1}\rangle\langle x_{N-1}|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x_{N-2}\rangle \dots \langle x_2|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x_1\rangle\langle x_1|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x_i\rangle \tag{2.3}$$

For each piece we have

$$\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle = \int \mathrm{d}p\,\langle x'|p\rangle\langle p|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle \tag{2.4}$$

If we stick to the simply case $\hat{H} = \hat{p}/2m + V(\hat{x})$ where we denote the kinetic operator as $T(\hat{p}) = -i\frac{\hat{p}^2}{2m\hbar}\delta t$ and the potential $U(\hat{x}) = -iV(\hat{x})\delta t/\hbar$, we can write

$$\langle p|\exp\{T(\hat{p})+U(\hat{x})\}\,|x\rangle = \langle p|\, e^{T(\hat{p})}e^{-T(\hat{p})}e^{T(\hat{p})+U(\hat{x})}e^{-U(\hat{x})}e^{U(\hat{x})}\,|x\rangle = e^{T(p)}e^{U(x)}\langle p|\, e^{C(\hat{p},\hat{x})}\,|x\rangle$$

where

$$e^{C(\hat{p},\hat{x})} = e^{-T(\hat{p})}e^{T(\hat{p})+U(\hat{x})}e^{-U(\hat{x})}$$

The operator C is given, using Baker-Campbell-Hausdorff formula [II] twice, as a series of commutators between T and U

$$C = \frac{1}{2}[T,U] + \frac{1}{6}([T,[T,U]] + [U,[U,T]]) + ...$$

If all commutators in the expression of C are $O(1)$, then all the terms of the expansion of C are $O((\delta t)^2)$ and therefore can be neglected. However, this assumption is not immediate, for example $[\hat{p}, V(\hat{x})] = -iV'(\hat{x})$ implies that in order to neglect C all derivatives of V must be bounded. Then, if the derivatives of V are bounded, the contribution of the operator $\hat{x}$ in the expansion of C is a bounded contribution and in the limit $\delta t \to 0$ vanishes.

The only remaining potential problem to concentrate on the $\delta t \to 0$ limit is represented by the integration over powers of p in (2.4). Indeed what we are integrating is a function that goes approximatively as the gaussian $\exp\left(-\delta t\, p^2\right)$, therefore the leading contribution to the p integral is the one with $p \sim \delta t^{-1/2}$, showing that p diverges in the small δt limit. Nevertheless one can actually prove[III] that for small δt:

$$\langle p|\, e^{C(\hat{p},\hat{x})}\,|x\rangle \simeq \langle p|x\rangle\,(1 + O(\delta t^{3/2}))$$

[II] Let $e^X e^Y = e^Z$ for some operators X,Y,Z, then

$$Z(X,Y) = \log(\exp X \exp Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) - \frac{1}{24}[Y,[X,[X,Y]]] + \dots$$

[III] See Rattazzi pag 14 for details.

and even if I consider all $N = 1/\delta t$ contributions of the form $\langle p|\, e^{C(\hat{p},\hat{x})}\, |x\rangle$ in (2.3), they can be neglected, since the final result is convergent to 1:

$$\lim_{\delta t\to 0}(1 + a\,\delta t^{3/2})^{1/\delta t} = 1$$

Therefore we can reasonably neglect contributions of C and then

$$\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle \simeq \lim_{\delta t\to 0}\int \mathrm{d}p \exp\left\{-i\frac{\delta t}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]\right\}\cdot \underbrace{\frac{\exp\{ip(x'-x)/\hbar\}}{2\pi\hbar}}_{\langle x'|p\rangle\langle p|x\rangle}$$

Now we introduce the variable $\dot{x} = (x' - x)/\delta t$:

$$\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle = \lim_{\delta t\to 0}\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left\{-i\frac{\delta t}{\hbar}\left(\frac{p^2}{2m} + V(x) - p\dot{x}\right)\right\}$$

and performing the change of variable $p' = p - m\dot{x}$ we obtain

$$\begin{aligned}\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle &= \lim_{\delta t\to 0}\int \frac{\mathrm{d}p'}{2\pi\hbar}\exp\left\{-i\frac{\delta t}{\hbar}\left(\frac{p'^2}{2m} + \underbrace{V(x) - \frac{1}{2}m\dot{x}^2}_{-\mathcal{L}}\right)\right\}\\ &= \lim_{\delta t\to 0}\underbrace{\sqrt{\frac{m}{2\pi i\hbar\,\delta t}}}_{\kappa}\exp\left\{i\frac{\delta t}{\hbar}\mathcal{L}(x,\dot{x})\right\}\end{aligned}$$

where we introduced the factor κ (that depends on δt) in order to simplify the notation. When I introduce this into eq. (2.3) we have the following formula for the transition amplitude ($x_0 = x_i$, $x_N = x_f$):

$$\begin{aligned}\langle x_f(t_f)|x_i(t_i)\rangle &= \lim_{\delta t\to 0}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\kappa^N\exp\left\{\frac{i}{\hbar}\sum_{m=0}^{N-1}\delta t\,\mathcal{L}(x_m,\dot{x}_m)\right\}\\ &= \lim_{\delta t\to 0}\kappa\int\prod_{j=1}^{N-1}(\mathrm{d}x_j\,\kappa)\exp\left\{\frac{i}{\hbar}S(x_f,x_i)\right\}\end{aligned}$$

We define the following functional measure over the space of trajectories[IV]:

$$\boxed{\int_{C_T[x_i,x_f]}\mathcal{D}x = \lim_{\delta t\to 0}\kappa\int\prod_{j=1}^{N-1}(\mathrm{d}x_j\,\kappa)} \tag{2.5}$$

Here $C_T[x_i, x_f]$ are all possible configurations on x that start in x_i and end in x_f over a time $T = t_f - t_i$, i.e. is the set of all possible path $x(t)$ such that $x(t_i) = x_i$ and $x(t_i + T) = x(t_f) = x_f$. We call $C_T[x_i, x_f]$ **space of configurations**. The weight $\exp\{\frac{i}{\hbar}S(x_f, x_i)\}$ is the phase related to eq.(2.2). Therefore I obtain

$$\boxed{\langle x_f(t_f)|x_i(t_i)\rangle = \int_{C_T[x_i,x_f]}\mathcal{D}x\,\exp\left\{\frac{i}{\hbar}S(x_f,x_i)\right\}} \tag{2.6}$$

This is just the analogous of eq.(2.1). Notice that eq.(2.6) is manifestly invariant under the symmetries of our theory.

The construction we made has no rigor, but clarify the idea behind Path Integrals theory. The result we obtained so far is the replacement of the sum into eq.(2.1) with an integral, hoping in a convergent expression. This is not obvious, indeed this does not happen in general.

This definition has some problematic points, that sometimes does not matter and one can skip on them in a straightforward way, but can also became crucial in order to obtain the results we obtained in

[IV] For the moment is not clear if this functional is well defined and is a measure, this is just an anticipation of following results.

operatorial approach. For example we assumed that trajectories were smooth and we can differentiate them, but this is not going to be true. Quite the opposite, the fact that trajectories may not be smooth leads to the contact between Path Integral approach and operatorial formalism of QM.

We won't examinate details of mathematical structure of functional formalism behind Path Integrals since it is not really interesting from the physical point of view, rather we focus on problematical aspects and special features of this formalism in order to obtain a deeper understanding of the physics.

Before proceeding with technical developments, it is worth assessing the role of the path integral in quantum mechanics. As it was hopefully highlighted so far, the path integral formulation is conceptually advantageous over the standard operatorial formulation of QM, in that the "good old" particle trajectories retain same role. The P.I. is however technically more involved. When working on simple quantum systems like the hydrogen atom, no technical profit is really given by path integrals. Nonetheless, after overcoming a few technical difficulties, the path integral offers a much more direct viewpoint on the semiclassical limit. Similarly, for issues involving topology (like the origin of Bose and Fermi statistics, the Aharonov-Bohm effect, charge quantization in the presence of a magnetic monopole, etc...) path integrals offer a much better viewpoint. Finally, for advanced issues like the quantization of gauge theories and for effects like instantons in quantum field theory it would be hard to think how to proceed without path integrals.

2.3 The Partition Function

Skinner [12] chap. 1

We want to generalize results of the previous section. What we have done, is to sum over all possible configurations of the field with a certain measure and weight them with a weight function that is essentially the action:

$$\boxed{\mathcal{Z}(\lambda, m, \dots) = \int_{\phi \in C} \mathcal{D}\phi \, \exp\left\{ \frac{i}{\hbar} S[\phi] \right\}} \tag{2.7}$$

This is the generic form for a **path integral** and in particular is the **partition function** for a certain theory. In order to describe the theory we have to specify:

(i) M: The manifold where out QFT theory lives, in particular we have to specify its dimension d and, if it exists, the space's metric g (we can also think about a metric with some degree of freedom, such in Quantum Gravity).

(ii) ϕ: Fields over the space, that are generically maps from M to some target space (e.g. $\mathbb{R}$, $\mathbb{C}$, vector fields, gauge group, gauge bundle, etc.)

(iii) C: Space of allowed field configurations (possibly with some boundary conditions)

(iv) λ: Other parameters of the theory (such as mass m etc.)

(v) $\mathcal{L}$: The Lagrangian of our theory

(vi) S: The action (depends on all the other parameters)

$$S[\phi] = \int_M \mathrm{d}^d x \sqrt{|g|} \mathcal{L}(\phi, \partial\phi, \lambda, m, \dots)$$

(vii) $\mathcal{D}\phi$: the measure of integration

Once I have specified this items i can obtain a well defined partition function form my theory eq.(2.7). Moreover starting with partition function we can compute correlation functions. On the other hand partition function can depends on other parameters λ' to which the action is independent. Notice that eq.(2.7) can be also a divergent, but this happens often and doesn't really matter. The important aspect is how the behaviour of the partition function depends on its parameters.

The name "partition function" comes from statistical mechanics[V]: consider for instance a map from the circle to another target space

$$x : S^1 \to N \qquad x(0) = x(T) = y \in N$$

When we go to the Euclidian space by sending[VI] $t \to i\tau$ ($\hbar = 1$) and we compute the path integral for some boundaries condition we obtain (notice that initial and final positions are the same)

$$\int_{x \in C_T[y,y]} \mathcal{D}x\, e^{-S[x]} = \langle y|\, e^{-HT}\, |y\rangle$$

and this is related to partition function of statistical mechanics, since this means that calculating the statistical partition function by tracing over all the Hilbert space $\mathcal{H}$

$$\mathrm{Tr}_{\mathcal{H}}(e^{-HT}) = \int \mathrm{d}y\, \langle y|\, e^{-HT}\, |y\rangle = \int \mathrm{d}y \int_{x \in C_T[y,y]} \mathcal{D}x e^{-S[x]}$$

is the same as taking path integral without boundaries, i.e. integrating over all possible configurations $y \in S^1$.

$$\mathrm{Tr}_{\mathcal{H}}(e^{-HT}) = \int_{x \in C_{S1}} \mathcal{D}x e^{-S[x]}$$

2.4 Operators and Time Ordering

Rattazzi [10] sec. 1.4, Skinner [12] sec. 1.2, 3.1; MacKenzie [5] chap. 6, 7; Zinn-Justin [18] sec. 2.4.2

Now we want to see how to use path integral in order to compute operator matrix elements and more in general correlation functions. Notice that this problem depends on which picture (Heisenberg or Schrödinger) we use. In the Heisenberg picture operators evolve with time according to

$$\hat{O}(t) = e^{\frac{i\hat{H}t}{\hbar}} \hat{O} e^{\frac{-i\hat{H}t}{\hbar}}$$

where $\hat{O}$ is the time independent operator in the Schröedinger picture, while states are time independent. In Schröedinger picture operators are time independent, while states evolves according to

$$|t\rangle = e^{\frac{i\hat{H}t}{\hbar}}\, |t = 0\rangle$$

For instance if we want to compute the value of position $\hat{x}$ on a state for a given time t we see that

$$\hat{x}(t)\, |x, t\rangle = e^{\frac{i\hat{H}t}{\hbar}} \hat{x} e^{-\frac{i\hat{H}t}{\hbar}} e^{\frac{i\hat{H}t}{\hbar}}\, |x, t = 0\rangle = x e^{\frac{i\hat{H}t}{\hbar}}\, |x, t = 0\rangle = x\, |x, t\rangle$$

We want to prove that matrix elements of *local* operators[VII] can be computed using path integrals using the following formula

$$\boxed{\int_{C_{t_f - t_i}[x_i, x_f]} \mathcal{D}x\, O(x(t)) \exp\left\{\frac{i}{\hbar} S[x]\right\} = \langle x_f, t_f|\, \hat{O}(\hat{x}(t))\, |x_i, t_i\rangle} \tag{2.8}$$

Therefore we have to relate matrix elements with the expression eq. (2.6). In order to do that I have to find a way to remove $\hat{O}(\hat{x}(t))$ by substituting it with some function that I can integrate. The way to do

[V] See also MacKenzie [5] chap. 5, Zinn-Justin [18] sec. 2.4 and Nakahara [6] sec. 1.3.1, 1.3.2 for a furthered description of this correspondence between path integral formulation and partition functions in statistical mechanics.

[VI] Notice that parametrizing S^1 into the Euclidian, we introduced an additional factor i, that multiplied by the factor i present in the definition of partition function gives a minus. This will happen often when we compute partition integrals explicitly.

[VII] Local operators are operators that depend on the value of the field (and perhaps finitely many derivatives) at a single point in M.

that is to compute each matrix element by evaluating operators on the states using eigenstates of the position:

$$\begin{aligned}
\langle x_f, t_f|\, \hat{O}(\hat{x}(t))\, |x_i, t_i\rangle &= \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t)}\hat{O}(\hat{x})e^{-i\frac{\hat{H}}{\hbar}(t-t_i)}\, |x_i\rangle \\
&= \int \mathrm{d}x\, \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t)}\hat{O}(\hat{x})\, |x\rangle\, \langle x|\, e^{-i\frac{\hat{H}}{\hbar}(t-t_i)}\, |x_i\rangle \\
&= \int \mathrm{d}x\, O(x)\, \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t)}\, |x\rangle\, \langle x|\, e^{-i\frac{\hat{H}}{\hbar}(t-t_i)}\, |x_i\rangle \\
&= \int \mathrm{d}x \int_{C_{t_f-t}[x,x_f]} \mathcal{D}x_a \int_{C_{t-t_i}[x_i,x]} \mathcal{D}x_b \exp\left\{\frac{i}{\hbar}(S[x_a]+S[x_b])\right\} O(x) \\
&= \int_{C_{t_f-t_i}[x_i,x_f]} \mathcal{D}x\, \exp\left\{\frac{i}{\hbar}S[x]\right\} O(x(t))
\end{aligned} \tag{2.9}$$

where in the last step we composed all possible paths.

Now we want to see how do compute correlators, i.e. products of operators. But the generalization of what we done is really simple. Consider two functions $O_1(x(t))$ and $O_2(x(t))$, then we will prove that

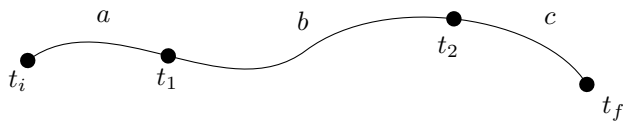
$$\boxed{\int_{C_{t_f-t_i}[x_i,x_f]} \mathcal{D}x\, O_1(x(t_1))O_2(x(t_2)) \exp\left\{\frac{i}{\hbar}S[x]\right\} = \langle x_f, t_f|\, T\left[\hat{O}_1(x(t_1))\hat{O}_2(x(t_2))\right]\, |x_i, t_i\rangle} \tag{2.10}$$

where we used the **time ordering product**

$$T\left[\hat{O}_1(t_1)\hat{O}_2(t_2)\right] = \theta(t_2-t_1)\hat{O}_2(t_2)\hat{O}_1(t_1) + \theta(t_1-t_2)\hat{O}_1(t_1)\hat{O}_2(t_2)$$

Now we show eq. (2.10). Let's assume that $t_2 > t_1$ for the time being. Then using composition of paths

$$\begin{aligned}
&\int_{C_{t_f-t_i}[x_i,x_f]} \mathcal{D}x\, O_1(x(t_1))O_2(x(t_2)) \exp\left\{\frac{i}{\hbar}S[x]\right\} = \\
&\quad= \int \mathrm{d}x_1\mathrm{d}x_2 O_1(x_1)O_2(x_2)\ \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t_2)}\, |x_2\rangle\, \langle x_2|\, e^{-i\frac{\hat{H}}{\hbar}(t_2-t_1)}\, |x_1\rangle\, \langle x_1|\, e^{-i\frac{\hat{H}}{\hbar}(t_1-t_i)}\, |x_i\rangle \\
&\quad= \int \mathrm{d}x_1\mathrm{d}x_2\ \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t_2)}\hat{O}_2(\hat{x})\, |x_2\rangle\, \langle x_2|\, e^{-i\frac{\hat{H}}{\hbar}(t_2-t_1)}\hat{O}_1(\hat{x})\, |x_1\rangle\, \langle x_1|\, e^{-i\frac{\hat{H}}{\hbar}(t_1-t_i)}\, |x_i\rangle \\
&\quad= \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t_2)}\hat{O}_2(\hat{x})e^{-i\frac{\hat{H}}{\hbar}(t_2-t_1)}\hat{O}_1(\hat{x})e^{-i\frac{\hat{H}}{\hbar}(t_1-t_i)}\, |x_i\rangle \\
&\quad= \langle x_f, t_f|\, \hat{O}_2(\hat{x}(t_2))\hat{O}_1(\hat{x}(t_1))\, |x_i, t_i\rangle
\end{aligned}$$



When we consider $t_2 < t_1$ the final result is

$$\langle x_f, t_f|\, \hat{O}_1(\hat{x}(t_1))O_2(\hat{x}(t_2))\, |x_i, t_i\rangle$$

therefore eq. (2.10) has been proved. Using eq. (2.10) we can compute matrix element of time ordering products using path integrals.

2.5 The Continuum Limit and Non-Commutativity

Skinner [12] sec. 3.2

Let's go back to an issue we considered in the introduction of the path integral. Now we analyze what happens if the paths we consider are not only the one someone expects, i.e. consider the case where paths are in general not differentiable. The non-commutativity of operators

$$[\hat{x}(t), \hat{p}(t)] = i\hbar \tag{2.11}$$

is somehow necessary in order for the path integral to work. And it is intimately related to the existence of not differentiable path that has to be taken into account in the path integral. We want to show that when we consider all possible path in the computation of the path integral, also non-differentiable path must be taken into account, and this non-differential paths are those to become important when we see non-commutativity of some operators in QM.

Let's consider only smooth paths, in particular where $\dot{x}(t)$ exists. Then for t_- slightly smaller than t we have[VIII]

$$\langle x_f, t_f | \, \hat{x}(t)\, \hat{p}(t_-) \, | x_i, t_i \rangle = \int_{C_{t_f - t_i}[x_i, x_f]} \mathcal{D}x \, x(t)\, \dot{x}(t_-) e^{\frac{i}{\hbar} S[x]}$$

Similarly, for t_+ slightly greater that t

$$\langle x_f, t_f | \, \hat{p}(t_+)\, \hat{x}(t) \, | x_i, t_i \rangle = \int_{C_{t_f - t_i}[x_i, x_f]} \mathcal{D}x \, x(t)\, \dot{x}(t_+) e^{\frac{i}{\hbar} S[x]}$$

Since we are considering only smooth path, we have

$$\begin{aligned} &\lim_{t_\pm \to t} \int_{C_{t_f - t_i}[x_i, x_f]} \mathcal{D}x \, x(t) (\dot{x}(t_-) - \dot{x}(t_+)) e^{\frac{i}{\hbar} S[x]} = \\ &\quad = \int_{C_{t_f - t_i}[x_i, x_f]} \mathcal{D}x \, x(t) \left(\lim_{t_\pm \to t} (\dot{x}(t_-) - \dot{x}(t_+)) \right) e^{\frac{i}{\hbar} S[x]} = 0 \end{aligned}$$

and then

$$\begin{aligned} &\langle x_f, t_f | \, [\hat{x}(t)\, \hat{p}(t)] \, | x_i, t_i \rangle = \\ &\quad = \lim_{t_\pm \to t} \left(\langle x_f, t_f | \, \hat{x}(t)\, \hat{p}(t_-) \, | x_i, t_i \rangle - \langle x_f, t_f | \, \hat{p}(t_+)\, \hat{x}(t) \, | x_i, t_i \rangle \right) = 0 \end{aligned}$$

which is clearly in contrast with eq. (2.11). The problem is the following: we wrote down the relation between operator formalism and path integral only in the discrete case for $\delta t = T/N = 0$ and then we took the continuous limit. We anticipated that this procedure may have some problematic points, that must be considered carefully.

Let's show this in the simply case of the free particle: $\hat{H} = \hat{p}^2/2m$. In order to simplify the notation we set $m = 1$. Then the propagator reads

$$K(x_f, t; x_i, 0) = \langle x_f, t | x_i, 0 \rangle = \langle x_f | \, e^{-i \frac{\hat{H}}{\hbar} t} \, | x_i \rangle$$

This satisfies the following relation

$$i\hbar \frac{\partial}{\partial t} K(x_f, t; x_i, 0) = \hat{H} K(x_f, t; x_i, 0) \tag{2.12}$$

therefore using eq. (2.12) and $\hat{p} = -i\hbar\partial_x$ we can verify the following explicit expression for the propagator[IX]

$$K(x_f, t; x_i, 0) = \frac{1}{\sqrt{2\pi i \hbar t}} \exp\left\{ \frac{i}{2\hbar t} (x_f - x_i)^2 \right\}$$

or, with a trivial time translation

$$K(x_f, t_f; x_i, t_i) = \frac{1}{\sqrt{2\pi i \hbar (t_f - t_i)}} \exp\left\{ \frac{i}{2\hbar} \frac{(x_f - x_i)^2}{t_f - t_i} \right\}$$

[VIII] We will write $\hat{p}(x(t)) = \dot{x}(t)$.This is not restrictive, since if we replace $\dot{x}(t)$ with any other sufficiently regular function the result does not change.

[IX] See Rattazzi sec. 1.2.2 for a derivation of this formula.

Therefore the propagator satisfies following relation

$$-i\hbar\frac{\partial}{\partial x}K(y,t_f;x,t_i)=\frac{x-y}{t_f-t_i}K(y,t_f;x,t_i)=i\hbar\frac{\partial}{\partial y}K(y,t_f;x,t_i) \tag{2.13}$$

The derivative expression of a path can be written as limit of the differential increase rate

$$\dot{x}(t)=\lim_{\delta t\to 0}\frac{x(t+\delta t)-x(t)}{\delta t}$$

If we set $t_+=t+\delta t$ and $t_-=t-\delta t$ we can write, event if $x(t)$ is not differentiable:

$$\begin{aligned}
&\lim_{\delta t\to 0}\left\langle x_{t_+},t_+\right|\left(\hat{x}(t)\,\hat{p}(t_-)-\hat{p}(t_+)\,\hat{x}(t)\right)\left|x_{t_-},t_-\right\rangle=\\
&\quad=\lim_{\delta t\to 0}\int \mathrm{d}x\left\langle x_{t_+},t_+\right|\left(\hat{x}(t)\,\hat{p}(t_-)-\hat{p}(t_+)\,\hat{x}(t)\right)|x\rangle\left\langle x\right|x_{t_-},t_-\rangle\\
&\quad=\lim_{\delta t\to 0}\int \mathrm{d}x\left\langle x_{t_+},t_+\right|x,t\rangle\,x(t)\left(\dot{x}(t_-)-\dot{x}(t_+)\right)\left\langle x,t\right|x_{t_-},t_-\rangle\\
&\quad=\lim_{\delta t\to 0}\int \mathrm{d}x\,K(x_{t+\delta t},t+\delta t;x,t)x(t)\left(\frac{x(t)-x(t-\delta t)}{\delta t}-\frac{x(t+\delta t)-x(t)}{\delta t}\right)K(x,t;x_{t-\delta t},t-\delta t)\\
&\quad=\lim_{\delta t\to 0}\int \mathrm{d}x\,x(t)\,(-i\hbar)\frac{\partial}{\partial x}\left(K(x_{t+\delta t},t+\delta t;x,t)K(x,t;x_{t-\delta t},t-\delta t)\right)\\
&\quad=\lim_{\delta t\to 0}\int \mathrm{d}x\,i\hbar K(x_{t+\delta t},t+\delta t;x,t)K(x,t;x_{t-\delta t},t-\delta t)\\
&\quad=\lim_{\delta t\to 0}i\hbar\,K(x_{t+\delta t},t+\delta t;x_{t-\delta t},t-\delta t)
\end{aligned}$$

Where in the second to last step we assumed vanishing boundary terms in the integration by parts. So using again composition of paths we obtain

$$\langle x_f,t_f|\left[\hat{x}(t)\,\hat{p}(t)\right]|x_i,t_i\rangle=i\hbar\,K(x_f,t_f;x_i,t_i)$$

This is exactly what we desired. It means that all the contributions to the matrix elements of commutators between non-commuting operators is given by non-differentiable paths

$$\lim_{t_+\to t}\dot{x}(t_+)\neq\lim_{t_-\to t}\dot{x}(t_-)$$

i.e. where velocity shows a discontinuity. If we consider only smooth path, then matrix elements of each commutators would be zero, and this is physically wrong.

Chapter 3

Perturbation Theory

In this chapter we will discuss perturbation theory using path integral approach. Also, we will see that path integrals contains more information than perturbation theory, since path integrals allows to study non-perturbative features, that are not involved in the perturbative approach. We will see that perturbative expansion has a limited fidelity to physics feature we want to analyze, even at highest order of expansion.

3.1 Correlators and scattering amplitudes

Skinner [12] sec. 1.2.2, 1.2.4

Primary object we will use to describe perturbative approach to QFT are the partition functions

$$\mathcal{Z}(g, m, \lambda, \dots) = \int_C \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]}$$

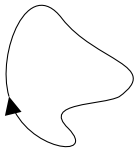
As we already mentioned, the partition function does not depend directly from fields , but only on the other parameters of the theory. Using partition functions, we want to compute correlation functions, which are genericay expectation values of products of operators of these fields

$$\boxed{\left\langle \prod_{i=1}^{n} O_i(\phi) \right\rangle = \frac{1}{\mathcal{Z}} \int_C \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} \prod_{i=1}^{n} O_i(\phi)} \tag{3.1}$$

Here we introduced the normalization factor $1/\mathcal{Z}$ so that the expectation value of operator 1 is unitary: $\langle 1 \rangle = 1$. We introduced this normalizations since we want to use correlation functions to compute scattering amplitude using LSZ formula.

Usually operators $O_i(\phi)$ are *local operators*, this means that they depend on the value of ϕ and/or its derivatives at a point of the spacetime $p \in M$. Some example of local operators may be $\phi^4(x)$, $\phi(x)\partial_\mu\phi(x)\partial^\mu\phi(x)$, etc. Other important operators are *integrated operators* over the full space time, for instance $\int_M \mathrm{d}^4x(\partial_\mu\phi\partial^\mu\phi)^2$. We can also integrate over some submanifold of M, for example this is what we do for *Wilson lines*, which is a fundamental object in gauge theories defined in terms of a given connections A on the gauge field and a certain closed path Γ

$$\mathcal{W}_\Gamma = \mathrm{Tr}\left(- \oint_\Gamma \mathrm{d}x^\mu A_\mu \right) \qquad \Gamma$$



All these operators can be computed directly using correlation functions.

One important point in the following, is that operators can be directly related to partition functions. Imagine for instance an action $S[\phi]$ that is given by

$$S[\phi] = \int_M \mathrm{d}^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{4!}\phi^4(x) \right]$$

Now we see that if we want to compute the expectation value of the operator O defined as

$$\langle O \rangle = \left\langle \frac{1}{4!} \int_M \mathrm{d}^4 x \, \phi^4(x) \right\rangle$$

then it can be written in terms of the partition function

$$\left\langle \frac{1}{4!} \int_M \phi^4(x) \mathrm{d}^4 x \right\rangle = \int_C \mathcal{D}\phi \left(\frac{1}{4!} \int \mathrm{d}^4 x \phi^4(x) \right) e^{\frac{i}{\hbar} S[\phi]} = -i\hbar \frac{\partial}{\partial \lambda} \int_C \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} = -i\hbar \frac{\partial}{\partial \lambda} \mathcal{Z}$$

Now, suppose that in my initial theory the action contains only the integral of $-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$ and we want to study the response of the theory when we add the additional term $\frac{\lambda}{4!}\phi^4(x)$ to the integrand of the action. Such a response is meant to be exactly the expectation value we just calculated. This implies that I can measure the response of the theory to the additional local operator by varying the partition function with respect to the coupling λ.

Let's generalize what we already stated. Image that I have an initial theory with action $S[\phi]$ (for example $S[\phi] = \int_M \mathrm{d}^4 x \, \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi\right)$) and I introduce a coupling term $\int_M \mathrm{d}^4 x J(x) O(x)$ as a perturbation to my initial theory. The function $J(x)$ is a called *local coupling*, or more traditionally "*source*", and generalize the parameter λ, while $O(x)$ is an operator (for instance $O(x) = \frac{1}{4!}\phi^4(x)$). I can measure the response of my QFT to the addition of the operator O by doing the following substitution

$$\boxed{S[\phi] \quad \to \quad S[\phi] + \int_M \mathrm{d}^4 x J(x) O(x)} \tag{3.2}$$

and then taking derivatives of the partition function:[I]

$$\boxed{\langle O_1(x_1) O_2(x_2) \dots O_n(x_n) \rangle = \frac{(-i\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J_i]}{\delta J_1(x_1) \delta J_2(x_2) \dots \delta J_n(x_n)} \bigg|_{J_i = 0}} \tag{3.3}$$

Notice that taking $J_i = 0$ at the end of the computation we removed the additional factor $\int_M \mathrm{d}^4 x J(x) O(x)$ we added before in the action. This is a generic way to obtain the value for the expectation value of products of operators using functional derivatives instead of direct computation. For instance

$$\langle \phi(x_1) \phi(x_2) \dots \phi(x_n) \rangle = \frac{(-i\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \delta J(x_2) \dots \delta J(x_n)} \bigg|_{J=0}$$

where we made the substitution $S \to S + \int \mathrm{d}^4 x \, J(x)\phi(x)$. Actually we can have even more complicated operators such as stress energy tensor of the general relativity.

Let's summarize our result: once we have our path integral and we defined our partition functions we can compute expectation values by using both eq. (3.1) and eq. (3.3), i.e. respectively either introducing operators in the partition function or introducing a local coupling in the action by mean of some function $J(x)$ and then taking variation with respects to this source.

This allows us to split our problem in a free theory for which we will be able to compute path integral explicitly, and some additional operators (that also may describe interactions) for which we can compute the contributions to our theory using perturbative approach by taking variations of the partition function with respect to the sources that couple to these operators.

In the special case of scattering amplitudes, we can obtain more specific forms for the previous results. Let's prescribe some asymptotic configuration for ϕ, namely ϕ_i for $t \to -\infty$ and ϕ_f for $t \to +\infty$. Then the scattering amplitude is

$$\langle \phi_f | \phi_i \rangle = \int_{C[\phi_f, \phi_i]} \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]}$$

In order to compute scattering amplitudes we will use correlators through the LSZ formula, and we will compute correlators by splitting the action in the quadratic part for which we can compute correlators in an exact way and additional contributions which are going to be treated perturbatively.

[I] We introduce normalization $1/\mathcal{Z}$ in order to normalize probabilities and we omit the time ordering product in the expectation value.

3.2 Free field theory

Skinner [12] sec. 2.1, 2.2; Vafa [4] 9.0, 9.1

Let's start from the free field theory. We start from the special case of QFT in zero dimensions. This means that the manifold of out theory is just a point $M = \{p\}$ (we assume M to be connected) and therefore there are no derivatives, we just have functions from a point to some target space (eg. $\mathbb{R}$, $\mathbb{C}$, etc.). Moreover, the space does not have any symmetry (for example, in such a theory spin doesn't exists). We can take fields as functions

$$\phi : \{p\} \longrightarrow \mathbb{R}$$

In this case the partition function is just the Lebesgue integral (we can consider it in the Euclidean form)

$$\mathcal{Z} = \int_{\mathbb{R}} \mathrm{d}\phi \, e^{-\frac{S[\phi]}{\hbar}}$$

were we used -1 instead of i because since we are computing a Lebesgue integral we hope that the minus factor will give to the integral a generic good behaviour (notice that for most theories kinetic terms have positive contributions to the action). Also we assume for simplicity that $S[\phi]$ is polinomial, in particular we consider the quadratic case. Then in this case the theory is characterized by an action

$$S = \frac{1}{2} M_{ij} \phi^i \phi^j \qquad i, j = 1, \dots, n$$

For simplify further the computation we take the matrix M_{ij} to be a real, symmetric, positive defined matrix (in this way this matrix has only real positive eigenvalues). The partition function for such a simple instance is

$$\mathcal{Z}_0 = \int_{\mathbb{R}^n} \mathrm{d}^n\phi \exp\left\{ -\frac{1}{2\hbar} M_{ij} \phi^i \phi^j \right\} = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}} \tag{3.4}$$

where we can computed easily the integral by diagonalization of the matrix M through orthogonal rotations $O(n)$. Since the Lebesgue measure is invariant under orthogonal rotations, we just have to compute the n-dimensional Gaussian integral we obtained in this way.

In general using Gaussian integrals we will always be able to compute quadratic terms. Hence we will call "free fields" theories which contains only quadratic terms of my action. In the perturbative approach possibly I will expand my action and I will do this in such a way that the first term that appears up to constant pieces is going to be a quadratic term in the field, and that is going to be the free theory.

Correlators in the free theory

Now we take into account correlators in the free theory:

$$\langle \phi^{a_1} \dots \phi^{a_n} \rangle = \frac{1}{\mathcal{Z}_0} \int_{\mathbb{R}^n} \mathrm{d}^n\phi \; \phi^{a_1} \dots \phi^{a_n} \exp\left\{ -\frac{1}{2\hbar} M_{ij} \phi^i \phi^j \right\}$$

We will compute this by means of introduction of sources:

$$\begin{aligned}
\langle \phi^{a_1} \dots \phi^{a_n} \rangle &= \frac{1}{\mathcal{Z}_0} \int_{\mathbb{R}^n} \mathrm{d}^n\phi \, \phi^{a_1} \dots \phi^{a_n} \exp\left\{ -\frac{1}{2\hbar} M_{ij} \phi^i \phi^j - \frac{1}{\hbar} J_i \phi^i \right\} \Bigg|_{J_i=0} \\
&= \frac{1}{\mathcal{Z}_0} \int_{\mathbb{R}^n} \mathrm{d}^n\phi \, (-\hbar)^n \frac{\delta^n}{\delta J_{a_1} \dots \delta J_{a_n}} \exp\left\{ -\frac{1}{2\hbar} M_{ij} \phi^i \phi^j - \frac{1}{\hbar} J_i \phi^i \right\} \Bigg|_{J_i=0} \\
&= \frac{(-\hbar)^n}{\mathcal{Z}_0} \frac{\delta^n}{\delta J_{a_1} \dots \delta J_{a_n}} \left(\int_{\mathbb{R}^n} \mathrm{d}^n\phi \exp\left\{ -\frac{1}{2\hbar} M_{ij} \phi^i \phi^j - \frac{1}{\hbar} J_i \phi^i \right\} \right) \Bigg|_{J_i=0}
\end{aligned}$$

We just followed the procedure described in the previous section. Starting from the free theory we added the coupling with the source, then we took derivatives that brings down fields. Computing the partition function for the theory with sources we can immediately obtain the correlation function by means of derivatives.

Now, we have in the exponential a term quadratic in fields and a term linear in fields. We can redefine fields in order to absorb the linear terms as follow:

$$\widetilde{\phi}^i = \phi^i + (M^{-1})^{ij} J_j$$

This is just a translation, therefore the measure does not change $\mathrm{d}^n\widetilde{\phi} = \mathrm{d}^n\phi$. The argument of the exponential become:

$$\begin{aligned} -\frac{1}{2}M_{ij}\phi^i\phi^j - J_i\phi^i &= -\frac{1}{2}M_{ij}\left(\widetilde{\phi}^i - (M^{-1})^{ik}J_k\right)\left(\widetilde{\phi}^j - (M^{-1})^{jl}J_l\right) - J_i\left(\widetilde{\phi}^i - (M^{-1})^{ik}J_k\right) \\ &= -\frac{1}{2}M_{ij}\widetilde{\phi}^i\widetilde{\phi}^j + \frac{1}{2}J_i(M^{-1})^{ij}J_j \end{aligned}$$

Then I can complete the calculation of the correlator

$$\langle\phi^{a_1}\dots\phi^{a_n}\rangle = \frac{(-\hbar)^n}{\mathcal{Z}_0}\frac{\delta^n}{\delta J_{a_1}\dots\delta J_{a_n}}\left(\int_{\mathbb{R}^n}\mathrm{d}^n\phi\,\exp\left\{-\frac{1}{2\hbar}M_{ij}\widetilde{\phi}^i\widetilde{\phi}^j + \frac{1}{2\hbar}J_i(M^{-1})^{ij}J_j\right\}\right)\Bigg|_{J_i=0}$$

and since the first term in the exponential cancel with normalization factor $\mathcal{Z}_0$ we finally end up with

$$\boxed{\langle\phi^{a_1}\dots\phi^{a_n}\rangle = (-\hbar)^n\frac{\delta^n}{\delta J_{a_1}\dots\delta J_{a_n}}\exp\left\{\frac{1}{2\hbar}J_i(M^{-1})^{ij}J_j\right\}\Bigg|_{J_i=0}} \tag{3.5}$$

From this expression I learn that

- The number of derivatives that I have to take must be even, otherwise all this expression vanishes when I take $J_i = 0$, therefore each correlator of an odd number of operator gets zero as result;
- For $n = 2$ the correlator corresponds to the *propagator*

$$\begin{aligned} \langle\phi^a\phi^b\rangle &= \hbar^2\frac{\delta^2}{\delta J_a\delta J_b}\exp\left\{\frac{1}{2\hbar}J_i(M^{-1})^{ij}J_j\right\}\Bigg|_{J_i=0} \\ &= \hbar\frac{\delta}{\delta J_b}\left((M^{-1})^{aj}J_j\exp\left\{\frac{1}{2\hbar}J_i(M^{-1})^{ij}J_j\right\}\right)\Bigg|_{J_i=0} = \hbar(M^{-1})^{ab} \end{aligned}$$

  I could have expected this feature since in zero dimensional theories propagators are just the inverse of quadratic forms. This will be different in higher dimensions since in that case propagators will be inverse of differential operators of the second order.

  Associated to each correlator there is a Feynman diagram, which in zero dimension is very simple

$$\langle\phi^a\phi^b\rangle = \quad \phi^a \text{ ———— } \phi^b \quad = \hbar(M^{-1})^{ab} \tag{3.6}$$

- Contributions of the initial partition function $\mathcal{Z}_0$ are normalized to 1, this implies that we don't have to calculate explicitly the partition function and our procedure is defined even if $\mathcal{Z}_0$ is divergent. Everything ends up to an easy computation given by the simple term eq. (3.5) that depends only on the sources.

Wick's theorem

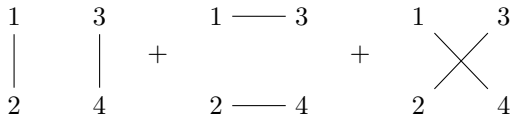
In the general case, when we want to compute the correlator between $2k$ fields we have to sum over products of all inequivalent ways of connecting pairs of ϕ^i using M^{-1}:

$$\langle\phi^{a_1}\dots\phi^{a_{2k}}\rangle = \hbar^k\sum_{\sigma\in\text{Pairings}}\prod_{i\in\sigma_{\text{Pairs}}}(M^{-1})^{i\sigma(i)} \tag{3.7}$$

This is just the statement of the *Wick's theorem*. For instance

$$\langle\phi^a\phi^b\phi^c\phi^d\rangle = \hbar^2\left[(M^{-1})^{ab}(M^{-1})^{cd} + (M^{-1})^{ac}(M^{-1})^{bd} + (M^{-1})^{ad}(M^{-1})^{bc}\right] \tag{3.8}$$

Formula eq. (3.7) comes from eq. (3.5): in order to have non vanishing terms, couples of derivatives must acts in such a way they brings down matrix elements $(M^{-1})^{ab}$ independent from sources. In other words eq. (3.7) states that contributions to $\langle \phi^{a_1} \dots \phi^{a_{2k}} \rangle$ are given by all possible products of contractions, and each contraction gives a factor $\hbar (M^{-1})^{ab}$, as shown in (3.6). For example, correlator eq. (3.8) is given by following pairings



All possible contractions for $2k$ fields are[II]

$$(2k-1)(2k-3)\dots(1) = \frac{2k}{2k}(2k-1)\frac{(2k-2)}{2(k-1)}(2k-3)\dots 1 = \frac{(2k)!}{2^k k!} \tag{3.9}$$

This is the number of all the Feynman diagrams I obtain through contractions when I do perturbation theory.

3.3 Perturbation theory

Skinner [12] sec. 2.3; Vafa [4] 9.0, 9.1

Now, we want to compute correlator functions or in general expectation values for operators (let's consider the simply case of one field ϕ):

$$\langle f(\phi) \rangle = \frac{1}{\mathcal{Z}} \int_{\mathbb{R}} \mathrm{d}\phi \, f(\phi) e^{-S[\phi]/\hbar} \tag{3.10}$$

Even in this simple instance, where we are considering zero dimensions and the path integral is a standard integral, the exact computation is in general impossible, neither if we consider f a simple function. For this reason is often very useful to use perturbation theory, indeed in many case it gives very good results. Usually when we consider perturbation theory we start from classical results and then we calculate quantum corrections. The general idea to do this is to take $\hbar \to 0$ to obtain the classical result and then expand in powers of $o(\hbar)$

We will see that the Taylor expansion in $\hbar$ that we will do is in general not well defined. Take for instance $\hbar > 0$, then the series we obtain from eq. (3.10) may converge, but for $\hbar < 0$ the series diverges for sure. We cannot assume simply $\hbar > 0$, since in order to use the Taylor expansion, we need to have some neighbourhood of the origin where the expansion in well defined, and with the restriction $\hbar > 0$ such neighbourhood have radius equal to 0.

The best we can do is to introduce an **asymptotic expansion**, i.e. we introduce a series $\sum_{n=0}^{\infty} a_n \hbar^n$ such that for each *finite* $N \in \mathbb{N}$ it approaches our function $F(\hbar)$:

$$\lim_{\hbar \to 0^+} \frac{1}{\hbar^N} \left| F(\hbar) - \sum_{n=0}^{N} a_n \hbar^n \right| = 0 \qquad \forall N \in \mathbb{N} \tag{3.11}$$

This means that for each N fixed we can choose a sufficiently small value for $\hbar$ such that the difference between the initial function and our asymptotic expansion is $O(\hbar^N)$. Raising the value for N we can obtain better reproductions of $F(\hbar)$, but notice that

- what we stated right now holds if $\hbar \to 0^+$, but this is not true for fixed (even if very small) values of $\hbar$, like in the physical case: an asymptotic expansion for finite values of $\hbar$ can't be obtained in general. Indeed, for fixed values of $\hbar$ we can use eq. (3.11) only for finite values of N, since over some value of N contributions in the order of $\hbar^N$ became relevant in our analysis;

[II] When I take the first field, I have $2k-1$ ways to couple it. Then I take one of the other $2k-2$ fields, and I have $2k-3$ way to couple this one, and so on.

- even if we trust our asymptotic expansion, we lose in any case all transcendental terms, i.e. terms that are sums of exponentials. For instance $F(\hbar)$ and $\widetilde{F}(\hbar) = F(\hbar) + Ae^{-B/\hbar^2}$ have same asymptotic expansion for $\hbar \to 0$, since $e^{-B/\hbar^2} \to 0$ when we take the limit. Usually these terms are small and can be neglected, anyhow in general one may be interested in these contributions, that cannot be obtained through asymptotic expansion.

Notice that these additional informations were contained in the initial path integral eq. (3.10), but are lost in the perturbative approach. As we already stated, path integral contains *more* information than perturbation theory.

Perturbation theory for an interacting model

Suppose we have an action $S[\phi]$ and we expand it around some minimum ϕ_0 in a constant term plus a quadratic piece

$$S(\phi) = S(\phi_0) + \frac{1}{2}\partial_\phi^2 S(\phi_0)\delta\phi^2 + \dots$$

The constant term is removed by the normalization into the partition function, then we can neglect it. We assumed that $\partial_\phi S(\phi) = 0$ but even if this is not true we can obtain it by redefine our field removing any linear field. The quadratic part was already analyzed in the last section, since it describes a free theory at the vacuum ϕ_0. Remaining terms are the one we will consider in our perturbative approach.

What we essentially do is to split the action into

$$S(\phi) = S_0(\phi) + S_I(\phi)$$

where $S_0(\phi)$ corresponds to the quadratic free theory and $S_I(\phi)$ is an interacting term. Let's start with a simple calculation of the expansion of the interacting part. Suppose that

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \quad , \quad S_0(\phi) = \frac{1}{2}m^2\phi^2 \quad \text{and} \quad S_I(\phi) = \frac{\lambda}{4!}\phi^4 \tag{3.12}$$

where $m^2 > 0$ and $\lambda > 0$. In such a way $\phi = 0$ is a global minimum, hence $S(\phi_0) = 0$ and $\partial_\phi S = 0$, and we have to expand around $\phi = 0$, i.e. ϕ can be interpreted as a fluctuation around zero. Then the partition function of the free theory depends only on the mass

$$\mathcal{Z}_0(m^2) = \int_{\mathbb{R}} \mathrm{d}\phi \exp\left\{-\frac{1}{2\hbar}m^2\phi^2\right\}$$

while the partition function depends also on the parameter λ:

$$\mathcal{Z}(m^2, \lambda) = \int_{\mathbb{R}} \mathrm{d}\phi \exp\left\{-\frac{1}{2\hbar}m^2\phi^2 - \frac{\lambda}{4!\hbar}\phi^4\right\}$$

We introduce the interaction as a response to the coupling with λ, using a source J

$$\mathcal{Z}(m^2, \lambda) = \int_{\mathbb{R}} \mathrm{d}\phi \exp\left\{-\frac{1}{2\hbar}m^2\phi^2 - \frac{\lambda}{4!\hbar}\phi^4 - \frac{1}{\hbar}J\phi\right\}\Bigg|_{J=0} \tag{3.13}$$

such that we recover the exponential of the interacting part as an operator inside the partition function of the free theory with the source:

$$\mathcal{Z}(m^2, \lambda) = \int_{\mathbb{R}} \mathrm{d}\phi\, O(\phi) \exp\left\{-\frac{1}{2\hbar}m^2\phi^2 - \frac{1}{\hbar}J\phi\right\}\Bigg|_{J=0} \qquad \text{with} \qquad O(\phi) = \exp\left\{-\frac{\lambda}{4!\hbar}\phi^4\right\}$$

Then we see that the result of this integral is

$$\mathcal{Z}(m^2, \lambda) = \langle O(\phi)\rangle_0\, \mathcal{Z}_0(m^2)$$

Using the procedure we described before we have

$$\begin{aligned}\mathcal{Z}(m^2,\lambda) &= \exp\left\{-\frac{\lambda}{4!\hbar}\left(-\hbar\frac{\partial}{\partial J}\right)^4\right\}\mathcal{Z}_0(m^2,J)\Big|_{J=0} \\ &= \sum_n \frac{1}{n!}\left(-\frac{\lambda}{4!\hbar}\right)^n\left(-\hbar\frac{\partial}{\partial J}\right)^{4n}\int_{\mathbb{R}}\mathrm{d}\phi\,\exp\left\{-\frac{1}{2\hbar}m^2\phi^2-\frac{1}{\hbar}J\phi\right\}\Big|_{J=0} \\ &= \int_{\mathbb{R}}\mathrm{d}\phi\,\exp\left\{-\frac{1}{2\hbar}m^2\phi^2\right\}\sum_n\frac{1}{n!}\left(-\frac{\lambda}{4!\hbar}\right)^n\phi^{4n}\end{aligned}$$

Even though eq. (3.13) could be computed directly using expansion in λ, this is not in general possible, while our procedure can be actually done also in more complex cases. In this simply case we can go on in the computation in an explicit way, in order to understand better the result we will obtain in general. If we set $\phi=\sqrt{2\hbar x/m^2}$ and $\mathrm{d}\phi=\sqrt{2\hbar/m^2}\,\mathrm{d}x/(2\sqrt{x})$, and we change sign of the field for $\phi\in(-\infty,0)$ (this gives us an additional factor 2 which cancel the factor $1/2$ in the Jacobian of the change of variable), we obtain

$$\begin{aligned}\mathcal{Z}(m^2,\lambda) &= \sqrt{\frac{2\hbar}{m^2}}\int_0^{+\infty}\mathrm{d}x\,\frac{e^{-x}}{\sqrt{x}}\sum_n\frac{1}{n!}\left(-\frac{\lambda}{4!\hbar}\right)^n\left(\frac{2\hbar}{m^2}\right)^{2n}x^{2n} \\ &= \frac{\sqrt{2\hbar}}{m}\sum_n\frac{1}{n!}\left(-\frac{\lambda\hbar}{3!m^4}\right)^n\underbrace{\int_0^{+\infty}\mathrm{d}x\,x^{2n+\frac{1}{2}-1}e^{-x}}_{\Gamma\left(2n+\frac{1}{2}\right)} \\ &= \underbrace{\frac{1}{m}\sqrt{2\pi\hbar}}_{\mathcal{Z}_0(m^2)}\sum_n\left(-\frac{\lambda\hbar}{m^4}\right)^n\frac{1}{(4!)^n n!}\frac{(4n)!}{4^n(2n)!} \\ &= \mathcal{Z}_0(m^2)\left(1-\frac{\lambda\hbar}{m^4}\frac{4!}{4!\,1!\,4\,2!}+\dots\right)\end{aligned}$$

where we used the Gamma function

$$\Gamma(z)=\int_0^{+\infty}e^{-x}x^{z-1}\mathrm{d}x \tag{3.14}$$

with proprieties

$$\Gamma(p)=p! \qquad \Gamma\left(p+\frac{1}{2}\right)=\sqrt{\pi}\,\frac{(2p)!}{4^p p!}$$

and equation eq. (3.4). The final result in the last step shows explicitly the perturbative contributions to the partition function.

We stress the fact that our result is not a Taylor expansion of the initial partition function, but is just an asymptotic expansion, as we stated in the beginning of this section. This is because, in our calculation, we were not really allowed to interchange the integral with the summation over n.

Notice also that the expansion is actually in terms of $(\lambda\hbar)^n$ and not only in terms of λ^n. This is because each correction in λ corresponds to a quantum correction, i.e. takes a factor $\hbar$. Moreover, we could have expected the factor $(-\lambda\hbar/m^4)^n$ simply by dimensional analysis, because the action has dimension $\hbar$, therefore by eq. (3.12) follows that λ and m^2 has both dimension $\hbar$.

Moreover, the term $1/((4!)^n n!)$ is fixed by the order of the expansion of the exponential. The other term $(4n)!/(4^n(2n)!)$ is going to be the number of ways we can join the 4^n fields given by each term of the expansion of my correlator, as described by eq. (3.9) (we have to replace $k=2n$).

Finally, notice that the result we obtain diverges. This is not a problem since we computed an asymptotic expansion, which not recovers the real initial partition function (anyhow, in general also the initial partition function may diverges, in this case we have to treat fields in correlators in such a way that divergent partition function is normalized at the end of the calculation). Recall Stirling approximation $n!\approx\exp\{n\log n\}$, this means that $(4n)!/((4!)^n n!)(4^n(2n)!)$ behaves like $\exp\{n\log n+\dots\}$ and this function has zero convergence radius. This number will corresponds to the number of Feynman

diagrams one obtain at each perturbative level, and this means that above some order the number of Feynman diagrams that contributes is so huge that overcomes the small pre-factor $(-\hbar\lambda/m^4)^n$ and gives divergent contribution to the full series. Hence, using perturbative approach to compute scattering amplitudes, up to some value N the introduction of higher order contributions gives more precise results, but beyond that value my scattering amplitude will start to diverge. Obviously this behaviour of the series leads to some doubt about the using of perturbative expansion but this is, again, a consequence of the use of asymptotic expansion instead of Taylor's one. Moreover, sometimes non-perturbative effects, allows series to converges, as we will see later on. Anyhow, these divergent contributions appears when we consider perturbative contributions many orders of magnitude away from the highest precision we will ever be able to archive experimentally. This means that asymptotic expansion have issues only from the mathematical point of view, while our results agree perfectly with physical results, and we are always be able to understand when our approach starts to show unphysical results.

3.3.1 Feynman Diagrams

Skinner [12] sec. 2.3.1

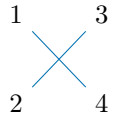
Now we want to express perturbative expansion in terms of Feynman diagrams. For this theory we immediately have

(3.15)

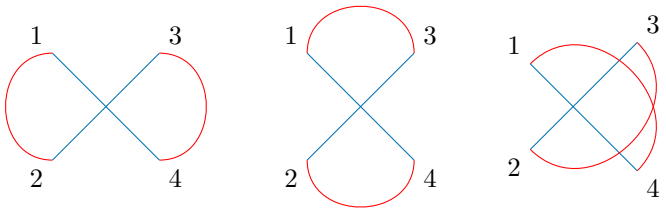
To compute the full partition function we just consider vacuum graph, i.e. without taking into account external legs, since we want to sum over all possible internal paths without boundary conditions. Therefore we just want to compute vacuum-vacuum correlators, expressed in terms of partition functions:

$$\frac{\mathcal{Z}(m^2,\lambda)}{\mathcal{Z}_0(m^2)} = 1 + A\lambda + B\lambda^2 + \dots$$

At the first order I have following interacting vertex



but I don't want external legs, so I have to connect them by means of propagators:



Obviously, all these diagrams are topologically equivalent, but using labels I obtain 3 distinguish graphs. Let's define $|D_n|$ the number of (labelled) graph at the n-th order, for instance $|D_1| = 3$. The symmetry group G_n for these graphs (for n vertices) is given by all possible permutations of indices, i.e. $G_n = (S_4)^n \times S_n$, where S_4 is the group of permutation for 4 indices (attached to each vertex), while S_n describes the permutations of n vertices. The dimension of this group is

$$|G_n| = (4!)^n n!$$

Recall our previous result:

$$\mathcal{Z}(m^2,\lambda) = \mathcal{Z}_0(m^2) \sum_n \left(-\frac{\lambda\hbar}{m^4}\right)^n \frac{1}{(4!)^n n!} \frac{(4n)!}{4^n (2n)!}$$

We see that the first combinatorial factor is exactly the inverse of $|G_n|$, while we recall that the second combinatorial factor is the number $|D_n|$ of ways to couple the fields given by eq. (3.9), i.e. the number of possibilities to connect legs that come out from each vertex. Then we can write

$$\mathcal{Z}(m^2, \lambda) = \mathcal{Z}_0(m^2) \sum_n \hbar^{l-1} \frac{(-\lambda)^v}{(m^2)^p} \frac{|D_n|}{|G_n|}$$

where

- v is the number of vertices ($= n$)
- p is the number of propagators ($= 2n$)
- l is the number of loops ($= p - v + 1$ by Euler's Theorem)

Using this expression is clear that even though we restricted our analysis to a very special case, the structure we obtained is the one that one expects by using perturbation theory using Feynman diagrams, since every time you do perturbation theory you would have

- an expansion in the coupling (that comes with a power equal to the number of vertices in our theory);
- inverse powers of masses (or in general propagators) in order to close diagrams and avoid external legs (in any case also external legs gives similar contribution);
- powers in $\hbar$ proportional to the number of loops;
- a combinatorial factor proportional to number of different possible labelling divided by the dimension of the symmetry group of permutations of labels in our graphs.

3.4 Borel resummation *

Spada [13] sec. 2.1, 2.2

Even though we saw that generic perturbative expansions are only asymptotic expansions that leads to some problems in the mathematical point of view. Moreover, in general one can be interested in non-perturbative features of our theory, that requires the correct partition function to be analyzed.

In the case of QED, where the coupling is $\alpha \sim e^2$, we can be prove easily that the convergence radius of partition function must be zero, since otherwise $\alpha = 0$ would be a regular point for any physical observable, and then values $\alpha < 0$ would be admitted. But $\alpha < 0$ is unacceptable: it would imply that e^- and e^+ reject each other, and then the destruction of the vacuum (pairs generated spontaneously from the vacuum cannot recombine), hence we cannot formalize any perturbation theory. We conclude that $\alpha = 0$ cannot be analytic. This is the physical reason behind the failure of perturbation theory over some point of perturbation.

In order to have an asymptotic series that describes in a good way our partition function we need

$$\mathcal{Z}(\lambda) - \sum_{n=0}^{N} \mathcal{Z}_n \lambda^n = O(\lambda^n)$$

Let's consider the value of N such that the asymptotic expansion give best results. For orders over N the predictions of our perturbative approach gives no more good results. For instance let's assume

$$\mathcal{Z}_n \sim n! a^n n^b$$

Using Stirling approximation $n! = \sqrt{2\pi n}\, n^n e^{-n}(1 + O(1/n))$ we can minimize the logarithm of $\mathcal{Z}_n \lambda^n$:

$$f(n) = \log(\mathcal{Z}_n \lambda^n) \sim -n + n \log(a\lambda n) + \left(b + \frac{1}{2}\right) \log n$$

which derivatives is

$$f'(n) = -1 + \log(a\lambda n) + 1 + \left(b + \frac{1}{2}\right)\frac{1}{n}$$

therefore we have a minimum for f when

$$n = \frac{1}{a\lambda}\left(1 + o\left(\frac{1}{n}\right)\right)$$

In the previous example we had

$$\mathcal{Z}_n \sim \frac{1}{(4!)^n n!}\frac{(4n)!}{4^n(2n)!} \sim 2\sqrt{2\pi n}\, e^{-n} n^n \underbrace{\left(\frac{4^4}{2^2 \cdot 4 \cdot 4!}\right)^n}_{(2/3)^n}$$

so $n = \frac{3}{2\lambda}$. This means that if for instance we take $\lambda = 0.1$ we would have problems for n higher that 15, which is very high respect to archiviable experimental precision.

We just seen that even though perturbative expansion have some mathematical issue it is perfectly consistent to describe physics. Anyhow, using Borel resummation in some cases we can resum the series corresponding the our asymptotic expansion in such a way that we get the correct partition function. This allows us also to recover non-perturbative features of our theory.

Starting from a divergent series given by an asymptotic expansion $\mathcal{Z}(\lambda) = \sum_n \mathcal{Z}_n \lambda^n$ we can define its **Borel transform**

$$\mathcal{B}\,\mathcal{Z}(\lambda) = \sum_n \frac{1}{n!}\mathcal{Z}_n \lambda^n \tag{3.16}$$

This series obviously has a finite radius of convergence. If the analytic continuation in the complex plane is free of singularities for $\lambda > 0$, then the following integral is well defined

$$\mathcal{Z}_{\mathcal{B}}(\lambda) = \int_0^\infty \mathrm{d}t\, e^{-t}\mathcal{B}\mathcal{Z}(t\lambda) \tag{3.17}$$

which has the same asymptotic expansion of $\mathcal{Z}(\lambda)$. Indeed, using Gamma function eq. (3.14) we obtain

$$\int_0^\infty \mathrm{d}t\, e^{-t} \sum_n \frac{1}{n!}\mathcal{Z}_n \lambda^n = \sum_n \frac{\mathcal{Z}_n \lambda^n}{n!}\Gamma(n+1) = \sum_n \mathcal{Z}_n \lambda^n$$

The interesting thing is that now we have the convergent series eq. (3.16) which can be used in eq. (3.17) in order to obtain same behaviour of asymptotic series. Now the question in whether $\mathcal{Z}_{\mathcal{B}}(\lambda)$ corresponds to the full partition function, since for the moment we just know that has same expansion as the asymptotic series corresponding to the partition function. In general the answer is not easy.

- If exists $\mathcal{Z}_{\mathcal{B}}$ (i.e. analytical continuation of $\mathcal{B}\,\mathcal{Z}$ has no zero for $\lambda > 0$) we say that $\mathcal{Z}$ is *Borel resummable*.
- If $\mathcal{Z}_{\mathcal{B}} = \mathcal{Z}$ we say that $\mathcal{Z}$ is *Borel resummable to the exact result*.

This does not happen in general, for instance for 4-dim Gauge theories this is not possible, but in some cases this is possible and Borel resummation gives exact results, for instance for $\lambda\phi^4$ theory in $d = 2, 3$ and also for various problems in QM (such as harmonic oscillator).

Finally notice that there are also other ways to obtain exact results in specific cases, and such as Borel resummation allow treatment of several theories fixing the perturbative approach.

3.5 Fermions and Grassmann variables

Hori [4] sec. 9.2, sec. 5.1, Skinner [12], sec. 2.5.0, 2.5.1; Cartier [2], sec. 3, 8

Realistic theories contains fermions. In higher dimensions, the spin-statistics theorem says that for a unitary theory, fermions must have half-integral spin. However, in $d = 0$ there is no notion of spin, much

less a spin-statistic theorem, and fermionic "fields" are simply **Grassmann numbers**. These are a set of n elements $\{\theta_i\}$ obeying the algebra

$$\{\theta_i\theta_j\} = 0 \quad \text{and} \quad [\theta_i, \phi_j] = 0 \quad \forall \phi_i \in \mathbb{C}$$

Notice that elements ϕ_i describes boson fields in zero dimensional spaces. Thus, Grassmann variables anticommute with each other and commute with any bosonic variable. In particular, this implies $\theta_i\theta_i = -\theta_i\theta_i = 0$ for each i. This property means that any function of a finite number of Grassmann variables has a finite expansion

$$f(\theta) = f_0 + \sum_i f_i\theta_i + \sum_{ij} f_{ij}\theta_i\theta_j + \cdots + \sum_{i_1,\ldots,i_n} f_{i_1,\ldots,i_n}\theta_{i_1}\ldots\theta_{i_n}$$

where we can take the coeficients to be totally antisymmetric, e.g. $f_{ij} = -f_{ji}$. For example, the exponential of a Grassmann number take a very simply form, since all terms $(\theta_i)^a$ vanishes for $a > 1$:

$$\exp(\theta) = 1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3!} + \cdots = 1 + \theta$$

One defines the differential operator $\frac{\mathrm{d}}{\mathrm{d}\theta}$ by means of

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta^i_j \qquad \left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0$$

In particular

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta^i_j - \theta_j\frac{\partial}{\partial\theta_i}(1) = \delta^i_j$$

and

$$\frac{\partial^2}{\partial\theta^2} = 0$$

Moreover, from the definition, we also obtain a relation equivalent to the Leibniz rule

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_z) = \delta^i_j\theta_k - \theta_j\delta^i_k = -\frac{\partial}{\partial\theta_i}(\theta_z\theta_j)$$

Regarding integration[III], we introduce the **Berezin integration**, which impose the following fundamental requirement on a definite integral: given an integral operator $\int$ and a derivative operator ∂ on a space of functions, then these operators satisfy following condition:

$$\partial\int = 0 = \int\partial$$

Moreover, we require that the integral operator does not depend on constant functions:

$$\partial f = 0 \qquad \Rightarrow \qquad \int(g\cdot f) = \left(\int g\right)\cdot f$$

The requirement $\partial\int = 0$ for functions of real variables says that the integral does not depend on the variable of integration, while the requirement $\int\partial = 0$ says that functions vanish on their domain boundaries.

These condition implies

$$\int \mathrm{d}\theta = \int \mathrm{d}\theta\, 1 = 0$$

and

$$\int \mathrm{d}\theta\,\theta = 1$$

[III] See `http://swc.math.arizona.edu/dls/DLSCartierCh9.pdf` for an additional reference regarding integration (and more in general analysis in Grassmann variables).

If we have n Grassmann variables θ_i , repeated applications of the above rules shows that the integral of an n parameter function coincides with derivation (in order to prove this, just expand the function f):

$$\int \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \, f(\theta_1, \dots, \theta_n) = \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\cdots\frac{\partial}{\partial\theta_n} f(\theta_1, \dots, \theta_n)$$

Obviously previous formula holds if we suppose to integrate starting with $\mathrm{d}\theta_n$, otherwise if we define multivariable integral starting from the first differential $\mathrm{d}\theta_1$ we have to reverse the order of differential operators.

The change of integration variable leads to a different rule respect to the usual one (which introduce inverse of the determinant of Jacobian): here differentiation and integration are the same, hence integration measure transforms as the derivative. If we consider some other coordinates $\tilde{\theta} = \tilde{\theta}(\theta)$ with[IV] $\tilde{\theta}_i = a_i^j \theta_j$ then

$$\begin{aligned}\int \mathrm{d}\tilde{\theta}_1 \dots \mathrm{d}\tilde{\theta}_n \, f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) &= \frac{\partial}{\partial\tilde{\theta}_1}\cdots\frac{\partial}{\partial\tilde{\theta}_n} f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) \\ &= \frac{\partial\theta_{i_1}}{\partial\tilde{\theta}_1}\cdots\frac{\partial\theta_{i_n}}{\partial\tilde{\theta}_n}\frac{\partial}{\partial\theta_{i_1}}\cdots\frac{\partial}{\partial\theta_{i_n}} f(\tilde{\theta}_1, \dots, \tilde{\theta}_n)\end{aligned}$$

Since $\frac{\partial}{\partial\theta_{i_1}}\cdots\frac{\partial}{\partial\theta_{i_n}}$ are anticommuting, then this term is proportional to the total antisymmetric tensor $\varepsilon_{i_1,\dots,i_n}$, and $\frac{\partial\theta_{i_1}}{\partial\tilde{\theta}_1}\cdots\frac{\partial\theta_{i_n}}{\partial\tilde{\theta}_n}\varepsilon_{i_1,\dots,i_n}$ gives the determinant of the Jacobian of $\tilde{\theta}(\theta)$:

$$\int \mathrm{d}\tilde{\theta}_1 \dots \mathrm{d}\tilde{\theta}_n \, f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) = \det\left(\frac{\partial\tilde{\theta}}{\partial\theta}\right)\int \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \, (f\circ\tilde{\theta})(\theta_1, \dots, \theta_n)$$

For future purposes, we now compute the Gaussian integral for Grassmann variables.

$$\int \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \, \exp\left(-\sum_{ij}\theta_i M_{ij}\theta_j\right)$$

Notice that if we have $n = 2k$ Grassmann variables, the expansion of the exponential may contain products of terms in the form $\theta_i M_{ij}\theta_j$ up to k-th order (in the $(k+1)$-th order we will have at least two variables θ_i^a with $a \geq 2$ that gives zero, same for higher orders). For $n = 2k+1$ variables, we would have again powers of $\theta_i M_{ij}\theta_j$ up to k-th order, since at higher orders at least one variable appears two times, giving zero contribution. Hence, when we perform Gaussian integral for $2k$ variables in general we have non-zero result, but when we integrate over $2k+1$ variables we obtain always zero, since we have to perform $2k+1$ derivation of products of $2k$ variables.
For an even number of variables, usually starting from $n = 2k$ initial variables one introduce k complex Grassmann variables, and in general the integral takes the form

$$\int \underbrace{\mathrm{d}\bar{\theta}_1\mathrm{d}\theta_1 \dots \mathrm{d}\bar{\theta}_k\mathrm{d}\theta_k}_{\mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta} \exp\left(-\sum_{ij}\bar{\theta}_i M_{ij}\theta_j\right)$$

Then we can redefine this further through an expansion of the exponential (in the right hand side we omit summations over i, j, moreover terms substituted with dots goes from the third order to the k-th order: the last term contains exactly k summations and $2k$ Grassmann variables):

$$\int \mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta \, \exp\left(-\sum_{ij}\bar{\theta}_i M_{ij}\theta_j\right) = \int \mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta \, \left(1 - M_{i_1 j_1}\bar{\theta}_{i_1}\theta_{j_1} + M_{i_1 j_1}M_{i_2 j_2}\bar{\theta}_{i_1}\theta_{j_1}\bar{\theta}_{i_2}\theta_{j_2} - \dots\right)$$

This integration keeps only the higher order terms, i.e. ones with $2k$ Grassman variables

$$\int \mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta \, \exp\left(-\sum_{ij}\bar{\theta}_i M_{ij}\theta_j\right) = (-)^k M_{i_1 j_1}M_{i_2 j_2}\dots M_{i_k j_k}\int \mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta \, \bar{\theta}_{i_1}\theta_{j_1}\bar{\theta}_{i_2}\theta_{j_2}\dots\bar{\theta}_{i_k}\theta_{j_k}$$

[IV] This is the most general relation between Grassmann numbers.

Recall that in order to perform latter integral, we have to move all θ on the left and all $\bar{\theta}$ on the right (obtaining again sign corrections). Finally we obtain

$$\int \mathrm{d}^k\bar{\theta}\,\mathrm{d}^k\theta\,\exp\left(-\sum_{ij}\bar{\theta}_i M_{ij}\theta_j\right) = \varepsilon_{i_1\dots i_k}\varepsilon_{j_1\dots j_k}M_{i_1 j_1}\dots M_{i_k j_k} = \det M$$

Let's compare this result with eq. (3.4) in the commuting case. The latter is proportional to the inverse of the square root of the determinant, while in this case the gaussian integral is directly proportional to the determinant. This is a consequence of the fact that for anticommuting variables the integral is the same as the derivation.

3.6 Localization *

Skinner [12] sec. 2.5.2; Cremonesi [3] sec. 3, Pestun [8] sec. 1-3

For a generic QFT, the asymptotic series is as good representation of the partition function (or correlation functions) as we can hope for, barring numerics. However, if the action is of a very special type, it may sometimes possible to evaluate the partition function and even certain correlation functions exactly. There are many mechanisms by which this might happen; this section gives a toy model of one of them, known as **localization** in supersymmetric theories.

Let's take a theory in zero-dimensional[V] space-time where there are one bosonic field ψ and two fermionic fields ψ_1 and ψ_2. In the zero-dimensionale spacetime, the space of fields is indicated with $\mathbb{R}^{1|2}$. Let's introduce the action ($\hbar = 1$)

$$S[\phi,\psi_1,\psi_2] = f(\phi) + g(\phi)\psi_1\psi_2$$

for some $\mathbb{R}$ valued functions of the boson field f and g. Notice that this is the most general expression for an action defined with fields ϕ,ψ_i: there can't be any terms in S involving only one of the fermion fields since this term would itself be fermionic; there also can't be higher order terms in the fermion fields since $\psi_i^2 = 0$ for a Grassmann variable.

Given the action, the partition function is

$$\mathcal{Z} = \int \mathrm{d}\phi\mathrm{d}\psi_1\mathrm{d}\psi_2\, e^{-S[\phi,\psi_i]} \tag{3.18}$$

Generically, we would have to be content with a perturbative evaluation of $\mathcal{Z}$, using Feynman diagrams formed from edges for the ϕ and ψ_i fields, together with vertices from all the different vertices that appears in our action. For a complicated action, even low orders of the perturbative expansion might be difficult to compute in general.

We can introduce the complexification of fermions fields

$$\psi = \frac{1}{2}(\psi_1 + i\psi_2) \qquad \bar{\psi} = \frac{1}{2}(\psi_1 - i\psi_2)$$

Let's suppose the action takes a special form

$$S[\phi,\psi,\bar{\psi}] = \frac{1}{2}\left(W'(\phi)\right)^2 - i\psi\bar{\psi}W''(\phi) \tag{3.19}$$

where $W'(\phi) = \partial_\phi W$ and $W''(\phi) = \partial_\phi^2 W$ for some real[VI] function W.
Now consider the transformations

$$\delta\phi = \varepsilon\psi + \bar{\varepsilon}\bar{\psi} \qquad \delta\psi = -i\bar{\varepsilon}W' \qquad \delta\bar{\psi} = i\varepsilon W' \tag{3.20}$$

[V] As we mentioned before, in zero dimension we distinguish between boson and fermions only by them statistics, i.e. commuting/anticommuting proprieties, since in zero dimension spin is not defined.

[VI] We require W to be real since ϕ is real.

where ε is a fermionic parameter. These are **fermionic symmetry** or **supersymmetry** transformations in this zero-dimensional context. Under transformations eq. (3.20) the action eq. (3.19) transforms as

$$\delta S = W'W''\delta\phi - i\delta\psi\bar{\psi}W'' + i\delta\bar{\psi}\psi W'' - iW''\delta\phi\psi\bar{\psi}$$

the term $\delta\phi\psi\bar{\psi}$ vanishes since $\delta\phi$ contains both ψ and $\bar{\psi}$, then

$$\delta S = W''[W'(\varepsilon\psi + \bar{\varepsilon}\bar{\psi}) - W'(\bar{\varepsilon}\bar{\psi} + \varepsilon\psi)] = 0$$

Thus the action is invariant under eq. (3.20), and this is a consequence of the proprieties of boson and fermions fields. Morover, the integration measure $\mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\bar{\psi}$ is invariant itself, this can be easily proved using Grassmann numbers proprieties and proprieties of Wedge product between $i\mathrm{d}\psi$ and $\mathrm{d}\bar{\psi}$.

Given an action with such special symmetry, we can prove that partition function $\mathcal{Z}$ (or any other correlator obtained by insertion of an operator into integral eq. (3.18)) is different from zero only in special points.

Let's describe the general argument. Consider a quantum field theory with fields collectively described by Φ, defined on a configuration space $\mathcal{C}$ over which we path integrate. Assume that there is a symmetry group G which acts freely on $\mathcal{C}$ (in particular the action S is invariant under G) and consider an operator O invariant under G. Then we can introduce collective coordinates for the G-action and integrate over them to get the volume of the group G. This volume factor multiplies a left-over path integral over $\mathcal{C}/G$, the space of orbits of G in the configuration space:

$$\int_{\mathcal{C}} \mathcal{D}\Phi\, O e^{-S[\Phi]} = \mathrm{Vol}(G) \int_{\mathcal{C}/G} \mathcal{D}\Phi\, O e^{-S[\Phi]} \tag{3.21}$$

For instance if we take

$$\langle O \rangle = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, O(x,y) e^{-S(x,y)}$$

where S and O (and obviously $\mathrm{d}x\mathrm{d}y$) are invariant under the orthogonal group $O(2)$: $\mathbf{x}' = R\mathbf{x}$, then in angular coordinates S and O depends only on radius r, therefore

$$\langle O \rangle = 2\pi \int_0^\infty O(r) e^{-S(r)} r\mathrm{d}r$$

where 2π is the volume of the group $O(2)$ respect to the measure $\mathrm{d}\varphi$ and $r\mathrm{d}r$ is the restriction of the two form $\mathrm{d}x\mathrm{d}y$ in the quotient subset $\mathcal{C}/G = \mathbb{R}^2/O(2) = \mathbb{R}_{\geq}$.

If the global symmetry group G is generated by a fermionic charge $\mathcal{Q}$ (a fermionic charge is called **supercharge**; since the action is invariant under G then the action of $\mathcal{Q}$ on S gives zero: $\mathcal{Q}S = 0$), the associate collective coordinate θ is a Grassmann variable and the volume vanishes by the rules of Berezin integrals: $\mathrm{Vol}(G) = \int \mathrm{d}\theta\, 1 = 0$. Of course a supercharge $\mathcal{Q}$ cannot act freely on the whole configuration space $\mathcal{C}$, otherwise eq. (3.21) would vanish even for the identity operator and we would not be able to normalize correlators. Hence the supercharge $\mathcal{Q}$ has fixed points, which form the so called **localization locus** of (bosonic) $\mathcal{Q}$-invariant configurations:

$$\mathcal{C}_{\mathcal{Q}} = \{[\Phi] \in \mathcal{C} |\, \text{fermions} = 0,\ \mathcal{Q}(\text{fermions}) = 0\}$$

The supercharge $\mathcal{Q}$ acts freely on the complement of the localization locus $\mathcal{C} \setminus \mathcal{C}_{\mathcal{Q}}$, and we can apply the above argument there to learn that the path integral with insertion of $\mathcal{Q}$-closed observables vanishes over $\mathcal{C} \setminus \mathcal{C}_{\mathcal{Q}}$. Therefore we conclude that the path integral over field space $\mathcal{C}$ localizes to a subspace, the localization locus $\mathcal{C}_{\mathcal{Q}}$.

For instance, localization locus of fermionic symmetry eq. (3.20) is given by values of ϕ and ψ such that $\delta\phi = 0 = \delta\psi$, for instance if we take a value of ϕ such that $W'(\phi) = 0$ and we take $\psi = 0$ then both $\delta\phi$ and $\delta\psi$ are zero.

There is another way to see that path integrals of supersymmetric field theories localize to $\mathcal{C}_{\mathcal{Q}}$. In this second localization argument we use the freedom to deform the path integrand of a supersymmetric quantum field theory by a $\mathcal{Q}$-exact term to force the path integral to localize to $\mathcal{C}_{\mathcal{Q}}$. Let $\mathcal{Q}$ be the charge associated to the symmetry of our theory. The value

$$B = \mathcal{Q}^2$$

is the bosonic charge associated to a bosonic symmetry. Let S be a $\mathcal{Q}$-invariant action, so that $\mathcal{Q}S = 0$. Consider a functional

$$V[\phi] = \int \mathrm{d}^d x \mathcal{V}(\phi)$$

which is invariant under B, so that $\mathcal{Q}^2 V = 0$. We can introduce a deformation of the action S by the introduction of a $\mathcal{Q}$-exact term[VII] $\mathcal{Q}V$ called **localized action term**[VIII]

In this part many statements may be not precise.

$$S_{loc} = \mathcal{Q}V = \int \mathrm{d}^d x \, \mathcal{Q}\mathcal{V}$$

Then, introducing Φ as the collection of all fields in the theory, we can define following partition function

$$\mathcal{Z}_t = \int \mathcal{D}\Phi \exp\left\{-S[\Phi] - t \int \mathcal{Q}V[\phi]\mathrm{d}^d x\right\} \tag{3.22}$$

When we take t derivative of this object, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_t = \int \mathcal{D}\Phi \left(\mathcal{Q}\int V[\phi]\mathrm{d}^d x\right) \exp\left\{-S[\Phi] - t \int \mathcal{Q}V[\phi]\mathrm{d}^d x\right\}$$

but since action is invariant ($\mathcal{Q}S = 0$) and $\mathcal{Q}^2 V = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_t = \int \mathcal{D}\Phi \, \mathcal{Q}\left(\int V[\phi]\mathrm{d}^d x \exp\left\{-S[\Phi] - t \int \mathcal{Q}V[\phi]\mathrm{d}^d x\right\}\right) \tag{3.23}$$

One can prove that since $\mathcal{Q}$ is the generator of the symmetry in our configuration space, then $\mathcal{Q}O$ is the infinitesimal variation of some operator O under transformations of G. Hence, by a generalization of Stokes theorem, the integral of such term gives the transformation of O at boundaries of the configuration space, where we assume a vanishing contribution from O:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_t = 0$$

Then the deformation of the action given by S_{loc} do not change corrlators, in particular $\mathcal{Z}_0 = \mathcal{Z}_t$ for any $t \in \mathbb{R}$. We can evaluate the partition function by taking $t \to \infty$:

$$\mathcal{Z}_0 = \mathcal{Z}_\infty = \lim_{t\to\infty} \int \mathcal{D}\Phi \exp\left\{-S[\Phi] - t \int \mathcal{Q}V[\phi]\mathrm{d}^d x\right\} = \lim_{t\to\infty} \int \mathcal{D}\Phi \exp\{-S[\Phi] - t\, S_{loc}\}$$

In this limit the integrand is dominate by the saddle points of S_{loc}. Let us expand the fields about the localization locus

$$\Phi = \Phi_0 + \frac{\delta\Phi}{\sqrt{t}} \tag{3.24}$$

In this way we obtain

$$S_{loc}[\Phi] = S_{loc}[\Phi_0] + \frac{1}{2}\frac{\delta^2 S_{loc}}{\delta\Phi^2}\frac{\delta\Phi^2}{t} + o\left(\frac{1}{t^{3/2}}\right) \tag{3.25}$$

where $S_{loc}[\phi_0]$ is a constant, and we can set it to zero, then performing the change of variables eq. (3.24) we have the substitution $\mathcal{D}\Phi \to \mathrm{d}\left(\frac{\delta\phi}{\sqrt{t}}\right)\mathrm{d}\left(\frac{\delta\psi}{\sqrt{t}}\right) = \frac{\mathrm{d}(\delta\phi)}{\sqrt{t}}\sqrt{t}\,\mathrm{d}(\delta\psi) = \mathrm{d}(\delta\phi)\mathrm{d}(\delta\psi)$ and we have

$$\mathcal{Z}_\infty = \lim_{t\to\infty} \int \mathcal{D}\delta\Phi \exp\left\{-S[\Phi_0] - \frac{1}{2}\frac{\delta^2 S_{loc}}{\delta\Phi^2}\delta\Phi^2 + o\left(\frac{1}{t^{1/2}}\right)\right\}$$

[VII] Recall that if $\mathcal{Q}$ is the charge associated to some symmetry G, then $\mathcal{Q}F$ is the infinitesimal transformation of F under the symmetry G. We say that some function is $\mathcal{Q}$-exact if it can be written as infinitesimal transformation of some function F under the symmetry G. This is analogous to exact functions in analysis, where some function f is exact is it can be written as $f = \mathrm{d}g$ for some other function g, i.e. it can be written as total derivative respect to its parameter $\frac{\mathrm{d}g}{\mathrm{d}x}$ times an infinitesimal variation of the parameter $\mathrm{d}x$.

[VIII] Usually $V = \sum_i \psi_i (\mathcal{Q}\psi_i)^\dagger$, where $\mathcal{Q}\psi_i$ is the action of the generator Q on the fermion ψ_i and the summation is over all the fermionic variables ψ_i. The symbol † means the complex conjugated term. In such a way $S_{loc} = \sum_i \mathcal{Q}(\sum_i \psi_i (\mathcal{Q}\psi_i)^\dagger)$ is a sum of squares of supersymmetric variations. Then the saddle point configurations of S_{loc} are given by $\mathcal{C}_\mathcal{Q}$.

where we substituted $S[\Phi]$ with $S[\Phi_0]$ plus additional terms included into $o\left(t^{-1/2}\right)$. The factor $S[\Phi_0]$ is constant and can be set to zero, hence in the limit we obtain an explicit formula

$$\mathcal{Z} = \mathcal{Z}_\infty = \lim_{t\to\infty} \int \mathcal{D}\delta\Phi \exp\left\{-\frac{1}{2}\frac{\delta^2 S_{loc}}{\delta\Phi^2}\delta\Phi^2\right\} = e^{-S[\Phi_0]}\frac{(2\pi\hbar)^{n/2}}{\det M_{bos}}\det M_{ferm}$$

where M_{bos} and M_{ferm} are operators appearing at quadratic order in bosonif nd fermionic fluctuations in eq. (3.24).

This means that under fermionic symmetries, there is a general procedure (not very easy) that allow to obtain $\mathcal{Z}$ with a simple gaussian integration of fluctuations around zero locus. Only configurations invariant under the symmetry contributes to the integral, and these contribution can be computed explicitly.

Let's go back to our previous example. In our case the fermionic symmetry is charaterized by $Q^2 \neq 0$, but we can anyway introduce a partition function $\mathcal{Z}_t$ that depends on a parameter t in such a way that the result does not depend on t, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_t = 0$, in such a way that I can compute

$$\mathcal{Z}_t = \int \mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\bar{\psi}\, \exp\{-S_t[\phi,\psi,\bar{\psi}]\}$$

by introducing a modified action

$$S_t = \frac{1}{2}t^2(W')^2 - itW''\psi\bar{\psi} \tag{3.26}$$

Modifying the action I have a different fermionic symmetry such that the theory is invariant, such a symmetry this time depends on t. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}S_t = t(W')^2 - iW''\psi\bar{\psi} = Q\left(\frac{i}{2}W'(\psi-\bar{\psi})\right) \tag{3.27}$$

where Q acts as

$$Q = (\psi+\bar{\psi})\frac{\partial}{\partial\phi} - itW'\left(\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\bar{\psi}}\right)$$

This means for instance

$$Q[\psi] = -itW'$$

and

$$Q[\phi] = \psi + \bar{\psi}$$

i.e. we are taking previous transformation for a hermitian parameter $\varepsilon = \bar{\varepsilon}$ and then we are rescaling the transformation in such a way that the action is invariant, $Q[S_t] = 0$. This is similar to what we done in eq. (3.22) where we introduced a boundary term $-t\int QV[\phi]\mathrm{d}^d x$ in such a way that the derivative eq. (3.23) gives a boundary term. In our case, instead of the partition function, we modified the action eq. (3.26) since in this way the derivative of the action itself is a boundary eq. (3.27). Indeed, also in this case

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}_t = \int \mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\bar{\psi}\, Q\left(\frac{1}{2}W'(\psi-\bar{\psi})e^{-S_t}\right) = 0$$

since the argument of the integrand is again a boundary. Then we can expand again fields about locus

$$\phi = \phi_0 + \frac{\delta\phi}{t} \qquad \psi = \frac{\delta\psi}{\sqrt{t}}$$

where we modified the exponents of t for bosonic fields in order to reproduce same degrees of freedom for both bosonic and fermionic parts, in such a way the action is invariant. Now the action can be expanded in the following way

$$S_t = \frac{t^2}{2}W'(\phi_0)^2 + \frac{t^2}{2}\left(2(W'')^2 + 2W'''W'(\phi_0)\right)\frac{\delta\phi^2}{t^2} - 2tW''(\phi_0)\frac{\delta\psi_1\delta\psi_2}{t} + o\left(\frac{1}{\sqrt{t}}\right)$$

Notice that we can set $W'(\phi_0) = 0$ and $W''' = 0$ since we are considering the localization locus where fermion symmetry vanishes, so the expansion has only quadratic pieces

$$\mathcal{Z}_t = \int \mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\bar{\psi}\, \exp\{(W'')^2\delta\phi^2 - 2W''\delta\psi_1\delta\psi_2\}$$

and up to coefficent this is proportional to the mass term of fermions W'' divided by the root of the mass squared of the boson $(W'')^2$

$$\mathcal{Z}_t \simeq \frac{W''}{\sqrt{(W'')^2}} = \frac{W''}{|W''|} = \mathrm{sgn} W''(\phi_0)$$

This means that the partition function simply counts maximum and minimum as explained in the next figure:

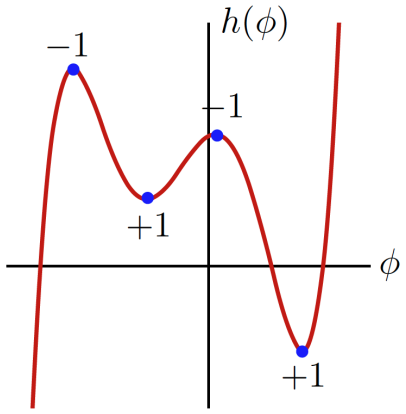


Figure 3.1: *The partition function receives contributions just from infinitesimal neighbourhoods of the critical points of $h(\phi) = W(\phi)$. These alternately contribute ± 1 accoarding to whether they are minima or maxima*

Now we can repeat this argument for expectation value of other operators $\langle O \rangle$ if $QO = O$. this means that also for such expectation values the only contributions are given by a specific set of points, defined by $W' = 0$

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