Contents

Chapter 1

Preliminaries

1.1 Quick review of Special Relativity

Here we expose a quick review of Special Relativity in order to set the notations.

Fundamental principles of Special Relativity are followings:

(i) All inertial reference frames are physically equivalent. There is no way to distinguish between different inertial frames in the sense that there is no preferred one.

(ii) There exists universal (dimensional) constant: $c \simeq 3 \times 10^8$m/s, i.e. the speed of massless particles.

In order to implement these features basic ingredients are

(i) Space and Time form a unique concept called **spacetime**.

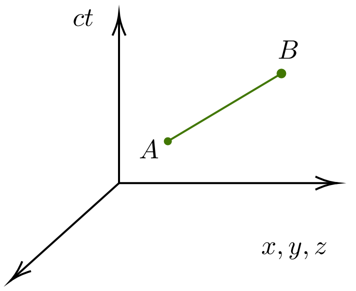
(ii) A spacetime is a collection of points called **event**.

(iii) Each inertial frame is associated with a set of **space time coordinates**. Each events is specified through coordinate system of a fixed initial frame.

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (x^0, x^i) \equiv (ct, x, y, z) \equiv (ct, \mathbf{x})$$

Usually x, y, z are assumed to be Cartesian coordinates.

Given 2 events A and B in spacetime



their distance is $\Delta x^\mu = x^\mu_B - x^\mu_A$. We introduce the (squared) **Minkowski distance**

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta^\nu \quad \text{where} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where $\eta_{\mu\nu}$ is **Minkowski metric**. This induces the **line element**

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

This element is scalar quantity and therefore does not depends on the specific inertial frame. The quantity Δs^2 has an intrinsic meaning

$$\begin{cases} \Delta s^2 > 0 & : \quad \Delta x^\mu \text{ is } \textbf{space-like} \text{ vector} \\ \Delta s^2 = 0 & : \quad \Delta x^\mu \text{ is } \textbf{time-like} \text{ vector} \\ \Delta s^2 < 0 & : \quad \Delta x^\mu \text{ is } \textbf{light-like/null} \text{ vector} \end{cases}$$

Space-like vector means that exists different frames were two events are simultaneous. Time-like vector means that exists different frames where two events have same space coordinates but they happen at different times. Light-like vectors means that two events may be connected by a light signal.

(iv) Allowed transformations for spacetime vectors must preserve the line element: $\Delta\tilde{s}^2 = \Delta s^2$. These transformations are the **Poincaré Transformations**

$$x^\mu \quad \to \quad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \text{with} \qquad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$$

Once we have reformulated notions of space and time, we have to reformulate law of physics in such a way they does not depends on the reference frame.

Trajectories of point like-particles are associated to curved **wordlines** in space time and described evolution of events. Mathematically they are described by maps from $\mathbb{R}$ into a set of four functions: $\lambda \in \mathbb{R} \to x^\mu(\lambda)$. Near if we consider nearby events separated by infinitesimal shift we can obtain infinitesimal variation of coordinates:

$$\mathrm{d}x^\mu(\lambda) \equiv x^\mu(\lambda + \mathrm{d}\lambda) - x^\mu(\lambda) = \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda}\mathrm{d}\lambda$$

Since no particles can move at a speed higher then light this implies that $\mathrm{d}s^2$ must be time-like. Notice that choice of parameter λ is free. One possible choice of this parameter is the **(differential) proper time**:

$$\mathrm{d}\tau \equiv \sqrt{-\mathrm{d}s^2} = \mathrm{d}\lambda\sqrt{-\eta_{\mu\nu}\dot{x}^\mu(\lambda)\dot{x}^\nu(\lambda)} = c\,\mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}} \equiv \frac{c\,\mathrm{d}t}{\gamma}$$

where third step holds if $\lambda \equiv t$. If we define $\beta \equiv v/c$ [I], then $\gamma = 1/\sqrt{1-\beta^2}$ is called **Lorentz factor**. Notice that last step implies time dilatation at higher velocities. For $\lambda = t$ we obtain

$$\tau = c\int \mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}}$$

i.e. with this definition the proper time has dimension of a length $[\tau] = L$. Physically the proper times it's the time measured by a clock moving along the trajectory.

Proper time allow us to define a vector called **4-velocity** that can be identified as relativistig generalization of velocity. Namely:

$$u^\mu(\tau) = \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau} = \left(\gamma, \gamma\frac{\mathbf{v}}{c}\right)$$

Notice

$$u^\mu u_\mu = -1$$

i.e. is a time-like vector. Moreover, this vector has only three degrees of freedom, since one component is fixed by previous propriety.

Now we can define the generalization of acceleration, **4-acceleration**, as follows

$$\alpha^\mu(\tau) = \frac{\mathrm{d}u^\mu(\tau)}{\mathrm{d}\tau}$$

[I] This is the speed in natural units, i.e. in units of β. If we set $c = 1$ then $v = \beta$.

Notice that, as we expected, 4-acceleration is orthogonal to 4-velocity

$$u_\mu \alpha^\mu = 0$$

and this implies that α^μ is a space-like since it is orthogonal to a time-like vector.

This proves a relativistic generalization of distance, speed and acceleration. Also laws of dynamic can be generalizated, in particular if we define the **four-force** f^μ as the generalization of force we can obtain the **Relativistic Second Newton's law**:

$$mc\alpha^\mu \equiv \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = f^\mu$$

where we used four acceleration or equivalently the generalization of newtonian momentum, **4-momentum**,

$$p^\mu \equiv mcu^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$$

For example, for Lorentz force

$$\mathbf{F}_L = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

can be generalizzated in a manifestly covariant way into[II]

$$f_L^\mu = \frac{e}{c} F^{\mu\nu} u_\nu \qquad \text{with} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

where $F^{\mu\nu}$ is the **EM-Tensor**.

We can also rewrite Maxwell equations into two covariant equation

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu \quad , \quad \partial_{[\mu} F_{\nu\rho]} = 0$$

where the former, inhomogeneous, shows the **4-current** $j^\mu = (c\rho, \mathbf{j})$. The second equation, homogeneous, exhibits total antisymmetrized indexes[III]. Each of these equations contains 2 independent equations.

We can conclude saying that all possible interactions can be written in a covariant way, except from gravitation. In order to include this force General Relativity has been developed.

1.2 Relativity and Gravitation

Before Einstein formulated General Relativity the accepted theory for Gravity was Newton's one.

In Newton theories particles interact according to **Newton's universal gravity law**:

$$\mathbf{F}_G = -\frac{GmM}{|\Delta\mathbf{r}|^3}\Delta\mathbf{r} \qquad G \simeq 6.67 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

where $\mathbf{F}_G$ is the (always) attractive gravitational force, $\Delta\mathbf{r}$ is the distance (at same time) between particles, and G is **Newton's gravitational constant**.

The point is that this law is not invariant under Poincaré. Practically this is evident since positions are evaluated at a certain time and therefore when a particle moves the corresponding formula for gravitational force changes instantaneously. This is unphysical since physical signal cannot travel with a velocity higher than speed of light. This instantaneous interaction between particles cannot be compatible with special relativity.

One possible strategy to way out is to look at an analogy with Coulomb force between two particles A and B:

$$\mathbf{F}_C = e_B\mathbf{E} = \frac{e_B e_A}{|\Delta\mathbf{r}|^3}\Delta\mathbf{r}$$

[II] Here is evident that this formula does not change under Poincaré transformations.

[III] $\partial_{[\mu} F_{\nu\rho]} = \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu}$.

These formulas are very analogous. The coulomb force is valid only in a static setting in which one put one particle in the electric field of the other. When particles moves this particles does not hold anymore since we have to consider magnetic field and Coulomb force must be substituted with the more general Lorentz force.

Let's go further with the analogy. In gravity we can introduce a potential which is completely analogous to electric static potential:

$$\begin{aligned} &\nabla^2\Phi = 4\pi G\rho_M && G\rho_M \leftrightarrow -\rho_{el} && \nabla^2\Phi_{el} = -\vec{\nabla}\cdot\mathbf{E} = -4\pi\rho_{el} \\ &\mathbf{F}_G = -m\vec{\nabla}\Phi && m \leftrightarrow e && \mathbf{F}_C = e\mathbf{E} = -e\vec{\nabla}\Phi_{el} \end{aligned}$$

where Φ describe potentials and ρ describes distributions.

In order to make Coulomb force compatible with special relativity we have to consider EM theory in a wider way, in order to express quantities in tensorial way:

$$\begin{aligned} \Phi_{el} &\quad\rightarrow\quad A^\mu = (\Phi, \mathbf{A}) \\ \mathbf{E} &\quad\rightarrow\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \rho_{el} &\quad\rightarrow\quad j^\mu = (c\rho_{el}, \mathbf{j}) \\ m\mathbf{a} = \mathbf{F}_c &\quad\rightarrow\quad mc\alpha^\mu = f_L^\mu = \frac{e}{c}F^{\mu\nu}u_\nu \end{aligned}$$

So in order to understand how to make gravity compatible with spacial relativity we should find gravitational analogous to previous completions of Coulomb theory. We will start from the last step, i.e. we have to understand what happen when we put a particle in an external gravitational field and then derive covariant relations, which in non relativistic approximation must leads to Newton's law

$$m\mathbf{a} = \mathbf{F}_G$$

In particular we have to find which is relativistic generalization of gravitational field Φ_G that allows us to build a covariant theory of gravitation.

First of all we have to highlight a deep difference between $\mathbf{F}_G$ and $\mathbf{F}_C$ that in principle we should

distinguish between two forces. For gravitational theory the force is proportional to the mass of the particle, while for Coulomb law the force is proportional to the charge. In order to make the analogy precise we have should distinguish between two different concepts of mass, that in principle may be differnet[IV]

$$\begin{aligned} &\textbf{interial mass } m_I && : \quad m_I\mathbf{a} = \mathbf{F}_G \\ &\textbf{gravitational mass } m_G && : \quad \mathbf{F}_G = -m_G\vec{\nabla}\cdot\Phi \qquad m_G \sim \text{gravitational charge} \end{aligned}$$

the fact that in Newton's law $m_G \equiv m_I$ is an highly not-obvious feature from theoretical point of view. Indeed, this is the Newtonian manifestation of the **(Weak) Equivalence principle**.

1.2.1 The Equivalence principle

The Equivalence principle is the consequence of the central observation that $m_G \equiv m_I$. This implies that

$$\cancel{m}_I\mathbf{a} = \mathbf{F}_G = -\cancel{m}_G\vec{\nabla}\Phi \quad\Rightarrow\quad \boxed{\mathbf{a} = -\vec{\nabla}\Phi}$$

i.e. acceleration of a mass is the same for each value of m (for example this does not happen for Coulomb interaction, where acceleration depends on the charge of the particle). Then for the same field Φ all bodies fall with the same acceleration.

This is a very important observation: if $\vec{\nabla}\Phi$ is considered approximatively constant in a chosen frame, then we cannot distinguish between gravitational force and an apparent force due to an acceleration in the opposite direction of this frame with respect to an inertial frame.

[IV] For the moment we consider the non-relativistic limit

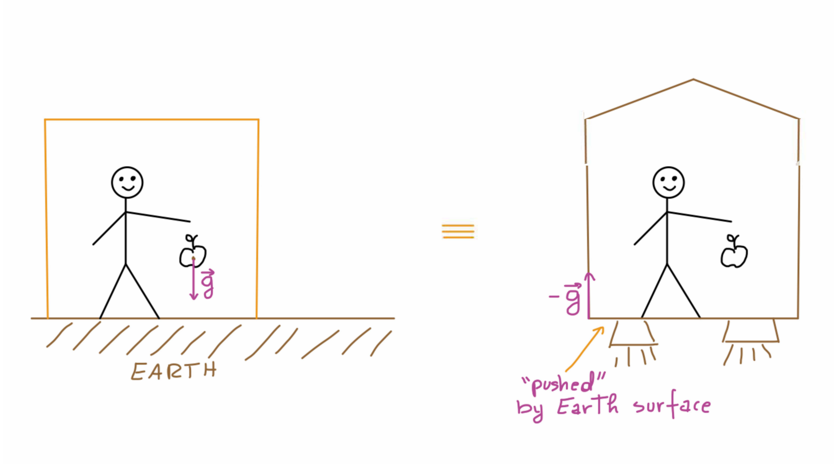


Figure 1.1: In the first figure the apple falls down, while in the second the rocket moves upward. Our Rocket Man can't fell any difference.

"The happiest thought of Einstein life":

> *"The gravitational field has only a* relative *existence . . . because for an observer freely falling from the roof of a house there exists - at least in the immediate surroundings - no gravitation"*

In other words, a freely falling system can be identified (up to some approximations) to an "inertial" frame[V] in the sense that within a freely falling system there is no way to distinguish between these two situations.

This leads to the formulation of **(Einstein) Equivalence "Principle"** (**EEP**)

> *In a small enough region of spacetime, the laws of physics reduce to those of Special Relativity: it is impossible to detect the existence of a gravitational field by means of local experiments.*

In other words though local experiments it is impossible to distinguish between a system accelerated and a system subjected to gravitational field. The caveat "small enough" refers to the *Tidal effects*, i.e. the previous statement holds only if the gravitational field can be considerated uniform and constant. Let l be the typical length scale of our experiment and L be the distance from the mass that origins the gravitational field, then "small enough" means

$$\left(\frac{l}{L}\right)^n \ll 1$$

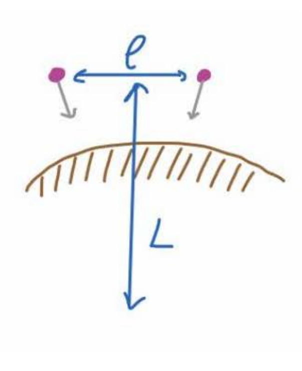


Figure 1.2: Tidal effect

[V]We will have to specify this concept in formalization of General Relativity.

There are 3 kinds of equivalence principles:[VI]

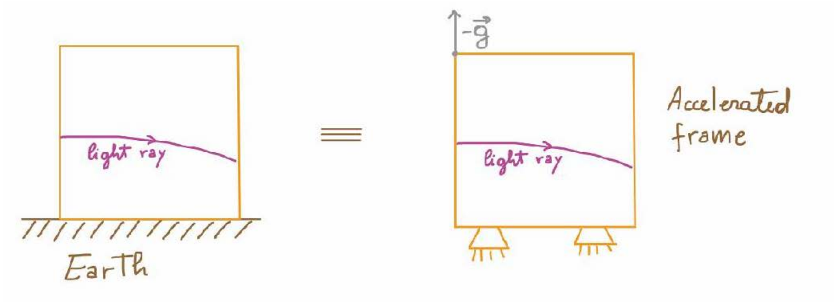
(i) **Weak Equivalence Principle (WEP)**: regards only experiments on freely falling non back-reacting test particles (direct consequence of $m_G = m_I$.

(ii) **Einstein Equivalence Principle (EEP)**: previously stated, include also other non-gravitational local experiments (no backreaction)

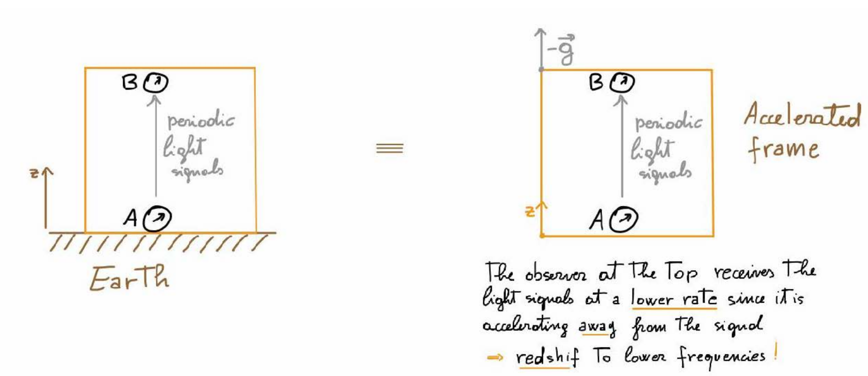
(iii) **Strong Equivalence Principle (SEP)**: includes also local gravitational effects (includes gravitational effects, i.e. back-reaction). For instance it consider also inertial masses in experiments involving them variation measure when accelerated.

One can think about these principles as heuristic ideas, which will be defined in a more precise way by General Relativity in a concrete framework.

There are some immediate implications of these principles. First *light is deflected* in presence of gravitation potential, this is obvious watching at the next image



Then, we will see *gravitational time dilatation and red-shift of light*:



The second frame is accelerated, then during the interval which the light signal need to go from a clock A to the clock B the clock B gets some additional velocity, so frequency measured by B is lower than the frequency measured by A:

$$v_B - v_a \simeq g\Delta t = g\frac{\Delta z}{c}$$

and we can observe a *Doppler effect*

$$\frac{\nu_B - \nu_A}{\nu_a} \simeq \frac{v_A - v_B}{c} = \frac{g(z_A - z_B)}{c^2}$$

If we take $\Phi \simeq gz$ we can obtain an explicit relation between redshift and acceleration

$$\frac{\nu_B - \nu_A}{\nu_A} \simeq \frac{1}{c^2}(\Phi_A - \Phi_B) < 0$$

[VI] In the paper *Di Casola, Liberati & Sonego*, 1310.7426, are described differences of these statements and experiments evidence for each principle.

Viceversa, since frequency is the inverse of time interval, this can be interpreted as a time dilatation. In other words, we can see that clock B "sees" clock A moving more slowly.[VII]

1.3 The constantly accelerated elevator

Blau sec. 1.3; 't Hooft chap. 3

Now we have to make more concrete what we introduced in the previous chapter, i.e. we want to extend in a covariant way the gravitational potential, using the equivalence principle. Up to the present, we obtained the equivalence principle using non-relativistic arguments, now we want to obtain same result using a suitable mathematical framework, which allows us to derive gravitational laws in a covariant way.

From now on we will use relativistic units for velocity, i.e. all velocities are expressed in units of c. This is the same as set $c = 1$. With this choice

$$[L] = [T], \qquad [E] = [M], \qquad \ldots$$

Now we have to look for a natural frame with its own natural coordinates which describe uniform accelerated frame. Recall that in SR a trajectory with constant acceleration will not make physical sense because any particle undergoing constant acceleration at a certain time would exceedes the speed of light, leading to a non-physical propagating signal.

On the other hand a proper definition of a uniformly accelerating trajectory is to impose that the proper acceleration is constant. For example for a rocket travelling with a constant proper acceleration a along x:

$$\alpha^\mu \alpha_\mu = a^2 \qquad \text{with} \quad \alpha^\mu = (\alpha^0, \alpha^1, 0, 0) = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}$$

The interpretation of this proper acceleration is that this is exactly the acceleration measured in the instantaneous rest-frame of the rocket. This means that when we consider the trajectory of the rocket we should think about a specific point of the rocket. Then this point will accelerate, but in any instant of time we can choose a rest frame S_I where the four velocity takes the form

$$u^\mu|_{S_I} = (1, 0, 0, 0)$$

and since $\alpha^\mu u_\mu = 0$ we have

$$\alpha^\mu|_{S_I} = (0, a, 0, 0)$$

where a is constant during the acceleration. Then we can explicitly write down the form of a trajectory that satisfies this relation in the following form

$$\begin{cases} t(\tau) \ = x^0(\tau) = \dfrac{1}{a}\sinh(a\tau) \ = X\sinh\left(\dfrac{\tau}{X}\right) \\ x(\tau) = x^1(\tau) = \dfrac{1}{a}\cosh(a\tau) = X\cosh\left(\dfrac{\tau}{X}\right) \end{cases} \tag{1.1}$$

where $X \equiv 1/a$. Notice that $(x^0, x^1)(0) = (0, X)$, as it's shown in the next figure:

[VII] See Hartle's book for the discussion of Redshift using time dilatation instead of moving clocks.

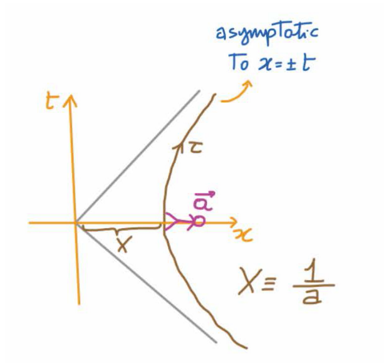


Figure 1.3: The green line describe the trajectory, parametrized by τ. In the picture are represented only first two coordinates.

A the notation suggests the parameter τ is the proper time, indeed we can check it computing $\mathrm{d}s^2$ over the trajectory:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = -\cosh^2(a\tau)\mathrm{d}\tau^2 + \sinh^2(a\tau)\mathrm{d}\tau^2 = -\mathrm{d}\tau^2$$

this implies

$$\int_0^\tau \sqrt{-\mathrm{d}s^2} = \int_0^\tau \mathrm{d}\tilde{\tau} = \tau$$

i.e. τ is exactly the proper time. It is also immediate to check that these trajectories satisfies our requirement

$$\begin{cases} u^0(\tau) = \cosh(a\tau) \\ u^1(\tau) = \sinh(a\tau) \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha^0(\tau) = a\sinh(a\tau) \\ \alpha^1(\tau) = a\cosh(a\tau) \end{cases} \quad \Rightarrow \quad \alpha^\mu \alpha_\mu = a^2$$

In order to simplify the notation in the following we will use also the (well known from SR) relation: $\gamma = \cosh(\tau/X)$.

Notice also that eq.(1.1) is a particular solution that satisfies the requirement of constant acceleration. We fixed integration constants so that any trajectory of this form asymptotically tends to the straight line $t = \pm x$ that describes the light-like signal passing through the origin. With this specific choice $X = 1/a$ can be identified with the x-position at $t = \tau = 0$.

Letting $X = 1/a$ change we get a family of hyperbolic trajectories

$$x^2 - t^2 = X^2 \equiv 1/a^2$$

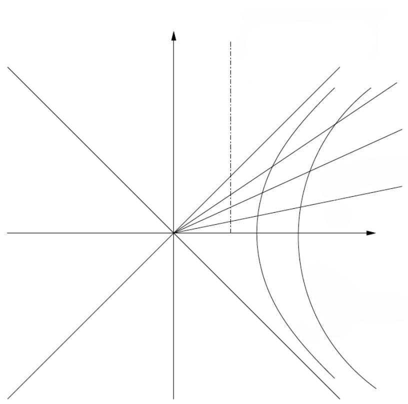
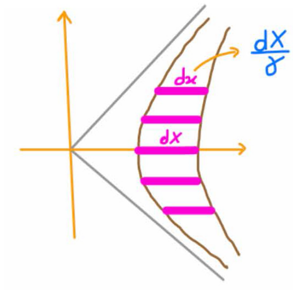


Figure 1.4: Here are shown all possible hyperbolic trajectories with $a > 0$. The vertical straight line is the world line of a stationary observer.

Notice that choosing negative acceleration $a < 0$ trajectories are in the left side of the space, i.e. trajectories for positive and negative acceleration lives in disconnected regions.

For $\tau = 0$ we have $u^0(\tau = 0) = \cosh(0) = 1$ and $u^1(\tau = 0) = \sinh(0) = 0$, i.e. we are in the rest frame for our system. Since X is the value of x for $\tau = 0$, then X can be interpreted as the proper x coordinate for a particle accelerating on the x direction (analogously as τ for the time). Equivalently, let $\mathrm{d}X$ be the infinitesimal distance between two close coordinates for the system in the rest frame, then $\mathrm{d}x$ for $\tau \neq 0$ is the contracted distance between the two coordinates when the system has non-zero velocity:

$$\mathrm{d}x = \frac{\mathrm{d}X}{\gamma}$$



Let's prove the last statement. For t =const., we have $\mathrm{d}t = [\sinh(\tau/X)\mathrm{d}X + \cosh(\tau/X)\mathrm{d}\tau] = 0$ and this means

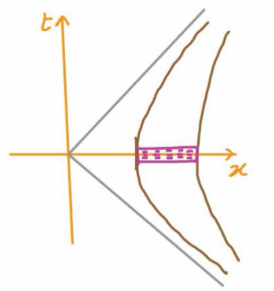
$$\mathrm{d}\tau = -\tanh\left(\frac{\tau}{X}\right)\mathrm{d}X$$

then we have

$$\mathrm{d}x = \cosh\left(\frac{\tau}{X}\right)\mathrm{d}X + \sinh\left(\frac{\tau}{X}\right)\mathrm{d}\tau = \frac{1}{\cosh(\tau/X)}\mathrm{d}X = \frac{\mathrm{d}X}{\gamma}$$

Therefore the family of hyperbolic trajectories describes motion of points of a rigid body accelerating on the x direction.

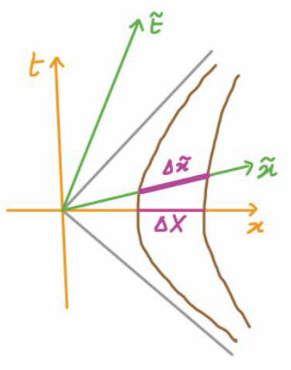
Notice that accelerations of points on the rigid body are all different, since different points belong to different trajectories, i.e. corresponds to different values for a. In our one-space-dimensional model this means that all points of the rod in the following picture has different acceleration[VIII]:



This is not true for higher dimensions since all points in a hyperplane orthogonal to the direction of motion have same acceleration. Anyhow points on different hyperplanes must have different accelerations.

Notice also that quantity $x^2 - t^2 = X^2$ is invariant under Lorentz transformation (i.e. 4-dimensional rotations), and then going from an inertial frame into another we have the same equation $\tilde{x}^2 - \tilde{t}^2 = X^2$. In particular for any inertial frame when $\tilde{t} = 0$ all points of the rod are at rest and $\Delta\tilde{x}(\tilde{t} = 0) = \Delta X$, i.e. all distances are equivalent in all rest frames.

[VIII] This can also be interpreted as the origin of the contraction of lengths.



Using last observation, we can think about X as a coordinate for (instead of a rod) a rigid lattice, parametrized by space coordinates X, Y, Z, where $Y = y$ and $Z = z$ are usual Euclidean coordinates. We consider only the connected lattice defined by $X > 0$ (i.e. $a > 0$). So far we used to parametrize each trajectory with its proper time τ measured by a clock moving on the trajectory. Therefore we can use as coordinate system of our lattice the set (τ, X, Y, Z).

In order to describe what is special with this reference frame, let's consider line elements:

$$\begin{cases} x^0 = X \sinh\left(\frac{\tau}{X}\right) \\ x^1 = X \cosh\left(\frac{\tau}{X}\right) \end{cases} \quad \Rightarrow \quad \begin{cases} \mathrm{d}x^0 = \left[\sinh(\tau/X) - \frac{\tau}{X}\cosh\left(\frac{\tau}{X}\right)\right]\mathrm{d}X + \cosh\left(\frac{\tau}{X}\right)\mathrm{d}\tau \\ \mathrm{d}x^1 = \left[\cosh(\tau/X) - \frac{\tau}{X}\sinh\left(\frac{\tau}{X}\right)\right]\mathrm{d}X + \sinh\left(\frac{\tau}{X}\right)\mathrm{d}\tau \end{cases}$$

and then the metric reads

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.2}$$

$$= -\mathrm{d}\tau^2 + \frac{2\tau}{X}\mathrm{d}\tau\,\mathrm{d}X + \left(1 - \frac{\tau^2}{X^2}\right)\mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \tag{1.3}$$

This non-trivial metric can be regarded as fully characterising the new frame, so is telling us how coordinates (identified as position on the lattice and proper time measured by accelerating clocks) enter into the definition of line elements.

So far we could simply think about this metric as the consequence of a simply change of coordinates, and not really matter. But now invoking the equivalence principle, we can say that the accelerated frame described by coordinates (τ, X, Y, Z) should be equivalent to a frame undergoing gravitational force. Therefore the existence of gravitational field could be revisited as the existence of a non trivial metric in our spacetime.

1.4 The Rindler spacetime

Blau sec. 1.3; 't Hooft chap. 3

In the previous section we constructed the rigid lattice with coordinates (τ, X, Y, Z) and non-trivial metric[IX] (1.2) where τ and X are respectively proper time and proper length in this lattice. Note that the components of the metric depend on (τ, X). In particular, the separation between simultaneous events A and B (with $\tau_A = \tau_B$) is not the Eucledian one ($\Delta l^2 = \Delta X^2 + \Delta Y^2 + \Delta Z^2$) and changes with time τ.

Despite the physical meaning of τ as proper time, a nicer coordinate T can be chosen for lattice's system of coordinates. Note that

$$\mathrm{d}s^2 = -\left(\mathrm{d}\tau - \frac{\tau}{X}\mathrm{d}X\right)^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2$$

therefore if we define following *adimensional* coordinate

$$T = \frac{\tau}{X}$$

[IX] Usually in these notes terms "metric" and "line element" are used equivalently, beside in the formal definition of metric that we will introduce in the next chapters.

and we use it instead of τ, following substitutions must be done

$$\tau = XT \qquad \Rightarrow \qquad \mathrm{d}\tau = X\mathrm{d}T + T\mathrm{d}X = X\mathrm{d}T + \frac{\tau}{X}\mathrm{d}X$$

and the line element takes the form

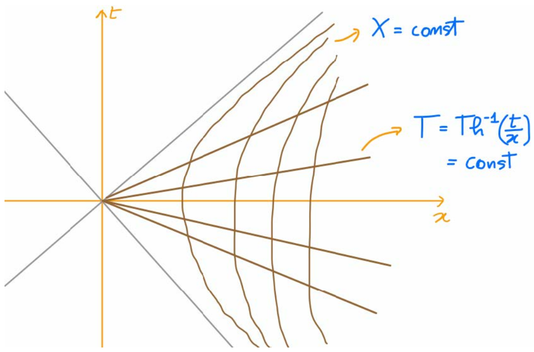
$$\mathrm{d}s^2 = -X^2\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \tag{1.4}$$

The spacetime equipped with metric eq.(1.4) is called **Rindler spacetime**. Notice that same metric can be obtained directly from the flat spacetime using following substitutions

$$\begin{cases} t = X \sinh T \\ x = X \cosh T \end{cases}$$

Proprieties of the coordinates system of Rindler space

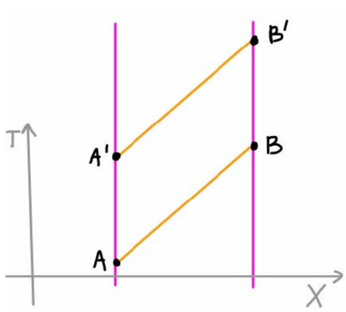
One of main advantages of the system of coordinates (T, X, Y, Z) is that if we consider simultaneous events ($\mathrm{d}\tau = 0 = \mathrm{d}T$) then metric eq.(1.4) is the Euclidean one. Also, if we draw the Rindler space as follows:



then points with constant X corresponds to hyperbolic trajectories with constant acceleration, and points with constant T are placed on the same straight line passing through the origin, in particular proper time is given by the relation

$$T = \tanh^{-1}\left(\frac{t}{x}\right)$$

Moreover, suppose that two clocks moving with constant acceleration are placed on the lattice, and one of them (namely "clock A") sends light signals to the other (called "clock B"). Let T_A be the proper time for A when the first clock sends the signal and T_B be the proper time for T_B when the second clock receives the signal. Then the difference $\Delta T = T_B - T_A$ does not depend on the time when the first clock sends it signal, i.e. ΔT is time-independent: if clock A send another light signal at the proper time T'_A, which is received by the clock B at proper time T'_B, then $T_B - T_A = T'_B - T'_A$.



In other words, if A sends light signals with a certain rate, then the clock B "sees"[X] clock A "clicking" with the same rate. We can prove this as follows: first of all for the light signal we have $\mathrm{d}s^2 = 0$, then

$$0 = \mathrm{d}s^2 = -X^2\mathrm{d}T^2 + \mathrm{d}X^2 \qquad \Rightarrow \qquad \mathrm{d}T = \frac{\mathrm{d}X}{X}$$

where in the second step we fixed the sign in order to choose the right direction of propagation (shown in the picture). If we consider the value of $\Delta T \equiv T_B - T_A$, i.e. the time between the click of A and the light signal seen by B, we have

$$\Delta T \equiv T_B - T_A = \int_A^B \mathrm{d}T = \int_A^B \frac{\mathrm{d}X}{X} = \log\frac{X_B}{X_A}$$

this means that ΔT depends only on the proper position on the lattice of the two clocks, hence does not depend on the time T_A when the first clock clicks:

$$T_B = T_A + \Delta T$$
$$T_{B'} = T_{A'} + \Delta T$$

We can also say that using time coordinate T then times of clocks are syncronized by light signals. This propriety is called propriety of **static** spacetime.

We seen that give EEP implies that this accelerating frame (the Rindler space) could be interpreted as a frame undergoing a gravitational field. This leads a non-trivial acceleration of free-falling objects. We know that acceleration experienced by points in the lattice is given by $\mathbf{a} = -(1/X)\hat{u}_x$, then just applying EEP we can obtain the formula for the gravitational field:

$$\mathbf{a}_G = -(1/X)\hat{u}_x = -\vec{\nabla}\Phi \qquad \Rightarrow \qquad \Phi = \log X$$

Notice that in this realization of EEP the gravitational field is not constant, since its strength is proportional to $-\vec{\nabla}\Phi$ and then decreases with X (in particular, respect to the last picture, the field strength is weaker in B than in A). Using $\Phi = \log X$ we can rewrite Rindler metric in the following form

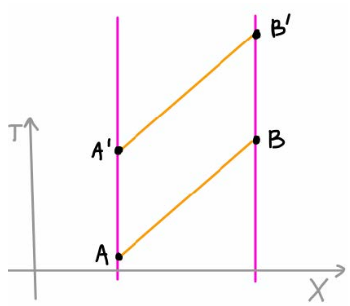
$$\boxed{\mathrm{d}s^2 = -e^{2\Phi}\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2} \tag{1.5}$$

where basically respect to the flat metric we changed the coefficient of the time component by a factor given by the gravitational potential. This confirms our suggestion[XI], namely by applying equivalence principle the potential associated to a specific gravitational field can be identified as a part of the metric specifying the frame in which the object experience such gravitational field. This will lead us to treat and complete the notion of gravitational potential with the metric itself characterizing a given spacetime.

Before starting to work on the complete theory of GR, let's consider other proprieties of Rindler space. Now we will discuss how Rindler spacetime already exhibits some effects that we will encounter in more general settings.

Time dilatation and red-shift

In previous sections we described time dilatation and red-shift using non-relativistic arguments, now we will analyze them using relativistic treatment. Consider again next picture:



[X]"Sees" mean seen by light signals.

[XI]By the analogy with EM we were looking for a way to rewrite gravitational laws in relativistic way using gravitational potential.

we know that the proper time is related with T by the relation $\tau = XT$. This means that the proper time interval between events B and B' can be written as follows

$$\tau_{B'} - \tau_B = X_B(T_{B'} - T_B) = X_B(T_{A'} - T_A) = \frac{X_B}{X_A}(\tau_{A'} - \tau_A)$$

and using gravitational potential we have:

$$\boxed{\Delta\tau_B = \frac{X_B}{X_A}\Delta\tau_A = e^{\Phi_B - \Phi_A}\Delta\tau_A}$$

If instead of clocks synchronized by light signals (i.e. measuring the value T) we consider 2 identical clocks measuring the proper time, then they "see" each other running differently:

- clock B "sees" clock A going *slower* by factor $X_A/X_B = e^{\Phi_A - \Phi_B} < 1$
- clock A "sees" clock B going *faster* by factor $X_B/X_A = e^{\Phi_B - \Phi_A} > 1$

Let's see this effect in terms of frequencies. If $\Delta\tau = 1/\nu$ then we have

$$\nu_B = \frac{X_A}{X_B}\nu_A \qquad \Rightarrow \qquad \nu_B < \nu_A$$

and then this shows *red-shift effect.* If $X_B = X_A + \delta X$ for $\delta X \ll 1$ then gravitational potential can be thought as linear and

$$\frac{\nu_B - \nu_A}{\nu_A} = \left(\frac{X_A}{X_B} - 1\right) \simeq -\frac{\delta X}{X_A} = -\delta\Phi \simeq -\frac{1}{c^2}(\Phi_B - \Phi_A)$$

where in the last step we restored the constant c. Energy of photons is proportional to them frequency ($E = h\nu$) so the difference between energy in A and energy B can be seen as the energy lost by the photon due to the fact that it is "climbing" the potential Φ (i.e. difference of energy is related to the difference of potential energy between A and B).

Event horizons

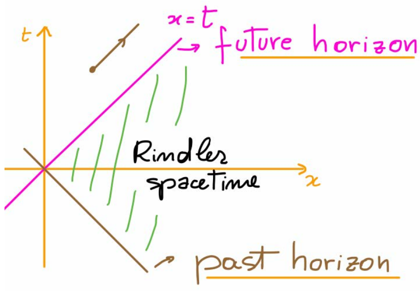
A second propriety of Rindler space is related to the presence of **event horizons**. Similar phenomena will appear in the treatment of black holes, but Rindler space can be used as toy model for it description and understanding. First observation is that X Rindler's coordinate is restricted to be positive, and we can see that the metric became degenerate for $X = 0$:

$$\mathrm{d}s^2 = -X^2\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2$$

i.e. we have a coordinate singularity for $X = 0$. In particular for $X = 0$ a purely time like difference ΔT between two events happening at different times becomes light-like. However, recovering original flat metric in a inertial frame (in particular we do not restrict only to the $X > 0$ case), then this metric is equivalent to a smooth metric well defined. The singularity is due to our specific choice of coordinates, and is not related to the geometry of the space itself.[XII] This somehow correspond to the $r = 0$ case for radial coordinates $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$: the flat space has its own well defined, smooth, metric, but with a specific choice of coordinates we could obtain a degenerate metric in some point, like the origin for radial coordinates or $X = 0$ for the Rindler space.

On the other hand the point corresponding to $X = 0$ has some special proprieties: recall $X \equiv \sqrt{x^2 - t^2}$, then $X = 0$ is the interception of two lines: $x = \pm t$. These two lines can be interpreted as horizons of Rindler spacetime, they separate Rindler space time to the remaining of the full Minkowskian spacetime:

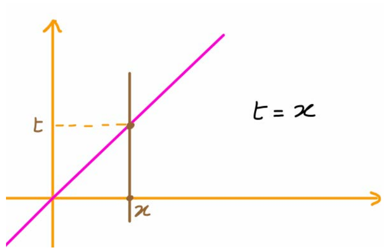
[XII] In Differential Geometry terminology, this means that our chart is well defined only on the open set corresponding to $X > 0$ local coordinates. On the other hand, we can find different charts and an atlas that covers all the $\mathbb{R}^4$ space with well defined coordinates for each point. For example, the Minkowski system of coordinates defines an atlas well defined on all the space. Rindler system of coordinates is just a system of coordinates comfortable for our description of trajectories when $X > 0$. This won't be true for singularities in black holes.



In particular, the right side of $x = t$ horizon is the boundary between events in the Rindler spacetime and events that cannot be reached by Rindler events. In other word, Rindler events cannot “see” beyond the future horizon line, i.e. events above the future horizon line cannot send signal to Rindler space. In the opposite, right side of the past horizon straight line represents the boundary between Rindler space and events that cannot receive any signal from Rindler space.

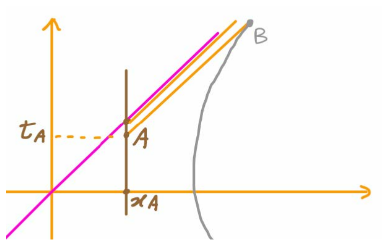
As we will see horizon will emerge in more general models and will actually characterize black holes.

Any observer freely “falling”[XIII] towards the horizon will cross it in a finite proper time (recall that t is the proper time if the observer is at rest), as it is pictorially represented in the next picture.



If, instead of measuring time using proper time, we use the coordinate T, then we know that $\tanh T_A = t_a/x_a$ and therefore when the observer cross the horizon we have $\tanh T_A = 1$, which implies that $T_A = \infty$. This means that if we try to describe a freely “falling” observer using Rindler coordinates, it can reach the horizon in a finite time, but then it needs an infinite amount of time in order to cross the boundary. If we suppose that, referencing to the next figure, an observer placed in A sends with a certain proper period a light signal to a Rindler observer (i.e. that movers with constant acceleration) placed in B and the latter measure the rate between signals, we can see that when A is crossing the horizon

$$\Delta\tau_B = X_B \Delta T_B = X_B \Delta T_A = \infty$$



This means that even though the first observer cross the horizon in a finite proper time, the observer B never “sees” the first observer crossing the horizon. This can be also understood by observing that the light signal when A approaches the horizon will take more and more time to reach B and in particular when A is crossing the horizon then the light signal will take an infinite time T to reach B.

[XIII] This means that he can freely move in Minkowski space.

1.5 From the Equivalence Principle to curved spacetime

Up to now we have seen that the EEP leads us to associate a gravitational field to a non-trivial metric. On the other hand the discussion leads us to the idea that the Rindler gravitational field may be considered "fake": we can make a global change into free-falling/inertial coordinates. In this way, we can also treat orther example of fake gravitational fields, that can be associated to other non-trivial metric in some more general coordinates X^μ:

$$g_{\mu\nu}(X) = \eta_{\rho\sigma}\frac{\partial x^\rho}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu} \tag{1.6}$$

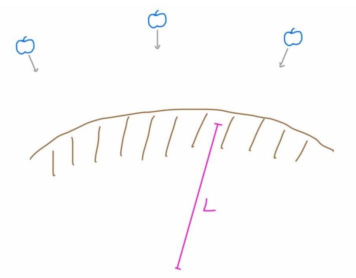
so that the line element takes the form

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}(X)\mathrm{d}X^\mu\mathrm{d}X^\nu$$

Vice versa, we can say that the gravitational field associated to some metric $g_{\mu\nu}$ is "fake" if we can find a global "freely-falling" frame $x^\mu(X)$ such that (1.6) holds. [XIV]

One can then consider more general metrics $g_{\mu\nu}(x)$ such that (1.6) does not hold. In such a case the metric can be identified with the "*genuine*" gravitational potetial, i.e. a potential that cannot be interpreted as a "freely-falling" frame. In analogy to the EM potential one can think of the "fake" gravitational potential as a vector potential which is a pure gauge potential (i.e. can be written as derivative of some function defining gauge transformation), while a "genuine" gravitational potential would correspond to a gauge field associated to a non trivial field strength.

We then need to study general metrics $g_{\mu\nu}(x)$ in general coordinate systems, and in order to do this we must consider more general space-times. For example we will consider the gravitational field produced by a spherical object



This is a (very important) example of a "genuine" gravitational potential. There is no way to define a global "freely falling" reference frame, this is possible only in a small neighbourhood of the object we would like to consider, where the potential is considered linear.

Recall our statement of EEP: "*In a small enough region of the space-time, the laws of physics reduce to those of SR*", this means that only in small regions of space-time we can go into "freely-falling" frames where the physics reduces to SR one. It translates into the existence of local frames, i.e. space-time coordinates x^μ in which

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + o\left(\frac{x^2}{L^2}\right)$$

so in a certain sense the typical length scales L "measure" how "honest" the gravitational field is, which is associated to the Tidal effect; L also parametrize the deviation from Minkowski spacetime, and in other words we will see that this corresponds to the curvature of the space-time.

We need a mathematical language in which

- there is no preferred globally defined coordinate system / frame x^μ (the inertial frames are not preferred respect to the others, they are just the coordinate system where the Minkowski metric is the flat one)
- intervals defined by a metric $g_{\mu\nu}(x)$ are associated to line elements in the form

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu$$

[XIV] An interesting example is given by an uniformly rotating frame in Minkowski space, described in Rindler sec. 9.7

- $g_{\mu\nu}(x)$ is the (*dynamical*) gravitational field
- the equation that describe the space time must be coordinate independent, i.e. we require a general covariance of physical laws

Notice also that in order to satisfy these requirements the global spacetime need no to be same of $\text{Mink}^4 \simeq \mathbb{R}^4$, but it will be determined by dynamics, i.e. from $g_{\mu\nu}(x)$. Differential geometry provides the natural mathematical framework/language to formulate such a theory.

Chapter 2

Spacetime

Refer to Carroll chap. 2 and Nakahara chap. 5,7 for this chapter.

2.1 Manifolds

Definition 2.1: Manifold

M is a d-dimensional differentiable manifold if

(i) M is a topological space

(ii) M is provided with a family of pairs $\{(U_\alpha, \phi_\alpha)\}$

(iii) $\{U_\alpha\}$ is a family of open sets which covers M, that is, $\bigcup_\alpha U_\alpha = M$. ϕ_α is a homeomorphism from U_α onto an open subset U'_α of $\mathbb{R}^d$;

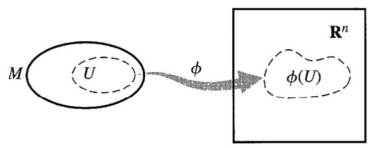


FIGURE 2.13 A coordinate chart covering an open subset U of M.

(iv) given U_α and U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, the map $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ from $\phi_\beta(U_\alpha \cap U_\beta)$ to $\phi_\alpha(U_\alpha \cap U_\beta)$ is infinitely differentiable.

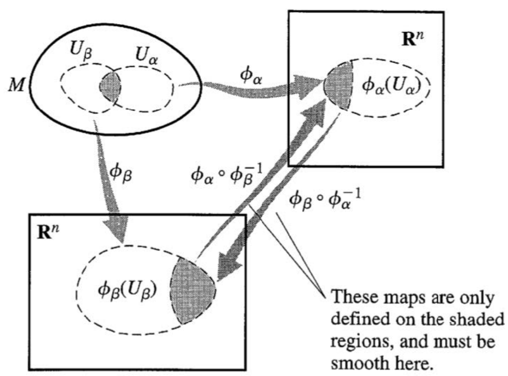


FIGURE 2.14 Overlapping coordinate charts.

The pair (U_α, ϕ_α) is called a **chart** while the whole family $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**. The subset U_α is called the **coordinate neighbourhood** while ϕ_α is the **coordinate function** or, simply, the **coordinate**. If U_α and U_β overlap, two coordinate systems are assigned to a point in $U_\alpha \cap U_\beta$. Axiom (iv) asserts that the transition from one coordinate system to another be *smooth* (C^∞). The map $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ is called **transition function**. The choice of an atlas is not unique, in particular for each manifold there are infinite equivalent choices of atlas. If the union of two atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \varphi_\beta)\}$ is again an atlas, these two atlases are said to be **compatible**. The compatibility is an equivalence relation, the equivalence class of which is called the **differentiable structure**.

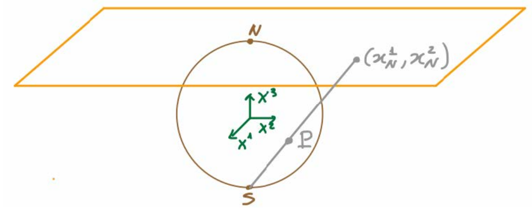
Example 1: S^2

The two dimensional sphere S^2 given by the subset of $\mathbb{R}^3$ defined by the equation

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2$$

is a differentiable manifold. Notice that no single chart is possible, since the sphere is a closed set, thus cannot be found any homeomorphism with an open set of $\mathbb{R}^2$, rather we need to define at least two charts. We can do this using *stereographic coordinates*:

$$\begin{aligned} \phi_N \ : U_N = &S^2 \setminus \{(0,0,-R)\} &\longrightarrow \quad &\phi_N(U_N) \simeq \mathbb{R}^2 \\ &(X^1, X^2, X^3) &\longmapsto \quad &\left(x_N^1 = \frac{2X^1}{R+X^3}, x_N^2 = \frac{2X^2}{R+X^3} \right) \end{aligned}$$



where the chart is defined for all points of S^2 beside the south pole. Analogously, removing the north pole:

$$\begin{aligned} \phi_S \ : U_S = &S^2 \setminus \{(0,0,R)\} &\longrightarrow \quad &\phi_S(U_S) \simeq \mathbb{R}^2 \\ &(X^1, X^2, X^3) &\longmapsto \quad &\left(x_S^1 = \frac{2X^1}{R-X^3}, x_S^2 = \frac{2X^2}{R-X^3} \right) \end{aligned}$$

We can see that transition functions are smooth:

$$\phi_S \circ \phi_N^{-1} \ : \ (x_N^1, x_N^2) \quad \longmapsto \quad \begin{cases} x_S^1 = \frac{4x_N^1}{(x_N^1)^2 + (x_N^2)^2} \\ x_S^2 = \frac{4x_N^1}{(x_N^2)^2 + (x_N^2)^2} \end{cases}$$

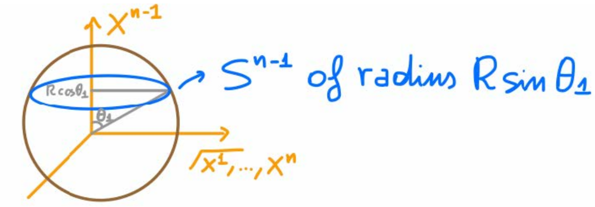
Example 2: S^n

The two dimensional sphere S^n given by the subset of $\mathbb{R}^{n+1}$ defined by the equation

$$(X^1)^2 + (X^2)^2 + \cdots + (X^{n+1})^2 = R^2$$

is a differentiable manifold. Notice that again no single chart is possible, rather we need to define several charts. This time, instead of stereographic coordinates (which works too) we use *angular* *coordinates*:

$$\begin{cases} X^{n+1} &= R\cos\theta_1 \\ X^n &= R\sin\theta_1\cos\theta_2 \\ X^{n-1} &= R\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ X^2 &= R\sin\theta_1\ldots\cos\theta_n \\ X^1 &= R\sin\theta_1\ldots\sin\theta_n \end{cases} \qquad \text{with} \quad 0<\theta_1,\ldots,\theta_{n-1}<\pi\,, \quad 0<\theta_n<2\pi$$



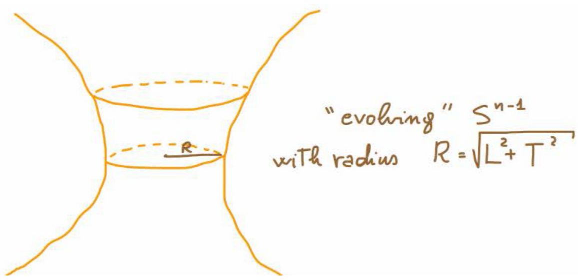
Notice that this coordinates degenerate at $\theta_1,\ldots,\theta_{n-1}=0,\pi$, therefore this chart do not cover the entire S^n. For example coordinates for the submanifold $S^{n-2}=S^n\cap\{X^1=X^2=0\}$ (i.e. $\theta_{n-1}=0$) are not well defined. In order to cover the full sphere we can use charts with "rotated" angular coordinates that covers subsets of S^n where the previous chart is not well defined.

Example 3: n-dim. de Sitter spacetime

The so called **n-dim. de Sitter spacetime**, indicated by d$\mathrm{S}n$, is the manifold defined as the subset of R^{n+1} with equation:

$$|\mathbf{X}|^2-T^2=L^2$$

where $\mathbf{X}=(X^1,\ldots,X^n)$ are n space coordinates while T is a time coordinate.



It's clear that this manifold takes the form of an hyperboloid. We may also introduce local coordinates on the manifold $x^\mu=(x^0,x^i)=(t,x^i)$. First we introduce coordinates t and $\hat{X}^I$ such that

$$T=L\sinh t$$
$$X^I=L\cosh\hat{X}^I$$

with $\sum_{I=1}^n(\hat{X}^I)^2=1$, i.e. coordinates $\{\hat{X}^I\}$ define a sphere $\hat{S}^{n-1}$ with radius $\hat{R}=1$. Then, $\hat{S}^{n-1}$ can be covered using some atlas (e.g. using stereographic or angular coordinates), defining local coordinates $\{x^i\}$ on $\hat{S}^{n-1}$. In this way, taking into account also $x^0=t$, we introduce coordinates x^μ over d$\mathrm{S}n$.

What we have done is just slice d$\mathrm{S}n$ into spheres (circumferences in the figure) and then parametrize them as we already know by previous examples.

Up to this point, we want to stress the fact that for these manifolds only pathches, topological and differential structure have been specified. No notion of distance has been introduced yet.

2.2 Calculus on Manifolds

Definition 2.2: Scalar field

A **scalar field** is defined as a function

$$\begin{aligned} F: M &\longrightarrow \mathbb{R} \qquad (\text{or } \mathbb{C}) \\ p &\longmapsto F(p) \end{aligned}$$

Let (U, ϕ) be a patch for M with local coordinates x^μ, then for each $p \in U$ we can define the *local form* of F

$$\begin{aligned} f = F \circ \phi^{-1}: \mathbb{R}^d &\longrightarrow \mathbb{R} \qquad (\text{or } \mathbb{C}) \\ x &\longmapsto F(\phi^{-1}(x)) \end{aligned}$$

If we consider a different patch $(\tilde{U}, \tilde{\phi})$ associated to local coordinates $\tilde{x}^\mu$, and the local form of f in this patch $\tilde{f} = F \circ \tilde{\phi}^{-1}$, then its immediate that following **transformation rule for a scalar field** holds:

$$\boxed{\tilde{f}(\tilde{x}) = f(x)}$$

where $\tilde{x}$ are the coordinates for M using the patch $(\tilde{U}, \tilde{\phi})$.

Pragmatically, we will not distinguish between $F(p)$ and its local form $f(x)$ and use the latter.

Definition 2.3: Vector field

We define a **vector field**

$$\begin{aligned} V: M &\longrightarrow TM \\ p &\longmapsto V^\mu(p)\partial_\mu\big|_p \end{aligned}$$

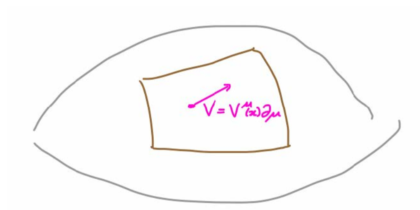
where V^μ is a set of functions called **components of the vector field** and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}|_p$ are partial derivatives defined respect to the local coordinates in p given by some patch, and they can be applied to any scalar field. The set TM is a set of derivative operators that will be defined later. If functions V^μ are differentiable then the vector field is said **smooth**, and this condition is independent from the choice of the patch. The element $V(p)$ for some point $p \in M$ is said to be a **vector in** p and the set of all vectors attached to a point p (for all possible vector fields) is called **tangent space in** p, denoted by T_pM. Each $V^\mu(p)\partial_\mu|_p$ can be interpreted as a vector in p with components $(V^1(p), V^2(p), \ldots, V^n(p))$.

Suppose that x^μ and $\tilde{x}^\mu$ are local coordinates on two subsets U and $\tilde{U}$. Notice that since transition functions for different charts are smooth, then in $U \cap \tilde{U}$ the Jacobian of the transition function $\frac{\partial \tilde{x}^\mu}{\partial x^\nu}(p)$ must be invertible, with inverse $\frac{\partial x^\mu}{\partial \tilde{x}^\nu}(p)$. The transformation rule for partial derivatives is then given by the differentiable application $\partial_\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\tilde{\partial}_\nu$ and similarly the **transformation rule**

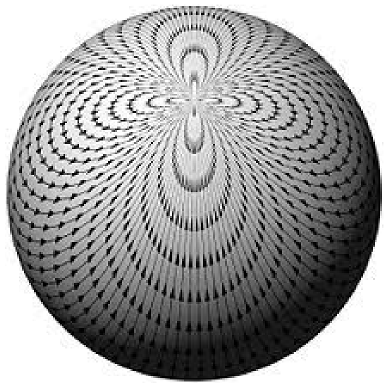
for a vector field is

$$V^{\mu}\partial_{\mu} = V^{\mu}\frac{\partial \tilde{x}^{\nu}}{\partial x^{\mu}}\tilde{\partial}_{\nu} = \tilde{V}^{\nu}\tilde{\partial}_{\mu}$$

$$\boxed{\tilde{V}^{\mu}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}V^{\nu}(x)}$$



Notice that this application reduce to the Minkowski rule for Poincaré transformations $\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + \alpha^{\mu}$. Here is a pictorial representation of a smooth vector field on a sphere:

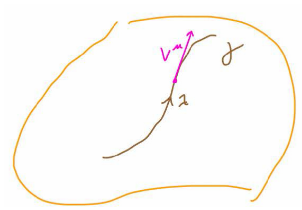


Starting from a manifold M and a vector field V, we can define **integral curves**

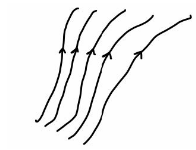
$$\begin{aligned} \gamma : \mathbb{R} \supset I \quad &\longrightarrow \quad \gamma(I) \subset M \\ \lambda \quad &\longmapsto \quad \gamma(\lambda) \end{aligned}$$

such that

$$\frac{\mathrm{d}\gamma^{\mu}}{\mathrm{d}\lambda}(\lambda) = V^{\mu}(\gamma(\lambda))$$



Notice that the latter equation describes a set of first order differential equations with unique solution for a given initial conditions. The set of all integral curves, for different initial coordinates, is called the **flow** of the vector field.



We can then regard $V(p)$ as defining a directional derivative in p of any smooth scalar field f along the integral curve of V that goes through $p = \gamma(\lambda_p)$:

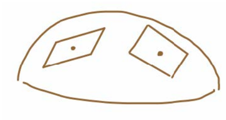
$$V(f)(p) \equiv V^\mu(p)\partial_\mu f(p) = \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}\lambda}(p)\partial_\mu f(p) = \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}\lambda}(\lambda_p)$$

and this can be interpreted as the derivative of the restriction of f along γ. This also allows a better interpretation of tangent vectors as derivative operators that can be applied to scalar fields.

At each point $p \in M$ the tangent space T_pM is a d-dimensional vector space and $\partial_\mu\big|_p \equiv \frac{\partial}{\partial x^\mu}\big|_p$ provide the coordinate basis associated with the local coordinates x^μ, called **coordinate basis**. However we could take any other basis of linearly independent vectors $e_a = e_a^\mu(x)\partial_\mu \in T_pM$, $a = 1, \dots, d$ so that $V(x) = V^a(x)e_a = V^\mu(x)\partial_\mu$ with $V^\mu(x) = e_a^\mu(x)V^a(x)$. Often indices a are called *"local"* (or *"flat"* in GR) *indices*, while μ are called *"curved" indices*.

The collection of the tangent spaces in each point $p \in M$ is called **tangent bundle** TM

$$TM = \bigcup_{p \in M} T_pM$$



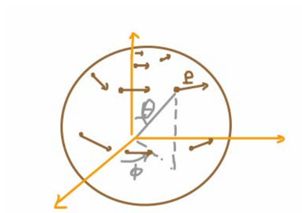
Notice that this is a $2d$-dimensional manifold. A vector field V can be also define as a function between the manifold and a bundle, i.e. a **section**. In particular a vector field is a section of the vector bundle

$$\begin{aligned} V\ : M &\longrightarrow TM \\ p &\longmapsto X_p \in T_pM \end{aligned}$$

Example 4

Let's consider the vector field $V = \sin\theta\partial_\phi$ where we used spherical coordinates $(x^1, x^2) = (\theta, \phi)$. Its components are

$$V^\mu = (V^1, V^2) \equiv (V^\theta, V^\phi) = (0, \sin\theta)$$



Definition 2.4: One-forms

One can consider another kind of vector fields α_μ, called **1-forms**, which transforms in the "dual" way, i.e. follows following **transformation rule of one-forms**:

$$\boxed{\tilde{\alpha}_\mu(\tilde{x}) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\alpha_\nu(x)}$$

Notice that since this transformation is the inverse of the one we defined for vector fields, then the product between a 1-form and a vector space is invariant under transformations, i.e. is a scalar

$$\alpha_\mu(x)V^\mu(x) = (x)$$

This proprieties suggests a more intrinsic definition of a one-form field, which assigns at any $p \in M$

an element α of the **cotangent space**

$$T_p^*M = \{\text{vector space } \textbf{dual} \text{ to } T_pM\}$$
$$= \{\text{space of linear functionals } \alpha \text{ on } T_pM\}$$

hence

$$\alpha : V \in T_pM \quad \longmapsto \quad \alpha(V) \in \mathbb{R}$$

such that

$$\alpha(aV_1 + bV_2) = a\alpha(V_1) + b\alpha(V_2)$$

The coordinate basis dual to basis $\partial_\mu \in T_pM$ is given by

$$\mathrm{d}x^\mu \in T_p^*M \quad \forall p \in U \quad \text{such that} \quad \mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$$

Respect to this basis a generic one-form we can write as $\alpha = \alpha_\mu(x)\mathrm{d}x^\mu$, so is action is described by

$$\alpha(V) = \alpha(V^\mu(x)\partial_\mu) = V^\mu(x)\alpha_\nu(x)\mathrm{d}x^\nu(\partial_\mu) = \alpha_\mu(x)V^\mu(x)$$

The last description in terms of the basis leads to an interpretation of one forms as linear combinations of infinitesimal coordinate variations. Indeed the transformation rule for one-forms is the same as the transformation rules for line elements

$$\mathrm{d}\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\mathrm{d}x^\nu \quad \Rightarrow \quad \tilde{\alpha}(\tilde{x}) = \tilde{\alpha}_\mu(\tilde{x})\mathrm{d}\tilde{x}^\mu = \alpha_\mu(x)\mathrm{d}x^\mu = \alpha(x)$$

One can also take arbitrary basis of linearly independent 1-forms:

$$\boxed{e^a = e^a_\mu(x)\mathrm{d}x^\mu \in T_p^*M \qquad a = 1, \ldots, \qquad \forall p \in M}$$

In order to be consistent with the representation in terms of coordinate basis we must have

$$\alpha(x) = \alpha_a(x)e^a = \alpha_\mu(x)\mathrm{d}x^\mu$$

therefore

$$\alpha_\mu(x) = e^a_\mu(x)\alpha_a(x)$$

A particular subclass of one forms is given by **exact 1-forms**

$$\alpha(x)\mathrm{d}f(x) = \partial_\mu f(x)\mathrm{d}x^\mu \qquad f(x) \text{ scalar field}$$

that in components implies $\alpha_\mu(x) = \partial_\mu f(x)$.

Observe that

$$\mathrm{d}f(V) = \partial_\mu f V^\mu$$

gives the **directional derivative** of f along V.

A one-form field α assigns an element of T_p^* at each $p \in M$. This can be considered as a section of the **cotangent bundle**

$$T^*M \equiv \bigcup_p T_p^*M$$

Definition 2.5: General tensors

Pragmatically, a **tensor field of type (n, m)** is a field characterized n upper indices and n lower indices

$$T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}(x)$$

In order to define such a field over an entire manifold one can define how this fields transforms under coordinate transformation, namely the **transformation rule for tensor fields**:

$$\boxed{\widetilde{T}^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}(\tilde{x}) = \frac{\partial\tilde{x}^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial\tilde{x}^{\mu_n}}{\partial x^{\rho_n}}\frac{\partial x^{\sigma_1}}{\partial\tilde{x}^{\nu_1}}\cdots\frac{\partial x^{\sigma_m}}{\partial\tilde{x}^{\nu_m}}T^{\rho_1\dots\rho_n}{}_{\sigma_1\dots\sigma_m}(x)}$$

More intrinsically, a tensor fields at each point $p \in M$ is an element of the vector space

$$(T_pM)^{\otimes n} \otimes (T_p^*M)^{\otimes m}$$

Then we can express the tensor fields in terms of coordinate basis or an arbitrary local basis either:

$$\begin{aligned} T(x) &= T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}(x)\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_n}\otimes \mathrm{d}x^{\nu_1}\otimes\cdots\otimes\mathrm{d}x^{\nu_m} \\ &= T^{a_1\dots a_n}{}_{b_1\dots b_m}(x)\,e_{a_1}\otimes\cdots\otimes e_{a_n}\otimes e^{b_1}\otimes\cdots\otimes e^{b_m} \end{aligned}$$

where this time $T^{\mu_1\dots\mu_n}{}_{\nu_1\dots\nu_m}(x)$ and $T^{a_1\dots a_n}{}_{b_1\dots b_m}(x)$ are a set of functions and are required to satisfy the equality between two transformations.

Some tensors may have specific index-symmetries, for instance

$$S^{\mu}{}_{\nu\rho} = S^{\mu}{}_{\rho\nu}$$

is **symmetric** in second and third indices, while

$$A^{\mu\nu\rho} = -A^{\mu\rho\nu}$$

is **antisymmetric** in second and third indices. Notice that tensors with $> d$ antisymmetric indices is identically vanishing.

Given a tensor, one can **symmetrize** or **antisymmetrize** any number of upper or lower indices

$$\begin{aligned} T_\rho{}^\sigma{}_{\mu_1\dots\mu_n} &\longrightarrow T_\rho{}^\sigma{}_{(\mu_1\dots\mu_n)} = \frac{1}{n!}(T_\rho{}^\sigma{}_{\mu_1\dots\mu_n} + \text{ permutations of } \mu_1\dots\mu_n) \\ T_\rho{}^\sigma{}_{\mu_1\dots\mu_n} &\longrightarrow T_\rho{}^\sigma{}_{[\mu_1\dots\mu_n]} = \frac{1}{n!}(T_\rho{}^\sigma{}_{\mu_1\dots\mu_n} \pm \text{ permutations of } \mu_1\dots\mu_n) \end{aligned}$$

where in the second case we take positive sign when we sum an element obtained by an even permutation of indices, while we take munis sign when we sum an element obtained by ad odd permutation of indices. For instance

$$\begin{aligned} T_{(\mu\nu)} &= \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \\ T_{[\mu\nu]} &= \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \end{aligned}$$

The factor $\frac{1}{n!}$ is inserted in such a way that if the initial tensor is already (anti)symmetric then its (anti)symmetrized tensor is the same as the initial one.

Definition 2.6: Differential forms

A distinguish subclass of tensor fields is given by the "forms"

$$A_{\mu_1\dots\mu_p}(x) = A_{[\mu_1\dots\mu_p]}(x)$$

i.e. total antisimmetric tensors, called p**-forms** where p is the degree of the form (recall that $p \leq d$, otherwise the form vanishes).

For this class of tensor is useful to introduce a special basis obtained by taking **wedge-product** of the one form basis

$$\begin{aligned}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} &= p!\, \mathrm{d}x^{[\mu_1} \otimes \mathrm{d}x^{\mu_2} \otimes \cdots \otimes \mathrm{d}x^{\mu_p]} \\ &= (\mathrm{d}x^{\mu_1} \otimes \mathrm{d}x^{\mu_2} \otimes \cdots \otimes \mathrm{d}x^{\mu_p} \; + \; \text{permutations with alternating signs })\end{aligned}$$

where $p!$ cancels the normalization factor we introduced in the definition of antisymmetrization. In this way p-forms can be written as

$$A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$$

for a set of scalar fields $A_{\mu_1 \ldots \mu_p}$.

We can define some operations over p-forms. The first one is the **Wedge product**, that is given by linearity extension of the wedge product between basis elements, defined as

$$A_p \wedge B_q = \frac{1}{p!}\frac{1}{q!} A_{\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q}$$

Notice that the result is a total antisymmetric $p+q$-tensor, hence in components

$$(A_p \wedge B_q)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p! \cdot q!} A_{[\mu_1 \ldots \mu_p} B_{\nu_{p+1} \ldots \nu_{p+q}]}$$

Another operation is the **exterior derivative**

$$\mathrm{d} \,:\, A_p \quad \longmapsto \quad (p+1)\text{-form } \mathrm{d}A_p$$

defined as

$$\begin{aligned}\mathrm{d}A_p &= \frac{1}{p!}\, \partial_\nu A_{\mu_1 \ldots \mu_p} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \\ &= \frac{1}{p!}\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+1}}\end{aligned}$$

or in components

$$(\mathrm{d}A_p)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}$$

We could prove that $\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}$ transforms tensorially. [a]

These operations have some important proprieties:

(i) (Weighted) **Leibniz rule**:

$$\mathrm{d}(A_p \wedge B_q) = \mathrm{d}A_p \wedge B_q + (-)^p A_p \wedge \mathrm{d}B_q$$

(ii) **Nilpotency**:

$$\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$$

Notice that this is a consequence of $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$.

Moreover we can check the consistency of the notation $\mathrm{d}x^\mu = \mathrm{d}(x^\mu)$.

Given a p-form A_p we say that it is

(i) **closed** if $\mathrm{d}A_p = 0$

(ii) **exact** if $A = \mathrm{d}B_{p-1}$ for some $(p-1)$-form B_{p-1}.

Notice that *any exact form is a closed form* as a consequence of the nilpotency. Viceversa, *Poincaré Lemma* states that *any closed form is locally exact.*

[a] Prove it as an exercize.

Example 5

The EM gauge field is given by the one form

$$A_1 = A_\mu \mathrm{d}x^\mu$$

Then we can define the **field-strength** the 2-form

$$F_2 = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}A_1 = \partial_{[\mu} A_{\nu]} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

We can check as exercize that the coefficient of the basis representation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

transforms covariantly.

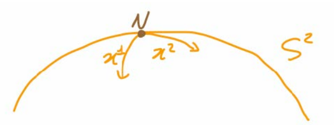
Example 6

Take S^2 with spherical coordinates $\tilde{x}^\mu = (\theta, \phi)$ and let's define the one form

$$\psi_1 = -\cos\theta \mathrm{d}\phi = \tilde{\psi}_\mu \mathrm{d}\tilde{x}^\mu \quad \to \tilde{\psi}_\mu = (0, -\cos\theta)$$

Notice that this one form is singular at the poles. To see this, pass to some coordinates well defined at the poles. If we focus to the North poles we can introduce angular coordinates

$$\begin{cases} x^1 = \sin\theta\cos\phi \\ x^2 = \sin\theta\sin\phi \end{cases} \quad \to \quad \begin{cases} \sin\theta = \sqrt{(x^1)^2 + (x^2)^2} \\ \tan\phi = \frac{x^2}{x^1} \end{cases}$$



In order to redefine the one form we can either using the Jacobian or notice that exterior derivative of $\tan\theta$ can be used to substitute $\mathrm{d}\phi$ with the new coordinates:

$$\mathrm{d}\tan\phi = (1 + \tan^2\phi)\mathrm{d}\phi = \left(1 + \left(\frac{x^2}{x^1}\right)^2\right) \mathrm{d}\phi = -\frac{x^2}{(x^1)^2}\mathrm{d}x^1 + \frac{1}{x^1}\mathrm{d}x^2$$

hence we obtain

$$\psi_1 = -\frac{\sqrt{1 - (x^1)^2 - (x^2)^2}}{(x^1)^2 + (x^2)^2}(-x^2 \mathrm{d}x^1 + x^1 \mathrm{d}x^2)$$

Now its clear that ψ_1 is singular for $(x^1, x^2) = (0, 0)$.

Let's consider now the 2-form

$$\omega_2 = \frac{1}{2}\tilde{\omega}_{\mu\nu} \mathrm{d}\tilde{x}^\mu \wedge \mathrm{d}\tilde{x}^\nu = \sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

We can easily prove that it is smooth on the poles. Notice that $\mathrm{d}\omega_2 = 0$ (is a 3-form in a 2-dim manifold) hence ω_2 is closed. By Poincaré lemma it is also locally exact, indeed in $S^2 \setminus \{\text{poles}\}$ we have

$$\mathrm{d}\psi_1 = \mathrm{d}(-\cos\,\theta \mathrm{d}\phi) = \sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\phi = \omega_2$$

But ψ_1 is not defined at poles, indeed ω_2 is only locally exact.

2.3 Manifolds with metric

Manifolds do not carry natural notions of

- length and volumes
- notions of spatial and temporal "directions"
- a way to distinguish the locally inertial/freely falling frames

All this notion are given by the introduction of a metric:

Definition 2.7: Metric

The **metric** is a tensor field

$$g(x) = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

which satisfy following proprieties: it is symmetric

$$g_{\mu\nu}(x) = g_{\nu\mu}(x) = g_{(\mu\nu)}(x)$$

and it is non-degenerate

$$\det(g_{\mu\nu}) \neq 0$$

Its content is equivalently encoded in the **line element**, since thanks to symmetry we can omit tensorial product symbol

$$\boxed{\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu}$$

therefore usually we use terms "metric" and "line element" with the same meaning.

The metric takes values in $T * M \otimes_S T * M$ and can be regarded as defining a scalar product

$$g(V, W) = g_{\mu\nu}(x)V^\mu(x)W^\nu(x)$$

which associate to two vector fields a scalar field. This also implies that the scalar product is invariant. We can also define the "lenght" of a vector, i.e. introduce a **norm**, given by $g(V, V)$. Generalizing what we stated for SR we say

(i) if $G(V, V) > 0$ then V^μ is **space-like**

(ii) if $G(V, V) < 0$ then V^μ is **time-like**

(iii) if $G(V, V) = 0$ then V^μ is **null** (or **light-like**)

An **Euclidean / Riemannian (metric) space** $\mathcal{M}_d$ is a manifold equipped to a metric such then all vectors are space-like, i.e. g is positive definite everywhere. A**Lorenzian / Minkowskian / pseudo-Riemannian (metric) space(-time)** $\mathcal{M}_d$ is a manifold equipped to a metric g which has $d-1$ positive eigenvalues and 1 negative eigenvalues (strictly, there are no zero eigenvalues). Negative eigenvalues corresponds to time direction, while positive eigenvalues corresponds to spatial directions. Can be proved that the characterization of the metric in terms of signature of its eigenvalues (this is called the **signature** of the metric) is invariant under change of coordinates, i.e. do not depends by the parametrization of the manifold.

Regarding $g_{\mu\nu}$ as a symmetric matrix, we can find an orthogonal matrix E ($E^T E = \mathbb{1}$) such that

$$E^T g E = \mathrm{diag}(\lambda_0, \ldots, \lambda_{d+1})$$

with $\lambda_i \neq 0$. Then previous condition implies

$$g_{\mu\nu} E^\mu{}_a E^\nu{}_b = \lambda_a \delta_{ab}$$

where in the last term there is no summation over index a. Once we identified the matrix E we can define basis

$$e^\mu{}_a = \frac{1}{\sqrt{|\lambda_a|}} E^\mu{}_a$$

so that

$$e^T g e = g_{\mu\nu} e^\mu{}_a e^\mu{}_b = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$$

and in particular for the Lorenzian spacetime we have $e^T g e = \mathrm{diag}(-1, 1, \dots, 1)$. Hence *a metric space(-time) is Lorenzian iff exists an orthonormal basis $e_a = e^\mu_a \partial_\mu$ such that*

$$\boxed{g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab} \qquad \text{with} \quad \eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1)}$$

By introducing a dual basis of 1-forms $e^a = e^a{}_\mu \mathrm{d}x^\mu$ such that $e^a_\mu e^\mu_b = \delta^a_b \Leftrightarrow e^a_\mu e^\nu_a = \delta^n_\mu u$ then we can equivalently say that *the metric space(-time) is Lorenzian iff exists an orthonormal basis $e^a = e^a_\mu \mathrm{d}x^\mu$ such that*

$$\boxed{g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \quad \Leftrightarrow \quad g = \eta_{ab} e^a \otimes e^b \quad \Leftrightarrow \quad \mathrm{d}s^2 = \eta_{ab} e^a e^b}$$

Basis e_a and e^a are often called **Lorentz / flat (co)frames** but notice that they exists only point-wise and, generically, they do not correspond to inertial coordinate systems $\hat{x}^\alpha$ such that

$$e^a = \delta^a_\alpha \mathrm{d}\hat{x}^\alpha \quad \Leftrightarrow \quad \hat{e}^a_\alpha(\hat{x}) = \delta^a_\alpha \quad \Leftrightarrow \quad \mathrm{d}s^2 = \eta_{\alpha\beta} \mathrm{d}\hat{x}^\alpha \mathrm{d}\hat{x}^\beta \tag{2.1}$$

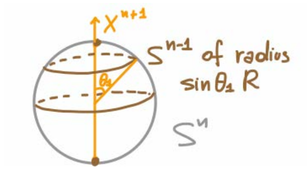
On the other hand we will see that for generic "curved" space-times condition eq. (2.1) can be satisfied only "locally" through a proper choice of coordinates

$$\boxed{\mathrm{d}s^2 = \left[\eta_{\alpha\beta} + O\left(\frac{\Delta\hat{x}^2}{L^2}\right)\right] \mathrm{d}\hat{x}^\alpha \mathrm{d}\hat{x}^\beta}$$

where the second order term cannot vanishes since it contains the information about the curvature.[I] Coordinates which satisfies this condition are called **"local" inertial coordinates**.

Example 7: (Euclidean) round metric on S^n

If we consider S^n embedded into $\mathbb{E}^{n+1}$



then the round metric is inherited from the ambient flat metric

$$\mathrm{d}s^2(\mathbb{E}^{n+1}) = (\mathrm{d}x^1)^2 + \dots + (\mathrm{d}x^{n+1})^2$$

We can introduce angular coordinates by iteration, starting from θ_1 we have

$$\mathrm{d}s^2(S^n) = R^2[\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}s^2(S^{n-1})]$$

In order to prove this (left as an exercize) it is easier to start with the explicit computation of $\mathrm{d}s^2(S^2)$ and $\mathrm{d}s^2(S^3)$.

In any metric space one can use the metric $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$ to lower or raise the indices of any tensor, for instance

$$T^\mu{}_\nu \quad \longrightarrow \quad T_{\mu\nu} \equiv g_{\mu\rho} T^\rho{}_\nu$$

In the following we will consider this identifications as implicit. However one should keep in mind that, being $G_{\mu\nu}(x)$ an elementary dynamical field in GR, $T^\mu{}_\nu(x)$ and $T_{\mu\nu}(x)$ may carry different physical content.

[I] See Carrol pag. 74 for an explanation about why this is the better result one can archive.

Interpretation of one forms and line elements

Let's give an interpretation of $\mathrm{d}x^\mu$. Consider an infinitesimal displacement

$$x^\mu \quad \longrightarrow \quad x^\mu + \delta x^\mu$$

We could set $\delta x^\mu = \varepsilon V^\mu$ with V^μ finite vector applied at x^μ, then we can identify an infinitesimal vector as

$$\varepsilon V = \varepsilon V^\mu \partial_\mu = \delta x^\mu \partial_\mu$$

Hence, when we apply the one form $\mathrm{d}x^\mu$ to such infinitesimal vector, we have

$$\mathrm{d}x^\mu(\varepsilon V) = \varepsilon V^\mu \equiv \delta x^\mu$$

Provided some infinitesimal vector εV, this suggest following identification

$$\mathrm{d}x^\mu \quad \sim \quad \delta x^\mu$$

where $\mathrm{d}x^\mu$ can be interpret as generic variation of the coordinate along a vector V.

When we generalize this analysis to one forms associated to scalar fields we have

$$\mathrm{d}f(\varepsilon V) = \partial_\mu f \mathrm{d}x^\mu(\varepsilon V) = \varepsilon V^\mu \partial_\mu f = \delta x^\mu \partial_\mu f = \delta f$$

then $\mathrm{d}f(\varepsilon V)$ coincides to the infinitesimal variation of the scalar field f corresponding to δx^μ. Again, provided some infinitesimal vector, we have the following identification

$$\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu \quad \sim \quad \delta f = \partial_\mu f \delta x^\mu$$

where $\mathrm{d}f$ is the generic variation of f while δf is the specific variation of f under δx^μ.

Let's see how this works for the metric. Given $\delta x^\mu = \varepsilon V^\mu$, its infinitesimal interval is given by

$$\delta s^2 = g(\varepsilon V, \varepsilon V) = \varepsilon^2 g_{\mu\nu} V^\mu V^\nu = g_{\mu\nu} \delta x^\mu \delta x^\nu$$

And, provided an infinitesimal vector, we have the identification

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \quad \sim \quad \delta s^2 = g_{\mu\nu} \delta x^\mu \delta x^\nu$$

where $\mathrm{d}s^2$ is a generic infinitesimal interval, wile δs^2 is the specific interval of $\delta x^\mu = \varepsilon V^\mu$.

2.3.1 Example: The evolving universe (I)

Example 8: The evolving universe (I)

Carroll, sec 2.6, 8.2, 8.4

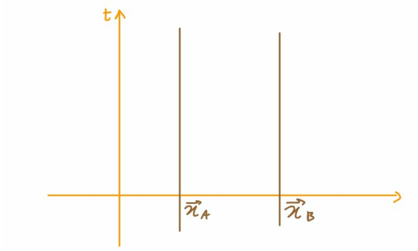
Let's consider a family of metrics that plays a special role in the description of expanding universe:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \tag{2.2}$$

Metrics in this forms are a subclass of the *Friedmann-Robertson-Walker (FRW) space-times.* Notice that t is the proper time measured by clock stuck at constant position $\mathbf{x}$, this is a consequence of the fact that the only term related to the time in the line element is $-\mathrm{d}t^2$ as in the flat metric.

On the other hand, the proper distance between simultaneous events scale as $a(t)$, which cannot be absorbed by change coordinates since it is a time dependent factor. The distance between two events in world line of observers placed in fixed positions x_a and x_B is

$$\Delta L = a(t)|\mathbf{x}_B - \mathbf{x}_A|$$



For this reason the factor $a(t)$ is called **scale factor**. In GR factor $a(t)$ is fixed by Einstein equations, and describes the expansion of universe. Anyhow just assuming homogeneous and isotropic distribution of matter and radiation one obtain a power law relation for $a(t)$:

$$a(t) = \left(\frac{t}{t_*}\right)^q \qquad 0 \leq q < 1 \qquad \begin{cases} q = \frac{2}{3} & \text{matter dominated universe} \\ q = \frac{1}{2} & \text{rafiation dominated universe} \end{cases}$$

where t_* is a typical time scale characterizing the phase of cosmological evolution. With such a a power low dependence, for $t \to 0$ we have $a(t) \to 0$, hence metric looks singular and this singularity does not depend on the choice of parametrization, it is intrinsic in the geometry of the manifold. For this reason is a *proper* singularity called *Big Bang*.

Nowadays this idea has been overcome, in particular in the early period of cosmological evolution is characterized by a **period of inflation**, and in such period one can approximate scale factor as

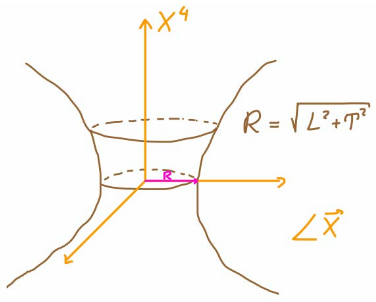
$$\boxed{a(t) = e^{Ht}} \tag{2.3}$$

where t_* is called **Hubble scale** and $H = 1/t*$ is the **Hubble parameter**. In this way the Big Bang singularity is pushed to $t = -\infty$.

The cosmological solution with $A(t) = e^{Ht}$ describes a patch of the 4-dimensional de Sitter space $\mathrm{dS}_4 \subset \mathbb{M}_5$ with equation

$$|\mathbf{X}|^2 - T^2 = L^2 \tag{2.4}$$

where $\mathbf{X} = (X^1, \ldots, X^4)$.



The flat metric $\mathbb{M}_5$ induces a metric on dS_4

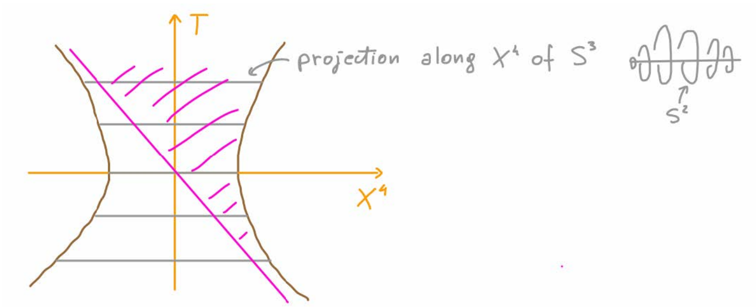
$$\mathrm{dS}_4(M_5) - \mathrm{d}T^2 + \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{X}$$

In order to solve eq. (2.4) one may set

$$T = L\sinh\left(\frac{t}{L}\right) + \frac{1}{2}\frac{|\mathbf{x}|^2}{L}e^{t/L}$$
$$X^4 = L\cosh\left(\frac{t}{L}\right) - \frac{1}{2}\frac{|\mathbf{x}|^2}{L}e^{t/L}$$
$$X^i = x^i e^{t/L} \qquad i = 1, 2, 3$$

where $\mathbf{x} = (x^1, x^2, x^3)$, then one obtain[a] that the line element induced in the de Sitter space is eq. (2.2) with eq. (2.3) and $H = \frac{1}{2}$.

Moreover, the patch covered by cosmological coordinates does not cover the full de Sitter space, but only half of it



[a]Prove it as exercise

Chapter 3

Motion in spacetime

Now that we introduced the framework of Differential Geometry, we can come back to the description of dynamics of physical systems. One one identified the metric of gravitational field, one can consider it as dynamic object.

Before discussing the EoM of the metric we first discuss the dynamics of a (probe) particle in a fixed metric (i.e. fixed gravitational fields. The EoM of the particle must be

(a) Invariant (in covariant form) under coordinate transformations

(b) Reduce to free rectilinear motion for (flat) Minkowski metric $g = \eta$

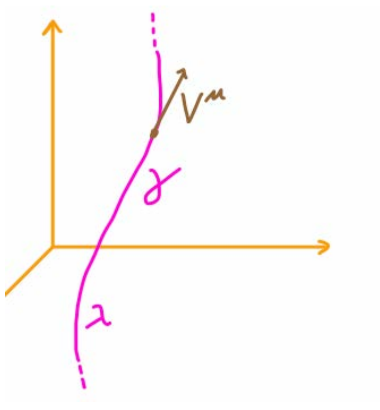
(c) Reproduce to Newton's universal law of gravitation in appropriate limit

We will first consider (a) and (b), exploiting the *least action principle.*

3.1 Particle's action in flat space-time

Let's focus to free particle in the flat space. The particle trajectories are described by world lines

$$X^\mu \;:\quad \lambda \in \mathbb{R} \longmapsto X^\mu(\lambda) \in \mathbb{R}^4$$



In general λ is arbitrary but if we chose it to be the proper time then the trajectory is given by a straight line, indeed for $\lambda = \tau$ and $m \neq 0$

$$\alpha^\mu = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0 \quad \Leftrightarrow \quad X^\mu = x^\mu_o + u^\mu \tau$$

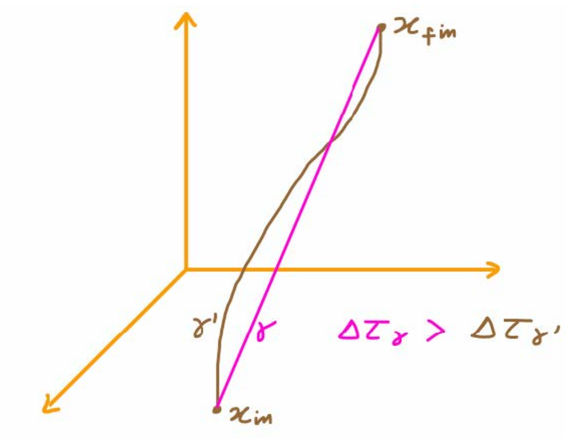
Now we want to derive this set of equation of motions (of for each μ) from a least action principle. Note that the proper time depedens on the trajectory itself, so it is not convenient to chose τ as parameter for the action. Since for a general λ

$$\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\lambda}=\frac{1}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu(\lambda)\dot{X}^\nu(\lambda)}}\frac{\mathrm{d}}{\mathrm{d}\lambda} \tag{3.1}$$

then we can rewrite $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}=0$ as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\eta_{\mu\nu}\dot{X}^\nu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\right)=0 \tag{3.2}$$

where we lowered index μ in $X\mu$ convenience. Notice that this equation is invariant under reparametrizations $\lambda\to\lambda'(\lambda)$. Hence we want to derive this equation from least action principle. A straight world line between two points x^μ_{in} and x^μ_{fin} in Mink$_4$ is the world line that maximize the proper time $\Delta\tau$ measured along the trajectory (since along a straight world line a clock runs faster, recall twin paradox)



This provides a natural candidate action, basically the proper time itself:

$$\boxed{S=-m\int \mathrm{d}\tau=-m\int_{\lambda_{in}}^{\lambda_{fin}}\mathrm{d}\lambda\,\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}} \tag{3.3}$$

In such a way when $\Delta\tau$ is maximum S is minimum. Moreover we added the mass term m in order to satisfy dimensional requirements.[I] Notice also that this action is invariant under reparametrizations $\lambda\to\lambda'$. If we consider a generical fluctuation $\delta X^\mu(\lambda)$ such that $\delta X^\mu(\lambda_{in})=\delta X^\mu(\lambda_{fin})=0$:

$$\begin{aligned}\delta S=-m\int_{\lambda_{in}}^{\lambda_{fin}}\mathrm{d}\lambda\,\delta\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}\qquad&=-m\int_{\lambda_{in}}^{\lambda_{fin}}\mathrm{d}\lambda\,\frac{\eta_{\mu\nu}\dot{X}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\delta\dot{X}^\nu\\ =-m\int_{\lambda_{in}}^{\lambda_{fin}}\mathrm{d}\lambda\,\frac{\eta_{\mu\nu}\dot{X}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\frac{\mathrm{d}}{\mathrm{d}\lambda}\delta X^\nu&=-m\int_{\lambda_{in}}^{\lambda_{fin}}\mathrm{d}\lambda\,\delta X^\nu(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\eta_{\mu\nu}\dot{X}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\right)\end{aligned}$$

where in the last step we performed integration by parts with vanishing boundaries conditions. Then imposing least action principle for any fluctuation $\delta X^\nu(\lambda)$ we have

$$\boxed{\delta S=0\quad\Leftrightarrow\quad\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\eta_{\mu\nu}\dot{X}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\right)=0} \tag{3.4}$$

Notice that in the massless case $m\to 0$ (e.g. for photons) eq. (3.3) doesn't work, indeed we know that photons travel along null trajectories:

$$\dot{X}^\mu\dot{X}_\mu=0\quad\Rightarrow\quad\Delta\tau=0\quad\Rightarrow\quad u^\mu=\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\ \text{is not well defined}$$

[I] In this way $[S]=EL=ET$ as we require for the action.

However, we can avoid this problem by reformulating least action principle introducing an action that reduces to eq. (3.3) in the massive case. In order to do that, we introduce a metric on the world-line, so that the trajectory can be consider as a space-time with zero space coordinates, so that line element takes the form

$$ds^2_\gamma = h(\lambda)\mathrm{d}\lambda^2 = -e^2(\lambda)\mathrm{d}\lambda^2$$

where $h(\lambda) = g_{00}(\lambda)$ and in the second step we rewritten the line element in terms of the *vielbein* one-form $e(\lambda)\mathrm{d}\lambda$, i.e. the basis such that the metric takes the diagonal form (for instance the basis given by eigenvectors for a matrix). In particular $[e(\lambda)\mathrm{d}\lambda] = \left[\frac{L}{M}\right]$ and $e(\lambda) = \tilde{e}(\tilde{\lambda})\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\lambda}$.

Then in terms of the world line metric, or equivalently in terms of the vielbein, we can construct an alternative parametrization-invariant action

$$\boxed{\begin{aligned}\tilde{S} &= -\frac{1}{2}\int \mathrm{d}\lambda\sqrt{-h}\left(h^{-1}\dot{X}^\mu\dot{X}_\mu + m^2\right) \\ &= \frac{1}{2}\int \mathrm{d}\lambda\left(e^{-1}\dot{X}^\mu\dot{X}_\mu - m^2 e\right)\end{aligned}} \tag{3.5}$$

In such action independent fields are not only X^μ, but also one-dimensional components of e^μ. This means that we promoted our time independent metric to a dynamical one by the introduction of the vielbein. On the other hand such dynamical metric, or equivalently the vielbein, appears only algebraically: it has no derivatives in the action, therefore if $m \neq 0$ we can minimize action respect to fluctuations of e, obtaining an exact expression for e in terms of coordinates X^μ:

$$\delta e \quad \Longrightarrow \quad \delta\tilde{S} = \frac{1}{2}\int \mathrm{d}\lambda\,\delta e\left(-\frac{1}{e^2}\dot{X}^\mu\dot{X}_\mu - m^2\right) \quad \Longrightarrow \quad e = \frac{-\sqrt{\dot{X}^\mu\dot{X}_\mu}}{m} \tag{3.6}$$

In this case ($m \neq 0$) the action $\tilde{S}$ reduces to eq. (3.3) :

$$\tilde{S}\big|_{e=\text{eq. (3.6)}} = -m\int \mathrm{d}\lambda\sqrt{-\dot{X}^\mu\dot{X}_\mu} = S$$

Therefore the modified action $\tilde{S}$ reduces to S when $m \neq 0$, and is a well defined action even for $m = 0$.

Before proving that $\tilde{S}$ provides the right EoM even for the massless case, notice that we can write $\tilde{S}$ in terms of the lagrangian $\tilde{L}$

$$\tilde{S} = \int \mathrm{d}\lambda\,\tilde{L}$$

and define the **4-momentum** p^μ as the conjugated field of $\dot{X}^\mu$

$$\boxed{P_\mu \equiv \frac{\partial\tilde{L}}{\partial\dot{X}^\mu} = e^{-1}(\lambda)\eta_{\mu\nu}\dot{X}^\nu} \tag{3.7}$$

Notice that the appearance of the vielbein component leads to the invariance under reparametrization of the 4-momentum. If $m \neq 0$ using eq. (3.6) with eq. (3.1) the 4-momentum reduces to the standard definition

$$P^\mu = m\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}$$

Anyhow, notice that eq. (3.7) is well defined even for $m = 0$. Such massless limit can be taken even in $\tilde{S}$, obtaining the well defined action for massless particles

$$\boxed{\tilde{S}\big|_{m\to 0} = \frac{1}{2}\int \mathrm{d}\lambda\,e^{-1}\dot{X}^\mu\dot{X}_\mu} \tag{3.8}$$

and the conditions we get by extremizing the action are

$$\delta e \quad : \quad \dot{X}^\mu\dot{X}_\mu = 0 \tag{3.9a}$$

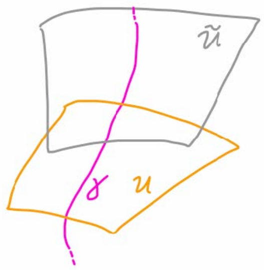
$$\delta X^\mu \quad : \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{-1}\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\right) \equiv \frac{\mathrm{d}P^\mu}{\mathrm{d}\lambda} = 0 \qquad \Rightarrow \qquad P^\mu(\lambda) \quad \text{is constant} \tag{3.9b}$$

Notice that first condition describes the null condition for trajectories, while second conditions describes the conservation of momentum. Both result were expected according with SR.

3.2 Particle's action in curved space-time

Now we consider a free particle in a general spacetime, i.e. a particle that interacts only with gravitational field. A particle's world line is now a curve in a more general 4 dimensional manifold $\mathcal{M}$

$$\gamma : \quad \mathbb{R} \supset I \quad \longrightarrow \quad \mathcal{M}$$



In each coordinate patch x^μ the world-line is described by 4 functions

$$X^\mu : \quad \lambda \quad \longmapsto \quad X^\mu(\lambda)$$

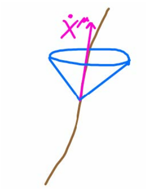
In other coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, new coordinates are given in the intersection of opens by

$$\tilde{X}^\mu(\lambda) \equiv \tilde{X}^\mu(X(\lambda))$$

Since space-time metric can be locally well approximated by the flat one, then (smooth enough) trajectories can be locally approximated by straight world-line, tangent to $\dot{X}^\mu$.

The natural requirement that no physical signal can travel faster than light translates into

$$g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \leq 0 \qquad (< 0 \quad \text{for } m \neq 0) \tag{3.10}$$



We can easily prove that this condition is invariant under reparametrization of $X^\mu(\lambda)$ (even for change on the direction) and also is invariant under change of patch for the manifold.

The natural covariant generalization of eq. (3.5) is

$$\tilde{S} = \frac{1}{2}\int \mathrm{d}\lambda \left(e^{-1} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2 e\right) \tag{3.11}$$

Notice that now also $g_{\mu\nu}(X)$ depends on λ, then the action is not purely quadratic as in the Minkowskian case. Again, this action is invariant under transformations of λ and x^μ, in particular the invariance under changes of patch can be proven easily through transformation proprieties of tensors.

Now we have to repeat same steps we did in the flat metric case. First let's assume $m \neq 0$, then we can integrate out the world-line vielbein component as we done in eq. (3.6):

$$\delta e \quad \Longrightarrow \quad \delta\tilde{S} = -\frac{1}{2}\int \mathrm{d}\lambda\, \delta e \left(\frac{1}{e^2} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu + m^2\right) \quad \Longrightarrow \quad e = \frac{\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}}{m} \tag{3.12}$$

hence in the massive case we obtain the generalization in the curved space of the action in the massive case eq. (3.3)

$$\boxed{S = -m\int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} = -m\int \mathrm{d}\tau = -m\Delta\tau} \tag{3.13}$$

with:

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2\big|_\gamma = g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \mathrm{d}\lambda^2$$

By the equivalence principle, we may again interpret $\mathrm{d}\tau$ and $\Delta\tau$ as proper time intervals: time intervals measured by the particle's clock. Hence, again, the particle's trajectory should be the one which (locally[II]) maximizes the proper time interval (between two fixed events). Notice that condition eq.(3.10) is required in order to have a well defined squared root in eq. (3.13).

Now we have to consider the massless case, and as in the Minkowskian case we can take the massless limit directly from the action eq. (3.11):

$$\tilde{S} = \frac{1}{2}\int \mathrm{d}\lambda \left(e^{-1} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu\right) \tag{3.14}$$

Taking the variation respect to the vielbein the least action principle leads to

$$\delta e \quad \Longrightarrow \quad g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu = 0 \tag{3.15}$$

hence particle moves along null trajectories. Again notice that in the massless case the vielbein is unfixed.

The EoM for the field $X^\mu(\lambda)$ given by action eq. (3.11) are given by the least action principle respect to variation $\delta X^\mu(\lambda)$:

$$\begin{aligned}
0 = \delta\tilde{S} &= \frac{1}{2}\int \mathrm{d}\lambda\, e^{-1}\delta\left(g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu\right) \\
&= \int \mathrm{d}\lambda\, e^{-1}\left(g_{\mu\nu}(X)\delta\dot{X}^\mu \dot{X}^\nu + \frac{1}{2}\delta X^\mu \partial_\mu g_{\nu\rho}(X)\dot{X}^\nu \dot{X}^\rho\right) \\
&= -\int \mathrm{d}\lambda\, \delta X^\mu \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{-1} g_{\mu\nu}(X)\dot{X}^\nu\right) - \frac{e^{-1}}{2}\partial_\mu g_{\nu\rho}(X)\dot{X}^\nu \dot{X}^\rho\right)
\end{aligned}$$

and we finally obtain the EoM in the curved spacetime (for both massive and massless case)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{-1} g_{\mu\nu}(X)\dot{X}^\nu\right) - \frac{e^{-1}}{2}\partial_\mu g_{\nu\rho}(X)\dot{X}^\nu \dot{X}^\rho = 0 \tag{3.16}$$

Let's try to rewrite these equation of motions in a more convenient way. Remember that the vielbein transforms in a non-trivial way under reparametrizations: $\tilde{e}(\tilde{\lambda})\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\lambda} = e(\lambda)$, then we can choose a parameter λ such that $e(\lambda)$ is constant, just solving the first order differential equation $\tilde{e}(\tilde{\lambda})\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\lambda} = e(\lambda) = \text{const.}$ respect to the variable λ. Such value of λ is said to be a **affine parameter**. For instance in the massive case $e(\lambda) = \frac{1}{m}$ corresponds to $\lambda = \tau$ (the proper time) as we can see from eq. (3.12), or, both in massive and massless case, we can always impose $e(\lambda) = 1$ for some other parameter λ (if $m \neq 0$ then $\lambda = \frac{\tau}{m}$). In case $e(\lambda) = 1$ we refer to the parameter λ as **proper** parameter.

Now, imposing λ to be an affine parameter[III] the EoM can be rewritten removing the vielbein

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}(X)\dot{X}^\nu\right) - \frac{1}{2}\partial_\mu g_{\nu\rho}(X)\dot{X}^\nu \dot{X}^\rho = 0$$

This is consistent with the fact that EoM should not depend on the field $e(\lambda)$ we introduced in order to using Lagrangian formalism in the massless case. Writing explicitly derivatives we have

$$g_{\mu\nu}(X)\ddot{X}^\nu + \partial_\rho\, g_{\nu\mu}(X)\dot{X}^\rho \dot{X}^\nu - \frac{1}{2}\partial_\mu g_{\nu\rho}\dot{X}^\nu \dot{X}^\rho = 0$$

Notice that we may exchange indices ρ and ν in the second term thanks to the symmetry of the tensor $\dot{X}^\mu \dot{X}^\nu$. Then only symmetric part of $\partial_\rho g_{\nu\mu}$ really contributes to the equation of motion

$$g_{\mu\nu}(X)\ddot{X}^\nu + \partial_{(\rho}\, g_{\nu)\mu}(X)\dot{X}^\rho \dot{X}^\nu - \frac{1}{2}\partial_\mu g_{\nu\rho}\dot{X}^\nu \dot{X}^\rho = 0$$

[II] For general non-trivial metrics, in particular with manifolds with non trivial topologies, may be different trajectories that maximize proper time, for instance around a cylinder one may take two paths with opposite directions that are both solutions of minimal action principle.

[III] This choice is said to be a **gauge choice**: we fixed a specific field $e(\lambda)$ among several equivalent choices $\tilde{e}(\tilde{\lambda})$. This is the same we do for EM potential A^μ invariant under gauge symmetry.

and taking into account this natural symmetrization we can write the EoM in the following form:

$$\boxed{\frac{\mathrm{d}^2X^\mu}{\mathrm{d}\lambda^2}+\Gamma^\mu_{\nu\rho}(X)\frac{\mathrm{d}X^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}X^\rho}{\mathrm{d}\lambda}=0} \tag{3.17}$$

where terms

$$\boxed{\Gamma^\mu_{\nu\rho}(X)\equiv\frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma}+\partial_\rho g_{\nu\sigma}-\partial_\sigma g_{\nu\rho})} \tag{3.18}$$

are known as **Christoffel symbols** or **Levi-Civita connection**. $\Gamma^\mu_{\nu\rho}(X)$ clearly is not a tensor, anyhow its transformation proprieties must be exactly the ones of $\ddot{X}^\mu$ in order to satisfy eq. (3.17) for any choice of coordinates X^μ. Moreover Christoffel symbols shows following symmetry

$$\Gamma^\mu_{\nu\rho}=\Gamma^\mu_{\rho\nu}$$

As we done for the flat space-time, we can set $\tilde{S}=\int \mathrm{d}\lambda\tilde{L}$ and define the four momentum in curved space as

$$P_\mu\equiv\frac{\partial L}{\partial\dot{X}^\mu}=e^{-1}\dot{X}^\mu \quad\rightarrow\quad P^\mu\equiv e^{-1}\dot{X}^\mu \tag{3.19}$$

This formula is general for any parameter λ, but when we chose λ to be the proper parameter ($e(\lambda)=1$) this formula reduces to

$$P^\mu(\lambda)=\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}$$

Recall that choice $e(\lambda)=1$ has dimension of $[\tau/m]$ i.e. a length over a mass, then P^μ has dimension of mass, which is the right dimension for the momentum when we set $c=1$. By using this identification we can rewrite EoM in terms of 4-momentum

$$\boxed{\frac{\mathrm{d}P^\mu}{\mathrm{d}\lambda}+\Gamma^\mu_{\nu\rho}(X)P^\nu P^\rho=0} \tag{3.20}$$

This formula holds both massive and massless case, but in the massive case we have $\lambda=\frac{\tau}{m}$ and the four momentum can be written in terms of velocity:

$$P^\mu=m\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}=mu^\mu$$

then the EoM becomes

$$\boxed{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}+\Gamma^\mu_{\nu\rho}(X)u^\nu u^\rho=0} \tag{3.21}$$

In general (even for massless particles) $\Gamma^\mu_{\nu\rho}\neq 0$ and then P^μ is not conserved:

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}\lambda}=-\Gamma^\mu_{\nu\rho}(X)P^\nu P^\rho$$

In order to obtain a conserved momentum we have to require additional symmetries in our manifold, as it happen in the Minkowski case, which is the maximally symmetric space-time (since $\eta_{\mu\nu}$ is invariant under a large number of symmetries given by the Poincaré group). We will see that the four momentum conservation is associated to the invariance of the theory under space translations. In general only the "lenght" of P^μ is preserved, indeed using eq. (3.13) and eq. (3.15) with $e(\lambda)=1$ we obtain

$$P^\mu P_\mu=\dot{X}^\mu\dot{X}_\mu=-m^2$$

We will see in the following that eq. (3.20) directly implies the conservation $P^\mu P_\mu$.

3.3 Covariant derivatives

Carroll sec. 3.1,3.2

Let's consider geometrical interpretation of eq. (3.17) and eq. (3.18) we derived in the previous lecture. All what we stated in this section hold in any metric space, not only Lorenzian ones.

First of all notice that $\Gamma^{\mu}_{\nu\rho}$ soes not transform as a tensor. Rather, they define a generalization of usual derivative, the so-called **covariant derivative** ∇_μ on vector fields, one forms, and in general on tensors

$$
\begin{aligned}
\nabla_\mu V^\mu &= \partial_\mu V^\nu + \Gamma^{\nu}_{\rho\mu} V^\rho \\
\nabla_\mu \alpha_\nu &= \partial_\mu \alpha_\nu - \Gamma^{\rho}_{\nu\mu}\alpha_\rho \\
\nabla_\mu T^{\nu\rho\dots}_{\sigma\dots} &= \partial_\mu T^{\nu\rho\dots}_{\sigma\dots} + \Gamma^{\nu}{}_{\tau\mu} T^{\tau\rho\dots}_{\sigma\dots} \\
&\quad + \Gamma^{\rho}_{\tau\mu} T^{\nu\tau\dots}_{\sigma\dots} + \dots - \Gamma^{\tau}_{\sigma\mu} T^{\nu\rho\dots}_{\tau\dots} - \dots
\end{aligned}
$$

Differently to usual partial derivative - which in general do not transform tensor field in tensor field - the covariant derivative ∇ applied to a tensor of type (n, m) produces a tensor of type $(n, m+1)$. Note that ∇_μ preserves (anti)symmetry of indices and satisfies Leibniz rule

$$
\nabla_\mu (T^\rho \dots W_{\sigma\dots}) = (\nabla_\mu T^{\rho\dots}) W_{\sigma\dots} + T^{\rho\dots} \nabla_\mu W_{\sigma\dots}
$$

In order to behave like a derivative (preserving tensorial nature of objects) $\Gamma^{\mu}_{\nu\rho}$ must transform in such a way to "correct" the non-covariance of $\partial_\mu T^{\nu\dots}_{\rho\dots}$. For instance, for vector fields

$$
\partial_\mu = A^{\nu}_{\mu} \tilde{\partial}_\nu \qquad V^\mu = (A^{-1})^{\mu}_{\nu} \tilde{V}^\nu
$$

with

$$
A^{\mu}_{\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \qquad (A^{-1})^{\mu}_{\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}
$$

when we make a coordinate transformation of $\partial_\mu V^\nu$ we see that traditional derivatives do not give another vector field

$$
\partial_\mu V^\nu = A^{\rho}_{\mu} \tilde{\partial}_\rho ((A^{-1})^{\nu}_{\sigma} \tilde{V}^\sigma) = A^{\rho}_{\mu} [(A^{-1})^{\nu}_{\sigma} \tilde{\partial}_\rho \tilde{V}^\sigma + (\tilde{\partial}_\rho (A^{-1})^{\nu}_{\tau}) \tilde{V}^\tau]
$$

I order to correct this, we have to use the covariant derivative

$$
\begin{aligned}
\nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^{\nu}_{\alpha\mu} V^\alpha \\
&= A^{\rho}_{\mu} (A^{-1})^{\nu}_{\sigma} [\tilde{\partial}_\rho \tilde{V}^\sigma \underbrace{\overbrace{A^{\sigma}_{\alpha} \tilde{\partial}_\rho (A^{-1})^{\alpha}_{\tau}}^{A^{\sigma}_{\alpha} (A^{-1})^{\beta}_{\rho} \partial_\beta (A^{-1})^{\alpha}_{\tau}} \tilde{V}^\tau + (A^{-1})^{\beta}_{\rho} A^{\sigma}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} (A^{-1})^{\alpha}_{\tau} \tilde{V}^\tau}_{\tilde{\Gamma}^{\sigma}_{\tau\rho} \tilde{V}^\tau}] \\
&= A^{\rho}_{\mu} (A^{-1})^{\nu}_{\sigma} (\tilde{\partial}_\rho \tilde{V}^\sigma + \tilde{\Gamma}^{\sigma}_{\tau\rho} \tilde{V} \tau) \\
&= A^{\rho}_{\mu} (A^{-1})^{\nu}_{\sigma} \tilde{\nabla}_\rho \tilde{V}^\sigma
\end{aligned}
$$

then in order to obtain a tensor as the result of the application covariant derivative to a vector field we require following transformation property under change of coordinates for Christoffel coefficients:

$$
\tilde{\Gamma}^{\mu}_{\nu\rho} = A^{\mu}_{\sigma} (A^{-1})^{\alpha}_{\nu} (A^{-1})^{\beta}_{\rho} \Gamma^{\sigma}_{\alpha\beta} + A^{\mu}_{\alpha} (A^{-1})^{\beta}_{\rho} \partial_\beta (A^{-1})^{\alpha}_{\nu} \tag{3.22}
$$

with

$$
A^{\mu}_{\nu} = \frac{\partial \tilde{x}^\nu}{\partial x^\nu} \qquad (A^{-1})^{\mu}_{\nu} = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}
$$

More explicitly, in coordinates

$$
\boxed{\tilde{\Gamma}^{\mu}_{\nu\rho} = \frac{\partial \tilde{x}^\mu}{\partial x^\sigma} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} \frac{\partial x^\beta}{\partial \tilde{x} \rho} \Gamma^{\sigma}_{\alpha\beta} + \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\rho \, \partial \tilde{x}^\nu}} \tag{3.23}
$$

This guarantees that any covariant derivative $\nabla_\mu T^{\rho\dots}_{\nu\dots}$ transforms covariantly. One can check that Levi-Civita connection transforms according to these rules.

Notice that

- any connection $\Gamma^\mu_{\nu\rho}$ must transform in this way, actually this is the defining property of a connection;
- the non-tensorial term in eq. (3.23) is Γ-independent: this means that differences of connections $\Delta\Gamma^\mu_{\nu\rho}$ transform as tensors;
- the antisymmetric components of coefficient of a connection[IV]

  $$T^\mu_{\nu\sigma} = 2\Gamma^\mu_{[\nu\rho]} = \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu}$$

  transforms as a tensor, and define the **torsion** tensor

  $$T = 2\Gamma^\mu_{[\nu\rho]}\partial_\mu \otimes \mathrm{d}x^\nu \mathrm{d}x^\rho$$

  Clearly, for Levi-Civita connection the torsion is vanishing

The Levi-Civita connection is the only connection that is

(i) torsion free ($\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$)

(ii) a metric connection such that $\nabla_\mu g_{\nu\rho} = 0$, such property is called **metricity**

One can prove that these two conditions, given some metric $g_{\mu\nu}$, define uniquely a connection, which is the Levi-Civita connection. Let's give a sketch of proof. Imposing the compatibility with the metric and the symmetry property given by $T = 0$ we have

$$0 = \nabla_\mu g_\nu\rho - \nabla_\nu g_{\rho\mu} - \nabla_\rho g_{\mu\nu} = \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + 2g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}$$

Note that a connection Γ is analogous to a non-abelian gauge field and indeed one can give a unified description in terms of principal and vector bundles. However, in the Standard Model the gauge fields are elementary, while in General Relativity the Levi-Civita connection $\Gamma^\mu_{\nu\rho}$ is composite, in terms of the elementary field $g_{\mu\nu}(x)$.

Another important remark is that combining $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu$, $\nabla_\rho\delta^\mu_\nu = 0$ (given by the fact that δ^μ_ν is constant) and the Leibniz rule we obtain that the covariant derivative of the inverse of the metric vanishes as well $\nabla_\mu g^{\nu\rho} = 0$.
Moreover, the metricity of LC connection, together with Leibniz rule, allows one to freely move $g_{\mu\nu}$ and $g^{\mu\nu}$ in and out-side ∇_μ.

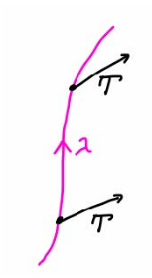
3.4 Parallel transport and geodesics

Carroll sec. 3.3, 3.4

Given a connection ∇ and a curve $\gamma \in \mathcal{M}$ with coordinates $X(\lambda)$, we can define the covariant derivative along the curve of some tensor T as

$$\frac{\mathrm{D}T}{\mathrm{d}\lambda} = \dot{X}^\mu(\lambda)(\nabla_\mu T)(X(\lambda)) = \frac{\mathrm{d}T(X(\lambda))}{\mathrm{d}\lambda} + \dot{X}^\mu\Gamma^\rho_{\sigma\mu}(X)T^{\sigma\cdots}_{\cdots} + \dots$$

which depends only on the values of the tensor on the curve. Indeed, $\frac{\mathrm{D}}{\mathrm{d}\lambda}$ is well defined also on tensor fields $T(\lambda)$ supported only on γ.



[IV] Levi-Civita coefficients are symmetric, but this does not holds in general for any connection

In particular, a vector field $V^\mu(\lambda)$ supported on a curve $\gamma : \lambda \mapsto X^\mu$, is **parallel-transported** along the curve iff it is covariantly constant along γ

$$\boxed{\frac{\mathrm{D}V^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}V^\mu(\lambda)}{\mathrm{d}\lambda} + \dot{X}^\rho(\lambda)\Gamma^\mu_{\sigma\rho}(X(\lambda))V^\sigma(\lambda) = 0} \tag{3.24}$$

This is a first order equation that can be always be integrated, then given $V^\mu(\lambda_0)$ one can define the corresponding parallel transported vector field uniquely through eq. (3.24).

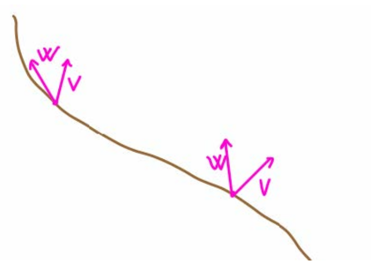
If the connection is compatible with the metric (for instance if it is Levi-Civita connection) the parallel transport preserves the scalar product between vectors: if

$$\frac{\mathrm{D}V^\mu(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{D}W^\mu(\lambda)}{\mathrm{d}\lambda} = 0$$

then

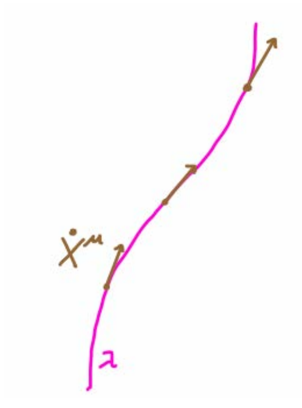
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(V^\mu W_\mu) = \frac{\mathrm{D}}{\mathrm{d}\lambda}(g_{\mu\nu}V^\mu W^\nu) = \frac{\mathrm{D}g_{\mu\nu}}{\mathrm{d}\lambda}V^\mu W^\nu + g_{\mu\nu}\left(\frac{\mathrm{D}V^\mu}{\mathrm{d}\lambda}W^\nu + V^\mu\frac{\mathrm{D}W^\nu}{\mathrm{d}\lambda}\right) = 0$$

i.e. $V^\mu W_\mu$ is constant along the curve if V^μ and W^μ are parallel transported along γ.



In a flat space a straight line is a path that parallel-transports its own tangent vector. We can generalize this concept for more general connection introducing a new object. In particular **geodesic** are curves that satisfies following relation, namely **geodesics equation**:

$$\boxed{\frac{\mathrm{D}\dot{X}(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2X^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho}(X)\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}X^\nu}{\mathrm{d}\lambda} = 0} \tag{3.25}$$



Such curves can be thought heuristically as straightest possible curve in a curved space, and are exactly the particle's EoM for the affine parameter λ. This makes further manifest the invariance of the particle's EoM, indeed trajectories are a geometrical object independent from bulk coordinate since covariant derivative of a vector is a tensor. Moreover, in the case of LC connection a time-like geodesic can be equivalently characterized as the curve between two fixed events that extremizes the proper time.

Another important point is that using the compatibility between parallel transport and scalar product we can conclude that under parallel transport

$$\dot{X}^\mu(\lambda)\dot{X}_\mu(\lambda) = g_{\mu\nu}(X(\lambda))\dot{X}^\mu(\lambda)\dot{X}^\nu(\lambda) = \text{ constant}$$

and then

$$\boxed{\dot{X}^\mu(\lambda_0)\,\text{is time-like/space-like/null} \quad \Rightarrow \quad \dot{X}^\mu(\lambda)\,\text{is time-like/space-like/null}\ \ \forall\lambda}$$

This means that we can distinguish between time-like, space-like and null geodesics. In particular, for massless particles we do no need to separately solve for $\dot{X}^2(\lambda) = 0$, but it is sufficient to impose it at $\lambda = \lambda_0$.

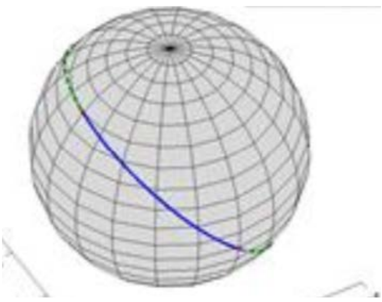
For space-like curves $X^\mu(\lambda)$, that is such that $g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu > 0$, one can define the **proper length**

$$\Delta l = \int \mathrm{d}\lambda \sqrt{g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu}$$

and by following the same steps as for the particle's action, one can show that Δs between two fixed points is extremized if the geodesic equation is satisfied, with Levi-Civita connection $\Gamma^\mu_{\nu\rho}$ and affine parameter in the form

$$\lambda = al + b$$

for some constant a and b (geodesic equation does not change under this reparametrization). In particular (locally) shortest curves between two points are space-like geodesics.



We said that geodesics parametrized by the affine parameter λ coincides with EoM of point-particles. However, we may want to use a more general parameter σ to describe the geodesics. Then the corresponding equation obtained from geodesic equation through a change of parameter is

$$\boxed{\frac{\mathrm{D}\dot{X}^\mu(\sigma)}{\mathrm{d}\sigma} = \frac{\mathrm{d}^2X^\mu}{\mathrm{d}\sigma^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}X^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}X^\rho}{\mathrm{d}\sigma} = f(\sigma)\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma}} \tag{3.26}$$

with $f(\sigma) = -\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right)^{-2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2}$ (prove this as an exercize). Viceversa, if this equation holds, then we can always fine an affine parameter λ such that $f(\sigma)\frac{\mathrm{d}X^\mu}{\mathrm{d}\sigma} = 0$. Then, the condition eq. (3.26) requires that $\dot{X}^\mu$ is parallely transported along γ, up to possible rescaling of its length. This gives a more general description of geodesics respect to eq. (3.25), indeed eq. (3.26) states that a geodesics is a curve such that the covariant derivative of the tangent vector is proportional to the tangent vector itself (previous definition states that tangent vector must be covariantly constant).

One may consider more general time-like trajectories parametrized by proper time and define **four-velocity** as

$$u^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}$$

Then using the notion of covariant derivative for a trajectory, define **four-acceleration** as

$$\alpha^\mu = \frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2X^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}X^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}X^\sigma}{\mathrm{d}\tau}$$

With the definition, as expected, geodesic condition is equivalent to

$$\alpha^\mu = 0$$

i.e. the 4-velocity is covariantly constant. This equation also define the EoM for free particles in a curved space. Whenever a force (not the gravitational one) is applied to a particle, the effect of this forces can be codified in a **four-force** defined as

$$f^\mu = m\alpha^\mu = m\frac{\mathrm{D}u^\mu}{\mathrm{d}\tau}$$

As in the special relativistic case, 4-velocity and 4-acceleration satisfies useful properties. First of all

$$u^\mu u_\mu = -1$$

Then, by applying the covariant derivative on both sides of this equation and using the metricity of the connection, we obtain

$$u_\mu \alpha^\mu = g_{\mu\nu} u^\mu \alpha^\nu = 0$$

and this means that α^μ is space-like. Finally, we can define **proper acceleration** as the acceleration in the proper instance frame of the particle itself

$$\alpha = \sqrt{\alpha^\mu \alpha_\mu}$$

3.5 The Newtonian limit

Carroll, sec. 4.1

We have started from the simplest possible action for particles and we have concluded that a probe particle follows geodesics, that is, the *strightest possible* path in space-time.

Regarding $g_{\mu\nu}(x)$ as a gravitational field, this provides the mathematical definition of a *freely falling* particle, then we should be able to recover the Newtonian description. This should hold in the so-called **Newtonian limit**.

The Newtonian limit is defined by three assumptions:

(1) **Weak field**: metric can be regarded to be a very small fluctuation about the Minkowskian one:

$$g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad \text{with} \quad h_{\mu\nu} \ll 1$$

(2) **Stationary field**: the metric is time independent up to higher corrections

$$h_{\mu\nu}(t, \mathbf{x}) = h_{\mu\nu}(\mathbf{x})$$

Notice that this implies the (approximate) break of the Lorentz symmetry (since there is a difference between space and time)

(3) **Non-reativistic limit**

$$v \ll c \qquad \Rightarrow \quad \text{slow massive particles}$$

More precisely, in units $c = 1$, we will assume $v^2 \lesssim h_{\mu\nu}$.

If we take the time t as world-line parameter (i.e. we use the *static gauge*) for the derivative of any vector we obtain $\dot{X}^\mu = (1, \dot{\mathbf{X}}(t))$, in particular in the Newtonian limit we have $|\dot{\mathbf{X}}| \sim v \ll 1$. Then the length of a tangent vector can be approximated as follows:

$$\begin{aligned}
g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu &= [\eta_{\mu\nu} + h_{\mu\nu}(\mathbf{X}) + O(h^2)]\dot{X}^\mu \dot{X}^\nu \\
&= \eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu + [h_{00}(\mathbf{X}) + O(h^{3/2})] \\
&= -[1 - |\dot{\mathbf{X}}|^2 - h_{00}(\mathbf{X}) + O(h^{3/2})]
\end{aligned}$$

Then we can perform the following expansion of the particle's action

$$\begin{aligned}
S &= -m\int \mathrm{d}t \sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \\
&\simeq -m\int \mathrm{d}t \sqrt{1 - |\dot{\mathbf{X}}|^2 - h_{00}(\mathbf{X}) + \dots} \\
&\simeq \int \mathrm{d}t \left[-m + \frac{1}{2}m|\dot{\mathbf{X}}|^2 + \frac{1}{2}m h_{00}(\mathbf{X}) + \dots\right]
\end{aligned}$$

We can see that, up to irrelevant constant terms, the expression into square brackets corresponds to the Lagrangian for a particle moving in a gravitational potential

$$L = \frac{1}{2}m|\dot{\mathbf{X}}|^2 - m\Phi(\mathbf{X}) \qquad \text{where} \qquad h_{00}(\mathbf{x}) := -2\Phi(\mathbf{x})$$

Such approximations holds as long as $\Phi(\mathbf{x}) \ll c^2$.

On the Earth surface $\Delta z \lesssim 10\text{km}$, we have $\frac{\Phi}{c^2} \lesssim 10^{-12}$ and $\left(\frac{v}{c}\right)^2 \lesssim h$ if $v \lesssim 10^{-6}c \simeq 3 \times 10^2\text{m/s} \simeq 10^3\text{km/h}$.

This matches what found by taking the weak-field limit ($\Phi \ll 1$) of Rindler's metric:

$$\mathrm{d}s^2 = -e^{2\Phi}\mathrm{d}t^2 + \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} \simeq -(1 + 2\Phi)\mathrm{d}t^2 + \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}$$

Exercise 1

Recover Newton's law $\mathbf{a} = -\vec{\nabla}\Phi$ by taking the Newtonian limit directly of the geodesic equation.

Hence we found that the geodesic equation reproduces the correct Newton's equation in the Newtonian limit.

3.5.1 Example: The evolving universe (II)

Example 9: The evolving universe (II)

Carroll, sec 3.5

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \qquad \Rightarrow \qquad g_{00} = -1 \quad g_{ij} = a(t)\delta_{ij} \quad g_{0i} = 0$$

The non-vanishing Christoffel symbols are $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$, $\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$ ($\dot{a} = \frac{\mathrm{d}a(t)}{\mathrm{d}t}$)
The geodesic equation becomes

$$\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2} + a(T)\dot{a}(T)\delta_{ij}\frac{\mathrm{d}X^i}{\mathrm{d}\lambda}\frac{\mathrm{d}X^j}{\mathrm{d}\lambda} = 0 \qquad \frac{\mathrm{d}^2X^i}{\mathrm{d}\lambda^2} + 2\frac{\dot{a}}{a}(T)\frac{\mathrm{d}T}{\mathrm{d}\lambda}\frac{\mathrm{d}X^i}{\mathrm{d}\lambda} = 0 \qquad (\text{with affine } \lambda)$$

$X^i(\lambda) \equiv x^i_o$ (constant) describes a time-like geodesic, provided

$$\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2} = 0 \qquad \Rightarrow \qquad T(\lambda) = a\lambda + b$$

A convenient choice is $T(\lambda) = \lambda \equiv \tau$ and then we fill the space-time with (probe) *freely falling* observers at fixed x^i coordinates and whose clocks measure the time t. However, notice that they are not "parallel" since their distance sales with $a(t)$.

More generic geodesics $X^i(\lambda) = x^i_0 + f(\lambda)n^i$ where x^i_0 and n_i are constant such that $\mathbf{n} \cdot \mathbf{n} = 1$ the geodesics equations becomes

$$\frac{\mathrm{d}^2T}{\mathrm{d}\lambda^2} + a(T)\dot{a}(T)\left(\frac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right)^2 = 0 \qquad \frac{\mathrm{d}^2f}{\mathrm{d}\lambda^2} + 2\frac{\dot{a}}{a}(T)\frac{\mathrm{d}T}{\mathrm{d}\lambda}\frac{\mathrm{d}f}{\mathrm{d}\lambda} = 0 \tag{3.27}$$

Hence geodesic equations reduce to equations for $T(\lambda)$ and $f(\lambda)$ only. Any other "freely falling" observer will travel along straight spatial trajectories, as expected from spatial homogeneity and isotropy.

Let's restrict ourself to null geodesics $X^i(\lambda) = x^i_0 + f(\lambda)n^i$. Then

$$0 = g_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}X^\nu}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}T}{\mathrm{d}\lambda}\right)^2 - a^2(T)\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}T}{\mathrm{d}\lambda}\right)^2 - a^2(T)\left(\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right)^2$$

and this implies

$$\frac{\mathrm{d}T}{\mathrm{d}\lambda} = a(T)\frac{\mathrm{d}f}{\mathrm{d}\lambda} \tag{3.28}$$

(the other sign can be reabsorbed by changing $f \to -f$).

Notice that we may use $t = T(\lambda)$ instead of λ to parametrize the world-line, hence

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{1}{a(t)} \tag{3.29}$$

and

$$X^i(t) = x_0^i(t) = x_0^i + n^i \int_{t_0}^{t} \frac{\mathrm{d}t'}{a(t')} \tag{3.30}$$

This is possible because of the isotropy, which allows us to reduce the problem to a 2-dimensional one.

On the other hand, both equations in eq. (3.27) reduce to

$$\frac{\mathrm{d}^2T(\lambda)}{\mathrm{d}\lambda^2} + \left(\frac{\dot{a}}{a}\right)(T)\left(\frac{\mathrm{d}T}{\mathrm{d}\lambda}\right)^2 = 0 \tag{3.31}$$

so

$$\frac{\mathrm{d}T}{\mathrm{d}\lambda} = \frac{E_0}{a(T)}$$

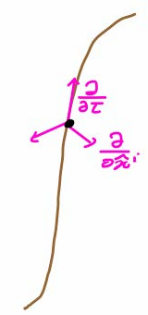
where E_0 is an integration constant.

Recall that $P^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}$ (with proper λ). In general, the energy of the particle measured by a (time-like observer with 4-velocity u^μ is then given by

$$E = -u_\mu P^\mu \tag{3.32}$$

(which is intrinsic but observer-dependent!) since in the observer's (instant) inertial rest frame

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} \qquad \hat{u}^\mu = (1,0,0,0) \quad \Rightarrow \quad E = \hat{P}^0$$



In cosmological space-times, consider (geodesic) coordinate observers at constant $\mathbf{x}$, this implies $u^\mu = (1,0,0,0)$ and

$$E = -u_\mu P^\mu = P^0 = \frac{\mathrm{d}T}{\mathrm{d}\lambda} = \frac{E_0}{a(T)}$$

The relation

$$\boxed{E = \frac{E_0}{a(T)}} \tag{3.33}$$

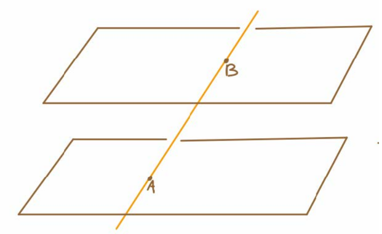
means the remarkable fact that in an evolving universe E is not constant. Indeed since $a(t) \sim t^p$ or e^{Ht} (expanding universe) E decreases.

By the QFT relation $P^m u = \hbar k^\mu$ (where k^μ is the wave 4-vector) which implies $E = \hbar\omega$ this implies

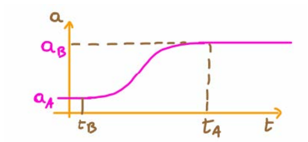
$$\omega = \frac{\omega_0}{a(t)} \tag{3.34}$$

Hence, in an expanding universe, there is a **cosmological red-shift**

$$\boxed{\frac{\omega_B}{\omega_A} = \frac{a_A}{a_B} < 1} \tag{3.35}$$



which is a purely cosmological effect, and cannot be interpreted as Doppler effect since, as we can see in the next picture, at times t_A and t_B the observers are not receeding:



Chapter 4

Curvature

4.1 Curvature

Carroll, sec. 3.6

The aim of this section is to give an answer to the following question: is there any *intrinsic* way to characterize a “honest”gravitational field? That is, a metric $g_{\mu\nu}(x)$ for which there does not exist any inertial frame $\hat{x}^\alpha$ in which

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}\hat{x}^\mu \mathrm{d}\hat{x}^\nu \qquad \text{and} \qquad \hat{\Gamma}^\mu_{\rho\sigma} \equiv 0$$

Let us first observe that a property of Minkowski space is that the parallel transport between two points does not depend on the path. This is obvious in flat (inertial) coordinates, in which $\hat{\Gamma}^\mu_{\rho\sigma} \equiv 0$

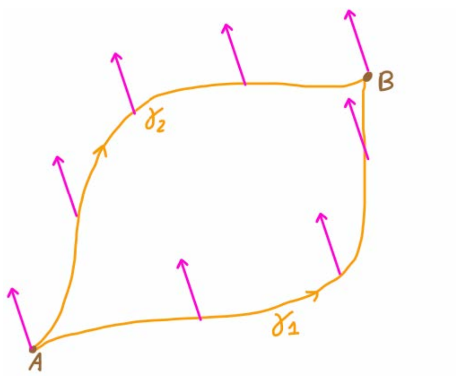
$$\frac{\mathrm{D}\hat{V}^\mu}{\mathrm{d}\lambda} \equiv \frac{\mathrm{d}\hat{V}^\mu}{\mathrm{d}\lambda} = 0 \qquad \Leftrightarrow \qquad \hat{V}^\mu(\lambda) = \text{constant}$$

which implies

$$\hat{V}^\mu_B|_{\gamma_1} = \hat{V}^\mu_A = \hat{V}^\mu_B|_{\gamma_2}$$

and

$$V^\mu_B|_{\gamma_1} = V^\mu_B|_{\gamma_2} \qquad \text{in any other coordinates } x^\mu$$



For a more generic metric, we generally have $V^\mu(\lambda_{\text{fin}}) \neq V^\mu(\lambda_{\text{in}})$. Indeed

$$\frac{\mathrm{d}V^\mu(s)}{\mathrm{d}s} = -\Gamma^\mu_{\nu\rho}(X(s))V^\nu(s)\dot{X}^\rho(s)$$

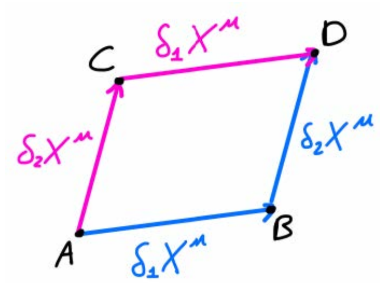
which implies

$$V^\mu(s+\delta s) = V^\mu(s) = \Gamma^\mu_{\nu\rho}(X(s))V^\nu(s)\overbrace{\dot{X}^\rho(s)\delta s}^{\delta X^\rho(s)} + \ldots$$

and then the infinitesimal change of V^μ parallely transported along δX^μ is given by

$$\delta V^\mu = -\Gamma^\mu_{\nu\rho}(x) V^\nu \delta X^\rho$$

Consider now two independent infinitesimal displacements $\delta_1 X^\mu$, $\delta_2 X^\mu$:



in formulas we have (in the second step we used $x_B = x_A + \delta_1 X$)

$$\begin{aligned} V^\mu_D|_{1-2} &= V^\mu_B - \Gamma^\mu_{\nu\rho}(x_B) V^\nu_B \delta_2 X^\rho \\ &= V^\mu_A - \Gamma^\mu_{\nu\rho}(x_A) V^\nu_A \delta_1 X^\rho - \Gamma^\mu_{\nu\rho}(x_A) V^\nu_A \delta_2 X^\rho \\ &\quad - \partial_\sigma \Gamma^\mu_{\nu\rho}(x_A) V^\nu_A \delta_1 X^\sigma \delta_2 X^\rho + \Gamma^\mu_{\nu\rho}(x_A) \Gamma^\nu_{\alpha\sigma}(x_A) V^\alpha_A \delta_1 X^\sigma \delta_2 X^\rho \\ V^\mu_D|_{2-1} &= (\text{the same with } 1 \leftrightarrow) \end{aligned}$$

Then $V^\mu_D|_{1-2} - V^\mu_D|_{2-1} = -R^\mu{}_{\nu\rho\sigma}(x_A) V^\nu_A \delta_1 X^\rho \delta_2 X^\sigma$ where we defined the **Riemann curvature**:

$$\boxed{R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho} \Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma} \Gamma^\alpha_{\nu\rho}} \tag{4.1}$$

Notice that this is a tensor. For any vector field $V^\mu(x)$ we have

$$[\nabla_\rho, \nabla_\sigma] V^\mu \equiv (\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho) V^\mu = R^\mu{}_{\nu\rho\sigma} V^\nu$$

(on the l.h.s. of the previous identity we omitted the factor $-T^\nu_{\rho\sigma} \nabla_\nu V^\mu$ with $T^\nu_{\rho\sigma} = 2\Gamma^\nu_{[\rho\sigma]}$ since it vanishes).

For more generic tensors $T^{\mu\cdots}_{\nu\cdots}$ we have

$$\begin{aligned} [\nabla_\rho, \nabla_\sigma] T^{\mu\cdots}_{\nu\cdots} &= R^\mu{}_{\tau\rho\sigma} T^{\tau\cdots}_{\nu\cdots} + (\text{similar action on } \textit{upper} \text{ indices}) \\ &\quad - R^\tau{}_{\nu\rho\sigma} T^{\mu\cdots}_{\tau\cdots} + (\text{similar action on } \textit{lower} \text{ indices}) \end{aligned}$$

Note also that $[\nabla_\mu, \nabla_\nu]\phi = 0$ for any scalar field.

Note that for $\mathrm{d}s^2 = \eta_{\alpha\beta} \mathrm{d}\hat{x}^\alpha \mathrm{d}\hat{x}^\beta$ we clearly have $\hat{R}^\mu{}_{\nu\rho\sigma} \equiv 0$, hence $R^\mu{}_{\nu\rho\sigma} \equiv 0$ in any other coordinate system x^μ (even if $\Gamma^\mu_{\nu\rho} \neq 0$). Viceversa, if $R^\mu{}_{\nu\rho\sigma} \equiv 0$ then one can find[I] a local coordinate system $\hat{x}^\alpha$ in which $\mathrm{d}s^2 = \hat{\eta}_{\alpha\beta} \mathrm{d}\hat{x}^\alpha \mathrm{d}\hat{x}^\beta$.

On the other hand, if $R^\mu{}_{\nu\rho\sigma} \not\equiv$ there do not exist inertial coordinates. Hence we arrived to the following important result:

$$\boxed{\begin{array}{rcl} \text{spacetime is } \textit{flat} \text{ (and locally like Mink)} & \Leftrightarrow & R^\mu{}_{\nu\rho\sigma} \equiv 0 \\ \text{spacetime is } \textit{curved} & \Leftrightarrow & R^\mu{}_{\nu\rho\sigma} \not\equiv 0 \end{array}} \tag{4.2}$$

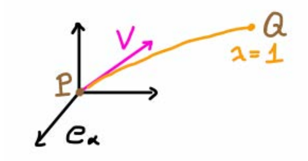
[I] See, for instance, Carroll pag. 124

Note that (if $[x^\mu] = L$) the curvature has dimension $[R^\mu{}_{\nu\rho\sigma}] = (\text{length})^{-2}$. In particular, $(R^\mu{}_{\nu\rho\sigma})^{-1/2}$ provides an estimate of the length scale L above which the space-time cannot be approximated by the flat one. This also set the scale below which the EEP holds.

Previous claim can be made precise by introducing *normal* coordinates.

4.1.1 Geodesic (or Riemann) normal coordinates

At any point p, choose orthonormal frame $e_\alpha = e^\mu_\alpha \partial_\mu$. Any $V \in T_pM$ can be decomposed as $V = \hat{x}^\alpha e_\alpha$



Throw a geodesic with affine parameter λ ($[\lambda] = 1$) such that $\frac{\mathrm{d}X^\mu}{\mathrm{dd}}\lambda|_{\lambda=0} = V^\mu$. The value $\lambda - 1$ identifies a point q, hence assign coordinates $\hat{x}^\alpha$ to q. In this way we obtained a defined coordinate system $\hat{X}^\alpha$ in a small enough neighbourhood of $\hat{x}^\alpha = 0$. In this coordinate system

$$\boxed{\hat{\Gamma}^\alpha_{\beta\gamma}|_p = 0 \qquad \hat{g}_{\alpha\beta}(\hat{x}) = \eta_{\alpha\beta} - \frac{1}{3}\hat{R}_{\alpha\gamma\beta\delta}\hat{x}^\gamma\hat{x}^\delta + \dots} \tag{4.3}$$

and $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta} + O\left(\frac{\hat{x}^2}{L^2}\right)$ with $L^2 \simeq (\text{Riem})^{-2}$ as claimed above and $\hat{x}^2$ which approximates the proper "distance" from p.

Such construction is the concrete realization of the EEP. The error is given by $\frac{(\text{distance})^2}{L^2}$ with $L^2 \sim (\text{Riem})^{-1}$.

4.1.2 Properties of the Riemann tensor

Carroll, sec. 3.7

It is convenient to introduce the tensor

$$R_{\mu\nu\rho\sigma} := g_{\mu\tau}R^\tau{}_{\nu\rho\sigma}$$

which satisfies following properties:

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\mu\nu[\rho\sigma]} \tag{4.4a}$$
$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = R_{[\mu\nu]\rho\sigma} \tag{4.4b}$$
$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{4.4c}$$
$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \qquad (\overset{a}{\Leftrightarrow} \quad R_{\mu[\nu\rho\sigma]} = 0) \tag{4.4d}$$

In order to prove above properties at any point p, we can use any locally inertial frame $\hat{x}^\alpha$:

$$\hat{\Gamma}^\alpha_{\beta\gamma} = 0 \quad \Leftrightarrow \quad \hat{\partial}_\alpha \hat{g}_{\beta\gamma} = 0 \quad \Leftrightarrow \quad \hat{\partial}_\alpha \hat{g}^{\beta\gamma} = 0 \qquad (\text{at } p)$$

Then, at p:

$$\begin{aligned}\hat{R}_{\alpha\beta\gamma\delta} &= \hat{g}_{\alpha\lambda}(\hat{\partial}_\gamma \hat{\Gamma}^\lambda_{\beta\delta} - \hat{\partial}_\delta \hat{\Gamma}^\lambda_{\beta\gamma}) \\ &= \frac{1}{2}(\hat{\partial}_\gamma\hat{\partial}_\beta\hat{g}_{\alpha\delta} - \hat{\partial}_\delta\hat{\partial}_\beta\hat{g}_{\alpha\gamma} - \hat{\partial}_\gamma\hat{\partial}_\alpha\hat{g}_{\beta\delta} + \hat{\partial}_\delta\hat{\partial}_\alpha\hat{g}_{\beta\gamma}\end{aligned}$$

In this way one can immediately check properties (a), (b) and (c) and, with slightly more work, (d).

The Riemann tensor obeys also a differential identity, called **Bianchi identity**:

$$\boxed{\nabla_{[\mu}R_{\nu\rho]\sigma\lambda} = \nabla_{[\mu|}R_{\sigma\lambda|\nu\rho]} = 0} \tag{4.5}$$

One may prove it in local flat coordinates (see Carroll, pag. 128) but it is easier to obtain it from the tautological **Jacobi identity**:

$$[\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] + [\nabla_\rho, [\nabla_\mu, \nabla_\nu]] = 0$$

and applying it to a generic vector field V^σ:

$$\begin{aligned}[\nabla_\mu, [\nabla_\nu, \nabla_\rho]]V^\sigma &= \nabla_\mu(R^\sigma{}_{\tau\nu\rho}V^\tau) - [\nabla_\nu, \nabla_\rho]\nabla_\mu V^\sigma \\ &= (\nabla_\mu R^\sigma{}_{\tau\nu\rho})V^\sigma + \underbrace{R^\sigma{}_{\tau\nu\rho}\nabla_\mu V^\tau - R^\sigma{}_{\tau\nu\rho}\nabla_\mu V^\tau}_{=0} + R^\tau{}_{\mu\nu\rho}\nabla_\tau V^\sigma\end{aligned}$$

Then antisymmetrizing on $\mu\nu\rho$ and using $R^\tau{}_{[\mu\nu\rho]} \equiv 0$ one gets the Bianchi identity in the form $\nabla_{[\mu|}R^\tau{}_{\sigma|\nu\rho]} \equiv 0$.

Some important quantities can be obtained through the contraction of indices in the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$:

(a) **Ricci tensor**:

$$\boxed{R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}} \tag{4.6}$$

Note that

$$R_{\nu\mu} = R^\rho{}_{\nu\rho\mu} = R_{\rho\mu}{}^\rho{}_\nu = R^\rho{}_{\mu\rho\nu} = R_{\mu\nu}$$

(b) **Scalar curvature**

$$\boxed{R \equiv g^{\mu\nu}R_{\mu\nu}} \tag{4.7}$$

with $[R] = L^{-2}$

(c) **Einstein tensor**

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R} \tag{4.8}$$

By contracting the Bianchi identity we get

$$\begin{aligned}0 &= g^{\nu\sigma}g^{\mu\lambda}(\nabla_\mu R_{\nu\rho\sigma\lambda} + \nabla_\nu R_{\rho\mu\sigma\lambda} + \nabla_\rho R_{\mu\nu\sigma\lambda}) \\ &= \nabla^\lambda R_{\rho\lambda} + \nabla^\sigma R_{\sigma\rho} - \nabla_\rho R\end{aligned}$$

which implies

$$\boxed{\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \qquad \Leftrightarrow \qquad \nabla^\mu G_{\mu\nu} = 0} \tag{4.9}$$

Notice that the previous relation for the Einstein tensor will be important in the future.

Example 10: S^2 with round metrix

Consider S^2 with a round metric of radius L:

$$ds^2 = L^2(d\theta^2 + \sin^2\theta d\phi^2)$$

i.e.

$$g_{\theta\theta} = L^2 \qquad g_{\phi\phi} = L^2\sin^2\theta \qquad g_{\theta\phi} = g_{\phi\theta} = 0$$

The connection is

$$\begin{aligned}\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\frac{1}{2}\left(\frac{1}{L^2}\right)2L^2\sin\theta\cos\theta = -\sin\theta\cos\theta \\ \Gamma^\phi_{\phi\theta} &= \Gamma^\phi_{\theta\phi} = \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{1}{2L^2\sin^2\theta}2L^2\sin\theta\cos\theta = \cot\theta \\ \Gamma^\theta_{\theta\theta} &= \Gamma^\theta_{\theta\phi} = \Gamma^\phi_{\phi\phi} = 0\end{aligned}$$

The Riemann tensor has only one independent component $R_{\theta\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi}$ where

$$R^{\theta}{}_{\phi\theta\phi} = \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\mu\theta}\Gamma^{\mu}_{\phi\phi} - \Gamma^{\theta}_{\mu\phi}\Gamma^{\mu}_{\phi\theta} = \sin^2\theta - \cos^2\theta + \cos^2\theta = \sin^2\theta$$

The Ricci tensor is

$$\begin{aligned} R_{\phi\phi} &= \sin^2\theta \\ &= R_{\theta\theta} = R^{\phi}{}_{\theta\phi\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = g^{\phi\phi} g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = 1 \end{aligned}$$

and so we obtained

$$\boxed{R_{\mu\nu} = \frac{1}{L^2} g_{\mu\nu}} \tag{4.10}$$

The scalar curvature is

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{1}{L^2}(1+1) = \frac{2}{L^2}$$

which is obviously constant because of natural invariance.

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Can be proved that the Einstein tensor is always zero in two dimension. Use normal coordinates to prove this statement.

Example 11: The evolving universe

The metric for the evolving universe is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

with connection

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \qquad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$$

The Riemann tensor has following components:

$$\begin{aligned} R^0{}_{i0j} &= \partial_0 \Gamma^0_{ij} - \partial_j \Gamma^0_{i0} + \Gamma^0_{k0}\Gamma^k_{ij} - \Gamma^0_{kj}\Gamma^k_{i0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a})\delta_{ij} - a\dot{a}\delta_{kj}\left(\frac{\dot{a}}{a}\right)\delta^k_i = a\dot{a}\delta_{ij} \quad \Rightarrow \quad \text{independent components} \\ R^0{}_{ijk} &= 0 - 0 + \Gamma^0_{lj}\Gamma^l_{ik} - \Gamma^0_{lk}\Gamma^l_{ij} = 0 = \Gamma^i_{jk}0 \\ R^i{}_{jkl} &= \Gamma^i_{0k}\Gamma^0_{jl} - \Gamma^i_{0l}\Gamma^0_{jk} = (\dot{a})^2(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk}) \end{aligned}$$

For the Ricci tensor

$$\begin{aligned} R_{00} &= R^i{}_{0i0} = g^{ij} R_{0i0j} = -\frac{1}{a^2}\delta^{ij}(a\dot{a}\delta_{ij}) = -3\frac{\ddot{a}}{a} \\ R_{ij} &= R^0{}_{i0j} + R^k{}_{ikj} = a\ddot{a}\delta_{ij} + \dot{a}^2(3\delta_{ij} - \delta_{ij}) = (2\dot{a}^2 + a\ddot{a})\delta_{ij} \\ R_{0i} &= R^\mu{}_{0\mu i} = 0 \end{aligned}$$

The scalar curvature is

$$R = g^{00}R_{00} + g_{ij}R_{ij} = 3\frac{\ddot{a}}{a} + \frac{1}{a^2}\delta^{ij}[(2\dot{a}^2 + a\ddot{a})\delta_{ij}] = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$$

The Einstein tensor is

$$G_{00} = -3\frac{\ddot{a}}{a} + \frac{1}{2}\cdot 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 3\left(\frac{\dot{a}}{a}\right)^2$$
$$G_{ij} = (2\dot{a}^2 + a\ddot{a})\delta_{ij} - 3(a\ddot{a} + \dot{a}^2)\delta_{ij} = -(\dot{a}^2 + 2a\ddot{a})\delta_{ij}$$

Take for instance the exponential of the de Sitter expansion

$$a(t) = e^{Ht} \qquad H = \frac{\dot{a}}{a}$$

Then $\frac{\ddot{a}}{a} = H^2$ and this implies

$$\boxed{R = 12H^2} \tag{4.11}$$

This implies that H^{-1} (Hubble scale) approximates the radius of curvature and

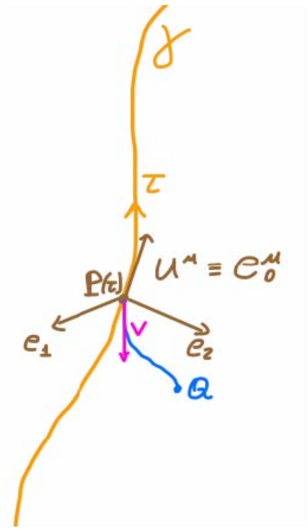
$$\boxed{G_{\mu\nu} = -3H^2 g_{\mu\nu}} \tag{4.12}$$

approximates the empty space-time with positive cosmological constant.

4.2 Geodesic deviation

Carroll, sec. 3.10

There are various possible normal coordinates (LIFs). For instance, the Fermi normal coordinates constructed around a freely falling observer provide the mathematical realisation of Einstein's freely-falling cabin.



In Fermi normal coordinates $\hat{x}^\alpha = (\hat{x}^0, \hat{x}^a)$ with $\hat{x}^0 = \tau$ te connection becomes

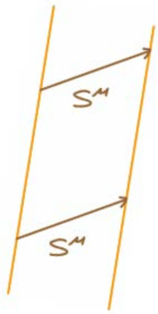
$$\boxed{\hat{\Gamma}^\alpha_{\beta\gamma}(\tau, \vec{0}) \equiv 0 \quad \Leftrightarrow \quad \hat{\partial}_\alpha \hat{g}_{\beta\gamma}|_\gamma = 0} \tag{4.13}$$

and in components

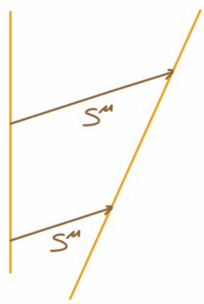
$$\boxed{\begin{aligned} \hat{g}_{00} &= -1 - \hat{R}_{0a0b}|_\gamma \hat{x}^\alpha \hat{x}^b + \dots \\ \hat{g}_{ab} &= \delta_{ab} - \frac{1}{3}\hat{R}_{abcd}|_\gamma \hat{x}^c \hat{x}^d + \dots \\ \hat{g}_{0a} &= \frac{2}{3}\hat{R}_{0bca}|_\gamma \hat{x}^b \hat{x}^c + \dots \end{aligned}} \tag{4.14}$$

The next question is how can we measure the effect of a non-trivial gravitational field. The answer is given measuring the relative motion of two bodies.

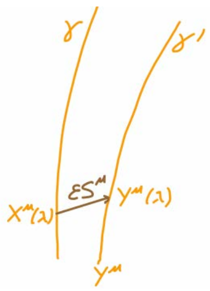
Let us start from the case of flat space, $\Gamma^{\mu}_{\nu\rho} \equiv 0$, and consider two geodesics $X^{\mu}(\lambda)$ and $Y^{\mu}(\lambda) = X^{\mu}(\lambda) + S^{\mu}(\lambda)$ for an affine parameter λ and with constant relative velocity $U^{\mu} \equiv \frac{\mathrm{d}X^{\mu}}{\mathrm{d}\lambda}$ and $V^{\mu} \equiv \frac{\mathrm{d}S^{\mu}}{\mathrm{d}\lambda}$. If $V^{\mu} = 0$ we have the following situation:



while if $V^{\mu} \neq$ constant, but $A^{\mu} \equiv \frac{\mathrm{d}V^{\mu}}{\mathrm{d}\lambda} \equiv 0$, we have vanishing relative acceleration:



For general coordinates and metrics the relative distance on nearby geodesics changes in a way dictated by the Riemann tensor. For instance, take two nearby geodesics $X^{\mu}(\lambda)$ and $Y^{\mu}(\lambda) = X^{\mu}(\lambda) + \varepsilon S^{\mu}(\lambda)$ for an affine parameter λ



In order to evaluate the relative acceleration at a point $p \in \gamma$ let us use normal coordinates $\hat{x}^{\alpha}$ around it, so that $\hat{\Gamma}^{\alpha}_{\beta\gamma}|_p = 0$ we have $\frac{\mathrm{d}^2 \hat{X}^{\alpha}(\lambda)}{\mathrm{d}\lambda^2}\Big|_p = 0$ which implies

$$0 = \left[\frac{\mathrm{d}^2 \hat{Y}^{\alpha}(\lambda)}{\mathrm{d}\lambda^2} + \hat{\Gamma}^{\alpha}_{\beta\gamma}(\hat{Y}) \frac{\mathrm{d}\hat{Y}^{\beta}}{\mathrm{d}\lambda} \frac{\mathrm{d}\hat{Y}^{\gamma}}{\mathrm{d}\lambda} \right] = \varepsilon \left[\frac{\mathrm{d}^2 \hat{S}^{\alpha}}{\mathrm{d}\lambda^2} + \hat{\partial}_{\delta} \hat{\Gamma}^{\alpha}_{\beta\gamma} \frac{\mathrm{d}\hat{X}^{\beta}}{\mathrm{d}\lambda} \frac{\mathrm{d}\hat{X}^{\gamma}}{\mathrm{d}\lambda} \hat{S}^{\delta} \right] \Bigg|_p + O(\varepsilon^2)$$

which implies

$$\frac{\mathrm{d}^2 \hat{S}^{\alpha}}{\mathrm{d}\lambda^2} \bigg|_p = - \hat{\partial}_{\delta} \hat{\Gamma}^{\alpha}_{\beta\gamma} \frac{\mathrm{d}\hat{X}^{\beta}}{\mathrm{d}\lambda} \frac{\mathrm{d}\hat{X}^{\gamma}}{\mathrm{d}\lambda} \hat{S}^{\delta} \bigg|_p \tag{4.15}$$

Then the acceleration is

$$\hat{A}^\alpha|_p = \frac{\mathrm{d}\hat{V}^\alpha}{\mathrm{d}\lambda}\Big|_p = \left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}\hat{S}^\alpha}{\mathrm{d}\lambda} + \hat{\Gamma}^\alpha_{\beta\gamma}\frac{\mathrm{d}\hat{X}^\gamma}{\mathrm{d}\lambda}\hat{S}^\beta\right)\right]\Bigg|_p$$
$$= \left(\frac{\mathrm{d}^2\hat{S}^\alpha}{\mathrm{d}\lambda^2} + \hat{\partial}_\delta\hat{\Gamma}^\alpha_{\beta\gamma}\frac{\mathrm{d}\hat{X}^\delta}{\mathrm{d}\lambda}\frac{\mathrm{d}\hat{X}^\gamma}{\mathrm{d}\lambda}\hat{S}^\beta\right)\Bigg|_p \overset{(4.15)}{=} \underbrace{\left(\hat{\partial}_\delta\hat{\Gamma}^\alpha_{\beta\gamma} - \hat{\partial}_\beta\hat{\Gamma}^\alpha_{\gamma\delta}\right)}_{\hat{R}^\alpha{}_{\gamma\delta\beta}}\frac{\mathrm{d}\hat{X}^\delta}{\mathrm{d}\lambda}\frac{\mathrm{d}\hat{X}^\gamma}{\mathrm{d}\lambda}\hat{S}^\beta$$

So we obtained that in any coordinate system the geodesic deviation is given by

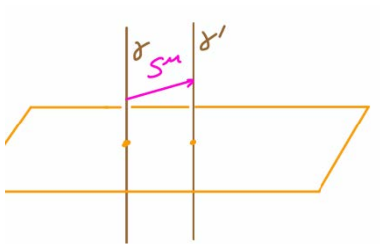
$$\boxed{A^\mu(\lambda) = \frac{\mathrm{D}^2 S^\mu}{\mathrm{d}\lambda^2} = R^\mu{}_{\nu\rho\sigma}(X)\dot{X}^\nu\dot{X}^\rho S^\sigma} \tag{4.16}$$

which is called **geodesic deviation equation**.

This quantifies the fact that the curvature determines how closely freely falling points accelerate towards or away each other, hence implies the gravitational "tidal effects" (for physical coordinates we have independent effects).

Exercise 2

Compute the relative acceleration between two nearby (comoving) coordinate observers in an expanding universe.



You will see a non-vanishing acceleration even though coordinates are constant.

4.3 The minimal coupling principle

Carroll, sec. 4.1

So far, we have only considered the effect of a non-trivial gravitational field, i.e. a curved metric, on the motions of a free particle, basically guided by the WEP ($\subset$ EEP). However, invoking the stronger EEP, we can extend the same logic to more general dynamical quantities. This leads us to the **minimal coupling principle**:

$$\eta \quad\to\quad g_{\mu\nu}(x) \qquad \partial_\mu \quad\to\quad \nabla_\mu$$

Notice that, since $[\nabla_\mu, \nabla_\nu] \sim$ (Riem), there may be ambiguites in presence of higher derivatives.

This prescription is particularly efficient for *Lagrangian densities* of the form

$$\text{Mink} \quad \mathcal{L}_\Phi(\Phi, \partial\Phi) \quad\longrightarrow\quad \mathcal{L}_\Phi(\Phi, \nabla\Phi)$$

where Φ is the set of all gauge and matter fields and $\partial\Phi$ indicates all first derivatives of these fields (all indices on the l.h.s. and on the r.h.s. are contracted respectively through $\eta_{\mu\nu}$ and $g_{\mu\nu}$).

For instance, for a real scalar field:

$$\mathcal{L}_\Phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \quad\longrightarrow\quad \mathcal{L}_\Phi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)$$

(notice that in this case $\nabla_\mu \phi$ is the same as $\partial_\mu \phi$, but this notation emphasises the covariance).

For the EM field:

$$\begin{cases} \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{cases} \longrightarrow \begin{cases} \mathcal{L}_A = -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + A_\mu J^\mu = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \\ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (\text{since } \Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}) \end{cases}$$

Having a scalar Lagrangian density $\mathcal{L}(x)$ we would like to integrate it over (a portion of) spacetime to get an action. We will see how to do this in the next section.

4.4 Integration over curved spaces

Carroll, sec. 2.10

Having a scalar Lagrangian density $\mathcal{L}(x)$ we would like to integrate it over (a portion of) spacetime to get an action. In this section we will see how to do this.

First of all, the naivest guess $\int \mathrm{d}^4 x \, \mathcal{L}(x)$ does not work. Indeed, in GR there is no "canonical" preferred choice of coordinates and then the spac-time *volume element* $\mathrm{d}^4 x$ is ambiguous, being coordinate dependent:

$$\mathrm{d}^4 \tilde{x} = \mathrm{d}^4 x \left| \det \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right|$$

So, we should identify, in a *coordinate-independent way* a volume element $\mathrm{d}vol_4$ to integrate over our space-time.

In an n-dimensional metric space there is natural choice of coordinate independent volume element

$$\boxed{\mathrm{d}vol_n \equiv \mathrm{d}^n x \sqrt{|\det g|} \equiv \mathrm{d}^n \sqrt{|g|}} \tag{4.17}$$

where $\det g(x) = \det(g_{\mu\nu})$. Indeed, $\tilde{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}$ and then $\det \tilde{g}(\tilde{x}) = \left[\det\left(\frac{\partial x}{\partial \tilde{x}}\right)\right]^2 \det g(x)$ and

$$\mathrm{d}\widetilde{vol}_n(\tilde{x}) = \mathrm{d}^n \tilde{x} \sqrt{|\det \tilde{g}(\tilde{x})|} = \mathrm{d}^n x \sqrt{|\det g(x)|} = \mathrm{d}vol_n(x)$$

Notice that if $g_{\mu\nu} = \delta_{\mu\nu}, \eta_{\mu\nu}$ (Euclidean and Minkowskian) then $\mathrm{d}vol_n = \mathrm{d}^n x$. Instead, if $\mathrm{d}s^2|_{\mathbb{E}_3} = \mathrm{d}^r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$ the volume element becomes the usual volume form $\mathrm{d}vol_{\mathbb{E}_3} = \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi \, r^2 \sin\theta$. Therefore eq. (4.17) is consistent with usual volume elements in the flat spaces.

Given canonical volume element eq. (4.17), we can define in an unambiguous and coordinate independent way the integral of any scalar function $f(x)$:

$$\int \mathrm{d}^n x \sqrt{|\det g|} f(x)$$

Notice that the above integral really makes sense if $f(x)$ has support on the coordinate patch. For more general cases, if M is covered by a set of patches U_α, the actual definition of integral of f over M is obtained by "weighted" contributions from different patches by a so-called "partition of unity".[II] For our purposes, this subtlety will be irrelevant and we will just write

$$I = \int_M \mathrm{d}^n x \sqrt{|\det g|} f(x)$$

For the integration by parts following formula is useful:

$$\boxed{\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu)} \tag{4.18}$$

Using this we obtain the formula for the integration by parts:

$$\boxed{\begin{aligned} \int_M \mathrm{d}^n x \sqrt{|\det g|} \nabla_\mu V^\mu f(x) &= \int_M \mathrm{d}^n x \, \partial_\mu (\sqrt{|\det g|} V^\mu) f(x) \\ &= -\int_M \mathrm{d}^n x \sqrt{|\det g|} V^\mu \nabla_\mu f(x) + \text{boundary terms} \end{aligned}} \tag{4.19}$$

[II] See [5] sec. 5.5 for a detailed description of this statement.

Take any (top) n-form in n-dimensional manifold[III]

$$\omega = \frac{1}{n!}\omega_{\mu_1\dots\mu_n}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_n} \equiv \omega_{1\dots n}\mathrm{d}x^1\wedge\dots\wedge\mathrm{d}x^n$$

Under a change of coordinates we have

$$\tilde{\omega}_{1\dots n} = \frac{\partial x^{\mu_1}}{\partial \tilde{x}^1}\dots\frac{\partial x^{\mu_n}}{\partial \tilde{x}^n}\omega_{\mu_1\dots\mu_n}(x) = \det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right)\omega_{1\dots n}(x)$$

If we fix an orientation on the manifold, so that $\det\left(\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right) > 0$, also $\mathrm{d}^n x\,\omega_{1\dots n}(x)$ is well defined. Hence, the natural definition of integral of a top form $\omega_{(n)}$ over an oriented manifold M is

$$\int_M \omega \equiv \int_M \mathrm{d}^n x\,\omega_{1\dots n}(x)$$

[III] Note the absence of the factor $1/n!$ when μ label is omitted.

Chapter 5

Gravitation

5.1 The action in the curved space

Carroll, sec. 4.3

Now we can obtain well defined actions for various kinds of matter fields Φ as follows [I]

$$S_\Phi = \int_M \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}(\Phi,\nabla\Phi)$$

From such action we expect to get fully covariant e.o.m.

For instance, take the Lagrangian for a real scalar field

$$S_\phi = \int \mathrm{d}^4x\,\sqrt{-g}\left[-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right]$$

If we use the least action principle in order to obtain the e.o.m. we obtain for a fixed $\delta\phi$:

$$\begin{aligned}\delta S_\phi &= \int \mathrm{d}^4x\,\sqrt{-g}\left(-\nabla_\mu\delta\phi\nabla^\mu\phi - \delta\phi\frac{\partial V}{\partial\phi}\right)\\ &= \int \mathrm{d}^4x\,\sqrt{-g}\left(\nabla_\mu\nabla^\mu\phi - \frac{\partial V}{\partial\phi}\right)\end{aligned}$$

and imposing this to be zero we obtain the e.o.m. for a real scalar field

$$\boxed{\nabla_\mu\nabla^\mu\phi = \frac{\partial V}{\partial\phi}} \tag{5.1}$$

which is a fully covariant equation.

For the EM field we obtain

$$S_A = \int \mathrm{d}^4x\,\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right) \qquad \text{with} \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

and for a fixed δA_μ we have

$$\begin{aligned}\delta S_A &= \int \mathrm{d}^4x\,\sqrt{-g}\left[-\left(\nabla_\mu\delta A_\nu\right)F^{\mu\nu} + \delta A_\mu J^\mu\right]\\ &= -\underbrace{\int \mathrm{d}4x\sqrt{-g}\nabla_\mu(\delta A_\nu F^{\mu\nu})}_{\int \mathrm{d}^4x\,\partial_\mu\left(\sqrt{-g}\delta A_\nu F^{\mu\nu}\right)=0} + \int \mathrm{d}^4x\,\sqrt{-g}\delta A_\nu\left(\nabla_\mu F^{\mu\nu} - J^\nu\right)\end{aligned}$$

[I]We use the compact notation $\sqrt{-g} := \sqrt{-\det g}$.

hence we obtain the e.o.m.

$$\boxed{\nabla_\mu F^{\mu\nu} = J^\nu \qquad \Rightarrow \qquad \nabla_\mu J^\mu = 0} \tag{5.2}$$

where the relation on the r.h.s. is given by properties of $F^{\mu\nu} = 0$. Again, these e.o.m. are fully covariant.

Clearly, these e.o.m.'s (approximately) reduce to the special relativistic ones by restricting to LIFs e.g. normal coordinates) $\hat{x}^\alpha$ in which $\Gamma \simeq 0$. They provides an explicit realization of the EEP and describe how the dynamics of matter field are affected by a non-trivial gravitational field.

5.1.1 Einstein-Hilbert action

Carroll, sec. 4.2-4.3

So far the metric has been considered non-dynamical, realizing the EEP. We now face the opposite problem:

- What are the e.o.m. of the metric?
- How is it affected by matter and gauge fields?

We will not follow a historical path, but again invoke the *least action principle*. In fact, the matter and gauge Lagrangians $\mathcal{L}(\Phi, \nabla\Phi)$ discussed above already contain interaction terms between gauga/matter fields and the metric.

We can however wonder whether there can be purely gravitational terms. In particular, we should look for a coordinate-independent 2-derivative action:

$$\int \mathrm{d}^4x \sqrt{-g}\mathcal{L}_G(x)$$

So the question is:

- Which 2-derivatives scalar Lagrangians $\mathcal{L}_G(x)$ can we construct out of $g_{\mu\nu}$?

There is only one possibility: $\mathcal{L}_G \propto R$. It is then natural to include in the action the **Einstein-Hilbert term**:

$$S_{\mathrm{EH}}[g] = \frac{1}{2\kappa}\int \mathrm{d}^4x \sqrt{-g}R$$

Notice that $[R] = L^{-2}$, $[\mathrm{d}^4x \sqrt{-g}] = L^4$, $[S] = ET$ and

$$[\kappa] = \left[\frac{L^2}{ET}\right] = \left[\frac{T}{M}\right] = \frac{[G]}{c^3}$$

Indeed we will see that $\kappa \sim \frac{G}{c^3}$.

We are then lead to consider 2-derivative actions of the form

$$S = S_{\mathrm{EH}}[g] + S_\Phi = \frac{1}{2\kappa}\int \mathrm{d}^4x \sqrt{-g}R + S_\Phi$$

Extremization with respect to Φ produces the equations written above. We instead still have to comput the *metric e.o.m.* obtained by extremizing with respect to $g_{\mu\nu}$.

We will do this in two steps, described in following section.

5.2 The Einstein equations

Let's start from the action

$$S = S_{\mathrm{EH}}[g] + S_\Phi = \frac{1}{2\kappa}\int \mathrm{d}^4x \sqrt{-g}R + S_\Phi$$

a) Variation of S_{EH}

We may extremize with respect to $\delta g_{\mu\nu}$. However this is completely equivaent to extremizing with respect to to $\delta g^{\mu\nu}$ since

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \qquad \Rightarrow \qquad \delta g^{\mu\nu} g_{\nu\rho} + g^{\mu\nu}\delta g_{\nu\rho} = 0$$

implies

$$\boxed{\delta g^{\mu\nu} = -\delta g_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu}} \tag{5.3}$$

hence $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ are in one-to-one correspondence.

We get two contributions

$$\delta \int \mathrm{d}^4 x \sqrt{-g} R = \delta \int \mathrm{d}^4 x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \int \mathrm{d}^4 x \,(\underbrace{\delta\sqrt{-g}}_{1} R + \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\, \underbrace{g^{\mu\nu}\delta R_{\mu\nu}}_{2})$$

Let's rewrite 1 and 2. For 1 we observe that $(g = g_{\mu\nu})$

$$\boxed{\log(\det g) = \mathrm{Tr}(\log g)} \tag{5.4}$$

indeed $\det g = \prod_a \lambda_a$ and $\mathrm{Tr}(\log g) = \sum_a \log \lambda_a$. Then

$$\begin{aligned}\delta \det g &= \delta e^{\log(\det g)} = \delta e^{\mathrm{Tr}\log g} = e^{\mathrm{Tr}\log g}\, \mathrm{Tr}(\delta \log g) \\ &= (\det g)\, \mathrm{Tr}\, g^{-1}\delta g = (\det g) g^{\mu\nu}\delta g_{\mu\nu} = -(\det g)\delta g^{\mu\nu} g_{\mu\nu}\end{aligned}$$

But $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta \det g$ and so

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\delta g^{\mu\nu} g_{\mu\nu}} \tag{5.5}$$

For 2 since $\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu}$ we have

$$\begin{aligned}\delta R^\rho{}_{\mu\sigma\nu} &= \delta[\partial_\sigma \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - (\sigma \leftrightarrow \nu)] \\ &= \partial_\delta \Gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu \delta\Gamma^\rho_{\mu\sigma} - \delta\Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\lambda\nu}\delta\Gamma^\lambda_{\mu\sigma}\end{aligned}$$

Recall that $\delta\Gamma^\mu_{\nu\rho}$ is a tensor

$$\nabla_\sigma \delta\Gamma^\rho_{\mu\nu} = \partial_\sigma \delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\delta\Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\nu\sigma}\delta\Gamma^\rho_{\mu\lambda}$$

And using the latter and the corresponding equation for $\nu \leftrightarrow \sigma$ we obtain

$$\delta R^\rho{}_{\mu\sigma\nu} = \nabla_\sigma \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\mu\sigma}$$

which implies

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\sigma_{\mu\sigma}$$

and

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\rho \left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\rho\mu}\delta\Gamma^\sigma_{\mu\sigma}\right) \equiv \nabla_\rho \delta V^\rho$$

Thanks to the results of sec. 4.4 we can write the result in the form of a covariant derivative whose integration vanishes because of vanishing boundary terms

$$\int \mathrm{d}^4 x \sqrt{-g}\nabla_\rho \delta V^\rho \equiv \int \mathrm{d}^4 x\, \partial_\rho \left(\sqrt{-g}\delta V^\rho\right) = 0$$

Putting these results together, we then arrive at the conclusion that

$$\boxed{\begin{aligned}\delta S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int \mathrm{d}^4 x \sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \\ &\equiv \frac{1}{2\kappa}\int \mathrm{d}^4 x \sqrt{-g}\delta g^{\mu\nu} G_{\mu\nu}\end{aligned}} \tag{5.6}$$

where we used the definition of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

b) Variation of S_Φ

In general we simply define the **energy-momentum** tensor (or **stress-energy** tensor) $T_{\mu\nu}$ by

$$\boxed{\delta S_\Phi = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu} \equiv \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\delta g_{\mu\nu}T^{\mu\nu}} \tag{5.7}$$

Notice that, by definition, $T_{\mu\nu}$ is *symmetric*

$$\boxed{T_{\mu\nu} = T_{\nu\mu}} \tag{5.8}$$

It does not always coincides with the "canonical" $\tilde{T}_{\mu\nu}$ but Belifante showed that one can "improve" $\tilde{T}_{\mu\nu}$ to get $\tilde{T}^{\mu\nu}$ to get $T^{\mu\nu}$.

Then the application of the least action principle $\delta S = \delta S_{\text{EH}} + \delta S_\Phi = 0$ leads to the **Einstein equations** (EE)

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}} \tag{5.9}$$

Notice that $\nabla^\mu G_{\mu\nu} = 0$ implies

$$\boxed{\nabla^\mu T_{\mu\nu} = 0} \tag{5.10}$$

5.3 The energy-momentum tensor

Carroll, sec. 1.9-1.10

Let's analyze the energy momentum tensor $T_{\mu\nu}$ in order to understand its meaning and properties.

Consider the Minkowski spacetime, we have

$$\boxed{P^\mu = \int \mathrm{d}^3x\,T^{0\mu}(x) \qquad \Rightarrow \qquad E = \int \mathrm{d}^3x\,T^{00} \qquad P^i = \int \mathrm{d}^3x\,T^{0i}} \tag{5.11}$$

hence $T^{00}(x)$ and $T^{0i}(x)$ should be meant respectively as "energy density" and "(linear) momentum density".

On the other side, the interpretation of T^{ij} requires more work. Let

$$P_V^j(t) := \int_V \mathrm{d}^3x\,T^{0j}(x) \tag{5.12}$$

then using Stoke's theorem we obtain[II]

$$\frac{\mathrm{d}P_V^j}{\mathrm{d}t} = -\int_V \mathrm{d}^3x\,\partial_i T^{ij} = -\int_S \mathrm{d}S_i\,T^{ii} =: F_V^j \tag{5.13}$$

with $S := \partial V$ is the boundary of V and $\mathrm{d}\mathbf{S}$ is the outward pointing surface form on S:

Hence we can interpret the object F_V^j as the **force exerted on V** and consequently T^{ij} as the force per unit area pushing in the j-direction surface portion. For this reason T^{ij} is called **(3D) stress tensor**. In particular $T^{ii} =: \mathcal{P}$ is the **pressure**, while T^{ij} for $i \neq j$ are **shear-terms**.

In more general spacetimes, this interpretation holds only locally and approximately in a LIF (e.g. of normal coordinates).

[II] The minus sign introduced is due to $\frac{\mathrm{d}}{\mathrm{d}t}T^{0j} = -\partial_i T^{ij}$ given by Equation (5.11) in the flat metric.

5.3.1 The point particle case

In fact, we will see that this is a subtle concept, once one takes into account the particle's backreaction. But let us for the moment ignore these issues. We already know the particle's action:

$$\begin{aligned} S_{\text{part}} &= \frac{1}{2}\int \mathrm{d}\lambda\, [e^{-1} g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu - m^2 e] \\ &= \frac{1}{2}\int \mathrm{d}\lambda \int \mathrm{d}^4x\, \delta^4(x - X(\lambda))[e^{-1} g_{\mu\nu}(x)\dot{X}^\mu \dot{X}^\nu - m^2 e] \end{aligned} \tag{5.14}$$

Now consider an infinitesimal metric variation $\delta g_{\mu\nu}$:

$$\begin{aligned} \delta S_{\text{part}} &= \frac{1}{2}\int \mathrm{d}^4x\, \delta g_{\mu\nu}(x) \int \mathrm{d}\lambda\, e^{-1} \delta^4(x - X(\lambda))\dot{X}^\mu \dot{X}^\nu \\ &=: \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu} \end{aligned} \tag{5.15}$$

i.e.

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = \int \mathrm{d}\lambda\, e^{-1}\dot{X}^\mu \dot{X}^\nu \frac{\delta^4(x - X(\lambda))}{\sqrt{-g(x)}}} \tag{5.16}$$

where the term $\frac{\delta^4(x-X(\lambda))}{\sqrt{-g(x)}}$ transform as a scalar.

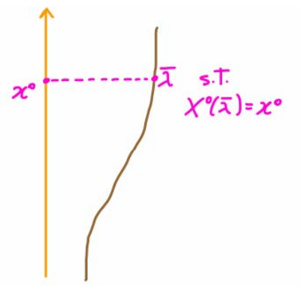
We can take proper λ: $e(\lambda) = 1$ so that $\dot{X}^\mu = p^\mu(\lambda)$ with $p^\mu p_\mu = -m^2$, in this case

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = \int \mathrm{d}\lambda\, p^\mu(\lambda) p^\nu(\lambda) \frac{\delta^4(x - X(\lambda))}{\sqrt{-g(x)}}} \tag{5.17}$$

is covariant. Furthermore we can rewrite

$$\delta^4(x - X(\lambda)) = \delta(x^0 - X^0(\lambda))\delta^3(\mathbf{x} - \mathbf{X}(\lambda)) = \frac{\delta(\lambda - \bar{\lambda})}{\dot{X}^0(\bar{\lambda})}\delta^3(\mathbf{x} - \mathbf{X}(x^0)) = \frac{\delta(\lambda - \lambda_0)}{p^0(x^0)}\delta^3(\mathbf{x} - \mathbf{X}(x^0)) \tag{5.18}$$

where in the last step we assumed $\dot{X}^0 > 0$:



Hence

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = p^\mu(t) p^\nu(t) \frac{\delta^3(\mathbf{x} - \mathbf{X}(t))}{p^0(t)\sqrt{-g(x, \mathbf{x})}}} \tag{5.19}$$

In a LIF $\hat{x}^\alpha = (\hat{x}^0, \hat{x}^a)$ we get

$$\begin{aligned} \hat{T}^{00} &= \hat{p}^0 \delta^3(\mathbf{x} - \mathbf{X}(t)) \quad \textbf{energy density} \\ \hat{T}^{0a} &= \hat{p}^a \delta^3(\mathbf{x} - \mathbf{X}(t)) \quad \textbf{momentum density} \end{aligned} \tag{5.20}$$

5.3.2 Perfect fluids

At large enough distances, many macroscopic systems can be approximated as **perfect fluids**, that is, such that any observer solidal with any fluid element, sees the fluid around him as isotropic. This happens if the free path of the interacting particles is much shorter than the relevant length scales.

In the rest LIF $\hat{x}^\alpha$ of each fluid element, we then have

$$\hat{T}^{00} = \varepsilon \quad , \quad \hat{T}^{ab} = \mathcal{P}\delta^{ab} = 0 \quad , \quad \hat{T}^{0a} = 0 \tag{5.21}$$

where ε is the **rest energy density** and $\mathcal{P}$ is the **rest pressure**. In matrix notation:

$$\hat{T}^{\alpha\beta} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix} \tag{5.22}$$

Since $\hat{u}^\alpha = (1, \mathbf{0})$ and $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$ then

$$\hat{T}^{\alpha\beta} = (\varepsilon + \mathcal{P})\hat{u}^\alpha\hat{u}^\beta + \mathcal{P}\eta^{\alpha\beta} \tag{5.23}$$

which can be immediately covariantized into

$$\boxed{T^{\mu\nu} = (\varepsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}} \tag{5.24}$$

Perfect fluids of particles

We already know that for point particles we have

$$g^{\mu\nu}(x) = \sum_I \frac{p_I^\mu p_I^\nu}{p_I^0}\frac{\delta^3(\mathbf{x} - \mathbf{X}(t))}{\sqrt{-g}} = \sum_I \int \mathrm{d}\lambda \, p_I^\mu(\lambda)p_I^\nu(\lambda)\frac{\delta^4(x - X_I(\lambda))}{\sqrt{-g}} \tag{5.25}$$

The perfect fluid regime then implies that, in the local fluid rest frame

$$\varepsilon = \sum_I E_I \delta^3(\hat{x} - \hat{X}_I(t)) \quad , \quad \mathcal{P} = \frac{1}{3}\sum_I \frac{|\mathbf{p}_I|^2}{E_I}\delta^3(\hat{x} - \hat{X}_I(t)) \tag{5.26}$$

Since $E_I = \sqrt{m_I^2 + |\mathbf{p}_I|^2}$ then we have $|\mathbf{p}_I|^2 \le E_I^2$ and hence

$$\boxed{0 \le \mathcal{P} \le \frac{1}{3}\varepsilon} \tag{5.27}$$

We can now go further in our analysis in some special regimes, which will be described in the following paragraphs.

Cool non-relativistic gas

Such regime correspond to the case $E_I \simeq m_I + \frac{\mathbf{p}_I^2}{2m}$, then we have

$$\begin{aligned} \mathcal{P} &\simeq \frac{1}{3}\sum_I \frac{|\mathbf{p}_I|^2}{m_I^2}\delta^3(\hat{x} - \hat{X}(t)) \\ \varepsilon &\simeq \sum_I (m_I + \frac{|\mathbf{p}_I|^2}{2m_I})\delta^3(x - \hat{X}_I(t)) = \rho(x) + \frac{3}{2}\mathcal{P}(x) \end{aligned} \tag{5.28}$$

where $\rho(x)$ denotes the **mass density in the fluid comoving frame**. Then the space-like matrix elements of $\hat{T}^{\mu\nu}$ are

$$\hat{T}^{ab} = \begin{pmatrix} \rho + \frac{3}{2}\mathcal{P} & 0 \\ 0 & \mathcal{P}\delta^{ab} \end{pmatrix} \tag{5.29}$$

and then

$$\boxed{T^{\mu\nu}_{\text{non-rel}} \simeq (\rho + \frac{5}{2}\mathcal{P})u^\mu u^\nu + \mathcal{P} g^{\mu\nu}} \tag{5.30}$$

In particular, the **extreme** non-relativistic limit $\mathcal{P} \simeq 0$ correspond to the **pressureless dust**

$$\boxed{g^{\mu\nu}_{\text{dust}} = \varepsilon u^\mu u^\nu} \tag{5.31}$$

with $\varepsilon = \rho$.

Ultra-relativistic regime

Such regime correspond to the case $E_I \simeq |\mathbf{p}_I|$, then we have

$$\boxed{\mathcal{P} \simeq \frac{1}{3}\varepsilon} \tag{5.32}$$

For instance, this is the case for the **gas of photons**. Then

$$\boxed{T^{\mu\nu} = \frac{4}{3}\varepsilon u^\mu u^\nu + \frac{1}{3}\varepsilon g^{\mu\nu}} \tag{5.33}$$

which implies $T^\mu_\mu = 0$.

A particular perfect fluid

Consider a **cosmological constant term**

$$-\frac{\Lambda}{k}\int \mathrm{d}^4x\,\sqrt{-g} \subset S_M \tag{5.34}$$

where $[\Lambda] = L^{-2}$ is known as **cosmological constant**. Then we see that

$$\boxed{g^{\mu\nu}_{\Lambda} = -\frac{\Lambda}{k} g^{\mu\nu}} \tag{5.35}$$

correspond to a perfect fluid with

$$\boxed{\varepsilon = \frac{\Lambda}{k}} \tag{5.36}$$

5.3.3 Example: The expanding universe

Example 12: The expanding universe

Consider the evolving universe described by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{5.37}$$

filled by a spatially homogeneous perfect fluid which is at rest in comoving coordinates $\mathbf{x}$. Then, since $u^\mu = (1, \mathbf{0})$ we get

$$T_{\mu\nu} = (\varepsilon + \mathcal{P})u_\mu u_\nu + \mathcal{P} g_{\mu\nu} = T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 \\ 0 & a^2\mathcal{P}\delta_{ij} \end{pmatrix} \tag{5.38}$$

with time-dependent energy and pressure, $\varepsilon = \varepsilon(t)$, $\mathcal{P} = \mathcal{P}(t)$. Then assume an **equation of state**

$$\boxed{\mathcal{P} = w\varepsilon} \tag{5.39}$$

where w is choosen depending on the following cases:

$$\begin{cases} w &= 0 \quad \text{matter dominated case} \\ w &= \frac{1}{3} \quad \text{radiation dominated case} \\ w &= -1 \quad \text{vacuum dominated case} \end{cases} \tag{5.40}$$

We already computed non-vanishing Christoffel symbols:

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \quad , \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \tag{5.41}$$

Then one can check (*exercize!*) than **continuity equation** $\nabla^\mu T_{\mu\nu} = 0$ gives

$$\boxed{\frac{\dot{\varepsilon}}{\varepsilon} = -3(1+w)\frac{\dot{a}}{a}} \tag{5.42}$$

We then get

$$\boxed{\varepsilon(t) = \varepsilon_0[a(t)]^{-3(1+w)}} \tag{5.43}$$

i.e.

$$\begin{aligned} &\varepsilon \propto \frac{1}{a^3} \text{ for matter domainated universe } (\sim \text{rescaling of physical volume}) \\ &\varepsilon \propto \frac{1}{a^4} \text{ for radiation domainated universe } (\sim \text{rescaling of physical volume + cosmological redshift}) \\ &\varepsilon = \varepsilon_0 \text{ for vacuum domainated universe (with } \Lambda = k\varepsilon_0) \end{aligned} \tag{5.44}$$

We also computed the Einstein tensor

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 \quad , \quad G_{ij} = -(\dot{a}^2 + 2a\ddot{a})\delta_{ij} \tag{5.45}$$

Then, from Einstein's equation $G_{\mu\nu} = kT_{\mu\nu}$ we get

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}k\varepsilon} \tag{5.46}$$

and

$$-(\dot{a}^2 + 2a\ddot{a}) = k\mathcal{P}a^2 \quad \Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = -\frac{k}{6}(\varepsilon + 3\mathcal{P})} \tag{5.47}$$

these are known as the **Friedmann equations** for (3d-flat) FRW universes.
Moreover assuming $\mathcal{P} = w\varepsilon$ we get

$$\begin{aligned} \mathcal{P} = w\varepsilon &\implies \varepsilon \propto a^{-3(1+w)} \implies \dot{a}^2 \propto a^{-1-3w} \\ &\implies a^{\frac{1}{2}+\frac{3}{2}w}\mathrm{d}a \propto \mathrm{d}t \implies a(t)^{\frac{3}{2}(1+w)} \propto t \\ &\implies a(t) \propto t^{\frac{2}{3(1+w)}} \end{aligned} \tag{5.48}$$

We also notice that the case $w = -1$ corresponds to $\varepsilon = -\mathcal{P} = \varepsilon_0$ which gives a **constant** Hubble parameter

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{1}{3}k\varepsilon_0} = \sqrt{\frac{\Lambda}{3}} \tag{5.49}$$

and then

$$a(t) = a_0 e^{Ht} = a_0 e^{\sqrt{\frac{\Lambda}{3}}t} \tag{5.50}$$

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