Chapter 1

Vortices

[**Shifman:2012**]

The second quantum soliton that we consider is the vortex in $2+1$ dimensions in a model that in high-energy *Abelian Higgs mode* and in $3+1$ dimensions was the first model proposed to make massive gauge fields in a gauge theory without losing gauge-invariance.
In its non-relativistic version in condensed matter it described by the *Landau-Ginzburg model* and in 3 space dimensions it was proposed as a phenomenological model for superconductors.
Our discussion will be performed in the Lagrangian formalism, starting from the classical model.

1.1 Classical treatment

The classical Lagrangian

The field content is made of a complex scalar field ϕ (whose complex conjugate is denote by ϕ^*) and a $U(1)$ gauge field A_μ. The classical relativistic Lagrangian is

$$\mathcal{L} = -\frac{1}{4e^2}F^2_{\mu\nu} + |D^\mu\phi|^2 - \lambda(|\phi|^2 - v^2)^2 \tag{1.1}$$

where e is the electric charge, the covariant derivative is defined by

$$D_\mu\phi = (\partial_\mu - in_e A_\mu)\phi \tag{1.2}$$

and n_e is the electric charge of ϕ in units of e.
The non-relativistic Euclidean version replaces $|D^0\phi|^2$ by a first order term

$$|D^0\phi|^2 \quad \rightarrow \quad \phi^*(\partial_0 - in_e A_0)\phi \tag{1.3}$$

For the model of superconductivity ϕ is a field representing the large distance behaviour of the Cooper pairs generated by phonon attraction and $n_e \equiv 2$. The vortices that will be discusse later in fact really appear in nature.

Application of vortices

A lattice version of such vortices, called *Abrikosov*[I] *vortices*, is the equilibrium state of a class of superconductors in the presence of a magnetic field, orthogonal to the surface of the superconductors, whose direction will be denoted by z. The z-dependence is then trivial, and in the gauge $A_z = 0$ the $3+1$ model reduces to a $2+1$ model.
Notice that a typical characteristics of superconductors is the expulsion of the magnetic field (*Meissner effect*), but there are two behaviours of superconducting materials in this respect, called *type I* and *type II*. In type I the magnetic flux is completely expelled from the bulk of the material, whereas in type II it penetrates in the superconductor in tubes whose two-dimensional cross section are the vortices, as shown

[I] Nobel prize in the 2003.

in fig. 1.1. Each of these tubes contains a flux $\frac{h}{e}$ and at equilibrium if they are sufficiently many[II] they are arranged in a triangular lattice, the *Abrikosov lattice*.

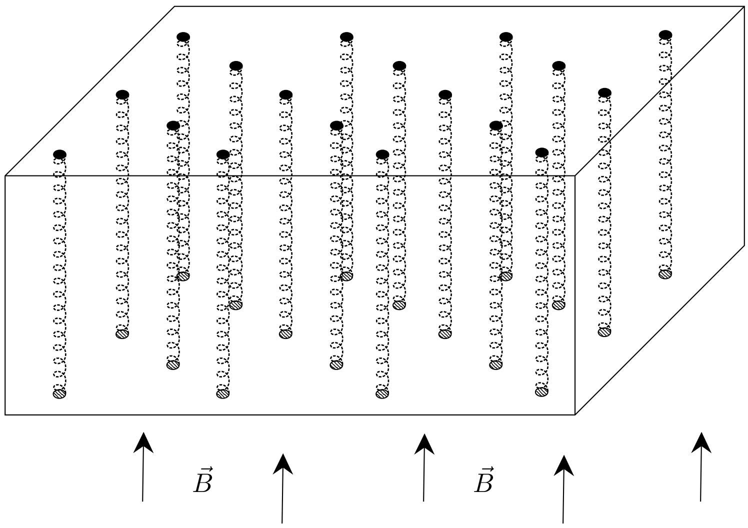

Figure 1.1: Vortices created by the magnetic field in a type II superconductor.

Gauge symmetry, energy density and Higgs mechanism

The model is invariant under the $U(1)$ gauge transformation

Mi è chiaro che nella trasformazione di gauge di A_μ lei voglia rendere esplicita la forma di Maurer-Cartan, ma non comprendo perché invece la forma $\partial_\mu \beta$ dovrebbe essere imprecisa.

$$\begin{cases} \phi(x) & \rightarrow \quad e^{i\beta(x)}\phi(x) \\ A_\mu(x) & \rightarrow \quad A_\mu(x) + \dfrac{1}{in_e} e^{-i\beta(x)}\partial_\mu e^{i\beta(x)} = A_\mu(x) + \dfrac{1}{n_e}\partial_\mu \beta(x) \end{cases} \tag{1.4}$$

Let us first consider static configurations in the *temporal* gauge $A_0 = 0$, so that there is no difference between relativistic and non-relativistic models. The energy is given by

$$\mathcal{E}\big(\boldsymbol{A}(\boldsymbol{x}), \phi(\boldsymbol{x})\big) = \int \mathrm{d}^2x \, \frac{1}{4e^2}F_{ij}^2 + |D_i\phi|^2 + \lambda\big(|\phi|^2 - v^2\big)^2 \tag{1.5}$$

and it has global minima at

$$\phi(\boldsymbol{x}) = ve^{i\theta} \quad \text{for} \quad \theta \in [0, 2\pi) \quad , \quad A_\mu(\boldsymbol{x}) = 0 \tag{1.6}$$

However by gauge invariance this configuration is physically equivalent to

$$\phi(\boldsymbol{x}) = ve^{i[\theta + \beta(\boldsymbol{x})]} \quad , \quad A_\mu(\boldsymbol{x}) = \frac{1}{in_e} e^{-i\beta(\boldsymbol{x})}\partial_\mu e^{i\beta(\boldsymbol{x})} \tag{1.7}$$

with $\beta(\boldsymbol{x})$ globally defined of compact support (indeed it cannot act on the boundary of the spacetime, otherwise it changes the boundary conditions).
The existence of degenerate global minima (up to gauge equivalence) labelled by θ suggests that the global $U(1)$ symmetry is spontaneously broken. Indeed the degeneracy of the vacuum cannot be regarded as a manifestation of the gauge symmetry of the theory, since gauge symmetries cannot act at boundaries of the spacetime, whereas the action of $e^{i\theta}$ extends also at the infinity. The presence of spontaneously symmetry breaking of the theory can be shown perturbatively (as in the SM), non-perturbatively in some specific gauge (e.g. in the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, for which a change of boundary conditions at ∞ is impossible), and using a non-local order parameter (which turns out to have a non zero expectation value independent from the gauge choice).

[II]In order to increase the number of tubes one can increase the strength of the magnetic field.

Due to the *Anderson-Higgs mechanism* (in high energy physics also called *BEH*, due to Brout-Englert-Higgs[III]), let $v = \langle \phi \rangle$ be the (real) expectation value of ϕ in the broken symmetry phase, the gauge field A_μ acquire a mass gap, whose inverse in condensed matter (in the Landau-Ginzburg model) is called *penetration depth*, given in the quadratic approximation by

Qui ho tolto il fattore e, i quanto non apparrebbe espandendo la lagrangiana rispetto al campo di Higgs.

$$m_A = \sqrt{2} v n_e \tag{1.8}$$

and also, writing

$$\phi(x) = (v + \chi(x))\, e^{i\theta(x)} \quad \text{with} \quad \chi(x), \theta(x) \text{ real fields} \tag{1.9}$$

the Higgs field $\chi(x)$ acquires a mass gap , whose inverse in condensed matter is called *coherence length*, which in the quadratic approximation is given by

$$m_H = 2\sqrt{\lambda} v \tag{1.10}$$

This can be easily proved: inserting the expansion eq. (1.9) in the Lagrangian eq. (1.1) we get

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4e^2} F_{\mu\nu}^2 + |(\partial_\mu - i n_e A_\mu)[(v+\chi)\, e^{i\theta}]|^2 - \lambda(|\,(v+\chi)\, e^{i\theta}|^2 - v^2)^2 \\ &= -\frac{1}{4e^2} F_{\mu\nu}^2 + |iv\partial_\mu \theta + \partial_\mu \chi - i n_e A_\mu (v+\chi)|^2 - \lambda(2v\chi + \chi^2)^2 \end{aligned} \tag{1.11}$$

In terms of eq. (1.9) the gauge transformation (1.4) become

$$\begin{cases} \chi(x) & \to & \chi(x) \\ \theta(x) & \to & \theta(x) + \beta(x) \\ A_\mu(x) & \to & A_\mu(x) + \dfrac{1}{n_e}\partial_\mu \beta(x) \end{cases} \tag{1.12}$$

hence acting on $\mathcal{L}$ with a transformation of the form eq. (1.12) with $\beta(x) = -\theta(x)$ we finally get

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4e^2} F_{\mu\nu}^2 + |\partial_\mu \chi - i n_e v A_\mu - i n_e A_\mu \chi|^2 - \lambda(2v\chi + \chi^2)^2 \\ &= -\frac{1}{4e^2} F_{\mu\nu}^2 + (\partial_\mu \chi)^2 + n_e^2 v^2 A_\mu^2 + 2n_e^2 v A_\mu^2 \chi - 4\lambda v^2 \chi^2 - 4\lambda v \chi^3 - \lambda \chi^4 \\ &= \underbrace{-\frac{1}{4e^2} F_{\mu\nu}^2 + \frac{1}{2} m_A^2 A_\mu^2 + (\partial_\mu \chi)^2 - \frac{1}{2} m_H^2 \chi^2}_{\text{kinetic term}} \underbrace{+ 2n_e^2 v A_\mu^2 \chi - 4\lambda v \chi^3 - \lambda \chi^4}_{\text{interaction term}} \end{aligned} \tag{1.13}$$

hence the masses of A_μ and the χ are exactly m_A and m_H, as we claimed above. Notice that the previous expression of the Lagrangian, eq. (1.13), holds in the specific gauge where θ vanishes, which is usually called *unitary gauge*.

Finite energy solutions - Vortices

Let us now impose the finiteness of the energy. The last term in eq. (1.5) forces

$$|\phi| \xrightarrow[|\boldsymbol{x}| \to \infty]{} v e^{if(\alpha)} \tag{1.14}$$

where α is the angle in polar coordinates of $\boldsymbol{x}$ at ∞, $\boldsymbol{x} = |\boldsymbol{x}| e^{i\alpha}$, and f is a real function satisfying $e^{if(2\pi)} = e^{if(0)}$. The first term in eq. (1.5) implies that at ∞, $F_{ij} = 0$, i.e. A_i is a pure gauge. The second term in eq. (1.5) implies that

$$|e^{if(\alpha)} \partial_j e^{if(\alpha)} - i n_e A_j| \xrightarrow[|\boldsymbol{x}| \to \infty]{} 0 \tag{1.15}$$

so A_j, asymptotically, is the pure gauge

$$A_j = \frac{1}{i n_e} e^{-if(\alpha)} \partial_j e^{if(\alpha)} = \frac{1}{i n_e} \partial_j \log e^{if(\alpha)} \quad \text{for} \quad |\boldsymbol{x}| \to \infty \tag{1.16}$$

[III] Nobel prizes in the 2013.

As a consequence of eq. (1.14) we have, asymptotically, the following map

$$e(\boldsymbol{x}) := \frac{\phi(\boldsymbol{x})}{|\phi(\boldsymbol{x})|} \quad : \quad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ \alpha & \longmapsto & e^{if(\alpha)} \end{array} \tag{1.17}$$

which, if continuous, define a homotopy class in $\pi_1(S^1) \simeq \mathbb{Z}$.[IV] These maps are classified by the number of times that $e^{if(\alpha)}$ covers the circle labelled by α. For example, if $f(\alpha) = n\alpha$, the homotopy class of $e^{if(\alpha)}$ is $[e^{if(\alpha)}] = n \in \pi_1(S^1)$.
In fact, using continuous gauge transformations one can always put $f(\alpha)$ in the form $n\alpha$ for some $n \in \mathbb{Z}$. Let's prove this. From eq. (1.16) we have that

$$\begin{aligned} \lim_{R\to\infty} \oint_{|\boldsymbol{x}|=R} A_j \mathrm{d}x^j &= \int_0^{2\pi} \mathrm{d}\alpha \, \frac{1}{in_e} \frac{\mathrm{d}x^j}{\mathrm{d}\alpha} \partial_j \log e^{if(\alpha)} \\ &= \frac{1}{in_e} \int_0^{2\pi} \mathrm{d}\alpha \, \frac{\mathrm{d}}{\mathrm{d}\alpha} \log e^{if(\alpha)} \\ &= \frac{1}{n_e} \big(f(2\pi) - f(0) \big) = 2\pi \frac{n}{n_e} \end{aligned} \tag{1.18}$$

hence we can compute n starting from A_j. If $\beta(x)$ labels a gauge transformation globally defined in $\mathbb{R}^2$ and its derivative is continuous (i.e. $\beta \in \mathcal{C}^1$) then by Stokes theorem

$$\oint_{|\boldsymbol{x}|=R} \partial_j \beta(x) \mathrm{d}x^j = 0 \tag{1.19}$$

hence, for a given $f(\alpha)$ and the corresponding n, we can take $\beta(x)$ such that

$$\beta(x) \xrightarrow[|\boldsymbol{x}|\to\infty]{} -f(\alpha) + n\alpha \tag{1.20}$$

so that we can replace $f(\alpha)$ by $n\alpha$.
Using Stokes theorem one can also derive

$$\lim_{R\to\infty} \oint_{|\boldsymbol{x}|=R} A_j \frac{\mathrm{d}x^j}{2\pi} = \int_{\mathbb{R}^2} \frac{\epsilon_{ij} F^{ij}}{2\pi} \mathrm{d}^2 x = \int_{\mathbb{R}^2} \frac{B(x)}{2\pi} \mathrm{d}^2 x \tag{1.21}$$

where $B(x)$ is the magnetic field associated to F^{ij}. Hence n can also be interpreted as the magnetic flux, and rescaling A_μ to give the standard form of the Maxwell free action $-\frac{1}{4}F_{\mu\nu}^2$, we get

$$\int B \, \mathrm{d}^2 x = n \frac{2\pi}{e} \hbar = n \frac{h}{e} \tag{1.22}$$

where we restored $\hbar$. Notice that $\frac{h}{e}$ is the unit of quantum flux. The number n is also called *vorticity* and its topological origin is at the basis of the stability of the vortices that we will now discuss in details. The configurations minimizing the energy for $n = \pm 1$ are called *vortex* and *anti-vortex* respectively. Plotting the complex field ϕ as a vector, we get fig. **??**, which shows clearly the analogy between our discussion and well known vortices in water.

Part of the text is omitted from the compiled subfile, in order to make the compilation faster. You can find it in the main file. To avoid this, remove " \onlyinmainfile{...}" from the code.

[IV] The homotopy class $\pi_1(S^1)$ is the set of equivalent maps from S^1 to S^1, where the equivalence is defined as follows: two maps are said equivalent if it is possible to deform one map into the other continuously, or better, if exists a continuous map interpolating between the two initial ones. The set $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ since to each equivalence class in $\pi_1(S^1)$ can be associated the number of windings of the map around the second circle.

The vortex solution

According to previous considerations, we want to see how the vortices are localized (which is a property required for solitons), at least qualitatively. Let us consider the case $n = 1$, $f(\alpha) = \alpha$, and in the usual notation for polar coordinates we set $r \equiv |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2}$ and $\varphi \equiv \alpha$. Using $\varphi = \arctan \frac{x^2}{x^1}$ we get the following asymptotic behaviours

$$\begin{aligned} A_i(\boldsymbol{x}) &= \frac{1}{n_e}\partial_i\varphi = -\frac{1}{n_e}\epsilon_{ij}\frac{x^j}{r^2} \\ \phi(\boldsymbol{x}) &= ve^{i\varphi} = v\frac{x^1 + ix^2}{r} \end{aligned} \qquad \text{for} \quad |\boldsymbol{x}| \to \infty \tag{1.23}$$

We can improve our approximation in finite regions of the spacetime introducing some corrections to our fields, in such a way that we can really obtain a solution of the equations of motion, in terms of some functions $g_A(r)$ and $g_R(r)$:[V]

$$\begin{aligned} A_i(\boldsymbol{x}) &= \frac{1}{n_e}\partial_i\varphi(1 - g_A(r)) = -\frac{1}{n_e}\epsilon_{ij}\frac{x^j}{r^2}(1 - g_A(r)) \\ \phi(\boldsymbol{x}) &= ve^{i\varphi}(1 - g_H(r)) = v\frac{x^1 + ix^2}{r}(1 - g_H(r)) \end{aligned} \tag{1.24}$$

We claim that a characteristic feature of vortex configurations is that F_{ij} is significantly different from 0 in a region of radius $O(m_A^{-1})$ from the center of the vortex and $\frac{|\phi|}{v}$ is significantly different from 1 in a region of radius $O(m_H^{-1})$, or more precisely

$$\begin{aligned} g_A(r) &\sim e^{-m_A r} \\ g_H(r) &\sim e^{-m_H r} \end{aligned} \tag{1.25}$$

We will show this with some simplifications. Furthermore, continuity at 0 implies that

$$g_A(0) = 1 = g_H(0) \tag{1.26}$$

so that

$$\phi(0) = 0 \tag{1.27}$$

In polar coordinates we can rewrite the first of eq. (1.24) as

$$A_r = 0 \quad \text{and} \quad A_\varphi = \frac{1}{n_e}(1 - g_A(r)) \tag{1.28}$$

Since

$$F_{r\varphi} = \partial_r A_\varphi - \cancel{\partial_\varphi A_r} \tag{1.29}$$

we get (recall that in polar coordinates the metric which we use to raise indices is not the flat one)

$$\partial^r F_{r\varphi} = \partial_r \frac{1}{r}\partial_r A_\varphi \tag{1.30}$$

so that the equation of motion for A_φ become

$$\partial_r \frac{1}{r}\partial_r A_\varphi = 2i\frac{n_e e^2}{r}\left(\phi^*\partial_\varphi\phi - in_e\phi^* A_\varphi\phi\right) \tag{1.31}$$

Introducing our ansatz eq. (1.24) we get

$$r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}g_A(r) - 2n_e^2e^2v^2(1 - g_H(r))^2 g_A(r) = 0 \tag{1.32}$$

and linearizing at large r (hence $g_H \sim 0$) we get

$$r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}g_A(r) - m_A^2 g_A(r) = 0 \tag{1.33}$$

[V] Such functions depends only on r in such a way that we preserve the behaviour of the vortex.

Putting $g_A(r) = r^\gamma \tilde{g}_A(r)$ we have

$$(2\gamma - 1)r^{\gamma-1}\frac{\mathrm{d}\tilde{g}_A(r)}{\mathrm{d}r} + r^\gamma\left[\frac{\mathrm{d}^2\tilde{g}_A(r)}{\mathrm{d}r^2} - m_A^2\tilde{g}_A(r)\right] = 0 \tag{1.34}$$

We see that the piece in the square brackets give us the exponential behaviour, and in order to compensate $r^{\gamma-1}$ and r^γ we necessarily have $\gamma = \frac{1}{2}$, so that the first factor cancels. Finally we have

$$g_A(r) \sim \sqrt{r}e^{-m_A r} \tag{1.35}$$

It worth to notice that $g_A(r)$ decay exponentially with the penetration depth m_A^{-1}.
Doing the same for the ϕ, we have the following equation of motion:

$$\Delta_A\phi + m_e^2e^2\phi(|\phi|^2 - v^2) = 0 \tag{1.36}$$

where the covariant laplacian Δ_A (recall that $A_r = 0$) can be written in polar coordinate as

$$\Delta_A = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\varphi} - iA_\varphi\right)^2 \tag{1.37}$$

so that introducing the ansatz eq. (1.24) we get

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}g_H(r) + \frac{1}{r^2}g_A^2(r)(1 - g_H(r)) + 2\lambda v^2(1 - g_H(r))(-2g_H(r) + g_H^2(r)) = 0 \tag{1.38}$$

Again linearizing at large r, we can neglect non-linear terms in g_A and g_H, so that the equation of motion become

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}g_H(r) - m_H^2g_H(r) = 0 \tag{1.39}$$

and, inserting $g_H(r) = r^\gamma\tilde{g}_H(r)$, we finally get the behaviour of g_H:

$$g_H(r) \sim \frac{1}{\sqrt{r}}e^{-m_H r} \tag{1.40}$$

Hence both the magnetic field associated to A_μ and the scalar field ϕ approach the vacuum outside of a compact region of radius $O(m_A^{-1}, m_H^{-1})$ around the center of the vortex, which therefore behaves as an "extended particle".

Interactions among vortices

In general these vortices interact among each other, but for $m_H = m_A$, i.e. $\lambda = \frac{1}{2}n_e^2e^2$ a clever inequality (again of the Bogomol'nyi type, sometimes called *Bogomol'nyi-Prasad-Sommerfield (BPS)* in this context) show that they are free, non-interacting.