Contents

Chapter 1

Exercise Sheet - week 1

Exercise 1.0.1. Let $X \subset \mathbf{A}^m$ and $Y \subset \mathbf{A}^n$ be algebraic sets. Prove that $X \times Y \subset \mathbf{A}^{m+n}$ is an algebraic set.
Solution: Let $a_1, \ldots, a_m$ be the coordinates over $\mathbf{A}^m$ and $b_1, \ldots, b_n$ be the coordinates over $\mathbf{A}^n$. There are two subsets $S_1 \subset k[a_1, \ldots, a_m]$ and $S_2 \subset k[b_1, \ldots, b_n]$ such then $\mathbb{V}(S_1) = X$ and $\mathbb{V}(S_2) = Y$. Then if we define $S = \{f \cdot g | f \in S_1, g \in S_2\} \subset k[a_1, \ldots, a_m, b_1, \ldots, b_n]$ we have $X \times Y = \mathbb{V}(S) \subset \mathbf{A}^{m+n}$.

Exercise 1.0.2. Let $X, Y \subset \mathbf{A}^n$ be algebraic sets. Prove the following equalities:

(a) $\mathbf{I}(X \cup Y) = \mathbf{I}(X) \cap \mathbf{I}(Y)$

(b) $\mathbf{I}(X \cap Y) = \sqrt{\mathbf{I}(X) + \mathbf{I}(Y)}$

where for two ideals $\mathcal{I}, \mathcal{J}$ we denote by $\mathcal{I} + \mathcal{J}$ the ideal generated by the union $\mathcal{I} \cup \mathcal{J}$.
Find an example where $\mathbf{I}(X \cap Y)$ and $\mathbf{I}(X) \cap \mathbf{I}(Y)$ are different. Can you give a geometric explenation of why we have an inequality in this case?
Solution: (a),"$\subset$" If $f \in \mathbf{I}(X \cup Y)$ then $f(p) = 0$ for each $p \in X \cup Y$. In particular $f(p) = 0$ for each $p \in X$, i.e. $f \in \mathbf{I}(X)$; and $f(p) = 0$ for each $p \in Y$, i.e. $f \in \mathbf{I}(Y)$; so $f \in \mathbf{I}(X) \cap \mathbf{I}(Y)$.
"$\supset$" If $f \in \mathbf{I}(X) \cap \mathbf{I}(Y)$ then $f(p) = 0$ for each $p \in X$ and $f(p) = 0$ for each $p \in Y$. Then $f(p) = 0$ for each $p \in X \cup Y$, i.e. $f \in \mathbf{I}(X \cup Y)$.
(b),"$\supset$" if $f \in \sqrt{\mathbf{I}(X) + \mathbf{I}(Y)}$, then exists $N \geq 1$ such that $f^N = g + h$ where $g \in \mathbf{I}(X)$ and $h \in \mathbf{I}(Y)$, and this implies that $f^N \in \mathbf{I}(X \cap Y)$. For any point p condition $f^N(p) = 0$ implies $f(p) = 0$, hence $f \in \mathbf{I}(X \cap Y)$.
"$\subset$" First, recall that $\sqrt{\mathbf{I}(X) + \mathbf{I}(Y)} = \mathbf{I}(\mathbb{V}(\mathbf{I}(X) + \mathbf{I}(Y)))$. If $p \in \mathbb{V}(\mathbf{I}(X) + \mathbf{I}(Y))$, then for each $f \in \mathbf{I}(X)$ and $g \in \mathbf{I}(Y)$ we have $f(p) = 0 = g(p)$, and this means $p \in X \cap Y$. We conclude using inclusion reversing propriety: $\mathbb{V}(\mathbf{I}(X) + \mathbf{I}(Y)) \subset X \cap Y$ implies $\mathbf{I}(X \cap Y) \subset \mathbf{I}(\mathbb{V}(\mathbf{I}(X) + \mathbf{I}(Y))) = \sqrt{\mathbf{I}(X) + \mathbf{I}(Y)}$.
The ideal $\mathbf{I}(X \cap Y)$ describes the intersection of algebraic sets X and Y while $\mathbf{I}(X) \cap \mathbf{I}(Y)$ describes them union. For example, if we consider two different sets $X = \{a\} \subset \mathbf{A}^1$ and $Y = \{b\} \subset \mathbf{A}^1$ and following polynomials: $f = x - a$, $g = x - b$ and $h = (x - a)(x - b)$. We have $f, h \in \mathbf{I}(X)$ and $g, h \in \mathbf{I}(Y)$, while only $h \in \mathbf{I}(X) \cap \mathbf{I}(Y)$. Since $X \cap Y = \emptyset$, we have $f, g, h \in \mathbf{I}(X \cap Y)$.

Exercise 1.0.3. Compute the ideals of the following algebraic sets in $\mathbf{A}^2(\mathbb{C})$:

(a) $X_1 = \{(1, 0), (0, 1)\}$

(b) $X_2 = \{(1, 0), (0, 1), (0, 0)\}$

(c) $X_3 = \{(1, 0), (0, 1), (\frac{1}{2}, \frac{1}{2})\}$

What is the minimal number of polynomials you need to generate $\mathbf{I}(X_1)$, $\mathbf{I}(X_2)$ and $\mathbf{I}(X_3)$ respectively?
Solution: Let's define following sets of plynomals:

(a) $S_1 = \{(x - 1), (x - i)\}$

(b) $S_2 = \{(x - 1), (x - i), (x)\}$

(c) $S_3 = \{(x - 1), (x - i), (x - \frac{1}{2} - \frac{1}{2}i)\}$

Then $\mathbf{I}(X_1) = (S_1)$, $\mathbf{I}(X_2) = (S_2)$ and $\mathbf{I}(X_3) = (S_3)$. It's clear that minimal number of polynomials required to generate ideals is 2 for X_1 and 3 for X_2 and X_3.

Exercise 1.0.4. Let X be the union of the coordinate axis in $\mathbf{A}^n$. Find generators for the ideal of X, how many polynomials do we need?
Solution: The union of the coordinate axes in $\mathbf{A}^n$ is the locus where at least one coordinate is zero, therefore $\mathbf{I}(X) = (\{(x_1), (x_2), \ldots, (x_n)\})$, where x_i is the coordinate of the i-th axis; hence we need only one generator for $\mathbf{I}(X)$.

Exercise 1.0.5. Let $X = \{(t, t^2, t^3) : t \in k\} \subset \mathbf{A}^3$. Prove that X is an irreducible algebraic set and find generators for its ideal $\mathbf{I}(X)$. Show that the dimension of X is one, i.e. X is an irreducible curve.
Solution: The algebraic set is the locus where $y = x^2$ and $z = x^3$, with $\{x, y, z\}$ coordinate system for $\mathbf{A}^3$. Then $\mathbf{I}(X) = (\{(y - x^2), (z - x^3)\})$. This ideal is principal and hence is prime, therefore X is irreducible. In order to find algebraic subsets of X, lets try to insert another polynomial in the generator of $\mathbf{I}(X)$, namely $f \in k[x, y, z]$. Notice first of all that coordinates y and z are already fixed by x, therefore in order to obtain some subset $X_0 = \mathbb{V}((\{(y - x^2), (z - x^3), f(x, y, z)\})) \neq \emptyset$, we require $f \in k[x]$. Since such function f will have a finite number of zeros, X_0 will be a collection of points, with zero dimension (since a point can't have any subset). We conclude that the maximum descending chain of subset is $X \subsetneqq X_0 \subsetneqq \emptyset$, where X_0 is a collection of points, hence X is an irreducible curve.

Exercise 1.0.6. Let $k = \mathbb{C}$. Decompose into irreducible components the following algebraic sets $X, Y \subset \mathbf{A}^3$ and determine the prime ideals of their irreducible components:

(a) X defined by $x_1^2 + x_2^2 + x_3^2 = x_1^2 - x_2^2 - x_3^2 + 1 = 0$

(b) Y defined by $x_1^2 - x_2 x_3 = x_1 x_3 - x_1 = 0$

Solution: (a) Let $\mathbf{I}(X) = (\{(x_1^2 + x_2^2 + x_3^2), (x_1^2 - x_2^2 - x_3^2 + 1)\})$. We can define subsets $X_1, X_2 \subset X$ through ideals $\mathbf{I}(X_1) = (\{(x_2^2 + x_3^2 - \frac{i}{2}), (x_1 - \frac{i}{\sqrt{2}})\})$ and $\mathbf{I}(X_2) = (\{(x_2^2 + x_3^2 + \frac{i}{2}), (x_1 - \frac{i}{\sqrt{2}})\})$. Notice that neither $(x_2^2 + x_3^2 + \frac{i}{2})$ nor $(x_1 \pm \frac{i}{\sqrt{2}})$ can be decomposed further in other polynomials with smaller degree. If we take $f \in k[x_1, x_2, x_3]/\mathbf{I}(X_1)$ and $g \in k[x_1, x_2, x_3]$ with $fg \in \mathbf{I}(X_1)$, then for some $a, b \in k[x_1, x_2, x_3]$ we have $fg = a(x_2^2 + x_3^2 - \frac{i}{2}) + b(x_1 - \frac{i}{\sqrt{2}})$ and $g = (a/f)(x_2^2 + x_3^2 - \frac{i}{2}) + (b/f)(x_1 - \frac{i}{\sqrt{2}})$ for some $(a/f), (b/f) \in k[x_1, x_2, x_3]$, and then $g \in \mathbf{I}(X_1)$. This means that $\mathbf{I}(X_1)$ is prime, and same proof holds also for $\mathbf{I}(X_2)$. Therefore X_1 and X_2 are affine varieties. Since $\mathbf{I}(X) = \mathbf{I}(X_1) \cap \mathbf{I}(X_2)$, we conclude $X = X_1 \cup X_2$, where X_1 and X_2 are irreducible components of X.
(b) Let $\mathbf{I}(Y) = (\{(x_1^2 - x_2 x_3), (x_1 x_3 - x_1)\})$. We can define subsets $Y_1, Y_2, Y_3 \subset Y$ through ideals $\mathbf{I}(Y_1) = (\{(x_1), (x_2)\})$, $\mathbf{I}(Y_2) = (\{(x_1), (x_3)\})$ and $\mathbf{I}(Y_3) = (\{(x_1^2 - x_2), (x_3 - 1)\})$. Analogously to (a), we can prove that $Y = Y_1 \cup Y_2 \cup Y_3$, where Y_1, Y_2 and Y_3 are irreducible components of Y.

Exercise 1.0.7. Let Y be a subset of a topological space X. Show that Y is irreducible if and only if the closure $\overline{Y}$ of Y in X is irreducible.
Solution: "$\Rightarrow$" Let's take $Y \subset X$ irreducible subset of the topological space X. Let $\overline{Y} \subset X$ be the smallest closed subset that contains Y. Suppose that $\overline{Y} = \overline{Y}_1 \cup \overline{Y}_2$, then $Y = (Y \cap \overline{Y}_1) \cup (Y \cap \overline{Y}_2)$, and since Y is irreducible then it must be contained either in $\overline{Y}_1$ or $\overline{Y}_2$. Assume that $Y \subset \overline{Y}_1$. Since $\overline{Y}_1 \subset \overline{Y}$ is a closed subset that contains Y and $\overline{Y}$ is the smallest closed subset that contains Y then $\overline{Y} = \overline{Y}_1$ and than $\overline{Y}$ is not reducible.
"$\Leftarrow$" Let's take $Y \subset X$ reducible subset of the topological space X, and let $Y = Y_1 \cup Y_2$ with $Y_i \subsetneqq Y$, $i = 1, 2$, subsets of Y closed in the subspace topology. Consider $\overline{Y}$, $\overline{Y}_1$ and $\overline{Y}_2$ to be the closure respectively of Y, Y_1 and Y_2; then $\overline{Y}_1 \cup \overline{Y}_2 \supset \overline{Y}$ because $\overline{Y}$ is the smallest closed set containing Y, and on the other side $\overline{Y}_1 \cup \overline{Y}_2 \subset \overline{Y}$ otherwise $\overline{Y} \cap \overline{Y}_i$ would be a closure of Y_i smaller that $\overline{Y}_i$ either for $i = 1$ or $i = 2$. Finally, notice that $Y_i = \overline{Y}_i \cap Y$ and $Y = \overline{Y} \cap Y$, so $Y_i \subsetneqq Y$ implies $\overline{Y}_i \subsetneqq \overline{Y}$. Then we can conclude that $\overline{Y} = \overline{Y}_1 \cup \overline{Y}_2$ is reducible.

Exercise 1.0.8. Let $f : X \to Y$ be a continuous map and let $W \subset X$ be an irreducible subset of X. Prove that the image of W is irreducible.
Solution: Using the result of the previous exercise, we just have to prove the statement in the case of W and $V = f(W)$ closed subsets. Then, suppose $V = V_1 \cup V_2$ and $W_i = (f^{-1}(V_i)) \cap W$ for $i = 1, 2$. Then W_1 and W_2 are closed subsets of W and $W = W_1 \cup W_2$, hence we can assume $W = W_1$. Then $V = f(W) = f(W_1) = V_1$, so V is irreducible.