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Chapter 1

The LSZ Reduction Formula

1.1 A new approach to Quantum Field Theory

In this chapter we will make a first contact between operator formalism and path integral formalism of QFT. QFT has main objective of compute:

(i) scattering amplitudes;

(ii) cross-section, where well defined and separated states interacts, creating new final states.

By now, these objective are obtained using canonical quantization formalism. This tool is useful for several situations but, as we will see, there is something more in QFT.

Another point to stress is that the approach we used so far is a perturbative method, which is a really good and precise method, and gave very good results. Nevertheless this method gives an incomplete information. With the standard approach of computing cross section using operator formalism, we missed some informations about the process we are studiying. So is important to have a different tool that allows us to obtain this non perturbative information. This tool is called **Path Integral formalism**.

With this method we will obtain all results we obtained with the perturbative method, but we will be able to understand better our previous results and also something more about QFT, such as non-perturbative effects. This is important because QFT plays, beside its role in Fundamental Interactions, a fundamental role also in Statistical Mechanics, Cosmology, Topology, Geometry and String Theory, where perturbative approach is not so efficient (or it is useless at all).

First of all we need to understand which are limits of operatorial formalism, and then which is the relation between the two formalism.

Limits of the Operatorial Formalism

In the operator formalism approach the step we done are

(i) pick a time variable t;

(ii) define an hamiltonian $H = H(\phi, \pi)$ and the field conjugated field

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)}$$

(iii) transform ϕ and π these into operators

(iv) impose that at equal time some (anti)commutation rules between fields are satisfied

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

Pros:

(i) Clear physical content (states, operators), expecially manifest unitariety (thanks to hamiltonian approach)

Cons:

(i) Covariance not manifest at intermediate stages

(ii) Derivation of Feynman rules can be complicated.
For example, consider a lagrangian with a term that mixes interaction and kinematic terms

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu - \lambda\phi^2\partial_\mu\phi\partial^\mu$$

The conjugated field is

$$\pi = \partial_0\phi(1 + 2\lambda\phi^2)$$

and the hamiltonian reads

$$H = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\frac{\pi^2}{1 + 2\lambda\phi^2} + \dots$$

So we don't have only π^2 but also an interaction term between π and ϕ. Assume $\lambda \ll 1$ then using perturbative approach

$$H = \frac{1}{2}\pi^2(1 - 2\lambda\phi^2 + 4\lambda^2\phi^4 + \dots)$$

so we obtain an infinite number of vertex related to terms $\lambda^n\phi^{2n}\pi$. In this case deriving Feynman rules is a really hard task.

(iii) In order to obtain a nice analysis of gauge theories (in particular non-abelian ones) at levels up to the tree level the perturbative approach is not so good. To obtain proprieties of these theories we must go beyond using use path integral approach.

1.2 Correlators and the LSZ reduction formula

Important: *Accoarding to the notation used in Srednicki, in this section we will use the metric*

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In this section we will obtain the connection between perturbative formalism and path integral formalism. Recall from previous courses that *correlators* or *Green functions* contains most of informations about QFT

$$G(x_1, x_2, \dots, x_n) = \langle 0|\, T[\phi(x_1), \dots, \varphi(x_n)]\, |0\rangle$$

Covariant convention for states

Srednicki, chap 3.

We start with a very simple theory, a free massive scalar field with lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

whose equation of motion is

$$\Box\phi = m^2\phi$$

and normalized solutions are in the form of plane waves[I]

$$\frac{1}{\sqrt{(2\pi)^3 2k^0}}e^{\pm ikx} \qquad \text{where} \quad k^2 = \mathbf{k}^2 - \omega^2 = -m^2$$

[I] $x^\mu = (t, \mathbf{x})$ and $k^\mu = (\omega, \mathbf{k})$. Depending on the contest, when k^0 is not used as integration variable we assume $k^0 = \omega$.

I can write scalar field in terms of α and $\alpha^\dagger$ operators:[II]

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3 2k^0}} \left(\alpha(\mathbf{k})e^{ikx} + \alpha^\dagger(\mathbf{k})e^{-ikx}\right)$$

When we quantize the theory we introduce the commutation relation

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = 0 = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})]$$

Equivalently, these relation can be imposed saying that α, $\alpha^\dagger$ are annihilation and creation operators that satisfy

$$[\alpha(\mathbf{k}), \alpha(\mathbf{k}')] = 0 = [\alpha(\mathbf{k}), \alpha(\mathbf{k}')]$$
$$[\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}')$$

Let's introduce an alternative notation in oder to obtain a covariant algebra for ladder operators and a covariant normalization for states.

We introduce the Lorentz invariant[III] measure (under $SO^+(3,1)$)

$$\mathrm{d}^4k\, \delta(k^2+m^2)\theta(k^0)$$

Notice that the theta function picks an arrow of time (i.e. particles or antiparticles).

Using proprieties of delta function, we know that $\delta(k^2+m^2)$ can be splitted as follows (here $k^0 \neq \omega$ in general)

$$\delta(k^2+m^2) = \frac{1}{2k^0}\left[\delta\left(k^0 - \sqrt{m^2+\mathbf{k}^2}\right) + \delta\left(k^0 + \sqrt{m^2+\mathbf{k}^2}\right)\right]$$

We then have (recall $\omega = \sqrt{m^2+\mathbf{k}^2}$)

$$\int \mathrm{d}k^0 \delta(k^2+m^2)\theta(k^0) = \frac{1}{2\omega}$$

thus we see that if we take $f(k) \propto \omega$ then $\mathrm{d}^3k/f(k)$ will be Lorentz invariant. If we take $f(k) = (2\pi)^3 2\omega$ then the field express with the corresponding Lorentz invariant differential is

$$\phi(x) = \frac{1}{(2\pi^3)} \int \frac{\mathrm{d}^3k}{2k^0} \left(a(k)e^{ikx} + a^\dagger(k)e^{-ikx}\right)$$

As a consequence of commutation relations, ladder operators obey the following covariant algebra (obviously $k^0 = \omega$)

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0$$
$$[a(k), a^\dagger(k')] = (2\pi)^3 2k^0 \delta^3(\mathbf{k}-\mathbf{k}')$$

We obtain a Fock space made of covariant normalized states

$$\langle 0|0\rangle = 1, \qquad |k\rangle = a^\dagger(k)\,|0\rangle$$

$$\langle k|k'\rangle = (2\pi)^3 2k^0 \delta^3(\mathbf{k}-\mathbf{k}')$$

In the free theory we can also write down a and $a^\dagger$ in function of the field, in particular

$$a^\dagger(k) = -i \int \mathrm{d}^3x (e^{ikx}\partial_0\phi(x) - \partial_0 e^{ikx}\phi(x)) \tag{1.1}$$

[II] If I consider a real field $\alpha(x) = \alpha^\dagger(x)$

[III] We don't consider full Lorentz group but only transformations that preserve time, i.e. transformations with determinant equal to 1 (proper Lorentz transformations).

Exercise 1

Prove explicitly equation (1.1)

insert solution

Scattering amplitude in interacting theory

Srednicki, chap 5.

When we do a scattering experiment (which are function of momenta) a and $a^\dagger$ will became time dependent operators. Consider a generic lagrangian in the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_0$ is the free field lagrangian while $\mathcal{L}_{int}$ is an interaction term.

Inital and final states takes the from

$$|i\rangle = \lim_{t\to-\infty} a^\dagger(k_1,t)\dots a^\dagger(k_n,t)\,|0\rangle$$

$$|f\rangle = \lim_{t\to+\infty} a^\dagger(k'_1,t)\dots a^\dagger(k'_n,t)\,|0\rangle$$

This states must be normalized, for istance

$$\langle i|i\rangle = 1 = \langle f|f\rangle$$

Notice that now a and $a^\dagger$ are ladder operators for the interacting theory, no more for the free theory. Now we look for an explicit expression for the scattering amplitude

$$S_{fi} = \langle f|i\rangle$$

in the interaction formalism. In order to obtain an useful expression for that we better try to express latter operators in terms of fields of the theory. Let's start from the free field, we can write

$$\begin{aligned}
a^\dagger(k,+\infty) - a^\dagger(k,-\infty) &= \int_{-\infty}^{+\infty} \mathrm{d}t\,\partial_0 a^\dagger(k,t) \\
&= \int \mathrm{d}t\,\partial_0 \left(-i\int_{-\infty}^{+\infty} \mathrm{d}^3x (e^{ikx}\partial_0\phi - \partial_0 e^{ikx}\phi)\right) \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(\partial_0^2 + (k^0)^2)\phi(x) \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(\partial_0^2 + m^2 + \mathbf{k}^2)\phi(x) \\
&= -i\int \mathrm{d}^4x\, [e^{ikx}(\partial_0^2 + m^2)\phi(x) - \phi(x)\vec{\nabla}^2 e^{ikx}] \\
&= -i\int \mathrm{d}^4x\, [e^{ikx}(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi(x)] + \text{surface term} \\
&= -i\int \mathrm{d}^4x\, e^{ikx}(-\partial_\mu\partial^\mu + m^2)\phi(x)
\end{aligned} \tag{1.2}$$

and vanishes in the free theory because of the e.o.m. as we expected since we are considering a free field.

Let assume that (1.1) holds also in the interaction case, then also the latter formula holds, but in this case the last term in general is not vanishing. Consider scattering amplitude

$$S_{fi} = \langle f|f\rangle\, i = \langle 0|\, a(k'_1,+\infty)\dots a(k'_n,+\infty)a^\dagger(k_1,-\infty)\dots a^\dagger(k_n,-\infty)\,|0\rangle$$

since operators are time ordered I can write

$$S_{fi} = \langle 0|\, T\left[a(k'_1,+\infty)\dots a(k'_n,+\infty)a^\dagger(k_1,-\infty)\dots a^\dagger(k_n,-\infty)\right]|0\rangle$$

Using (1.2) and its complex conjugated, i can rewrite:

$$S_{fi} = i^{m+n} \int \prod_{i=1}^{m} \mathrm{d}^4 x_i \, e^{ik_i x_i} (-\Box_{x_i} + m^2) \prod_{j=1}^{n} \mathrm{d}^4 y_j \, e^{ig_j k_j} (-\Box_{y_j} + m^2) \times \\ \times \underbrace{\langle 0| \, T \left[\phi(x_1) \dots \phi(x_m) \phi(y_1) \dots \phi(y_n)\right] |0\rangle}_{G(x_1,\dots,x_n,y_1,\dots,y_n)} \tag{1.3}$$

Eq. (1.3) is the **Lehmann-Symanzik-Zimmermann reduction formula**, it express the scattering amplitude in terms of Green functions. Here is clear the importance of the Green functions.

Let's recall a crucial assumption we made: formula (1.1) must holds in the interacting case. The equation (1.1) holds for the interacting theory only if a couple of conditions are satisfied:

(i) $\langle 0| \, \phi(x) \, |0\rangle = 0$

This means that whenever you start from the state $|0\rangle$ the operator $a^\dagger$ must create some particle state, orthogonal to the vacuum. This is obvious for free theories, but does not holds in general for interacting theories. In some cases $\phi(x) \, |0\rangle$ is a linear combination of one particle states and the vacuum state.

So, if $v = \langle 0| \, \phi(x) \, |0\rangle$ is a Lorentz invariant number different to zero, we will shift the field $\phi(x)$ by the constant v:

$$\phi(x) \to \phi(x) + v = \tilde{\phi}(x)$$

This is just a change in the name of the operator of interest, and does not affect the physics. However, the shifted $\tilde{\phi}(x)$ obeys, by costruction, $\langle 0| \, \tilde{\phi}(x) \, |0\rangle = 0$.

(ii) $\langle k| \, \phi(x) \, |0\rangle = e^{-ikx}$

This is what it is in the free theory, and we know that in free theory, $a^\dagger(\pm\infty)$ creates a correctly normalized one-particle state. Thus, for $a^\dagger(\pm\infty)$ creates a correctly normalized one-particle state in the interacting theory, we must have $\langle k| \, \phi(x) \, |0\rangle = \langle k| \, e^{-i\hat{k}x} \phi(0) e^{i\hat{k}x} \, |0\rangle = e^{-ikx} \, \langle k| \, \phi(0) \, |0\rangle = e^{-ikx}$.

So, if $\langle k| \, \phi(0) \, |0\rangle$ is a Lorentz invariant number different to one, we will rescale (or, *renormalize* $\phi(x)$ by a multiplicative constant:

$$\phi(x) \to Z_\phi^{1/2} \phi = \phi_{int}$$

This is just a change of the name of the operator of interest, and does not affect the physics. However, the rescaled $\phi(x)$ obeys, by costruction, $\langle k| \, \phi(0) \, |0\rangle = 1$.

Let us recap. The basic formula for scattering amplitude in terms of the fields of an interacting quantum field theory is the LSZ formula (1.3). The LSZ formula is valid *provided* that the fields obeys

$$\langle 0| \, \phi(x) \, |0\rangle = 0 \quad \text{and} \quad \langle k| \, \phi(x) \, |0\rangle = e^{-ikx} \tag{1.4}$$

These normalization conditions may conflict with our original choice of field and parameter normalization in the lagrangian. Consider, for example, a lagrangian originally specified as

$$\mathcal{L} = \underbrace{-\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2}_{\mathcal{L}_0} + \underbrace{\frac{1}{6} g \phi^3}_{\mathcal{L}_{int}}$$

After shifting and rescaling (and renaming some parameters), we will have instead

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_{CT} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{6} Z_g g \phi^3 + Y\phi$$

where

$$\mathcal{L}_{CT} = -\frac{1}{2}(Z_\phi - 1)\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}(Z_m - 1) m^2 \phi^2 + \frac{1}{6}(Z_g - 1) g \phi^3 + Y\phi$$

is a new term called *countive (?) lagrangian*. Here Z_ϕ, Z_m, Z_g and Y are yet unknown constant. They must be chosen to ensure the validity of eq.(1.4); this gives us two conditions in four unknown. We fix the parameter $Z_m m^2$ by requiring to be equal to the actual mass of the particle (equivalently, the energy

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of the first excited state relative to the ground state), and we fix the parameter $Z_g g$ by requiring some particular scattering cross section to depend on the coupling term in some particular way. So we have four conditions in four unknown, and it is possible to calculate Y and the three Zs order by order in powers of g.

Next, we must develop the tools needed to compute the correlation functions $\langle 0| T [\phi(x_1) \dots] |0\rangle$ in an interacting quantum field theory.

Chapter 2

The Path integral in Quantum Mechanics

Path Integral will be the main tool we will use in this course in order to study QFT. We will start introducing this tool in special case of Quantum Mechanics.

2.1 Intuitive Introduction to Path Integrals

Rattazzi, sec. 1.1.1 - 1.1.3

One of the important experiments that show the fundamental difference between Quantum and Classical Mechanics is the double slit experiment. It is interesting with respect to the path integral formalism because it leads to a conceptual motivation for introducing it.

Consider a source S of approximatively mono-energetic particles, electrons for instance, placed at position (x_i, y_i). The flux of electrons is measured on a screen facing the source. Imagine now placing a third screen in between the ohers, with two slits on it, which can be opened or closed. When the first is open and the second closed we measure the flux F_1, when the first slit is closed and the second open we measure a flux F_2 and when both slits are open we measure the flux F.

One finds in general $F = F_1 + F_2 + F_{int}$, and the structure of F_{int} precisely corresponds to the interference between two waves passing respectively through 1 and 2:

$$F = |\phi_1 + \phi_2|^2 = \underbrace{|\phi_1|^2}_{F_1} + \underbrace{|\phi_2|^2}_{F_2} + \underbrace{\phi_1\phi_2^* + \phi_2\phi_1^*}_{F_{int}}$$

where ϕ_i is the probability amplitude for a point-like particle position and $|\phi_i|^2$ the corresponding probability density.

The idea behind the path integral approach to QM is to take the implications of the double slit experiment to its extreme consequences. One can immagine adding extra screens and drilling more and more holes through them, generalising the result of the double slit experiment by the superposition principle.

Let's denote as follows the superposition of N slits fluxes:

$$\Phi = \sum_{i=1}^{N} \phi(y_A^i)$$

where $\phi(y_A^i)$ denotes the flux of the particle whose trajectory goes through the i-th slit positioned in the coordinate y_A^i of the screen placed in x_A. Nothing stops us from taking the ideal limit where $N \to \infty$ and the holes fill all the surface. The sum $\sum_i$ becomes now an integral over y_A:

$$\Phi = \int \mathrm{d}y_A \phi(y_A)$$

Fai un disegno. Le coordinate x sono quelle dove sono posizionati i vari schermi, le y sono le posizioni su ogni schermo

We can go on and further refine our trajectories by adding more and more screens between the source and the final screen. In the limit in which the added screens become infinitesimally close, we have specified all possible paths $y(x)$:

$$\Phi = \int \mathrm{d}x \int \mathrm{d}y_x \phi(y(x))$$

where $\phi(y(x))$ is the flux corresponding to the path $y(x)$. We then arrived at a formal representation of the probability amplitude as a sum over all possible trajectories $g(x)$:

$$\Phi = \sum_{\substack{\text{All trajectories} \\ \{x(t),y(t)\}}} \phi(g(x)) \tag{2.1}$$

How do I make sense of this? What Φ is and how do I make sense of this sum? Moreover, I'd like that for $\hbar \to 0$ I should go back to Classical Mechanics. This implies that ϕ must depends on $\hbar$ in someway.

Since $\hbar$ has dimensionality [Energy]×[Time], one can guess that

$$\phi(\gamma) = e^{i\frac{S[\gamma]}{\hbar}} \tag{2.2}$$

where S is the action which describes the classical trajectories via the principle of least action. This means that we associate to each trajectory γ a phase related to the action $S[\gamma]$. Recall that the classical trajectories are given by the stationary points of $S[\gamma]$ ($\delta S[\gamma] = 0$).

Let's analyze our guess. The choice $\phi(\gamma) = f(S[\gamma]/\hbar)$ is natural, since in this way the argument is adimensional. The choice of the exponential function seems promising for two reasons:

(i) The requirement $\delta S[\gamma] = 0$ for $\hbar \to 0$ is heuristically seen to hold. In a macroscopic, classical, situation the gradient $\delta S/\delta\gamma$ is for most trajectories much greater than $\hbar$. Around such trajectories the phase $e^{iS/\hbar}$ oscillates extremely rapidly and the sum over neighbouring trajectories will tend to cancel.

On the other hand, on a classical trajectory γ_{cl} the action S is stationary. Therefore in the neighbourhood of γ_{cl}, S varies very little, so that all trajectories in a tube centred around γ_{cl} add up coherently in the sum over trajectories.

Indeed, this means that in the exact limit $\hbar \to 0$ these effects becomes dramatic and only the classical trajectory survives.

(ii) Eq. (2.2) leads to crucial composition property. Indeed the action for a path γ_{12} obtained by joining two subsequent paths γ_1 and γ_2 satisfies the simple additive relation

$$S[\gamma_{12}] = S[\gamma_1] + S[\gamma_2]$$

Thanks to eq.(2.2) the additivity of S translates into a factorization property for the amplitude:

$$\phi[\gamma_{12}] = \phi[\gamma_1]\phi[\gamma_2]$$

This feature is required for our theory.

2.2 From Schroedinger Equation to the Path Integral

Rattazzi, sec. 1.1.4

The transition amplitude for a particle in QM reads

$$\langle x_f(t_f)|x_i(t_i)\rangle = \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}(t_f-t_i)}\, |x_i\rangle$$

In order to evaluate this quantity we split $t_f - t_i$ in N pieces with N large. Let $\delta t = (t_f - t_i)/N$. Recall that $\int \mathrm{d}x\, |x\rangle\langle x| = 1 = \int \mathrm{d}p\, |p\rangle\langle p|$. Then

$$\langle x_f(t_f)|x_i(t_i)\rangle = \int \prod_{j=1}^{N-1} \mathrm{d}x_j \, \langle x_f|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\, |x_{N-1}\rangle \langle x_{N-1}|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\, |x_{N-2}\rangle \dots \tag{2.3}$$

$$\langle x_2|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\, |x_1\rangle \langle x_1|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\, |x_i\rangle \tag{2.4}$$

For each piece we have

$$\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle = \int \mathrm{d}p\, \langle x'|p\rangle\, \langle p|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle \tag{2.5}$$

If we stick to the simply case $\hat{H} = \hat{p}/2m + V(\hat{x})$ and we denote the kinetic operator as $T(\hat{p}) = -i\frac{\hat{p}^2}{2m\hbar}\delta t$ and the potential $U(\hat{x}) = -iV(\hat{x})\delta t/\hbar$, we can write

$$\langle p|\exp\{T(\hat{p}) + U(\hat{x})\}\,|x\rangle = \langle p|\, e^{T(\hat{p})}e^{-T(\hat{p})}e^{T(\hat{p})+U(\hat{x})}e^{-U(\hat{x})}e^{U((\hat{x}))}\,|x\rangle = e^{T(p)}e^{U(x)}\,\langle p|\, e^{C(\hat{p},\hat{x})}\,|x\rangle$$

where

$$e^{C(\hat{p},\hat{x})} = e^{-T(\hat{p})}e^{T(\hat{p})+U(\hat{x})}e^{-U(\hat{x})}$$

The operator C is given, using Baker-Campbell-Hausdorff formula twice, as a series of commutators between T and U

$$C = \frac{1}{2}[T,V] + \frac{\delta t}{6}([T,[T,V]] + [V,[V,T]]) + ...$$

If all commutators in the expression of C are $o(1)$, then all the terms of the expansion of C are $o((\delta t)^2)$ and therefore can be neglected. However, this assumption is not immediate, for example $[\hat{p}, V(\hat{x})] = -iV'(\hat{x})]$ implies that in order to neglect C all derivatives of V must be bounded. Then, if the derivatives of V are bounded, the contribution of the operator $\hat{x}$ in the expansion of C is a bounded contribution and in the limit $\delta t \to 0$ vanishes. The only potential problem to concentrate on the $\delta t \to 0$ limit is represented by the integration over powers of p in (2.5). Essentially what we are integrating is a function that goes approximatively as the gaussian $\exp\big(-\delta t\, p^2\big)$, therefore the leading contribution to the p integral is the one with $p \sim \delta t^{-1/2}$. Thus we have

$$\langle p|\, e^{C(\hat{p},\hat{x})}\,|x\rangle \simeq \langle p|x\rangle\,(1 + O(\delta t^{3/2}))$$

Even if I consider all $N = 1/\delta t$ contributions in (2.3) they can be neglected, since the final result is convergent to 1:

$$\lim_{\delta t\to 0}(1 + a\,\delta t^{3/2})^{1/\delta t} = 1$$

Therefore we can reasonably neglect contributions of C and then

$$\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle \simeq \lim_{\delta t\to 0}\int \mathrm{d}p \exp\left\{-i\frac{\delta t}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]\right\}\cdot \underbrace{\frac{\exp\{ip(x'-x)/\hbar\}}{2\pi\hbar}}_{\langle x'|p\rangle\langle p|x\rangle}$$

Now we introduce the variable $\dot{x} = (x' - x)/\delta t$, then

$$\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle = \lim_{\delta t\to 0}\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left\{-i\frac{\delta t}{\hbar}\left(\frac{p^2}{2m} + V(x) - p\dot{x}\right)\right\}$$

and performing the change of variable $p' = p - m\dot{x}$ we obtain

$$\begin{aligned}\lim_{\delta t\to 0}\langle x'|\, e^{-i\frac{\hat{H}}{\hbar}\delta t}\,|x\rangle &= \lim_{\delta t\to 0}\int \frac{\mathrm{d}p'}{2\pi\hbar}\exp\left\{-i\frac{\delta t}{\hbar}\left(\frac{p'^2}{2m} + \underbrace{V(x) - \frac{1}{2}m\dot{x}^2}_{-\mathcal{L}}\right)\right\} \\ &= \lim_{\delta t\to 0}\underbrace{\left(\frac{m}{2\pi i\hbar\,\delta t}\right)^{1/2}}_{k}\exp\left\{i\frac{\delta t}{\hbar}\mathcal{L}(x,\dot{x})\right\}\end{aligned}$$

where we introduced the factor k (that depends on δt) in order to simplify the notation. When I introduce this into eq.(2.3) we have ($x_0 = x_i$ $x_N = x_f$):

$$\lim_{\delta t\to 0}\langle x_f(t_f)|x_i(t_i)\rangle = \lim_{\delta t\to 0}\int\prod_{j=1}^{N-1}\mathrm{d}x_j k^N \exp\left\{\frac{i}{\hbar}\sum_{m=0}^{N-1}\delta t\mathcal{L}(x_m,\dot{x}_m)\right\}$$
$$= \lim_{\delta t\to 0}\int\prod_{j=1}^{N-1}\mathrm{d}x_j k \exp\left\{\frac{i}{\hbar}S(x_f,x_i)\right\}$$
$$= \int_{C_T[x_i,x_f]}\mathcal{D}x\,\exp\left\{\frac{i}{\hbar}S(x_f,x_i)\right\} \tag{2.6}$$

where we defined the following functional measure over the space of trajectories[I]:

$$\int_{C_T[x_i,x_f]}\mathcal{D}x = \lim_{\delta t\to 0} k\int\prod_{j=1}^{N-1}\mathrm{d}x_j$$

Here $C_T[x_i,x_f]$ are all possible configurations on x that start in x_i and end in x_f over a time $T=t_f-t_i$, i.e. is the set of all possible path $x(t)$ such that $x(t_i)=x_i$ and $x(t_i+T)=x(t_f)=x_f$. We call $C_T[x_i,x_f]$ **space of configurations**. The weight $\exp\{\frac{i}{\hbar}S(x_f,x_i)\}$ is the phase related to eq.(2.2). Eq.(2.6) is just the analogous of eq.(2.1).

The construction we made has no rigor, but clarify the idea behind Path Integrals theory. The result we obtained so far is the replacement of the sum into eq.(2.1) with an integral, hoping in a convergent expression. This is not obvious, indeed this does not happen in general.

This definition has some problematic points, that sometimes does not matter and we can skip on them in a straightforward way, but can also became crucial in order to obtain the results we obtained in operatorial approach. For example we assumed that trajectories were smooth and we can differentiate them, but this is not going to be true. Quite the opposite, the fact that trajectories may not be smooth leads to the contact between Path Integral approach and operatorial formalism of QM.

We won't exhaminate details of mathematical structure behind Path Integrals since it is not really interesting from the physical point of view, rather we focus on problematical aspects and special features of this formalism in order to obtain a deeper understanding of the physics.

Before proceeding with technical developments, it is worth assessing the role of the path integral in quantum mechanics. As it was hopefully highlighted so far, the path integral formulation is conceptually advantageous over the standard operatorial formulation of QM, in that the "good old" particle trajectories retain same role. The P.I. is however technically more involved. When working on simple quantum systems like the hydrogen atom, no technical profit is really given by path integrals. Nonetheless, after overcoming a few technical difficulties, the path integral offers a much more direct viewpoint on the semiclassical limit. Similarly, for issues involving topology (like the origin of Bose and Fermi statistics, the Aharonov-Bohm effect, charge quantization in the presence of a magnetic monopole, etc...) path integrals offer a much better viewpoint. Finally, for advanced issues like the quantization of gauge theories and for effects like instantons in quantum field theory it would be hard to think how to proceed without path integrals.

2.3 The Partition Function

Rattazzi, chap. 1

We want to generalize results of the previous section. What we have done, is to sum over all possible configurations of the field with a certain measure and weight them with a weight function that is essentially the action:

$$\boxed{\mathcal{Z}(\lambda,m,\dots)=\int_{\phi\in C}\mathcal{D}[\phi]\exp\left\{\frac{i}{\hbar}S[\phi]\right\}} \tag{2.7}$$

this integral is generic form of **path integral** and in particular is the **partition function** for a certain theory. In order to specify the theory we have to specify:

[I] For the moment is not clear if this functional is well defined and is a measure, this is just an anticipation of following results.

(i) M: The space where out QFT theory lives and of course its metric, in particular we have to specify its dimension d and, if it exists, the space's metric g (we can also think about a metric with some degree of freedom, such in Quantum Gravity).

(ii) ϕ: Fields over the space, that are generically maps from M to some target space (e.g. $\mathbb{R}$, $\mathbb{C}$, vector fields, gauge group, gauge bundle, etc.)

(iii) C: Space of allowed field configurations (possibly with some boundary conditions)

(iv) λ: Other parameters of the theory (such as mass m etc.)

(v) $\mathcal{L}$: The lagrangian of our theory

(vi) S: The action (depends on all the other parameters)

$$S[\phi] = \int_M \mathrm{d}^d x \sqrt{|g|} \mathcal{L}(\phi, \partial\phi, \lambda, m, \dots)$$

(vii) $\mathcal{D}\phi$: the measure of integration

Once I have specified this items i can obtain a well defined partition function form my theory eq.(2.7). Moreover starting with partition function we can compute correlation functions. On the other hand partition function can depends on other parameters λ' to which the action is independent. Notice that eq.(2.7) can be also a divergent, but this happens often and doesn't really matter. The important aspect is how the behaviour of the partition function depends on its parameters.

The name "partition function" comes from statistical mechanics: consider for instance a map from the circle to another target space

Verify what stated here

$$x : S^1 \to N \qquad x(0) = x(T) = y \in N$$

When we go to the Euclidian space by sending[II] $t \to i\tau$ ($\hbar = 1$) and we compute the path integral for some boundaries condition we obtain

$$\int_{x \in C_T[y,y]} \mathcal{D}x \exp\{-S[x]\} = \langle y | \exp\{-HT\} | y \rangle$$

and this is related to partition function of statistical mechanics, since this means that calculating the statistical partition function by tracing over all the Hilbert space $\mathcal{H}$,

$$\mathrm{Tr}_{\mathcal{H}}(e^{-HT}) = \int \mathrm{d}y \, \langle y | \, e^{-HT} \, | y \rangle = \int \mathrm{d}y \int_{x \in C_T[y,y]} \mathcal{D}x e^{-S[x]}$$

is the same as taking path integral without boundaries, i.e. integrating over all possible configurations $y \in S^1$.

$$\mathrm{Tr}_{\mathcal{H}}(e^{-HT}) = \int_{x \in C_{S1}} \mathcal{D}x e^{-S[x]}$$

2.4 Operators and Time Ordering

2.5 The Continuum Limit and Non-Commutativity

[II] Notice that parametrizing S^1 into the Euclidian, we introduced an additional factor i, that multiplied by the factor i present in the definition of partition function gives a minus. This will happen often when we compute partition integrals explicitly.

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