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Chapter 1

Introduction

Quantum Field Theory (QFT) was born from an attempt to solve inconsistencies of the Dirac's relativistic quantum mechanics (RQM) when the interaction with the electromagnetic field is introduced. However, very soon it become a common framework in many branches of physics, exhibiting an unexpected unity in the description of elementary quantum processes that deeply modifies our view of the physical reality, mostly with a "pictorial"representation in terms of Feynman diagrams. In fact, quite amazingly, elementary QFT processes can be described qualitatively in terms of very few ingredients:

- *propagators*, describing the virtual propagation of quantum particle excitations and drawn as lines, typically oriented;
- *vertices*, describing the process of emission and absorption of particle excitations, possibly changing the nature of the original particle, and drawn as a point from which the propagators of emitted and absorbed particle emerge

Propagators and vertices are then embodied in diagrams, describing the quantum processes. Clearly, a "change of nature" of the particle during emission/absorption is not allowed in standard quantum mechanics (QM).
At the same time for a process describing an electron decelerating by emitting photons (which took away the electron kinetic energy) the number of photons emitted can be arbitrarily high, again QM of finite degrees of freedom is insufficient to describe such process.
Another case in which QM turns out to be insufficient arise in the so called thermodynamic limit in solid state systems. In real physical systems the number of electrons and ions is finite, although usually very big, $N \sim 10^{23}$, and the volume V is finte (infrared (IR) cutoff). Furthermore in a crystal the lattice constant a is finite (ultraviolet (UV) cutoff). However, usually we are interested in universal properties, independent of detail$V$ and a. Therefore it is convenient also in these case to consider the limit $N \to \infty$, $V \to \infty$ with N/V (or its expectation value) constant (thermodynamic IR limit) and $a \to 0$ (continuum UV limit). These limits are not only technically useful, for the non-analyticity appearing the phase transitions, or the appearance of Euclidean invariance, but e.g. the thermodynamic limit guaranteers that the theory does not depend on specific details of V and N.
Notice that if the removal of the IR or the UV cutoff is impossible and we assumed that the theory without the cutoff is the physical one (hence effective field theories in the modern sense are excluded), then the non-existence of the limit implies that the physical theory depends on details at infinite distances (IR) or infinite momentum (UV) in a manner not controllable by regularization.
Furthermore in relativistic QFT (RQFT) a cutoff breaks the Poincaré invariance, and the only possible regularization which does not break such symmetry, the dimensional regularization, has no non-perturbative realization.
Many of the key results of QFT are obtained through a perturbative expansion, which are serious mathematical problems, and there are crucial ares of applications that do not rely on perturbative methods.
The aim of this course is to provide a view of some results in these areas, with examples both from elementary particle and condensed matter physics, emphasising the underlying common features.

Chapter 2

Review of QFT

2.1 Fock space

Ref. [GR96, Chapters 3, 4]; [BS80, Chapters 1, 2]

In $d = 3$ space dimensions, quantum particles are either bosons or fermions (in lower dimensions other braid statistics may arise, but we'll not discuss them here).

Fixed number of particles

The Hilbert space of states for N identical particles $\mathcal{H}_N$ is constructed as follows: let $\mathcal{H}_1$ be the single-particle Hilbert space, Σ_N the permutations group of N objects, $\pi \in \Sigma_N$ and P_π the corresponding operators, $\sigma(\pi)$ the number of exchanges made by π, ε a constant which takes the value $+1$ for bosons and -1 for fermions, then define

$$P^\varepsilon := \frac{1}{N!} \sum_{\pi \in \Sigma_N} (\varepsilon)^{\sigma(\pi)} P_\pi$$

then

$$\mathcal{H}_N^\pm := P^\pm(\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_1)$$

More concretely let A be a complete set of compatible observables in the one-particle Hilbert space of an elementary quantum particle $\mathcal{H}_1$ (we assume for the momenta discrete spectrum for these observables), and $\{|\alpha_i\rangle, i \in I\}$ the corresponding eigenstates, where α_i is the set of common eigenvalues of A. Then an orthonormal basis in $\mathcal{H}_N^\pm$ is given by

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle^\varepsilon := \sqrt{\frac{N!}{\prod_i n_i!}} P^\varepsilon |\alpha_{i_1}\rangle \otimes \ldots \otimes |\alpha_{i_N}\rangle$$

where n_i is the number of one-particle states with eigenvalue α_i in $|\alpha_{i_1}\rangle, \ldots, |\alpha_{i_N}\rangle$, satisfying $\sum_{i\in I} n_i = N$. If the values of n_i are the same for both the sets $\alpha_{i_1} \ldots \alpha_{i_N}$ and $\alpha_{j_1} \ldots \alpha_{j_N}$ then

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle^\varepsilon = \pm |\alpha_{j_1} \ldots \alpha_{j_N}\rangle^\varepsilon$$

and we can label states by their **occupation numbers** $\{n_i\}_{i\in I}$, once the one-particle basis $\{|\alpha_i\rangle, i \in I\}$ has been fixed:

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle =: |n_1, \ldots, n_i, \ldots\rangle = |\{n_i, i \in I\}\rangle$$

By antisymmetry (Pauli principle) for fermions $n_i = 0, 1$, whereas for bosons $n_i \in \mathbb{N}$.
A generic vector in $\mathcal{H}_N^\varepsilon$, then can be written as the linear combination

$$|\Psi_N\rangle = \sum_{\{n_i\}} \Psi(\{n_i\}) |\{n_i\}\rangle$$

with the conditions

$$\sum_{\{n_i\}} |\Psi(\{n_i\})|^2 < \infty \quad , \quad \sum_{i\in I} = N$$

Variable number of particles, $N \to \infty$ limit

Let's consider now the case in which N is not fixed, and may be take $N \to \infty$. Define the **vacuum sector** $\mathcal{H}_0^\varepsilon = \mathbb{C}$ and the corresponding normalized vector $|\Psi_0\rangle$ (or $|0\rangle$) is called the **vacuum**. Formally set

$$\mathcal{F}^\varepsilon := \mathcal{H}_0^\varepsilon \oplus \mathcal{H}_1^\varepsilon \oplus \ldots \oplus \mathcal{H}_N^\varepsilon \oplus \cdots = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^\varepsilon$$

Notice that the direct sum implies that there is no interference between the different sectors $\mathcal{H}_N^\varepsilon$.
We would like to explain better the meaning of the previous formal direct sum. Let

$$\mathcal{D} = \left\{ \bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle \right\}$$

with $N_{\max}$ arbitrary but finite. In this space of direct sum of finite sequences of vectors we define an *inner product* by

$$\left(\bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle , \bigoplus_{N'=0}^{N'_{\max}} |\Phi_{N'}\rangle \right) := \sum_{N=0}^{\infty} \langle \Psi_N | \Phi_N \rangle$$

where the sum is formally extended to infinity as only a finite number of terms is non zero. Together with this inner product the space $\mathcal{D}$ is pre-Hilbert, and then $\mathcal{F}^\varepsilon$ is defined as the Hilbert space obtained by completion od $\mathcal{D}$, i.e. the space of sequences $\bigoplus_{N=0}^{\infty} |\Psi_N\rangle$ such that $\sum_{N=0}^{\infty} \langle \Psi_N | \Psi_N \rangle < \infty$. Notice that $\mathcal{D}$ is dense in $\mathcal{F}^\varepsilon$, as required by the definition of Hilbert space.
Then, the space $\mathcal{F}^\varepsilon$ allows the description of processes with non-conserved number of particles.
In D we can define the **annihilation and creation operators**

$$\begin{aligned} a_i \,|\{n_j\}\rangle^+ &:= \sqrt{n_i}\,|\{n_{j\neq i}, n_i - 1\}\rangle^+ \\ a_i^\dagger \,|\{n_j\}\rangle^+ &:= \sqrt{n_i + 1}\,|\{n_{j\neq i}, n_i + 1\}\rangle^+ \end{aligned}$$

for the bosons and

$$\begin{aligned} a_i \,|\{n_j\}\rangle^- &:= (-1)^{(\sum_{k<i} n_k)} n_i \,|\{n_{j\neq i}, n_i - 1\}\rangle^- \\ a_i^\dagger \,|\{n_j\}\rangle^- &:= (-1)^{(\sum_{k<i} n_k)} (1 - n_i)\,|\{n_{j\neq i}, n_i + 1\}\rangle^- \end{aligned}$$

for fermions.
In follows from the definition that

(i) $a_i^\dagger$ is the adjoint of a_i;

(ii) $a_i \,|\Psi_0\rangle = 0$;

(iii) for bosons hold the **canonical commutation relations** (CCR)

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] \quad , \quad [a_i, a_j^\dagger] = \delta_{ij}$$

while for fermions hold the **canonical anticommutation relations** (CAR)

$$\{a_i, a_j\} = 0 = \{a_i^\dagger, a_j^\dagger\} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij}$$

(iv) defining $\hat{N}_i := a_i^\dagger a_i$ we have

$$\hat{N}_i \,|\{n_j\}\rangle = n_i \,|\{n_j\}\rangle$$

and $\hat{N} := \sum_{i\in I} \hat{N}_i$ is well defined in $\mathcal{D}$;

(v) any vector can be constructed by means of applications of creation operators

$$|\{n_j\}\rangle = \frac{1}{\sqrt{\prod_j n_j!}} \prod_j (a_j^\dagger)^{n_j} \,|0\rangle$$

Annihilation and creation operators corresponding to another basis (i.e. another complete set of commuting observables with discrete spectrum) $\{|\beta_j\rangle, j \in J\}$ can be obtained by applying the Dirac completeness

$$\sum_{i\in I} |\alpha_i\rangle \langle\alpha_i| = \mathbb{1}_{\mathcal{H}_1}$$

as follows

$$b_j^\dagger |0\rangle = |\beta_j\rangle = \sum_{i\in I} |\alpha_i\rangle \langle\alpha_i|\beta_j\rangle = \sum_{i\in I} a_i^\dagger |0\rangle \langle\alpha_i|\beta_j\rangle$$

implying

$$b_j^\dagger = \sum_{i\in I} \langle\alpha_i|\beta_j\rangle\, a_i^\dagger$$

These ideas extends to the case of complete sets of commuting observables with continuum (or mixed) spectrum in $\mathcal{H}_1$ using

$$\int \mathrm{d}\alpha\ |\alpha\rangle \langle\alpha| = \mathbb{1}_{\mathcal{H}_1}$$

Formally we can set $a^\dagger(\alpha)|0\rangle = |\alpha\rangle$ and then

$$a^\dagger(\alpha) = \sum_{i\in I} \langle\alpha_i|\alpha\rangle\, a_i^\dagger$$

but since $|\alpha\rangle$ is only an improper state ($\langle\alpha|\alpha'\rangle \sim \delta(\alpha-\alpha')$) then $a^\dagger(\alpha)$ is not a true operator, it is an *operator valued distribution*, i.e. is a true operator only if smeared out with a test function f:

$$a(f) = \int a(\alpha) f(\alpha) \mathrm{d}\alpha$$

A typical example is given for an elementary particle with classical analogue (in $d=3$) by $|\alpha\rangle = |\boldsymbol{x}\rangle$ or $|\alpha\rangle = |\boldsymbol{p}\rangle$. If we define $a^\dagger(\boldsymbol{p})|0\rangle = |\boldsymbol{p}\rangle$ with $[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p}-\boldsymbol{p}')$, then is clear that this is not an ordinary operator, rather an operator valued distribution. When we turn to the $\boldsymbol{x}$-representation of a (i.e. we smear $a(\boldsymbol{p})$ with $f(\boldsymbol{p}) = \langle\boldsymbol{x}|\boldsymbol{p}\rangle$) we get

$$\psi(\boldsymbol{x}) = \int \mathrm{d}^3p\ \langle\boldsymbol{x}|\boldsymbol{p}\rangle\, a(\boldsymbol{p}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{p}} a(\boldsymbol{p})$$

and $\psi(\boldsymbol{x})$ is called a **quantum field operator** and it satisfies (for bosons)

$$[\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{y})] = \delta(\boldsymbol{x}-\boldsymbol{y})$$

where all the trivial (anti)commutation relation will be always omitted from now on. Notice that the last relation can be interpreted as a form of "locality": the effect of a field in a point cannot affect the effect of a simultaneous field in a different point.
The operator $\psi^\dagger(\boldsymbol{x})$ formally creates a particle with wave function a Dirac δ with support on $\boldsymbol{x}$. More precisely if $f(\boldsymbol{x}) \in L^2(\mathbb{R}^3, \mathrm{d}^3x)$ then

$$\psi^\dagger(f) = \int \mathrm{d}^3x\, \psi^\dagger(\boldsymbol{x}) f(\boldsymbol{x})$$

creates a particle with wave function $f(\boldsymbol{x})$. Notice that if in $\mathcal{H}_1$ we can use an orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$, then setting $a(f_i) =: a_i$ it satisfies the CCR because

$$[a(f_i), a^\dagger(f_j)] = (f_i, f_j) = \delta_{ij}$$

If the particle we are interested in is not elementary the situation is lightly nore complex, but we will not discuss it.

Relativistic case, $N \to \infty$ limit

Notice that moving from non-relativistic context (where space coordinates and time are treated differently) to the relativistic one, some problems arises. Dirac formulation of relativistic quantum mechanics treats time and space in the same way, making the space coordinate a label like the time rather than an observable. Indeed we have to take care of the consequences of the Heisenberg principle: due to uncertainty on the energy we are able to produce particle-antiparticle pairs, but if we are interested in measuring the position as better as possible, then the uncertainty on the momentum become huge and lot of pairs are produced, and we are no more able to identify which is the particle we want to measure. Therefore the position is no more an observable in the relativistic framework, and can only be used as a label to describe the evolution of a state, as we do for the time.
Since we cannot characterize particles using position as an observable in a complete set of compatible observables (as we do in the non-relativistic case, where position, energy and spin provides an irreducible set of compatible observables), we need to understand how to choose a new complete set of compatible observables which allows us to build an Fock space for elementary particles using eigenvectors.
A celebrated theorem of Wigner states that the one-particle Hilbert space of an elementary particle should be the representation space of an irreducible unitary representation of space-time symmetries (i.e. the universal covering of the restricted Poincaré group) and internal symmetries (e.g. EM charge conservation, we do not discuss them here).
Irreducible unitary representations of space-time symmetries, according to Wigner's theorem, are characterized by the mass $m \in \mathbb{R}_+$ and either the spin $s \in \mathbb{N}/2$ if $m > 0$ or the helicity (projection of the spin in the direction of the motion[I]) $h \in \mathbb{Z}/2$ if $m = 0$.
Let's consider for simplicity the case of a massive ($m > 0$) scalar ($s = 0$) particle. We know that for a relativistic particle the dispersion relation is

$$p_\mu p^\mu = m^2 \quad , \quad p^0 > 0$$

i.e. the momentum should be contained in the *positive hyperboloid of mass m*, denoted by V_m^+. Notice that if $m = 0$ we have that the momentum should be contained in the *forward light cone*.
Since we want to have a representation of the (covering of the) Poincaré group all the points in the hyperboloid should be weighted by the same weight, and we cannot use the position as an observable, then a natural choice for our Hilbert space for one particle is provided by

$$\mathcal{H}_1 = L^2\left(\mathbb{R}^4, \frac{\mathrm{d}^4 p}{(2\pi)^4}\, 2\pi\delta(p_\mu p^\mu - m^2)\theta(p^0)\right) = L^2\left(V_m^+, \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}}\right)$$

Let $a^\dagger(p)\left|0\right\rangle =: \left|p\right\rangle$ be the (generalized) eigenvector of the four-momentum operator $\hat{p}^\mu$ for (generalized) eigenvalue $p^\mu =: p \in \mathbb{R}^4$. The Dirac completness relation in $\mathcal{H}_1$ is given by

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\theta(p^0)\left|p\right\rangle\left\langle p\right| = \mathbb{1}_{\mathcal{H}_1}$$

It immediately follows multiplying by $\left|p'\right\rangle$, $p' \in V_m^+$ that

$$\begin{aligned}
\left|p'\right\rangle = \left|\boldsymbol{p'}, \sqrt{\boldsymbol{p'}^2 + m^2}\right\rangle &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)\left|p\right\rangle\left\langle p|p'\right\rangle \\
&= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)\left|p\right\rangle\left\langle 0\right| a(p)a^\dagger(p')\left|0\right\rangle \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}}\left|\boldsymbol{p}, \sqrt{\boldsymbol{p}^2 + m^2}\right\rangle\left\langle 0\right|[a(p)a^\dagger(p')]\left|0\right\rangle
\end{aligned}$$

therefore in order to get consistency the following commutation relation should be satisfied:

$$[a(p), a^\dagger(p')] = (2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2}\,\delta(\boldsymbol{p} - \boldsymbol{p'}) \tag{2.1}$$

[I]Notice that helicity is well defined only if the particle moves at the speed of light, otherwise through a change of frame one can reverse the projection direction.

Notice that if $\{f_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}_1$, then we define

$$a_i := \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0)f_i(p)a(p)$$

and it satisfies the canonical $[a_i, a_j^\dagger] = \delta_{ij}$, hence also in the relativistic case it is possible to obtain canonical commutation relation by smearing a and $a^\dagger$ with an orthonormal basis of the Hilbert space. The new factor $2\sqrt{\boldsymbol{p}^2+m^2}$ in eq. (2.1), which wasn't present in the non-relativistic case, lead to non-locality problems if one proceeds defining quantum field operators as in the non-relativistic case. If we try to apply the previous idea to define quantum field operators, i.e.

$$\psi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0)a(p)e^{ip\cdot x}$$

then

$$\begin{aligned}
[\psi(\boldsymbol{x},0),\psi^\dagger(\boldsymbol{y},0)] &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\frac{\mathrm{d}^4 p'}{(2\pi)^3}\delta(p^2-m^2)\delta(p'^2-m^2)\theta(p^0)\theta(p'^0)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}e^{i\boldsymbol{p'}\cdot\boldsymbol{y}}[a(p),a^\dagger(p')] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3\, 2\sqrt{\boldsymbol{p}^2+m^2}}\frac{\mathrm{d}^3 p'}{(2\pi)^3\, 2\sqrt{\boldsymbol{p'}^2+m^2}}e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p'}\cdot\boldsymbol{y})}\,(2\pi)^3 2\sqrt{\boldsymbol{p}^2+m^2}\delta(\boldsymbol{p}-\boldsymbol{p'}) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3\, 2\sqrt{\boldsymbol{p}^2+m^2}}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \neq 0
\end{aligned}$$

even for $\boldsymbol{x}\neq\boldsymbol{y}$. This violates the "locality" even in the weak non-relativistic form we showed before. Actually, since the concept of "present" (or equivalently of "simultaneity") is not universal in relativity, the vanishing of the commutator (for observable fields at least) should hold for x and y space-like separated (we denoted space-like separated coordinates x and y by $x \times y$).
Actually in general the fields are not observable (e.g. the charged scalar field is not, since it is not self-adjoint) but a remarkable theorem of Doplicher-Roberts (essentially) shows that in a massive RQFT if the observables commute at space-like distances, then the fields of the corresponding QFT either commute or anticommute at space-like distances in $d = 3+1$. Moreover the spin-statistics theorem proves that fields with integer spin are bosons and those with half-integer spin are fermions. Hence we know that in the massive case fields with integer spin commute and those with half-integer spin anticommute.
Therefore we need to impose such (anti)commutation relations for our relativistic fields at space-like separated coordinates. Since $a(p)\left|0\right\rangle = 0$, the state obtained applying $\psi^\dagger(x)$ to $\left|0\right\rangle$ is the same if one add an additional contribution $\sim a$ to $\psi^\dagger(x)$. Let's define

$$\phi(x) = \psi(x) + \psi^\dagger(x)$$

where the presence of both ψ and $\psi^\dagger$ is reminiscent of the fact that the dispersion relation $p_\mu p^\mu = m^2$ has two solutions $\pm\sqrt{\boldsymbol{p}^2+m^2}$, one to one associated to ψ and $\psi^\dagger$. Then $\phi^\dagger\left|0\right\rangle = \psi^\dagger\left|0\right\rangle$, ϕ is a self-adjoint field operator and as we desired

$$[\phi(x),\phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0$$

or more precisely

$$[\phi(f),\phi(g)] = 0 \quad \text{if} \quad \operatorname{supp} f \times \operatorname{supp} g$$

It is however clear that even if the above sense of $\phi(x)$ is "localized in x" (so that it cannot affect points space like separated) or better the sense of $\phi(f)$ is "localized in $\operatorname{supp} f$" (using Doplicher-Roberts theorem), conversely the state $\phi(x)\left|0\right\rangle$ is not localized in x and $\phi(f)\left|0\right\rangle$ is not localized in $\operatorname{supp} f$. In fact the *two point correlation function* for a real field reads

$$\begin{aligned}
\left\langle 0\right|\phi(x)\phi(y)\left|0\right\rangle &= \left\langle 0\right|\psi(x)\psi^\dagger(y)\left|0\right\rangle = \left\langle 0\right|[\psi(x)\psi^\dagger(y)]\left|0\right\rangle = \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\sqrt{\boldsymbol{p}^2+m^2}}e^{ip\cdot(x-y)} \sim \frac{m^{1/2}}{|\boldsymbol{x}-\boldsymbol{y}|^{3/2}}e^{-m|\boldsymbol{x}-\boldsymbol{y}|} \neq 0
\end{aligned}$$

where in the second line $p^0 := \sqrt{\boldsymbol{p}^2 + m^2}$ and the approximation holds for $(x-y)^2 \ll -1$.[II]
Differently from NRQFT case, where

$$\langle 0|\,\psi(\boldsymbol{x})\psi^\dagger(\boldsymbol{y})\,|0\rangle = \delta(\boldsymbol{x}-\boldsymbol{y})$$

holds, in RQFT $a^\dagger(p)$ applied at the vacuum creates a one-particle state with 4-momentum p^μ, but $\phi(x)$ applied to the vacuum does not create a particle localized in x^μ (i.e. with "wave function" given by a δ localized in x^μ).
Now it's very clear that $\hat{x}^\mu$ is not a good observable in RQFT, whereas $\hat{p}^\mu$ is a good observable. It is indeed possible to create a particle with well defined momentum p^μ but is impossible to create a particle with well defined momentum x^μ.
The physical underlying reason as we already mentioned is that the measure of $\hat{x}^\mu$ would produce a diverging fluctuation in p due to Heisenberg principle $\Delta x^\mu \Delta p^\mu \gtrsim \hbar$, allowing the production of particle-hole pairs. Thus the "space-time coordinate x" looses its meaning, since it is impossible to understand to which particle it refers.
Finally notice that from its definition $\phi(x)$ satisfies the homogeneous Klein-Gordon equation

$$(\Box + m^2)\phi(x) := \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi(x) = 0$$

If we replace c with the phonon velocity this is the same equation appearing in Condensed Matter for optical phonon at small (quasi)-momentum (in the lattice p^μ is periodic), showing a first example that some excitations in material exhibits "relativistic features" in some regime of parameters.

2.2 Characteristic features of Fock space

Ref. [BR97, Section 5.2]

The aim of this section is to understand how CCR and CAR affects the properties of the Fock space we built up. We have seen that the creation and annihilation operators for bosons and fermions satisfy the CCR and the CAR respectively

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij}$$

Since such relations are purely algebraic, can be thought as the caracterizing rules for an algebra (i.e. a vector space where we defined the multiplication) endowed with an involution denoted by $\dagger$ (which

[II] Let's see how the result is obtained. First replace $(x-y) \mapsto x$ and then set $\boldsymbol{p}\cdot\boldsymbol{x} = |\boldsymbol{p}||\boldsymbol{x}|\cos\theta =: |\boldsymbol{p}||\boldsymbol{x}|u$ (here $u := \cos\theta$). Then

$$\int \frac{\mathrm{d}^3 p}{\sqrt{\boldsymbol{p}^2+m^2}} e^{ip^0x^0} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \sim \int \frac{|\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}|\,\mathrm{d}u\,\mathrm{d}\phi}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} \sim \int_0^\infty \frac{|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} \int_{-1}^{1} \mathrm{d}u\, e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} =$$

$$= \int_0^\infty \frac{|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} \frac{1}{-i|\boldsymbol{p}||\boldsymbol{x}|}\left(e^{-i|\boldsymbol{p}||\boldsymbol{x}|} - e^{i|\boldsymbol{p}||\boldsymbol{x}|}\right) = \frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^{\infty} \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i(p^0x^0+|\boldsymbol{p}||\boldsymbol{x}|)}$$

Due to $x^2 \ll -1$, i.e. $|\boldsymbol{x}| \gg x^0$ and $|\boldsymbol{x}| \gg 1$, we have that

$$\frac{\partial}{\partial|\boldsymbol{p}|}(p^0x^0 + |\boldsymbol{p}||\boldsymbol{x}|) = \frac{|\boldsymbol{p}|x^0}{p^0} + |\boldsymbol{x}| \approx |\boldsymbol{x}|$$

hence the contribution of p^0x^0 to the variation of the total phase respect to $|\boldsymbol{p}|$ is negligible, and it will give just a small correction to the final result

$$\frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^{\infty} \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i(p^0x^0+|\boldsymbol{p}||\boldsymbol{x}|)} \approx \frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^{\infty} \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i|\boldsymbol{p}||\boldsymbol{x}|} = \frac{1}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}|\boldsymbol{p}|\, \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2+m^2}}$$

Substituting $p \mapsto m\sinh(t)$, $\mathrm{d}p = \sqrt{|\boldsymbol{p}|^2+m^2}\mathrm{d}t$, we get

$$\frac{1}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}|\boldsymbol{p}|\, \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2+m^2}} = \frac{m}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}t\, \sinh(t)\sin(m|\boldsymbol{x}|\sinh(t)) = \frac{m}{i|\boldsymbol{x}|}\mathrm{K}_1(m|\boldsymbol{x}|)$$

where $\mathrm{K}_n(x)$ is the modified Bessel function, which satisfies

$$\mathrm{K}_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \quad \text{for} \quad x \gg n$$

promote the algebra to a *-algebra), generated by $\{a_i\}_{i\in I}$. In this point of view a and $a^\dagger$ are then just representations of this algebra as operators acting on $\mathcal{F}^\pm$ ($\mathcal{F}^\pm$ are representation spaces of the algebra). If the set I is finite, i.e. the algebra is finitely generated, then the *von Neumann uniqueness theorem* ensures that all the representation of CCR and CAR are unitarily equivalent, hence all possible representations live in the same abstract Hilbert space.
For instance this occur in QM for an N particles system in $\mathbb{R}^d$ (finite degrees of freedom (d.o.f.)), where we can put $a_i = \frac{1}{\sqrt{2}}(q_i + p_i)$, $i = 1, \dots, d$, with representation space $L^2((\mathbb{R}^d)^N, \mathrm{d}^d x_1 \dots \mathrm{d}^d x_N)$.
This is not true anymore for infinitely generated algebras (infinite d.o.f.). Roughly speaking, a unitary operator mapping one representation of the algebra into another one can be thought in zero dimensions as $e^{iN} \in U(1)$ for finite N but as $N \to \infty$ it vanishes. Infinitely generated CCR and CAR algebras have infinite inequivalent representations acting in completely disjoint Hilbert spaces. Moreover each of these possible representations describes completely different physics.
The point is now understand what actually characterizes a specific $\mathcal{F}$, since CCR and CAR are the same also for infinitely many different Hilbert spaces. The answer is given by the number operator $\hat{N}$: indeed it is a well defined observable if and only if the elements of $\mathcal{F}$ are created by a specific representation of $a_i^\dagger$, or in other words if the excitations created out from the vacuum by the $a_i^\dagger$ are the ones that can be counted (even if infinitely many) by a specific $\hat{N}$.
Mathematically, $\hat{N} := \sum_{i\in I} a_i^\dagger a_i$ is well defined in $\mathcal{H}$ if exists a domain D dense in $\mathcal{H}$ containing $|0\rangle$ in which $\hat{N}$ is self-adjoint. This also implies that its spectrum is $\sigma(\hat{N}) = \mathbb{N}$ by the standard argument seen for the harmonic oscillator in QM.
Representations of CCR or CAR on which $\hat{N}$ is not well defined are called *non-Fock representation.* We know wonder whether is possible or not to create non-Fock representations. A simple example of this is given by *Bogoliubov transformations*: starting from given a_i, $a_i^\dagger$, we define

$$a'_i := \alpha_i a_i + \beta_i a_i^\dagger$$
$$a'^\dagger_i := \alpha_i^* a_i^\dagger + \beta_i^* a_i$$

for $\alpha_i, \beta_i \in \mathbb{C}$. If $\{a_i\}_i$ generates a representation of CCR then

$$[a'_i, a'^\dagger_j] = (|\alpha_i|^2 - |\beta_i|^2)\delta_{ij}$$

from which we see that $\{a'_i\}_i$ gives a new representation of CCR provided that $|\alpha_i|^2 - |\beta_i|^2 = 1$. Conversely if $\{a_i\}_i$ generates a representation of CAR then

$$\{a'_i, a'^\dagger_j\} = (|\alpha_i|^2 + |\beta_i|^2)\delta_{ij}$$

from which we see that $\{a'_i\}_i$ gives a new representation of CAR provided that $|\alpha_i|^2 + |\beta_i|^2 = 1$. Let $|0\rangle$ be the vacuum in the a_i representation, then

$$\langle 0|\, \hat{N}'\, |0\rangle = \sum_{i\in I} \langle 0|\, (\alpha_i^* a_i^\dagger + \beta_i^* a_i)(\alpha_i a_i + \beta_i a_i^\dagger)\, |0\rangle = \sum_{i\in N} |\beta_i|^2$$

If $\sum_i |\beta_i|^2 < \infty$ then such transformation is allowed and since it is unitary then the Fock space of a_i is a Fock space for a'_i too (even if states, including the vacuum, may not be the same). Otherwise, for $\sum_i |\beta_i|^2 = \infty$, the vacuum $|0\rangle$ in the a_i representation is not even an element of $D(\hat{N}') := \{|\psi\rangle \text{ s.t. } ||\hat{N}'\,|\psi\rangle\,|| < \infty\}$, nor is any $a_i^\dagger$ excitation created from $|0\rangle$.[III] Therefore if for $\sum_i |\beta_i|^2 = \infty$ the Fock space for $\{a_i\}_{i\in I}$ is not the right space for the excitations described by a'_i. Actually we proved something more, we proved that the Fock space for a_i and $a_i^\dagger$ are completely disjoint since no state of one of the two spaces is contained in the other one.
This actually happens in practice also in some simple situations such as for *Fermi gas*.[IV] Let's consider N fermions in a finite volume V and zero temperature $T = 0$. In such conditions the density of states is

$$n(\boldsymbol{k}) = \begin{cases} 2 & \text{for} \quad |\boldsymbol{k}| \le k_F \\ 0 & \text{for} \quad |\boldsymbol{k}| > k_F \end{cases}$$

[III] Indeed using CCR and CAR one obtains

$$\hat{N}' a_i^\dagger = \alpha_i a'^\dagger_i \pm \beta_i^* a'_i + a_i^\dagger \hat{N}'$$

hence for any polynomial in $a_i^\dagger$ eventually $\hat{N}'$ hit the vacuum producing a divergence.
[IV] This example is presented in **[Strocchi:1985aa]**.

where we have two particles for each momentum state due to the Fermi statistic. Recall that k_F is the Fermi momentum. Clearly the ground state $|\psi_0\rangle$ corresponding to such configuration $u(\boldsymbol{k})$ is not a vacuum for the annihilation operators $a(\boldsymbol{k},s)$ when $|\boldsymbol{k}| \leq k_F$ because $a(\boldsymbol{k},s)\,|\psi_0\rangle \neq 0$, since it just annihilate a particle which certainly exists in the whole ensemble where all configurations for $\boldsymbol{k} \leq k_F$ are occupied.
Nevertheless if one defines the Bogoliubov transformation

$$a'(\boldsymbol{k},s) = \alpha(\boldsymbol{k},s)a(\boldsymbol{k},s) + \beta(-\boldsymbol{k},-s)a^\dagger(-\boldsymbol{k},-s)$$

with

$$\begin{cases} \alpha(\boldsymbol{k},s) = 1 & \beta(-\boldsymbol{k},-s) = 0 \qquad \text{for} \quad |\boldsymbol{k}| > k_F \\ \alpha(\boldsymbol{k},s) = 0 & \beta(-\boldsymbol{k},-s) = 1 \qquad \text{for} \quad |\boldsymbol{k}| \leq k_F \end{cases}$$

in such a way that the ground state is a Fock vacuum for the $a'(\boldsymbol{k},s)$:

$$a'(\boldsymbol{k},s)\,|\psi_0\rangle = 0$$

According to the previous argument we know that $|\psi_0\rangle \in \mathcal{F}$ (for $a(\boldsymbol{k},s)$) if and only if $\sum_{\boldsymbol{k},s} |\beta(\boldsymbol{k},s)|^2 < \infty$. This is clearly true if $N < \infty$, but it is false in the limit $N \to \infty$, $V \to \infty$, N/V fixed. Therefore, in the thermodynamic limit the Hilbert space of a quantum Fermi gas (the one built with $a'(\boldsymbol{k},s)$ is not in the Fock space of the non-relativistic fermions (the one built with $a(\boldsymbol{k},s)$).
Another example of non-Fock space is provided by the radiation emitted by a charged particle changing its momentum $\boldsymbol{p} \mapsto \boldsymbol{p'}$. A simple qualitative argument goes as follows: the energy emitted by radiation goes like

$$\mathcal{E} = \int \mathrm{d}^3k\, |\mathbf{E}_{\text{rad}}(\boldsymbol{k})|^2$$

and we know that is is a finite number, moreover since photons asymptotically (where they are free) have energy $\omega(\boldsymbol{k}) = c|\boldsymbol{k}|$ then the number of emitted photons is

$$N = \int \mathrm{d}^3k \frac{|\mathbf{E}_{\text{rad}}(\boldsymbol{k})|^2}{c|\boldsymbol{k}|}$$

The problem is that the electric field of the radiation behaves in the infrared ($|\boldsymbol{x}| \to \infty$) as $\mathbf{E}_{\text{rad}}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-2}$ implying $\mathbf{E}(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1}$ as $|\boldsymbol{k}| \to 0$, i.e. is logarithmically divergent. Hence

$$N \sim \int \mathrm{d}^3k\, \frac{1}{|\boldsymbol{k}|}\left(\frac{1}{|\boldsymbol{k}|}\right)^2 = \infty$$

therefore the Hilbert space of photons at the end of the process is not the fock space $\mathcal{F}_{\text{in}}$ of the initial photons.

2.3 Interacting fields

Ref. [GR96, Chapter 8, 9]; [Kle15, Section 1.9, Chapter 10]; [BS80, Chapter 4]

Up to now we have considered only "free" fields whose Hilbert space is a fock space, however to obtain physical informations we need interactions. Now, starting from the particle physics case (zero temperature $T = 0$, zero density $n = 0$, relativistic case), and then moving to the condensed matter case, we will introduce the perturbative calculation of the interactions. We are mainly interested in understand which are the problems that arise in the perturbative approach and in general the need of non-perturbative techniques.
In particle physics most of the informations are extracted from scattering experiments. As a very simple model we can consider the ϕ^4 model, which in its complex version describe the (low energy of the) Higgs field with only quartic self-interactions taken into account, in the unbroken symmetry phase (which we presume was present in the early universe).
To be concrete we consider a real field $\phi(x)$ with canonical lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \text{and} \quad \mathcal{L}_I \quad = -\frac{\lambda}{4}\phi^4$$

The corresponding Hamiltonian is

$$H = H_0 + H_I = \int \mathrm{d}\boldsymbol{x}\ (\mathcal{H}_0(\boldsymbol{x}) + \mathcal{H}_I(\boldsymbol{x}))$$

with

$$\mathcal{H}_0(\boldsymbol{x}) = \frac{\pi^2}{2}(\boldsymbol{x}) + \frac{(\boldsymbol{\nabla}\phi)^2(\boldsymbol{x})}{2} + \frac{m^2}{2}\phi^2(\boldsymbol{x}) \quad \text{and} \quad \mathcal{H}_I(\boldsymbol{x}) = \frac{\lambda}{4}\phi^4(\boldsymbol{x})$$

where the canonical momentum is $\pi(x) = \dot{\phi}(x)$ and the canonical quantization would give

$$[\hat{\phi}(\boldsymbol{x}, t), \dot{\hat{\phi}}(\boldsymbol{y}, t)] = i\hbar\delta(\boldsymbol{x} - \boldsymbol{y})$$

Now one may wonder in which Hilbert space $\hat{\phi}$ is defined as an operator-valued distribution, and if such space is the same as the Fock space of the free scalar field. Such question is actually very non-trivial. To gain the maximum support from our knowledge of the free field let us first suppost that we switch-off the interaction term H_I for large $|t|$, i.e. we replace λ by a C^∞ function of t, $\lambda_\varepsilon(t)$, such that

$$\lambda_\varepsilon(t) = \begin{cases} \lambda & |t| \leq \varepsilon^{-1} \\ 0 & |t| \to \infty \end{cases}$$

and we denoted the new interacting Hamiltonian with H^I_ε. Then we will take the limit $\varepsilon \to 0$ to restore the physical situation. For any finite $\varepsilon > 0$ in the limit $t \to -\infty$ the field $\hat{\phi}(\boldsymbol{x}, t)$ tends to the free field $\hat{\phi}_{\text{in}}$, defined in the Fock space $\mathcal{F}_{\text{in}}$ and in the limit $t \to +\infty$ to the free field $\hat{\phi}_{\text{out}}$ defined in $\mathcal{F}_{\text{out}}$.
Notice that in general (as we will comment later on) to have a well defined expression we need to cutoff also in the spatial direction, so that ε become an infrared regulator of space time, and possibly we need to introduce also ultraviolet counterterms.
Since we *assumed* that $\hat{\phi}$ (using the Heisenberg picture) satisfies the CCR (but this assumption is not guaranteed) and by definition CCR are obeyed also by $\hat{\phi}_{\text{in}}$ and $\hat{\phi}_{\text{out}}$ since they are free, then it exists a unitary one-parameter group $U^I_\varepsilon(t)$ such that

$$\begin{aligned} U^{I\dagger}_\varepsilon(t)\hat{\phi}_{\text{in}}(\boldsymbol{x}, t)U^I_\varepsilon(t) = \hat{\phi}(\boldsymbol{x}, t) \\ U^{I\dagger}_\varepsilon(t)\hat{\pi}_{\text{in}}(\boldsymbol{x}, t)U^I_\varepsilon(t) = \hat{\pi}(\boldsymbol{x}, t) \end{aligned} \tag{2.2}$$

If this is true then

$$\begin{aligned} U^I_\varepsilon(t) \xrightarrow[t\to-\infty]{} \mathbb{1} \\ U^I_\varepsilon(t) \xrightarrow[t\to+\infty]{} S_\varepsilon \end{aligned}$$

where (since U^I_ε is unitary) S_ε is a unitary operator called *Scattering matrix (with cutoff ε)* such that

$$S^\dagger_\varepsilon\hat{\phi}_{\text{in}}S_\varepsilon = \hat{\phi}_{\text{out}}$$

Notice that by consistency, if $U(t) = \exp\{-\frac{itH}{\hbar}\}$ denotes the unitary evolution in Heisenberg picture of $\hat{\phi}$ using the complete Hamiltonian, then eq. (2.2) implies

$$U^{I\dagger}_\varepsilon(t)e^{+\frac{itH_0}{\hbar}}\hat{\phi}_{\text{in}}(\boldsymbol{x}, 0)e^{-\frac{itH_0}{\hbar}}U^I_\varepsilon(t) = U^\dagger(t)\hat{\phi}(\boldsymbol{x}, 0)U(t)$$

must hold, so that

$$U^I_\varepsilon(t) = e^{\frac{itH_0}{\hbar}}U(t) \tag{2.3}$$

This provides the evolution operator in the *interaction picture* for the states, while as we know the fields in such picture evolve according to Heisenberg picture with the free Hamiltonian, so that

$$\langle\chi|\,\hat{\phi}(\boldsymbol{x}, t)\,|\psi\rangle = \underbrace{\langle\chi|\,U^{I\dagger}_\varepsilon(t)}_{\substack{\text{Sch. evol. of } |\chi\rangle \\ \text{using } U^I_\varepsilon}}\ \underbrace{e^{\frac{itH_0}{\hbar}}\hat{\phi}_{\text{in}}(\boldsymbol{x}, 0)e^{-\frac{itH_0}{\hbar}}}_{\substack{\text{Heis. evol. of } \hat{\phi} \\ \text{using } H_0.}}\ \underbrace{U^I_\varepsilon(t)\,|\psi\rangle}_{\substack{\text{Sch. evol. of } |\psi\rangle \\ \text{using } U^I_\varepsilon}}$$

In order to determine $U^I_\varepsilon(t)$, we can differentiate respect to t in eq. (2.3) and we get

$$\begin{aligned}\frac{\mathrm{d}U^I_\varepsilon(t)}{\mathrm{d}t} &= \frac{i}{\hbar}H_0 e^{\frac{itH_0}{\hbar}}U(t) - \frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}(H_0 + H^I_\varepsilon)U(t) \\ &= -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H^I_\varepsilon U(t) = -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H^I_\varepsilon e^{-\frac{itH_0}{\hbar}}U^I_\varepsilon(t)\end{aligned}$$

and using the boundary condition $U^I_\varepsilon(-\infty) = \mathbb{1}$ we finally get the integral equation

$$U^I_\varepsilon(t) = \mathbb{1} - \frac{i}{\hbar}\int_{-\infty}^t H^I_\varepsilon(t')U^I_\varepsilon(t')\mathrm{d}t' \tag{2.4}$$

and by successively re-inserting the l.h.s. of eq. (2.4) we get

$$U^I_\varepsilon(t) = \sum_{n=0}^\infty \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^t \mathrm{d}t_1 \ldots \int_{-\infty}^t \mathrm{d}t_n\, \mathrm{T}[H^I_\varepsilon(t_1)\ldots H^I_\varepsilon(t_n)] =: \mathrm{T}\exp\Big[-\frac{i}{\hbar}\int_{-\infty}^t H^I_\varepsilon(t')\mathrm{d}t'\Big]$$

where $\mathrm{T}[A_1(t_1)\ldots A_n(t_n)]$ is the *time-ordering* defined by

$$\mathrm{T}[A_1(t_1)\ldots A_n(t_n)] = \sum_{\pi\in\Sigma_n}\Theta(t_{\pi(1)},\ldots,t_{\pi(n)})\epsilon^{\sigma(\pi)}A_{\pi(1)}(t_{\pi(1)})\ldots A_{\pi(n)}(t_{\pi(n)})$$

with

$$\Theta(t_{\pi(1)},\ldots,t_{\pi(n)}) = \begin{cases}1 & \text{if} \quad t_{\pi(1)} \geq t_{\pi(2)} \geq \cdots \geq t_{\pi(n)} \\ 0 & \text{otherwise}\end{cases}$$

By construction $\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}} \subset \mathcal{H}$ (the space of $\hat{\phi}$) and if $|\text{in}\rangle \in \mathcal{F}_{\text{in}}$ is the state of free particles prepared at $t = -\infty$ and $|\text{out}'\rangle \in \mathcal{F}_{\text{out}}$ the state of free particles found at $t = +\infty$, then the probability to get this transition is given, according to the rules of QM, by

$$|\langle\text{in}|\text{out}'\rangle|^2 = |\langle\text{in}|S_\varepsilon|\text{in}'\rangle|^2$$

Hence the scattering is analyzed in terms of these matrix elements.
These matrix elements are in turn related to the *correlation or Green functions* of $\hat{\phi}$

$$G^{(n)}(x_1,\ldots,x_n) := \langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle$$

by the LSZ (Lehmann-Symanzik-Zimmermann) formula: if the in-state is given by particles with momenta $q_1,\ldots,q_m$ and the out state by particles with momenta $p_1,\ldots,p_n$, then

$$\begin{aligned}&\langle q_1,\ldots,q_m\,\text{in}|p_1,\ldots,p_n\,\text{out}\rangle = \\ &\quad = i^{m+n}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_m\int\mathrm{d}^4y_1\ldots\mathrm{d}^4y_n\times \\ &\qquad \times e^{-i(q_1x_1+\ldots+q_mx_m)}e^{i(p_1y_1+\ldots+p_ny_n)}\times \\ &\qquad \times(\Box_{x_1}+m^2)\ldots(\Box_{x_m}+m^2)(\Box_{y_1}+m^2)\ldots(\Box_{y_n}+m^2)\times \\ &\qquad \times\langle 0|\mathrm{T}[\hat{\phi}(y_1)\ldots\hat{\phi}(y_n)\hat{\phi}(x_1)\ldots\hat{\phi}(x_m)]|0\rangle + \\ &\qquad + \text{disconnected terms without interactions not contributing to the cross section}\end{aligned}$$

The perturbative approach tries to compute the correlation functions $\langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle$ of interacting Heisenberg fields in terms of the free in-fields $\hat{\phi}_{\text{in}}(x)$. Using the expression of $U^I_\varepsilon(t_1,t_2) := U^I_\varepsilon(t_1)U^{I\,-1}_\varepsilon(t_2)$ one can prove the Gell-Mann - Low formula providing the above connection. Let $|0\rangle$ be the vacuum state in $\mathcal{H}$, then

$$\langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle = \frac{\langle 0_{\text{in}}|\mathrm{T}[\hat{\phi}_{\text{in}}(x_1)\ldots\hat{\phi}_{\text{in}}(x_n)e^{-\frac{i}{\hbar}\int\mathrm{d}t\,H^I_\varepsilon(t)}]|0_{\text{in}}\rangle}{\langle 0_{\text{in}}|\mathrm{T}[e^{-\frac{i}{\hbar}\int\mathrm{d}t\,H^I_\varepsilon(t)}]|0_{\text{in}}\rangle} \tag{2.5}$$

This formula connects interacting Heisenberg fields $\hat{\phi}$ with in-fields $\hat{\phi}_{\text{in}}$ assuming that we have a finite infrared cutoff ε in time.

We now give a sketch of the proof of this result.[V] Assume $t \gg t_1 > t_2 > \ldots > t_n$, then using unitarity

$$U_\varepsilon^{I\dagger}(t_1) = U_\varepsilon^{I\,-1}(t_1) = U_\varepsilon^{I\,-1}(t)U_\varepsilon^I(t,t_1) = U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1) \quad , \quad U_\varepsilon^I(t_n) = U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t)$$

and

$$\begin{aligned}
\langle 0|\,\mathrm{T}[\hat\phi(x_1)\ldots\hat\phi(x_n)]\,|0\rangle &= \langle 0|\,\hat\phi(x_1)\ldots\hat\phi(x_n)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t_1)\hat\phi_{\mathrm{in}}(x_1)U_\varepsilon^I(t_1)\ldots U_\varepsilon^{I\dagger}(t_n)\hat\phi_{\mathrm{in}}(x_n)U_\varepsilon^I(t_n)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1)\hat\phi_{\mathrm{in}}(x_1)\ldots\hat\phi_{\mathrm{in}}(x_n)U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t)\mathrm{T}[\hat\phi_{\mathrm{in}}(x_1)\ldots\hat\phi_{\mathrm{in}}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\,H_\varepsilon^I(t)}]U_\varepsilon^I(-t)\,|0\rangle
\end{aligned}$$

where in the last step we moved all the unitary operators on the right (since they are inside the time ordering the result is the same) and we used

$$U_\varepsilon^I(t,t_1)U_\varepsilon^I(t_1)U_\varepsilon^{I\dagger}(t_2)U_\varepsilon^I(t_2)\ldots U_\varepsilon^{I\dagger}(t_n)U_\varepsilon^I(t_n,-t) = U_\varepsilon^I(t)U_\varepsilon^{I\dagger}(-t) = \mathrm{T}[e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\,H_\varepsilon^I(t)}]$$

Now since

$$U_\varepsilon^I(t)\,|0\rangle \xrightarrow[t\to-\infty]{} |0_{\mathrm{in}}\rangle \quad , \quad U_\varepsilon^I(t)\,|0\rangle \xrightarrow[t\to+\infty]{} |0_{\mathrm{out}}\rangle$$

we have

$$\langle 0|\,\mathrm{T}[\hat\phi(x_1)\ldots\hat\phi(x_n)]\,|0\rangle = \langle 0_{\mathrm{out}}|\,\mathrm{T}[\hat\phi_{\mathrm{in}}(x_1)\ldots\hat\phi_{\mathrm{in}}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\,H_\varepsilon^I(t)}]\,|0_{\mathrm{in}}\rangle$$

If we assume that the vacuum is non degenerate, we expect that by adiabatic evolution (i.e. changing the value of ε) it cannot go in another state, hence

$$|0_{\mathrm{out}}\rangle = e^{iL}\,|0_{\mathrm{in}}\rangle$$

for some operator L. Now $\langle 0_{\mathrm{in}}|0_{\mathrm{out}}\rangle = e^{iL}$ therefore

$$\langle 0_{\mathrm{out}}| = \langle 0_{\mathrm{in}}|\,e^{-iL} = \frac{\langle 0_{\mathrm{in}}|}{\langle 0_{\mathrm{in}}|0_{\mathrm{out}}\rangle} = \frac{\langle 0_{\mathrm{in}}|}{\langle 0_{\mathrm{in}}|\,S_\varepsilon\,|0_{\mathrm{in}}\rangle}$$

and taking $t \to +\infty$ we finally get eq. (2.5).
In general is very hard to evaluate expressions like eq. (2.5). In order to make this effort, the perturbative approach consists in replacing

$$\langle 0_{\mathrm{in}}|\,\mathrm{T}[\hat\phi_{\mathrm{in}}(x_1)\ldots\hat\phi_{\mathrm{in}}(x_n)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\,H_\varepsilon^I(t')\right)^\ell]\,|0_{\mathrm{in}}\rangle$$

$$\downarrow$$

$$\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\,\langle 0_{\mathrm{in}}|\,\mathrm{T}[\hat\phi_{\mathrm{in}}(x_1)\ldots\hat\phi_{\mathrm{in}}(x_n)\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\,H_\varepsilon^I(t')\right)^\ell]\,|0_{\mathrm{in}}\rangle$$

i.e. moving the infinite series outside the expectation value. The terms inside the sum can now be computed in terms of in-fields using Wick theorem or equivalently Feynman diagrams.
Resumming our steps, we expressed the scattering amplitudes in terms of correlators of interacting fields using LSZ formula, then Gell-Mann - Low formula allows us to express such correlators in terms of correlators of in-fields (free). Finally the perturbative approach prescripts the extraction of the infinite series outside the expectation value, in such a way that we can obtain our result computing expectation values of free fields through Feynman diagrams. We'll see in the following how dangerous is perturbative approach prescription.

2.4 Condensed matter systems

[Kle15, Sections 1.16, 2.17–2.19, 10.9]; [Fjæ13]

[V] Reference: https://authors.library.caltech.edu/60474/1/PhysRev.84.350.pdf

Let us turn to condensed matter systems. A source of information on the physical properties (typically transport properties) are the correlation functions of local observables, such as spin, density, current, etc; in particular two-points functions. Hence in this case physical informations are much more involved than just scattering amplitudes in particle physics. These two-points functions appear naturally in particular when one studies the *linear response*, i.e. the response to a test (i.e. infinitesimal) perturbation.
Let us consider a system with Hamiltonian H (which might include the classical potential term μN) and we want to see the effect on the mean value of an observable O_1 or a field (in general not self-adjoint) of a test perturbation typically generated by another observable O_2 in the form

$$V^\varepsilon(t) := \xi_\varepsilon(t) O_2^H(t)$$

with $\xi_\varepsilon(t)$ vanishing as $|t| \to \infty$ as the previous considered $\lambda_\varepsilon(t)$ and $O_2^H(t) = e^{itH} O_2 e^{-itH}$. Notice that however we do not assume H to be free as in the scattering case.
Let $\{|n\rangle\}$ be the set of eigenfunctions of H ($H|n\rangle = \mathcal{E}_n |n\rangle$) generating a Dirac completeness in $\mathcal{H}$ and consider the first order modification induced by the perturbation in

$$\langle n| O_1^H(t) |n\rangle = \langle n| e^{itH} O_1 e^{-itH} |n\rangle$$

The time evolution in the presence of the perturbation of O_1 is

$$O_1^{H+V_\varepsilon}(t) = U^{H+V_\varepsilon \dagger}(t) O_1 U^{H+V_\varepsilon}(t)$$

and we want to rewrite this in terms of the evolution generated by H in an "interaction picture", i.e.

$$O_1^{H+V_\varepsilon}(t) = U_\varepsilon^{V\dagger}(t) O_1^H(t) U_\varepsilon^V(t)$$

where $U_\varepsilon^V(t)$ gives the contribution of the interaction $V_\varepsilon(t)$ to the free evolution described by $O_1^H(t)$.
The situation in this respect is similar to the one considered before in RQFT, just replacing of H_I^ε by V_ε we get that

$$U_\varepsilon^V(t) = \mathrm{T}[e^{-i\int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t'}] \tag{2.6}$$

We calculate the response to the perturbation through a variation, using the perturbative ansatz and then taking only first order contribution of V_ε in the perturbative expansion:

$$\begin{aligned}
\delta \langle n| O_1^{H+V_\varepsilon}(t) |n\rangle &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon'} \langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle \big|_{\varepsilon'=0} = \lim_{\varepsilon'\to 0} \frac{\langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle - \langle n| O_1^H(t) |n\rangle}{\varepsilon'} = \\
&= \lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'} \langle n| \left[\left(1 + i\varepsilon' \int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t' + O(\varepsilon')^2\right) O_1^H(t) \left(1 - i\varepsilon' \int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t' + O(\varepsilon')^2\right) - O_1^H(t)\right] |n\rangle = \\
&= i \int_{-\infty}^t \mathrm{d}t'\, \xi_\varepsilon(t') \langle n| [O_2^H(t'), O_1^H(t)] |n\rangle = -i \int_{-\infty}^{+\infty} \mathrm{d}t'\, \xi_\varepsilon(t')\theta(t-t') \langle n| [O_1^H(t'), O_2^H(t)] |n\rangle
\end{aligned}$$

One can now just sum over $\{|n\rangle\}$ to perform the thermal expectation value

$$\langle (\bullet) \rangle_{\beta := \frac{1}{kT}} = \frac{\sum_n \langle n| (\bullet) |n\rangle e^{-\beta \mathcal{E}_n}}{\sum_n e^{-\beta\mathcal{E}_n}} \tag{2.7}$$

obtaining the expectation value at a given temperature T

$$\delta \langle O_1^{H+V_\varepsilon}(t)\rangle_\beta = \int \mathrm{d}t'\, \xi_\varepsilon(t') \left(-i\theta(t-t')\langle [O_1^H(t), O_2^H(t')]\rangle_\beta\right) = \int_{-\infty}^{+\infty} \mathrm{d}t'\, \xi_\varepsilon(t') G_{\mathrm{ret}}^{O_1 O_2}(t,t') \tag{2.8}$$

where

$$G_{\mathrm{ret}}^{O_1O_2}(t,t') := -i\theta(t-t')\langle [O_1^H(t), O_2^H(t')]\rangle_\beta \tag{2.9}$$

is called *retarded correlation function*. Also in this case, similarly to what happens in RQFT, that experimental data are obtained from particular correlation functions.
Of curse if O_1, O_2 and ξ_ε depend also on space coordinates then eq. (2.8) becomes:

$$\begin{aligned}
\delta \langle O_1^{H+V_\varepsilon}(\boldsymbol{x}, t)\rangle_T &= \int \mathrm{d}^3x'\, \mathrm{d}t'\, \xi_\varepsilon(\boldsymbol{x'}, t')(-i\theta(t-t')\langle [O_1^H(\boldsymbol{x'}, t), O_2^H(\boldsymbol{x'}, t')]\rangle_\beta) = \\
&= \int \mathrm{d}^3x'\, \mathrm{d}t'\, \xi_\varepsilon(\boldsymbol{x'}, t')\, G_{\mathrm{ret}}^{O_1O_2}(\boldsymbol{x}, t, \boldsymbol{x'}, t')
\end{aligned} \tag{2.10}$$

where

$$G_{\text{ret}}^{O_1 O_2}(\boldsymbol{x}, t, \boldsymbol{x'}, t') := -i\theta(t-t')\langle [O_1^H(\boldsymbol{x}, t), O_2^H(\boldsymbol{x'}, t')]\rangle_\beta \tag{2.11}$$

Assuming translational invariance, i.e. $G_{\text{ret}}^{O_1 O_2}$ depends only on t, t', and the space difference $\boldsymbol{x} - \boldsymbol{x'}$, we can easily perform the Fourier transform, since eq. (2.10) corresponds to a convolution:

$$\delta\langle O_1^{H+V_\varepsilon}(\boldsymbol{q}, \omega)\rangle_\beta = \tilde{\xi}_\varepsilon(\boldsymbol{q}, \omega)\, \tilde{G}_{\text{ret}}^{O_1 O_2}(\boldsymbol{q}, \omega)$$

The retarded correlators are typically directed connected to experiments. For example suppose to measure the magnetization of a spin system, with spin $\boldsymbol{S}(\boldsymbol{x})$ in presence of a test magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. The coupling between the magnetic field and the spin is given by

$$\int \mathrm{d}^3 x\, \boldsymbol{B}(\boldsymbol{x}, t) \cdot \boldsymbol{S}(\boldsymbol{x})$$

The linear response to the perturbation is determined by

$$-i\theta(t_1 - t_2)\langle [\boldsymbol{S}^H(\boldsymbol{x_1}, t_1), \boldsymbol{S}^H(\boldsymbol{x_2}, t_2)]\rangle_\beta$$

whose Fourier transform is precisely the dynamic magnetic susceptibility $\chi_s(\boldsymbol{q}, \omega)$ measurable by neutrons. Analogously charged particles are coupled to the electromagnetic field (in the gauge $A_0 = 0$) by

$$\int \mathrm{d}^3 x\, \boldsymbol{A}(\boldsymbol{x}, t) \cdot \boldsymbol{j}(\boldsymbol{x})$$

where in the free case the current $\boldsymbol{j}(\boldsymbol{x})$ is given by

$$\boldsymbol{j}(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x}) \frac{\overleftrightarrow{\boldsymbol{\nabla}}}{2mi} \psi(\boldsymbol{x}) := \psi^\dagger(\boldsymbol{x}) \left(\frac{\boldsymbol{\nabla}}{2mi} \psi(\boldsymbol{x}) \right) - \left(\frac{\boldsymbol{\nabla}}{2mi} \psi^\dagger(\boldsymbol{x}) \right) \psi(\boldsymbol{x})$$

and the *conductivity* $\sigma_{\alpha\beta}$ is directly related to the Fourier transform of

$$-i\theta(t_1 - t_2)\langle [\boldsymbol{j}_\alpha^H(\boldsymbol{x_1}, t_1), \boldsymbol{j}_\beta^H(\boldsymbol{x_2}, t_2)]\rangle_\beta$$

Notice that if O_1 and O_2 are fermionic fields, since the Hamiltonian is a scalar we get that $\xi_\varepsilon(t)$ must be an anticommuting function so that

$$O_1^H \xi_\varepsilon O_2^H - \xi_\varepsilon O_2^H O_1^H = -\xi_\varepsilon (O_1^H O_2^H + O_2^H O_1^H)$$

and in fact the retarded correlation function for fermionic fields O_1 and O_2 is

$$G_{\text{ret}}^{O_1 O_2} = i\theta(t_1 - t_2)\langle \{O_1^H(t_1), O_2^H(t_2)\}\rangle_\beta$$

For example the intensity of response in metals by high frequency photons (if the direction of the photon is chosen then the determination of such intensity is called *Angle-Resolved Photoemission Spectroscopy* (*ARPES*)) is related to the immaginary part of the Fourier transform of

$$i\theta(t_1 - t_2)\langle \{\psi^\dagger(\boldsymbol{x_1}, t_1), \psi(\boldsymbol{x_2}, t_2)\}\rangle_\beta$$

Matsubara formalism

If H is not free the next question is how to compute the correlation functions. Let $H = H_0 + H_I$, with H_0 "free" (typically this means that contains only the quadratic terms in the fields, with no mixed components). For $T = 0$ one just have to replace

$$\langle (\bullet) \rangle_\beta \to \langle 0 |\, (\bullet)\, | 0 \rangle$$

and the Gell-Mann Low formula applies as before with adiabatic switching as in the relativistic case and the perturbative approach is completely similar.
For $T > 0$ the situation is more complicated because H_I would appear in two places: in the time evolution eq. (2.6) as in $T = 0$ but also in the Boltzmann weight $e^{-\beta(H_0 + H_I)}$. This makes the standard perturbative

treatment inapplicable, since we need to disentangle a perturbation in these two places. The way to solve this issue is due to Matsubara, and for this reason called the *Matsubara formalism*. Let's see how it works.
We define an "evolution" of the operators by a new parameter τ:

$$O^H(\tau) = e^{\tau H} O e^{-\tau H} \quad \text{with} \quad 0 \leq \tau \leq \beta$$

and then instead of computing correlation functions at ordinary time t, we compute them using the modified evolution and the parameters τ_i:

$$\langle O_1^H(\tau_1) \dots O_n^H(\tau_n) \rangle_\beta$$

Suppose that $H = H_0 + H_I$ (notice that the cutoff ε is not required here since the domain of τ is already finite, $\tau \in [0, \beta]$) and write again the evolution separating the contribution of H_0 and H_I as in the interaction picture

$$O^H(\tau) = U^I(\tau) O^{H_0}(\tau) U^I(\tau)$$

getting as before

$$U^I(\tau) = e^{\tau H_0} e^{-\tau H} \quad \text{with} \quad U^I(0) = \mathbb{1}$$

and again

$$U^I(\tau) = \mathrm{T}_\tau[e^{-\int_0^\tau \mathrm{d}\tau'\, H_I(\tau')}]$$

We define *Matsubara correlator / Green function* the object

$$G_M^{O_1 \dots O_n}(\tau_1, \dots, \tau_n) := -\langle \mathrm{T}_\tau[O_1^H(\tau_1) \dots O_n^H(\tau_n)] \rangle_\beta$$

Now for the analogue of the Gell-Mann Low formula

$$\langle \mathrm{T}_\tau[O_1^H(\tau_1) \dots O_n^H(\tau_n)] \rangle_\beta = \frac{\langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}] \rangle_\beta^0}{\langle \mathrm{T}_\tau[e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}] \rangle_\beta^0}$$

with $\langle \bullet \rangle_\beta^0$ the thermal average computed using H_0 as Hamiltonian. In comparison with the Gell-Mann Low formula for particles physics, in this case the vacuum expectation values are replaced by the Boltzmann-weighted thermal traces, and the vacuum expectation value of the S-matrix operator $U_\varepsilon^I(+\infty)$ in the denominator is replaced by the Boltzmann-weighted trace of the interaction operator $U^I(\beta)$ along the euclidean time axis τ.
Finally perturbative treatment then prescripts

$$\langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}] \rangle_\beta^0 = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) \Big(\int_0^\beta \mathrm{d}\tau'\, H_I(\tau') \Big)^\ell] \rangle_\beta^0$$

This leads to a series of thermally averaged products of many fields which move according to the free field equations. Therefore Wick's theorem can be applied and we obtain an expansion of the Matsubara Green function completely analogous to the field theoretic one. The only difference is the finite-time interaction. Now everything is again expressed in terms of the free fields and can be computed using Feynman diagrams.
To be precise in most cases one also need to use an infrared cutoff ε, which in this case (the domain of τ is finite) is required only in the space coordinates, and will be removed at the end of the calculation.
Now the question is how this thermal averaged operator can be related to the retarded correlators needed to compute the linear response of our system. Using thermal average formula eq. (2.7) and $Z := \mathrm{Tr}_\mathcal{H}\, e^{-\beta H}$ the *partition function* computed taking the trace over the whole Hilbert space of the system $\mathcal{H}$[VI] (not only the 1-particle Hilbert space $\mathcal{H}_1$), the 2-points Matsubara correlator (the only needed in the perturbative approach) reads:

$$\begin{aligned} G_M^{O_1 O_2}(\tau_1, \tau_2) = -\frac{1}{Z} \big\{ &\mathrm{Tr}_\mathcal{H}[e^{-\beta H} e^{\tau_1 H} O_1 e^{-\tau_1 H} e^{\tau_2 H} O_2 e^{-\tau_2 H}] \theta(\tau_1 - \tau_2) \\ &\pm \mathrm{Tr}_\mathcal{H}[e^{-\beta H} e^{\tau_2 H} O_2 e^{-\tau_2 H} e^{\tau_1 H} O_1 e^{-\tau_1 H}] \theta(\tau_2 - \tau_1) \big\} = \\ = -\frac{1}{Z} \big\{ &\mathrm{Tr}_\mathcal{H}[e^{-\beta H} O_1 e^{-(\tau_1 - \tau_2) H} O_2 e^{(\tau_1 - \tau_2) H}] \theta(\tau_1 - \tau_2) \\ &\pm \mathrm{Tr}_\mathcal{H}[e^{-\beta H} O_2 e^{-(\tau_2 - \tau_1) H} O_1 e^{(\tau_2 - \tau_1) H}] \theta(\tau_2 - \tau_1) \big\} \end{aligned}$$

[VI] Notice that at this point we do not know whether this space is Fock or non-Fock.

where in the second step we used the cyclicity of the trace and the choice of the sign depends on the commutation relation between O_1^H and O_2^H ($+$ if they commute or $-$ if they anticommute). This computation shows that the two points correlator is just a function of the difference $\tau := \tau_1 - \tau_2$.

$$G_M^{O_1 O_2}(\tau) = -\frac{1}{Z}\left\{ \operatorname{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-\tau H} O_2 e^{\tau H}]\theta(\tau) \pm \operatorname{Tr}_{\mathcal{H}}[e^{-\beta H} O_2 e^{\tau H} O_1 e^{-\tau H}]\theta(-\tau)\right\} \tag{2.12}$$

By construction $0 \leq \tau_i \leq \beta$, $i = 1, 2$, hence we get that $\tau \in [-\beta, \beta]$. Notice that for $\tau \in [-\beta, 0]$, $\tau + \beta \geq 0$

$$\operatorname{Tr}_{\mathcal{H}}[e^{-\beta H} O_2 e^{\tau H} O_1 e^{-\tau H}] = \operatorname{Tr}_{\mathcal{H}}[O_2 e^{(\tau+\beta)H} e^{-\beta H} O_1 e^{-(\tau+\beta)H}] = \operatorname{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-(\tau+\beta)H} O_2 e^{(\tau+\beta)H}]$$

hence $G_M^{O_1 O_2}(\tau) = \pm G_M^{O_1 O_2}(\tau + \beta)$ and we can regard $G_M^{O_1 O_2}(\tau)$ as an (anti)periodic function of period β defined for any $\beta \in \mathbb{R}$.

Notice that in general traces can be infinite or the limit $\varepsilon \to 0$ of the cutoff might not exists, making the perturbative approach not working (notice that an infinite trace is the thermal-equivalent of a divergent vacuum expectation value of the theoretical field case). Nevertheless the periodicity of the Matsubara correlator is a non-perturbative result which hold anyway, even if the definition of the trace lose its meaning. Such relation is called *Kubo-Martin-Schwinger (KMS) condition*:

$$G_M(\tau) = \pm G_M(\tau + \beta) \tag{2.13}$$

which is independent on the choice of ε and works also for $\varepsilon \to 0$.

Due to periodicity, $G_M(\tau)$ can be represented in terms of Fourier coefficients

$$G_M(\tau) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} e^{-i\omega_n \tau} G_M(\omega_n) \tag{2.14}$$

with

$$\omega_n = \begin{cases} \dfrac{2n\pi}{\beta} & \text{if} \quad G_M(\tau) = +G_M(\tau + \beta) \\ \dfrac{(2n+1)\pi}{\beta} & \text{if} \quad G_M(\tau) = -G_M(\tau + \beta) \end{cases} \tag{2.15}$$

called *Matsubara frequencies* and

$$G_M(\omega_n) = \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n \tau} G_M(\tau) \tag{2.16}$$

Notice that as $T \to 0$ Matsubara coefficients become continuous and eq. (2.14) become a Fourier transformation.

Lehmann spectral representation of the correlators ($T > 0$)

The Wick rotation of the t-axis in eq. (2.11) exactly coincides with the axis on which τ is defined, and the Wick rotation send the energies in the field theoretic Green functions eq. (2.11) to the axis along which Matsubara frequencies are situated. The retarded Green functions are related to the imaginary-time Matsubara Green functions by an analytical continuation.

To see this relation we introduce a Dirac completeness of eigenstates $\{|n\rangle\}$ of H inside the coefficients of the Fourier series eq. (2.16) (we change the label for the Fourier coefficients from n to s) using eq. (2.12):

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z} \int_0^\beta \mathrm{d}\tau\, e^{i\omega_s \tau} \operatorname{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-\tau H} O_2 e^{\tau H}] \\ &= -\frac{1}{Z} \int_0^\beta \mathrm{d}\tau \sum_{m,n} e^{i\omega_s \tau} e^{-\beta \mathcal{E}_n} e^{\tau(\mathcal{E}_n - \mathcal{E}_m)} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \end{aligned} \tag{2.17}$$

Using $\int_0^\beta \mathrm{d}\tau\, e^{\alpha\tau} = \frac{1}{\alpha}(e^{\alpha\beta} - 1)$ we can integrate:

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z} \sum_{m,n} e^{-\beta\mathcal{E}_n} \frac{e^{i\omega_s \beta + \beta(\mathcal{E}_n - \mathcal{E}_m)} - 1}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \\ &= -\frac{1}{Z} \sum_{m,n} \frac{e^{i\omega_s\beta - \beta\mathcal{E}_m} - e^{-\beta\mathcal{E}_n}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \end{aligned}$$

Finally using the explicit values of Matsubara frequencies eq. (2.15), one gets

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z}\sum_{m,n}\frac{\pm e^{-\beta\mathcal{E}_m}-e^{-\beta\mathcal{E}_n}}{i\omega_s+\mathcal{E}_n-\mathcal{E}_m}\langle n|\,O_1\,|m\rangle\,\langle m|\,O_2\,|n\rangle = \\ &= \frac{1}{Z}\sum_{m,n}\frac{e^{-\beta\mathcal{E}_n}\mp e^{-\beta\mathcal{E}_m}}{i\omega_s+\mathcal{E}_n-\mathcal{E}_m}\langle n|\,O_1\,|m\rangle\,\langle m|\,O_2\,|n\rangle \quad \text{for} \quad G_M(\tau)=\pm G_M(\tau+\beta) \end{aligned} \tag{2.18}$$

Such expression for the Green function is called *Lehmann representation* (of G_M, in this case). Notice that in this representation the Fourier series coefficients can be computed in terms of H eigenvalues and the matrix elements of O_1 and O_2 only (for this reason such representation is also called "spectral" representation). Notice that these coefficients have poles along the imaginary axis.
Let's try to find the Lehmann representation of the retarded Green function we introduced before. Actually a more generic definition respect to eq. (2.9) of the retarded correlator is used:

$$G_{\text{ret}}^{O_1O_2}(t,t') := -i\theta(t-t')\langle[O_1^H(t),O_2^H(t')]_{\mp}\rangle_\beta \tag{2.19}$$

taking the commutator or the anticommutator depending on the label after the square bracket. Rewriting eq. (2.19) introducing a Dirac completeness as in eq. (2.17) we get

$$\begin{aligned} G_{\text{ret}}(t_1,t_2) &= -i\theta(t_1-t_2)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}[e^{it_1H}O_1e^{-it_1H},e^{it_2H}O_2e^{-it_2H}]_{\mp}] \\ &= -\frac{i}{Z}\theta(t_1-t_2)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}(e^{it_1H}O_1e^{-it_1H}e^{it_2H}O_2e^{-it_2H}\mp e^{it_2H}O_2e^{-it_2H}e^{it_1H}O_1e^{-it_1H})] \\ &= -\frac{i}{Z}\theta(t)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}e^{itH}O_1e^{-itH}O_2\mp e^{itH}O_1e^{-\beta H}e^{-itH}O_2] \\ &= -\frac{i}{Z}\theta(t)\sum_{m,n}(e^{-\beta\mathcal{E}_n}\mp e^{-\beta\mathcal{E}_m})e^{it(\mathcal{E}_n-\mathcal{E}_m)}\langle n|\,O_1\,|m\rangle\,\langle m|\,O_2\,|n\rangle \end{aligned}$$

where $t := t_1-t_2$ and in the third step we used the cyclicity of the trace. Using the Fourier Transform of the following functions (the limit $\delta\to 0$, $\delta>0$, is understood)

$$\mathcal{F}[\theta(t)](\omega)=\frac{i}{\omega+i\delta} \quad , \quad \mathcal{F}[e^{i\alpha t}](\omega)=\delta(\omega+\alpha)$$
$$\mathcal{F}[f(t)g(t)](\omega)=\int \mathrm{d}\omega'\,\tilde{f}(\omega-\omega')\tilde{g}(\omega')$$

one get the Fourier transform of $G_{\text{ret}}(t)$:

$$\begin{aligned} G_{\text{ret}}(\omega) &= -\frac{i}{Z}\int\mathrm{d}\omega'\,\frac{i}{\omega-\omega'+i\delta}\sum_{m,n}(e^{-\beta\mathcal{E}_n}\mp e^{-\beta\mathcal{E}_m})\delta(\omega'+\mathcal{E}_n-\mathcal{E}_m)\,\langle n|\,O_1\,|m\rangle\,\langle m|\,O_2\,|n\rangle \\ &= \frac{1}{Z}\sum_{m,n}\frac{e^{-\beta\mathcal{E}_n}\mp e^{-\beta\mathcal{E}_m}}{\omega+\mathcal{E}_n-\mathcal{E}_m+i\delta}\langle n|\,O_1\,|m\rangle\,\langle m|\,O_2\,|n\rangle \end{aligned} \tag{2.20}$$

which is the Lehmann representation of G_{ret}. Notice that the prescription $i\delta$ is needed to move slightly the poles of the functions from the real axis, in such a way that the Wick rotation of the function is allowed.
Is now clear comparing eq. (2.18) and eq. (2.20) that the retarded correlator is just the analytical continuation of the Matsubara correlator

$$G_{\text{ret}}(\omega)=G_M(i\omega_s\to\omega+i\delta)$$

Can be proved[VII] using Carleman's theorem that such relation is well-defined and can be used to reconstruct uniquely $G_{\text{ret}}(\omega)$ by analytical continuation of $G_M(\omega_n)$. The only requirement on the Matsubara Green function is that it satisfy the KMS condition eq. (2.13). Therefore it is possible to compute G_M as in the high energy case and then obtain G_{ret} from analytical continuation.

[VII] Original proof by Baym and Mermin (1960): https://doi.org/10.1063/1.1703704, useful reference: Appendix A in https://fks.sk/~bzduso/physics/master/thesis.pdf.

Chapter 3

The need of a non-perturbative approach

We saw in the previous chapter that all the interesting physical quantities can be computed, both in high-energy physics and in condensed matter physics, using the perturbative approach. In this chapter we'll see why what we said doesn't really work so good, and in some instances we need a non-perturbative approach. We already anticipated some problems in the introduction, now we'll go more in detail.

3.1 The asymptotic series

Divergent quantities

Let's start from the simple example provided by a massive theory with quartic coupling in $d = 0$. Consider the following quantity, which naturally emerges in perturbative computations:

$$\frac{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} e^{-\lambda x^4}}{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \quad \text{for} \quad \lambda > 0 \quad , \quad \mathrm{Re}\,\alpha > 0$$

Such quantity is clearly smaller than 1, since the numerator contains the integral of a function which is everywhere smaller than the function in the integral present in the denominator. Nevertheless if one applies the perturbative prescription, the previous quantity is substitute by

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} (-\lambda x^4)^n}{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \tag{3.1}$$

But if one tries to compute explicitly the coefficient of the series

$$\frac{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} (\lambda x^4)^n}{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} = \lambda^n \frac{(4n)!}{(2n)!} \frac{1}{2^{2n+1}} \frac{1}{\alpha^{2n}} \geq \frac{\lambda^n}{2} \frac{((2n)!)^2}{(2n)!} \frac{1}{(2\alpha)^{2n}} = \frac{\lambda^n}{2} \frac{(2n)!}{(2\alpha)^{2n}} \geq \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2}\right)^n (n!)^2$$

one gets that the series eq. (3.1) is absolutely divergent

$$\sum_{n=0}^{\infty} \left| \frac{(-\lambda)^n}{n!} \frac{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} x^{4n}}{\displaystyle\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \right| \geq \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2}\right)^n n! = +\infty$$

Obviously we saw that the perturbative prescription in this case didn't worked, but can be proved that the same issue appear also in higher dimension. Such problem is actually more general, indeed all the perturbation series needed to compute the collerator using Gell-Mann Low formula are divergent.
Nevertheless is well known that usually the perturbative approach works (for instance in QED it works very well, and allows to obtain extremely precise predictions) hence we wonder if it is somehow possible to obtain the right expression of the original correlator using the coefficients of the perturbative series. In other words we want to know if, although divergent, could the perturbative series at least determine uniquely the corresponding correlation function.

The asymptotic series

One says that a series $\sum_{n=0}^{\infty} a_n \lambda^n$, $\lambda > 0$, is *asymptotic to a function* $f(\lambda)$ if

$$\lim_{\lambda \to 0^+} \frac{\left| f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n \right|}{\lambda^N} = 0 \quad \text{for all} \quad N > 0$$

This means that the absolute difference between $f(\lambda)$ and the truncated series at order N is $O(\lambda^{N+1})$ so that for $\lambda \ll 1$ small enough we can make the difference as mild as we prefer.
Apparently with the increase of the order of perturbation the truncated series approximate better and better $f(\lambda)$. This naive idea is obviously wrong, due to the fact that increasing the perturbation order one should also decrease the value of λ. Indeed if one fixes the value of λ and increases the perturbation order N, the absolute difference between $f(\lambda)$ and the truncated series initially decreases, but eventually it reaches its minimum and starts to increase again, diverging as $N \to \infty$. At higher perturbative orders one should take values of λ smaller and smaller to make the correction negligible. The fact that we can take $\lambda \ll 1$ to make the difference negligible doesn't solve our problem, since in physical applications the role of λ is done by $\hbar$, and decreasing its value below its physical value is meaningless.

Non-analytical contributions

The second problem is that any series asymptotic to a function is also asymptotic to infinitely many distinguished functions. For instance, if a series is asymptotic to some $f(\lambda)$ it is also asymptotic to $f(\lambda) + e^{-\frac{1}{\lambda \alpha}}$, since

$$\lim_{\lambda \to 0^+} \frac{e^{-\frac{1}{\lambda \alpha}}}{\lambda^N} = 0 \quad \text{for any} \quad N \in \mathbb{N} \quad \text{and} \quad a > 0$$

This is due to the fact that the series expansion of $e^{-\frac{1}{\lambda \alpha}}$ for $\lambda \ll 1$ has all the coefficients equal to zero. Unfortunately terms in the form $e^{-\frac{1}{\lambda \alpha}}$ are exactly the contributions that arises if there are non-trivial topological configurations of the fields in the correlation function.
This makes completely impossible any unique reconstruction of the original function starting from its asymptotic series, since in any case the reconstructed function would be defined up to non-analytical terms which didn't appear in its asymptotic series.

Is the perturbative series asymptotic to the correlation function?

Even if it's impossible to reconstruct the correlator from its asymptotic series, at lest it has been shown that in many QFT where the ultraviolet renormalization does not involve coupling constants (the so called *super-renormalizable theories*) the renormalized perturbation series is asymptotic to the non-perturbative defined correlation function (e.g. QED for $d < 3+1$ and ϕ^4 in $d < 3+1$).
Nevertheless if our theory is *renormalizable* (hence not super-renormalizable) in the only case we have almost completely rigorous control of the RQFT, i.e. ϕ^4 in $d = 3+1$, it has been proved that the renormalized perturbation series is not asymptotic to the non-perturbative correlator. Even if we don't have a rigorous proof, this seems the case also for QED in $d = 3+1$.

Borel resummation

In some cases, called *Borel resummable theories*, even if the perturbative series is divergent it is possible to resum it via *Borel resummation* obtaining a finite result.
The idea is ot introduce the identity

$$1 = \frac{1}{n!}\int_0^\infty x^n e^{-x}\mathrm{d}x$$

inside the perturbative series

$$\sum_{n=0}^\infty a_n\lambda^n = \int_0^\infty \sum_{n=0}^\infty \frac{a_n}{n!}(x\lambda)^n e^{-x}\mathrm{d}x$$

In some instances, the introduction of the factor $\frac{1}{n!}$ inside the perturbative coefficients makes the new series converge: $\sum_n \frac{1}{n!}a_n(x\lambda)^n$ might converge even if $\sum_n a_n\lambda^n$ does not. If the new coefficients are smooth enough to make the integral, we are then able to obtain the resummed series.
The first problem is that the resummed series is still not sensible to non-analytical terms (they cannot be reconstructed just resumming the analytical contributions), moreover a-posteriori one has to check that the resummed series is asymptotic to the non-perturbative result (up to non-analytical terms), since the right convergence is not ensured in general.

Resurgence

We just mention that there is a very recent technique called *resurgence* which applies to Quantum Mechanics and allows to obtain non-perturbative results just using perturbative techniques. Is still unknown whether such technique can be implemented or not also in QFT.

3.2 The Källen-Lehmann representation and the Lehmann representation

[GR96, Section 9.3]

The second problem of the perturbative approach is related to the limit $\varepsilon \to 0$ of the IR cutoff of the relation

$$U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\text{in}}(\boldsymbol{x},t)U_\varepsilon^I(t) = \hat{\phi}(\boldsymbol{x},t)$$

Indeed the limit $\varepsilon \to 0$ should be taken in a way compatible with physical properties of the interacting theory, in particular translational invariance and, in high energy physics, the Poincaré invariance.
In order to exploit the problem we first need to introduce the spectral representation of the Green functions (the same we introduced for Matsubara and reatarded correlators, but from a more general and deep point of view).

Källen-Lehmann representation (relativistic case)

If $\mathcal{H}$ is the Hilbert space of the interacting theory, in order to have translational and Poincaré invariance, some properties has to be satisfied: in particular we need in $\mathcal{H}$

(1) A unitary representation of the covering of the restricted Poincaré group $\tilde{\mathcal{P}}_+^\uparrow$. Let's denote by $\hat{P}^\mu$ the corresponding generators of space-time translations.

(2) A vacuum vector $|0\rangle$ invariant under the representation $U(a)$, $a \in \mathbb{R}^{d+1}$, of spacetime translations ($d = 3$) (due to homogeneity of spacetime).

(3) The spectrum of $\hat{P}^\mu$ is contained in the forward light cone, $\sigma(\hat{P}^\mu) \subseteq V_0^+$.

(4) The operator $\hat{\phi}$ should transform with an irreducible representation of $\tilde{\mathcal{P}}_+^\uparrow$ under $U(a)$. In particular assuming that $\hat{\phi}$ is scalar we have

$$U(a)\hat{\phi}(x)U(a)^\dagger = \hat{\phi}(x-a)$$

From (1) we get that exists a Dirac completeness $|\alpha\rangle$ of (generalized) eigenvectors of $\hat{P}^\mu$ with eigenvalues p_α (for simplicity we write $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$ also if $\hat{P}^\mu$ has continuum spectrum).
Consider the 2-points function of a scalar RQFT

$$\begin{aligned}\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle &= \sum_\alpha \langle 0|\hat{\phi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}(y)|\alpha\rangle \overset{(4)}{=} \sum_\alpha \langle 0|U(x)\hat{\phi}(0)U^\dagger(x)|\alpha\rangle\langle\alpha|U(y)\hat{\phi}(0)U^\dagger(y)|0\rangle \\ &= \sum_\alpha e^{-ip_\alpha(x-y)}|\langle\alpha|\hat{\phi}(0)|0\rangle|^2 \overset{d=3}{=} \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)}\end{aligned} \tag{3.2}$$

where in the last step we used the identity $1 = \int \mathrm{d}q\,\delta(q-p_\alpha)$ and we defined the Fourier transform (up to a factor 2π) of the 2-point function

$$\rho_+(q) = (2\pi)^3\sum_\alpha \delta(q-p_\alpha)|\langle 0|\hat{\phi}(0)|\alpha\rangle|^2 \tag{3.3}$$

which has some interesting properties:

(a) $\rho_+(q) \geq 0$;

(b) $\rho_+(q) = 0$ if $q \notin \overline{V}_0^+$, thanks to (3), the bar over the forward light cone indicate its closure;

(c) $\rho_+(\Lambda q) = \rho_+(q)$, for $\Lambda \in L_+^\uparrow$, thanks to (4).

Therefore from all these conditions we get that the most general form for ρ_+ is

$$\rho_+(q) = \sigma(q^2)\theta(q^0) + c\,\delta(q) \quad \text{with} \quad \sigma(q^2) = 0 \quad \text{if} \quad q^2 < 0 \quad \text{and} \quad c \text{ constant} \tag{3.4}$$

The delta function is introduced to compensate the ambiguity due to the sign of q^0 in $\theta(q^0)$ when $q^0 = 0$. Notice that property (a), $\rho_+(q) \geq 0$, implies that $\sigma(q^2)$ is a semi-definite positive function, i.e. $\sigma(q^2) \geq 0$ for any value of $q \in \overline{V}_0^+$.
Let's introduce the *spectral function*[I]

$$\rho(q) = \rho_+(q) - \rho_+(-q)$$

which due to eq. (3.2) is the Fourier transform (up to a factor 2π) of $\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle$: using eq. (3.2) we get

$$\begin{aligned}\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle &= \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle - \langle 0|\hat{\phi}(y)\hat{\phi}(x)|0\rangle \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(y-x)} \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(-q)e^{-iq(x-y)} \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho(q)e^{-iq(x-y)}\end{aligned} \tag{3.6}$$

Using expression eq. (3.4) we get

$$\rho(q) = \rho_+(q) - \rho_+(-q) = \sigma(q^2)\theta(q^0) + \cancel{c\,\delta(q)} - \sigma(q^2)\theta(-q^0) - \cancel{c\,\delta(-q)} = \mathrm{sign}(q^0)\sigma(q^2) \tag{3.7}$$

If $\hat{\phi}$ obeys the CCR $[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)] = i\delta(\boldsymbol{x}-\boldsymbol{y})$ then ρ satisfies the *sum sule*

$$\int_{-\infty}^{+\infty} \mathrm{d}q^0\, q^0\rho(q) = 1 \tag{3.8}$$

[I]Notice that sometimes $\rho(q)$ is defined with an additional factor 2π in such a way that it is exactly the Fourier transform of $\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle$. In some other cases it is defined (equivalently to eq. (3.5)) as 2 times the immaginary part of the retarded correlator, using the relations

$$\mathcal{F}(-i\theta(t))(\omega) = \frac{1}{\omega+i\delta} \quad \text{and} \quad \mathrm{Im}\,\frac{1}{\omega+i\delta} = \pi\delta(\omega) \tag{3.5}$$

where $\omega = q_0$.

Indeed consider the identity

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \delta(\boldsymbol{x}-\boldsymbol{y}) = -i\,\langle 0|\,[\hat\phi(\boldsymbol{x},t),\dot{\hat\phi}(\boldsymbol{y},t)]\,|0\rangle = -i\partial_{y^0}\,\langle 0|\,[\hat\phi(x),\hat\phi(y)]\,|0\rangle\,\Big|_{\substack{x_0=t\\ y_0=t}} =$$
$$= -i\partial_{y^0}\int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,\rho(q)e^{-iq(x-y)}\Big|_{\substack{x_0=t\\ y_0=t}} = \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,q^0\rho(q)e^{-iq(x-y)}\Big|_{\substack{x_0=t\\ y_0=t}} =$$
$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,q^0\rho(q)e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

then comparing the left and the right side of the previous identity one gets exactly eq. (3.8).
Moreover using eq. (3.7) one can rewrite

$$1 = \int_{-\infty}^{+\infty}\mathrm{d}q^0\,q^0\rho(q) = \int_{-\infty}^{+\infty}\mathrm{d}q^0\,|q^0|\sigma(q^2) = 2\int_0^{+\infty}\mathrm{d}q^0\,q^0\sigma(q^2) = \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2) \tag{3.9}$$

where using $m^2 = q^2 = (q^0)^2 - \boldsymbol{q}^2$ and the fact that in our integration $\boldsymbol{q}$ is fixed, we applied the change of variable $q^0 \mapsto m^2$, $2q^0\mathrm{d}q^0 \mapsto \mathrm{d}m^2$.
Recalling that $\sigma(q^2) \geq 0$, we have that $q^0\rho(q) = |q^0|\sigma(q^2) \geq 0$ too, hence using eq. (3.8) and eq. (3.9) we get that both

$$A(\omega,\boldsymbol{q}) := \omega\rho(\omega,\boldsymbol{q}) = q^0\rho(q)$$

and $\sigma(m^2)$ are probability densities. In particular can be proved that $\sigma(m^2)$ is the probability density to find the state $\int e^{iqx}\hat\phi(x)\,|0\rangle$ in a state of mass m^2, while $A(\omega,\boldsymbol{q})$ is the probability density to find the same state $\int e^{iqx}\hat\phi(x)\,|0\rangle$ in a state of energy ω for fixed value $\boldsymbol{q}$.
We just mention that for fermions the same sum rule eq. (3.8) holds equivalently provided that we have the CAR $\{\hat\phi(\boldsymbol{x},t),\dot{\hat\phi}(\boldsymbol{y},t)\} = i\delta(\boldsymbol{x}-\boldsymbol{y})$.
Now, suppose that we do not know if our interacting field satisfies CCR, but we are sure that the associated free theory does. Denoting by $\Delta_+(x-y;m)$ the 2-points function of the free scalar field of mass m, then inserting inside eq. (3.2) the equality $\int_0^{+\infty}\mathrm{d}m^2\,\delta(q^2-m^2) = 1$ we get

$$\begin{aligned}
\langle 0|\,\phi(x)\phi(y)\,|0\rangle &= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\sigma(q^2)\theta(q^0)e^{-iq(x-y)} + |\,\langle 0|\,\hat\phi(0)\,|0\rangle\,|^2 \\
&= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\int\frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(q^2-m^2)\theta(q^0)e^{-iq(x-y)} + |\,\langle 0|\,\hat\phi(0)\,|0\rangle\,|^2 \\
&= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\Delta_+(x-y;m) + |\,\langle 0|\,\hat\phi(0)\,|0\rangle\,|^2
\end{aligned}$$

where $|\,\langle 0|\,\hat\phi(0)\,|0\rangle\,|^2$ is an irrelevant constant which comes from eq. (3.3) when evaluated at $q=0$ due to the term $\delta(q)$ in eq. (3.4), and can be removed just shifting $\hat\phi(x)$. This means that even in the interacting theory we can write the 2-points function as an integral of free 2-points functions of the free fields with varying mass weighted by a suitable measure $\mathrm{d}m^2\,\sigma(m^2)$. This fact is a purely relativistic effect, and has no analogue for NRQFT, due to Lorentz invariance, and is no more true if we broke Lorentz symmetry.
One of the consequences is that, since the 2-point function of the interacting theory can be written in terms of weighted two point functions of free theories, which satisfies CCR, also the interacting theory should satisfy CCR, up to some constant factor c which can be reabsorbed in the field:

$$\begin{aligned}
\langle 0|\,[\hat\phi(\boldsymbol{x},t),\dot{\hat\phi}(\boldsymbol{y},t)]\,|0\rangle &= \partial_{y^0}\,\langle 0|\,[\hat\phi(x),\hat\phi(y)]\,|0\rangle\,\Big|_{\substack{x_0=t\\ y_0=t}} \\
&= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\,\partial_{y^0}\big(\Delta_+(x-y;m) - \Delta_+(y-x;m)\big)\Big|_{\substack{x_0=t\\ y_0=t}} \\
&= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\,i\delta(\boldsymbol{x}-\boldsymbol{y}) = c\,i\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned} \tag{3.10}$$

Non sono molto sicuro di quanto scritto qui, l'ho aggiunto per motivare quanto detto a fine capitolo riguardo a RQFT con Z=0. Non posso assumere c=1 senza richiedere la sum rule, che vale solo se già so che la teoria interagente soddisfa le CCR. Inoltre, se la sum rule non vale, c potrebbe essere divergente per quanto ne so, o uguale a Z.

with $c = \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)$.

Representation of correlation functions given by eq. (3.2) and eq. (3.6) are called *Källen-Lehmann representations* since they was firstly derived by these two physicist.[II] Actually Lehmann applied such representation also to the non-relativistic case, as we now show.

Lehmann representation (non-relativistic case)

Let's see how to generalize the previous representation without assuming Lorentz invariance. In this case in the Hilbert space $\mathcal{H}$ of our theory we need:

(1') A unitary representation U of spacetime translations (one can also add rotational invariance).

(2') The field $\hat{\phi}$ should transform under an irreducible representation of transformations under U.[III] We assume that $\hat{\phi}$ transforms as a scalar.

As in the relativistic case, (1') implies that exists a Dirac completeness for the generators $\hat{\boldsymbol{P}}$, H of $U(a)$ with eigenvalue p_α. Applying the same strategy as in the relativistic case (the symbol $\pm$ in $\langle\bullet\rangle_{\beta\,\pm}$ indicates whether the operators are bosons or fermions when we compute the average at finite temperature T, while $[\bullet,\bullet]_+$ is the anticommutator and $[\bullet,\bullet]_-$ is the commutator)

$$
\begin{aligned}
\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_{\beta\,\pm} &= \frac{\mathrm{Tr}\Big[e^{-\beta H}[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\Big]}{\mathrm{Tr}\big[e^{-\beta H}\big]} \\
&= \frac{1}{Z}\sum_{\alpha,\alpha'}\big(\langle\alpha'|\,e^{-\beta H}\hat{\phi}(x)\,|\alpha\rangle\,\langle\alpha|\,\hat{\phi}^\dagger(y)\,|\alpha'\rangle \pm \langle\alpha|\,e^{-\beta H}\hat{\phi}^\dagger(y)\,|\alpha'\rangle\,\langle\alpha'|\,\hat{\phi}(x)\,|\alpha\rangle\big) \\
&= \frac{1}{Z}\sum_{\alpha,\alpha'}\big(e^{-\beta\mathcal{E}_{\alpha'}} \pm e^{-\beta\mathcal{E}_\alpha}\big)e^{i(p_\alpha-p_{\alpha'})(x-y)}|\,\langle\alpha'|\,\hat{\phi}(0)\,|\alpha\rangle\,|^2 \\
&\overset{d=3}{=} \int\frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho^\pm(q)e^{-q(x-y)}
\end{aligned}
\tag{3.11}
$$

with *spectral function*

$$
\rho^\pm(q) = \frac{(2\pi)^3}{Z}\sum_{\alpha,\alpha'}\big(e^{-\beta\mathcal{E}_{\alpha'}} \pm e^{-\beta\mathcal{E}_\alpha}\big)\delta(q-p_\alpha+p_{\alpha'})|\,\langle\alpha'|\,\hat{\phi}(0)\,|\alpha\rangle\,|^2
$$

Eq. (3.11) is called *Lehmann representation* of $\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_{\beta\,\pm}$.
For fermions, provided that CAR $\{\hat{\phi}(\boldsymbol{x},t),\hat{\phi}^\dagger(\boldsymbol{y},t)\} = i\delta(\boldsymbol{x}-\boldsymbol{y})$ is satisfied[IV] and $T=0$, we get that $A(\omega,\boldsymbol{q}) := \rho^+(q)$ (without q^0 in the non relativistic case) is a probability density, and it describes the probability density to find the state $\int e^{-iqx}\hat{\phi}(x)\,|0\rangle$ in a state of energy ω.

The constraint $0 \le Z(\boldsymbol{q}) \le 1$

Let's see what these spectral representations have to do with the inconsistency of the perturbation theory as $\varepsilon \to 0$.
In general in a QFT (relativistic or not) if there is a particle (or a quasi-particle) excitation with dispersion $\omega = \omega(\boldsymbol{q})$ (in the relativistic case $\omega = \sqrt{m^2+\boldsymbol{q}^2}$) (we assume $\langle 0|\,\hat{\phi}(0)\,|0\rangle = 0$, otherwise we can shift the field) then in $A(\omega,\boldsymbol{q})$ there should be a term

$$
\begin{aligned}
&Z(\boldsymbol{q})\delta(\omega-\omega(\boldsymbol{q})) \quad \text{in the non-relativistic case} \\
&Z\delta(q^2-m^2)\theta(q^0) \quad \text{in the relativistic case}
\end{aligned}
\tag{3.12}
$$

Notice that in the relativistic case there cannot be any dependence on $\boldsymbol{q}$ in Z due to Lorentz invariance.

[II] Källen: https://doi.org/10.5169%2Fseals-112316, Lehmann: https://doi.org/10.1007%2Fbf02783624.
[III] For lattice theories U is restricted to discrete spatial lattice translations and typically the time lattice translations are represented contractively, i.e. through terms of the form e^{-tH} (continuum variation of the transfer matrix).
[IV] Notice that in the non-relativistic case there is no time derivative in the CAR.

Suppose that we have only one particle in the system (no other particles excitations), then in the non-relativistic case[V] since $A(\omega, \boldsymbol{q})$ describe the probability density to find the state $\int e^{-iqx}\hat{\phi}(x)\left|0\right\rangle$ in a state of energy ω we have that in general

$$A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q})) + A_{\text{inc}}(\omega, \boldsymbol{q}) \tag{3.13}$$

where A_{inc} is a regular (differently from δ) "featureless" positive function describing contributions of multi-particle states in the two points function (we still allow our particle to generate some other particles which eventually recombine in the original one due to energy uncertainty, if there are interactions in our theory). These give a (small) non-zero probability to find a particle in an energy state different from $\omega(\boldsymbol{q})$.

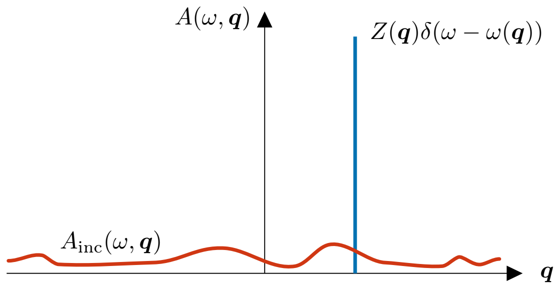

Figure 3.1: One-dimensional pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a stable particle.

Just notice that since $A(\omega, \boldsymbol{q})$ has an explicitly physical meaning as probability amplitude, the plot shown in Fig. 3.1 can be obtained directly in experiments, up to uncertainties which turn the δ-function into a smooth but very localized function.
From the sum sule we get

$$1 = \int \mathrm{d}\omega\, A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q}) + \underbrace{\int \mathrm{d}\omega\, A_{\text{inc}}(\omega, \boldsymbol{q})}_{\geq 0}$$

which implies

$$0 \leq Z(\boldsymbol{q}) \leq 1$$

Notice that the case $Z(\boldsymbol{q}) = 1$ the theory is free, indeed for a free particle $A(\omega, \boldsymbol{q}) = \delta(\omega - \omega(\boldsymbol{q}))$. Our problem is that, as we'll see, is very difficult to satisfy $0 \leq Z < 1$ in an interacting situation.
We can weak a little the above requirement of a stable particle excitation: let's see what happen if we do not have a stable particle but a long-life resonance (i.e. a particle with a lifetime large respect to the typical time of the system). The δ-function in eq. (3.12) can be rewritten as

$$\delta(\omega - \omega(\boldsymbol{q})) \quad \mapsto \quad \frac{1}{\mathrm{Im}}\frac{1}{\pi}\omega - \omega(\boldsymbol{q}) + i\delta \quad \text{where the limit } \delta \to 0^+ \text{ is understood}$$

and this implies that the retarded correlator has a pole at $\omega = \omega(\boldsymbol{q}) - i\delta$. Notice that δ essentially correspond to the inverse lifetime τ^{-1} of the particle, hence the limit $\delta \to 0^+$ correspond to the infinite lifetime of the particle.
A resonance is described through a Lorentzian distribution, by replacing the above pole by a complex pole at $\omega = f(\omega, \boldsymbol{q})$ where f is a complex function with $\operatorname{Im} f(\omega, \boldsymbol{q})$ (in such a way that $A(\omega, \boldsymbol{q}) \geq 0$ is satisfied). Then

$$\operatorname{Im} \frac{1}{\omega - f(\omega, \boldsymbol{q})} = \frac{-\operatorname{Im} f(\omega, \boldsymbol{q})}{(\omega - \operatorname{Re} f(\omega, \boldsymbol{q}))^2 + (\operatorname{Im} f(\omega, \boldsymbol{q}))^2}$$

where $\omega = \operatorname{Re} f(\omega, \boldsymbol{q})$ describes the dispersion relation of the unstable particle and $\operatorname{Im} f(\omega, \boldsymbol{q})$ its inverse lifetime.

[V] The relativistic one is obtained imposing $q^2 = m^2$.

Then if we take $\operatorname{Im} f(\omega, \boldsymbol{q}) \to 0^-$ then $\operatorname{Im}(\omega - f(\omega, \boldsymbol{q})^{-1} \to \delta(\omega - f(\omega, \boldsymbol{q}))$ with $\omega(\boldsymbol{q})$ solution of the equation $\omega - \operatorname{Re} f(\omega, \boldsymbol{q}) = 0$:

$$\delta(\omega - f(\omega, \boldsymbol{q})) = \frac{\delta(\omega - \omega(\boldsymbol{q}))}{\left|\frac{\partial f}{\partial \omega}(\omega, \boldsymbol{q})\right| \Big|_{\omega=\omega(\boldsymbol{q})}} = Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q}))$$

This means that in the case of the resonance we just broaden the δ-function in the spectral function, as shown in fig. 3.2.

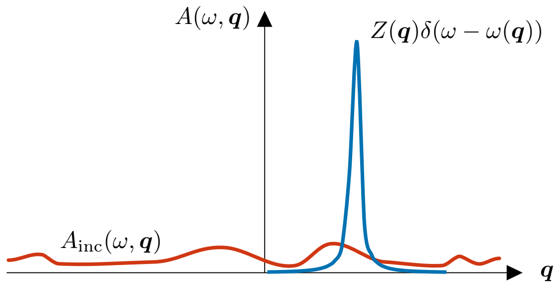

Figure 3.2: One-dimensional pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a unstable particle. The width of the blue peak is proportional to the inverse lifetime τ^{-1} of the particle.

Again, provided CCR or CAR (depending on the physical situation) we again have the contstraint

$$0 \leq Z \leq 1$$

hence the constraint applies even if we consider unstable particles.

3.3 The incompatibility between CCR and U^I

Let's see where the problem with the removal of the cutoff ε arises, first considering the relativistic case. Let's inspect the relation between Z and asymptotic fields. With the introduction of the infrared cutoff ε (at least in t) we got that the free field is related to the interacting field by a unitary operator

$$\hat{\phi}(\boldsymbol{x}, t) = U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\mathrm{in}}(\boldsymbol{x}, t)U_\varepsilon^I(t)$$

such that

$$\hat{\phi}(\boldsymbol{x}, t) \xrightarrow[t \to -\infty]{} \hat{\phi}_{\mathrm{in}}(\boldsymbol{x}, t) \quad \text{and} \quad U_\varepsilon^I(t) \xrightarrow[t \to +\infty]{} \mathbb{1}$$

Let's see if such unitary relation between free and interacting field is possible preserving translational invariance.
Consider a one particle system, then from eq. (3.13) we can identify $\langle 0| \hat{\phi}_{\mathrm{in}}\hat{\phi}_{\mathrm{in}} |0\rangle$ as responsible for the contribution

$$Z\delta(q^2 - m^2)\theta(q^0) \tag{3.14}$$

in the spectral function $A(q)$ of $\hat{\phi}$. Since we want that $\hat{\phi}$ describe an interacting field we need to assume $Z \neq 1$ (otherwise $\hat{\phi}$ is free), conversely for $\hat{\phi}_{\mathrm{in}}$ we already know that eq. (3.14) is satisfied for $Z = 1$. Therefore it is impossible to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} \hat{\phi}_{\mathrm{in}}$, one can at best hope to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} Z^{1/2}\hat{\phi}_{\mathrm{in}}$, i.e.

$$\hat{\phi}(\boldsymbol{x}, t) = Z^{1/2}U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\mathrm{in}}(\boldsymbol{x}, t)U_\varepsilon^I(t) \tag{3.15}$$

But if $\hat{\phi}_{\mathrm{in}}$ and $\hat{\phi}$ are canonical, i.e. satisfy CCR, then eq. (3.15) must be wrong, indeed from eq. (3.15) we get

$$[\hat{\phi}(\boldsymbol{x}, t), \dot{\hat{\phi}}(\boldsymbol{y}, t)] \xrightarrow[t \to -\infty]{} Z[\hat{\phi}_{\mathrm{in}}(\boldsymbol{x}, t), \dot{\hat{\phi}}_{\mathrm{in}}(\boldsymbol{y}, t)] \tag{3.16}$$

but then using CCR for $\hat{\phi}_{\text{in}}$ and $\hat{\phi}$ we get

$$\delta(\boldsymbol{x}-\boldsymbol{y}) \overset{\text{CCR}}{=} \lim_{x^0\to-\infty} \langle 0| [\hat{\phi}(\boldsymbol{x},x^0), \dot{\hat{\phi}}(\boldsymbol{y},y^0)] |0\rangle \,|_{x^0=y^0} =$$

$$= Z \lim_{x^0\to-\infty} \langle 0| [\hat{\phi}_{\text{in}}(\boldsymbol{x},x^0), \dot{\hat{\phi}}_{\text{in}}(\boldsymbol{y},y^0)] |0\rangle \,|_{x^0=y^0} \overset{\text{CCR}}{=} Z\delta(\boldsymbol{x}-\boldsymbol{y})$$

which give $Z=1$, but this is wrong since we assumed that $\hat{\phi}$ is interacting.
Notice that the above argument about the inconsistency between CCR and eq. (3.15) requires that both x^0 and y^0 are sent to $-\infty$ contemporaneily in eq. (3.16). Conversely it is still possible that $\hat{\phi}$ approaches $Z^{1/2}\hat{\phi}_{\text{in}}$ as $t\to-\infty$ in the weak sense $\hat{\phi} \xrightharpoonup[t\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, i.e. in matrix elements: taken a sequence A_n of operators the *weak limit*

$$A_n \xrightharpoonup[n\to+\infty]{} A$$

means that

$$|\langle\psi,(A_n-A)\phi\rangle| \xrightarrow[n\to+\infty]{} 0$$

Indeed taken $A_n \rightharpoonup A$ and $B_n \rightharpoonup B$ is not ensured that $A_nB_n \rightharpoonup AB$, therefore $\hat{\phi}(x) \xrightharpoonup[x^0\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$ and $\dot{\hat{\phi}}(y) \xrightharpoonup[y^0\to-\infty]{} Z^{1/2}\dot{\hat{\phi}}_{\text{in}}(y)$ does not imply $[\hat{\phi}(x),\dot{\hat{\phi}}(y)] \xrightarrow[x^0,y^0\to-\infty]{} Z[\hat{\phi}_{\text{in}}(x),\dot{\hat{\phi}}_{\text{in}}(y)]$ and eq. (3.16) does not hold anymore. Therefore we don't have any inconsistency between CCR and eq. (3.15) if $\hat{\phi}(x) \xrightharpoonup[x^0\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$.[VI]
Notice that LSZ formula uses precisely such weak limit for $t\to-\infty$, so there is no inconstistence with it.[VII]
The serious problem arises when in perturbative approach is used the Gell-Mann Low formula, which requires the existence of a unitary operator $U^I=\lim_{\varepsilon\to 0^+}U^I_\varepsilon$ interpolating between $\hat{\phi}$ and $\hat{\phi}_{\text{in}}$, but if $\hat{\phi}$ obeys CCR and translation invariance is recovered (i.e. we take the limit $\varepsilon\to 0^+$) we get from eq. (3.15) that such unitary operator does not exist (in order to get unitarity we should have $Z=1$, which is forbitten provided that $\hat{\phi}$ is interacting).

3.4 The Haag's theorem

[EF05, Section 3]; [SW00, Section 4.5]; [Str13, Pages 39-40, 95–96]

The important argument from Haag[VIII] proves that the situation does not improves even if we do not require $\hat{\phi}$ to be canonical. Except very special non-relativistic cases, the Hilbert space of the free field is disjoint from the Hilbert space of the interacting field (i.e. don't exist any unitary operator which maps the states of one Hilbert spaces into the states of the other one).

Theorem 3.1 (Haag, [EF05, Page 8]). *Assume that the unique translationally invariant state of an interacting scalar QFT with Hilbert space $\mathcal{H}$ and field $\hat{\phi}\in\mathcal{H}$ is the vacuum $|0\rangle\in\mathcal{H}$ [IX] (for lattice theories only discrete translations are considered). Let $U(\boldsymbol{x})$, $\boldsymbol{x}\in\mathbb{R}^d$, be the unitary representation of translations in $\mathcal{H}$. Suppose that the Hamiltonian of the theory can be written as $H=H_0+H_I$, with H_0 free Hamiltonian. Then there is no state $|0_F\rangle\in\mathcal{H}$ such that $H_0|0_F\rangle=0$ i.e. the vacuum of the free theory $|0_F\rangle$ is not an element of $\mathcal{H}$.*

Proof. By contradiction, assume that the Fock vacuum $|0_F\rangle$ of the free theory with Hamiltonian H_0 is an element of $\mathcal{H}$. Denote by $a_F(\boldsymbol{q})$ the annihilation operator of H_0, then

$$U^\dagger(\boldsymbol{x})a_F(\boldsymbol{q})U(\boldsymbol{x}) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}a_F(\boldsymbol{q})$$

[VI] More about this fact in [GR96, Section 9.2].
[VII] The derivation of the LSZ formula for the scalar theory can be found in [GR96, Section 9.4].
[VIII] Original paper: [Haa55].
[IX] It is reasonable to assume that the only translationally invariant state should be the vacuum, as the presence of any particle should broke translational invariance. This is ensured if the underlying free theory has a mass gap, i.e. all particles of our theory are massive. In general these assumptions does not hold if the free theory has massless particles. See [EF05, Section 3] for further comments about this fact.

therefore

$$0 = U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) |0_F\rangle = U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) U(\boldsymbol{x}) U^\dagger(\boldsymbol{x}) |0_F\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} a_F(\boldsymbol{q})) U^\dagger(\boldsymbol{x}) |0_F\rangle$$

and this means that $)U^\dagger(\boldsymbol{x}) |0_F\rangle$ annihilates $U(\boldsymbol{x}) |0_F\rangle$, hence $|0_F\rangle$ is translationally invariant. By uniqueness of the translational invariant state $|0_F\rangle = c|0\rangle$ for some constan c (actually if our states are normalized c is just a phase). Then

$$0 = H|0\rangle = (H_0 + H_I) c |0_F\rangle = c H_I |0_F\rangle$$

which give us $H_I |0_F\rangle = 0$ and $H|0_F\rangle = 0$. But the typical Hamiltonians for interacting fields does not annihilate $|0_F\rangle$ (i.e. H "polarizes the vacuum") since H_I describes interactions of the field with itself.[X] Thus, we arrive at a contradiction: it follows from the announced assumptions that $H|0_F\rangle = 0$, but for typical interaction Hamiltonians $H|0_F\rangle \neq 0$. □

The relative simplicity of this first version of Haag's theorem is purchased at the expense of generality since it appeals to the fact that typical Hamiltonians for interacting fields do not annihilate $|0_F\rangle$. It was the ambition of the second version of his theorem to dispense with any reference to the particular form of the interacting field Hamiltonian. However, Haag's attempted proof of such general version was found "inconclusive". The gap was filled by a pair of theorems presented by Hall and Wightman[XI], which together constitutes the so called *Haag-Hall-Weightman (HHW) theorem*, and apply to any pair to neutral scalar field.

Theorem 3.2 (HHW, Part 1, [HW57, Section 3, Thm. 1], [SW00, Thm. 4.14])**.** *Consider two QFTs with fields $\hat\phi_1$, $\hat\phi_2$ and Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and assume that*

(i) the Euclidean group $\mathbb{E}_d$ (roto-traslations) is represented unitarily by U_1 and U_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively:

$$U_j(\boldsymbol{a},\boldsymbol{R}) \hat\phi_j(\boldsymbol{x},t) U_j^\dagger(\boldsymbol{a},\boldsymbol{R}) = \hat\phi_j(\boldsymbol{R}\,\boldsymbol{x} + \boldsymbol{a}, t)$$

(ii) the representation of the fields is irreducible, i.e. if an operator $\hat A$ defined in $\mathcal{H}_j$ commutes with all the fields $\{\hat\phi_j\} \subset \mathcal{H}_j$ then it is a multiple of the identity, $\hat A = c\,\mathbb{1}$;

(iii) the fields are related at some time t by a unitary transformation $V(t)$

$$\hat\phi_2(\boldsymbol{x},t) = V(t)\hat\phi_1(\boldsymbol{x},t)V^\dagger(t)$$

Then

$$U_2(\boldsymbol{a},\boldsymbol{R}) = V(t) U_1(\boldsymbol{a},\boldsymbol{R}) V^\dagger(t) \tag{3.17}$$

Further, if there are unique states $|0_j\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a},\boldsymbol{R})|0_j\rangle = |0_j\rangle$, then

$$V(t)|0_1\rangle = c\,|0_2\rangle \quad \textit{with} \quad c \in \mathbb{C} \tag{3.18}$$

In particular if $|0_j\rangle$, $j = 1, 2$ are normalized to 1 then $|c| = 1$.

Notice that this part of the theorem holds also in the non-relativistic case. The operator V is meant to be the unitary operator interpolating between the free and the interacting fields in the Gell-Mann Low formula.

[X] As an illustration, consider the $\hat\phi^4$ scalar field in $\mathbb{R}^{3+1}$ with (formal) Hamiltonian

$$H = H_0 + \lambda \int \mathrm{d}^3x \, : \hat\phi^4 : - C$$

where C is a c-number constant chosen to give to the ground state energy zero, and : : indicates the Wick normal product. This H will not annihilate the bare vacuum $|0_F\rangle$ since the factor that follows the coupling constat λ contains a term with a product of four creation operator $\sim (a^\dagger)^4$ (and there is only one term of this form, so it will not be canceled by another term).

[XI] Original paper: [HW57]

Proof. Consider the operator $U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)$, this commutes with all $\hat{\phi}_1 \in \mathcal{H}_1$, indeed:

$$\begin{aligned}
&U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x},t)V^\dagger(t)U_2^\dagger(\boldsymbol{a},\boldsymbol{R})V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_2(\boldsymbol{x},t)U_2^\dagger(\boldsymbol{a},\boldsymbol{R})V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)\hat{\phi}_2(\boldsymbol{R}\,\boldsymbol{x}+\boldsymbol{a},t)V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_1(\boldsymbol{R}\,\boldsymbol{x}+\boldsymbol{a},t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})U_1(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_1(\boldsymbol{x},t)U_1(\boldsymbol{a},\boldsymbol{R})U_1^\dagger(\boldsymbol{a},\boldsymbol{R}) = \\
&= \hat{\phi}_1(\boldsymbol{x},t)
\end{aligned}$$

which implies

$$U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x},t) = \hat{\phi}_1(\boldsymbol{x},t)U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)$$

Due to the irreducibility of the representation we get that

$$U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t) = \omega(\boldsymbol{a},\boldsymbol{R})\mathbb{1}$$

for some complex function $\omega(\boldsymbol{a},\boldsymbol{R})$ which depend on the element $(\boldsymbol{a},\boldsymbol{R}) \in \mathbb{E}_d$ and then

$$\omega(\boldsymbol{a},\boldsymbol{R})U_1(\boldsymbol{a},\boldsymbol{R}) = V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t) \tag{3.19}$$

Notice that in order to preserve unitarity we must have $|\omega(\boldsymbol{a},\boldsymbol{R})| = 1$ for all $(\boldsymbol{a},\boldsymbol{R}) \in \mathbb{E}_d$. By definition $\omega(\boldsymbol{a},\boldsymbol{R})$ provides a continuous one-dimensional representation (on $\mathbb{C}$) of a non-compact group ($\mathbb{E}_d$), and it is a general mathematical result that there is no such a representation appart from the trivial one; then, one may take $\omega(\boldsymbol{a},\boldsymbol{R}) = 1$. If there are unique states $|0_j\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a},\boldsymbol{R})\,|0_j\rangle = |0_j\rangle$, then from $U_1(\boldsymbol{a},\boldsymbol{R})\,|0_1\rangle = |0_1\rangle$ one gets using eq. (3.19)

$$V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\,|0_1\rangle = |0_1\rangle$$

and

$$U_2(\boldsymbol{a},\boldsymbol{R})V(t)\,|0_1\rangle = V(t)\,|0_1\rangle$$

but the only state in $\mathcal{H}_2$ invariant under $U_2(\boldsymbol{a},\boldsymbol{R})$ is $|0_2\rangle$, therefore we must have

$$V(t)\,|0_1\rangle = c\,|0_2\rangle$$

for some constant $c \in \mathbb{C}$, which should be a phase, $|c| = 1$, if $|0_1\rangle$ and $|0_2\rangle$ are normalized to 1. □

Corollary 3.2.1. *In any two theories satisfying the hypothesis of theorem 3.2 the equal time correlation functions of $\hat{\phi}_1$ and $\hat{\phi}_2$ coincides*

$$\langle 0_1|\,\hat{\phi}_1(\boldsymbol{x}_1,t)\ldots\hat{\phi}_1(\boldsymbol{x}_n,t)\,|0_1\rangle = \langle 0_2|\,\hat{\phi}_2(\boldsymbol{x}_1,t)\ldots\hat{\phi}_2(\boldsymbol{x}_n,t)\,|0_2\rangle$$

Proof. The statement directly follows from eq. (3.17) and eq. (3.18). □

In principle an operator V satisfying the hypothesis of Gell-Mann Low formula can still exists, and corollary 3.2.1 still admits different dynamics for $\hat{\phi}_1$ and $\hat{\phi}_2$ since it applies only for equal time correlators. As we consider correlators at different time one should use time evolution operators involving the Hamiltonian, and no useful conclusion can be stated at this point if there is no vacuum polarization. We need one more result, which however works only in the relativistic case:

Theorem 3.3 (HHW, Part 2, [HW57, Section 3, Thm. 2], [SW00, Thm. 4.16])**.** *Under the hypothesis of theorem 3.2 assume further that the (covering of the) restricted Poincaré group $\mathcal{P}_+^\uparrow$ is represented unitarily by U_1 and U_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, i.e.*

$$U_j(\Lambda,a)\hat{\phi}_j(x)U_j^\dagger(\Lambda,a) = \hat{\phi}_j(\Lambda\,x+a)$$

Then

$$\langle 0_1|\,\hat{\phi}_1(x)\hat{\phi}_1(y)\,|0_1\rangle = \langle 0_2|\,\hat{\phi}_2(x)\hat{\phi}_2(y)\,|0_2\rangle$$

and if $\hat{\phi}_1$ is a free field of mass $m > 0$, then $\hat{\phi}_2$ is a free field of mass $m > 0$ too.

Proof. The proof is much more involved than the previous one, since requires a preliminary result of R. Jost and B. Schroer, which can be found in [SW00, Thm. 4.15]. We just sketch the argument for the proof: by Poincaré covariance through a boost transformation we can bring any two points in the space-time to equal time, in such a way that corollary 3.2.1 applies. Then the correlator we obtained can be extended to the initial points by analiticity, preserving the value of Z and the position of the pole in the Källen-Lehmann representation. This is enough to prove the statement of the theorem. $\square$

The immediate consequence of the latter is that, if the hypothesis holds, any interacting theory is actually free. The conclusion that all the fields are free is unacceptable because the intention is to represent an interacting field, therefore at least one of the assumptions of the theorem must be dropped. A n obvious candidate is the assumption that at some t there exists a unitary transformation $V(t)$ relating the fields. Therefore the Heisenberg dynamics of the interacting field do not exists in the Hilbert space of the free field, but requires a different Hilbert space, inequivalent to the free one.
The second part of the HHW theorem applies only to relativistic QFT, nevertheless the unitary operator U^I interpolating between free and interacting fields does not exists also in most of the non-relativistic QFT, except for very special cases. Therefore almost always the perturbative approach is not consistent.

3.5 Ultraviolet singularities and canonical quantization

[Str13, Section 2.4]

It is worth to notice that for a RQFT such that $Z = 0$ (recall that Z is the constant connecting the renormalized and the free field, $\phi \xrightarrow[t\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, $\hat{\phi}_{\text{ren}} := Z^{-1/2}\hat{\phi} \xrightarrow[t\to-\infty]{} \hat{\phi}_{\text{in}}$) then in perturbation theory one usually gets

$$Z = 1 - \lambda\infty \sim \frac{1}{1+\lambda\infty} = 0$$

where ∞ simply denotes some divergent quantity which has been removed through the renormalization procedure. Then at some fixed time the field itself does not exist as an operator. In order to have a well defined operator, one needs to smear $\hat{\phi}(x)$ not only in space

$$\hat{\phi}(x^0, f) = \int \mathrm{d}^3x\, f(\boldsymbol{x})\hat{\phi}(x^0, \boldsymbol{x})$$

but also in time

$$\hat{\phi}(g) = \int \mathrm{d}^4x\, g(x)\hat{\phi}(x)$$

Indeed suppose that $\hat{\phi}$ is the Heisenberg interacting field, and suppose that it is well defined at fixed time x^0, then according to Källen-Lehmann representation (recall eq. (3.10)) we get

Ho aggiunto questa formula ai paragrafi precedenti, non son sicuro sia giusta, ho qualche dubbio. Altrimenti non saprei come motivare che il campo interagente soddisfa le CCR.

$$\langle 0|\,[\hat{\phi}(x^0, \boldsymbol{x}), \dot{\hat{\phi}}(x^0, \boldsymbol{y})]\,|0\rangle = i\delta(\boldsymbol{x} - \boldsymbol{y})$$

because and interacting 2-points function can be written in terms of free 2-points functions of different masses satisfying CCR. Therefore the "renormalized" field $\hat{\phi}_{\text{ren}} = Z^{-1/2}\hat{\phi}$ which approaches weakly the $\hat{\phi}_{\text{in}}$ as $t \to -\infty$, satisfies

$$\langle 0|\,[\hat{\phi}_{\text{ren}}(x^0, \boldsymbol{x}), \dot{\hat{\phi}}_{\text{ren}}(x^0, \boldsymbol{y})]\,|0\rangle = \frac{i}{Z}\delta(\boldsymbol{x} - \boldsymbol{y})$$

Clearly if $Z = 0$ this expression is ill-defined, but no assumption has been made except that $\hat{\phi}$ is well defined at fixed time, hence if $Z = 0$ this assumption should be wrong, i.e. the renormalized field cannot be defined at fixed time. A Schrödinger picture requires that we have a definite space at a fixed time, hence we also proved that it is impossible to defined a picture of dynamics using Schrödinger idea if $Z = 0$.

Chapter 4

The reconstruction theory

4.1 How to avoid Haag's theorem

[Str13, Section 2.5]

Let's see how to avoid problems raised by Haag theorem. The idea is to start by introducing IR (volume) and UV cutoffs (the IR one is not needed for lattice models) so that the cutoff model describes a system of finite degrees of freedom, such that the interaction picture is still correct, thanks to von Neumann uniqueness theorem, which guarantees that free and interacting fields are unitarily equivalent and defined in the same Hilbert space. Moreover one can also require that free fields (and then also interacting ones due to Källen-Lehmann representation) satisfy CCR (or CAR) since no problem arises in this framework. Consider a $T = 0$ RQFT (the restriction $T = 0$ is not really needed, but simplify the traction) and let $\hat{\phi}_0(\boldsymbol{x}) := \hat{\phi}(x^0 = 0, \boldsymbol{x})$ be the $t = 0$ field, then the cutoff field at arbitrary time is given by

$$\hat{\phi}^{\Lambda}_{\#}(x^0, \boldsymbol{x}) = e^{ix^0 H_{\#}} \hat{\phi}_0(\boldsymbol{x}) e^{-ix^0 H_{\#}}$$

where $\#$ denotes either the free of the interacting theory and Λ denotes the cutoffs. The previous time evolution is ensured by the von Newman theorem, assuming that the free and the interacting fields coincides at the initial time $t = 0$. Let $\left|0^{\Lambda}_I\right\rangle$ be the vacuum of the interacting Hamiltonian, which is still in the Fock space of the free field.
We construct the following functions (actually they are distributions, but with abuse of terminology, we call "functions" also these expectation values)

$$W^{\Lambda}_n(x_1, \ldots, x_n) := \left\langle 0^{\Lambda}_I \middle| \hat{\phi}^{\Lambda}_I(x_1) \cdots \hat{\phi}^{\Lambda}_I(x_n) \middle| 0^{\Lambda}_I \right\rangle$$

We stress the fact that since translational invariance (and thus also Poincaré invariance) is broken, Haag's theorem does not apply.
Then we identify the necessary counterterms to be added to the Hamiltonian (or to the Lagrangian if the path integral approach is used) and the field renormalization Z_Λ (called *wave-function renormalization constant*) leading to "*renormalized fields*", for which the correlation functions have a well-defined limit when IR an UV cutoffs are removed from outside[I] of the correlation function, i.e. such that the following limit

$$\lim_{\substack{\Lambda_{\mathrm{UV}} \to \infty \\ \Lambda_{\mathrm{IR}} \to 0}} Z^{-n}_{\Lambda_{\mathrm{UV}}} \left\langle 0^{\Lambda}_I \middle| \hat{\phi}^{\Lambda}_I(x_1) \cdots \hat{\phi}^{\Lambda}_I(x_n) \middle| 0^{\Lambda}_I \right\rangle$$

exists in a suitable space of functions or distributions, e.g. $\mathcal{S}'(\mathbb{R}^d)$. This is possible since the previous limit is a functions limit, hence much weaker than the operator limit (for the fields) which cannot exists due to Haag's theorem.
The crucial fact is that if the limiting functions satisfy certain properties we can reconstruct quantum fields acting in an appropriate Hilbert space, such that their expectation values are precisely the functions we started from. The reconstructed field $\hat{\phi}$ now has no reasons to be equivalent to a free field, nor the Hilbert space of $\hat{\phi}$ has reason to be the same Fock space we started from (actually in general it could even not be a Fock space), hence the reconstruction is not in conflict with Haag's argument.

[I] It is important to notice that the limit should be removed from outside, otherwise Haag's theorem apply and we are stuck.

4.2 Wightman's reconstruction theorem

[SW00, Chapter 3]; [Jos65, Chapter 3]; [Str13, Chapter 3]; [Str93, Sections 1.3-1.4]

As anticipated, provided some properties for the limiting functions

$$W_n(x_1,\ldots,x_n) := \lim_{\substack{\Lambda_{\mathrm{UV}}\to\infty\\ \Lambda_{\mathrm{IR}}\to 0}} Z_{\Lambda_{\mathrm{UV}}}^{-n} W_n^{\Lambda}(x_1,\ldots,x_n)$$

these functions themselves determine states and fields of the interacting theory, avoiding Haag's theorem. In this section we first describe how this reconstruction works for a RQFT at $T=0$, then we'll sketch how to adapt the formalism to more general situations.
We give the statement and the complete proof of the *reconstruction theorem* for a real scalar field at $T=0$ for a relativistic theory.

Theorem 4.1 (Wightman's reconstruction theorem, [Wig56], [SW00, Thm. 3.7], [Jos65, Section 3.4, Thm. 1])**.** *Let $\{W_n\}_{n=0}^{\infty}$ be a sequence of (tempered) distributions, $W_n \in \mathcal{S}'(\mathbb{R}^{(d+1)n})$ for all $n>0$, $W_0 := 1$. Suppose that they satisfy the following properties:*

La condizione di hermiticità non è necessaria per definire un campo reale?

*(1) (*Positive Definiteness*) For any finite sequence of N test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1,x_2),\ldots)$, $f_n \in \mathcal{S}(\mathbb{R}^{(d+1)n})$, then*

Qui ho usato la definizione di Streater-Wight e Jost, differente da quella data a lezione, non mi sembra che le due definizioni siano equivalenti (quella data a lezione sembra più forte, visto che la serie diventa infinita).

$$\sum_{j,k=0}^{N} \int \mathrm{d}x_1\ldots\mathrm{d}x_j\mathrm{d}y_1\ldots\mathrm{d}y_k\, f_n^*(x_1,\ldots,x_j) W_{j+k}(x_1,\ldots,x_j,y_1,\ldots,y_k) f_k(y_1,\ldots,y_k) \geq 0$$

or in more compact notation $(\underline{f}, W\underline{f}) \geq 0$.[II]

*(2) (*Poincaré covariance*) For any $(a,\Lambda)\in\mathcal{P}_+^{\uparrow}$, $\Lambda \in L_+^{\uparrow}$ and $a\in\mathbb{R}^{d+1}$,*

$$W_n(x_1,\ldots,x_n) = W_n(\Lambda x_1 + a,\ldots,\Lambda x_n + a)$$

for all $n\in\mathbb{N}$.

*(3) (*Spectral condition*) For each W_n, $n>0$, exists $\widetilde{W}_n$ such that*

$$W_n(x_1,\ldots,x_n) = \int \mathrm{d}p_1\ldots\mathrm{d}p_n\, \widetilde{W}_{n-1}(p_1,\ldots,p_{n-1})\, e^{i(x_n-x_{n-1})p_{n-1}}\ldots e^{i(x_2-x_1)p_1} \tag{4.1}$$

with $\widetilde{W}_{n-1}(p_1,\ldots,p_{n-1}) = 0$ if $p_j \notin \overline{V}^+$ for some j, $1\le j\le n-1$.

*(4) (*Local commutativity condition / Symmetry*) If x_i and x_{i+1} are space-like separated for some i, that is $(x_i - x_{i+1})^2 < 0$, then for all $n>i$*

$$W_n(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = W_n(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)$$

*(5) (*Cluster property*) For any $n\ge 2$, $j<n$,*

$$\lim_{\substack{a\to\infty\\ a^2<0}} W_n(x_1,\ldots x_j, x_{j+1}+a,\ldots,x_n+a) = W_j(x_1,\ldots,x_j)W_{n-j}(x_{j+1},\ldots,x_n)$$

Then exists

(1') a separable Hilbert space $\mathcal{H}$,

(2') a continuous unitary representation $U(a,\Lambda)$ of $\mathcal{P}_+^{\uparrow}$[III] *in $\mathcal{H}$,*

(3') a state $\Omega\in\mathcal{H}$, translationally invariant, called vacuum*,*

[II] Notice that the proof of this property is not easy, since one should also prove the existence of the limit $W_n^{\Lambda}\to W_n$.
[III] In the case of fields with half-integer spin $\mathcal{P}_+^{\uparrow}$ is replaced by $\tilde{\mathcal{P}}_+^{\uparrow}$.

(4') an operator-valued distribution field $\hat{\phi}(x)$ such that, given a test function f, $\hat{\phi}(f) := \int \mathrm{d}x\, f(x)\hat{\phi}(x)$ is an operator with domain D, D dense in $\mathcal{H}$ and $|\Omega\rangle \in D$, and denoting by $\mathcal{P}(\hat{\phi}(f))$ the set of polynomials of $\hat{\phi}(f)$[IV]*, the subspace of $\mathcal{H}$ generated applying polynomials of the smeared fields to the vacuum, $\mathcal{P}(\hat{\phi}(f))\,|\Omega\rangle$, is dense in $\mathcal{H}$*

and the following properties hold:

(5') the field $\hat{\phi}(x)$ transforms covariantly under $\mathcal{P}_+^\uparrow$:

$$U(a,\Lambda)\hat{\phi}(x)U^\dagger(a,\Lambda) = \hat{\phi}(\Lambda x + a)$$

(6') let $\sigma(\hat{P}^\mu)$ be the spectrum of the generator $\hat{P}^\mu$ of the subgroup of translations, then $\sigma(\hat{P}^\mu) \subseteq \overline{V}^+$, i.e. $\sigma(\hat{P}^\mu\hat{P}_\mu) \geq 0$ and $\sigma(\hat{P}^0) \geq 0$;

*(7') (*Locality*) fields in space-like separated regions commute*

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad \textit{if} \quad (x-y)^2 < 0$$

or more precisely, in terms of smeared fields f, g,

$$[\hat{\phi}(f), \hat{\phi}(g)] = 0 \quad \textit{if} \quad \operatorname{supp} f \times \operatorname{supp} g$$

where $\times$ denotes that the two regions are space-like separated,

*(8') (*Uniqueness of the vacuum*) The vector Ω is the unique vector in $\mathcal{H}$ translationally invariant;*

(9') for all $n \in \mathbb{N}$[V]

$$(\Omega, \hat{\phi}(x_1)\dots\hat{\phi}(x_n)\Omega) = W_n(x_1,\dots,x_n) \tag{4.2}$$

Proof. (*sketch*)
(1') Consider the vector space $\underline{\mathcal{S}}$ of sequences of test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1,x_2), \dots)$, $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{(d+1)k})$, with finite number of non-vanishing elements. Addition and multiplication by complex scalars are definite by

Ho cercato di completare la dimostrazione, ora mi sembra completa, non più “abbozzata”.

$$(f_0, f_1, \dots) + (g_0, g_1, \dots) = (f_0+g_0, f_1+g_1, \dots)$$
$$\alpha(f_0, f_1, \dots) = (\alpha f_1, \alpha f_2, \dots)$$

Next we introduce a scalar product defined on pairs of vectors of the vector space

$$(\underline{f}, \underline{g}) := \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \dots \mathrm{d}x_j \mathrm{d}y_1 \dots \mathrm{d}y_k f_j^*(x_1,\dots,x_j) W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k) g_k(y_1,\dots,y_k) \tag{4.3}$$

or using the previous compact notation $(\underline{f}, \underline{g}) := (\underline{f}, W\underline{g})$. Recall $W_0 = 1$ by definition. The linearity in $\underline{g}$ and anti-linearity in $\underline{f}$ are evident from the definition, furthermore from *(1)* we get that the norm $\|\underline{f}\|^2 = (\underline{f}, \underline{f})$ is semi-definite positive, i.e. $\|\underline{f}\| \geq 0$ for any $\underline{f}$. Therefore the operation $(-,-)$ is a well defined inner product in $\underline{\mathcal{S}}$, but $\underline{\mathcal{S}}$ is not a pre-Hilbert space yet since it may contain vectors of zero norm. In order to get a pre-Hilbert, note that the set

In caso, aggiungere hemiticità del prodotto scalare, vedi Streater-Wight.

$$\underline{\mathcal{S}_0} := \{\underline{f} \in \underline{\mathcal{S}} \text{ s.t. } \|\underline{f}\| = 0\} \subseteq \underline{\mathcal{S}}$$

is an isotropic subspace of $\underline{\mathcal{S}}$, that is, a subspace in which each vector is orthogonal to every other vector, indeed

$$|(\underline{f}, \underline{g})| \leq \|\underline{f}\|\,\|\underline{g}\| = 0$$

by the Schwartz inequality (which is valid as long as the scalar product is non-negative). Thus, if $\underline{f}$ and $\underline{g}$ are of zero norm, then $\underline{f}$ is orthogonal to $\underline{g}$ and $\alpha\underline{f} + \beta\underline{g}$ is of zero norm. We now form equivalence

[IV] It is important to notice that $\mathcal{P}(\hat{\phi}(f))$ denotes polynomials of fields smeared (in general) with different test functions, not only one test function f, for instance $\hat{\phi}(f_1)\hat{\phi}(f_2)\dots\hat{\phi}(f_n) \in \mathcal{P}(\hat{\phi}(f))$.

[V] Usually in the axiomatic approach due to Wightman the Dirac notation is avoided, and the scalar product is denoted by $(-,-)$.

classes of sequences $\underline{f}$, two sequences being equivalent if they differ by a sequence of zero norm. These equivalence classes form in a natural way a vector space, denoted by $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, on which the scalar product induced by $\underline{\mathcal{S}}$ is well-defined and positive definite, since $\|[\,\underline{f}\,]\| = 0$ implies $[\,\underline{f}\,] = [\,\underline{0}\,]$. Therefore $\underline{\mathcal{S}}$ is a pre-Hilbert space. A generic element of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ is of the form

$$\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0} \ni [\,\underline{f}\,] = \{\underline{g} \in \underline{\mathcal{S}} \mid \underline{f} - \underline{g} \in \underline{\mathcal{S}_0}\}$$

Then we can finally define the Hilbert space $\mathcal{H}$ by completion[VI] of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ respect to the induced norm:

$$\mathcal{H} := \overline{\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}}$$

that is, let $\mathfrak{h}$ be the space of Cauchy sequences $F = \{[\,\underline{f_1}\,], [\,\underline{f_2}\,], \ldots\}$ with scalar product

$$(F, G) := \lim_{n\to\infty} ([\,\underline{f_n}\,], [\,\underline{g_n}\,]) \tag{4.4}$$

and $\mathfrak{h}_0$ its subspace given by vectors of zero norm, then $\mathcal{H} := \mathfrak{h}/\mathfrak{h}_0$. In the following we'll denote by $([\,\underline{f_1}\,], [\,\underline{f_2}\,], \ldots)$ the elements of $\mathcal{H}$ (i.e. as elements of $\mathfrak{h}$) rather then $[([\,\underline{f_1}\,], [\,\underline{f_2}\,], \ldots)]$, we give as understood the fact that we refer to the associated equivalence class.
Being $\mathcal{S}$ separable so is $\underline{\mathcal{S}}$, and since the latter is dense in $\mathcal{H}$ also $\mathcal{H}$ is separable.

(2') We define the linear transformation $U(a, \Lambda)$ in $\underline{\mathcal{S}}$ by

$$U(a, \Lambda)(f_0, f_1, f_2, \ldots) := (f_0, \{a, \Lambda\} f_1, \{a, \Lambda\} f_2, \ldots)$$

where

$$\{a, \Lambda\} f_k(x_1, \ldots, x_k) := f_k(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_k - a))$$

The operator $U(a, \Lambda)$ leaves the scalar product of $\underline{\mathcal{S}}$ invariant by virtue of *(2)*:

$$\begin{aligned}
&(U(a,\Lambda)\underline{f}, U(a,\Lambda)\underline{g}) = \\
&\quad = \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k f_j^*(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_j - a)) \times \\
&\qquad\qquad \times W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) g_k(\Lambda^{-1}(y_1 - a), \ldots, \Lambda^{-1}(y_k - a)) \\
&\quad = \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k f_j^*(x_1, \ldots, x_j) W_{j+k}(\Lambda x_1 + a, \ldots, \Lambda y_k + a) g_k(y_1, \ldots, y_k) \\
&\quad \overset{(2)}{=} \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) g_k(y_1, \ldots, y_k) \\
&\quad = (\underline{f}, \underline{g})
\end{aligned}$$

thus $U(a, \Lambda)$ leaves invariant also $\underline{\mathcal{S}_0}$, since $\|\underline{f}\| = 0$ implies $\|U(a, \Lambda)\underline{f}\| = 0$. We have to check that $U(a, \Lambda)$ is actually a mapping of equivalence classes in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. But due to the invariance of the scalar product if $[\,\underline{f}\,] = [\,\underline{g}\,]$ then $[\,U(a, \Lambda)\underline{f}\,] = [\,U(a, \Lambda)\underline{g}\,]$, hence $U(a, \Lambda)$ is well defined also in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. Being bounded it extends by continuity to $\mathcal{H}$, preserving the scalar product in $\mathcal{H}$. Recall that by Wigner's theorem linear operators that preserves the scalar products in a Hilbert space are unitary, then $U(a, \Lambda)$ is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$.

Onestamente non mi ricordo bene questa parte sugli spazi di Hilbert, spero di non aver scritto scorrettezze.

(3') Notice that

$$U(a, \Lambda)(1, 0, \ldots, 0, \ldots) = (1, 0, \ldots, 0, \ldots) =: \underline{\mathbb{1}}$$

hence $\underline{\mathbb{1}} \in \underline{\mathcal{S}}$ is translational invariant in $\underline{\mathcal{S}}$ and the same holds for $[\,\underline{\mathbb{1}}\,]$ in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. Finally, the vacuum in $\mathcal{H}$ is given by the Cauchy sequence $\Omega := ([\,\underline{\mathbb{1}}\,], [\,\underline{\mathbb{1}}\,], \ldots)$.

[VI] The completion procedure is described in [SW00, Pages 121-122].

(4') We introduce in $\underline{\mathcal{S}}$ a linear operator $\hat{\phi}(h)$ for each test function $h \in \mathcal{S}(\mathbb{R}^{d+1})$ by the equation

$$\hat{\phi}(h)\underline{f} := (0, hf_0, h \otimes f_1, h \otimes f_2, \ldots)$$

or in compact notation $\hat{\phi}(h)\underline{f} := h \times \underline{f}$, where

$$(h \otimes f_k)(x_1, \ldots, x_{k+1}) := h(x_1) f_k(x_2, x_3, \ldots, x_{k+1})$$

is clearly a test function. As functionals of h the matrix elements $(\underline{f}, \hat{\phi}(h)\underline{g})$ are tempered distributions since they are finite sums of W's, and further

$$(\underline{f}, \hat{\phi}(h)\underline{g}) = (\hat{\phi}(h^*)\underline{f}, \underline{g}) \tag{4.5}$$

We have to check that $\hat{\phi}(h)$ is a mapping of equivalence classes in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. That $\|\underline{f}\| = 0$ implies $\|\hat{\phi}(h)\underline{f}\| = 0$ follows from previous definition and Schwartz inequality:

$$(\hat{\phi}(h)\underline{f}, \hat{\phi}(h)\underline{f}) = (\underline{f}, \hat{\phi}(h^*)\hat{\phi}(h), \underline{f}) \leq \|\underline{f}\| \, \|\hat{\phi}(h^*)\hat{\phi}(h)\underline{f}\| = 0$$

if $\|\underline{f}\| = 0$. Therefore $\hat{\phi}(h)$ is well-defined also in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, $\hat{\phi}(h)[\underline{f}] := [\hat{\phi}(h)\underline{f}]$. For each $[\underline{f}] \in \underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, $\hat{\phi}(h)$ is well-defined also for the associated vectors in $\mathcal{H}$, $\hat{\phi}(h)([\underline{f}], [\underline{f}], \ldots) := (\hat{\phi}(h)[\underline{f}], \hat{\phi}(h)[\underline{f}], \ldots)$, hence $\hat{\phi}$ is defined in a dense subset D of $\mathcal{H}$, $D \cong \underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. In particular, being $|\Omega\rangle = ([\underline{\mathbb{1}}], [\underline{\mathbb{1}}], \ldots) \in D$, this defines $\hat{\phi}(h)$ on $\mathcal{P}(\hat{\phi}(h))\,|\Omega\rangle$. Notice that

$$\hat{\phi}(h_1)\cdots\hat{\phi}(h_n)\underline{\mathbb{1}} = \hat{\phi}(h_1)\cdots\hat{\phi}(h_{n-1})(0, h_n, 0, 0, \ldots) = (\underbrace{0, \ldots, 0}_{n}, h_1 \otimes h_2 \otimes \ldots \otimes h_n, 0, 0, \ldots) \tag{4.6}$$

hence $\mathcal{P}(\hat{\phi}(h))\,\underline{\mathbb{1}} \cong \underline{\mathcal{S}}$ and then $\mathcal{P}(\hat{\phi}(h))\,|\Omega\rangle \cong D$ is dense in $\mathcal{H}$.

(9') Notice that the vacuum expectation value in $\mathcal{H}$ reads

$$\begin{aligned}
(\Omega, \hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) &= ([\underline{\mathbb{1}}], \hat{\phi}(h_1)\ldots\hat{\phi}(h_n)[\underline{\mathbb{1}}]) && \text{scalar product in } \underline{\mathcal{S}}/\underline{\mathcal{S}_0} \\
&= (\underline{\mathbb{1}}, \hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\underline{\mathbb{1}}) && \text{scalar product in } \underline{\mathcal{S}} \\
&= \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, W_n(x_1, \ldots, x_n) h_1(x_1) \ldots h_n(x_n) \\
&= W_n(h_1, \ldots, h_n)
\end{aligned}$$

where in the first step we computed the scalar product in $\mathcal{H}$ using eq. (4.4), in the second step we chosed the representative $\underline{\mathbb{1}}$ for the equivalence class $[\underline{\mathbb{1}}]$ and then we computed the scalar product in $\underline{\mathcal{S}}$ according to the definition eq. (4.3) using eq. (4.6) to compute $\hat{\phi}(h_1)\cdots\hat{\phi}(h_n)\underline{\mathbb{1}}$. Then we noted that the result we obtained coincide with the smearing of the Wightman function. Finally eq. (4.2) follows immediately from $\langle\Omega|\,\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\,|\Omega\rangle = W_n(h_1, \ldots, h_n)$.

(5') That $\hat{\phi}(h)$ satisfies the transformation law

$$U(a,\Lambda)\hat{\phi}(h)U^\dagger(a,\Lambda) = \hat{\phi}(\{a,\Lambda\}h)$$

can be easy proved in $\underline{\mathcal{S}}$:

$$\begin{aligned}
U(a,\Lambda)\hat{\phi}(h)(f_0, f_1, \ldots) &= U(a,\Lambda)(0, hf_0, h \otimes f_1, \ldots) \\
&= (0, \{a,\Lambda\}hf_0, \{a,\Lambda\}h \otimes \{a,\Lambda\}f_1, \ldots) \\
&= \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda)(f_0, f_1, \ldots)
\end{aligned}$$

then since the previous computation holds for each $\underline{f} \in \underline{\mathcal{S}}$ and both $\hat{\phi}(h)$ and $U(a,\Lambda)$ are mapping of equivalence classes then

$$U(a,\Lambda)\hat{\phi}(h)[\underline{f}] = \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda)[\underline{f}]$$

and finally we can extend it by continuity to $\mathcal{H}$, since $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ is dense in it:

$$U(a,\Lambda)\hat{\phi}(h)([\,\underline{f}_1\,],[\,\underline{f}_2\,],\ldots) = \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda)([\,\underline{f}_1\,],[\,\underline{f}_2\,],\ldots)$$

Since this holds for any element of $\mathcal{H}$ we get

$$U(a,\Lambda)\hat{\phi}(h) = \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda) \quad \Rightarrow \quad \hat{\phi}(\{a,\Lambda\}h) = U(a,\Lambda)\hat{\phi}(h)U^{-1}(a,\Lambda)$$

and we are done, since "unsmearing" the fields we get

$$\int \mathrm{d}x\, h(\Lambda^{-1}(x-a))\hat{\phi}(x) = U(a,\Lambda)\int \mathrm{d}x\, h(x)\hat{\phi}(x)U^{-1}(a,\Lambda)$$

and by a change of variable on the r.h.s.:

$$\int \mathrm{d}x\, h(x)\hat{\phi}(\Lambda x + a) = \int \mathrm{d}x\, h(x)U(a,\Lambda)\hat{\phi}(x)U^{-1}(a,\Lambda)$$

(6') Consider the unitary group of translations $\{U(a)\} := \{U(a,\mathbb{1})\}$. Stone's theorem[VII] apply to such operators (the group can be decomposed in $(d+1)$ one-parameter unitary groups), then exists a unique operator $\hat{P}: D_P \to \mathcal{H}$, self-adjoint in the dense domain D_P, such that $U(a) = e^{ia\hat{P}}$ for any $a \in \mathbb{R}^{d+1}$. Moreover thanks to *(5')* we have

$$\hat{\phi}(x) = e^{ix\hat{P}}\hat{\phi}(0)e^{-ix\hat{P}} \tag{4.7}$$

Then, using *(9')* together with *(3')* we have

$$\begin{aligned}
W_n(x_1,\ldots,x_n) &= \\
&= \langle\Omega|\,\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\,|\Omega\rangle \\
&= \langle\Omega|\,\hat{\phi}(0)e^{i\hat{P}(x_2-x_1)}\hat{\phi}(0)\ldots\hat{\phi}(0)e^{i\hat{P}(x_n-x_{n-1})}\hat{\phi}(0)\,|\Omega\rangle \\
&= \int \mathrm{d}p_1\ldots\mathrm{d}p_{n-1}\ \langle\Omega|\,\hat{\phi}(0)e^{i\hat{P}(x_2-x_1)}\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\ldots\hat{\phi}(0)e^{i\hat{P}(x_n-x_{n-1})}\,|p_{n-1}\rangle\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle \\
&= \int \mathrm{d}p_1\ldots\mathrm{d}p_{n-1}\, e^{ip_1(x_2-x_1)}\ldots e^{ip_{n-1}(x_n-x_{n-1})}\,\langle\Omega|\,\hat{\phi}(0)\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\,|p_2\rangle\ldots\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle
\end{aligned}$$

Comparing this expression with eq. (4.1) we get

$$\widetilde{W}_{n-1}(p_1,\ldots,p_{n-1}) = \langle\Omega|\,\hat{\phi}(0)\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\,|p_2\rangle\ldots\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle$$

Then the claim follows using property *(3)*. Indeed, take $p_2 \notin \overline{V}^+$ and $n=4$, then from *(3)* we get

$$\widetilde{W}_3(p_1,p_2,p_3) = \langle\Omega|\,\hat{\phi}(0)\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\,|p_2\rangle\langle p_2|\,\hat{\phi}(0)\,|p_3\rangle\langle p_3|\,\hat{\phi}(0)\,|\Omega\rangle = 0 \quad \text{for all} \quad p_1,p_3 \in \mathbb{R}^{d+1}$$

This is possible only if

$$\langle p|\,\hat{\phi}(0)\,|p_2\rangle = 0 \quad \text{for all} \quad p \in \mathbb{R}^{d+1} \quad \Leftrightarrow \quad \hat{\phi}(0)\,|p_2\rangle = 0 \quad \overset{(4.7)}{\Leftrightarrow} \quad \hat{\phi}(x)\,|p_2\rangle = 0$$

Hence for each $p \notin \overline{V}^+$ we have $\hat{\phi}(x)\,|p\rangle = 0$. Applying $\hat{P}$ to any element of $\mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle$ we get then

$$\hat{P}\big(\mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle\,\big) = \int \mathrm{d}p\ |p\rangle\langle p|\,\big(\mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle\,\big) = \int_{\overline{V}^+} \mathrm{d}p\ |p\rangle\langle p|\,\big(\mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle\,\big)$$

which give the following spectral representation of $\hat{P}$

$$\hat{P} = \int_{\overline{V}^+} \mathrm{d}p\ |p\rangle\langle p| \quad \text{in} \quad \mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle$$

but since $\mathcal{P}(\hat{\phi}(x))\,|\Omega\rangle$ is dense in $\mathcal{H}$ the previous spectral representation for $\hat{P}$ holds in the whole $\mathcal{H}$ (actually, in $D_P \subseteq \mathcal{H}$). The claim follows immediately.

Mi sembra che l'argomentazione data a lezione non fosse sufficiente a dimostrare questo punto del teorema. Da qui in poi la dimostrazione l'ho fatta di sana pianta, quindi potrebbe essere molto sbagliata (spero di no).

(7') Take $f, g \in \mathcal{S}(\mathbb{R}^{d+1})$ space-like separated, that is $\operatorname{supp} f \times \operatorname{supp} g$, and $\{h_i\}_{i=0}^{\infty}$. Then from *(4)* we know that, for each value of n,

Anche qui ho cercato di completare la dimostrazione, mi faccia sapere se è giusta o devo correggerla.

$$W_{n+2}(f, g, h_1, \ldots, h_n) = W_{n+2}(g, f, h_1, \ldots, h_n)$$

and *(9')* tells us that

$$(\Omega, \hat{\phi}(f)\hat{\phi}(g)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = (\Omega, \hat{\phi}(g)\hat{\phi}(f)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega)$$

Then, by linearity

$$(\Omega, [\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = 0$$

An analogous procedure together with eq. (4.5) lead us also to

$$([\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega, [\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = \|[\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega\|^2 = 0$$

Since the scalar product is non-degenerate, this implies that $[\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega = 0$ is the zero vector in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. The only way to satisfy this for any choice of n and any set of smearing test functions $\{h_i\}_{i=1}^{\infty}$ is that in $\mathcal{P}(\hat{\phi}(h))\Omega$ we have

$$[\hat{\phi}(f), \hat{\phi}(g)] = 0$$

meaning that the operator $[\hat{\phi}(f), \hat{\phi}(g)]$ maps all the elements of $\mathcal{P}(\hat{\phi}(h))\Omega$ into the zero vector. But then the claim extends to the whole Hilbert space $\mathcal{H}$ due to the denseness of $\mathcal{P}(\hat{\phi}(h))\Omega$.

(8') Assume that, beside $\Omega = ([\underline{\mathbb{1}}], [\underline{\mathbb{1}}], \ldots)$ there is another translationally invariant state Ω'. Without loss of generality we can orthonormalize Ω' respect to Ω, i.e. we take $(\Omega', \Omega) = 0$ and $(\Omega', \Omega') = 1$. If $\Omega' \in \underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, $\Omega = [\,\underline{f}\,]$, then we would have an immediate contradiction because,

$$\begin{aligned}
1 = (\Omega', \Omega') &\overset{(a)}{=} \lim_{\substack{a\to\infty\\a^2<0}} (\Omega', U(a)\Omega') = \lim_{\substack{a\to\infty\\a^2<0}} ((f_0, f_1, \ldots), (f_0, \{a, \mathbb{1}\}f_1, \{a, \mathbb{1}\}f_2, \ldots)) \\
&\overset{(4.3)}{=} \lim_{\substack{a\to\infty\\a^2<0}} \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) f_k(y_1 - a, \ldots, y_k - a) \\
&= \lim_{\substack{a\to\infty\\a^2<0}} \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1 + a, \ldots, y_k + a) f_k(y_1, \ldots, y_k) \\
&\overset{(5)}{=} \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, f_j^*(x_1, \ldots, x_j) W_j(x_1, \ldots, x_j) W_k(y_1, \ldots, y_k) f_k(y_1, \ldots, y_k) \\
&= \left(\sum_{j=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \, f_j^*(x_1, \ldots, x_j) W_j(x_1, \ldots, x_j) \right) \left(\sum_{k=0}^{\infty} \int \mathrm{d}y_1 \ldots \mathrm{d}y_k \, W_k(y_1, \ldots, y_k) f_k(y_1, \ldots, y_k) \right) \\
&= (\Omega', \Omega)(\Omega, \Omega') = 0
\end{aligned}$$

where in (a) we used the translational invariance of Ω'. In general Ω' is not an element of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, but because the latter is dense then Ω' can be approximated by elements of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ by arbitrary accuracy, then the contradiction easily extends for $\Omega' \in \mathcal{H}$, see [SW00, Page 124] for the details. □

The most important consequence of the theorem is that in order to exhibit a relativistic quantum field theory model, it is enough to give a set of Wightman functions satisfying properties *(1)-(5)*. These properties provide a non-perturbative substitute for canonical quantization, since allow to construct a quantum field theory without use the canonical quantization procedure and CCR (or CAR) which are inconsistent in the interacting case due to Haag's theorem.
Properties *(1') - (8')* are called *Wightman axioms*[VIII], and define axiomatically a RQFT at $T = 0$ for a real scalar field. Provided appropriated versions of properties *(1)-(5)*, more general forms of the

[VII] https://doi.org/10.2307%2F1968538

[VIII] These axioms are discussed in [SW00, Section 3.1] and [Jos65, Section 3.2].

reconstruction theorem allow to reconstruct also theories with complex fields and more general spins, satisfying appropriate Wightman axioms.
Notice that property *(9')* tells us that the vacuum expectation values of the reconstructed theory are exactly the Wightman function we started from.
Conversely, the vacuum expectation values of any theory satisfying Wightman axioms satisfy themselves the properties required in the reconstruction theorem.[IX] In particular, if $\hat{\phi}_F$ is a free theory defined in a Fock space, then the vacuum expectation values $\langle 0| \hat{\phi}_F(x_1) \dots \hat{\phi}_F(x_n) |0\rangle$ satisfy properties *(1)-(5)* and the reconstructed theory coincide with the initial one, $\hat{\phi}_{\text{rec}} = \hat{\phi}_F$. On the other side, a reconstructed interacting theory is defined in a Hilbert space disjoint from the Fock space of the free theory, due to Haag's theorem.
Can also be proved that the reconstructed theory is unique, i.e. any other theory satisfying properties *(1') - (9')* (in particular with same vacuum expectation values $\{W_n\}_{n=0}^{\infty}$) is unitarily equivalent to the one reconstructed in the proof of the theorem.[X]
Finding a set of Wightman functions satisfying properties *(1)-(5)* turned out to be a very hard problem, apart from the non-interacting case, also because it is difficult to satisfy the positivity condition, which has a non-linear structure, in contrast to the other properties, which have a linear structure. We will see in the following how this problem has been solved by computing Wightman functions in the Euclidean space, where properties *(1)-(5)* take a more simple form.
In the proof of the theorem we constructed the Hilbert space of the theory as (the completion of) finite sequences of test functions. In order to give a better interpretation of the Hilbert space, notice that if we define

Non sono molto convinto di quanto scritto da qui in poi.

$$\hat{\phi}(\underline{f}) := f_0 + \int \mathrm{d}x_1\, \hat{\phi}(x_1) f_1(x_1) + \int \mathrm{d}x_1 \mathrm{d}x_2\, \hat{\phi}(x_1)\hat{\phi}(x_2) f_2(x_1, x_2) + \dots$$

such that

$$\hat{\phi}(\underline{f})\underline{g} := (f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 \otimes g_1 + f_2 g_0, \dots) \quad , \quad \hat{\phi}(\underline{f})[\,\underline{g}\,] := [\,\hat{\phi}(\underline{f})\underline{g}\,]$$

and in particular (let $\Omega := [\,\underline{\mathbb{1}}\,]$ in this instance, since we do not consider the completion of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$)

$$\hat{\phi}(\underline{f})\underline{\mathbb{1}} = (f_0, f_1, f_2, \dots) = \underline{f} \quad , \quad \hat{\phi}(\underline{f})\Omega = [\,\underline{f}\,]$$

Then the scalar product in $\underline{\mathcal{S}}$ and $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ can be rewritten as[XI]

$$(\underline{f}, \underline{g}) = \langle \hat{\phi}(\underline{f})\underline{\mathbb{1}}, \hat{\phi}(\underline{g})\underline{\mathbb{1}} \rangle \quad , \quad ([\,\underline{f}\,], [\,\underline{g}\,]) = \langle \hat{\phi}(\underline{f})\Omega, \hat{\phi}(\underline{g})\Omega \rangle$$

Hence we can identify by unitary equivalence (i.e. up to representations) $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ with $\{\hat{\phi}(\underline{f})\Omega \mid \underline{f} \in \underline{\mathcal{S}}\}$, since the scalar products defined in the two spaces are the same. This is related with the denseness of $\mathcal{P}(\hat{\phi}(f))\Omega$ in $\mathcal{H}$. Then we are done: one can forget the structure underlying $\underline{f}$ and Ω and define the Hilbert space by applying the operator $\hat{\phi}(\underline{f})$ on the vacuum Ω, as usual.

4.3 Additional remarks

[IX] See [SW00, Theorems 3.1-3.4] and [Jos65, Section 3.3].
[X] Actually this claim is included in the reconstruction theorem, see [SW00, Thm. 3.7] and [Jos65, Section 3.4, Thm. 1].
[XI] The bilinear $(-,-)$ denotes as usual the scalar product in $\underline{\mathcal{S}}$, whereas $\langle -,- \rangle$ denotes the scalar product in $\{\hat{\phi}(\underline{f})\underline{\mathbb{1}} \mid \underline{f} \in \underline{\mathcal{S}}\}$. Analogous notations hold for the quotient sets.

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