Chapter 1

Vortices

[Shifman:2012]

The second quantum soliton that we consider is the vortex in $2+1$ dimensions in a model that in high-energy *Abelian Higgs mode* and in $3+1$ dimensions was the first model proposed to make massive gauge fields in a gauge theory without losing gauge-invariance.
In its non-relativistic version in condensed matter it described by the *Landau-Ginzburg model* and in 3 space dimensions it was proposed as a phenomenological model for superconductors.
Our discussion will be performed in the Lagrangian formalism, starting from the classical model.

1.1 Classical treatment

The field content is made of a complex scalar field ϕ (whose complex conjugate is denote by ϕ^*) and a $U(1)$ gauge field A_μ. The classical relativistic Lagrangian is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + |F^\mu\phi|^2 - \lambda(|\phi|^2 - v^2)^2 \tag{1.1}$$

where e is the electric charge, the covariant derivative is defined by

$$D_\mu\phi = (\partial_\mu - in_e A_\mu)\phi \tag{1.2}$$

and n_e is the electric charge of ϕ in units of e.
The non-relativistic Euclidean version replaces $|D^0\phi|^2$ by a first order term

$$|D^0\phi|^2 \quad \rightarrow \quad \phi^*(\partial_0 - in_e A_0)\phi \tag{1.3}$$

For the model of superconductivity ϕ is a field representing the large distance behaviour of the Cooper pairs generated by phonon attraction and $n_e \equiv 2$. The vortices that will be discusse later in fact really appear in nature.
A lattice version of such vortices, called *Abrikosov*[I] *vortices*, is the equilibrium state of a class of superconductors in the presence of a magnetic field, orthogonal to the surface of the superconductors, whose direction will be denoted by z. The z-dependence is then trivial, and in the gauge $A_z = 0$ the $3+1$ model reduces to a $2+1$ model.
Notice that a typical characteristics of superconductors is the expulsion of the magnetic field (*Meissner effect*), but there are two behaviours of superconducting materials in this respect, called *type I* and *type II*. In type I the magnetic flux is completely expelled from the bulk of the material, whereas in type II it penetrates in the superconductor in tubes whose two-dimensional cross section are the vortices, as shown in fig. 1.1. Each of these tubes contains a flux $\frac{h}{e}$ and at equilibrium if they are sufficiently many[II] they are arranged in a triangular lattice, the *Abrikosov lattice*.

[I] Nobel prize in the 2003.
[II] In order to increase the number of tubes one can increase the strength of the magnetic field.

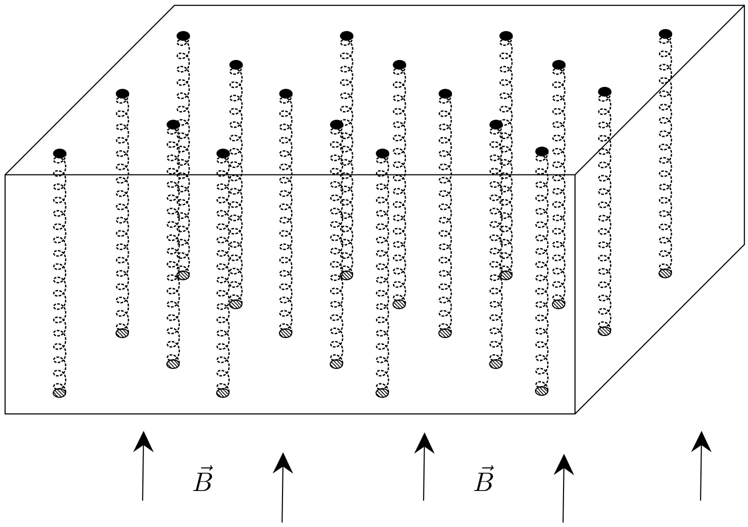

Figure 1.1: Vortices created by the magnetic field in a type II superconductor.

The model is invariant under the $U(1)$ gauge transformation

$$\begin{cases} \phi(x) & \rightarrow & e^{i\beta(x)}\phi(x) \\ A_\mu(x) & \rightarrow & A_\mu(x) + \dfrac{1}{in_e} e^{-i\beta(x)}\partial_\mu e^{i\beta(x)} = A_\mu(x) + \dfrac{1}{n_e}\partial_\mu \beta(x) \end{cases} \tag{1.4}$$

Let us first consider static configurations in the *temporal* gauge $A_0 = 0$, so that there is no difference between relativistic and non-relativistic models. The energy is given by

$$\mathcal{E}\big(\boldsymbol{A}(\boldsymbol{x}), \phi(\boldsymbol{x})\big) = \int \mathrm{d}^2 x \, \frac{1}{4e^2} F_{ij}^2 + |D_i\phi|^2 + \lambda\big(|\phi|^2 - v^2\big)^2 \tag{1.5}$$

and it has a global minimum for

$$\phi(\boldsymbol{x}) = v e^{i\theta} \quad \text{for} \quad \theta \in [0, 2\pi) \quad , \quad A_\mu(\boldsymbol{x}) \tag{1.6}$$

However by gauge invariance this configuration is physically equivalent to

$$\phi(\boldsymbol{x}) = v e^{i[\theta + \beta(\boldsymbol{x})]} \quad , \quad A_\mu(\boldsymbol{x}) = \frac{1}{in_e} e^{-i\beta(x)} \partial_\mu e^{i\beta(x)} \tag{1.7}$$

with $\beta(\boldsymbol{x})$ globally defined of compact support (indeed it cannot act on the boundary of the spacetime, otherwise it changes the boundary conditions).