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# Part I

# Formal introduction to QFT

# Chapter 1

# Introduction

Quantum Field Theory (QFT) was born from an attempt to solve inconsistencies of the Dirac's relativistic quantum mechanics (RQM) when the interaction with the electromagnetic field is introduced. However, very soon it become a common framework in many branches of physics, exhibiting an unexpected unity in the description of elementary quantum processes that deeply modifies our view of the physical reality, mostly with a "pictorial"representation in terms of Feynman diagrams. In fact, quite amazingly, elementary QFT processes can be described qualitatively in terms of very few ingredients:

- *propagators*, describing the virtual propagation of quantum particle excitations and drawn as lines, typically oriented;

- *vertices*, describing the process of emission and absorption of particle excitations, possibly changing the nature of the original particle, and drawn as a point from which the propagators of emitted and absorbed particle emerge

Propagators and vertices are then embodied in diagrams, describing the quantum processes. Clearly, a "change of nature" of the particle during emission/absorption is not allowed in standard quantum mechanics (QM).

At the same time for a process describing an electron decelerating by emitting photons (which took away the electron kinetic energy) the number of photons emitted can be arbitrarily high, again QM of finite degrees of freedom is insufficient to describe such process.

Another case in which QM turns out to be insufficient arise in the so called thermodynamic limit in solid state systems. In real physical systems the number of electrons and ions is finite, although usually very big, $N \sim 10^{23}$, and the volume $V$ is finte (infrared (IR) cutoff). Furthermore in a crystal the lattice constant $a$ is finite (ultraviolet (UV) cutoff). However, usually we are interested in universal properties, independent of detail$V$ and $a$. Therefore it is convenient also in these case to consider the limit $N \to \infty$, $V \to \infty$ with $N/V$ (or its expectation value) constant (thermodynamic IR limit) and $a \to 0$ (continuum UV limit). These limits are not only technically useful, for the non-analyticity appearing the phase transitions, or the appearance of Euclidean invariance, but e.g. the thermodynamic limit guaranteers that the theory does not depend on specific details of $V$ and $N$.

Notice that if the removal of the IR or the UV cutoff is impossible and we assumed that the theory without the cutoff is the physical one (hence effective field theories in the modern sense are excluded), then the non-existence of the limit implies that the physical theory depends on details at infinite distances (IR) or infinite momentum (UV) in a manner not controllable by regularization.

Furthermore in relativistic QFT (RQFT) a cutoff breaks the Poincaré invariance, and the only possible regularization which does not break such symmetry, the dimensional regularization, has no non-perturbative realization.

Many of the key results of QFT are obtained through a perturbative expansion, which are serious mathematical problems, and there are crucial ares of applications that do not rely on perturbative methods. The aim of this course is to provide a view of some results in these areas, with examples both from elementary particle and condensed matter physics, emphasising the underlying common features.

# Chapter 2

# Review of QFT

## 2.1 Fock space

Ref. [GR96, Chapters 3, 4]; [BS80, Chapters 1, 2]

In $d = 3$ space dimensions, quantum particles are either bosons or fermions (in lower dimensions other braid statistics may arise, but we'll not discuss them here).

**Fixed number of particles**

The Hilbert space of states for $N$ identical particles $\mathcal{H}_N$ is constructed as follows: let $\mathcal{H}_1$ be the single-particle Hilbert space, $\Sigma_N$ the permutations group of $N$ objects, $\pi \in \Sigma_N$ and $P_\pi$ the corresponding operators, $\sigma(\pi)$ the number of exchanges made by $\pi$, $\varepsilon$ a constant which takes the value $+1$ for bosons and $-1$ for fermions, then define

$$P^\varepsilon := \frac{1}{N!} \sum_{\pi \in \Sigma_N} (\varepsilon)^{\sigma(\pi)} P_\pi$$

then

$$\mathcal{H}_N^\pm := P^\pm (\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_1)$$

More concretely let $A$ be a complete set of compatible observables in the one-particle Hilbert space of an elementary quantum particle $\mathcal{H}_1$ (we assume for the momenta discrete spectrum for these observables), and $\{|\alpha_i\rangle, i \in I\}$ the corresponding eigenstates, where $\alpha_i$ is the set of common eigenvalues of $A$. Then an orthonormal basis in $\mathcal{H}_N^\pm$ is given by

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle^\varepsilon := \sqrt{\frac{N!}{\prod_i n_i!}} P^\varepsilon |\alpha_{i_1}\rangle \otimes \ldots \otimes |\alpha_{i_N}\rangle$$

where $n_i$ is the number of one-particle states with eigenvalue $\alpha_i$ in $|\alpha_{i_1}\rangle, \ldots, |\alpha_{i_N}\rangle$, satisfying $\sum_{i \in I} n_i = N$. If the values of $n_i$ are the same for both the sets $\alpha_{i_1} \ldots \alpha_{i_N}$ and $\alpha_{j_1} \ldots \alpha_{j_N}$ then

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle^\varepsilon = \pm |\alpha_{j_1} \ldots \alpha_{j_N}\rangle^\varepsilon$$

and we can label states by their **occupation numbers** $\{n_i\}_{i \in I}$, once the one-particle basis $\{|\alpha_i\rangle, i \in I\}$ has been fixed:

$$|\alpha_{i_1} \ldots \alpha_{i_N}\rangle =: |n_1, \ldots, n_i, \ldots\rangle = |\{n_i, i \in I\}\rangle$$

By antisymmetry (Pauli principle) for fermions $n_i = 0, 1$, whereas for bosons $n_i \in \mathbb{N}$.
A generic vector in $\mathcal{H}_N^\varepsilon$, then can be written as the linear combination

$$|\Psi_N\rangle = \sum_{\{n_i\}} \Psi(\{n_i\}) |\{n_i\}\rangle$$

with the conditions

$$\sum_{\{n_i\}} |\Psi(\{n_i\})|^2 < \infty \quad , \quad \sum_{i \in I} = N$$

**Variable number of particles, $N \to \infty$ limit**

Let's consider now the case in which $N$ is not fixed, and may be take $N \to \infty$. Define the **vacuum sector** $\mathcal{H}_0^\varepsilon = \mathbb{C}$ and the corresponding normalized vector $|\Psi_0\rangle$ (or $|0\rangle$) is called the **vacuum**. Formally set

$$\mathcal{F}^\varepsilon := \mathcal{H}_0^\varepsilon \oplus \mathcal{H}_1^\varepsilon \oplus \ldots \oplus \mathcal{H}_N^\varepsilon \oplus \cdots = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^\varepsilon$$

Notice that the direct sum implies that there is no interference between the different sectors $\mathcal{H}_N^\varepsilon$. We would like to explain better the meaning of the previous formal direct sum. Let

$$\mathcal{D} = \left\{ \bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle \right\}$$

with $N_{\max}$ arbitrary but finite. In this space of direct sum of finite sequences of vectors we define an *inner product* by

$$\left( \bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle, \bigoplus_{N'=0}^{N'_{\max}} |\Phi_{N'}\rangle \right) := \sum_{N=0}^{\infty} \langle \Psi_N | \Phi_N \rangle$$

where the sum is formally extended to infinity as only a finite number of terms is non zero. Together with this inner product the space $\mathcal{D}$ is pre-Hilbert, and then $\mathcal{F}^\varepsilon$ is defined as the Hilbert space obtained by completion od $\mathcal{D}$, i.e. the space of sequences $\bigoplus_{N=0}^{\infty} |\Psi_N\rangle$ such that $\sum_{N=0}^{\infty} \langle \Psi_N | \Psi_N \rangle < \infty$. Notice that $\mathcal{D}$ is dense in $\mathcal{F}^\varepsilon$, as required by the definition of Hilbert space.

Then, the space $\mathcal{F}^\varepsilon$ allows the description of processes with non-conserved number of particles. In $D$ we can define the **annihilation and creation operators**

$$a_i |\{n_j\}\rangle^+ := \sqrt{n_i} |\{n_{j \neq i}, n_i - 1\}\rangle^+$$
$$a_i^\dagger |\{n_j\}\rangle^+ := \sqrt{n_i + 1} |\{n_{j \neq i}, n_i + 1\}\rangle^+$$

for the bosons and

$$a_i |\{n_j\}\rangle^- := (-1)^{(\sum_{k<i} n_k)} n_i |\{n_{j \neq i}, n_i - 1\}\rangle^-$$
$$a_i^\dagger |\{n_j\}\rangle^- := (-1)^{(\sum_{k<i} n_k)} (1 - n_i) |\{n_{j \neq i}, n_i + 1\}\rangle^-$$

for fermions.
In follows from the definition that

(i) $a_i^\dagger$ is the adjoint of $a_i$;

(ii) $a_i |\Psi_0\rangle = 0$;

(iii) for bosons hold the **canonical commutation relations** (CCR)

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] \quad , \quad [a_i, a_j^\dagger] = \delta_{ij}$$

while for fermions hold the **canonical anticommutation relations** (CAR)

$$\{a_i, a_j\} = 0 = \{a_i^\dagger, a_j^\dagger\} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij}$$

(iv) defining $\hat{N}_i := a_i^\dagger a_i$ we have

$$\hat{N}_i |\{n_j\}\rangle = n_i |\{n_j\}\rangle$$

and $\hat{N} := \sum_{i \in I} \hat{N}_i$ is well defined in $\mathcal{D}$;

(v) any vector can be constructed by means of applications of creation operators

$$|\{n_j\}\rangle = \frac{1}{\sqrt{\prod_j n_j!}} \prod_j (a_j^\dagger)^{n_j} |0\rangle$$

Annihilation and creation operators corresponding to another basis (i.e. another complete set of commuting observables with discrete spectrum) $\{|\beta_j\rangle, j \in J\}$ can be obtained by applying the Dirac completeness

$$\sum_{i \in I} |\alpha_i\rangle\langle\alpha_i| = \mathbb{1}_{\mathcal{H}_1}$$

as follows

$$b_j^\dagger |0\rangle = |\beta_j\rangle = \sum_{i \in I} |\alpha_i\rangle \langle\alpha_i|\beta_j\rangle = \sum_{i \in I} a_i^\dagger |0\rangle \langle\alpha_i|\beta_j\rangle$$

implying

$$b_j^\dagger = \sum_{i \in I} \langle\alpha_i|\beta_j\rangle a_i^\dagger$$

These ideas extends to the case of complete sets of commuting observables with continuum (or mixed) spectrum in $\mathcal{H}_1$ using

$$\int \mathrm{d}\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1}_{\mathcal{H}_1}$$

Formally we can set $a^\dagger(\alpha) |0\rangle = |\alpha\rangle$ and then

$$a^\dagger(\alpha) = \sum_{i \in I} \langle\alpha_i|\alpha\rangle a_i^\dagger$$

but since $|\alpha\rangle$ is only an improper state ($\langle\alpha|\alpha'\rangle \sim \delta(\alpha - \alpha')$) then $a^\dagger(\alpha)$ is not a true operator, it is an *operator valued distribution*, i.e. is a true operator only if smeared out with a test function $f$:

$$a(f) = \int a(\alpha) f(\alpha) \mathrm{d}\alpha$$

A typical example is given for an elementary particle with classical analogue (in $d = 3$) by $|\alpha\rangle = |\boldsymbol{x}\rangle$ or $|\alpha\rangle = |\boldsymbol{p}\rangle$. If we define $a^\dagger(\boldsymbol{p}) |0\rangle = |\boldsymbol{p}\rangle$ with $[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p} - \boldsymbol{p}')$, then is clear that this is not an ordinary operator, rather an operator valued distribution. When we turn to the $\boldsymbol{x}$-representation of $a$ (i.e. we smear $a(\boldsymbol{p})$ with $f(\boldsymbol{p}) = \langle\boldsymbol{x}|\boldsymbol{p}\rangle$) we get

$$\psi(\boldsymbol{x}) = \int \mathrm{d}^3 p \, \langle\boldsymbol{x}|\boldsymbol{p}\rangle \, a(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{\frac{i}{\hbar} \boldsymbol{x} \cdot \boldsymbol{p}} a(\boldsymbol{p})$$

and $\psi(\boldsymbol{x})$ is called a **quantum field operator** and it satisfies (for bosons)

$$[\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{y})] = \delta(\boldsymbol{x} - \boldsymbol{y})$$

where all the trivial (anti)commutation relation will be always omitted from now on. Notice that the last relation can be interpreted as a form of "locality": the effect of a field in a point cannot affect the effect of a simultaneous field in a different point.

The operator $\psi^\dagger(\boldsymbol{x})$ formally creates a particle with wave function a Dirac $\delta$ with support on $\boldsymbol{x}$. More precisely if $f(\boldsymbol{x}) \in L^2(\mathbb{R}^3, \mathrm{d}^3 x)$ then

$$\psi^\dagger(f) = \int \mathrm{d}^3 x \, \psi^\dagger(\boldsymbol{x}) f(\boldsymbol{x})$$

creates a particle with wave function $f(\boldsymbol{x})$. Notice that if in $\mathcal{H}_1$ we can use an orthonormal basis $\{f_i\}_{i \in \mathbb{N}}$, then setting $a(f_i) =: a_i$ it satisfies the CCR because

$$[a(f_i), a^\dagger(f_j)] = (f_i, f_j) = \delta_{ij}$$

If the particle we are interested in is not elementary the situation is lightly nore complex, but we will not discuss it.

**Relativistic case, $N \to \infty$ limit**

Notice that moving from non-relativistic context (where space coordinates and time are treated differently) to the relativistic one, some problems arises. Dirac formulation of relativistic quantum mechanics treats time and space in the same way, making the space coordinate a label like the time rather than an observable. Indeed we have to take care of the consequences of the Heisenberg principle: due to uncertainty on the energy we are able to produce particle-antiparticle pairs, but if we are interested in measuring the position as better as possible, then the uncertainty on the momentum become huge and lot of pairs are produced, and we are no more able to identify which is the particle we want to measure. Therefore the position is no more an observable in the relativistic framework, and can only be used as a label to describe the evolution of a state, as we do for the time.

Since we cannot characterize particles using position as an observable in a complete set of compatible observables (as we do in the non-relativistic case, where position, energy and spin provides an irreducible set of compatible observables), we need to understand how to choose a new complete set of compatible observables which allows us to build an Fock space for elementary particles using eigenvectors.

A celebrated theorem of Wigner states that the one-particle Hilbert space of an elementary particle should be the representation space of an irreducible unitary representation of space-time symmetries (i.e. the universal covering of the restricted Poincaré group) and internal symmetries (e.g. EM charge conservation, we do not discuss them here).

Irreducible unitary representations of space-time symmetries, according to Wigner's theorem, are characterized by the mass $m \in \mathbb{R}_+$ and either the spin $s \in \mathbb{N}/2$ if $m > 0$ or the helicity (projection of the spin in the direction of the motion[I]) $h \in \mathbb{Z}/2$ if $m = 0$.

Let's consider for simplicity the case of a massive ($m > 0$) scalar ($s = 0$) particle. We know that for a relativistic particle the dispersion relation is

$$p_\mu p^\mu = m^2 \quad , \quad p^0 > 0$$

i.e. the momentum should be contained in the *positive hyperboloid of mass $m$*, denoted by $V_m^+$. Notice that if $m = 0$ we have that the momentum should be contained in the *forward light cone*.

Since we want to have a representation of the (covering of the) Poincaré group all the points in the hyperboloid should be weighted by the same weight, and we cannot use the position as an observable, then a natural choice for our Hilbert space for one particle is provided by

$$\mathcal{H}_1 = L^2 \left( \mathbb{R}^4, \frac{\mathrm{d}^4 p}{(2\pi)^4} \, 2\pi \delta(p_\mu p^\mu - m^2) \theta(p^0) \right) = L^2 \left( V_m^+, \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}} \right)$$

Let $a^\dagger(p) |0\rangle =: |p\rangle$ be the (generalized) eigenvector of the four-momentum operator $\hat{p}^\mu$ for (generalized) eigenvalue $p^\mu =: p \in \mathbb{R}^4$. The Dirac completness relation in $\mathcal{H}_1$ is given by

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^3} \, \delta(p^2 - m^2) \theta(p^0) |p\rangle\langle p| = \mathbb{1}_{\mathcal{H}_1}$$

It immediately follows multiplying by $|p'\rangle$, $p' \in V_m^+$ that

$$
\begin{aligned}
|p'\rangle = |\boldsymbol{p}', \sqrt{\boldsymbol{p}'^2 + m^2}\rangle &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) |p\rangle \, \langle p|p'\rangle \\
&= \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0) |p\rangle \langle 0| \, a(p) a^\dagger(p') |0\rangle \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}} |\boldsymbol{p}, \sqrt{\boldsymbol{p}^2 + m^2}\rangle \langle 0| \, [a(p) a^\dagger(p')] |0\rangle
\end{aligned}
$$

therefore in order to get consistency the following commutation relation should be satisfied:

$$[a(p), a^\dagger(p')] = (2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2} \delta(\boldsymbol{p} - \boldsymbol{p}') \tag{2.1}$$

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[I]Notice that helicity is well defined only if the particle moves at the speed of light, otherwise through a change of frame one can reverse the projection direction.

Notice that if $\{f_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}_1$, then we define

$$a_i := \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0) f_i(p) a(p)$$

and it satisfies the canonical $[a_i, a_j^\dagger] = \delta_{ij}$, hence also in the relativistic case it is possible to obtain canonical commutation relation by smearing $a$ and $a^\dagger$ with an orthonormal basis of the Hilbert space. The new factor $2\sqrt{\boldsymbol{p}^2 + m^2}$ in eq. (2.1), which wasn't present in the non-relativistic case, lead to non-locality problems if one proceeds defining quantum field operators as in the non-relativistic case. If we try to apply the previous idea to define quantum field operators, i.e.

$$\psi(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0) a(p) e^{ip \cdot x}$$

then

$$[\psi(\boldsymbol{x}, 0), \psi^\dagger(\boldsymbol{y}, 0)] = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \frac{\mathrm{d}^4 p'}{(2\pi)^3} \delta(p^2 - m^2)\delta(p'^2 - m^2)\theta(p^0)\theta(p'^0) e^{-i\boldsymbol{p} \cdot \boldsymbol{x}} e^{i\boldsymbol{p}' \cdot \boldsymbol{y}} [a(p), a^\dagger(p')]$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2\sqrt{\boldsymbol{p}^2 + m^2}} \frac{\mathrm{d}^3 p'}{(2\pi)^3 \, 2\sqrt{\boldsymbol{p}'^2 + m^2}} e^{-i(\boldsymbol{p} \cdot \boldsymbol{x} - \boldsymbol{p}' \cdot \boldsymbol{y})} (2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2}\delta(\boldsymbol{p} - \boldsymbol{p}')$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \, 2\sqrt{\boldsymbol{p}^2 + m^2}} e^{-i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})} \neq 0$$

even for $\boldsymbol{x} \neq \boldsymbol{y}$. This violates the "locality" even in the weak non-relativistic form we showed before. Actually, since the concept of "present" (or equivalently of "simultaneity") is not universal in relativity, the vanishing of the commutator (for observable fields at least) should hold for $x$ and $y$ space-like separated (we denoted space-like separated coordinates $x$ and $y$ by $x \times y$).

Actually in general the fields are not observable (e.g. the charged scalar field is not, since it is not self-adjoint) but a remarkable theorem of Doplicher-Roberts (essentially) shows that in a massive RQFT if the observables commute at space-like distances, then the fields of the corresponding QFT either commute or anticommute at space-like distances in $d = 3 + 1$. Moreover the spin-statistics theorem proves that fields with integer spin are bosons and those with half-integer spin are fermions. Hence we know that in the massive case fields with integer spin commute and those with half-integer spin anticommute.

Therefore we need to impose such (anti)commutation relations for our relativistic fields at space-like separated coordinates. Since $a(p)|0\rangle = 0$, the state obtained applying $\psi^\dagger(x)$ to $|0\rangle$ is the same if one add an additional contribution $\sim a$ to $\psi^\dagger(x)$. Let's define

$$\phi(x) = \psi(x) + \psi^\dagger(x)$$

where the presence of both $\psi$ and $\psi^\dagger$ is reminiscent of the fact that the dispersion relation $p_\mu p^\mu = m^2$ has two solutions $\pm\sqrt{\boldsymbol{p}^2 + m^2}$, one to one associated to $\psi$ and $\psi^\dagger$. Then $\phi^\dagger|0\rangle = \psi^\dagger|0\rangle$, $\phi$ is a self-adjoint field operator and as we desired

$$[\phi(x), \phi(y)] = 0 \quad \text{if} \quad (x - y)^2 < 0$$

or more precisely

$$[\phi(f), \phi(g)] = 0 \quad \text{if} \quad \operatorname{supp} f \times \operatorname{supp} g$$

It is however clear that even if the above sense of $\phi(x)$ is "localized in $x$" (so that it cannot affect points space like separated) or better the sense of $\phi(f)$ is "localized in supp $f$" (using Doplicher-Roberts theorem), conversely the state $\phi(x)|0\rangle$ is not localized in $x$ and $\phi(f)|0\rangle$ is not localized in supp $f$. In fact the *two point correlation function* for a real field reads

$$\langle 0| \phi(x)\phi(y) |0\rangle = \langle 0| \psi(x)\psi^\dagger(y) |0\rangle = \langle 0| [\psi(x)\psi^\dagger(y)] |0\rangle =$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2}} e^{ip \cdot (x - y)} \sim \frac{m^{1/2}}{|\boldsymbol{x} - \boldsymbol{y}|^{3/2}} e^{-m|\boldsymbol{x} - \boldsymbol{y}|} \neq 0$$

where in the second line $p^0 := \sqrt{\boldsymbol{p}^2 + m^2}$ and the approximation holds for $(x-y)^2 \ll -1$.[II]
Differently from NRQFT case, where

$$\langle 0| \, \psi(\boldsymbol{x})\psi^\dagger(\boldsymbol{y}) \, |0\rangle = \delta(\boldsymbol{x} - \boldsymbol{y})$$

holds, in RQFT $a^\dagger(p)$ applied at the vacuum creates a one-particle state with 4-momentum $p^\mu$, but $\phi(x)$ applied to the vacuum does not create a particle localized in $x^\mu$ (i.e. with "wave function" given by a $\delta$ localized in $x^\mu$).

Now it's very clear that $\hat{x}^\mu$ is not a good observable in RQFT, whereas $\hat{p}^\mu$ is a good observable. It is indeed possible to create a particle with well defined momentum $p^\mu$ but is impossible to create a particle with well defined momentum $x^\mu$.

The physical underlying reason as we already mentioned is that the measure of $\hat{x}^\mu$ would produce a diverging fluctuation in $p$ due to Heisenberg principle $\Delta x^\mu \Delta p^\mu \gtrsim \hbar$, allowing the production of particle-hole pairs. Thus the "space-time coordinate $x$" looses its meaning, since it is impossible to understand to which particle it refers.

Finally notice that from its definition $\phi(x)$ satisfies the homogeneous Klein-Gordon equation

$$(\Box + m^2)\phi(x) := \left( \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + m^2 \right)\phi(x) = 0$$

If we replace $c$ with the phonon velocity this is the same equation appearing in Condensed Matter for optical phonon at small (quasi)-momentum (in the lattice $p^\mu$ is periodic), showing a first example that some excitations in material exhibits "relativistic features" in some regime of parameters.

## 2.2 Characteristic features of Fock space

Ref. [BR97, Section 5.2]

The aim of this section is to understand how CCR and CAR affects the properties of the Fock space we built up. We have seen that the creation and annihilation operators for bosons and fermions satisfy the CCR and the CAR respectively

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij}$$

Since such relations are purely algebraic, can be thought as the caracterizing rules for an algebra (i.e. a vector space where we defined the multiplication) endowed with an involution denoted by † (which

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[II]Let's see how the result is obtained. First replace $(x-y) \mapsto x$ and then set $\boldsymbol{p} \cdot \boldsymbol{x} = |\boldsymbol{p}||\boldsymbol{x}|\cos\theta =: |\boldsymbol{p}||\boldsymbol{x}|u$ (here $u := \cos\theta$). Then

$$\int \frac{\mathrm{d}^3 p}{\sqrt{\boldsymbol{p}^2 + m^2}} e^{ip^0 x^0} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \sim \int \frac{|\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}| \, \mathrm{d}u \, \mathrm{d}\phi}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{ip^0 x^0} e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} \sim \int_0^\infty \frac{|\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{ip^0 x^0} \int_{-1}^1 \mathrm{d}u \, e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} =$$

$$= \int_0^\infty \frac{|\boldsymbol{p}|^2 \mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{ip^0 x^0} \frac{1}{-i|\boldsymbol{p}||\boldsymbol{x}|} \left( e^{-i|\boldsymbol{p}||\boldsymbol{x}|} - e^{i|\boldsymbol{p}||\boldsymbol{x}|} \right) = \frac{1}{i|\boldsymbol{x}|} \int_{-\infty}^\infty \frac{|\boldsymbol{p}| \, \mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{i(p^0 x^0 + |\boldsymbol{p}||\boldsymbol{x}|)}$$

Due to $x^2 \ll -1$, i.e. $|\boldsymbol{x}| \gg x^0$ and $|\boldsymbol{x}| \gg 1$, we have that

$$\frac{\partial}{\partial|\boldsymbol{p}|}(p^0 x^0 + |\boldsymbol{p}||\boldsymbol{x}|) = \frac{|\boldsymbol{p}|x^0}{p^0} + |\boldsymbol{x}| \approx |\boldsymbol{x}|$$

hence the contribution of $p^0 x^0$ to the variation of the total phase respect to $|\boldsymbol{p}|$ is negligible, and it will give just a small correction to the final result

$$\frac{1}{i|\boldsymbol{x}|} \int_{-\infty}^\infty \frac{|\boldsymbol{p}| \, \mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{i(p^0 x^0 + |\boldsymbol{p}||\boldsymbol{x}|)} \approx \frac{1}{i|\boldsymbol{x}|} \int_{-\infty}^\infty \frac{|\boldsymbol{p}| \, \mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2 + m^2}} e^{i|\boldsymbol{p}||\boldsymbol{x}|} = \frac{1}{i|\boldsymbol{x}|} \int_0^\infty \mathrm{d}|\boldsymbol{p}| \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2 + m^2}}$$

Substituting $p \mapsto m\sinh(t)$, $\mathrm{d}p = \sqrt{|\boldsymbol{p}|^2 + m^2}\mathrm{d}t$, we get

$$\frac{1}{i|\boldsymbol{x}|} \int_0^\infty \mathrm{d}|\boldsymbol{p}| \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2 + m^2}} = \frac{m}{i|\boldsymbol{x}|} \int_0^\infty \mathrm{d}t \, \sinh(t)\sin(m|\boldsymbol{x}|\sinh(t)) = \frac{m}{i|\boldsymbol{x}|}\mathrm{K}_1(m|\boldsymbol{x}|)$$

where $\mathrm{K}_n(x)$ is the modified Bessel function, which satisfies

$$\mathrm{K}_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \quad \text{for} \quad x \gg n$$

promote the algebra to a *-algebra), generated by $\{a_i\}_{i \in I}$. In this point of view $a$ and $a^\dagger$ are then just representations of this algebra as operators acting on $\mathcal{F}^\pm$ ($\mathcal{F}^\pm$ are representation spaces of the algebra). If the set $I$ is finite, i.e. the algebra is finitely generated, then the *von Neumann uniqueness theorem* ensures that all the representation of CCR and CAR are unitarily equivalent, hence all possible representations live in the same abstract Hilbert space.

For instance this occur in QM for an $N$ particles system in $\mathbb{R}^d$ (finite degrees of freedom (d.o.f.)), where we can put $a_i = \frac{1}{\sqrt{2}}(q_i + p_i)$, $i = 1, \ldots, d$, with representation space $L^2((\mathbb{R}^d)^N, \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_N)$.

This is not true anymore for infinitely generated algebras (infinite d.o.f.). Roughly speaking, a unitary operator mapping one representation of the algebra into another one can be thought in zero dimensions as $e^{iN} \in U(1)$ for finite $N$ but as $N \to \infty$ it vanishes. Infinitely generated CCR and CAR algebras have infinite inequivalent representations acting in completely disjoint Hilbert spaces. Moreover each of these possible representations describes completely different physics.

The point is now understand what actually characterizes a specific $\mathcal{F}$, since CCR and CAR are the same also for infinitely many different Hilbert spaces. The answer is given by the number operator $\hat{N}$: indeed it is a well defined observable if and only if the elements of $\mathcal{F}$ are created by a specific representation of $a_i^\dagger$, or in other words if the excitations created out from the vacuum by the $a_i^\dagger$ are the ones that can be counted (even if infinitely many) by a specific $\hat{N}$.

Mathematically, $\hat{N} := \sum_{i \in I} a_i^\dagger a_i$ is well defined in $\mathcal{H}$ if exists a domain $D$ dense in $\mathcal{H}$ containing $|0\rangle$ in which $\hat{N}$ is self-adjoint. This also implies that its spectrum is $\sigma(\hat{N}) = \mathbb{N}$ by the standard argument seen for the harmonic oscillator in QM.

Representations of CCR or CAR on which $\hat{N}$ is not well defined are called *non-Fock representation*. We know wonder whether is possible or not to create non-Fock representations. A simple example of this is given by *Bogoliubov transformations*: starting from given $a_i$, $a_i^\dagger$, we define

$$a_i' := \alpha_i a_i + \beta_i a_i^\dagger$$
$$a_i'^\dagger := \alpha_i^* a_i^\dagger + \beta_i^* a_i$$

for $\alpha_i, \beta_i \in \mathbb{C}$. If $\{a_i\}_i$ generates a representation of CCR then

$$[a_i', a_j'^\dagger] = (|\alpha_i|^2 - |\beta_i|^2)\delta_{ij}$$

from which we see that $\{a_i'\}_i$ gives a new representation of CCR provided that $|\alpha_i|^2 - |\beta_i|^2 = 1$. Conversely if $\{a_i\}_i$ generates a representation of CAR then

$$\{a_i', a_j'^\dagger\} = (|\alpha_i|^2 + |\beta_i|^2)\delta_{ij}$$

from which we see that $\{a_i'\}_i$ gives a new representation of CAR provided that $|\alpha_i|^2 + |\beta_i|^2 = 1$. Let $|0\rangle$ be the vacuum in the $a_i$ representation, then

$$\langle 0| \hat{N}' |0\rangle = \sum_{i \in I} \langle 0| (\alpha_i^* a_i^\dagger + \beta_i^* a_i)(\alpha_i a_i + \beta_i a_i^\dagger) |0\rangle = \sum_{i \in N} |\beta_i|^2$$

If $\sum_i |\beta_i|^2 < \infty$ then such transformation is allowed and since it is unitary then the Fock space of $a_i$ is a Fock space for $a_i'$ too (even if states, including the vacuum, may not be the same). Otherwise, for $\sum_i |\beta_i|^2 = \infty$, the vacuum $|0\rangle$ in the $a_i$ representation is not even an element of $D(\hat{N}') := \{|\psi\rangle \text{ s.t. } ||\hat{N}' |\psi\rangle|| < \infty\}$, nor is any $a_i^\dagger$ excitation created from $|0\rangle$.[III] Therefore if for $\sum_i |\beta_i|^2 = \infty$ the Fock space for $\{a_i\}_{i \in I}$ is not the right space for the excitations described by $a_i'$. Actually we proved something more, we proved that the Fock space for $a_i$ and $a_i^\dagger$ are completely disjoint since no state of one of the two spaces is contained in the other one.

This actually happens in practice also in some simple situations such as for *Fermi gas*.[IV] Let's consider $N$ fermions in a finite volume $V$ and zero temperature $T = 0$. In such conditions the density of states is

$$n(\boldsymbol{k}) = \begin{cases} 2 & \text{for} \quad |\boldsymbol{k}| \leq k_F \\ 0 & \text{for} \quad |\boldsymbol{k}| > k_F \end{cases}$$

---

[III] Indeed using CCR and CAR one obtains

$$\hat{N}' a_i^\dagger = \alpha_i a_i'^\dagger \pm \beta_i^* a_i' + a_i^\dagger \hat{N}'$$

hence for any polynomial in $a_i^\dagger$ eventually $\hat{N}'$ hit the vacuum producing a divergence.

[IV] This example is presented in [**Strocchi:1985aa**].

where we have two particles for each momentum state due to the Fermi statistic. Recall that $k_F$ is the Fermi momentum. Clearly the ground state $|\psi_0\rangle$ corresponding to such configuration $u(\boldsymbol{k})$ is not a vacuum for the annihilation operators $a(\boldsymbol{k}, s)$ when $|\boldsymbol{k}| \leq k_F$ because $a(\boldsymbol{k}, s) |\psi_0\rangle \neq 0$, since it just annihilate a particle which certainly exists in the whole ensemble where all configurations for $\boldsymbol{k} \leq k_F$ are occupied.

Nevertheless if one defines the Bogoliubov transformation

$$a'(\boldsymbol{k}, s) = \alpha(\boldsymbol{k}, s)a(\boldsymbol{k}, s) + \beta(-\boldsymbol{k}, -s)a^\dagger(-\boldsymbol{k}, -s)$$

with

$$\begin{cases} \alpha(\boldsymbol{k}, s) = 1 & \beta(-\boldsymbol{k}, -s) = 0 & \text{for} \quad |\boldsymbol{k}| > k_F \\ \alpha(\boldsymbol{k}, s) = 0 & \beta(-\boldsymbol{k}, -s) = 1 & \text{for} \quad |\boldsymbol{k}| \leq k_F \end{cases}$$

in such a way that the ground state is a Fock vacuum for the $a'(\boldsymbol{k}, s)$:

$$a'(\boldsymbol{k}, s) |\psi_0\rangle = 0$$

According to the previous argument we know that $|\psi_0\rangle \in \mathcal{F}$ (for $a(\boldsymbol{k}, s)$) if and only if $\sum_{\boldsymbol{k},s} |\beta(\boldsymbol{k}, s)|^2 < \infty$. This is clearly true if $N < \infty$, but it is false in the limit $N \to \infty$, $V \to \infty$, $N/V$ fixed. Therefore, in the thermodynamic limit the Hilbert space of a quantum Fermi gas (the one built with $a'(\boldsymbol{k}, s)$ is not in the Fock space of the non-relativistic fermions (the one built with $a(\boldsymbol{k}, s)$).

Another example of non-Fock space is provided by the radiation emitted by a charged particle changing its momentum $\boldsymbol{p} \mapsto \boldsymbol{p}'$. A simple qualitative argument goes as follows: the energy emitted by radiation goes like

$$\mathcal{E} = \int \mathrm{d}^3k \, |\mathbf{E}_{\mathrm{rad}}(\boldsymbol{k})|^2$$

and we know that is is a finite number, moreover since photons asymptotically (where they are free) have energy $\omega(\boldsymbol{k}) = c|\boldsymbol{k}|$ then the number of emitted photons is

$$N = \int \mathrm{d}^3k \frac{|\mathbf{E}_{\mathrm{rad}}(\boldsymbol{k})|^2}{c|\boldsymbol{k}|}$$

The problem is that the electric field of the radiation behaves in the infrared ($|\boldsymbol{x}| \to \infty$) as $\mathbf{E}_{\mathrm{rad}}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-2}$ implying $\mathbf{E}(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1}$ as $|\boldsymbol{k}| \to 0$, i.e. is logarithmically divergent. Hence

$$N \sim \int \mathrm{d}^3k \frac{1}{|\boldsymbol{k}|} \left(\frac{1}{|\boldsymbol{k}|}\right)^2 = \infty$$

therefore the Hilbert space of photons at the end of the process is not the fock space $\mathcal{F}_{\mathrm{in}}$ of the initial photons.

## 2.3 Interacting fields

Ref. [GR96, Chapter 8, 9]; [Kle15, Section 1.9, Chapter 10]; [BS80, Chapter 4]

Up to now we have considered only "free" fields whose Hilbert space is a fock space, however to obtain physical informations we need interactions. Now, starting from the particle physics case (zero temperature $T = 0$, zero density $n = 0$, relativistic case), and then moving to the condensed matter case, we will introduce the perturbative calculation of the interactions. We are mainly interested in understand which are the problems that arise in the perturbative approach and in general the need of non-perturbative techniques.

In particle physics most of the informations are extracted from scattering experiments. As a very simple model we can consider the $\phi^4$ model, which in its complex version describe the (low energy of the) Higgs field with only quartic self-interactions taken into account, in the unbroken symmetry phase (which we presume was present in the early universe).

To be concrete we consider a real field $\phi(x)$ with canonical lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \text{and} \quad \mathcal{L}_I = -\frac{\lambda}{4}\phi^4$$

The corresponding Hamiltonian is

$$H = H_0 + H_I = \int \mathrm{d}\boldsymbol{x} \ (\mathcal{H}_0(\boldsymbol{x}) + \mathcal{H}_I(\boldsymbol{x}))$$

with

$$\mathcal{H}_0(\boldsymbol{x}) = \frac{\pi^2}{2}(\boldsymbol{x}) + \frac{(\boldsymbol{\nabla}\phi)^2(\boldsymbol{x})}{2} + \frac{m^2}{2}\phi^2(\boldsymbol{x}) \quad \text{and} \quad \mathcal{H}_I(\boldsymbol{x}) = \frac{\lambda}{4}\phi^4(\boldsymbol{x})$$

where the canonical momentum is $\pi(x) = \dot{\phi}(x)$ and the canonical quantization would give

$$[\hat{\phi}(\boldsymbol{x}, t), \dot{\hat{\phi}}(\boldsymbol{y}, t)] = i\hbar\delta(\boldsymbol{x} - \boldsymbol{y})$$

Now one may wonder in which Hilbert space $\hat{\phi}$ is defined as an operator-valued distribution, and if such space is the same as the Fock space of the free scalar field. Such question is actually very non-trivial. To gain the maximum support from our knowledge of the free field let us first suppost that we switch-off the interaction term $H_I$ for large $|t|$, i.e. we replace $\lambda$ by a $C^\infty$ function of $t$, $\lambda_\varepsilon(t)$, such that

$$\lambda_\varepsilon(t) = \begin{cases} \lambda & |t| \leq \varepsilon^{-1} \\ 0 & |t| \to \infty \end{cases}$$

and we denoted the new interacting Hamiltonian with $H^I_\varepsilon$. Then we will take the limit $\varepsilon \to 0$ to restore the physical situation. For any finite $\varepsilon > 0$ in the limit $t \to -\infty$ the field $\hat{\phi}(\boldsymbol{x}, t)$ tends to the free field $\hat{\phi}_{\text{in}}$, defined in the Fock space $\mathcal{F}_{\text{in}}$ and in the limit $t \to +\infty$ to the free field $\hat{\phi}_{\text{out}}$ defined in $\mathcal{F}_{\text{out}}$.
Notice that in general (as we will comment later on) to have a well defined expression we need to cutoff also in the spatial direction, so that $\varepsilon$ become an infrared regulator of space time, and possibly we need to introduce also ultraviolet counterterms.
Since we *assumed* that $\hat{\phi}$ (using the Heisenberg picture) satisfies the CCR (but this assumption is not guaranteed) and by definition CCR are obeyed also by $\hat{\phi}_{\text{in}}$ and $\hat{\phi}_{\text{out}}$ since they are free, then it exists a unitary one-parameter group $U^I_\varepsilon(t)$ such that

$$\begin{aligned} U^{I\dagger}_\varepsilon(t)\hat{\phi}_{\text{in}}(\boldsymbol{x}, t)U^I_\varepsilon(t) &= \hat{\phi}(\boldsymbol{x}, t) \\ U^{I\dagger}_\varepsilon(t)\hat{\pi}_{\text{in}}(\boldsymbol{x}, t)U^I_\varepsilon(t) &= \hat{\pi}(\boldsymbol{x}, t) \end{aligned} \tag{2.2}$$

If this is true then

$$U^I_\varepsilon(t) \xrightarrow[t\to-\infty]{} \mathbb{1}$$
$$U^I_\varepsilon(t) \xrightarrow[t\to+\infty]{} S_\varepsilon$$

where (since $U^I_\varepsilon$ is unitary) $S_\varepsilon$ is a unitary operator called *Scattering matrix (with cutoff $\varepsilon$)* such that

$$S^\dagger_\varepsilon\hat{\phi}_{\text{in}}S_\varepsilon = \hat{\phi}_{\text{out}}$$

Notice that by consistency, if $U(t) = \exp\left\{-\frac{itH}{\hbar}\right\}$ denotes the unitary evolution in Heisenberg picture of $\hat{\phi}$ using the complete Hamiltonian, then eq. (2.2) implies

$$U^{I\dagger}_\varepsilon(t)e^{+\frac{itH_0}{\hbar}}\hat{\phi}_{\text{in}}(\boldsymbol{x}, 0)e^{-\frac{itH_0}{\hbar}}U^I_\varepsilon(t) = U^\dagger(t)\hat{\phi}(\boldsymbol{x}, 0)U(t)$$

must hold, so that

$$U^I_\varepsilon(t) = e^{\frac{itH_0}{\hbar}}U(t) \tag{2.3}$$

This provides the evolution operator in the *interaction picture* for the states, while as we know the fields in such picture evolve according to Heisenberg picture with the free Hamiltonian, so that

$$\langle\chi|\,\hat{\phi}(\boldsymbol{x}, t)\,|\psi\rangle = \underbrace{\langle\chi|\,U^{I\dagger}_\varepsilon(t)}_{\substack{\text{Sch. evol. of }|\chi\rangle \\ \text{using } U^I_\varepsilon}}\ \underbrace{e^{\frac{itH_0}{\hbar}}\hat{\phi}_{\text{in}}(\boldsymbol{x}, 0)e^{-\frac{itH_0}{\hbar}}}_{\substack{\text{Heis. evol. of }\hat{\phi} \\ \text{using } H_0.}}\ \underbrace{U^I_\varepsilon(t)\,|\psi\rangle}_{\substack{\text{Sch. evol. of }|\psi\rangle \\ \text{using } U^I_\varepsilon}}$$

In order to determine $U_\varepsilon^I(t)$, we can differentiate respect to $t$ in eq. (2.3) and we get

$$\frac{\mathrm{d}U_\varepsilon^I(t)}{\mathrm{d}t} = \frac{i}{\hbar}H_0 e^{\frac{itH_0}{\hbar}}U(t) - \frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}(H_0 + H_\varepsilon^I)U(t)$$

$$= -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H_\varepsilon^I U(t) = -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H_\varepsilon^I e^{-\frac{itH_0}{\hbar}}U_\varepsilon^I(t)$$

and using the boundary condition $U_\varepsilon^I(-\infty) = \mathbb{1}$ we finally get the integral equation

$$U_\varepsilon^I(t) = \mathbb{1} - \frac{i}{\hbar}\int_{-\infty}^{t} H_\varepsilon^I(t')U_\varepsilon^I(t')\mathrm{d}t' \tag{2.4}$$

and by successively re-inserting the l.h.s. of eq. (2.4) we get

$$U_\varepsilon^I(t) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{t}\mathrm{d}t_1 \ldots \int_{-\infty}^{t}\mathrm{d}t_n\, \mathrm{T}[H_\varepsilon^I(t_1)\ldots H_\varepsilon^I(t_n)] =: \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_{-\infty}^{t}H_\varepsilon^I(t')\mathrm{d}t'\right]$$

where $\mathrm{T}[A_1(t_1)\ldots A_n(t_n)]$ is the *time-ordering* defined by

$$\mathrm{T}[A_1(t_1)\ldots A_n(t_n)] = \sum_{\pi \in \Sigma_n}\Theta(t_{\pi(1)}, \ldots, t_{\pi(n)})\epsilon^{\sigma(\pi)}A_{\pi(1)}(t_{\pi(1)})\ldots A_{\pi(n)}(t_{\pi(n)})$$

with

$$\Theta(t_{\pi(1)}, \ldots, t_{\pi(n)}) = \begin{cases} 1 & \text{if } t_{\pi(1)} \geq t_{\pi(2)} \geq \cdots \geq t_{\pi(n)} \\ 0 & \text{otherwise} \end{cases}$$

By construction $\mathcal{F}_{\mathrm{in}}, \mathcal{F}_{\mathrm{out}} \subset \mathcal{H}$ (the space of $\hat{\phi}$) and if $|\mathrm{in}\rangle \in \mathcal{F}_{\mathrm{in}}$ is the state of free particles prepared at $t = -\infty$ and $|\mathrm{out}'\rangle \in \mathcal{F}_{\mathrm{out}}$ the state of free particles found at $t = +\infty$, then the probability to get this transition is given, according to the rules of QM, by

$$|\langle\mathrm{in}|\mathrm{out}'\rangle|^2 = |\langle\mathrm{in}|S_\varepsilon|\mathrm{in}'\rangle|^2$$

Hence the scattering is analyzed in terms of these matrix elements.

These matrix elements are in turn related to the *correlation or Green functions* of $\hat{\phi}$

$$G^{(n)}(x_1, \ldots, x_n) := \langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle$$

by the LSZ (Lehmann-Symanzik-Zimmermann) formula: if the in-state is given by particles with momenta $q_1, \ldots, q_m$ and the out state by particles with momenta $p_1, \ldots, p_n$, then

$$\langle q_1, \ldots, q_m\,\mathrm{in}|p_1, \ldots, p_n\,\mathrm{out}\rangle =$$

$$= i^{m+n}\int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_m \int \mathrm{d}^4 y_1 \ldots \mathrm{d}^4 y_n \times$$

$$\times\, e^{-i(q_1 x_1 + \ldots + q_m x_m)}e^{i(p_1 y_1 + \ldots + p_n y_n)} \times$$

$$\times\, (\Box_{x_1} + m^2)\ldots(\Box_{x_m} + m^2)(\Box_{y_1} + m^2)\ldots(\Box_{y_n} + m^2) \times$$

$$\times\, \langle 0|\mathrm{T}[\hat{\phi}(y_1)\ldots\hat{\phi}(y_n)\hat{\phi}(x_1)\ldots\hat{\phi}(x_m)]|0\rangle +$$

$$+\, \text{disconnected terms without interactions not contributing to the cross section}$$

The perturbative approach tries to compute the correlation functions $\langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle$ of interacting Heisenberg fields in terms of the free in-fields $\hat{\phi}_{\mathrm{in}}(x)$. Using the expression of $U_\varepsilon^I(t_1, t_2) := U_\varepsilon^I(t_1)U_\varepsilon^{I-1}(t_2)$ one can prove the Gell-Mann - Low formula providing the above connection. Let $|0\rangle$ be the vacuum state in $\mathcal{H}$, then

$$\langle 0|\mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle = \frac{\langle 0_{\mathrm{in}}|\mathrm{T}[\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)e^{-\frac{i}{\hbar}\int \mathrm{d}t\, H_\varepsilon^I(t)}]|0_{\mathrm{in}}\rangle}{\langle 0_{\mathrm{in}}|\mathrm{T}[e^{-\frac{i}{\hbar}\int \mathrm{d}t\, H_\varepsilon^I(t)}]|0_{\mathrm{in}}\rangle} \tag{2.5}$$

This formula connects interacting Heisenberg fields $\hat{\phi}$ with in-fields $\hat{\phi}_{\mathrm{in}}$ assuming that we have a finite infrared cutoff $\varepsilon$ in time.

We now give a sketch of the proof of this result.[V] Assume $t \gg t_1 > t_2 > \ldots > t_n$, then using unitarity

$$U_\varepsilon^{I\dagger}(t_1) = U_\varepsilon^{I-1}(t_1) = U_\varepsilon^{I-1}(t)U_\varepsilon^I(t,t_1) = U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1) \quad , \quad U_\varepsilon^I(t_n) = U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t)$$

and

$$\begin{aligned}
\langle 0 | \, \mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)] \, |0\rangle &= \langle 0 | \, \hat{\phi}(x_1)\ldots\hat{\phi}(x_n) \, |0\rangle \\
&= \langle 0 | \, U_\varepsilon^{I\dagger}(t_1)\hat{\phi}_{\mathrm{in}}(x_1)U_\varepsilon^I(t_1)\ldots U_\varepsilon^{I\dagger}(t_n)\hat{\phi}_{\mathrm{in}}(x_n)U_\varepsilon^I(t_n) \, |0\rangle \\
&= \langle 0 | \, U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1)\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t) \, |0\rangle \\
&= \langle 0 | \, U_\varepsilon^{I\dagger}(t)\mathrm{T}[\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\, H_\varepsilon^I(t)}]U_\varepsilon^I(-t) \, |0\rangle
\end{aligned}$$

where in the last step we moved all the unitary operators on the right (since they are inside the time ordering the result is the same) and we used

$$U_\varepsilon^I(t,t_1)U_\varepsilon^I(t_1)U_\varepsilon^{I\dagger}(t_2)U_\varepsilon^I(t_2)\ldots U_\varepsilon^{I\dagger}(t_n)U_\varepsilon^I(t_n,-t) = U_\varepsilon^I(t)U_\varepsilon^{I\dagger}(-t) = \mathrm{T}[e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\, H_\varepsilon^I(t)}]$$

Now since

$$U_\varepsilon^I(t) \, |0\rangle \xrightarrow[t\to-\infty]{} |0_{\mathrm{in}}\rangle \quad , \quad U_\varepsilon^I(t) \, |0\rangle \xrightarrow[t\to+\infty]{} |0_{\mathrm{out}}\rangle$$

we have

$$\langle 0 | \, \mathrm{T}[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)] \, |0\rangle = \langle 0_{\mathrm{out}} | \, \mathrm{T}[\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\, H_\varepsilon^I(t)}] \, |0_{\mathrm{in}}\rangle$$

If we assume that the vacuum is non degenerate, we expect that by adiabatic evolution (i.e. changing the value of $\varepsilon$) it cannot go in another state, hence

$$|0_{\mathrm{out}}\rangle = e^{iL} \, |0_{\mathrm{in}}\rangle$$

for some operator $L$. Now $\langle 0_{\mathrm{in}}|0_{\mathrm{out}}\rangle = e^{iL}$ therefore

$$\langle 0_{\mathrm{out}} | = \langle 0_{\mathrm{in}} | \, e^{-iL} = \frac{\langle 0_{\mathrm{in}} |}{\langle 0_{\mathrm{in}}|0_{\mathrm{out}}\rangle} = \frac{\langle 0_{\mathrm{in}} |}{\langle 0_{\mathrm{in}} | \, S_\varepsilon \, |0_{\mathrm{in}}\rangle}$$

and taking $t \to +\infty$ we finally get eq. (2.5).

In general is very hard to evaluate expressions like eq. (2.5). In order to make this effort, the perturbative approach consists in replacing

$$\langle 0_{\mathrm{in}} | \, \mathrm{T}[\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\, H_\varepsilon^I(t')\right)^{\ell}] \, |0_{\mathrm{in}}\rangle$$

$$\downarrow$$

$$\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\langle 0_{\mathrm{in}} | \, \mathrm{T}[\hat{\phi}_{\mathrm{in}}(x_1)\ldots\hat{\phi}_{\mathrm{in}}(x_n)\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\, H_\varepsilon^I(t')\right)^{\ell}] \, |0_{\mathrm{in}}\rangle$$

i.e. moving the infinite series outside the expectation value. The terms inside the sum can now be computed in terms of in-fields using Wick theorem or equivalently Feynman diagrams.

Resumming our steps, we expressed the scattering amplitudes in terms of correlators of interacting fields using LSZ formula, then Gell-Mann - Low formula allows us to express such correlators in terms of correlators of in-fields (free). Finally the perturbative approach prescripts the extraction of the infinite series outside the expectation value, in such a way that we can obtain our result computing expectation values of free fields through Feynman diagrams. We'll see in the following how dangerous is perturbative approach prescription.

## 2.4 Condensed matter systems

[Kle15, Sections 1.16, 2.17–2.19, 10.9]; [Fjæ13]

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[V]Reference: https://authors.library.caltech.edu/60474/1/PhysRev.84.350.pdf

Let us turn to condensed matter systems. A source of information on the physical properties (typically transport properties) are the correlation functions of local observables, such as spin, density, current, etc; in particular two-points functions. Hence in this case physical informations are much more involved than just scattering amplitudes in particle physics. These two-points functions appear naturally in particular when one studies the *linear response*, i.e. the response to a test (i.e. infinitesimal) perturbation.

Let us consider a system with Hamiltonian $H$ (which might include the classical potential term $\mu N$) and we want to see the effect on the mean value of an observable $O_1$ or a field (in general not self-adjoint) of a test perturbation typically generated by another observable $O_2$ in the form

$$V^\varepsilon(t) := \xi_\varepsilon(t) O_2^H(t)$$

with $\xi_\varepsilon(t)$ vanishing as $|t| \to \infty$ as the previous considered $\lambda_\varepsilon(t)$ and $O_2^H(t) = e^{itH} O_2 e^{-itH}$. Notice that however we do not assume $H$ to be free as in the scattering case.

Let $\{|n\rangle\}$ be the set of eigenfunctions of $H$ ($H |n\rangle = \mathcal{E}_n |n\rangle$) generating a Dirac completeness in $\mathcal{H}$ and consider the first order modification induced by the perturbation in

$$\langle n| O_1^H(t) |n\rangle = \langle n| e^{itH} O_1 e^{-itH} |n\rangle$$

The time evolution in the presence of the perturbation of $O_1$ is

$$O_1^{H+V_\varepsilon}(t) = U^{H+V_\varepsilon \dagger}(t) O_1 U^{H+V_\varepsilon}(t)$$

and we want to rewrite this in terms of the evolution generated by $H$ in an "interaction picture", i.e.

$$O_1^{H+V_\varepsilon}(t) = U_\varepsilon^{V \dagger}(t) O_1^H(t) U_\varepsilon^V(t)$$

where $U_\varepsilon^V(t)$ gives the contribution of the interaction $V_\varepsilon(t)$ to the free evolution described by $O_1^H(t)$. The situation in this respect is similar to the one considered before in RQFT, just replacing of $H_I^\varepsilon$ by $V_\varepsilon$ we get that

$$U_\varepsilon^V(t) = \mathrm{T}[e^{-i \int_{-\infty}^t V_\varepsilon(t') \mathrm{d}t'}] \tag{2.6}$$

We calculate the response to the perturbation through a variation, using the perturbative ansatz and then taking only first order contribution of $V_\varepsilon$ in the perturbative expansion:

$$\delta \langle n| O_1^{H+V_\varepsilon}(t) |n\rangle = \frac{\mathrm{d}}{\mathrm{d}\varepsilon'} \langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle \big|_{\varepsilon'=0} = \lim_{\varepsilon' \to 0} \frac{\langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle - \langle n| O_1^H(t) |n\rangle}{\varepsilon'} =$$

$$= \lim_{\varepsilon' \to 0} \frac{1}{\varepsilon'} \langle n| \left[ \left( 1 + i\varepsilon' \int_{-\infty}^t V_\varepsilon(t') \mathrm{d}t' + O(\varepsilon')^2 \right) O_1^H(t) \left( 1 - i\varepsilon' \int_{-\infty}^t V_\varepsilon(t') \mathrm{d}t' + O(\varepsilon')^2 \right) - O_1^H(t) \right] |n\rangle =$$

$$= i \int_{-\infty}^t \mathrm{d}t' \, \xi_\varepsilon(t') \langle n| [O_2^H(t'), O_1^H(t)] |n\rangle = -i \int_{-\infty}^{+\infty} \mathrm{d}t' \, \xi_\varepsilon(t') \theta(t-t') \langle n| [O_1^H(t'), O_2^H(t)] |n\rangle$$

One can now just sum over $\{|n\rangle\}$ to perform the thermal expectation value

$$\langle (\bullet) \rangle_{\beta := \frac{1}{kT}} = \frac{\sum_n \langle n| (\bullet) |n\rangle e^{-\beta \mathcal{E}_n}}{\sum_n e^{-\beta \mathcal{E}_n}} \tag{2.7}$$

obtaining the expectation value at a given temperature $T$

$$\delta \langle O_1^{H+V_\varepsilon}(t) \rangle_\beta = \int \mathrm{d}t' \, \xi_\varepsilon(t') \left( -i\theta(t-t') \langle [O_1^H(t), O_2^H(t')] \rangle_\beta \right) = \int_{-\infty}^{+\infty} \mathrm{d}t' \, \xi_\varepsilon(t') G_{\mathrm{ret}}^{O_1 O_2}(t, t') \tag{2.8}$$

where

$$G_{\mathrm{ret}}^{O_1 O_2}(t, t') := -i\theta(t-t') \langle [O_1^H(t), O_2^H(t')] \rangle_\beta \tag{2.9}$$

is called *retarded correlation function*. Also in this case, similarly to what happens in RQFT, that experimental data are obtained from particular correlation functions.

Of curse if $O_1$, $O_2$ and $\xi_\varepsilon$ depend also on space coordinates then eq. (2.8) becomes:

$$\delta \langle O_1^{H+V_\varepsilon}(\boldsymbol{x}, t) \rangle_T = \int \mathrm{d}^3 x' \, \mathrm{d}t' \, \xi_\varepsilon(\boldsymbol{x}', t') (-i\theta(t-t') \langle [O_1^H(\boldsymbol{x}', t), O_2^H(\boldsymbol{x}', t')] \rangle_\beta) =$$

$$= \int \mathrm{d}^3 x' \, \mathrm{d}t' \, \xi_\varepsilon(\boldsymbol{x}', t') G_{\mathrm{ret}}^{O_1 O_2}(\boldsymbol{x}, t, \boldsymbol{x}', t') \tag{2.10}$$

where

$$G_{\text{ret}}^{O_1 O_2}(\boldsymbol{x}, t, \boldsymbol{x}', t') := -i\theta(t - t')\langle[O_1^H(\boldsymbol{x}, t), O_2^H(\boldsymbol{x}', t')]\rangle_\beta \qquad (2.11)$$

Assuming translational invariance, i.e. $G_{\text{ret}}^{O_1 O_2}$ depends only on $t, t'$, and the space difference $\boldsymbol{x} - \boldsymbol{x}'$, we can easily perform the Fourier transform, since eq. (2.10) corresponds to a convolution:

$$\delta\langle O_1^{H+V_\varepsilon}(\boldsymbol{q}, \omega)\rangle_\beta = \tilde{\xi}_\varepsilon(\boldsymbol{q}, \omega)\, \tilde{G}_{\text{ret}}^{O_1 O_2}(\boldsymbol{q}, \omega)$$

The retarded correlators are typically directed connected to experiments. For example suppose to measure the magnetization of a spin system, with spin $\boldsymbol{S}(\boldsymbol{x})$ in presence of a test magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. The coupling between the magnetic field and the spin is given by

$$\int \mathrm{d}^3 x\, \boldsymbol{B}(\boldsymbol{x}, t) \cdot \boldsymbol{S}(\boldsymbol{x})$$

The linear response to the perturbation is determined by

$$-i\theta(t_1 - t_2)\langle[\boldsymbol{S}^H(\boldsymbol{x_1}, t_1), \boldsymbol{S}^H(\boldsymbol{x_2}, t_2)]\rangle_\beta$$

whose Fourier transform is precisely the dynamic magnetic susceptibility $\chi_s(\boldsymbol{q}, \omega)$ measurable by neutrons. Analogously charged particles are coupled to the electromagnetic field (in the gauge $A_0 = 0$) by

$$\int \mathrm{d}^3 x\, \boldsymbol{A}(\boldsymbol{x}, t) \cdot \boldsymbol{j}(\boldsymbol{x})$$

where in the free case the current $\boldsymbol{j}(\boldsymbol{x})$ is given by

$$\boldsymbol{j}(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\frac{\overset{\leftrightarrow}{\boldsymbol{\nabla}}}{2mi}\psi(\boldsymbol{x}) := \psi^\dagger(\boldsymbol{x})\left(\frac{\boldsymbol{\nabla}}{2mi}\psi(\boldsymbol{x})\right) - \left(\frac{\boldsymbol{\nabla}}{2mi}\psi^\dagger(\boldsymbol{x})\right)\psi(\boldsymbol{x})$$

and the *conductivity* $\sigma_{\alpha\beta}$ is directly related to the Fourier transform of

$$-i\theta(t_1 - t_2)\langle[\boldsymbol{j}_\alpha^H(\boldsymbol{x_1}, t_1), \boldsymbol{j}_\beta^H(\boldsymbol{x_2}, t_2)]\rangle_\beta$$

Notice that if $O_1$ and $O_2$ are fermionic fields, since the Hamiltonian is a scalar we get that $\xi_\varepsilon(t)$ must be an anticommuting function so that

$$O_1^H \xi_\varepsilon O_2^H - \xi_\varepsilon O_2^H O_1^H = -\xi_\varepsilon(O_1^H O_2^H + O_2^H O_1^H)$$

and in fact the retarded correlation function for fermionic fields $O_1$ and $O_2$ is

$$G_{\text{ret}}^{O_1 O_2} = i\theta(t_1 - t_2)\langle\{O_1^H(t_1), O_2^H(t_2)\}\rangle_\beta$$

For example the intensity of response in metals by high frequency photons (if the direction of the photon is chosen then the determination of such intensity is called *Angle-Resolved Photoemission Spectroscopy* (*ARPES*)) is related to the immaginary part of the Fourier transform of

$$i\theta(t_1 - t_2)\langle\{\psi^\dagger(\boldsymbol{x_1}, t_1), \psi(\boldsymbol{x_2}, t_2)\}\rangle_\beta$$

## Matsubara formalism

If $H$ is not free the next question is how to compute the correlation functions. Let $H = H_0 + H_I$, with $H_0$ "free" (typically this means that contains only the quadratic terms in the fields, with no mixed components). For $T = 0$ one just have to replace

$$\langle(\bullet)\rangle_\beta \to \langle 0|\,(\bullet)\,|0\rangle$$

and the Gell-Mann Low formula applies as before with adiabatic switching as in the relativistic case and the perturbative approach is completely similar.

For $T > 0$ the situation is more complicated because $H_I$ would appear in two places: in the time evolution eq. (2.6) as in $T = 0$ but also in the Boltzmann weight $e^{-\beta(H_0 + H_I)}$. This makes the standard perturbative

treatment inapplicable, since we need to disentangle a perturbation in these two places. The way to solve this issue is due to Matsubara, and for this reason called the *Matsubara formalism*. Let's see how it works.

We define an "evolution" of the operators by a new parameter $\tau$:

$$O^H(\tau) = e^{\tau H} O e^{-\tau H} \quad \text{with} \quad 0 \le \tau \le \beta$$

and then instead of computing correlation functions at ordinary time $t$, we compute them using the modified evolution and the parameters $\tau_i$:

$$\langle O_1^H(\tau_1) \dots O_n^H(\tau_n) \rangle_\beta$$

Suppose that $H = H_0 + H_I$ (notice that the cutoff $\varepsilon$ is not required here since the domain of $\tau$ is already finite, $\tau \in [0, \beta]$) and write again the evolution separating the contribution of $H_0$ and $H_I$ as in the interaction picture

$$O^H(\tau) = U^I(\tau) O^{H_0}(\tau) U^I(\tau)$$

getting as before

$$U^I(\tau) = e^{\tau H_0} e^{-\tau H} \quad \text{with} \quad U^I(0) = \mathbb{1}$$

and again

$$U^I(\tau) = \mathrm{T}_\tau [e^{-\int_0^\tau \mathrm{d}\tau' \, H_I(\tau')}]$$

We define *Matsubara correlator / Green function* the object

$$G_M^{O_1 \dots O_n}(\tau_1, \dots, \tau_n) := -\langle \mathrm{T}_\tau [O_1^H(\tau_1) \dots O_n^H(\tau_n)] \rangle_\beta$$

Now for the analogue of the Gell-Mann Low formula

$$\langle \mathrm{T}_\tau [O_1^H(\tau_1) \dots O_n^H(\tau_n)] \rangle_\beta = \frac{\langle \mathrm{T}_\tau [O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) e^{-\int_0^\beta \mathrm{d}\tau' \, H_I(\tau')}] \rangle_\beta^0}{\langle \mathrm{T}_\tau [e^{-\int_0^\beta \mathrm{d}\tau' \, H_I(\tau')}] \rangle_\beta^0}$$

with $\langle \bullet \rangle_\beta^0$ the thermal average computed using $H_0$ as Hamiltonian. In comparison with the Gell-Mann Low formula for particles physics, in this case the vacuum expectation values are replaced by the Boltzmann-weighted thermal traces, and the vacuum expectation value of the $S$-matrix operator $U_\varepsilon^I(+\infty)$ in the denominator is replaced by the Boltzmann-weighted trace of the interaction operator $U^I(\beta)$ along the euclidean time axis $\tau$.

Finally perturbative treatment then prescripts

$$\langle \mathrm{T}_\tau [O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) e^{-\int_0^\beta \mathrm{d}\tau' \, H_I(\tau')}] \rangle_\beta^0 = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \langle \mathrm{T}_\tau [O_1^{H_0}(\tau_1) \dots O_n^{H_0}(\tau_n) \left( \int_0^\beta \mathrm{d}\tau' \, H_I(\tau') \right)^\ell] \rangle_\beta^0$$

This leads to a series of thermally averaged products of many fields which move according to the free field equations. Therefore Wick's theorem can be applied and we obtain an expansion of the Matsubara Green function completely analogous to the field theoretic one. The only difference is the finite-time interaction. Now everything is again expressed in terms of the free fields and can be computed using Feynman diagrams.

To be precise in most cases one also need to use an infrared cutoff $\varepsilon$, which in this case (the domain of $\tau$ is finite) is required only in the space coordinates, and will be removed at the end of the calculation.

Now the question is how this thermal averaged operator can be related to the retarded correlators needed to compute the linear response of our system. Using thermal average formula eq. (2.7) and $Z := \mathrm{Tr}_\mathcal{H} \, e^{-\beta H}$ the *partition function* computed taking the trace over the whole Hilbert space of the system $\mathcal{H}$[VI] (not only the 1-particle Hilbert space $\mathcal{H}_1$), the 2-points Matsubara correlator (the only needed in the perturbative approach) reads:

$$\begin{aligned} G_M^{O_1 O_2}(\tau_1, \tau_2) = -\frac{1}{Z} \Big\{ & \mathrm{Tr}_\mathcal{H} [e^{-\beta H} e^{\tau_1 H} O_1 e^{-\tau_1 H} e^{\tau_2 H} O_2 e^{-\tau_2 H}] \theta(\tau_1 - \tau_2) \\ & \pm \mathrm{Tr}_\mathcal{H} [e^{-\beta H} e^{\tau_2 H} O_2 e^{-\tau_2 H} e^{\tau_1 H} O_1 e^{-\tau_1 H}] \theta(\tau_2 - \tau_1) \Big\} = \\ = -\frac{1}{Z} \Big\{ & \mathrm{Tr}_\mathcal{H} [e^{-\beta H} O_1 e^{-(\tau_1 - \tau_2) H} O_2 e^{(\tau_1 - \tau_2) H}] \theta(\tau_1 - \tau_2) \\ & \pm \mathrm{Tr}_\mathcal{H} [e^{-\beta H} O_2 e^{-(\tau_2 - \tau_1) H} O_1 e^{(\tau_2 - \tau_1) H}] \theta(\tau_2 - \tau_1) \Big\} \end{aligned}$$

---

[VI]Notice that at this point we do not know whether this space is Fock or non-Fock.

where in the second step we used the cyclicity of the trace and the choice of the sign depends on the commutation relation between $O_1^H$ and $O_2^H$ (+ if they commute or − if they anticommute). This computation shows that the two points correlator is just a function of the difference $\tau := \tau_1 - \tau_2$.

$$G_M^{O_1 O_2}(\tau) = -\frac{1}{Z}\left\{ \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-\tau H} O_2 e^{\tau H}]\theta(\tau) \pm \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_2 e^{\tau H} O_1 e^{-\tau H}]\theta(-\tau)\right\} \tag{2.12}$$

By construction $0 \le \tau_i \le \beta$, $i = 1, 2$, hence we get that $\tau \in [-\beta, \beta]$. Notice that for $\tau \in [-\beta, 0]$, $\tau + \beta \ge 0$

$$\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_2 e^{\tau H} O_1 e^{-\tau H}] = \mathrm{Tr}_{\mathcal{H}}[O_2 e^{(\tau+\beta)H} e^{-\beta H} O_1 e^{-(\tau+\beta)H}] = \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-(\tau+\beta)H} O_2 e^{(\tau+\beta)H}]$$

hence $G_M^{O_1 O_2}(\tau) = \pm G_M^{O_1 O_2}(\tau + \beta)$ and we can regard $G_M^{O_1 O_2}(\tau)$ as an (anti)periodic function of period $\beta$ defined for any $\beta \in \mathbb{R}$.

Notice that in general traces can be infinite or the limit $\varepsilon \to 0$ of the cutoff might not exists, making the perturbative approach not working (notice that an infinite trace is the thermal-equivalent of a divergent vacuum expectation value of the theoretical field case). Nevertheless the periodicity of the Matsubara correlator is a non-perturbative result which hold anyway, even if the definition of the trace lose its meaning. Such relation is called *Kubo-Martin-Schwinger (KMS) condition*:

$$G_M(\tau) = \pm G_M(\tau + \beta) \tag{2.13}$$

which is independent on the choice of $\varepsilon$ and works also for $\varepsilon \to 0$.

Due to periodicity, $G_M(\tau)$ can be represented in terms of Fourier coefficients

$$G_M(\tau) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} e^{-i\omega_n \tau} G_M(\omega_n) \tag{2.14}$$

with

$$\omega_n = \begin{cases} \dfrac{2n\pi}{\beta} & \text{if } G_M(\tau) = +G_M(\tau + \beta) \\[2ex] \dfrac{(2n+1)\pi}{\beta} & \text{if } G_M(\tau) = -G_M(\tau + \beta) \end{cases} \tag{2.15}$$

called *Matsubara frequencies* and

$$G_M(\omega_n) = \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n \tau} G_M(\tau) \tag{2.16}$$

Notice that as $T \to 0$ Matsubara coefficients become continuous and eq. (2.14) become a Fourier transformation.

**Lehmann spectral representation of the correlators $(T > 0)$**

The Wick rotation of the $t$-axis in eq. (2.11) exactly coincides with the axis on which $\tau$ is defined, and the Wick rotation send the energies in the field theoretic Green functions eq. (2.11) to the axis along which Matsubara frequencies are situated. The retarded Green functions are related to the imaginary-time Matsubara Green functions by an analytical continuation.

To see this relation we introduce a Dirac completeness of eigenstates $\{|n\rangle\}$ of $H$ inside the coefficients of the Fourier series eq. (2.16) (we change the label for the Fourier coefficients from $n$ to $s$) using eq. (2.12):

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z} \int_0^\beta \mathrm{d}\tau\, e^{i\omega_s \tau} \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-\tau H} O_2 e^{\tau H}] \\ &= -\frac{1}{Z} \int_0^\beta \mathrm{d}\tau \sum_{m,n} e^{i\omega_s \tau} e^{-\beta \mathcal{E}_n} e^{\tau(\mathcal{E}_n - \mathcal{E}_m)} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \end{aligned} \tag{2.17}$$

Using $\int_0^\beta \mathrm{d}\tau\, e^{\alpha \tau} = \frac{1}{\alpha}(e^{\alpha \beta} - 1)$ we can integrate:

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z} \sum_{m,n} e^{-\beta \mathcal{E}_n} \frac{e^{i\omega_s \beta + \beta(\mathcal{E}_n - \mathcal{E}_m)} - 1}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \\ &= -\frac{1}{Z} \sum_{m,n} \frac{e^{i\omega_s \beta - \beta \mathcal{E}_m} - e^{-\beta \mathcal{E}_n}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle \langle m| O_2 |n\rangle \end{aligned}$$

Finally using the explicit values of Matsubara frequencies eq. (2.15), one gets

$$
G_M(\omega_s) = -\frac{1}{Z} \sum_{m,n} \frac{\pm e^{-\beta \mathcal{E}_m} - e^{-\beta \mathcal{E}_n}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle\langle m| O_2 |n\rangle =
$$
$$
= \frac{1}{Z} \sum_{m,n} \frac{e^{-\beta \mathcal{E}_n} \mp e^{-\beta \mathcal{E}_m}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n| O_1 |m\rangle\langle m| O_2 |n\rangle \quad \text{for} \quad G_M(\tau) = \pm G_M(\tau + \beta)
$$

(2.18)

Such expression for the Green function is called *Lehmann representation* (of $G_M$, in this case). Notice that in this representation the Fourier series coefficients can be computed in terms of $H$ eigenvalues and the matrix elements of $O_1$ and $O_2$ only (for this reason such representation is also called "spectral" representation). Notice that these coefficients have poles along the imaginary axis.

Let's try to find the Lehmann representation of the retarded Green function we introduced before. Actually a more generic definition respect to eq. (2.9) of the retarded correlator is used:

$$
G_{\text{ret}}^{O_1 O_2}(t, t') := -i\theta(t - t')\langle [O_1^H(t), O_2^H(t')]_{\mp}\rangle_\beta
$$

(2.19)

taking the commutator or the anticommutator depending on the label after the square bracket. Rewriting eq. (2.19) introducing a Dirac completeness as in eq. (2.17) we get

$$
G_{\text{ret}}(t_1, t_2) = -i\theta(t_1 - t_2) \text{Tr}_\mathcal{H}[e^{-\beta H}[e^{it_1 H} O_1 e^{-it_1 H}, e^{it_2 H} O_2 e^{-it_2 H}]_{\mp}]
$$
$$
= -\frac{i}{Z}\theta(t_1 - t_2) \text{Tr}_\mathcal{H}[e^{-\beta H}(e^{it_1 H} O_1 e^{-it_1 H} e^{it_2 H} O_2 e^{-it_2 H} \mp e^{it_2 H} O_2 e^{-it_2 H} e^{it_1 H} O_1 e^{-it_1 H})]
$$
$$
= -\frac{i}{Z}\theta(t) \text{Tr}_\mathcal{H}[e^{-\beta H} e^{itH} O_1 e^{-itH} O_2 \mp e^{itH} O_1 e^{-\beta H} e^{-itH} O_2]
$$
$$
= -\frac{i}{Z}\theta(t) \sum_{m,n}(e^{-\beta \mathcal{E}_n} \mp e^{-\beta \mathcal{E}_m}) e^{it(\mathcal{E}_n - \mathcal{E}_m)} \langle n| O_1 |m\rangle\langle m| O_2 |n\rangle
$$

where $t := t_1 - t_2$ and in the third step we used the cyclicity of the trace. Using the Fourier Transform of the following functions (the limit $\delta \to 0$, $\delta > 0$, is understood)

$$
\mathcal{F}[\theta(t)](\omega) = \frac{i}{\omega + i\delta} \quad , \quad \mathcal{F}[e^{i\alpha t}](\omega) = \delta(\omega + \alpha)
$$
$$
\mathcal{F}[f(t)g(t)](\omega) = \int d\omega' \, \tilde{f}(\omega - \omega')\tilde{g}(\omega')
$$

one get the Fourier transform of $G_{\text{ret}}(t)$:

$$
G_{\text{ret}}(\omega) = -\frac{i}{Z} \int d\omega' \frac{i}{\omega - \omega' + i\delta} \sum_{m,n}(e^{-\beta \mathcal{E}_n} \mp e^{-\beta \mathcal{E}_m})\delta(\omega' + \mathcal{E}_n - \mathcal{E}_m)\langle n| O_1 |m\rangle\langle m| O_2 |n\rangle
$$
$$
= \frac{1}{Z} \sum_{m,n} \frac{e^{-\beta \mathcal{E}_n} \mp e^{-\beta \mathcal{E}_m}}{\omega + \mathcal{E}_n - \mathcal{E}_m + i\delta} \langle n| O_1 |m\rangle\langle m| O_2 |n\rangle
$$

(2.20)

which is the Lehmann representation of $G_{\text{ret}}$. Notice that the prescription $i\delta$ is needed to move slightly the poles of the functions from the real axis, in such a way that the Wick rotation of the function is allowed.

Is now clear comparing eq. (2.18) and eq. (2.20) that the retarded correlator is just the analytical continuation of the Matsubara correlator

$$
G_{\text{ret}}(\omega) = G_M(i\omega_s \to \omega + i\delta)
$$

Can be proved[VII] using Carleman's theorem that such relation is well-defined and can be used to reconstruct uniquely $G_{\text{ret}}(\omega)$ by analytical continuation of $G_M(\omega_n)$. The only requirement on the Matsubara Green function is that it satisfy the KMS condition eq. (2.13). Therefore it is possible to compute $G_M$ as in the high energy case and then obtain $G_{\text{ret}}$ from analytical continuation.

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[VII] Original proof by Baym and Mermin (1960): https://doi.org/10.1063/1.1703704, useful reference: Appendix A in https://fks.sk/~bzduso/physics/master/thesis.pdf.

## 2.5  Path integral formalism

[Zha99], [Zin02], [Kle15, Chapter 14]

In this section we just sketch in an "heuristic way", without details, how the path-integral is defined, with some more details in the non relativistic setting and $T > 0$.
The key ingredients of Feynman path-integrals both in real and imaginary time are

(1) a Dirac completeness like $\int \mathrm{d}x \, |x\rangle\langle x| = \mathbb{1}$, appropriately generalized for fields;

(2) the representation of the kernel of infinitesimal time evolution in terms of the action.

**Path integral for bosonic fields**

For bosonic fields the completeness employees the *coherent states*. We just recall that for the harmonic QM oscillator the coherent states $|z\rangle$, $z \in \mathbb{C}$, are of the form (with $a$ annihilation operator)

$$a \, |z\rangle = z \, |z\rangle \quad , \quad |z\rangle = e^{za^\dagger} \, |0\rangle \tag{2.21}$$

One easily verifies that

$$\langle z| \, a^\dagger = z^* \langle z| \quad , \quad \langle z|z'\rangle = e^{z^* z'} \quad , \quad \int_{\mathbb{C}} \mathrm{d}z \, \mathrm{d}z^* \, |z\rangle\langle z| \, e^{-|z|^2} = \mathbb{1} \tag{2.22}$$

The concept of coherent states easily generalizes to fields, just replacing the single annihilation operator in QM with a set of annihilation operators $\{a_\alpha\}_{\alpha \in A}$. Indeed a generic field can be written as[VIII]

$$\hat{\phi}(\boldsymbol{x}) = \sum_{\alpha \in A} \phi_\alpha(\boldsymbol{x}) a_\alpha \tag{2.23}$$

The set $\{a_\alpha\}_{\alpha \in A}$ is commuting, hence all the operators admits a common eigenbasis $\{ |\phi^\beta\rangle \}_{\beta \in B}$, $|\phi^\beta\rangle :=$ $|\{\phi_{\alpha'}^\beta\}_{\alpha' \in A}\rangle$, such that for any $\alpha \in A$

$$a_\alpha \, |\phi^\beta\rangle = \phi_\alpha^\beta \, |\phi^\beta\rangle \quad \text{with eigenvalues} \quad \phi_\alpha^\beta \in \mathbb{C}$$

Moreover, using commutation relations for ladder operators, this implies

$$|\phi^\beta\rangle = e^{\sum_\alpha \phi_\alpha^\beta a_\alpha^\dagger} \, |0\rangle \quad \text{and} \quad \langle \phi^\beta | \phi^{\beta'} \rangle = e^{\sum_\alpha \phi_\alpha^{\beta\,*} \phi_\alpha^{\beta'}}$$

It is clear that vectors $\{ |\phi^\beta\rangle \}$ are eigenvectors also for the field operator

$$\hat{\phi}(\boldsymbol{x}) \, |\phi^\beta\rangle = \sum_{\alpha \in A} \phi_\alpha(\boldsymbol{x}) \phi_\alpha^\beta \, |\phi^\beta\rangle = \phi^\beta(\boldsymbol{x}) \, |\phi^\beta\rangle$$

with eigenvalue $\phi^\beta(\boldsymbol{x}) := \sum_{\alpha \in A} \phi_\alpha(\boldsymbol{x}) \phi_\alpha^\beta$ (is a function). In particular, for a RQFT we have the operator

$$\hat{\phi}(\boldsymbol{x}) = \sum_\alpha \phi_\alpha(\boldsymbol{x}) a_\alpha + \phi_\alpha^*(\boldsymbol{x}) a_\alpha^\dagger$$

and correspondingly we have the eigenvalues

$$\phi^\beta(\boldsymbol{x}) = \sum_\alpha \phi_\alpha(\boldsymbol{x}) \phi_\alpha^\beta + \phi_\alpha^*(\boldsymbol{x}) \phi_\alpha^{\beta\,*}$$

Notice that up to now we have properly generalized eq. (2.21) in the case of fields.
Let's see how the last identity in eq. (2.22) generalizes in this setting. The following Dirac completeness can be proved:

$$\int \frac{\mathrm{d}\phi \, \mathrm{d}\phi^*}{2\pi i} e^{-\phi^* \phi} \, |\phi\rangle\langle\phi| := \sum_{\beta \in B} \frac{1}{2\pi i} e^{-\sum_\alpha \phi_\alpha^{\beta\,*} \phi_\alpha^\beta} \, |\phi^\beta\rangle\langle\phi^\beta| = \mathbb{1} \tag{2.24}$$

where on the l.h.s. we defined $\phi^{\beta*}\phi^{\beta} := \langle \phi^{\beta}|\phi^{\beta}\rangle$ and suppressed the label $\beta$ by replacing $\sum_{\beta} f(\phi^{\beta}, \phi^{\beta*}) \to \int \mathrm{d}\phi\,\mathrm{d}\phi^* \, f(\phi, \phi^*)$.

For each operator $A$ and a basis $\{|n\rangle\}$ satisfying Dirac completeness $\sum_n |n\rangle\langle n| = \mathbb{1}$ we have

$$
\begin{aligned}
\operatorname{Tr} A &= \sum_n \langle n|\, A\, |n\rangle = \sum_n \langle n|\, A \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi} |\phi\rangle \langle\phi|n\rangle \\
&= \sum_n \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi} \langle\phi|n\rangle \langle n|\, A\, |\phi\rangle \\
&= \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi} \langle\phi|\, A\, |\phi\rangle
\end{aligned}
$$

Beside the fields at fixed time $\hat{\phi}(\boldsymbol{x})$ in the Gell-Mann - Low formula or in the Matsubara formalism (to which we now address our effort) there are also the time-evolution operators. Let us consider an infinitesimal one: for $H$ the Hamiltonian in terms of free fields, $\Delta\tau$ infinitesimal "immaginary time" interval, the infinitesimal time-evolution operator is

$$
e^{-\Delta\tau H(a_\alpha^\dagger, a_\alpha)}
$$

We rewrite it in "normal order" : : form, i.e. with all the creation operators on the left of the annihilation ones:

$$
e^{-\Delta\tau H(a_\alpha^\dagger, a_\alpha)} = \; :\, e^{-\Delta\tau H(a_\alpha^\dagger, a_\alpha)}\, : \; + O((\Delta\tau)^2) \tag{2.25}
$$

Let us now study the partition function of the system. It reads (from now on $\beta$ is no more a label, rather it is the Boltzmann factor $\beta := 1/(k_b T)$)

$$
Z := \operatorname{Tr} e^{-\beta H} = \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi} \langle\phi|\, e^{-\beta H}\, |\phi\rangle
$$

We now split the interval $[0, \beta]$ into $N$ small interval of length $\beta/N =: \Delta\tau$ and at the "immaginary times" multiples of $\Delta\tau$ we insert the Dirac completeness eq. (2.24). Finally we take the limit $N \to \infty$ so that the intervals becomes arbitrarily small. We get

$$
Z = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j\,\mathrm{d}\phi_j^*}{2\pi i} e^{-\phi_j^*\phi_j} \langle\phi_j|\, e^{-\Delta\tau H}\, |\phi_{j-1}\rangle \quad \text{where} \quad \phi_0 := \phi_N
$$

For each $|\phi_j\rangle$, $j = 1, \dots, N$, the integral runs over all possible states of the completeness eq. (2.24). Due to the limit $N \to \infty$ and eq. (2.25) we have

$$
\begin{aligned}
\langle\phi_j|\, e^{-\Delta\tau H(a_\alpha^\dagger, a_\alpha)}\, |\phi_{j-1}\rangle &\simeq \langle\phi_j|\; :\, e^{-\Delta\tau H(a_\alpha^\dagger, a_\alpha)}\, :\; |\phi_{j-1}\rangle = \\
&= e^{-\Delta\tau H(\phi_j^*, \phi_{j-1})} \langle\phi_j|\phi_{j-1}\rangle = e^{-\Delta\tau H(\phi_j^*, \phi_{j-1})} e^{\sum_\alpha \phi_{\alpha\,j}^* \phi_{\alpha\,j-1}}
\end{aligned}
$$

Putting things together we have

$$
Z = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j\,\mathrm{d}\phi_j^*}{2\pi i} e^{-S_N(\phi^*, \phi)}
$$

with

$$
S_N(\phi^*, \phi) := \Delta\tau \left( \sum_{j=1}^{N} \sum_\alpha \phi_{\alpha\,j}^* \frac{\phi_{\alpha\,j} - \phi_{\alpha\,j-1}}{\Delta\tau} + H(\phi_j^*, \phi_{j-1}) \right) \quad \text{where} \quad \phi_0 := \phi_N
$$

We now define the *path integral measure* $\mathcal{D}\phi(\tau)$ such that

$$
\int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau) \dots := \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j\,\mathrm{d}\phi_j^*}{2\pi i} \dots
$$

---

[VIII] We consider the field at $t = 0$ in analogy with QM, the evolution can be recovered according to Heisenberg picture.

and taking the limit on $S_N$ as a Riemann integral, the label $j$ is replaced by a continuous variable $\tau$:

$$S_N \xrightarrow[N\to\infty]{} \int_0^\beta \mathrm{d}\tau \left( \phi^*(\tau) \frac{\partial}{\partial\tau} \phi(\tau) + H(\phi^*(\tau), \phi(\tau)) \right) =: S(\phi^*, \phi) \quad \text{where} \quad \phi(0) = \phi(\beta)$$

In this way we obtained the *Euclidean action* $S$ with time $\tau$ of period $\beta$, since $\phi_0 := \phi_N$ has been replaced by the condition $\phi(0) = \phi(\beta)$. Hence for the partition function we get

$$Z = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\, e^{-S(\phi^*,\phi)} \tag{2.26}$$

We can interpret $e^{-S(\phi^*,\phi)}$ as a Boltzmann weight for field configurations. In general expression eq. (2.26) has not a rigorous mathematical definition. One exception is provided by free fields, for which one can give a rigorous meaning to the condensed expression

$$\mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\, e^{-S(\phi^*,\phi)}$$

as a gaussian measure (notice that for free fields the action is quadratic), and then the whole integral is well defined.

The original non-commutativity of the field operators $\hat{\phi}$ is replaced here by the integration over all possible "alternatives" of the function fields $\phi$, without the classical constraint of the equation of motion.

It worth to notice that in general the expansion in annihilation operators eq. (2.23) is performed in momentum space, hence the labels $\{\alpha \in A\}$ are the three momentum vectors $\boldsymbol{p} \in \mathbb{R}^3$. Since each element $\phi(\tau)$ should be meant as a collection $\{\phi_\alpha(\tau)\}_{\alpha\in A} = \{\phi_{\boldsymbol{p}}(\tau)\}_{\boldsymbol{p}\in\mathbb{R}^3}$, then we can define a function $\phi(\tau, \boldsymbol{p})$ defined by $\phi(\tau, \boldsymbol{p}) := \phi_{\boldsymbol{p}}(\tau)$. Similarly, the expansion of $\hat{\phi}$ can be done in the standard space of elements $\{\boldsymbol{x} \in \mathbb{R}^3\}$, defining in this way the functions $\phi(\tau, \boldsymbol{x})$. Therefore the path-integral measure can be thought as an integration over all possible functions defined on $\mathbb{R}^4$, periodic in $\tau$, and the summation $\sum_\alpha$ can always be replaced by $\int \mathrm{d}^3x$ or $\int \mathrm{d}^3p$.

**Path integral for fermionic fields**

If we have fermions the above procedure needs a little modification since now annihilation states satisfy a different algebra

$$\{a_\alpha, a_{\alpha'}\} = 0$$

so that for each common eigenstate $|\phi^\beta\rangle$ of the annihilation operators we have

$$\{a_\alpha, a_{\alpha'}\}|\phi^\beta\rangle = \{\phi_\alpha^\beta, \phi_{\alpha'}^\beta\}|\phi^\beta\rangle \neq 0$$

hence the eigenvalues $\phi_\alpha^\beta, \phi_{\alpha'}^\beta$ of the annihilation operators cannot be complex numbers, and we need to introduce some *Grassmann variables*, i.e. anticommuting objects such that

$$\xi_\alpha \xi_{\alpha'} + \xi_{\alpha'} \xi_\alpha = 0$$

A strategy similar to the one previously sketched replacing bosonic operators $\hat{\phi}$ by complex functions $\phi^\beta$ can be adapted, replacing fermionic operators $\hat{\phi}$ by Grassmann functions $\xi$. For instance the trace of an operator now reads

$$\operatorname{Tr} A = \int \mathrm{d}\xi^* \mathrm{d}\xi\, e^{\sum_\alpha \xi_\alpha^* \xi_\alpha} \langle -\xi | A | \xi \rangle$$

so that

$$Z = \lim_{N\to\infty} \int \prod_{j=1}^N \mathrm{d}\xi_j^* \mathrm{d}\xi_j\, e^{-S_N(\xi^*,\xi)} =: \int_{\xi(0)=-\xi(\beta)} \mathcal{D}\xi^*(\tau)\mathcal{D}\xi(\tau)\, e^{-S(\xi^*,\xi)}$$

with the *Euclidean action* defined in terms of Grassmann fields

$$S(\xi^*, \xi) := \int_0^\beta \left( \xi^*(\tau) \frac{\partial}{\partial\tau} \xi(\tau) + H(\xi^*(\tau), \xi(\tau)) \right)$$

Again one can give a rigorous meaning to the above formulas for free by proving that $\mathcal{D}\xi^*(\tau)\mathcal{D}\xi(\tau)\, e^{-S(\xi^*,\xi)}$ is a "Grassmann Gaussian integral".

In the following we denote by $\phi$ both the complex fields in the bosonic case and Grassmann fields in the fermionic case.

## Computation of Matsubara Green functions

As an example, let's see how to compute Matsubara Green functions using the path integral formalism in the Fock space in the finite volume case. Since we are working in the Fock space we can use for the free field the ladder operators $\{a_\alpha, a_\alpha^\dagger\}_{\alpha \in A}$, and using the Matsubara formalism we can introduce a dependence on $\tau$ for these operators:

$$a_\alpha(\tau) = e^{\tau H} a_\alpha e^{-\tau H}$$

Then in Fock space the Matsubara green functions are defined by

$$\langle T[a_{\alpha_1}(\tau_1) \ldots a_{\alpha_n} a_{\alpha_{n+1}}^\dagger \ldots a_{\alpha_{2n}}]\rangle_\beta := \frac{\mathrm{Tr}_{\mathcal{F}^\pm} T\left[e^{-\beta H} T[a_{\alpha_1}(\tau_1) \ldots a_{\alpha_n} a_{\alpha_{n+1}}^\dagger \ldots a_{\alpha_{2n}}]\right]}{\mathrm{Tr}_{\mathcal{F}^\pm} e^{-\beta H}}$$

where the time ordering $T[-]$ is considered in the Euclidean time $\tau$, and we consider $\mathcal{F}^+$ for bosonic fields and $\mathcal{F}^-$ for fermionic fields. The previous formula holds provided that $\mathrm{Tr}_{\mathcal{F}^\pm} e^{-\beta H}$ is finite. The requirement of finite volume makes this assumption more easily satisfied, but obviously this can raise easily problems in the thermodynamic limit.

The time ordering product can be removed by means of a permutation $\pi$ that puts the immaginary times in chronological order, $\tau_{\pi(i)} > \tau_{\pi(i+1)}$, then using notation

$$\tilde{a}_{\alpha_i}(\tau_i) = \begin{cases} a_{\alpha_i}(\tau_i) & \text{if} \quad i \leq n \\ a_{\alpha_i}^\dagger(\tau_i) & \text{if} \quad i < n \end{cases}$$

then

$$\langle T[a_{\alpha_1}(\tau_1) \ldots a_{\alpha_n} a_{\alpha_{n+1}}^\dagger \ldots a_{\alpha_{2n}}]\rangle_\beta =$$

$$= \frac{1}{Z} \epsilon^{\sigma(\pi)} \mathrm{Tr}\left[e^{-\beta H} \tilde{a}_{\alpha_{\pi(1)}}(\tau_{\pi(1)}) \ldots \tilde{a}_{\alpha_{\pi(2n)}}(\tau_{\pi(2n)})\right]$$

$$= \frac{1}{Z} \epsilon^{\sigma(\pi)} \int_{\phi(0) = \pm\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\, e^{-S(\phi^*,\phi)} \tilde{\phi}_{\alpha_{\pi(1)}}(\tau_{\pi(1)}) \ldots \tilde{\phi}_{\alpha_{\pi(2n)}}(\tau_{\pi(2n)})$$

$$= \frac{1}{Z} \int_{\phi(0) = \pm\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\, e^{-S(\phi^*,\phi)} \tilde{\phi}_{\alpha_1}(\tau_1) \ldots \tilde{\phi}_{\alpha_{2n}}(\tau_{2n})$$

where in the last step we put the operators in the original order, killing in this way the factor $\epsilon^{\sigma(\pi)}$. We saw in this way that the path integral is very useful to avoid problems related to the time ordering inside the Gell-Mann Low formula.

## Relativistic case

Completely analogour formulas holds in the relativistic case, where $S(\phi^*, \phi)$ is replaced by the Euclidean relativistic action. For instance for the massive bosonic field it is given by

$$S(\phi^*, \phi) = \frac{1}{\hbar} \int \mathrm{d}^{d+1}x \, \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi^* + m^2\phi^*\phi)(x, \tau)$$

# Chapter 3

# The need of a non-perturbative approach

We saw in the previous chapter that all the interesting physical quantities can be computed, both in high-energy physics and in condensed matter physics, using the perturbative approach. In this chapter we'll see why what we said doesn't really work so good, and in some instances we need a non-perturbative approach. We already anticipated some problems in the introduction, now we'll go more in detail.

## 3.1 The asymptotic series

**Divergent quantities**

Let's start from the simple example provided by a massive theory with quartic coupling in $d = 0$. Consider the following quantity, which naturally emerges in perturbative computations:

$$\frac{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2} e^{-\lambda x^4}}{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}} \quad \text{for} \quad \lambda > 0 \quad , \quad \operatorname{Re}\alpha > 0$$

Such quantity is clearly smaller than 1, since the numerator contains the integral of a function which is everywhere smaller than the function in the integral present in the denominator. Nevertheless if one applies the perturbative prescription, the previous quantity is substitute by

$$\frac{\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}(-\lambda x^4)^n}{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}} \tag{3.1}$$

But if one tries to compute explicitly the coefficient of the series

$$\frac{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}(\lambda x^4)^n}{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}} = \lambda^n \frac{(4n)!}{(2n)!} \frac{1}{2^{2n+1}} \frac{1}{\alpha^{2n}} \geq \frac{\lambda^n}{2} \frac{((2n)!)^2}{(2n)!} \frac{1}{(2\alpha)^{2n}} = \frac{\lambda^n}{2} \frac{(2n)!}{(2\alpha)^{2n}} \geq \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2}\right)^n (n!)^2$$

one gets that the series eq. (3.1) is absolutely divergent

$$\sum_{n=0}^{\infty} \left| \frac{(-\lambda)^n}{n!} \frac{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2} x^{4n}}{\int \mathrm{d}x \, e^{-\frac{\alpha}{2}x^2}} \right| \geq \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2}\right)^n n! = +\infty$$

Obviously we saw that the perturbative prescription in this case didn't worked, but can be proved that the same issue appear also in higher dimension. Such problem is actually more general, indeed all the perturbation series needed to compute the collerator using Gell-Mann Low formula are divergent.

Nevertheless is well known that usually the perturbative approach works (for instance in QED it works very well, and allows to obtain extremely precise predictions) hence we wonder if it is somehow possible to obtain the right expression of the original correlator using the coefficients of the perturbative series. In other words we want to know if, although divergent, could the perturbative series at least determine uniquely the corresponding correlation function.

### The asymptotic series

One says that a series $\sum_{n=0}^{\infty} a_n \lambda^n$, $\lambda > 0$, is *asymptotic to a function $f(\lambda)$* if

$$\lim_{\lambda \to 0^+} \frac{\left| f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n \right|}{\lambda^N} = 0 \quad \text{for all} \quad N > 0$$

This means that the absolute difference between $f(\lambda)$ and the truncated series at order $N$ is $O(\lambda^{N+1})$ so that for $\lambda \ll 1$ small enough we can make the difference as mild as we prefer.

Apparently with the increase of the order of perturbation the truncated series approximate better and better $f(\lambda)$. This naive idea is obviously wrong, due to the fact that increasing the perturbation order one should also decrease the value of $\lambda$. Indeed if one fixes the value of $\lambda$ and increases the perturbation order $N$, the absolute difference between $f(\lambda)$ and the truncated series initially decreases, but eventually it reaches its minimum and starts to increase again, diverging as $N \to \infty$. At higher perturbative orders one should take values of $\lambda$ smaller and smaller to make the correction negligible. The fact that we can take $\lambda \ll 1$ to make the difference negligible doesn't solve our problem, since in physical applications the role of $\lambda$ is done by $\hbar$, and decreasing its value below its physical value is meaningless.

### Non-analytical contributions

The second problem is that any series asymptotic to a function is also asymptotic to infinitely many distinguished functions. For instance, if a series is asymptotic to some $f(\lambda)$ it is also asymptotic to $f(\lambda) + e^{-\frac{1}{\lambda \alpha}}$, since

$$\lim_{\lambda \to 0^+} \frac{e^{-\frac{1}{\lambda \alpha}}}{\lambda^N} = 0 \quad \text{for any} \quad N \in \mathbb{N} \quad \text{and} \quad a > 0$$

This is due to the fact that the series expansion of $e^{-\frac{1}{\lambda \alpha}}$ for $\lambda \ll 1$ has all the coefficients equal to zero. Unfortunately terms in the form $e^{-\frac{1}{\lambda \alpha}}$ are exactly the contributions that arises if there are non-trivial topological configurations of the fields in the correlation function.

This makes completely impossible any unique reconstruction of the original function starting from its asymptotic series, since in any case the reconstructed function would be defined up to non-analytical terms which didn't appear in its asymptotic series.

### Is the perturbative series asymptotic to the correlation function?

Even if it's impossible to reconstruct the correlator from its asymptotic series, at lest it has been shown that in many QFT where the ultraviolet renormalization does not involve coupling constants (the so called *super-renormalizable theories*) the renormalized perturbation series is asymptotic to the non-perturbative defined correlation function (e.g. QED for $d < 3 + 1$ and $\phi^4$ in $d < 3 + 1$).

Nevertheless if our theory is *renormalizable* (hence not super-renormalizable) in the only case we have almost completely rigorous control of the RQFT, i.e. $\phi^4$ in $d = 3 + 1$, it has been proved that the renormalized perturbation series is not asymptotic to the non-perturbative correlator. Even if we don't have a rigorous proof, this seems the case also for QED in $d = 3 + 1$.

**Borel resummation**

In some cases, called *Borel resummable theories*, even if the perturbative series is divergent it is possible to resum it via *Borel resummation* obtaining a finite result.
The idea is ot introduce the identity

$$1 = \frac{1}{n!} \int_0^\infty x^n e^{-x} \mathrm{d}x$$

inside the perturbative series

$$\sum_{n=0}^\infty a_n \lambda^n = \int_0^\infty \sum_{n=0}^\infty \frac{a_n}{n!}(x\lambda)^n e^{-x} \mathrm{d}x$$

In some instances, the introduction of the factor $\frac{1}{n!}$ inside the perturbative coefficients makes the new series converge: $\sum_n \frac{1}{n!} a_n (x\lambda)^n$ might converge even if $\sum_n a_n \lambda^n$ does not. If the new coefficients are smooth enough to make the integral, we are then able to obtain the resummed series.
The first problem is that the resummed series is still not sensible to non-analytical terms (they cannot be reconstructed just resumming the analytical contributions), moreover a-posteriori one has to check that the resummed series is asymptotic to the non-perturbative result (up to non-analytical terms), since the right convergence is not ensured in general.

**Resurgence**

We just mention that there is a very recent technique called *resurgence* which applies to Quantum Mechanics and allows to obtain non-perturbative results just using perturbative techniques. Is still unknown whether such technique can be implemented or not also in QFT.

## 3.2 The Källen-Lehmann representation and the Lehmann representation

[GR96, Section 9.3]

The second problem of the perturbative approach is related to the limit $\varepsilon \to 0$ of the IR cutoff of the relation

$$U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\mathrm{in}}(\boldsymbol{x},t)U_\varepsilon^I(t) = \hat{\phi}(\boldsymbol{x},t)$$

Indeed the limit $\varepsilon \to 0$ should be taken in a way compatible with physical properties of the interacting theory, in particular translational invariance and, in high energy physics, the Poincaré invariance.
In order to exploit the problem we first need to introduce the spectral representation of the Green functions (the same we introduced for Matsubara and reatarded correlators, but from a more general and deep point of view).

**Källen-Lehmann representation (relativistic case)**

If $\mathcal{H}$ is the Hilbert space of the interacting theory, in order to have translational and Poincaré invariance, some properties has to be satisfied: in particular we need in $\mathcal{H}$

(1) A unitary representation of the covering of the restricted Poincaré group $\tilde{\mathcal{P}}_+^\uparrow$. Let's denote by $\hat{P}^\mu$ the corresponding generators of space-time translations.

(2) A vacuum vector $|0\rangle$ invariant under the representation $U(a)$, $a \in \mathbb{R}^{d+1}$, of spacetime translations ($d = 3$) (due to homogeneity of spacetime).

(3) The spectrum of $\hat{P}^\mu$ is contained in the forward light cone, $\sigma(\hat{P}^\mu) \subseteq V_0^+$.

(4) The operator $\hat{\phi}$ should transform with an irreducible representation of $\tilde{\mathcal{P}}_+^\uparrow$ under $U(a)$. In particular assuming that $\hat{\phi}$ is scalar we have

$$U(a)\hat{\phi}(x)U(a)^\dagger = \hat{\phi}(x-a)$$

From (1) we get that exists a Dirac completeness $|\alpha\rangle$ of (generalized) eigenvectors of $\hat{P}^\mu$ with eigenvalues $p_\alpha$ (for simplicity we write $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$ also if $\hat{P}^\mu$ has continuum spectrum).
Consider the 2-points function of a scalar RQFT

$$\langle 0| \hat{\phi}(x)\hat{\phi}(y) |0\rangle = \sum_\alpha \langle 0| \hat{\phi}(x) |\alpha\rangle\langle\alpha| \hat{\phi}(y) |\alpha\rangle \overset{(4)}{=} \sum_\alpha \langle 0| U(x)\hat{\phi}(0)U^\dagger(x) |\alpha\rangle\langle\alpha| U(y)\hat{\phi}(0)U^\dagger(y) |0\rangle$$

$$= \sum_\alpha e^{-ip_\alpha(x-y)}|\langle\alpha| \hat{\phi}(0) |0\rangle|^2 \overset{d=3}{=} \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho_+(q)e^{-iq(x-y)}$$

(3.2)

where in the last step we used the identity $1 = \int \mathrm{d}q\, \delta(q - p_\alpha)$ and we defined the Fourier transform (up to a factor $2\pi$) of the 2-point function

$$\rho_+(q) = (2\pi)^3 \sum_\alpha \delta(q - p_\alpha)|\langle 0| \hat{\phi}(0) |\alpha\rangle|^2$$

(3.3)

which has some interesting properties:

(a) $\rho_+(q) \geq 0$;

(b) $\rho_+(q) = 0$ if $q \notin \overline{V}_0^+$, thanks to (3), the bar over the forward light cone indicate its closure;

(c) $\rho_+(\Lambda q) = \rho_+(q)$, for $\Lambda \in L_+^\uparrow$, thanks to (4).

Therefore from all these conditions we get that the most general form for $\rho_+$ is

$$\rho_+(q) = \sigma(q^2)\theta(q^0) + c\,\delta(q) \quad \text{with} \quad \sigma(q^2) = 0 \quad \text{if} \quad q^2 < 0 \quad \text{and} \quad c \text{ constant}$$

(3.4)

The delta function is introduced to compensate the ambiguity due to the sign of $q^0$ in $\theta(q^0)$ when $q^0 = 0$. Notice that property (a), $\rho_+(q) \geq 0$, implies that $\sigma(q^2)$ is a semi-definite positive function, i.e. $\sigma(q^2) \geq 0$ for any value of $q \in \overline{V}_0^+$.
Let's introduce the *spectral function*[I]

$$\rho(q) = \rho_+(q) - \rho_+(-q)$$

which due to eq. (3.2) is the Fourier transform (up to a factor $2\pi$) of $\langle 0| [\hat{\phi}(x), \hat{\phi}(y)] |0\rangle$: using eq. (3.2) we get

$$\langle 0| [\hat{\phi}(x), \hat{\phi}(y)] |0\rangle = \langle 0| \hat{\phi}(x)\hat{\phi}(y) |0\rangle - \langle 0| \hat{\phi}(y)\hat{\phi}(x) |0\rangle$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho_+(q)e^{-iq(y-x)}$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho_+(-q)e^{-iq(x-y)}$$

$$= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, \rho(q)e^{-iq(x-y)}$$

(3.6)

Using expression eq. (3.4) we get

$$\rho(q) = \rho_+(q) - \rho_+(-q) = \sigma(q^2)\theta(q^0) + c\,\delta(q) - \sigma(q^2)\theta(-q^0) - c\,\delta(-q) = \text{sign}(q^0)\sigma(q^2)$$

(3.7)

If $\hat{\phi}$ obeys the CCR $[\hat{\phi}(\boldsymbol{x}, t), \dot{\hat{\phi}}(\boldsymbol{y}, t)] = i\delta(\boldsymbol{x} - \boldsymbol{y})$ then $\rho$ satisfies the *sum sule*

$$\int_{-\infty}^{+\infty} \mathrm{d}q^0\, q^0 \rho(q) = 1$$

(3.8)

---

[I]Notice that sometimes $\rho(q)$ is defined with an additional factor $2\pi$ in such a way that it is exactly the Fourier transform of $\langle 0| [\hat{\phi}(x), \hat{\phi}(y)] |0\rangle$. In some other cases it is defined (equivalently to eq. (3.5)) as 2 times the immaginary part of the retarded correlator, using the relations

$$\mathcal{F}(-i\theta(t))(\omega) = \frac{1}{\omega + i\delta} \quad \text{and} \quad \text{Im}\frac{1}{\omega + i\delta} = \pi\delta(\omega)$$

(3.5)

where $\omega = q_0$.

Indeed consider the identity

$$
\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \delta(\boldsymbol{x}-\boldsymbol{y}) = -i\langle 0|\,[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)]\,|0\rangle = -i\partial_{y^0}\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle\big|_{\substack{x_0=t\\y_0=t}} =
$$

$$
= -i\partial_{y^0}\int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,\rho(q)e^{-iq(x-y)}\big|_{\substack{x_0=t\\y_0=t}} = \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,q^0\rho(q)e^{-iq(x-y)}\big|_{\substack{x_0=t\\y_0=t}} =
$$

$$
= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,q^0\rho(q)e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}
$$

then comparing the left and the right side of the previous identity one gets exactly eq. (3.8).
Moreover using eq. (3.7) one can rewrite

$$
1 = \int_{-\infty}^{+\infty} \mathrm{d}q^0\, q^0\rho(q) = \int_{-\infty}^{+\infty} \mathrm{d}q^0\,|q^0|\sigma(q^2) = 2\int_0^{+\infty} \mathrm{d}q^0\, q^0\sigma(q^2) = \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2) \tag{3.9}
$$

where using $m^2 = q^2 = (q^0)^2 - \boldsymbol{q}^2$ and the fact that in our integration $\boldsymbol{q}$ is fixed, we applied the change of variable $q^0 \mapsto m^2$, $2q^0\mathrm{d}q^0 \mapsto \mathrm{d}m^2$.
Recalling that $\sigma(q^2) \geq 0$, we have that $q^0\rho(q) = |q^0|\sigma(q^2) \geq 0$ too, hence using eq. (3.8) and eq. (3.9) we get that both

$$
A(\omega,\boldsymbol{q}) := \omega\rho(\omega,\boldsymbol{q}) = q^0\rho(q)
$$

and $\sigma(m^2)$ are probability densities. In particular can be proved that $\sigma(m^2)$ is the probability density to find the state $\int e^{iqx}\hat{\phi}(x)\,|0\rangle$ in a state of mass $m^2$, while $A(\omega,\boldsymbol{q})$ is the probability density to find the same state $\int e^{iqx}\hat{\phi}(x)\,|0\rangle$ in a state of energy $\omega$ for fixed value $\boldsymbol{q}$.
We just mention that for fermions the same sum rule eq. (3.8) holds equivalently provided that we have the CAR $\{\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)\} = i\delta(\boldsymbol{x}-\boldsymbol{y})$.
Now, suppose that we do not know if our interacting field satisfies CCR, but we are sure that the associated free theory does. Denoting by $\Delta_+(x-y;m)$ the 2-points function of the free scalar field of mass $m$, then inserting inside eq. (3.2) the equality $\int_0^{+\infty} \mathrm{d}m^2\,\delta(q^2-m^2) = 1$ we get

$$
\langle 0|\,\phi(x)\phi(y)\,|0\rangle = \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\sigma(q^2)\theta(q^0)e^{-iq(x-y)} + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2
$$

$$
= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(q^2-m^2)\theta(q^0)e^{-iq(x-y)} + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2
$$

$$
= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\Delta_+(x-y;m) + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2
$$

where $|\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2$ is an irrelevant constant which comes from eq. (3.3) when evaluated at $q=0$ due to the term $\delta(q)$ in eq. (3.4), and can be removed just shifting $\hat{\phi}(x)$. This means that even in the interacting theory we can write the 2-points function as an integral of free 2-points functions of the free fields with varying mass weighted by a suitable measure $\mathrm{d}m^2\,\sigma(m^2)$. This fact is a purely relativistic effect, and has no analogue for NRQFT, due to Lorentz invariance, and is no more true if we broke Lorentz symmetry. One of the consequences is that, since the 2-point function of the interacting theory can be written in terms of weighted two point functions of free theories, which satisfies CCR, also the interacting theory should satisfy CCR, up to some constant factor $c$ which can be reabsorbed in the field:

$$
\langle 0|\,[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)]\,|0\rangle = \partial_{y^0}\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle\big|_{\substack{x_0=t\\y_0=t}}
$$

$$
= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\,\partial_{y^0}\big(\Delta_+(x-y;m) - \Delta_+(y-x;m)\big)\big|_{\substack{x_0=t\\y_0=t}} \tag{3.10}
$$

$$
= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\,i\delta(\boldsymbol{x}-\boldsymbol{y}) = c\,i\delta(\boldsymbol{x}-\boldsymbol{y})
$$

with $c = \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)$.

Representation of correlation functions given by eq. (3.2) and eq. (3.6) are called *Källen-Lehmann representations* since they was firstly derived by these two physicist.[II] Actually Lehmann applied such representation also to the non-relativistic case, as we now show.

**Lehmann representation (non-relativistic case)**

Let's see how to generalize the previous representation without assuming Lorentz invariance. In this case in the Hilbert space $\mathcal{H}$ of our theory we need:

(1') A unitary representation $U$ of spacetime translations (one can also add rotational invariance).

(2') The field $\hat{\phi}$ should transform under an irreducible representation of transformations under $U$.[III] We assume that $\hat{\phi}$ transforms as a scalar.

As in the relativistic case, (1') implies that exists a Dirac completeness for the generators $\hat{\boldsymbol{P}}$, $H$ of $U(a)$ with eigenvalue $p_\alpha$. Applying the same strategy as in the relativistic case (the symbol $\pm$ in $\langle\bullet\rangle_{\beta\pm}$ indicates whether the operators are bosons or fermions when we compute the average at finite temperature $T$, while $[\bullet,\bullet]_+$ is the anticommutator and $[\bullet,\bullet]_-$ is the commutator)

$$
\begin{aligned}
\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_{\beta\pm} &= \frac{\mathrm{Tr}\left[e^{-\beta H}[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\right]}{\mathrm{Tr}\left[e^{-\beta H}\right]} \\
&= \frac{1}{Z}\sum_{\alpha,\alpha'}\left(\langle\alpha'|e^{-\beta H}\hat{\phi}(x)|\alpha\rangle\langle\alpha|\hat{\phi}^\dagger(y)|\alpha'\rangle \pm \langle\alpha|e^{-\beta H}\hat{\phi}^\dagger(y)|\alpha'\rangle\langle\alpha'|\hat{\phi}(x)|\alpha\rangle\right) \\
&= \frac{1}{Z}\sum_{\alpha,\alpha'}\left(e^{-\beta\mathcal{E}_{\alpha'}} \pm e^{-\beta\mathcal{E}_\alpha}\right)e^{i(p_\alpha-p_{\alpha'})(x-y)}|\langle\alpha'|\hat{\phi}(0)|\alpha\rangle|^2 \\
&\overset{d=3}{=} \int\frac{\mathrm{d}^4 q}{(2\pi)^3}\rho^\pm(q)e^{-q(x-y)}
\end{aligned}
\tag{3.11}
$$

with *spectral function*

$$
\rho^\pm(q) = \frac{(2\pi)^3}{Z}\sum_{\alpha,\alpha'}\left(e^{-\beta\mathcal{E}_{\alpha'}} \pm e^{-\beta\mathcal{E}_\alpha}\right)\delta(q-p_\alpha+p_{\alpha'})|\langle\alpha'|\hat{\phi}(0)|\alpha\rangle|^2
$$

Eq. (3.11) is called *Lehmann representation* of $\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_{\beta\pm}$.
For fermions, provided that CAR $\{\hat{\phi}(\boldsymbol{x},t),\hat{\phi}^\dagger(\boldsymbol{y},t)\} = i\delta(\boldsymbol{x}-\boldsymbol{y})$ is satisfied[IV] and $T=0$, we get that $A(\omega,\boldsymbol{q}) := \rho^+(q)$ (without $q^0$ in the non relativistic case) is a probability density, and it describes the probability density to find the state $\int e^{-iqx}\hat{\phi}(x)|0\rangle$ in a state of energy $\omega$.

**The constraint $0 \leq Z(\boldsymbol{q}) \leq 1$**

Let's see what these spectral representations have to do with the inconsistency of the perturbation theory as $\varepsilon \to 0$.
In general in a QFT (relativistic or not) if there is a particle (or a quasi-particle) excitation with dispersion $\omega = \omega(\boldsymbol{q})$ (in the relativistic case $\omega = \sqrt{m^2+\boldsymbol{q}^2}$) (we assume $\langle 0|\hat{\phi}(0)|0\rangle = 0$, otherwise we can shift the field) then in $A(\omega,\boldsymbol{q})$ there should be a term

$$
\begin{aligned}
&Z(\boldsymbol{q})\delta(\omega-\omega(\boldsymbol{q})) \quad \text{in the non-relativistic case} \\
&Z\delta(q^2-m^2)\theta(q^0) \quad \text{in the relativistic case}
\end{aligned}
\tag{3.12}
$$

Notice that in the relativistic case there cannot be any dependence on $\boldsymbol{q}$ in Z due to Lorentz invariance.

---

[II]Källen: https://doi.org/10.5169%2Fseals-112316, Lehmann: https://doi.org/10.1007%2Fbf02783624.
[III]For lattice theories $U$ is restricted to discrete spatial lattice translations and typically the time lattice translations are represented contractively, i.e. through terms of the form $e^{-tH}$ (continuum variation of the transfer matrix).
[IV]Notice that in the non-relativistic case there is no time derivative in the CAR.

Suppose that we have only one particle in the system (no other particles excitations), then in the non-relativistic case[V] since $A(\omega, \boldsymbol{q})$ describe the probability density to find the state $\int e^{-iqx} \hat{\phi}(x) \left|0\right\rangle$ in a state of energy $\omega$ we have that in general

$$A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q})) + A_{\text{inc}}(\omega, \boldsymbol{q}) \tag{3.13}$$

where $A_{\text{inc}}$ is a regular (differently from $\delta$) "featureless" positive function describing contributions of multi-particle states in the two points function (we still allow our particle to generate some other particles which eventually recombine in the original one due to energy uncertainty, if there are interactions in our theory). These give a (small) non-zero probability to find a particle in an energy state different from $\omega(\boldsymbol{q})$.
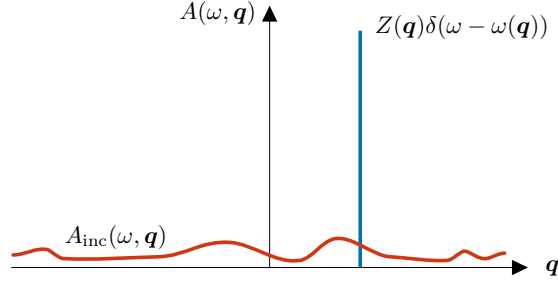


Figure 3.1: One-dimensional pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a stable particle.

Just notice that since $A(\omega, \boldsymbol{q})$ has an explicitly physical meaning as probability amplitude, the plot shown in Fig. 3.1 can be obtained directly in experiments, up to uncertainties which turn the $\delta$-function into a smooth but very localized function.
From the sum sule we get

$$1 = \int \mathrm{d}\omega\, A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q}) + \underbrace{\int \mathrm{d}\omega\, A_{\text{inc}}(\omega, \boldsymbol{q})}_{\geq 0}$$

which implies

$$0 \leq Z(\boldsymbol{q}) \leq 1$$

Notice that the case $Z(\boldsymbol{q}) = 1$ the theory is free, indeed for a free particle $A(\omega, \boldsymbol{q}) = \delta(\omega - \omega(\boldsymbol{q}))$. Our problem is that, as we'll see, is very difficult to satisfy $0 \leq Z < 1$ in an interacting situation.
We can weak a little the above requirement of a stable particle excitation: let's see what happen if we do not have a stable particle but a long-life resonance (i.e. a particle with a lifetime large respect to the typical time of the system). The $\delta$-function in eq. (3.12) can be rewritten as

$$\delta(\omega - \omega(\boldsymbol{q})) \quad \mapsto \quad \frac{1}{\text{Im}}\frac{1}{\pi}\omega - \omega(\boldsymbol{q}) + i\delta \quad \text{where the limit } \delta \to 0^+ \text{ is understood}$$

and this implies that the retarded correlator has a pole at $\omega = \omega(\boldsymbol{q}) - i\delta$. Notice that $\delta$ essentially correspond to the inverse lifetime $\tau^{-1}$ of the particle, hence the limit $\delta \to 0^+$ correspond to the infinite lifetime of the particle.
A resonance is described through a Lorentzian distribution, by replacing the above pole by a complex pole at $\omega = f(\omega, \boldsymbol{q})$ where $f$ is a complex function with $\text{Im}\, f(\omega, \boldsymbol{q})$ (in such a way that $A(\omega, \boldsymbol{q}) \geq 0$ is satisfied). Then

$$\text{Im} \frac{1}{\omega - f(\omega, \boldsymbol{q})} = \frac{-\text{Im}\, f(\omega, \boldsymbol{q})}{(\omega - \text{Re}\, f(\omega, \boldsymbol{q}))^2 + (\text{Im}\, f(\omega, \boldsymbol{q}))^2}$$

where $\omega = \text{Re}\, f(\omega, \boldsymbol{q})$ describes the dispersion relation of the unstable particle and $\text{Im}\, f(\omega, \boldsymbol{q})$ its inverse lifetime.

---

[V]The relativistic one is obtained imposing $q^2 = m^2$.

Then if we take $\text{Im} f(\omega, \boldsymbol{q}) \to 0^-$ then $\text{Im}(\omega - f(\omega, \boldsymbol{q})^{-1} \to \delta(\omega - f(\omega, \boldsymbol{q}))$ with $\omega(\boldsymbol{q})$ solution of the equation $\omega - \text{Re} f(\omega, \boldsymbol{q}) = 0$:

$$\delta(\omega - f(\omega, \boldsymbol{q})) = \frac{\delta(\omega - \omega(\boldsymbol{q}))}{\left|\dfrac{\partial f}{\partial \omega}(\omega, \boldsymbol{q})\right|\Big|_{\omega = \omega(\boldsymbol{q})}} = Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q}))$$

This means that in the case of the resonance we just broaden the $\delta$-function in the spectral function, as shown in fig. 3.2.
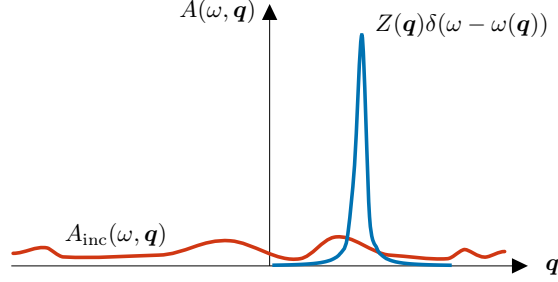


Figure 3.2: One-dimensional pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a unstable particle. The width of the blue peak is proportional to the inverse lifetime $\tau^{-1}$ of the particle.

Again, provided CCR or CAR (depending on the physical situation) we again have the contstraint

$$0 \leq Z \leq 1$$

hence the constraint applies even if we consider unstable particles.

## 3.3 The incompatibility between CCR and $U^I$

Let's see where the problem with the removal of the cutoff $\varepsilon$ arises, first considering the relativistic case. Let's inspect the relation between $Z$ and asymptotic fields. With the introduction of the infrared cutoff $\varepsilon$ (at least in $t$) we got that the free field is related to the interacting field by a unitary operator

$$\hat{\phi}(\boldsymbol{x}, t) = U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\text{in}}(\boldsymbol{x}, t)U_\varepsilon^I(t)$$

such that

$$\hat{\phi}(\boldsymbol{x}, t) \xrightarrow[t \to -\infty]{} \hat{\phi}_{\text{in}}(\boldsymbol{x}, t) \quad \text{and} \quad U_\varepsilon^I(t) \xrightarrow[t \to +\infty]{} \mathbb{1}$$

Let's see if such unitary relation between free and interacting field is possible preserving translational invariance.
Consider a one particle system, then from eq. (3.13) we can identify $\langle 0| \hat{\phi}_{\text{in}}\hat{\phi}_{\text{in}} |0\rangle$ as responsible for the contribution

$$Z\delta(q^2 - m^2)\theta(q^0) \tag{3.14}$$

in the spectral function $A(q)$ of $\hat{\phi}$. Since we want that $\hat{\phi}$ describe an interacting field we need to assume $Z \neq 1$ (otherwise $\hat{\phi}$ is free), conversely for $\hat{\phi}_{\text{in}}$ we already know that eq. (3.14) is satisfied for $Z = 1$. Therefore it is impossible to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} \hat{\phi}_{\text{in}}$, one can at best hope to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, i.e.

$$\hat{\phi}(\boldsymbol{x}, t) = Z^{1/2}U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\text{in}}(\boldsymbol{x}, t)U_\varepsilon^I(t) \tag{3.15}$$

But if $\hat{\phi}_{\text{in}}$ and $\hat{\phi}$ are canonical, i.e. satisfy CCR, then eq. (3.15) must be wrong, indeed from eq. (3.15) we get

$$[\hat{\phi}(\boldsymbol{x}, t), \dot{\hat{\phi}}(\boldsymbol{y}, t)] \xrightarrow[t \to -\infty]{} Z[\hat{\phi}_{\text{in}}(\boldsymbol{x}, t), \dot{\hat{\phi}}_{\text{in}}(\boldsymbol{y}, t)] \tag{3.16}$$

but then using CCR for $\hat{\phi}_{\text{in}}$ and $\hat{\phi}$ we get

$$\delta(\boldsymbol{x}-\boldsymbol{y}) \stackrel{\text{CCR}}{=} \lim_{x^0 \to -\infty} \langle 0| \, [\hat{\phi}(\boldsymbol{x},x^0), \dot{\hat{\phi}}(\boldsymbol{y},y^0)] \, |0\rangle \big|_{x^0=y^0} =$$

$$= Z \lim_{x^0 \to -\infty} \langle 0| \, [\hat{\phi}_{\text{in}}(\boldsymbol{x},x^0), \dot{\hat{\phi}}_{\text{in}}(\boldsymbol{y},y^0)] \, |0\rangle \big|_{x^0=y^0} \stackrel{\text{CCR}}{=} Z\delta(\boldsymbol{x}-\boldsymbol{y})$$

which give $Z = 1$, but this is wrong since we assumed that $\hat{\phi}$ is interacting.

Notice that the above argument about the inconsistency between CCR and eq. (3.15) requires that both $x^0$ and $y^0$ are sent to $-\infty$ contemporaneily in eq. (3.16). Conversely it is still possible that $\hat{\phi}$ approaches $Z^{1/2}\hat{\phi}_{\text{in}}$ as $t \to -\infty$ in the weak sense $\hat{\phi} \xrightarrow[t \to -\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, i.e. in matrix elements: taken a sequence $A_n$ of operators the *weak limit*

$$A_n \xrightarrow[n \to +\infty]{} A$$

means that

$$|\langle \psi, (A_n - A)\phi \rangle| \xrightarrow[n \to +\infty]{} 0$$

Indeed taken $A_n \rightharpoonup A$ and $B_n \rightharpoonup B$ is not ensured that $A_n B_n \rightharpoonup AB$, therefore $\hat{\phi}(x) \xrightarrow[x^0 \to -\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$ and $\dot{\hat{\phi}}(y) \xrightarrow[y^0 \to -\infty]{} Z^{1/2}\dot{\hat{\phi}}_{\text{in}}(y)$ does not imply $[\hat{\phi}(x), \dot{\hat{\phi}}(y)] \xrightarrow[x^0,y^0 \to -\infty]{} Z[\hat{\phi}_{\text{in}}(x), \dot{\hat{\phi}}_{\text{in}}(y)]$ and eq. (3.16) does not hold anymore. Therefore we don't have any inconsistency between CCR and eq. (3.15) if $\hat{\phi}(x) \xrightarrow[x^0 \to -\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$.[VI]

Notice that LSZ formula uses precisely such weak limit for $t \to -\infty$, so there is no inconstistence with it.[VII]

The serious problem arises when in perturbative approach is used the Gell-Mann Low formula, which requires the existence of a unitary operator $U^I = \lim_{\varepsilon \to 0^+} U_\varepsilon^I$ interpolating between $\hat{\phi}$ and $\hat{\phi}_{\text{in}}$, but if $\hat{\phi}$ obeys CCR and translation invariance is recovered (i.e. we take the limit $\varepsilon \to 0^+$) we get from eq. (3.15) that such unitary operator does not exist (in order to get unitarity we should have $Z = 1$, which is forbitten provided that $\hat{\phi}$ is interacting).

## 3.4 The Haag's theorem

[EF05, Section 3]; [SW00, Section 4.5]; [Str13, Pages 39-40, 95–96]

The important argument from Haag[VIII] proves that the situation does not improves even if we do not require $\hat{\phi}$ to be canonical. Except very special non-relativistic cases, the Hilbert space of the free field is disjoint from the Hilbert space of the interacting field (i.e. don't exist any unitary operator which maps the states of one Hilbert spaces into the states of the other one).

**Theorem 3.1** (Haag, [EF05, Page 8]). *Assume that the unique translationally invariant state of an interacting scalar QFT with Hilbert space $\mathcal{H}$ and field $\hat{\phi} \in \mathcal{H}$ is the vacuum $|0\rangle \in \mathcal{H}$ [IX] (for lattice theories only discrete translations are considered). Let $U(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, be the unitary representation of translations in $\mathcal{H}$. Suppose that the Hamiltonian of the theory can be written as $H = H_0 + H_I$, with $H_0$ free Hamiltonian. Then there is no state $|0_F\rangle \in \mathcal{H}$ such that $H_0 |0_F\rangle = 0$ i.e. the vacuum of the free theory $|0_F\rangle$ is not an element of $\mathcal{H}$.*

*Proof.* By contradiction, assume that the Fock vacuum $|0_F\rangle$ of the free theory with Hamiltonian $H_0$ is an element of $\mathcal{H}$. Denote by $a_F(\boldsymbol{q})$ the annihilation operator of $H_0$, then

$$U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) U(\boldsymbol{x}) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} a_F(\boldsymbol{q})$$

---

[VI]More about this fact in [GR96, Section 9.2].

[VII]The derivation of the LSZ formula for the scalar theory can be found in [GR96, Section 9.4].

[VIII]Original paper: [Haa55].

[IX]It is reasonable to assume that the only translationally invariant state should be the vacuum, as the presence of any particle should broke translational invariance. This is ensured if the underlying free theory has a mass gap, i.e. all particles of our theory are massive. In general these assumptions does not hold if the free theory has massless particles. See [EF05, Section 3] for further comments about this fact.

therefore
$$0 = U^\dagger(\boldsymbol{x})a_F(\boldsymbol{q})\left|0_F\right\rangle = U^\dagger(\boldsymbol{x})a_F(\boldsymbol{q})U(\boldsymbol{x})U^\dagger(\boldsymbol{x})\left|0_F\right\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}a_F(\boldsymbol{q}))U^\dagger(\boldsymbol{x})\left|0_F\right\rangle$$

and this means that $)U^\dagger(\boldsymbol{x})\left|0_F\right\rangle$ annihilates $U(\boldsymbol{x})\left|0_F\right\rangle$, hence $\left|0_F\right\rangle$ is translationally invariant. By uniqueness of the translational invariant state $\left|0_F\right\rangle = c\left|0\right\rangle$ for some constan $c$ (actually if our states are normalized $c$ is just a phase). Then

$$0 = H\left|0\right\rangle = (H_0 + H_I)c\left|0_F\right\rangle = c\,H_I\left|0_F\right\rangle$$

which give us $H_I\left|0_F\right\rangle = 0$ and $H\left|0_F\right\rangle = 0$. But the typical Hamiltonians for interacting fields does not annihilate $\left|0_F\right\rangle$ (i.e. $H$ "polarizes the vacuum") since $H_I$ describes interactions of the field with itself.[X] Thus, we arrive at a contradiction: it follows from the announced assumptions that $H\left|0_F\right\rangle = 0$, but for typical interaction Hamiltonians $H\left|0_F\right\rangle \neq 0$. $\qquad\square$

The relative simplicity of this first version of Haag's theorem is purchased at the expense of generality since it appeals to the fact that typical Hamiltonians for interacting fields do not annihilate $\left|0_F\right\rangle$. It was the ambition of the second version of his theorem to dispense with any reference to the particular form of the interacting field Hamiltonian. However, Haag's attempted proof of such general version was found "inconclusive". The gap was filled by a pair of theorems presented by Hall and Wightman[XI], which together constitutes the so called *Haag-Hall-Weightman (HHW) theorem*, and apply to any pair to neutral scalar field.

**Theorem 3.2** (HHW, Part 1, [HW57, Section 3, Thm. 1], [SW00, Thm. 4.14]). *Consider two QFTs with fields $\hat{\phi}_1$, $\hat{\phi}_2$ and Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and assume that*

*(i) the Euclidean group $\mathbb{E}_d$ (roto-traslations) is represented unitarily by $U_1$ and $U_2$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively:*
$$U_j(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_j(\boldsymbol{x},t)U_j^\dagger(\boldsymbol{a},\boldsymbol{R}) = \hat{\phi}_j(\boldsymbol{R}\,\boldsymbol{x}+\boldsymbol{a},t)$$

*(ii) the representation of the fields is irreducible, i.e. if an operator $\hat{A}$ defined in $\mathcal{H}_j$ commutes with all the fields $\{\hat{\phi}_j\} \subset \mathcal{H}_j$ then it is a multiple of the identity, $\hat{A} = c\,\mathbb{1}$;*

*(iii) the fields are related at some time $t$ by a unitary transformation $V(t)$*
$$\hat{\phi}_2(\boldsymbol{x},t) = V(t)\hat{\phi}_1(\boldsymbol{x},t)V^\dagger(t)$$

*Then*
$$U_2(\boldsymbol{a},\boldsymbol{R}) = V(t)U_1(\boldsymbol{a},\boldsymbol{R})V^\dagger(t) \tag{3.17}$$
*Further, if there are unique states $\left|0_j\right\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a},\boldsymbol{R})\left|0_j\right\rangle = \left|0_j\right\rangle$, then*
$$V(t)\left|0_1\right\rangle = c\left|0_2\right\rangle \quad with \quad c \in \mathbb{C} \tag{3.18}$$

*In particular if $\left|0_j\right\rangle$, $j = 1, 2$ are normalized to 1 then $|c| = 1$.*

Notice that this part of the theorem holds also in the non-relativistic case. The operator $V$ is meant to be the unitary operator interpolating between the free and the interacting fields in the Gell-Mann Low formula.

---

[X]As an illustration, consider the $\hat{\phi}^4$ scalar field in $\mathbb{R}^{3+1}$ with (formal) Hamiltonian
$$H = H_0 + \lambda \int \mathrm{d}^3 x \; : \hat{\phi}^4 : - C$$
where $C$ is a $c$-number constant chosen to give to the ground state energy zero, and $:\;:$ indicates the Wick normal product. This $H$ will not annihilate the bare vacuum $\left|0_F\right\rangle$ since the factor that follows the coupling constat $\lambda$ contains a term with a product of four creation operator $\sim (a^\dagger)^4$ (and there is only one term of this form, so it will not be canceled by another term).

[XI]Original paper: [HW57]

*Proof.* Consider the operator $U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t)$, this commutes with all $\hat{\phi}_1 \in \mathcal{H}_1$, indeed:

$$\begin{aligned}
U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})&V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x}, t)V^\dagger(t)U_2^\dagger(\boldsymbol{a}, \boldsymbol{R})V(t)U_1(\boldsymbol{a}, \boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})\hat{\phi}_2(\boldsymbol{x}, t)U_2^\dagger(\boldsymbol{a}, \boldsymbol{R})V(t)U_1(\boldsymbol{a}, \boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)\hat{\phi}_2(\boldsymbol{R}\boldsymbol{x} + \boldsymbol{a}, t)V(t)U_1(\boldsymbol{a}, \boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})\hat{\phi}_1(\boldsymbol{R}\boldsymbol{x} + \boldsymbol{a}, t)U_1(\boldsymbol{a}, \boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})U_1(\boldsymbol{a}, \boldsymbol{R})\hat{\phi}_1(\boldsymbol{x}, t)U_1(\boldsymbol{a}, \boldsymbol{R})U_1^\dagger(\boldsymbol{a}, \boldsymbol{R}) = \\
&= \hat{\phi}_1(\boldsymbol{x}, t)
\end{aligned}$$

which implies

$$U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x}, t) = \hat{\phi}_1(\boldsymbol{x}, t)U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t)$$

Due to the irreducibility of the representation we get that

$$U_1^\dagger(\boldsymbol{a}, \boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t) = \omega(\boldsymbol{a}, \boldsymbol{R})\mathbb{1}$$

for some complex function $\omega(\boldsymbol{a}, \boldsymbol{R})$ which depend on the element $(\boldsymbol{a}, \boldsymbol{R}) \in \mathbb{E}_d$ and then

$$\omega(\boldsymbol{a}, \boldsymbol{R})U_1(\boldsymbol{a}, \boldsymbol{R}) = V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t) \tag{3.19}$$

Notice that in order to preserve unitarity we must have $|\omega(\boldsymbol{a}, \boldsymbol{R})| = 1$ for all $(\boldsymbol{a}, \boldsymbol{R}) \in \mathbb{E}_d$. By definition $\omega(\boldsymbol{a}, \boldsymbol{R})$ provides a continuous one-dimensional representation (on $\mathbb{C}$) of a non-compact group ($\mathbb{E}_d$), and it is a general mathematical result that there is no such a representation appart from the trivial one; then, one may take $\omega(\boldsymbol{a}, \boldsymbol{R}) = 1$. If there are unique states $|0_j\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a}, \boldsymbol{R})|0_j\rangle = |0_j\rangle$, then from $U_1(\boldsymbol{a}, \boldsymbol{R})|0_1\rangle = |0_1\rangle$ one gets using eq. (3.19)

$$V^\dagger(t)U_2(\boldsymbol{a}, \boldsymbol{R})V(t)|0_1\rangle = |0_1\rangle$$

and

$$U_2(\boldsymbol{a}, \boldsymbol{R})V(t)|0_1\rangle = V(t)|0_1\rangle$$

but the only state in $\mathcal{H}_2$ invariant under $U_2(\boldsymbol{a}, \boldsymbol{R})$ is $|0_2\rangle$, therefore we must have

$$V(t)|0_1\rangle = c|0_2\rangle$$

for some constant $c \in \mathbb{C}$, which should be a phase, $|c| = 1$, if $|0_1\rangle$ and $|0_2\rangle$ are normalized to 1. $\square$

**Corollary 3.2.1.** *In any two theories satisfying the hypothesis of theorem 3.2 the equal time correlation functions of $\hat{\phi}_1$ and $\hat{\phi}_2$ coincides*

$$\langle 0_1|\hat{\phi}_1(\boldsymbol{x}_1, t)\ldots\hat{\phi}_1(\boldsymbol{x}_n, t)|0_1\rangle = \langle 0_2|\hat{\phi}_2(\boldsymbol{x}_1, t)\ldots\hat{\phi}_2(\boldsymbol{x}_n, t)|0_2\rangle$$

*Proof.* The statement directly follows from eq. (3.17) and eq. (3.18). $\square$

In principle an operator $V$ satisfying the hypothesis of Gell-Mann Low formula can still exists, and corollary 3.2.1 still admits different dynamics for $\hat{\phi}_1$ and $\hat{\phi}_2$ since it applies only for equal time correlators. As we consider correlators at different time one should use time evolution operators involving the Hamiltonian, and no useful conclusion can be stated at this point if there is no vacuum polarization. We need one more result, which however works only in the relativistic case:

**Theorem 3.3** (HHW, Part 2, [HW57, Section 3, Thm. 2], [SW00, Thm. 4.16])**.** *Under the hypothesis of theorem 3.2 assume further that the (covering of the) restricted Poincaré group $\mathcal{P}_+^\uparrow$ is represented unitarily by $U_1$ and $U_2$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, i.e.*

$$U_j(\Lambda, a)\hat{\phi}_j(x)U_j^\dagger(\Lambda, a) = \hat{\phi}_j(\Lambda x + a)$$

*Then*

$$\langle 0_1|\hat{\phi}_1(x)\hat{\phi}_1(y)|0_1\rangle = \langle 0_2|\hat{\phi}_2(x)\hat{\phi}_2(y)|0_2\rangle$$

*and if $\hat{\phi}_1$ is a free field of mass $m > 0$, then $\hat{\phi}_2$ is a free field of mass $m > 0$ too.*

*Proof.* The proof is much more involved than the previous one, since requires a preliminary result of R. Jost and B. Schroer, which can be found in [SW00, Thm. 4.15]. We just sketch the argument for the proof: by Poincaré covariance through a boost transformation we can bring any two points in the space-time to equal time, in such a way that corollary 3.2.1 applies. Then the correlator we obtained can be extended to the initial points by analiticity, preserving the value of $Z$ and the position of the pole in the Källen-Lehmann representation. This is enough to prove the statement of the theorem. $\qquad\square$

The immediate consequence of the latter is that, if the hypothesis holds, any interacting theory is actually free. The conclusion that all the fields are free is unacceptable because the intention is to represent an interacting field, therefore at least one of the assumptions of the theorem must be dropped. A n obvious candidate is the assumption that at some $t$ there exists a unitary transformation $V(t)$ relating the fields. Therefore the Heisenberg dynamics of the interacting field do not exists in the Hilbert space of the free field, but requires a different Hilbert space, inequivalent to the free one.

The second part of the HHW theorem applies only to relativistic QFT, nevertheless the unitary operator $U^I$ interpolating between free and interacting fields does not exists also in most of the non-relativistic QFT, except for very special cases. Therefore almost always the perturbative approach is not consistent.

## 3.5    Ultraviolet singularities and canonical quantization

[Str13, Section 2.4]

It is worth to notice that for a RQFT such that $Z = 0$ (recall that $Z$ is the constant connecting the renormalized and the free field, $\phi \xrightarrow[t\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, $\hat{\phi}_{\text{ren}} := Z^{-1/2}\hat{\phi} \xrightarrow[t\to-\infty]{} \hat{\phi}_{\text{in}}$) then in perturbation theory one usually gets

$$Z = 1 - \lambda\infty \sim \frac{1}{1 + \lambda\infty} = 0$$

where $\infty$ simply denotes some divergent quantity which has been removed through the renormalization procedure. Then at some fixed time the field itself does not exist as an operator. In order to have a well defined operator, one needs to smear $\hat{\phi}(x)$ not only in space

$$\hat{\phi}(x^0, f) = \int \mathrm{d}^3x\, f(\boldsymbol{x})\hat{\phi}(x^0, \boldsymbol{x})$$

but also in time

$$\hat{\phi}(g) = \int \mathrm{d}^4x\, g(x)\hat{\phi}(x)$$

Indeed suppose that $\hat{\phi}$ is the Heisenberg interacting field, and suppose that it is well defined at fixed time $x^0$, then according to Källen-Lehmann representation (recall eq. (3.10) ) we get

$$\langle 0|\,[\hat{\phi}(x^0, \boldsymbol{x}), \dot{\hat{\phi}}(x^0, \boldsymbol{y})]\,|0\rangle = i\delta(\boldsymbol{x} - \boldsymbol{y})$$

because and interacting 2-points function can be written in terms of free 2-points functions of different masses satisfying CCR. Therefore the "renormalized" field $\hat{\phi}_{\text{ren}} = Z^{-1/2}\hat{\phi}$ which approaches weakly the $\hat{\phi}_{\text{in}}$ as $t \to -\infty$, satisfies

$$\langle 0|\,[\hat{\phi}_{\text{ren}}(x^0, \boldsymbol{x}), \dot{\hat{\phi}}_{\text{ren}}(x^0, \boldsymbol{y})]\,|0\rangle = \frac{i}{Z}\delta(\boldsymbol{x} - \boldsymbol{y})$$

Clearly if $Z = 0$ this expression is ill-defined, but no assumption has been made except that $\hat{\phi}$ is well defined at fixed time, hence if $Z = 0$ this assumption should be wrong, i.e. the renormalized field cannot be defined at fixed time. A Schrödinger picture requires that we have a definite space at a fixed time, hence we also proved that it is impossible to defined a picture of dynamics using Schrödinger idea if $Z = 0$.

# Chapter 4

# The reconstruction theory

Beside the literature present at the beginning of each section, have a look at [Kaz99] for this chapter and the following one.

## 4.1 How to avoid Haag's theorem

[Str13, Section 2.5]

Let's see how to avoid problems raised by Haag theorem. The idea is to start by introducing IR (volume) and UV cutoffs (the IR one is not needed for lattice models) so that the cutoff model describes a system of finite degrees of freedom, such that the interaction picture is still correct, thanks to von Neumann uniqueness theorem, which guarantees that free and interacting fields are unitarily equivalent and defined in the same Hilbert space. Moreover one can also require that free fields (and then also interacting ones due to Källen-Lehmann representation) satisfy CCR (or CAR) since no problem arises in this framework. Consider a $T = 0$ RQFT (the restriction $T = 0$ is not really needed, but simplify the traction) and let $\hat{\phi}_0(\boldsymbol{x}) := \hat{\phi}(x^0 = 0, \boldsymbol{x})$ be the $t = 0$ field, then the cutoff field at arbitrary time is given by

$$\hat{\phi}_\#^\Lambda(x^0, \boldsymbol{x}) = e^{ix^0 H_\#} \hat{\phi}_0(\boldsymbol{x}) e^{-ix^0 H_\#}$$

where $\#$ denotes either the free of the interacting theory and $\Lambda$ denotes the cutoffs. The previous time evolution is ensured by the von Newman theorem, assuming that the free and the interacting fields coincides at the initial time $t = 0$. Let $|0_I^\Lambda\rangle$ be the vacuum of the interacting Hamiltonian, which is still in the Fock space of the free field.

We construct the following functions (actually they are distributions, but with abuse of terminology, we call "functions" also these expectation values)

$$W_n^\Lambda(x_1, \ldots, x_n) := \langle 0_I^\Lambda | \hat{\phi}_I^\Lambda(x_1) \cdots \hat{\phi}_I^\Lambda(x_n) | 0_I^\Lambda \rangle$$

We stress the fact that since translational invariance (and thus also Poincaré invariance) is broken, Haag's theorem does not apply.

Then we identify the necessary counterterms to be added to the Hamiltonian (or to the Lagrangian if the path integral approach is used) and the field renormalization $Z_\Lambda$ (called *wave-function renormalization constant*) leading to "*renormalized fields*", for which the correlation functions have a well-defined limit when IR an UV cutoffs are removed from outside[I] of the correlation function, i.e. such that the following limit

$$\lim_{\substack{\Lambda_{\mathrm{UV}} \to \infty \\ \Lambda_{\mathrm{IR}} \to 0}} Z_{\Lambda_{\mathrm{UV}}}^{-n} \langle 0_I^\Lambda | \hat{\phi}_I^\Lambda(x_1) \cdots \hat{\phi}_I^\Lambda(x_n) | 0_I^\Lambda \rangle$$

exists in a suitable space of functions or distributions, e.g. $\mathcal{S}'(\mathbb{R}^d)$. This is possible since the previous limit is a functions limit, hence much weaker than the operator limit (for the fields) which cannot exists due to Haag's theorem.

---

[I]It is important to notice that the limit should be removed from outside, otherwise Haag's theorem apply and we are stuck.

The crucial fact is that if the limiting functions satisfy certain properties we can reconstruct quantum fields acting in an appropriate Hilbert space, such that their expectation values are precisely the functions we started from. The reconstructed field $\hat{\phi}$ now has no reasons to be equivalent to a free field, nor the Hilbert space of $\hat{\phi}$ has reason to be the same Fock space we started from (actually in general it could even not be a Fock space), hence the reconstruction is not in conflict with Haag's argument.

## 4.2 Wightman's reconstruction theorem

[SW00, Chapter 3]; [Jos65, Chapter 3]; [Str13, Chapter 3]; [Str93, Sections 1.3-1.4]

As anticipated, provided some properties for the limiting functions

$$W_n(x_1, \ldots, x_n) := \lim_{\substack{\Lambda_{\mathrm{UV}} \to \infty \\ \Lambda_{\mathrm{IR}} \to 0}} Z_{\Lambda_{\mathrm{UV}}}^{-n} W_n^{\Lambda}(x_1, \ldots, x_n)$$

these functions themselves determine states and fields of the interacting theory, avoiding Haag's theorem. In this section we first describe how this reconstruction works for a RQFT at $T = 0$, then we'll sketch how to adapt the formalism to more general situations.

We give the statement and the complete proof of the *reconstruction theorem* for a real scalar field at $T = 0$ for a relativistic theory.

**Theorem 4.1** (Wightman's reconstruction theorem, [Wig56], [SW00, Thm. 3.7], [Jos65, Section 3.4, Thm. 1]). *Let $\{W_n\}_{n=0}^{\infty}$ be a sequence of (tempered) distributions, $W_n \in \mathcal{S}'(\mathbb{R}^{(d+1)n})$ for all $n > 0$, $W_0 := 1$. Suppose that they satisfy the following properties:*

*(1)* (*Positive Definiteness*) *For any finite sequence of $N$ test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1, x_2), \ldots)$, $f_n \in \mathcal{S}(\mathbb{R}^{(d+1)n})$, then*

$$\sum_{j,k=0}^{N} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, f_n^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) f_k(y_1, \ldots, y_k) \geq 0$$

*or in more compact notation $(\underline{f}, W\underline{f}) \geq 0$.*[II]

*(2)* (*Poincaré covariance*) *For any $(a, \Lambda) \in \mathcal{P}_+^{\uparrow}$, $\Lambda \in L_+^{\uparrow}$ and $a \in \mathbb{R}^{d+1}$,*

$$W_n(x_1, \ldots, x_n) = W_n(\Lambda x_1 + a, \ldots, \Lambda x_n + a)$$

*for all $n \in \mathbb{N}$.*

*(3)* (*Spectral condition*) *For each $W_n$, $n > 0$, exists $\widetilde{W}_n$ such that*

$$W_n(x_1, \ldots, x_n) = \int \mathrm{d}p_1 \ldots \mathrm{d}p_n \, \widetilde{W}_{n-1}(p_1, \ldots, p_{n-1}) \, e^{i(x_n - x_{n-1})p_{n-1}} \ldots e^{i(x_2 - x_1)p_1} \tag{4.1}$$

*with $\widetilde{W}_{n-1}(p_1, \ldots, p_{n-1}) = 0$ if $p_j \notin \overline{V}^+$ for some $j$, $1 \leq j \leq n-1$.*

*(4)* (*Local commutativity condition / Symmetry*) *If $x_i$ and $x_{i+1}$ are space-like separated for some $i$, that is $(x_i - x_{i+1})^2 < 0$, then for all $n > i$*

$$W_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = W_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

*(5)* (*Cluster property*) *For any $n \geq 2$, $j < n$,*

$$\lim_{\substack{a \to \infty \\ a^2 < 0}} W_n(x_1, \ldots x_j, x_{j+1} + a, \ldots, x_n + a) = W_j(x_1, \ldots, x_j) W_{n-j}(x_{j+1}, \ldots, x_n)$$

*Then exists*

---

[II]Notice that the proof of this property is not easy, since one should also prove the existence of the limit $W_n^{\Lambda} \to W_n$.

*(1')* a separable Hilbert space $\mathcal{H}$,

*(2')* a continuous unitary representation $U(a, \Lambda)$ of $\mathcal{P}_+^\uparrow$ [III] in $\mathcal{H}$,

*(3')* a state $\Omega \in \mathcal{H}$, translationally invariant, called vacuum,

*(4')* an operator-valued distribution field $\hat{\phi}(x)$ such that, given a test function $f$, $\hat{\phi}(f) := \int \mathrm{d}x\, f(x)\hat{\phi}(x)$ is an operator with domain $D$, $D$ dense in $\mathcal{H}$ and $|\Omega\rangle \in D$, and denoting by $\mathcal{P}(\hat{\phi}(f))$ the set of polynomials of $\hat{\phi}(f)$[IV], the subspace of $\mathcal{H}$ generated applying polynomials of the smeared fields to the vacuum, $\mathcal{P}(\hat{\phi}(f))|\Omega\rangle$, is dense in $\mathcal{H}$

and the following properties hold:

*(5')* the field $\hat{\phi}(x)$ transforms covariantly under $\mathcal{P}_+^\uparrow$:

$$U(a, \Lambda)\hat{\phi}(x)U^\dagger(a, \Lambda) = \hat{\phi}(\Lambda x + a)$$

*(6')* let $\sigma(\hat{P}^\mu)$ be the spectrum of the generator $\hat{P}^\mu$ of the subgroup of translations, then $\sigma(\hat{P}^\mu) \subseteq \overline{V}^+$, i.e. $\sigma(\hat{P}^\mu \hat{P}_\mu) \geq 0$ and $\sigma(\hat{P}^0) \geq 0$;

*(7')* (*Locality*) fields in space-like separated regions commute

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad if \quad (x-y)^2 < 0$$

or more precisely, in terms of smeared fields $f, g$,

$$[\hat{\phi}(f), \hat{\phi}(g)] = 0 \quad if \quad \mathrm{supp}\, f \,\mathsf{X}\, \mathrm{supp}\, g$$

where $\mathsf{X}$ denotes that the two regions are space-like separated,

*(8')* (*Uniqueness of the vacuum*) The vector $\Omega$ is the unique vector in $\mathcal{H}$ translationally invariant;

*(9')* for all $n \in \mathbb{N}$[V]

$$(\Omega, \hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\Omega) = W_n(x_1, \ldots, x_n) \tag{4.2}$$

*Proof.* (*sketch*)

*(1')* Consider the vector space $\underline{\mathcal{S}}$ of sequences of test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1, x_2), \ldots)$, $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{(d+1)k})$, with finite number of non-vanishing elements. Addition and multiplication by complex scalars are definite by

$$(f_0, f_1, \ldots) + (g_0, g_1, \ldots) = (f_0 + g_0, f_1 + g_1, \ldots)$$
$$\alpha(f_0, f_1, \ldots) = (\alpha f_1, \alpha f_2, \ldots)$$

Next we introduce a scalar product defined on pairs of vectors of the vector space

$$(\underline{f}, \underline{g}) := \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) g_k(y_1, \ldots, y_k) \tag{4.3}$$

or using the previous compact notation $(\underline{f}, \underline{g}) := (\underline{f}, W\underline{g})$. Recall $W_0 = 1$ by definition. The linearity in $\underline{g}$ and anti-linearity in $\underline{f}$ are evident from the definition, furthermore from *(1)* we get that the norm $\|\underline{f}\|^2 = (\underline{f}, \underline{f})$ is semi-definite positive, i.e. $\|\underline{f}\| \geq 0$ for any $\underline{f}$. Therefore the operation $(-, -)$ is a well defined inner product in $\underline{\mathcal{S}}$, but $\underline{\mathcal{S}}$ is not a pre-Hilbert space yet since it may contain vectors of zero norm. In order to get a pre-Hilbert, note that the set

$$\underline{\mathcal{S}}_0 := \{\underline{f} \in \underline{\mathcal{S}} \text{ s.t. } \|\underline{f}\| = 0\} \subseteq \underline{\mathcal{S}}$$

---

[III] In the case of fields with half-integer spin $\mathcal{P}_+^\uparrow$ is replaced by $\tilde{\mathcal{P}}_+^\uparrow$.

[IV] It is important to notice that $\mathcal{P}(\hat{\phi}(f))$ denotes polynomials of fields smeared (in general) with different test functions, not only one test function $f$, for instance $\hat{\phi}(f_1)\hat{\phi}(f_2)\ldots\hat{\phi}(f_n) \in \mathcal{P}(\hat{\phi}(f))$.

[V] Usually in the axiomatic approach due to Wightman the Dirac notation is avoided, and the scalar product is denoted by $(-, -)$.

is an isotropic subspace of $\underline{\mathcal{S}}$, that is, a subspace in which each vector is orthogonal to every other vector, indeed

$$|(\underline{f},\underline{g})| \le \|\underline{f}\|\,\|\underline{g}\| = 0$$

by the Schwartz inequality (which is valid as long as the scalar product is non-negative). Thus, if $\underline{f}$ and $\underline{g}$ are of zero norm, then $\underline{f}$ is orthogonal to $\underline{g}$ and $\alpha\underline{f} + \beta\underline{g}$ is of zero norm. We now form equivalence classes of sequences $\underline{f}$, two sequences being equivalent if they differ by a sequence of zero norm. These equivalence classes form in a natural way a vector space, denoted by $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, on which the scalar product induced by $\underline{\mathcal{S}}$ is well-defined and positive definite, since $\|[\,\underline{f}\,]\| = 0$ implies $[\,\underline{f}\,] = [\,\underline{0}\,]$. Therefore $\underline{\mathcal{S}}$ is a pre-Hilbert space. A generic element of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ is of the form

$$\underline{\mathcal{S}}/\underline{\mathcal{S}_0} \ni [\,\underline{f}\,] = \{\underline{g} \in \underline{\mathcal{S}} \mid \underline{f} - \underline{g} \in \underline{\mathcal{S}_0}\}$$

Then we can finally define the Hilbert space $\mathcal{H}$ by completion[VI] of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ respect to the induced norm:

$$\mathcal{H} := \overline{\underline{\mathcal{S}}/\underline{\mathcal{S}_0}}$$

that is, let $\mathfrak{h}$ be the space of Cauchy sequences $F = \{[\,\underline{f_1}\,],[\,\underline{f_2}\,],\dots\}$ with scalar product

$$(F,G) := \lim_{n\to\infty}([\,\underline{f_n}\,],[\,\underline{g_n}\,]) \tag{4.4}$$

and $\mathfrak{h}_0$ its subspace given by vectors of zero norm, then $\mathcal{H} := \mathfrak{h}/\mathfrak{h}_0$. In the following we'll denote by $([\,\underline{f_1}\,],[\,\underline{f_2}\,],\dots)$ the elements of $\mathcal{H}$ (i.e. as elements of $\mathfrak{h}$) rather then $[([\,\underline{f_1}\,],[\,\underline{f_2}\,],\dots)]$, we give as understood the fact that we refer to the associated equivalence class.
Being $\mathcal{S}$ separable so is $\underline{\mathcal{S}}$, and since the latter is dense in $\mathcal{H}$ also $\mathcal{H}$ is separable.

*(2')* We define the linear transformation $U(a,\Lambda)$ in $\underline{\mathcal{S}}$ by

$$U(a,\Lambda)(f_0, f_1, f_2, \dots) := (f_0, \{a,\Lambda\}f_1, \{a,\Lambda\}f_2, \dots)$$

where

$$\{a,\Lambda\}f_k(x_1,\dots,x_k) := f_k(\Lambda^{-1}(x_1 - a),\dots,\Lambda^{-1}(x_k - a))$$

The operator $U(a,\Lambda)$ leaves the scalar product of $\underline{\mathcal{S}}$ invariant by virtue of *(2)*:

$$(U(a,\Lambda)\underline{f}, U(a,\Lambda)\underline{g}) =$$
$$= \sum_{j,k=0}^{\infty} \int dx_1 \dots dx_j dy_1 \dots dy_k\, f_j^*(\Lambda^{-1}(x_1 - a),\dots,\Lambda^{-1}(x_j - a)) \times$$
$$\times W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k) g_k(\Lambda^{-1}(y_1 - a),\dots,\Lambda^{-1}(y_k - a))$$
$$= \sum_{j,k=0}^{\infty} \int dx_1 \dots dx_j dy_1 \dots dy_k\, f_j^*(x_1,\dots,x_j) W_{j+k}(\Lambda x_1 + a,\dots,\Lambda y_k + a) g_k(y_1,\dots,y_k)$$
$$\overset{(2)}{=} \sum_{j,k=0}^{\infty} \int dx_1 \dots dx_j dy_1 \dots dy_k\, f_j^*(x_1,\dots,x_j) W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k) g_k(y_1,\dots,y_k)$$
$$= (\underline{f},\underline{g})$$

thus $U(a,\Lambda)$ leaves invariant also $\underline{\mathcal{S}_0}$, since $\|\underline{f}\| = 0$ implies $\|U(a,\Lambda)\underline{f}\| = 0$. We have to check that $U(a,\Lambda)$ is actually a mapping of equivalence classes in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. But due to the invariance of the scalar product if $[\,\underline{f}\,] = [\,\underline{g}\,]$ then $[U(a,\Lambda)\underline{f}] = [U(a,\Lambda)\underline{g}]$, hence $\overline{U}(a,\Lambda)$ is well defined also in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. Being bounded it extends by continuity to $\mathcal{H}$, preserving the scalar product in $\mathcal{H}$. Recall that by Wigner's theorem linear operators that preserves the scalar products in a Hilbert space are unitary, then $U(a,\Lambda)$ is a unitary representation of $\mathcal{P}_+^{\uparrow}$ on $\mathcal{H}$.

*(3')* Notice that

$$U(a,\Lambda)(1,0,\dots,0,\dots) = (1,0,\dots,0,\dots) =: \mathbb{1}$$

---

[VI]The completion procedure is described in [SW00, Pages 121-122].

hence $\underline{1} \in \underline{\mathcal{S}}$ is translational invariant in $\underline{\mathcal{S}}$ and the same holds for $[\underline{1}]$ in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. Finally, the vacuum in $\mathcal{H}$ is given by the Cauchy sequence $\Omega := \overline{([\underline{1}], [\underline{1}], \ldots)}$.

*(4')* We introduce in $\underline{\mathcal{S}}$ a linear operator $\hat{\phi}(h)$ for each test function $h \in \mathcal{S}(\mathbb{R}^{d+1})$ by the equation

$$\hat{\phi}(h)\underline{f} := (0, hf_0, h \otimes f_1, h \otimes f_2, \ldots)$$

or in compact notation $\hat{\phi}(h)\underline{f} := h \times \underline{f}$, where

$$(h \otimes f_k)(x_1, \ldots, x_{k+1}) := h(x_1)f_k(x_2, x_3, \ldots, x_{k+1})$$

is clearly a test function. As functionals of $h$ the matrix elements $(\underline{f}, \hat{\phi}(h)\underline{g})$ are tempered distributions since they are finite sums of $W$'s, and further

$$(\underline{f}, \hat{\phi}(h)\underline{g}) = (\hat{\phi}(h^*)\underline{f}, \underline{g}) \tag{4.5}$$

We have to check that $\hat{\phi}(h)$ is a mapping of equivalence classes in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. That $\|\underline{f}\| = 0$ implies $\|\hat{\phi}(h)\underline{f}\| = 0$ follows from previous definition and Schwartz inequality:

$$(\hat{\phi}(h)\underline{f}, \hat{\phi}(h)\underline{f}) = (\underline{f}, \hat{\phi}(h^*)\hat{\phi}(h), \underline{f}) \leq \|\underline{f}\| \|\hat{\phi}(h^*)\hat{\phi}(h)\underline{f}\| = 0$$

if $\|\underline{f}\| = 0$. Therefore $\hat{\phi}(h)$ is well-defined also in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, $\hat{\phi}(h)[\underline{f}] := [\hat{\phi}(h)\underline{f}]$. For each $[\underline{f}] \in \underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, $\hat{\phi}(h)$ is well-defined also for the associated vectors in $\mathcal{H}$, $\hat{\phi}(h)([\underline{f}], [\underline{f}], \ldots) := (\hat{\phi}(h)[\underline{f}], \hat{\phi}(h)[\underline{f}], \ldots)$, hence $\hat{\phi}$ is defined in a dense subset $D$ of $\mathcal{H}$, $D \cong \underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. In particular, being $|\Omega\rangle = ([\underline{1}], [\underline{1}], \ldots) \in D$, this defines $\hat{\phi}(h)$ on $\mathcal{P}(\hat{\phi}(h)) |\Omega\rangle$. Notice that

$$\hat{\phi}(h_1) \cdots \hat{\phi}(h_n)\underline{1} = \hat{\phi}(h_1) \cdots \hat{\phi}(h_{n-1})(0, h_n, 0, 0, \ldots) = (\underbrace{0, \ldots, 0}_{n}, h_1 \otimes h_2 \otimes \ldots \otimes h_n, 0, 0, \ldots) \tag{4.6}$$

hence $\mathcal{P}(\hat{\phi}(h))\underline{1} \cong \underline{\mathcal{S}}$ and then $\mathcal{P}(\hat{\phi}(h)) |\Omega\rangle \cong D$ is dense in $\mathcal{H}$.

*(9')* Notice that the vacuum expectation value in $\mathcal{H}$ reads

$$\begin{aligned}
(\Omega, \hat{\phi}(h_1) \ldots \hat{\phi}(h_n)\Omega) &= ([\underline{1}], \hat{\phi}(h_1) \ldots \hat{\phi}(h_n)[\underline{1}]) && \text{scalar product in } \underline{\mathcal{S}}/\underline{\mathcal{S}_0} \\
&= (\underline{1}, \hat{\phi}(h_1) \ldots \hat{\phi}(h_n)\underline{1}) && \text{scalar product in } \underline{\mathcal{S}} \\
&= \int dx_1 \ldots dx_n \, W_n(x_1, \ldots, x_n)h_1(x_1) \ldots h_n(x_n) \\
&= W_n(h_1, \ldots, h_n)
\end{aligned}$$

where in the first step we computed the scalar product in $\mathcal{H}$ using eq. (4.4), in the second step we chose the representative $\underline{1}$ for the equivalence class $[\underline{1}]$ and then we computed the scalar product in $\underline{\mathcal{S}}$ according to the definition eq. (4.3) using eq. (4.6) to compute $\hat{\phi}(h_1) \cdots \hat{\phi}(h_n)\underline{1}$. Then we noted that the result we obtained coincide with the smearing of the Wightman function. Finally eq. (4.2) follows immediately from $\langle\Omega| \hat{\phi}(h_1) \ldots \hat{\phi}(h_n) |\Omega\rangle = W_n(h_1, \ldots, h_n)$.

*(5')* That $\hat{\phi}(h)$ satisfies the transformation law

$$U(a, \Lambda)\hat{\phi}(h)U^\dagger(a, \Lambda) = \hat{\phi}(\{a, \Lambda\}h)$$

can be easy proved in $\underline{\mathcal{S}}$:

$$\begin{aligned}
U(a, \Lambda)\hat{\phi}(h)(f_0, f_1, \ldots) &= U(a, \Lambda)(0, hf_0, h \otimes f_1, \ldots) \\
&= (0, \{a, \Lambda\}hf_0, \{a, \Lambda\}h \otimes \{a, \Lambda\}f_1, \ldots) \\
&= \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda)(f_0, f_1, \ldots)
\end{aligned}$$

then since the previous computation holds for each $\underline{f} \in \underline{\mathcal{S}}$ and both $\hat{\phi}(h)$ and $U(a, \Lambda)$ are mapping of equivalence classes then

$$U(a, \Lambda)\hat{\phi}(h)[\underline{f}] = \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda)[\underline{f}]$$

and finally we can extend it by continuity to $\mathcal{H}$, since $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ is dense in it:

$$U(a, \Lambda)\hat{\phi}(h)([\underline{f_1}], [\underline{f_2}], \dots) = \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda)([\underline{f_1}], [\underline{f_2}], \dots)$$

Since this holds for any element of $\mathcal{H}$ we get

$$U(a, \Lambda)\hat{\phi}(h) = \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda) \quad \Rightarrow \quad \hat{\phi}(\{a, \Lambda\}h) = U(a, \Lambda)\hat{\phi}(h)U^{-1}(a, \Lambda)$$

and we are done, since "unsmearing" the fields we get

$$\int \mathrm{d}x\, h(\Lambda^{-1}(x - a))\hat{\phi}(x) = U(a, \Lambda) \int \mathrm{d}x\, h(x)\hat{\phi}(x)U^{-1}(a, \Lambda)$$

and by a change of variable on the r.h.s.:

$$\int \mathrm{d}x\, h(x)\hat{\phi}(\Lambda x + a) = \int \mathrm{d}x\, h(x)U(a, \Lambda)\hat{\phi}(x)U^{-1}(a, \Lambda)$$

*(6')* Consider the unitary group of translations $\{U(a)\} := \{U(a, \mathbb{1})\}$. Stone's theorem[VII] apply to such operators (the group can be decomposed in $(d+1)$ one-parameter unitary groups), then exists a unique operator $\hat{P} : D_P \to \mathcal{H}$, self-adjoint in the dense domain $D_P$, such that $U(a) = e^{ia\hat{P}}$ for any $a \in \mathbb{R}^{d+1}$. Moreover thanks to *(5')* we have

$$\hat{\phi}(x) = e^{ix\hat{P}}\hat{\phi}(0)e^{-ix\hat{P}} \tag{4.7}$$

Then, using *(9')* together with *(3')* we have

$$
\begin{aligned}
W_n(x_1, \dots, x_n) &= \\
&= \langle\Omega|\, \hat{\phi}(x_1) \dots \hat{\phi}(x_n)\, |\Omega\rangle \\
&= \langle\Omega|\, \hat{\phi}(0)e^{i\hat{P}(x_2 - x_1)}\hat{\phi}(0) \dots \hat{\phi}(0)e^{i\hat{P}(x_n - x_{n-1})}\hat{\phi}(0)\, |\Omega\rangle \\
&= \int \mathrm{d}p_1 \dots \mathrm{d}p_{n-1}\, \langle\Omega|\, \hat{\phi}(0)e^{i\hat{P}(x_2 - x_1)}\, |p_1\rangle\langle p_1|\, \hat{\phi}(0) \dots \hat{\phi}(0)e^{i\hat{P}(x_n - x_{n-1})}\, |p_{n-1}\rangle\langle p_{n-1}|\, \hat{\phi}(0)\, |\Omega\rangle \\
&= \int \mathrm{d}p_1 \dots \mathrm{d}p_{n-1}\, e^{ip_1(x_2 - x_1)} \dots e^{ip_{n-1}(x_n - x_{n-1})}\langle\Omega|\, \hat{\phi}(0)\, |p_1\rangle\langle p_1|\, \hat{\phi}(0)\, |p_2\rangle \dots \langle p_{n-1}|\, \hat{\phi}(0)\, |\Omega\rangle
\end{aligned}
$$

Comparing this expression with eq. (4.1) we get

$$\widetilde{W}_{n-1}(p_1, \dots, p_{n-1}) = \langle\Omega|\, \hat{\phi}(0)\, |p_1\rangle\langle p_1|\, \hat{\phi}(0)\, |p_2\rangle \dots \langle p_{n-1}|\, \hat{\phi}(0)\, |\Omega\rangle$$

Then the claim follows using property *(3)*. Indeed, take $p_2 \notin \overline{V}^+$ and $n = 4$, then from *(3)* we get

$$\widetilde{W}_3(p_1, p_2, p_3) = \langle\Omega|\, \hat{\phi}(0)\, |p_1\rangle\langle p_1|\, \hat{\phi}(0)\, |p_2\rangle\langle p_2|\, \hat{\phi}(0)\, |p_3\rangle\langle p_3|\, \hat{\phi}(0)\, |\Omega\rangle = 0 \quad \text{for all} \quad p_1, p_3 \in \mathbb{R}^{d+1}$$

This is possible only if

$$\langle p|\, \hat{\phi}(0)\, |p_2\rangle = 0 \quad \text{for all} \quad p \in \mathbb{R}^{d+1} \quad \Leftrightarrow \quad \hat{\phi}(0)\, |p_2\rangle = 0 \overset{(4.7)}{\Leftrightarrow} \hat{\phi}(x)\, |p_2\rangle = 0$$

Hence for each $p \notin \overline{V}^+$ we have $\hat{\phi}(x)\, |p\rangle = 0$. Applying $\hat{P}$ to any element of $\mathcal{P}(\hat{\phi}(x))\, |\Omega\rangle$ we get then

$$\hat{P}\big(\mathcal{P}(\hat{\phi}(x))\, |\Omega\rangle\big) = \int \mathrm{d}p\, |p\rangle\langle p|\, \big(\mathcal{P}(\hat{\phi}(x))\, |\Omega\rangle\big) = \int_{\overline{V}^+} \mathrm{d}p\, |p\rangle\langle p|\, \big(\mathcal{P}(\hat{\phi}(x))\, |\Omega\rangle\big)$$

which give the following spectral representation of $\hat{P}$

$$\hat{P} = \int_{\overline{V}^+} \mathrm{d}p\, |p\rangle\langle p| \quad \text{in} \quad \mathcal{P}(\hat{\phi}(x))\, |\Omega\rangle$$

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but since $\mathcal{P}(\hat{\phi}(x))|\Omega\rangle$ is dense in $\mathcal{H}$ the previous spectral representation for $\hat{P}$ holds in the whole $\mathcal{H}$ (actually, in $D_P \subseteq \mathcal{H}$). The claim follows immediately.

*(7')* Take $f, g \in \mathcal{S}(\mathbb{R}^{d+1})$ space-like separated, that is $\operatorname{supp} f \times \operatorname{supp} g$, and $\{h_i\}_{i=0}^{\infty}$. Then from *(4)* we know that, for each value of $n$,

$$W_{n+2}(f, g, h_1, \ldots, h_n) = W_{n+2}(g, f, h_1, \ldots, h_n)$$

and *(9')* tells us that

$$(\Omega, \hat{\phi}(f)\hat{\phi}(g)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = (\Omega, \hat{\phi}(g)\hat{\phi}(f)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega)$$

Then, by linearity

$$(\Omega, [\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = 0$$

An analogous procedure together with eq. (4.5) lead us also to

$$([\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega, [\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega) = \|[\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega\|^2 = 0$$

Since the scalar product is non-degenerate, this implies that $[\hat{\phi}(f), \hat{\phi}(g)]\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\Omega = 0$ is the zero vector in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. The only way to satisfy this for any choice of $n$ and any set of smearing test functions $\{h_i\}_{i=1}^{\infty}$ is that in $\mathcal{P}(\hat{\phi}(h))\Omega$ we have

$$[\hat{\phi}(f), \hat{\phi}(g)] = 0$$

meaning that the operator $[\hat{\phi}(f), \hat{\phi}(g)]$ maps all the elements of $\mathcal{P}(\hat{\phi}(h))\Omega$ into the zero vector. But then the claim extends to the whole Hilbert space $\mathcal{H}$ due to the denseness of $\mathcal{P}(\hat{\phi}(h))\Omega$.

*(8')* Assume that, beside $\Omega = ([\underline{\mathbb{1}}], [\underline{\mathbb{1}}], \ldots)$ there is another translationally invariant state $\Omega'$. Without loss of generality we can orthonormalize $\Omega'$ respect to $\Omega$, i.e. we take $(\Omega', \Omega) = 0$ and $(\Omega', \Omega') = 1$. If $\Omega' \in \underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, $\Omega = [\underline{f}]$, then we would have an immediate contradiction because,

$$1 = (\Omega', \Omega') \overset{\text{(a)}}{=} \lim_{\substack{a \to \infty \\ a^2 < 0}} (\Omega', U(a)\Omega') = \lim_{\substack{a \to \infty \\ a^2 < 0}} ((f_0, f_1, \ldots), (f_0, \{a, \underline{\mathbb{1}}\}f_1, \{a, \underline{\mathbb{1}}\}f_2, \ldots))$$

$$\overset{(4.3)}{=} \lim_{\substack{a \to \infty \\ a^2 < 0}} \sum_{j,k=0}^{\infty} \int dx_1 \ldots dx_j dy_1 \ldots dy_k\, f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) f_k(y_1 - a, \ldots, y_k - a)$$

$$= \lim_{\substack{a \to \infty \\ a^2 < 0}} \sum_{j,k=0}^{\infty} \int dx_1 \ldots dx_j dy_1 \ldots dy_k\, f_j^*(x_1, \ldots, x_j) W_{j+k}(x_1, \ldots, x_j, y_1 + a, \ldots, y_k + a) f_k(y_1, \ldots, y_k)$$

$$\overset{(5)}{=} \sum_{j,k=0}^{\infty} \int dx_1 \ldots dx_j dy_1 \ldots dy_k\, f_j^*(x_1, \ldots, x_j) W_j(x_1, \ldots, x_j) W_k(y_1, \ldots, y_k) f_k(y_1, \ldots, y_k)$$

$$= \left( \sum_{j=0}^{\infty} \int dx_1 \ldots dx_j\, f_j^*(x_1, \ldots, x_j) W_j(x_1, \ldots, x_j) \right) \left( \sum_{k=0}^{\infty} \int dy_1 \ldots dy_k\, W_k(y_1, \ldots, y_k) f_k(y_1, \ldots, y_k) \right)$$

$$= (\Omega', \Omega)(\Omega, \Omega') = 0$$

where in (a) we used the translational invariance of $\Omega'$. In general $\Omega'$ is not an element of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, but because the latter is dense then $\Omega'$ can be approximated by elements of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ by arbitrary accuracy, then the contradiction easily extends for $\Omega' \in \mathcal{H}$, see [SW00, Page 124] for the details. $\qquad\square$

The most important consequence of the theorem is that in order to exhibit a relativistic quantum field theory model, it is enough to give a set of Wightman functions satisfying properties *(1)-(5)*. As we'll discuss in the next section, these properties provide a non-perturbative substitute for canonical quantization, since allow to construct a quantum field theory without use the canonical quantization procedure and CCR (or CAR) which are inconsistent in the interacting case due to Haag's theorem.

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[VII] https://doi.org/10.2307%2F1968538

Properties *(1')* - *(8')* are called *Wightman axioms*[VIII], and define axiomatically a RQFT at $T = 0$ for a real scalar field. Provided appropriated versions of properties *(1)-(5)*, more general forms of the reconstruction theorem allow to reconstruct also theories with complex fields and more general spins, satisfying appropriate Wightman axioms.

Notice that property *(9')* tells us that the vacuum expectation values of the reconstructed theory are exactly the Wightman function we started from.

Conversely, the vacuum expectation values of any theory satisfying Wightman axioms satisfy themselves the properties required in the reconstruction theorem.[IX] In particular, if $\hat{\phi}_F$ is a free theory defined in a Fock space, then the vacuum expectation values $\langle 0| \hat{\phi}_F(x_1) \ldots \hat{\phi}_F(x_n) |0\rangle$ satisfy properties *(1)-(5)* and the reconstructed theory coincide with the initial one, $\hat{\phi}_{\mathrm{rec}} = \hat{\phi}_F$. On the other side, a reconstructed interacting theory is defined in a Hilbert space disjoint from the Fock space of the free theory, due to Haag's theorem.

Can also be proved that the reconstructed theory is unique, i.e. any other theory satisfying properties *(1')* - *(9')* (in particular with same vacuum expectation values $\{W_n\}_{n=0}^{\infty}$) is unitarily equivalent to the one reconstructed in the proof of the theorem.[X]

Finding a set of Wightman functions satisfying properties *(1)-(5)* turned out to be a very hard problem, apart from the non-interacting case, also because it is difficult to satisfy the positivity condition, which has a non-linear structure, in contrast to the other properties, which have a linear structure. We will see in the following how this problem has been solved by computing Wightman functions in the Euclidean space, where properties *(1)-(5)* take a more simple form.

In the proof of the theorem we constructed the Hilbert space of the theory as (the completion of) finite sequences of test functions. In order to give a better interpretation of the Hilbert space, notice that if we define

$$\hat{\phi}(\underline{f}) := f_0 + \int \mathrm{d}x_1\, \hat{\phi}(x_1)f_1(x_1) + \int \mathrm{d}x_1\mathrm{d}x_2\, \hat{\phi}(x_1)\hat{\phi}(x_2)f_2(x_1, x_2) + \ldots$$

such that

$$\hat{\phi}(\underline{f})\underline{g} := (f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 \otimes g_1 + f_2 g_0, \ldots) \quad , \quad \hat{\phi}(\underline{f})[\underline{g}] := [\hat{\phi}(\underline{f})\underline{g}]$$

and in particular (let $\Omega := [\underline{\mathbb{1}}]$ in this instance, since we do not consider the completion of $\underline{\mathcal{S}}/\underline{\mathcal{S}}_0$)

$$\hat{\phi}(\underline{f})\underline{\mathbb{1}} = (f_0, f_1, f_2, \ldots) = \underline{f} \quad , \quad \hat{\phi}(\underline{f})\Omega = [\underline{f}]$$

Then the scalar product in $\underline{\mathcal{S}}$ and $\underline{\mathcal{S}}/\underline{\mathcal{S}}_0$ can be rewritten as[XI]

$$(\underline{f}, \underline{g}) = \langle \hat{\phi}(\underline{f})\underline{\mathbb{1}}, \hat{\phi}(\underline{g})\underline{\mathbb{1}}\rangle \quad , \quad ([\underline{f}], [\underline{g}]) = \langle \hat{\phi}(\underline{f})\Omega, \hat{\phi}(\underline{g})\Omega\rangle$$

Hence we can identify by unitary equivalence (i.e. up to representations) $\underline{\mathcal{S}}/\underline{\mathcal{S}}_0$ with $\{\hat{\phi}(\underline{f})\Omega \mid \underline{f} \in \underline{\mathcal{S}}\}$, since the scalar products defined in the two spaces are the same. This is related with the denseness of $\mathcal{P}(\hat{\phi}(f))\Omega$ in $\mathcal{H}$. Then we are done: one can forget the structure underlying $\underline{f}$ and $\Omega$ and define the Hilbert space by applying the operator $\hat{\phi}(\underline{f})$ on the vacuum $\Omega$, as usual.

## 4.3 Additional remarks

[Str13, Sections 3.4, 4.1–4.4]; [Jos65, Chapters 4, 5]; [SW00, Chapter 4]

By exploiting the analyticity properties of the Wightman's functions, general results have been derived. In this section we will focus on some of the main results which have been derived in the specific case of relativistic QFT's.

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[VIII]These axioms are discussed in [SW00, Section 3.1] and [Jos65, Section 3.2].

[IX]See [SW00, Theorems 3.1-3.4] and [Jos65, Section 3.3].

[X]Actually this claim is included in the reconstruction theorem, see [SW00, Thm. 3.7] and [Jos65, Section 3.4, Thm. 1].

[XI]The bilinear $(-, -)$ denotes as usual the scalar product in $\underline{\mathcal{S}}$, whereas $\langle -, -\rangle$ denotes the scalar product in $\{\hat{\phi}(\underline{f})\underline{\mathbb{1}} \mid \underline{f} \in \underline{\mathcal{S}}\}$. Analogous notations hold for the quotient sets.

**Spin-statistics theorem**

The connection between spin and statistics, which is at the basis of Pauli principle (namely, in the alternative of anti/commutation relations at space-like points, fields carrying half-integer/integer spin must anti-commute/commute), have been first derived by Pauli for free theories in Fock space. It's generalization to interacting theories holds in the Wightman axiomatic formalism, and is a consequence of Lorentz covariance, as has been proved in *spin-statistics theorem*:

**Theorem 4.2** (Spin-statistics, [Str13, Section 4.2], [SW00, Section 4.4], [Jos65, Section 5.3]). *Let $\psi_{\alpha,\dot{\beta}}$ be a spinor field*[XII] *transforming as $\mathcal{D}^{j/2,k/2}$, and therefore carrying a integer/half-integer spin, corresponding to $j+k =$even/odd, then the wrong connection between spin and statistics, i.e., for integer/half-integer spin the field anti-commutes/commutes at space-like separations, implies that $\psi\Omega = 0$ and, if locality holds, $\psi = 0$.*

**PCT theorem**

The (up to now experimentally established) *PCT symmetry* also follows from Wightman functions properties, being in particular related to local commutativity:

**Theorem 4.3** (PCT, [Str13, Section 4.3-4.4], [SW00, Section 4.3], [Jos65, Section 5.2]). *If a spinor field satisfies covariance, spectral condition, and locality, then the corresponding Wightman functions are PCT symmetric.*

**Reeh-Schlieder theorem**

Wightman's functions analyticity can be proved[XIII] and imply that the knowledge of the Wightman functions of the fields localized in a given open set $\mathcal{O} \subset \mathbb{R}^4$, however small, completely determines the theory. In fact, if $\mathcal{P}(\mathcal{O})$ denotes the polynomial algebra of the fields smeared with test functions with support in $\mathcal{O}$, one has

**Theorem 4.4** (Reeh-Schlieder, [SW00, Section 4.2], [Jos65, Section 4.8], [Str13, Section 3.4.4], [Haa96, Section II.5.3]). *For any $\mathcal{O} \subset \mathbb{R}$, the vacuum is cyclic with respect to the algebra $\mathcal{P}(\mathcal{O})$, i.e.*

$$\{\mathcal{P}(\hat{\phi}(f)) \text{ s.t. } \mathrm{supp}(f) \subset \mathcal{O}\}\Omega$$

*is dense in $\mathcal{H}$. Moreover, if $A \in \mathcal{P}(\mathcal{O})$ and $A\Omega = 0$ then $A = 0$.*

This means that the information contained in fields localized in any, arbitrarily small, open subset of $\mathbb{R}^4$, can be used to reconstruct the entire RQFT.

**Non-commutativity of the algebra of interacting fields**

As we already pointed out equal-time canonical (anti-)commutation relations cannot be used to quantize field equations for interacting theories. A natural question is whether and where a quantization condition is contained in the general requirements of Wightman's axioms, equivalently in the properties of Wightman functions. To this purpose, one may show that, as a consequence of the spectral condition, the field algebra of the reconstructed theory cannot be commutative, hence properties *(1)-(5)* imply quantization. Since can be proved[XIV] that field obeying free fields equations satisfy either CCR of CAR, Wightman's properties qualify as he way of defining quantum fields by providing the correct substitute of canonical quantization in the interacting case.

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[XII]Here we consider a spinor field since it can describe the most general case, i.e. both integer and half integer fields. We will do the same also for the spin-statistics theorem.
[XIII][Str13, Sections 3.4.1-3.4.3], [SW00, Chapter 4]
[XIV][Str13, Section 4.1]

## 4.4   Haag-Ruelle scattering theory (massive case) and LSZ reduction formula

[Rue62]; [Hep65]; [Haa96, Chapter II.4]; [Jos65, Chapters 6]; [Str13, Sections 6.1-6.2]

Another important consequence of properties *(1)-(5)* for relativistic QFT's is the existence of asymptotic fields which provide a non-perturbative definition of the $S$-matrix, which is unitary if asymptotic completeness holds. Also the LSZ asymptotic condition and the LSZ reduction formulas can be derived from the same properties, as proved by Hepp; furthermore, one can prove dispersion relations for scattering amplitudes, yielding experimentally measurable relations.

One should notice that the axioms of Wightman do not introduce the notion of a particle. The particle notion is, however, of crucial importance if any connection of the theory with realities of physics is to be achieved. The principal observable of elementary particle physics, the $S$-matrix, depends essentially on the possibility of defining asymptotic incoming and outgoing states, which themselves can be described by free particles and their corresponding free fields.
It is highly gratifying that, under natural conditions, a Wightman field theory allows the precise definition of such asymptotic fields. This was proved by Ruelle, who applied Haag's idea of construct asymptotic ingoing and outgoing states as strong limits in Hilbert space, if certain "space like asymptotic condition" is verified by the vacuum expectation values of products of field operators.
In his theory, Ruelle has to make some natural additional assumptions which go beyond axioms of Wightman. This is not astonishing, it is easily seen that in a theory which allows a complete particle interpretation the spectrum of the energy momentum vector must have special properties.

We shall not present the general theory in this section. The special case to which we shall restrict ourselves, however, shows most of the interesting aspects of the general case. Let's assume that our theory have a *mass gap*. This means that the spectrum of the generators of the spacetime translations, as determined by the Wightman functions and the reconstruction theorem, satisfies the following:
*Mass gap condition*: Above the isolated point $p = 0$ (corresponding to the vacuum state) there is a gap up to an isolated hyperboloid $p^2 = m^2$ (*mass gap*) denoted by $V_m^+ \subset \sigma(\hat{P}^\mu) \subseteq \overline{V}^+$, with finite multiplicity (one-particle state), followed by a continuous spectrum starting at $p^2 = (2m)^2$ (two-particles states). It suffices that such a mass gap condition holds in the subspace characterized by the conserved quantum numbers of the asymptotic states under investigation.
Notice that such condition cannot be satisfied if theories admits massless particles, for instance it does not apply to QED.

Let us restrict to the one-body problem, i.e. the description of the asymptotic sate for a one particle system. The existence of free asymptotic fields of mass $m$ should be related to the existence of an isolated hyperboloid $p^2 = m^2$ in the two-point spectral function of the field $\hat{\phi}$.
Let $\mathcal{H}_m$ be the improper[XV] subspace of $\mathcal{H}$, which is eigenspace of the operator $\hat{P}^\mu \hat{P}_\mu$ associated to the eigenvalue $m^2$. Suppose that our (possibly renormalized, if needed) field $\hat{\phi}$ when applied to the vacuum has a non-empty intersection with $\mathcal{H}_m$, that is $\hat{\phi}(x)\Omega \not\perp \mathcal{H}_m$. Let $h(x)$ be a test function whose Fourier transform satisfies $\text{supp}\,\tilde{h}(p) \cap \sigma(\hat{P}^\mu) \subseteq V_m^+$, or equivalently, its support in momentum space does not intersect the continuum region above $V_m^+$ of $\overline{V}^+$ (the one associated to multi-particles states).
Let's define the operator

$$\hat{\phi}_m(h) := \int \mathrm{d}p\, \tilde{h}(p)\hat{\tilde{\phi}}(p)$$

and consider the two-points function

$$(\Omega, \hat{\phi}_m(h)\hat{\phi}_m(h')\Omega)$$

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[XV]It is an improper subspace since it is made of generalized vectors, in the sense of Dirac's formalism.

where $h'$ satisfies same properties as $h$. Such two-point function in the Källen-Lehmann representation contains only an isolated term like $\delta(p^2 - m^2)\theta(p^0)$, since the localization of $h$ on the one-particle hyperboloid acts as a cutoff for the continuum part in the spectral representation of the correlator.

The field $\hat{\phi}_m(h)$ is not a free field, since in general $\mathrm{supp}\, h \not\subset \sigma(\hat{P}^\mu)$ and $(p^2 - m^2)\tilde{\hat{\phi}}_m(p) \neq 0$, however $(p^2 - m^2)\tilde{\hat{\phi}}_m(p)\Omega = 0$, hence we selected at the field level a neighbourhood of the mass shell.

Let's see how to construct more than one. Suppose that we have $n$ test functions, $\{h_i\}_{i=1}^n$, $\mathrm{supp}\,\tilde{h}(p) \cap \sigma(\hat{P}^\mu) \subseteq V_m^+$, such that their supports in terms of velocities are disjoint, that is

$$\frac{\boldsymbol{p}_i}{p_i^0} \neq \frac{\boldsymbol{p}_j}{p_j^0} \quad \text{for any} \quad p_i \in \mathrm{supp}\,\tilde{h}_i \quad , \quad p_j \in \mathrm{supp}\,\tilde{h}_j \quad , \quad i \neq j \tag{4.8}$$

Due to condition (4.8), for very large values of $t$ the operators

$$e^{-itH}\hat{\phi}_m(h_i)e^{itH} \quad , \quad i = 1, \ldots, n$$

have their supports separated by large distances, hence in the asymptotic limit $t \to \pm\infty$ they are free (since interactions decay exponentially for $m > 0$, see [Str13, Section 6.2.2]). Then we can "bring them back" using free evolution, i.e. using the standard dispersion relation $\sqrt{\boldsymbol{p}^2 + m^2}$:

$$\hat{\phi}_m(h, t) := \int \mathrm{d}p\, e^{it\sqrt{\boldsymbol{p}^2 + m^2}}\tilde{h}(p)e^{-itH}\hat{\phi}_m(p)e^{itH}$$

Such field behaves for $t \to \pm\infty$ as a free field.
The final result is provided by the following theorem

**Theorem 4.5** (Haag-Ruelle, [Rue62, Section 4]; [Str13, Theorem 6.2.1], [Haa96, Theorem II.4.2.1], [Jos65, Theorems 1, 2 section 6.3]). *The following asymptotic limits*

$$\lim_{t \to -\infty} \hat{\phi}_m(h_1, t) \cdots \hat{\phi}_m(h_n, t)\,|\Omega\rangle =: |h_1, \ldots, h_n\rangle_{\mathrm{in}}$$

$$\lim_{t \to +\infty} \hat{\phi}_m(h_1, t) \cdots \hat{\phi}_m(h_n, t)\,|\Omega\rangle =: |h_1, \ldots, h_n\rangle_{\mathrm{out}}$$

*exist in the strong sense and define asymptotic in/out scattering states. The closure of the space of scattering states $\mathcal{H}_{\mathrm{in/out}}$ is contained in $\mathcal{H}$. In $\mathcal{H}_{\mathrm{in/out}}$ are defined the operators $\hat{\phi}_{\mathrm{in/out}}$ such that*

$$\hat{\phi}_{\mathrm{in}}(h)\,|h_1, \ldots, h_n\rangle = \lim_{t \to -\infty} \hat{\phi}_m(h, t)\,|h_1, \ldots, h_n\rangle_{\mathrm{in}}$$

$$\hat{\phi}_{\mathrm{out}}(h)\,|h_1, \ldots, h_n\rangle = \lim_{t \to +\infty} \hat{\phi}_m(h, t)\,|h_1, \ldots, h_n\rangle_{\mathrm{out}}$$

Let's see what happens if abandon the restriction on the support of the test function, i.e. if we define

$$\hat{\phi}(f, t) = \int \mathrm{d}p\, f(p)e^{itH}\tilde{\hat{\phi}}(p)e^{-itH}e^{it\sqrt{\boldsymbol{p}^2 + m^2}}$$

for some (generic) test function $f$. Hepp proved that it admits a weak limit $t \to \pm\infty$ on scattering states, and can be used as asymptotic fields in the LSZ formula

**Theorem 4.6** (Hepp, [Hep65]). *The following weak limits exist:*

$$\lim_{t \to -\infty} {}_{\mathrm{in}}\langle h_1, \ldots, h_n|\hat{\phi}(f, t)\,|h_1', \ldots, h_n'\rangle_{\mathrm{in}} = {}_{\mathrm{in}}\langle h_1, \ldots, h_n|\hat{\phi}_{\mathrm{in}}(f)\,|h_1', \ldots, h_n'\rangle_{\mathrm{in}}$$

$$\lim_{t \to +\infty} {}_{\mathrm{out}}\langle h_1, \ldots, h_n|\hat{\phi}(f, t)\,|h_1', \ldots, h_n'\rangle_{\mathrm{out}} = {}_{\mathrm{out}}\langle h_1, \ldots, h_n|\hat{\phi}_{\mathrm{out}}(f)\,|h_1', \ldots, h_n'\rangle_{\mathrm{out}}$$

*and defined scattering fields $\hat{\phi}_{\mathrm{in/out}}$. These fields can be used inside the LSZ reduction formula, hence the reduction formalism work even in this axiomatic framework starting from Wightman functions.*

Hence within Wightman theory all the formalism of asymptotic states and fields is mathematically well defined.

Non capisco il senso di "portare indietro" il campo utilizzando l'evoluzione libera, non potevamo utilizzare direttamente $e^{-itH}\hat{\phi}_m(h_i)e^{itH}$. Inizialmente pensavo che servisse a definire un campo libero per $t$ generico, ma $\hat{\phi}_m(h, t)$ è libero solo per $t \to \pm\infty$... Suppongo allora che tale definizione serva per dimostrare il teorema di Ruelle. Mi chiedo inoltre se $\hat{\phi}_m(h, t)$ in genere possa considerarsi un elemento di uno spazio di Fock per $t \to \pm\infty$ o abbia qualche peculiarità che non lo permette, in particolare nel teorema diciamo che $\mathcal{H}_{\mathrm{in/out}} \subseteq \mathcal{H}$ quindi immagino di no.

# Chapter 5

# Constructive QFT

[SW00, Pages 185-191], [Sum16], [Wig76], [Str13, Section 5.1], [GJ87, Section 6.1, Chapter 19]

We now have to deal with the construction of the Wightman functions or their non relativistic counterpart, which as we have seen are the cornerstone of our formalism. It turns out that the easiest way to construct them is the following

(1) go back to the Gell-Man - Low formula and introduce an IR cutoff (an in the relativistic case also an UV one) so that we avoid Haag's no-go theorem;

(2) rewrite such formula using the path-integral formalism as described in Sec. 2.5 (when the cutoffs are introduce, we'll denote the action by $S_\Lambda$), in particular

   (a) for a relativistic theory the path integral is defined in an Euclidean space-time,

   (b) for a non relativistic theory the path integral is defined in an Euclidean space-time for an "immaginary" time;

(3) remove the cutoffs from outside the path integral, we assume that this is possible, otherwise the procedure fails;

(4) provided some properties for the correlation functions, analytically continue them to the appropriate space, in particular

   (a) for a relativistic theory, the correlation functions in the Euclidean space are called *Schwinger functions*, denoted by $\{S_n\}$

   $$S_n(x_1, \ldots, x_n) = \lim_{\substack{\Lambda_{\mathrm{UV}} \to \infty \\ \Lambda_{\mathrm{IR}} \to 0}} \frac{\int \mathcal{D}\phi \mathcal{D}\phi^* \, e^{-S_\Lambda(\phi^*, \phi)} \phi(x_1) \ldots \phi(x_n)}{\int \mathcal{D}\phi \mathcal{D}\phi^* \, e^{-S_\Lambda(\phi^*, \phi)}}$$

   they determine Wightman function by analytical continuation in the Minkowski space if a set of properties, called *Osterwalder-Schrader (OS) axioms*, are satisfied,

   (b) for a non relativistic theory with $T > 0$ the Matsubara Green functions we obtained for an "immaginary" time can be analytically continued to the real time provided that other suitable axioms are satisfied, defining in this way the retarded functions for the interacting theory.

We first discuss the relativistic case and later we make some comments on the non relativistic $T > 0$ and the lattice QFT cases.

Let's heuristically motivate why it is easier to work in the Euclidean space rather than in the Minkowski one. In the Euclidean space the exponential inside the path integral has a real argument, $e^{-S}$, whereas in the Minkowski one the argument is imaginary: $e^{iS}$. In the Euclidean space one can use more powerful techniques such as saddle point method, moreover in general the path integral measure can be defined with a real exponential, while the definition in the Minkowski space is much more difficult and up to now it is not really understood if the path integral can be defined in a rigorous way. Without enter in the details, notice for instance that $\int e^{-x^2} \mathrm{d}x$ clearly converges, whereas $\int e^{ix^2} \mathrm{d}x$ should be regularized. For the same reason, in the non relativistic case is better to work with imaginary time rather than with the real one.

## 5.1 Osterwalder-Schrader reconstruction

[OS73], [OS75], [Ost73], [Str13, Chapter 5], [Str93, Chapter 3]

We now list the properties of Schwinger functions needed to determine Wightman functions of an interacting (scalar) RQFT. Such properties are called *Osterwalder-Schrader axioms*:

(0) (*Regularity*) The $\{S_n(x_1, \ldots, x_n)\}_{n=0}^{\infty}$, $x_i \in \mathbb{R}^{d+1}$ with Euclidean metric are distributions in $\mathcal{S}(\mathbb{R}^{d+1})$ and "should not grow too fast in $n$" (we omit the details about this requirement).

(1) (*Euclidean covariance*) They transform covariantly under the Euclidean $d + 1$-dimensional group $\mathbb{E}^{d+1}$. In particular for a scalar field $\phi$, $(a, R) \in \mathbb{E}^{d+1}$, we have

$$S_n(Rx_1 + a, \ldots, Rx_n + a) = S_n(x_1, \ldots, x_n) \tag{5.1}$$

(2) (*Osterwalder-Schrader positivity / Reflection positivity*) Let $F(\phi)$ be a function of the field $\phi$ with support in the upper half-space in the Euclidean time direction (i.e. the support is completely contained in the $\tau > 0$ region). Let $\theta$ denote the convolution corresponding to reflection with respect to the time-zero $d$-dimensional space, together with complex conjugation.[I] Then OS positivity is the inequality (in the bosonic case)
$$\langle F(\phi)\theta F(\phi) \rangle \geq 0$$
when the (vacuum) expectation value is written in terms of Schwinger functions.

(3) (*Symmetry*) Schwinger functions of bosonic (fermionic) fields are symmetric (antisymmetric) under any exchange of the arguments. For a scalar bosonic theory we have $S_n(x_1, \ldots, x_n) = S_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation $\pi$.

(4) (*Cluster property*) Schwinger functions asymptotically factorize when two sets of the arguments are taken apart to $\infty$:

$$\lim_{a \to \infty} S_n(x_1 + a, \ldots, x_j + a, x_{j+1}, \ldots, x_n) = S_j(x_1, \ldots, x_j) S_{n-j}(x_{j+1}, \ldots, x_n)$$

Notice that it is much more easy to verify this axioms (beside OS positivity) rather than properties of Schwinger functions. For instance in this case no spectral condition, since it is very cleverly hidden inside OS positivity.

To Schwinger functions satisfying OS axioms, one can apply a reconstruction theorem which is a variant of Wightman reconstruction theorem. Then Schwinger functions are directly related to Wightman functions (whose existence is now guaranteed by such theorem) by analytical continuation[II] in the time variable.

**Theorem 5.1** (Osterwalder-Schrader). *Assume that the Schwinger functions $\{S_n(x_1, \ldots, x_n)\}_{n=0}^{\infty}$ satisfy the OS axioms. Then they uniquely determine*

*(1') a Hilbert space of states $\mathcal{H}$;*

*(2') a continuous representation $U(a, \Lambda)$ of $\tilde{\mathcal{P}}_+^{\uparrow}$ on $\mathcal{H}$;*

*(3') a unique vector $|\Omega\rangle \in \mathcal{H}$ invariant under $U$;*

*(4') a field operator (in the bosonic case) $\hat{\phi}(\boldsymbol{x})$ such that, for $t_i < t_{i+1}$,*

$$S_n(x_1, \ldots, x_n) = \langle \Omega | \prod_{j=1}^{n-1} \hat{\phi}(\boldsymbol{x}_j) e^{-(t_{j+1} - t_j)H} \hat{\phi}(\boldsymbol{x}_n) | \Omega \rangle$$

$$= \langle \Omega | e^{-t_1 H} \hat{\phi}(\boldsymbol{x}_1) e^{t_1 H} e^{-t_2 H} \hat{\phi}(\boldsymbol{x}_2) \ldots e^{-t_n H} \hat{\phi}(\boldsymbol{x}_n) e^{t_n H} | \Omega \rangle$$

*where $H$ is the generator of $U(a^0)$ and $H \geq 0$.[III]*

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[I]For instance, let $x = (\tau, \boldsymbol{x})$, $\tau > 0$, then $\theta f(\phi(\tau, \boldsymbol{x})) = f^*(\phi^*(-\tau, \boldsymbol{x}))$.

[II]Notice that in order to make an analytical continuation from immaginary time to real time one should prove that no obstruction in the complex plane such as poles or cuts.

[III]Since $H$ is semidefinite positive is reasonable that Schwinger functions can be analytically continued to the real time obtaing Wightman functions, since this provides the right sign inside the time evolution exponentials.

*Furthermore such* $S_n(x_1, \ldots, x_n)$ *are the restriction of functions* $W_n(\boldsymbol{x}_1, z_1, \ldots, \boldsymbol{x}_n, z_n)$, *analytic in* $\{z_i\}_{i=1}^n$ *on the domain* $\operatorname{Im} z_i < \operatorname{Im} z_{i+1}$, *to the region* $z_j = it_j$, *for* $j = 1, \ldots, n$. *Formally*[IV]

$$S_n(x_1, \ldots, x_n) = W_n(\boldsymbol{x}_1, it_1, \ldots, \boldsymbol{x}_n, it_n) \quad \text{where} \quad x_j = (\boldsymbol{x}_j, t_j)$$

*The functions* $W_n(\boldsymbol{x}_1, t_1, \ldots, \boldsymbol{x}_n, t_n)$ *are the Wightman functions of a RQFT. In other words the Wightman functions are obtained by the Schwinger functions by analytical continuation in time.*
*Properties (5')-(9') in Wightman reconstruction theorem are satisfied.*
*The Hilbert space* $\mathcal{H}$, *the representation* $U$, *the vacuum* $|\Omega\rangle$ *and the field* $\hat{\phi}$ *constructed in the theorem are the same obtained by Wightman reconstruction starting from the Wightman functions* $\{W_n\}$ *obtained by analytical continuation.*

Notice that the fact that we reconstruct the theory directly from the Euclidean space, without using Wightman functions, turns out to be crucial in different situations, for instance OS reconstruction theorem apply to the quantization of solitons, whereas Wightman reconstruction theorem doesn't.
Moreover we already know that the path integral can be used for built Schwinger functions for the theory regularized by cutoffs (since they are simply vacuum expectation values and we can apply previous formulas, which certainly give well-defined results thanks to the cutoffs we introduced), whereas it is not well defined in the Minkowski space, where Wightman functions are defined.
After the construction of functions using the path integral for the regularized theory one has to check that after the removal of the cutoffs the OS axioms are satisfied.

*Proof. (sketch) (1')* Let $\underline{\mathcal{S}}_+$ be the space of finite sequences $\underline{f}$ of thest functions with support in the positive euclidean time half-space. Define a semi-definite inner product by

$$(\underline{f}, \underline{g}) := \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, (\theta f(x_1, \ldots, x_j)) g(y_1, \ldots, y_k) S_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) \quad (5.2)$$

also denoted in compact notation by $S(\theta f \times g)$, which is semi-definite positive by OS positivity.
The construction of $\mathcal{H}$ then is performed like in the Wightman reconstruction theorem:

$$\mathcal{H} := \underline{\mathcal{S}}_+ / \mathcal{N} \quad \text{where} \quad \mathcal{N} = \{\underline{f} \in \underline{\mathcal{S}}_+ \mid \|\underline{f}\| = 0\}$$

We denote by $|\underline{f}\rangle$ the vectors in $\mathcal{H}$ corresponding to $\underline{f}$.
*(2')* Given a sequence $\underline{f}$ let $\underline{f}_t$ be the sequence with all argument translated in the euclidean time by $t > 0$ and $\underline{f}_{\boldsymbol{a},R}$ the sequence with arguments rotated by $R$ and then translated spatially by $\boldsymbol{a}$. Set also the compact notation $\underline{f}_{\boldsymbol{a}} := \underline{f}_{\boldsymbol{a},\mathbb{1}}$.
Spatial roto-translations are well-defined by the same arguments that we used for $U(a, \Lambda)$ in the Wightman reconstruction theorem, with a unitary operator $U(\boldsymbol{a}, R)$ such that:

$$(\underline{f}, \underline{g}_{\boldsymbol{a},R}) = (\underline{f}_{(\boldsymbol{a},R)^{-1}}, \underline{g}) = (\underline{f}, U(\boldsymbol{a}, R)\underline{g}) \quad \text{and} \quad (\underline{f}, \underline{g}_{\boldsymbol{a}}) = (\underline{f}_{-\boldsymbol{a}}, \underline{g}) = (\underline{f}, e^{i\boldsymbol{P}\boldsymbol{a}}\underline{g})$$

Different arguments are required for time translations, since restricting ourself to $\tau > 0$ we broke time translational invariance. We want to prove that

$$(\underline{f}, \underline{g}_t) = (\underline{f}_t, \underline{g}) = (\underline{f}, e^{-tH}\underline{g}) \quad \text{with} \quad H \geq 0 \quad (5.3)$$

The continuity in $t$ of $(\underline{f}, \underline{g}_t)$ is obvious, and

$$(\underline{f}, \underline{g}_t) = (\underline{f}_t, \underline{g}) \quad (5.4)$$

follows directly from the definition eq. (5.2) using eq. (5.1) and the definition of $\theta$. Let's define the

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[IV]Notice that the relation between Wightman functions and Schwinger functions is very similar to the relation between retarded correlators and Matsubara functions.

operator $P_t : \underline{g} \mapsto \underline{g}_t$, then it is symmetric thanks to eq. (5.4). Moreover

$$\|P_t \underline{g}\|^2 = \|\underline{g}_t\|^2 = |(\underline{g}_t, \underline{g}_t)| = |(\underline{g}, \underline{g}_{2t})| \le |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}_{2t}, \underline{g}_{2t})|^{1/2} = |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}, \underline{g}_{4t})|^{1/2}$$
$$\le |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}, \underline{g})|^{1/4} |(\underline{g}_{4t}, \underline{g}_{4t})|^{1/4}$$
$$\le \ldots$$
$$\le |(\underline{g}, \underline{g})|^{\sum_{n=1}^{N} \frac{1}{2^n}} |(\underline{g}_{2^N t}, \underline{g}_{2^N t})|^{1/2^N}$$
$$\le |(\underline{g}, \underline{g})|^{\sum_{n=1}^{\infty} \frac{1}{2^n}} |(\underline{g}_{\infty}, \underline{g}_{\infty})|^0$$
$$= |(\underline{g}, \underline{g})| = \|\underline{g}\|^2$$

where inequalities are due to Schwartz. Due to continuity and $\|P_t \underline{g}\|^2 \le \|\underline{g}\|^2$ an analogue of Stone's theorem proves the existence of $H \ge 0$ such that $P_t = e^{-tH}$. In this way we proved eq. (5.3).

Moreover since $H$ is positive, this suggest us that analytical continuation of $e^{-tH}$ to $e^{-itH}$ may be possible.

For boosts the situation is much more complicated and it will not be discussed here. We just say that they can be built starting from representations of spatial roto-translations and time translations.

(3') The vacuum $|\Omega\rangle$ is then given (up to quotienting and completion) by $\underline{1} = (1, 0, \ldots, 0, \ldots)$. The proof of uniqueness is omitted.

(4') We just sketch how to construct the field $\hat{\phi}$, without proving the well-definiteness and its properties. Define the domain

$$D_+ = \bigcup_{\varepsilon > 0} e^{-\varepsilon H} \underline{\mathcal{S}}_+ / \mathcal{N}$$

given by sequences shifted in the positive time direction by arbitrary $\varepsilon > 0$. In such domain we can define the field at time $t = 0$, $\hat{\phi}_0$, as the quadratic form[V] in $D_+ \times D_+$.[VI] In particular, let $\underline{f}, \underline{h} \in D_+$ and $g \in \mathcal{S}(\mathbb{R}^d)$, then the field at $t = 0$ is defined by

$$\langle \underline{f}| \, \hat{\phi}_0(g) \, |\underline{h}\rangle := \big(\underline{f}, (g \otimes \delta(\tau)) \times \underline{h}\big) \tag{5.5}$$

This is well-defined since all sequences in $D_+$ are moved away from the $\tau = 0$ axis, and then the previous object is always finite provided that both $\underline{f}, \underline{g} \in D_+$. Notice that the r.h.s. of eq. (5.5) is exactly the generalization of smeared field

$$\hat{\phi}(g) = \int \mathrm{d}\boldsymbol{x} \, \hat{\phi}(\boldsymbol{x}, 0) g(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x} \, \mathrm{d}\tau \, \hat{\phi}(\boldsymbol{x}, \tau) g(\boldsymbol{x}) \delta(\tau)$$

where we defined it as a quadratic function (i.e. using matrix elements) in an appropriate domain such that the delta function gives always a finite contribution.

Notice that for $\tau = 0$ the field is identical to its analytic continuation to the real time, hence $\hat{\phi}_0$ provides a well defined field for $t = 0$ where $t$ is the real time. We can then generalize the definition to arbitrary real time just using the unitary time evolution defined in Minkowski, and this can be done in such a way that the final field is an operator rather than just a quadratic form, just by appropriately smearing in

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[V]A *quadratic form* is a quantity defined in matrix elements only when both the vectors are in a specified dense domain. This provides a generalization of the concept of self-adjoint operator in some cases in which an operator would be unbounded.

Indeed, given an operator $O$ and its domain $D(O)$, $|\psi\rangle \in D(O)$, then $O|\psi\rangle \in \mathcal{H}$ and $\langle\phi|O|\psi\rangle$ is well defined for any $|\phi\rangle \in \mathcal{H}$. Conversely, a quadratic form $Q$ has domain $D(Q)$ such that $\langle\phi|Q|\psi\rangle$ is well defined for $|\phi\rangle, |\psi\rangle \in D(Q)$. Comparing the two definitions, the latter is weaker that the former, since in general $Q|\psi\rangle$ is not defined for $|\psi\rangle \in D(Q)$.

For instance, consider the formal map $Q : g \mapsto \delta(x) g(x)$ with the scalar product $(f, Qg) = \int \mathrm{d}x f(x) \delta(x) g(x)$. But the multiplication by $\delta(x)$ is not an operator, hence $Q$ can not be defined as an operator, but $Q$ can be defined as a quadratic form. Indeed one should require that the function $g(x)$ is not singular in $x = 0$, but this is not enough, one should also require that the same property is satisfied also by $f(x)$.

This problems can be avoided when one consider a system in QM with potential $V(x) = \delta(x)$. Indeed if we consider the operator $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ associated to the Schrödinger equation this is well-defined since the ground state of the wave function has a jump in the first derivative, so that the double derivative produces a delta function which cancels with the one produced by the potential.

Reference: [RS80, Section 8.6].

[VI]We are forced to use a quadratic form since in general it is not guaranteed that the field exists as an operator at a fixed time. For instance we have seen that this it is not possible if the wave function renormalization is vanishing $Z = 0$, as we previously discussed.

time the resulting field. In particular, let $f \in \mathcal{S}(\mathbb{R}^{d+1})$, $f(\boldsymbol{x}, t) = f_t(\boldsymbol{x})$, be a test function in Minkowski, then the following is a well-defined operator in a dense domain:

$$\hat{\phi}(f) = \int_{-\infty}^{+\infty} \mathrm{d}t \, e^{itH} \int \mathrm{d}\boldsymbol{x} \, \hat{\phi}_0(\boldsymbol{x}) f_t(\boldsymbol{x}) e^{-itH}$$

Indeed for any fixed value of $t$, the function $f_t(\boldsymbol{x})$ is an element of $\mathcal{S}(\mathbb{R}^d)$ such as $g$ in eq. (5.5), such that $\int \mathrm{d}\boldsymbol{x} \, \hat{\phi}_0(\boldsymbol{x}) f_t(\boldsymbol{x})$ is a well defined quadratic form. The function $f(\boldsymbol{x}, t)$ should be seen as a set of function of $\boldsymbol{x}$ labelled by $t$, and then smearing in $t$ we promote the quadratic form to an operator defined in a dense domain.

*(Locality)* We just make a comment about the proof of the locality for the field we constructed. It can be proved analogously to what we have done for the Wightman reconstruction theorem.

Let's consider two points $x, y$ space-like separated, it always exists a boost putting them at equal time and by time translational invariance we ban put both of them at $t = 0$. We denote the resulting space-like component by $\boldsymbol{x}'$ and $\boldsymbol{y}'$ respectively, $\boldsymbol{x}' \neq \boldsymbol{y}'$.

Now consider $\underline{f}, \underline{g} \in D_+$ and construct the following matrix elements

$$\langle \underline{f}| \, \hat{\phi}(\boldsymbol{x}', 0) e^{-\varepsilon H} \phi(\boldsymbol{y}', 0) \, |\underline{g}\rangle \quad \text{and} \quad \langle \underline{f}| \, \hat{\phi}(\boldsymbol{y}', 0) e^{-\varepsilon H} \phi(\boldsymbol{x}', 0) \, |\underline{g}\rangle$$

for some $\varepsilon > 0$. If one consider the matrix element on the l.h.s., the exponential $e^{-\varepsilon H}$ raised slightly the argument of $\phi(\boldsymbol{y}', 0) |\underline{g}\rangle$ above the $t = 0$ hyperplane. Looking at this matrix element from the right to the left, $\phi(\boldsymbol{y}', 0) |\underline{g}\rangle$ is well defined since $\underline{g} \in D_+$, but then also $e^{-\varepsilon H} \phi(\boldsymbol{y}', 0) |\underline{g}\rangle$ is an element of $D_+$ due to the effect of $e^{-\varepsilon H}$, and this finally imply that such matrix element is well defined since both the vectors on which the quadratic form $\hat{\phi}(\boldsymbol{x}', 0)$ acts on are elements of $D_+$. Similarly, also the matrix elements on the r.h.s. is well defined, just inverting the role of $\boldsymbol{x}'$ and $\boldsymbol{y}'$.

But then
$$\lim_{\varepsilon \to 0} \langle \underline{f}| \, \hat{\phi}(\boldsymbol{x}', 0) e^{-\varepsilon H} \phi(\boldsymbol{y}', 0) \, |\underline{g}\rangle = \lim_{\varepsilon \to 0} \langle \underline{f}| \, \hat{\phi}(\boldsymbol{y}', 0) e^{-\varepsilon H} \phi(\boldsymbol{x}', 0) \, |\underline{g}\rangle$$

due to symmetry of the Schwinger functions (we are still in the Euclidean space), which implies that, on matrix elements,
$$[\hat{\phi}(\boldsymbol{x}', 0), \hat{\phi}(\boldsymbol{y}', 0)] = 0$$

Now using again boosts (and Lorentz invariance) this vanishing condition apply also for the initial space-like separated points, and locality is proven.

The proof of all the other points of the theorem (e.g. uniqueness of the vacuum and analytical continuation of Schwinger functions) are omitted. □

## 5.2 Generalizations of OS reconstruction theorem

[Sum16], [GJ87, Section 6.1]

There are several variants of the OS reconstruction theorem, e.g. applied to lattice theories, to fields not supported on points (like Wilson loops in gauge theories, which in abelian case take the form $\exp\left(-\oint_C A\right)$, to finite temperature and also to soliton fields. We'll see in the following some examples. All these variants are based on the same idea, just changing appropriately the OS axioms depending on the situation.

For this reason, let us now pinpoint where the properties of OS axioms have been used:

(1) the OS positivity was used to construct $\mathcal{H}$ and $\Omega$;

(2)  (i) time translational invariance was used to construct the time evolution $e^{-tH}$ and the field operator $\hat{\phi}(x)$;

   (ii) space roto-translational invariance was used to construct $U(\boldsymbol{a}, R)$;

   (iii) euclidean symmetry was used to construct the full representation of the Poincaré group and, combined with OS positivity, to prove that $\sigma(\hat{P}^\mu) \subseteq \overline{V}^+$;

(3) cluster property was used to ensure the uniqueness of $\Omega$ as translational invariant state;

(4) symmetry was used to prove the locality of $\hat{\phi}$.

### 5.2.1 Lattice and gauge theories

[Frö80, Section 4], [OS78], [Sei82], [Str93, Section 6.4]

In view of this remarks is not difficult to understand the variants of OS reconstruction for lattice theories involving also non-local fields, such as the Wilson loop in gauge theories. For abelian theories with gauge potential $A_\mu$ the Wilson loop is defined as follows. Let $\mathcal{C}$ be a loop, then the *Wilson loop* is defined by[VII]

$$W(\mathcal{C}) = e^{i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu} \tag{5.6}$$

One can prove that the the collection of all $W(\mathcal{C})$ for each loop $\mathcal{C}$ completely determines a massless gauge theory. Therefore Wilson loops are those objects that contains all the information about a (massless) gauge theory in a gauge invariant way.

In the non-abelian case we need to take also a path ordering P($-$), which is just the time-ordering for the "time" parametrizing the loop, combined with the trace:

$$W(\mathcal{C}) = \operatorname{Tr} \operatorname{P} \left( e^{i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu} \right)$$

Consider a lattice whose sites are labelled by $i, j$, then for each pair of neighboring sites (neighboring sites are denoted by $\langle i, j \rangle$) we can define (for the abelian case)

$$U_{\langle i,j \rangle} := \operatorname{P} \left( e^{i \int_i^j A_\mu \mathrm{d}x^\mu} \right)$$

Then given a loop in the lattice let's define

$$W(\mathcal{C}) := \prod_{\langle i,j \rangle \in \mathcal{C}} U_{\langle i,j \rangle}$$

such that this object is exactly the lattice regularization of eq. (5.6).
We can now take

$$\langle W(\mathcal{C}_1) \ldots W(\mathcal{C}_n) \rangle$$

as the correlator for the lattice theory.
Using such regularization it has been computed (using computers) some very interesting results for QED and QCD, which gave very good approximations of experimental results.

**OS axioms for lattice theories**

Let $O_\alpha(\mathcal{C}_i)_{\alpha \in A}$ be lattice fields, each of them with support on a compact connected set of cells $\{\mathcal{C}_i\}_{i=1}^n$ of the lattice (the supports $\mathcal{C}_i$ can be points, lines, surfaces, etc, in the case of Wilson loops the support is given by a closed line). Let's define the correlation functions as expectation values of these fields

$$S_n(\alpha_1, \mathcal{C}_1, \ldots, \alpha_n, \mathcal{C}_n) := \langle O_{\alpha_1}(\mathcal{C}_1) \ldots O_{\alpha_n}(\mathcal{C}_n) \rangle$$

Notice that in general is not required that all the $\{\mathcal{C}_i\}$ are of the same type, for instance we can have both lines and surfaces. Let $F(\{O_{\alpha_i}(\mathcal{C}_i)\}) \in \mathcal{F}$ where $\mathcal{F}$ is the space of functions in these lattice fields with $\{\mathcal{C}_i\}$ supported on the positive "euclidean-time" half-lattice. Let $\theta$ denote the OS reflection, i.e. the reflection with respect to the euclidean time 0 space together with (for bosonic fields) complex conjugation. Moreover, set the lattice spacing to 1.

**Theorem 5.2** (OS reconstruction theorem for lattice theories). *Given a set of correlation functions $\{S_n(\alpha_1, \mathcal{C}_1, \ldots, \alpha_n, \mathcal{C}_n)\}_{n=0}^\infty$ satisfying*

*(1) lattice translational invariance,*

---

[VII]Recall that in QM not only the field strength is an observable. Due to Aharonov-Bohm effect also objects of the form eq. (5.6) are observables.

(2) *OS positivity: for every $F \in \mathcal{F}$*

$$\langle FOF \rangle \geq 0$$

*when expressed in terms of correlation functions $\{S_n\}$.*

*Then one can reconstruct from $\{S_n\}$*

(1') *a separable Hilbert space $\mathcal{H}$,*

(2') *a vector $|\Omega\rangle \in \mathcal{H}$, the vacuum,*

(3') *a self adjoint transfer matrix $T = e^{-H}$ and unitary space translation operators $U_\mu$, $\mu = 1, \ldots, d$, acting on $\mathcal{H}$ [VIII] such that $T|\Omega\rangle = |\Omega\rangle$, $U_\mu|\Omega\rangle = |\Omega\rangle$,*

*Moreover if cluster property holds, i.e.*

$$\lim_{a \to \infty} S_n(\alpha_1, \mathcal{C}_1 + a, \ldots, \alpha_j, \mathcal{C}_j + a, \alpha_{j+1}, \mathcal{C}_{j+1}, \ldots, \alpha_n, \mathcal{C}_n) =$$
$$= S_j(\alpha_1, \mathcal{C}_1, \ldots, \alpha_j, \mathcal{C}_j) S_{n-j}(\alpha_{j+1}, \mathcal{C}_{j+1}, \ldots, \alpha_n, \mathcal{C}_n)$$

*then*

(4') *$|\Omega\rangle$ is the unique translationally (both in space and time direction) invariant vector in $\mathcal{H}$,*

(5') *there is a set of vectors in $\mathcal{H}$, $\{|\alpha_1, \mathcal{C}_1, \ldots, \alpha_n, \mathcal{C}_n\rangle\}$ such that their linear combination is dense in $\mathcal{H}$ such that[IX]*

$$\langle \alpha_1, \mathcal{C}_1, \ldots, \alpha_m, \mathcal{C}_m | \alpha'_1, \mathcal{C}'_1, \ldots, \alpha_n, \mathcal{C}'_n \rangle =$$
$$= \langle O_{\alpha'_1}(\mathcal{C}'_1) \ldots O_{\alpha'_n}(\mathcal{C}'_n) \theta[O_{\alpha_1}(\mathcal{C}_1) \ldots O_{\alpha_m}(\mathcal{C}_m)] \rangle$$

(6') *there are some "field operators" $\hat{O}_\alpha(\mathcal{C})$, with $\mathcal{C}$ contained in the strip $[0, \tau]$, $\tau \in \mathbb{N} \setminus \{0\}$, such that*

$$\hat{O}_\alpha(\mathcal{C}) T(\tau) |\alpha_1, \mathcal{C}_1, \ldots, \alpha_n, \mathcal{C}_n\rangle = |\alpha, \mathcal{C}, \alpha_1, \mathcal{C}_1 + \tau, \ldots, \alpha_n, \mathcal{C}_n + \tau\rangle$$

*with $T(\tau) := T^\tau$ representing a time translation by $\tau$ time units.*

### 5.2.2 Reconstruction at finite temperature

[BF02], [Fro75]

Suppose that we have a set of Matsubara functions $S_n$ depending on creation and annihilation operators $a^\#_{\alpha_i}$, where $^\#$ denotes either annihilation $a_{\alpha_i}$ or creation $a^\dagger_{\alpha_i}$ operators, and euclidean times $\tau_i$, time ordered.

This are constructed starting from some expectation values of ladder operators at finite volume $V$, where we have finite degrees of freedom,

$$\langle a^\#_{\alpha_1}(\tau_1) \ldots a^\#_{\alpha_n}(\tau_n) \rangle^\beta_V$$

and then taking them thermodynamic limit

$$S^\beta_n(a^\#_{\alpha_1}, \tau_1, \ldots a^\#_{\alpha_n}, \tau_n) := \lim_{V \to \infty} \langle a^\#_{\alpha_1}(\tau_1) \ldots a^\#_{\alpha_n}(\tau_n) \rangle^\beta_V$$

Indeed in general functions $S^\beta_n$ cannot be defined as expectation values of operators defined in the infinite volume limit since such operators cannot exists due to Haag's theorem.

Due to usual boundary conditions operators $a_{\alpha_i}$ should be thought as operators defined on a circle of circumference $\beta$.

**Theorem 5.3** (OS reconstruction theorem for finite temperature). *Suppose that functions $\{S^\beta_n\}^\infty_{n=0}$ satisfy the following properties:*

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[VIII]The matrix $T$ describes the time evolution by a unit of time, while $U_\mu$ describe a space translation o f a unit of space in the direction given by $\mu$.

[IX]Notice that the scalar product in $\mathcal{H}$ is defined through the convolution $\theta$.

*(1) Regularity conditions in $a_{\alpha_i}$ and $\{\tau_i\}$.*

*(2) Time translational invariance*

$$S_n^\beta(a_{\alpha_1}^\#, \tau_1, \ldots a_{\alpha_n}^\#, \tau_n) = S_n^\beta(a_{\alpha_1}^\#, \tau_1 + \tau, \ldots a_{\alpha_n}^\#, \tau_n + \tau)$$

*for arbitrary $\tau \in \mathbb{R}$.*

*(3) KMS condition*

$$S_n^\beta(a_{\alpha_1}^\#, \tau_1, \ldots a_{\alpha_n}^\#, \tau_n) = S_n^\beta(a_{\alpha_{j+1}}^\#, \tau_1, \ldots a_{\alpha_n}^\#, \tau_n, a_{\alpha_1}^\#, \tau_1 + \beta, \ldots a_{\alpha_n}^\#, \tau_n + \beta)$$

*for any $j = 1, \ldots, n-1$.*

*(4) OS positivity: Consider all polynomials in $\{a_i^\#(\tau_i)\}$ variables, $\mathcal{P}(\{a_i^\#(\tau_i)\})$, such that the time variables are ordered and bounded by $\beta/2$, that is*

$$\tau_1 < \tau_2, \ldots \tau_n < \beta/2$$

*For such polynomial we define the convolution $\theta$ by*[X]

$$\theta\mathcal{P}(\{a_i^\#(\tau_i)\}) = \mathcal{P}(\{a_i^\#(\beta - \tau_i)\})^\dagger$$

*with again $\beta - \tau_i$ ordered. Then the OS positivity is the requirement*

$$\langle \mathcal{P}(\{a_i^\#(\tau_i)\})\theta\mathcal{P}(\{a_j^\#(\tau_j)\}) \geq 0$$

*for any couple of these polynomials, when the expectation value is written in terms of $\{S_n^\beta\}$.*

*Then the $\{S_n^\beta\}$ uniquely determine*

*(1') a separable Hilbert space $\mathcal{H}_\beta$,*

*(2') a vector $\Omega_\beta \in \mathcal{H}_\beta$,*

*(3') a continuous unitary one-parameter group $\{e^{itL}, t \in \mathbb{R}\}$ (t is now the real time, not the Euclidean one) leaving $\Omega_\beta$ invariant,*

*(4') a representation in terms of operators in $\mathcal{H}$ of the algebra of $a_{\alpha_i}^\#$, $\Pi(a_{\alpha_i}^\#)$ such that the functions $S_n^\beta$ are obtained from the real-time "Wightman" functions*

$$W_n(a_{\alpha_1}^\#, t_1, \ldots, a_{\alpha_n}^\#, t_n) = \left\langle \prod_{j=1}^n e^{it_j L}\Pi(a_{\alpha_i}^\#)e^{-it_j L}\Omega_\beta \middle| \Omega_\beta \right\rangle$$

*(the expression in the r.h.s. denotes the scalar product between elements of $\mathcal{H}_\beta$) by analytic continuation in the time variable $t_1, \ldots, t_n$ to the domain $(z_1, \ldots, z_n)$, $z_i \in \mathbb{C}$, with $\operatorname{Im} z_1 < \operatorname{Im} z_i < \operatorname{Im} z_{i+1} < \operatorname{Im} z_n < \operatorname{Im} z_1 + \beta$ and then restricting $(z_1, \ldots, z_n)$ to $(i\tau_1, \ldots, i\tau_n)$ with $0 < \tau_i < \tau_{i+1} < \beta$.*

From the real-time "Wightman" functions one can of course construct the retarded Green functions just using $\theta(t)$ and (anti)commutators.

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[X]The definition of such convolution is just what we expect adapting the definition of $\theta$ that we gave for $T = 0$. First it reflects the time variables in the circle of circumference $\beta$ respect to the line passing through $\tau = 0$ and $\tau = \beta/2$. Then takes the adjoint of the resulting polynomial, such that the order of $a^\#$ is inverted and each $a^\#$ is substituted with the corresponding adjoint. In some sense this is the appropriate adjustment of the reflection $\tau \to -\tau$.

**A comment on the Hilbert space $\mathcal{H}_\beta$**

For finite volume $\mathcal{H}_\beta$ constructed in the proof is the Hilbert space of operators $A$ such that $\operatorname{Tr} A^\dagger A < \infty$ with inner product and vacuum defined by

$$\langle A|B\rangle := \operatorname{Tr} A^\dagger B \quad \text{and} \quad |\Omega_\beta\rangle := \frac{|e^{-\beta H/2}\rangle}{\sqrt{\operatorname{Tr} e^{-\beta H}}}$$

where the denominator in the definition $|\Omega_\beta\rangle$ appropriately normalizes the state to 1.

Heuristically, the reason beside this strange structure of the Hilbert space is due to the fact that in finite temperature we are not considering an isolated system anymore, but a system in a thermal bath. Such thermal bath is an infinite volume system, and now we have to take care of the fluctuations coming from the thermal bath. This fluctuations induces an infinite renormalization of the energy, so if one proceeds as before defining the Hamiltonian starting from the temporal evolution would obtain a divergent result in the infinite volume limit. Therefore one should define a new operator $H$ in such a way that fluctuations coming from the thermal bath are killed. This turns out to be impossible using a Fock space, whereas is possible if one uses such strange Hilbert space.

This is also related to the fact that in statistical mechanics the Hamiltonian of a system in a thermal bath cannot be positive definite, so one has to consider the Liouvillian $L$ instead of the Hamiltonian $H$. In this case these objects are just related by a renormalization constant.

More formally, in $\mathcal{H}_\beta$ at finite volume we have two commuting representation of observables $A(\{a_\alpha^\#\})$:

$$\Pi(A)|B\rangle = |AB\rangle \quad \text{and} \quad \Pi_r(A)|B\rangle = |BA^\dagger\rangle \tag{5.7}$$

where the first one is the one used in the statement of the theorem. It turns out that the generator of time evolution is

$$L = \Pi(H) - \Pi_r(H) \tag{5.8}$$

The subtraction of $\Pi_r(H)$, since $\Pi_r$ and $\Pi$ commutes, does not change the equations of motion of the observable in $\Pi$. Indeed $\Pi_r(H)$ acts as a constant, and cancel the infinities appearing in $H$ in the infinite volume limit.

Let us see the relation with the standard formalism with the Boltzmann weight $e^{-\beta H}$. For a finite system at finite $T$ we have two equivalent descriptions: in the Fock space $\mathcal{F}$ the equilibrium state is a mixed states of the form

$$\rho = \frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \tag{5.9}$$

whereas in the Hilbert space $\mathcal{H}_\beta$ defined above the equilibrium state $\Omega_\beta$ is a vector of the form $|\sqrt{\rho}\rangle$ (instead of a mixed state) which is a eigenvector of $L = \Pi(H) - \Pi_r(H)$ of zero eigenvalue, indeed using eq. (5.7) ad eq. (5.8) one gets

$$L|\sqrt{\rho}\rangle = \frac{|He^{-\beta H/2}\rangle - |e^{-\beta H/2}H^\dagger\rangle}{\sqrt{\operatorname{Tr} e^{-\beta H/2}}} \overset{H=H^\dagger}{=} 0 \tag{5.10}$$

since the two vectors in the numerator cancel each other.

Notice that as $V \to \infty$ the expectation value of the energy in $\rho$ diverges

$$\frac{\operatorname{Tr} He^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \to \infty$$

thus $\rho$ in eq. (5.9) diverges as operator as $V \to \infty$. Conversely (the symbol $(\bullet)$ should be substituted with some operator)

$$\langle \Omega_\beta|\,(\bullet)\,|\Omega_\beta\rangle_V = \frac{\operatorname{Tr}_V(\bullet)e^{-\beta H}}{\operatorname{Tr}_V e^{-\beta H}}$$

provides an expectation value, respect to the vector $|\Omega_\beta\rangle \in \mathcal{H}_\beta$, well defined even in the infinite volume limit (for operators in an appropriate domain, from which $H$ is excluded since is unbounded) which still satisfies $L|\Omega_\beta\rangle > 0$ (thanks to eq. (5.10)) and the KMS condition.

**Summary**

The procedure to construct retarded Green functions is the following: first one constructs the Fock space for the theory at finite temperature appropriately regularized (lattice and finite volume regularization), then one computes the Matsubara Green functions in the Fock space using the path integral method and assumes that the thermodynamic limit exists. If Matsubara functions in the thermodynamic limit exists and satisfy the axioms then one can reconstruct completely the theory even in the thermodynamic limit. Obviously the Hilbert space we constructed has nothing to do with the original Fock space, and in particular is not the generalization of the original Fock space but instead is the generalization of the finite volume $\mathcal{H}_\beta$, which is a space of operators. By analytical continuation one defines an analogue of Wightman functions and combining them with Heaviside functions and commutators one finally gets the desired retarded Green functions.

# Part II

# Topological structures in QFT

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