Chapter 1

Gravitation

1.1 The action in the curved space

Carroll, sec. 4.3

Now we can obtain well defined actions for various kinds of matter fields Φ as follows [I]

$$S_\Phi = \int_M \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}(\Phi, \nabla\Phi)$$

From such action we expect to get fully covariant e.o.m.

For instance, take the Lagrangian for a real scalar field

$$S_\phi = \int \mathrm{d}^4x\,\sqrt{-g}\left[-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right]$$

If we use the least action principle in order to obtain the e.o.m. we obtain for a fixed $\delta\phi$:

$$\begin{aligned}\delta S_\phi &= \int \mathrm{d}^4x\,\sqrt{-g}\left(-\nabla_\mu\delta\phi\nabla^\mu\phi - \delta\phi\frac{\partial V}{\partial\phi}\right)\\ &= \int \mathrm{d}^4x\,\sqrt{-g}\left(\nabla_\mu\nabla^\mu\phi - \frac{\partial V}{\partial\phi}\right)\end{aligned}$$

and imposing this to be zero we obtain the e.o.m. for a real scalar field

$$\boxed{\nabla_\mu\nabla^\mu\phi = \frac{\partial V}{\partial\phi}} \tag{1.1}$$

which is a fully covariant equation.

For the EM field we obtain

$$S_A = \int \mathrm{d}^4x\,\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right) \qquad \text{with} \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

and for a fixed δA_μ we have

$$\begin{aligned}\delta S_A &= \int \mathrm{d}^4x\,\sqrt{-g}\left[-\left(\nabla_\mu\delta A_\nu\right)F^{\mu\nu} + \delta A_\mu J^\mu\right]\\ &= -\underbrace{\int \mathrm{d}4x\sqrt{-g}\nabla_\mu(\delta A_\nu F^{\mu\nu})}_{\int \mathrm{d}^4x\,\partial_\mu\left(\sqrt{-g}\delta A_\nu F^{\mu\nu}\right)=0} + \int \mathrm{d}^4x\,\sqrt{-g}\delta A_\nu\left(\nabla_\mu F^{\mu\nu} - J^\nu\right)\end{aligned}$$

[I] We use the compact notation $\sqrt{-g} := \sqrt{-\det g}$.

hence we obtain the e.o.m.

$$\boxed{\nabla_\mu F^{\mu\nu} = J^\nu \qquad \Rightarrow \qquad \nabla_\mu J^\mu = 0} \tag{1.2}$$

where the relation on the r.h.s. is given by properties of $F^{\mu\nu} = 0$. Again, these e.o.m. are fully covariant.

Clearly, these e.o.m.'s (approximately) reduce to the special relativistic ones by restricting to LIFs e.g. normal coordinates) $\hat{x}^\alpha$ in which $\Gamma \simeq 0$. They provides an explicit realization of the EEP and describe how the dynamics of matter field are affected by a non-trivial gravitational field.

1.1.1 Einstein-Hilbert action

Carroll, sec. 4.2-4.3

So far the metric has been considered non-dynamical, realizing the EEP. We now face the opposite problem:

- What are the e.o.m. of the metric?
- How is it affected by matter and gauge fields?

We will not follow a historical path, but again invoke the *least action principle*. In fact, the matter and gauge Lagrangians $\mathcal{L}(\Phi, \nabla\Phi)$ discussed above already contain interaction terms between gauga/matter fields and the metric.

We can however wonder whether there can be purely gravitational terms. In particular, we should look for a coordinate-independent 2-derivative action:

$$\int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}_G(x)$$

So the question is:

- Which 2-derivatives scalar Lagrangians $\mathcal{L}_G(x)$ can we construct out of $g_{\mu\nu}$?

There is only one possibility: $\mathcal{L}_G \propto R$. It is then natural to include in the action the **Einstein-Hilbert term**:

$$S_{\mathrm{EH}}[g] = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}R$$

Notice that $[R] = L^{-2}$, $[\mathrm{d}^4x\,\sqrt{-g}] = L^4$, $[S] = ET$ and

$$[\kappa] = \left[\frac{L^2}{ET}\right] = \left[\frac{T}{M}\right] = \frac{[G]}{c^3}$$

Indeed we will see that $\kappa \sim \frac{G}{c^3}$.

We are then lead to consider 2-derivative actions of the form

$$S = S_{\mathrm{EH}}[g] + S_\Phi = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}R + S_\Phi$$

Extremization with respect to Φ produces the equations written above. We instead still have to comput the *metric e.o.m.* obtained by extremizing with respect to $g_{\mu\nu}$.

We will do this in two steps, described in following section.

1.2 The Einstein equations

Let's start from the action

$$S = S_{\mathrm{EH}}[g] + S_\Phi = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}R + S_\Phi$$

a) Variation of $S_{\mathbf{EH}}$

We may extremize with respect to $\delta g_{\mu\nu}$. However this is completely equivaent to extremizing with respect to to $\delta g^{\mu\nu}$ since

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \qquad \Rightarrow \qquad \delta g^{\mu\nu} g_{\nu\rho} + g^{\mu\nu}\delta g_{\nu\rho} = 0$$

implies

$$\boxed{\delta g^{\mu\nu} = -\delta g_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu}} \tag{1.3}$$

hence $\delta g_{\mu\nu}$ and $\delta g^{\mu\nu}$ are in one-to-one correspondence.

We get two contributions

$$\delta \int \mathrm{d}^4x\, \sqrt{-g} R = \delta \int \mathrm{d}^4x\, \sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \int \mathrm{d}^4x\, (\underbrace{\delta\sqrt{-g}}_{1}\, R + \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\, \underbrace{g^{\mu\nu}\delta R_{\mu\nu}}_{2})$$

Let's rewrite 1 and 2. For 1 we observe that ($g = g_{\mu\nu}$)

$$\boxed{\log(\det g) = \mathrm{Tr}(\log g)} \tag{1.4}$$

indeed $\det g = \prod_a \lambda_a$ and $\mathrm{Tr}(\log g) = \sum_a \log \lambda_a$. Then

$$\begin{aligned} \delta \det g &= \delta e^{\log(\det g)} = \delta e^{\mathrm{Tr}\log g} = e^{\mathrm{Tr}\log g}\, \mathrm{Tr}(\delta \log g) \\ &= (\det g)\, \mathrm{Tr}\, g^{-1}\delta g = (\det g) g^{\mu\nu}\delta g_{\mu\nu} = -(\det g)\delta g^{\mu\nu} g_{\mu\nu} \end{aligned}$$

But $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta \det g$ and so

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\delta g^{\mu\nu} g_{\mu\nu}} \tag{1.5}$$

For 2 since $\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu}$ we have

$$\begin{aligned} \delta R^\rho{}_{\mu\sigma\nu} &= \delta[\partial_\sigma \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - (\sigma \leftrightarrow \nu)] \\ &= \partial_\delta \Gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\delta\Gamma^\rho_{\mu\sigma} - \delta\Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\lambda\nu}\delta\Gamma^\lambda_{\mu\sigma} \end{aligned}$$

Recall that $\delta\Gamma^\mu_{\nu\rho}$ is a tensor

$$\nabla_\sigma \delta\Gamma^\rho_{\mu\nu} = \partial_\sigma\delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\delta\Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\nu\sigma}\delta\Gamma^\rho_{\mu\lambda}$$

And using the latter and the corresponding equation for $\nu \leftrightarrow \sigma$ we obtain

$$\delta R^\rho{}_{\mu\sigma\nu} = \nabla_\sigma\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\mu\sigma}$$

which implies

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\sigma_{\mu\sigma}$$

and

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\rho\mu}\delta\Gamma^\sigma_{\mu\sigma}\right) \equiv \nabla_\rho \delta V^\rho$$

Thanks to the results of sec. **??** we can write the result in the form of a covariant derivative whose integration vanishes because of vanishing boundary terms

$$\int \mathrm{d}^4x\, \sqrt{-g}\nabla_\rho\delta V^\rho \equiv \int \mathrm{d}^4x\, \partial_\rho\left(\sqrt{-g}\delta V^\rho\right) = 0$$

Putting these results together, we then arrive at the conclusion that

$$\boxed{\begin{aligned} \delta S_{\mathrm{EH}} &= \frac{1}{2\kappa}\int \mathrm{d}^4x\, \sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) \\ &\equiv \frac{1}{2\kappa}\int \mathrm{d}^4x\, \sqrt{-g}\delta g^{\mu\nu} G_{\mu\nu} \end{aligned}} \tag{1.6}$$

where we used the definition of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

b) Variation of S_Φ

In general we simply define the **energy-momentum** tensor (or **stress-energy** tensor) $T_{\mu\nu}$ by

$$\boxed{\delta S_\Phi = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu} \equiv \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\delta g_{\mu\nu}T^{\mu\nu}} \tag{1.7}$$

Notice that, by definition, $T_{\mu\nu}$ is *symmetric*

$$\boxed{T_{\mu\nu} = T_{\nu\mu}} \tag{1.8}$$

It does not always coincides with the "canonical" $\tilde{T}_{\mu\nu}$ but Belifante showed that one can "improve" $\tilde{T}_{\mu\nu}$ to get $\tilde{T}^{\mu\nu}$ to get $T^{\mu\nu}$.

Then the application of the least action principle $\delta S = \delta S_{\text{EH}} + \delta S_\Phi = 0$ leads to the **Einstein equations** (EE)

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu}} \tag{1.9}$$

Notice that $\nabla^\mu G_{\mu\nu} = 0$ implies

$$\boxed{\nabla^\mu T_{\mu\nu} = 0} \tag{1.10}$$

1.3 The energy-momentum tensor

Carroll, sec. 1.9-1.10

Let's analyze the energy momentum tensor $T_{\mu\nu}$ in order to understand its meaning and properties.

Consider the Minkowski spacetime, we have

$$\boxed{P^\mu = \int \mathrm{d}^3x\,T^{0\mu}(x) \qquad \Rightarrow \qquad E = \int \mathrm{d}^3x\,T^{00} \qquad P^i = \int \mathrm{d}^3x\,T^{0i}} \tag{1.11}$$

hence $T^{00}(x)$ and $T^{0i}(x)$ should be meant respectively as "energy density" and "(linear) momentum density".

On the other side, the interpretation of T^{ij} requires more work. Let

$$P_V^j(t) := \int_V \mathrm{d}^3x\,T^{0j}(x) \tag{1.12}$$

then using Stoke's theorem we obtain[II]

$$\frac{\mathrm{d}P_V^j}{\mathrm{d}t} = -\int_V \mathrm{d}^3x\,\partial_i T^{ij} = -\int_S \mathrm{d}S_i\,T^{ii} =: F_V^j \tag{1.13}$$

with $S := \partial V$ is the boundary of V and $\mathrm{d}\mathbf{S}$ is the outward pointing surface form on S:

Hence we can interpret the object F_V^j as the **force exerted on V** and consequently T^{ij} as the force per unit area pushing in the j-direction surface portion. For this reason T^{ij} is called **(3D) stress tensor**. In particular $T^{ii} =: \mathcal{P}$ is the **pressure**, while T^{ij} for $i \neq j$ are **shear-terms**.

In more general spacetimes, this interpretation holds only locally and approximately in a LIF (e.g. of normal coordinates).

[II] The minus sign introduced is due to $\frac{\mathrm{d}}{\mathrm{d}t}T^{0j} = -\partial_i T^{ij}$ given by Equation (1.11) in the flat metric.

1.3.1 The point particle case

In fact, we will see that this is a subtle concept, once one takes into account the particle's backreaction. But let us for the moment ignore these issues. We already know the particle's action:

$$\begin{aligned} S_{\text{part}} &= \frac{1}{2}\int \mathrm{d}\lambda\,[e^{-1}g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu - m^2 e] \\ &= \frac{1}{2}\int \mathrm{d}\lambda \int \mathrm{d}^4x\,\delta^4(x - X(\lambda))[e^{-1}g_{\mu\nu}(x)\dot{X}^\mu\dot{X}^\nu - m^2 e] \end{aligned} \tag{1.14}$$

Now consider an infinitesimal metric variation $\delta g_{\mu\nu}$:

$$\begin{aligned} \delta S_{\text{part}} &= \frac{1}{2}\int \mathrm{d}^4x\,\delta g_{\mu\nu}(x)\int \mathrm{d}\lambda\,e^{-1}\delta^4(x - X(\lambda))\dot{X}^\mu\dot{X}^\nu \\ &=: \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\delta g_{\mu\nu}T^{\mu\nu} \end{aligned} \tag{1.15}$$

i.e.

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = \int \mathrm{d}\lambda\,e^{-1}\dot{X}^\mu\dot{X}^\nu\frac{\delta^4(x - X(\lambda))}{\sqrt{-g(x)}}} \tag{1.16}$$

where the term $\frac{\delta^4(x-X(\lambda))}{\sqrt{-g(x)}}$ transform as a scalar.

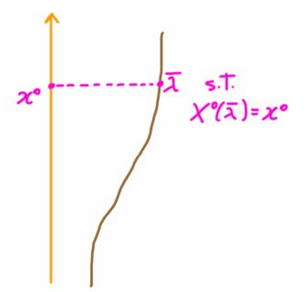
We can take proper λ: $e(\lambda) = 1$ so that $\dot{X}^\mu = p^\mu(\lambda)$ with $p^\mu p_\mu = -m^2$, in this case

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = \int \mathrm{d}\lambda\,p^\mu(\lambda)p^\nu(\lambda)\frac{\delta^4(x - X(\lambda))}{\sqrt{-g(x)}}} \tag{1.17}$$

is covariant. Furthermore we can rewrite

$$\delta^4(x - X(\lambda)) = \delta(x^0 - X^0(\lambda))\delta^3(\mathbf{x} - \mathbf{X}(\lambda)) = \frac{\delta(\lambda - \bar{\lambda})}{\dot{X}^0(\bar{\lambda})}\delta^3(\mathbf{x} - \mathbf{X}(x^0)) = \frac{\delta(\lambda - \lambda_0)}{p^0(x^0)}\delta^3(\mathbf{x} - \mathbf{X}(x^0)) \tag{1.18}$$

where in the last step we assumed $\dot{X}^0 > 0$:



Hence

$$\boxed{T^{\mu\nu}_{(\text{part})}(x) = p^\mu(t)p^\nu(t)\frac{\delta^3(\mathbf{x} - \mathbf{X}(t))}{p^0(t)\sqrt{-g(x,\mathbf{x})}}} \tag{1.19}$$

In a LIF $\hat{x}^\alpha = (\hat{x}^0, \hat{x}^a)$ we get

$$\begin{aligned} \hat{T}^{00} &= \hat{p}^0\delta^3(\mathbf{x} - \mathbf{X}(t)) \quad \textbf{energy density} \\ \hat{T}^{0a} &= \hat{p}^a\delta^3(\mathbf{x} - \mathbf{X}(t)) \quad \textbf{momentum density} \end{aligned} \tag{1.20}$$

1.3.2 Perfect fluids

At large enough distances, many macroscopic systems can be approximated as **perfect fluids**, that is, such that any observer solidal with any fluid element, sees the fluid around him as isotropic. This happens if the free path of the interacting particles is much shorter than the relevant length scales.

In the rest LIF $\hat{x}^\alpha$ of each fluid element, we then have

$$\hat{T}^{00} = \varepsilon \quad , \quad \hat{T}^{ab} = \mathcal{P}\delta^{ab} = 0 \quad , \quad \hat{T}^{0a} = 0 \tag{1.21}$$

where ε is the **rest energy density** and $\mathcal{P}$ is the **rest pressure**. In matrix notation:

$$\hat{T}^{\alpha\beta} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix} \tag{1.22}$$

Since $\hat{u}^\alpha = (1, \mathbf{0})$ and $\hat{g}_{\alpha\beta} = \eta_{\alpha\beta}$ then

$$\hat{T}^{\alpha\beta} = (\varepsilon + \mathcal{P})\hat{u}^\alpha \hat{u}^\beta + \mathcal{P}\eta^{\alpha\beta} \tag{1.23}$$

which can be immediately covariantized into

$$\boxed{T^{\mu\nu} = (\varepsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}} \tag{1.24}$$

Perfect fluids of particles

We already know that for point particles we have

$$g^{\mu\nu}(x) = \sum_I \frac{p_I^\mu p_I^\nu}{p_I^0} \frac{\delta^3(\mathbf{x} - \mathbf{X}(t))}{\sqrt{-g}} = \sum_I \int \mathrm{d}\lambda\, p_I^\mu(\lambda) p_I^\nu(\lambda) \frac{\delta^4(x - X_I(\lambda))}{\sqrt{-g}} \tag{1.25}$$

The perfect fluid regime then implies that, in the local fluid rest frame

$$\varepsilon = \sum_I E_I \delta^3(\hat{x} - \hat{X}_I(t)) \quad , \quad \mathcal{P} = \frac{1}{3}\sum_I \frac{|\mathbf{p}_I|^2}{E_I}\delta^3(\hat{x} - \hat{X}_I(t)) \tag{1.26}$$

Since $E_I = \sqrt{m_I^2 + |\mathbf{p}_I|^2}$ then we have $|\mathbf{p}_I|^2 \leq E_I^2$ and hence

$$\boxed{0 \leq \mathcal{P} \leq \frac{1}{3}\varepsilon} \tag{1.27}$$

We can now go further in our analysis in some special regimes, which will be described in the following paragraphs.

Cool non-relativistic gas

Such regime correspond to the case $E_I \simeq m_I + \frac{\mathbf{p}_I^2}{2m}$, then we have

$$\begin{aligned} \mathcal{P} &\simeq \frac{1}{3}\sum_I \frac{|\mathbf{p}_I|^2}{m_I^2}\delta^3(\hat{x} - \hat{X}(t)) \\ \varepsilon &\simeq \sum_I (m_I + \frac{|\mathbf{p}_I|^2}{2m_I})\delta^3(x - \hat{X}_I(t)) = \rho(x) + \frac{3}{2}\mathcal{P}(x) \end{aligned} \tag{1.28}$$

where $\rho(x)$ denotes the **mass density in the fluid comoving frame**. Then the space-like matrix elements of $\hat{T}^{\mu\nu}$ are

$$\hat{T}^{ab} = \begin{pmatrix} \rho + \frac{3}{2}\mathcal{P} & 0 \\ 0 & \mathcal{P}\delta^{ab} \end{pmatrix} \tag{1.29}$$

and then

$$\boxed{T^{\mu\nu}_{\text{non-rel}} \simeq (\rho + \frac{5}{2}\mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}} \tag{1.30}$$

In particular, the **extreme** non-relativistic limit $\mathcal{P} \simeq 0$ correspond to the **pressureless dust**

$$\boxed{g^{\mu\nu}_{\text{dust}} = \varepsilon u^\mu u^\nu} \tag{1.31}$$

with $\varepsilon = \rho$.

Ultra-relativistic regime

Such regime correspond to the case $E_I \simeq |\mathbf{p}_I|$, then we have

$$\boxed{\mathcal{P} \simeq \frac{1}{3}\varepsilon} \tag{1.32}$$

For instance, this is the case for the **gas of photons**. Then

$$\boxed{T^{\mu\nu} = \frac{4}{3}\varepsilon u^\mu u^\nu + \frac{1}{3}\varepsilon g^{\mu\nu}} \tag{1.33}$$

which implies $T^\mu_\mu = 0$.

A particular perfect fluid

Consider a **cosmological constant term**

$$-\frac{\Lambda}{k}\int \mathrm{d}^4x\,\sqrt{-g} \subset S_M \tag{1.34}$$

where $[\Lambda] = L^{-2}$ is known as **cosmological constant**. Then we see that

$$\boxed{g^{\mu\nu}_\Lambda = -\frac{\Lambda}{k}g^{\mu\nu}} \tag{1.35}$$

correspond to a perfect fluid with

$$\boxed{\varepsilon = \frac{\Lambda}{k}} \tag{1.36}$$

1.3.3 Example: The expanding universe

Example 1: The expanding universe

Consider the evolving universe described by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1.37}$$

filled by a spatially homogeneous perfect fluid which is at rest in comoving coordinates $\mathbf{x}$. Then, since $u^\mu = (1, \mathbf{0})$ we get

$$T_{\mu\nu} = (\varepsilon + \mathcal{P})u_\mu u_\nu + \mathcal{P}g_{\mu\nu} = T_{\mu\nu} = \begin{pmatrix} \varepsilon & 0 \\ 0 & a^2\mathcal{P}\delta_{ij} \end{pmatrix} \tag{1.38}$$

with time-dependent energy and pressure, $\varepsilon = \varepsilon(t)$, $\mathcal{P} = \mathcal{P}(t)$. Then assume an **equation of state**

$$\boxed{\mathcal{P} = w\varepsilon} \tag{1.39}$$

where w is choosen depending on the following cases:

$$\begin{cases} w &= 0 \quad \text{matter dominated case} \\ w &= \dfrac{1}{3} \quad \text{radiation dominated case} \\ w &= -1 \quad \text{vacuum dominated case} \end{cases} \tag{1.40}$$

We already computed non-vanishing Christoffel symbols:

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij} \quad , \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j \tag{1.41}$$

Then one can check (*exercize!*) that **continuity equation** $\nabla^\mu T_{\mu\nu} = 0$ gives

$$\boxed{\frac{\dot{\varepsilon}}{\varepsilon} = -3(1+w)\frac{\dot{a}}{a}} \tag{1.42}$$

We then get

$$\boxed{\varepsilon(t) = \varepsilon_0[a(t)]^{-3(1+w)}} \tag{1.43}$$

i.e.

$$\begin{aligned} &\varepsilon \propto \frac{1}{a^3} \text{ for matter domainated universe } (\sim \text{rescaling of physical volume}) \\ &\varepsilon \propto \frac{1}{a^4} \text{ for radiation domainated universe } (\sim \text{rescaling of physical volume + cosmological redshift}) \\ &\varepsilon = \varepsilon_0 \text{ for vacuum domainated universe (with } \Lambda = k\varepsilon_0) \end{aligned} \tag{1.44}$$

We also computed the Einstein tensor

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 \quad , \quad G_{ij} = -(\dot{a}^2 + 2a\ddot{a})\delta_{ij} \tag{1.45}$$

Then, from Einstein's equation $G_{\mu\nu} = kT_{\mu\nu}$ we get

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}k\varepsilon} \tag{1.46}$$

and

$$-(\dot{a}^2 + 2a\ddot{a}) = k\mathcal{P}a^2 \quad \Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = -\frac{k}{6}(\varepsilon + 3\mathcal{P})} \tag{1.47}$$

these are known as the **Friedmann equations** for (3d-flat) FRW universes.
Moreover assuming $\mathcal{P} = w\varepsilon$ we get

$$\begin{aligned} \mathcal{P} = w\varepsilon &\implies \varepsilon \propto a^{-3(1+w)} \implies \dot{a}^2 \propto a^{-1-3w} \\ &\implies a^{\frac{1}{2}+\frac{3}{2}w}\mathrm{d}a \propto \mathrm{d}t \implies a(t)^{\frac{3}{2}(1+w)} \propto t \\ &\implies a(t) \propto t^{\frac{2}{3(1+w)}} \end{aligned} \tag{1.48}$$

We also notice that the case $w = -1$ corresponds to $\varepsilon = -\mathcal{P} = \varepsilon_0$ which gives a **constant** Hubble parameter

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{1}{3}k\varepsilon_0} = \sqrt{\frac{\Lambda}{3}} \tag{1.49}$$

and then

$$a(t) = a_0 e^{Ht} = a_0 e^{\sqrt{\frac{\Lambda}{3}}t} \tag{1.50}$$

1.4 Weak field gravitational physics