Contents

Chapter 1

Prerequisites

1.1 Topological Spaces

Definion 1.1.1. Let X be a set and $\mathcal{T} = \{U_i | i \in I, U_i \in X\}$. $(X, \mathcal{T})$ is a **topological space** if

1. $\phi, X \in \mathcal{T}$
2. $\forall J \subset I$, $\bigcup_{j \in J} U_j \in \mathcal{T}$
3. $\forall K \subset I$, K finite, $\bigcap_{k \in K} U_k \in \mathcal{T}$

(X is the topological space, $\mathcal{T}$ its topology, U_i the open sets)

Example 1.1.2.

(a) X is a set and $\mathcal{T}$ the collection of all subset of X
This is called the *Discrete Topology.*
(b) X is a set and $\mathcal{T} = \{\phi, X\}$. This is called the *Trivial Topology.*
(c) $X = \mathbb{R}$, $\mathcal{T}$ all open intervals and their unions (the usual topology).
Notice that $\bigcap_{\substack{a<-1 \\ b>+1}} (a, b) = [-1, +1]$ so we have an example of infinite intersection of open sets that does not give an open.
(d) X is set, $d : X \times X \to \mathbb{R}$ a metric, that is, $\forall x, y \in X$:
   (a) $d(x, y) = d(y, x)$
   (b) $d(x, y) \geq 0$ with $d(x, y) = 0 \Leftrightarrow x = y$
   (c) $d(x, y) + d(y, x) \geq d(x, z)$

   $\mathcal{T}$ open discs $U_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$ and unions. This is called the *Metric Topology.*
(e) $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ topological spaces
$X \times Y$, $\mathcal{T}" = \{\text{unions of } U \times V | U \in \mathcal{T}, V \in \mathcal{T}'\}$
This is called the *Product Topology*

1.1.1 Continuous maps

Definion 1.1.3. X,Y topological spaces. $f : X \to Y$ is **continuous** if $\forall\, V \underset{\text{open}}{\subset} Y$, $f^{-1}(V) \underset{\text{open}}{\subset} X$.

Remark 1.1.4. Previous definition doesn't work with direct images U and $f(U)$. For example take $f(x) = x^2$, then $U = (-1, +1)$ is open but $f(U) = [0, 1)$ doesn't. Nevertheless f is a continuous function.

1.1.2 Neighbourhoods and Haudorff spaces

Definion 1.1.5. $(X, \mathcal{T})$ topological space. $N \subset X$ is a **neighbourhood** of $x \in X$ if $\exists U \in \mathcal{T}$ such that $x \in U \subset N$

Example 1.1.6. $X = \mathbb{R}$ with the usual topology. $[-1, +1]$ is a neighbourhood of $0 \in \mathbb{R}$.

Definion 1.1.7. $(X, \mathcal{T})$ topological space is **Hausdorff** if $\forall x, y \in X$ such that $x \neq y$ $\exists N_1$ neighbourhood of x, $\exists N_2$ neighbourhood of y, such that $N_1 \cap N_2 = \emptyset$.

Exercise 1.1.8. $X = \{\text{John, Paul, Ringo, George}\}$, $U_0 = \emptyset$, $U_1 = \{\text{John}\}$, $U_2 = \{\text{John, Paul}\}$, $U_3 = \{\text{John, Paul, Ringo, George}\}$. Show $\mathcal{T} = \{U_0, U_1, U_2, U_3\}$ is a topology and that $(X, \mathcal{T})$ is not Hausdorff

Exercise 1.1.9. Show $\mathbb{R}^n$ is Hausdorff. Show any metric space is Hausdorff.

We will always assume all our topological spaces to be Hausdorff.

1.1.3 Closed sets

Definion 1.1.10. $(X, \mathcal{T})$ topologocal space. $A \in X$ is **closed** iff $X \setminus A$ is open.

Example 1.1.11. $\emptyset$, X are both open and closed

Definion 1.1.12. The **closure** $\overline{A}$ of any subset $A \in X$ is the smallest closed set containing A.
The **interior** A^0 of any subset $A \in X$ is the largest open subset of A.
The **boundary** $b(A)$ or ∂A of any subset $A \in X$ is $\overline{A} \setminus A^0 =: b(A)$

1.1.4 Compactness

Definion 1.1.13. $(X, \mathcal{T})$ topological space. A family $\{A_i\}_{i \in I}$ is called a **covering** of X if $\bigcup_{i \in I} A_i = X$. If all of the A_i $\forall i \in I$ are open, $\{A_i\}_{i \in I}$ is an **open covering**

Definion 1.1.14. $(X, \mathcal{T})$ topological space. X is **compact** if $\forall$ open covering $\{A_i\}_{i \in I}$ $\exists J \subset I$ finite such that $\{A_j\}_{j \in J}$ is also an open covering

Theorem 1.1.15. *$X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

Exercise 1.1.16. (a) A point is compact.
(b) Let $(a, b) \subset \mathbb{R}$ and the open covering $U_n = (a, b - \frac{1}{n})$, $n \in \mathbb{N}$. Indeed $\bigcup_{n \in \mathbb{N}} U_n = (a, b)$. If no finite subfamilies of $\{U_n\}_{n \in \mathbb{N}}$ is a covering then (a, b) is not compact.
(c) $S^n = \{x_1^2 + \dots x_n^2 = 1 | (x_1, \dots, x_n) \in \mathbb{R}^n\}$ is compact since it is closed and bounded.

1.1.5 Connectedness

Definion 1.1.17. (a) X topological space is **connected** if it cannot be written as $X = X_1 \cup X_2$ where X_1, X_2 are open and $X_1 \cap X_2 = \emptyset$. Otherwise X is called **disconnected**.
(b) X is **arcwise connected** if, $\forall x, y \in X$, $\exists f : [0,1] \to X$ continuous with $f(0) = x$, $f(1) = y$.
(c) A **loop** in a topological space is a continuous map $f : [0,1] \to X$ with $f(0) = f(1)$. X is **simply connected** if, for any loop $f : [0,1] \to X$, there exists a continuous map $g : [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} &g(0,t) = g(1,t) && \text{(i.e. } g(\cdot, t) =: f_t(\cdot) \text{ is a family of loops)} \\ &g(s,0) = f(s) && \\ &g(s,1) = \bar{x} \in X && (f_t \text{ shrikes } f_0 = f \text{ to a point } f_1 = \bar{x}) \end{aligned}$$

Example 1.1.18. (a) $\mathbb{R}$ is arwise connected while $\mathbb{R} \setminus \{0\}$ is not. $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, are both arwise connected.
(b) S^n is arwise connected $\forall n \geq 1$ but simply connected only for $n \geq 2$.
(c) $T^n := \underbrace{S^1 \times \dots \times S^1}_{n\text{-times}}$ is arcwise but not simply connected
(d) $\mathbb{R}^2 \setminus \mathbb{R}$ is not arcwise connected. $\mathbb{R}^2 \setminus \{0\}$ is arcwise but not simply connected. $\mathbb{R}^3 \setminus \{0\}$ is both arcwise and simply connected.

1.1.6 Homeomorphisms