Notes on Non-Perturbative Quantum Field Theory

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Compiled: July 10, 2021

University of Padua

Academic year 2020-2021

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Chapter 1

Introduction

Quantum Field Theory (QFT) was born from an attempt to solve inconsistencies of the Dirac's relativistic quantum mechanics (RQM) when the interaction with the electromagnetic field is introduced. However, very soon it became a common framework in many branches of physics, exhibiting an unexpected unity in the description of elementary quantum processes that deeply modifies our view of the physical reality, mostly with a "pictorial" representation in terms of Feynman diagrams. In fact, quite amazingly, elementary QFT processes can be described qualitatively in terms of very few ingredients:

- *propagators*, describing the virtual propagation of quantum particle excitations and drawn as lines, typically oriented;
- *vertices*, describing the process of emission and absorption of particle excitations, possibly changing the nature of the original particle, and drawn as a point from which the propagators of emitted and absorbed particle emerge

Propagators and vertices are then embodied in diagrams, describing the quantum processes. Clearly, a "change of nature" of the particle during emission/absorption is not allowed in standard quantum mechanics (QM).
At the same time for a process describing an electron decelerating by emitting photons (which took away the electron kinetic energy) the number of photons emitted can be arbitrarily high, again QM of finite degrees of freedom is insufficient to describe such process.
Another case in which QM turns out to be insufficient arises in the so called thermodynamic limit in solid state systems. In real physical systems the number of electrons and ions is finite, although usually very big, $N \sim 10^{23}$, and the volume V is finite (infrared (IR) cutoff). Furthermore e.g. in a crystal the lattice constant a is finite (ultraviolet (UV) cutoff). However, usually we are interested in universal properties, independent on details of V and a. Therefore it is convenient also in these case to consider the limit $N \to \infty$, $V \to \infty$ with N/V (or its expectation value) constant (thermodynamic IR limit) and $a \to 0$ (continuum UV limit). These limits are not only technically useful, for the non-analyticity appearing in the phase transitions, or the appearance of Euclidean invariance, but e.g. the thermodynamic limit guarantees that the theory does not depend on specific details of V and N.
Notice that if the removal of the IR or the UV cutoff is impossible and we assumed that the theory without the cutoff is the physical one (hence effective field theories in the modern sense are excluded), then the non-existence of the limit implies that the physical theory depends on details at infinite distances (IR) or infinite momentum (UV) in a manner not controllable by regularization.
Furthermore in relativistic QFT (RQFT) a cutoff breaks the Poincaré invariance, and the only possible regularization which does not break such symmetry, the dimensional regularization, has no non-perturbative realization.
Many of the key results of QFT are obtained through a perturbative expansion, which has serious mathematical problems, and there are crucial ares of applications that do not rely on perturbative methods.
The aim of this course is to provide a view of some results in these areas, with examples both from elementary particle and condensed matter physics, emphasizing the underlying common features.

Chapter 2

Review of QFT

2.1 Fock space

[NO98]; [GR96, Chapters 3, 4]; [BS80, Chapters 1, 2]

In $d = 3$ space dimensions, quantum particles are either bosons or fermions (in lower dimensions other braid statistics may arise, but we'll not discuss them here).

Fixed number of particles

The Hilbert space of states for N identical particles $\mathcal{H}_N$ is constructed as follows: let $\mathcal{H}_1$ be the single-particle Hilbert space, Σ_N the permutations group of N objects, $\pi \in \Sigma_N$ and P_π the corresponding operators, $\sigma(\pi)$ the number of exchanges made by π, ε a constant which takes the value $+1$ for bosons and -1 for fermions, then define

$$P^\varepsilon := \frac{1}{N!} \sum_{\pi \in \Sigma_N} \varepsilon^{\sigma(\pi)} P_\pi \tag{2.1}$$

then

$$\mathcal{H}_N^\pm := P^\pm (\underbrace{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{N \text{ times}}) \tag{2.2}$$

More concretely, let A be a complete set of compatible observables in the one-particle Hilbert space of an elementary quantum particle $\mathcal{H}_1$ (we assume for the moment discrete spectrum for these observables), and $\{ |\alpha_i\rangle, i \in I\}$ the corresponding eigenstates, where α_i is the set of common eigenvalues of A. Then an orthonormal basis in $\mathcal{H}_N^\pm$ is given by

$$|\alpha_{i_1} \dots \alpha_{i_N}\rangle^\varepsilon := \sqrt{\frac{N!}{\prod_i n_i!}} P^\varepsilon |\alpha_{i_1}\rangle \otimes \dots \otimes |\alpha_{i_N}\rangle \tag{2.3}$$

where n_i is the number of one-particle states with eigenvalue α_i in $|\alpha_{i_1}\rangle, \dots, |\alpha_{i_N}\rangle$, satisfying $\sum_{i \in I} n_i = N$. If the values of n_i are the same for both the sets $\alpha_{i_1} \dots \alpha_{i_N}$ and $\alpha_{j_1} \dots \alpha_{j_N}$ then

$$|\alpha_{i_1} \dots \alpha_{i_N}\rangle^\varepsilon = \pm |\alpha_{j_1} \dots \alpha_{j_N}\rangle^\varepsilon \tag{2.4}$$

and we can label states by their *occupation numbers* $\{n_i\}_{i \in I}$, once the one-particle basis $\{ |\alpha_i\rangle, i \in I\}$ has been fixed:

$$|\alpha_{i_1} \dots \alpha_{i_N}\rangle =: |n_1, \dots, n_i, \dots\rangle = |\{n_i, i \in I\}\rangle \tag{2.5}$$

By antisymmetry (Pauli principle) for fermions $n_i = 0, 1$, whereas for bosons $n_i \in \mathbb{N}$.
A generic vector in $\mathcal{H}_N^\varepsilon$, then can be written as the linear combination

$$|\Psi_N\rangle = \sum_{\{n_i\}} \Psi(\{n_i\}) |\{n_i\}\rangle \tag{2.6}$$

with the conditions

$$\sum_{\{n_i\}} |\Psi(\{n_i\})|^2 < \infty \quad , \quad \sum_{i \in I} n_i = N \tag{2.7}$$

Variable number of particles, $N \to \infty$ limit

Let's consider now the case in which N is not fixed or we take the $N \to \infty$ limit. Define the **vacuum sector** $\mathcal{H}_0^\varepsilon = \mathbb{C}$ and the corresponding normalized vector $|\Psi_0\rangle$ (or $|0\rangle$) is called the **vacuum**. Formally set

$$\mathscr{F}^\varepsilon := \mathcal{H}_0^\varepsilon \oplus \mathcal{H}_1^\varepsilon \oplus \ldots \oplus \mathcal{H}_N^\varepsilon \oplus \cdots = \bigoplus_{N=0}^{\infty} \mathcal{H}_N^\varepsilon \tag{2.8}$$

Notice that the direct sum implies that there is no interference between the different sectors $\mathcal{H}_N^\varepsilon$.
We would like to explain better the meaning of the previous formal direct sum. Let

$$\mathcal{D} = \left\{ \bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle \right\} \tag{2.9}$$

with $N_{\max}$ arbitrary but finite. In this space of direct sum of finite sequences of vectors we define an *inner product* by

$$\left(\bigoplus_{N=0}^{N_{\max}} |\Psi_N\rangle, \bigoplus_{N'=0}^{N'_{\max}} |\Phi_{N'}\rangle \right) := \sum_{N=0}^{\infty} \langle \Psi_N | \Phi_N \rangle \tag{2.10}$$

where the sum is formally extended to infinity as only a finite number of terms is non zero. Together with this inner product the space $\mathcal{D}$ is pre-Hilbert, and then $\mathscr{F}^\varepsilon$ is defined as the Hilbert space obtained by completion of $\mathcal{D}$, i.e. the space of sequences $\bigoplus_{N=0}^{\infty} |\Psi_N\rangle$ such that $\sum_{N=0}^{\infty} \langle \Psi_N | \Psi_N \rangle < \infty$. Notice that $\mathcal{D}$ is dense in $\mathscr{F}^\varepsilon$, as required by the definition of Hilbert space.
Then, the space $\mathscr{F}^\varepsilon$ allows the description of processes with non-conserved number of particles, and will be called *Fock space*.
In $\mathcal{D}$ we can define the *annihilation and creation operators*

$$\begin{aligned} a_i \, |\{n_j\}\rangle^+ &:= \sqrt{n_i} \, |\{n_{j\neq i}, n_i - 1\}\rangle^+ \\ a_i^\dagger \, |\{n_j\}\rangle^+ &:= \sqrt{n_i + 1} \, |\{n_{j\neq i}, n_i + 1\}\rangle^+ \end{aligned} \tag{2.11}$$

for the bosons and

$$\begin{aligned} a_i \, |\{n_j\}\rangle^- &:= (-1)^{(\sum_{k<i} n_k)} n_i \, |\{n_{j\neq i}, n_i - 1\}\rangle^- \\ a_i^\dagger \, |\{n_j\}\rangle^- &:= (-1)^{(\sum_{k<i} n_k)} (1 - n_i) \, |\{n_{j\neq i}, n_i + 1\}\rangle^- \end{aligned} \tag{2.12}$$

for fermions.
It follows from the definition that

(i) $a_i^\dagger$ is the adjoint of a_i;

(ii) $a_i \, |\Psi_0\rangle = 0$;

(iii) for bosons the *canonical commutation relations* (CCR) hold:

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] \quad , \quad [a_i, a_j^\dagger] = \delta_{ij} \tag{2.13}$$

while for fermions the *canonical anticommutation relations* (CAR) hold:

$$\{a_i, a_j\} = 0 = \{a_i^\dagger, a_j^\dagger\} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij} \tag{2.14}$$

(iv) defining $\hat{N}_i := a_i^\dagger a_i$ we have

$$\hat{N}_i \, |\{n_j\}\rangle = n_i \, |\{n_j\}\rangle \tag{2.15}$$

and $\hat{N} := \sum_{i \in I} \hat{N}_i$ is well defined in $\mathcal{D}$;

(v) any vector can be constructed by means of applications of creation operators

$$|\{n_j\}\rangle = \frac{1}{\sqrt{\prod_j n_j!}} \prod_j (a_j^\dagger)^{n_j} \, |0\rangle \tag{2.16}$$

Annihilation and creation operators corresponding to another basis (i.e. another complete set of commuting observables with discrete spectrum) $\{\,|\beta_j\rangle, j \in J\}$ can be obtained by applying the Dirac completeness

$$\sum_{i\in I} |\alpha_i\rangle\langle\alpha_i| \,= \mathbb{1}_{\mathcal{H}_1} \tag{2.17}$$

as follows

$$b_j^\dagger\,|0\rangle = \,|\beta_j\rangle = \sum_{i\in I} |\alpha_i\rangle\,\langle\alpha_i|\beta_j\rangle = \sum_{i\in I} a_i^\dagger\,|0\rangle\,\langle\alpha_i|\beta_j\rangle \tag{2.18}$$

implying

$$b_j^\dagger = \sum_{i\in I} \langle\alpha_i|\beta_j\rangle\, a_i^\dagger \tag{2.19}$$

These ideas extends to the case of complete sets of commuting observables with continuum (or mixed, with suitable modifications) spectrum in $\mathcal{H}_1$ using

$$\int \mathrm{d}\alpha\; |\alpha\rangle\langle\alpha| \,= \mathbb{1}_{\mathcal{H}_1} \tag{2.20}$$

Formally we can set $a^\dagger(\alpha)\,|0\rangle = \,|\alpha\rangle$ and then

$$a^\dagger(\alpha) = \sum_{i\in I} \langle\alpha_i|\alpha\rangle\, a_i^\dagger \tag{2.21}$$

but since $|\alpha\rangle$ is only an improper state ($\langle\alpha|\alpha'\rangle \sim \delta(\alpha-\alpha')$) then $a^\dagger(\alpha)$ is not a true operator, it is an *operator valued distribution*, i.e. is a true operator only if smeared out with a test function f:

$$a(f) := \int a(\alpha) f(\alpha) \mathrm{d}\alpha \tag{2.22}$$

A typical example is given for an elementary particle with classical analogue (in $d = 3$) by $|\alpha\rangle = |\boldsymbol{x}\rangle$ or $|\alpha\rangle = |\boldsymbol{p}\rangle$. If we define $a^\dagger(\boldsymbol{p})\,|0\rangle = \,|\boldsymbol{p}\rangle$ with $[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta(\boldsymbol{p}-\boldsymbol{p}')$, then is clear that $a(\boldsymbol{p})$ is not an ordinary operator, rather an operator valued distribution. When we turn to the $\boldsymbol{x}$-representation of a (i.e. we smear $a(\boldsymbol{p})$ with $f(\boldsymbol{p}) = \langle\boldsymbol{x}|\boldsymbol{p}\rangle$) we get

$$\psi(\boldsymbol{x}) := \int \mathrm{d}^3p\; \langle\boldsymbol{x}|\boldsymbol{p}\rangle\, a(\boldsymbol{p}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{p}} a(\boldsymbol{p}) \tag{2.23}$$

and $\psi(\boldsymbol{x})$ is called a *quantum field operator* and it satisfies (for bosons)

$$[\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{y})] = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.24}$$

where all the trivial (anti)commutation relation will be always omitted from now on. Notice that the last relation can be interpreted as a form of "locality": the effect of a field in a point cannot affect the effect of a simultaneous field in a different point.
The operator $\psi^\dagger(\boldsymbol{x})$ formally creates a particle with wave function a Dirac δ with support on $\boldsymbol{x}$. More precisely if $f(\boldsymbol{x}) \in L^2(\mathbb{R}^3, \mathrm{d}^3x)$ then

$$\psi^\dagger(f) = \int \mathrm{d}^3x\, \psi^\dagger(\boldsymbol{x}) f(\boldsymbol{x}) \tag{2.25}$$

creates a particle with wave function $f(\boldsymbol{x})$. Notice that if in $\mathcal{H}_1$ we can use an orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$, then setting $a(f_i) =: a_i$ it satisfies the CCR because

$$[a(f_i), a^\dagger(f_j)] = (f_i, f_j) = \delta_{ij} \tag{2.26}$$

where (f_i, f_j) denotes the scalar product defined in $L^2(\mathbb{R}^3, \mathrm{d}^3x)$.
If the particle we are interested in is not elementary the situation is lightly nore complex, but we will not discuss it.

Relativistic case, $N \to \infty$ limit

Notice that moving from non-relativistic context (where space coordinates and time are treated differently) to the relativistic one, some problems arises. Dirac formulation of relativistic quantum mechanics treats time and space in the same way, making also the time an observable. However, in presence of interactions we have to take care of the consequences of the Heisenberg principle: due to uncertainty on the energy we are able to produce particle-antiparticle pairs, but if we are interested in measuring the position in the best possible way, then the uncertainty on the momentum become huge and lot of pairs are produced, and we are no more able to identify which is the particle we want to measure. Therefore the position is no more an observable in the relativistic framework, and can only be used as a label to describe the evolution of a state, as we do for the time in quantum mechanics.
Since we cannot characterize elementary particles using position as an observable in an irreducible set of observables (as we do in the non-relativistic case, where position, momentum and spin provide an irreducible set of observables), we need to understand how to choose a new set of observables which allows us to build a Fock space for elementary particles using eigenvectors.
A celebrated theorem of Wigner states that in a relativistic QFT the one-particle Hilbert space of an elementary particle should be the representation space of an irreducible unitary representation of space-time symmetries (i.e. the universal covering of the restricted Poincaré group) and internal symmetries (e.g. EM charge, but we do not discuss them here).
Irreducible unitary representations of space-time symmetries, according to Wigner's theorem, are characterized by the mass $m \in \mathbb{R}_+$ and either the spin $s \in \frac{1}{2}\mathbb{N}$ if $m > 0$ or the helicity (projection of the spin in the direction of the motion[I]) $h \in \frac{1}{2}\mathbb{Z}$ if $m = 0$.
Let's consider for simplicity the case of a massive ($m > 0$) scalar ($s = 0$) particle. We know that for a relativistic particle the dispersion relation is

$$p_\mu p^\mu = m^2 \quad , \quad p^0 > 0 \tag{2.27}$$

i.e. the momentum should be contained in the *positive hyperboloid of mass* m, denoted by V_m^+. Notice that if $m = 0$ we have that the momentum should be contained in the *forward light cone*.
Since we want to have a representation of the (covering of the) Poincaré group all the points in the hyperboloid should be weighted by the same weight by the measure defined in the representation space, then a natural choice for our one-particle Hilbert space is provided by

$$\mathcal{H}_1 = L^2\left(\mathbb{R}^4, \frac{\mathrm{d}^4 p}{(2\pi)^4}\, 2\pi\delta(p_\mu p^\mu - m^2)\theta(p^0)\right) = L^2\left(V_m^+, \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}}\right) \tag{2.28}$$

Let $a^\dagger(p)\,|0\rangle =: |p\rangle$ be the (generalized) eigenvector of the four-momentum operator $\hat{p}^\mu$ for (generalized) eigenvalue $p^\mu =: p \in \mathbb{R}^4$. The Dirac completness relation in $\mathcal{H}_1$ is given by

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^3}\, \delta(p^2 - m^2)\theta(p^0)\, |p\rangle\langle p| \, = \mathbb{1}_{\mathcal{H}_1} \tag{2.29}$$

It immediately follows multiplying by $|p'\rangle$, $p' \in V_m^+$ that

$$\begin{aligned} |p'\rangle = |\sqrt{\boldsymbol{p'}^2 + m^2}, \boldsymbol{p'}\rangle &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)\, |p\rangle\, \langle p|p'\rangle \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0)\, |p\rangle\langle 0|\, a(p)a^\dagger(p')\,|0\rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2\sqrt{\boldsymbol{p}^2 + m^2}}\, |\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p}\rangle\langle 0|\, [a(p)a^\dagger(p')]\,|0\rangle \end{aligned} \tag{2.30}$$

where from the second to the third line we integrated over p^0 and we used $\langle 0|\, a^\dagger(p')a(p)\,|0\rangle = 0$. Therefore in order to get consistency the following commutation relation should be satisfied:

$$[a(p), a^\dagger(p')] = (2\pi)^3 2\sqrt{\boldsymbol{p}^2 + m^2}\delta(\boldsymbol{p} - \boldsymbol{p'}) \tag{2.31}$$

[I] Notice that helicity is well defined only if the particle moves at the speed of light, otherwise through a change of frame one can reverse the projection direction.

Notice that if $\{f_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}_1$ and we define

$$a_i := \int \frac{\mathrm{d}^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0)f_i(p)a(p) \tag{2.32}$$

then eq. (2.31) implies that CCR are satisfied, $[a_i, a_j^\dagger] = \delta_{ij}$, hence also in the relativistic case it is possible to obtain canonical commutation relation by smearing a and $a^\dagger$ with an orthonormal basis of the Hilbert space.
The new factor $2\sqrt{\boldsymbol{p}^2+m^2}$ in eq. (2.31), which wasn't present in the non-relativistic case, lead to non-locality problems if one proceeds defining quantum field operators as in the non-relativistic case. If we try to apply the previous idea to define quantum field operators, i.e. we adapt eq. (2.23)

$$\psi(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0)a(p)e^{ip\cdot x} \tag{2.33}$$

then

$$\begin{aligned}[\psi(\boldsymbol{x},0),\psi^\dagger(\boldsymbol{y},0)] &= \int \frac{\mathrm{d}^4p}{(2\pi)^3}\frac{\mathrm{d}^4p'}{(2\pi)^3}\delta(p^2-m^2)\delta(p'^2-m^2)\theta(p^0)\theta(p'^0)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}e^{i\boldsymbol{p'}\cdot\boldsymbol{y}}[a(p),a^\dagger(p')] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2\sqrt{\boldsymbol{p}^2+m^2}}\frac{\mathrm{d}^3p'}{(2\pi)^3\,2\sqrt{\boldsymbol{p'}^2+m^2}}e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{p'}\cdot\boldsymbol{y})}\,(2\pi)^3 2\sqrt{\boldsymbol{p}^2+m^2}\delta(\boldsymbol{p}-\boldsymbol{p'}) \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\,2\sqrt{\boldsymbol{p}^2+m^2}}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \neq 0\end{aligned} \tag{2.34}$$

even for $\boldsymbol{x}\neq\boldsymbol{y}$. This violates the "locality" even in the weak non-relativistic form of eq. (2.24). In fact, according to the concept of "present" (or equivalently of "simultaneity") in relativity, the vanishing of the commutator (for observable fields at least) should hold for x and y space-like separated (we denoted space-like separated coordinates x and y by $x \mathsf{X} y$).
Actually in general the fields are not observable (e.g. the charged scalar field is not, since it is not self-adjoint) hence one may argue that they are not forced to satisfy locality, nevertheless a remarkable theorem of Doplicher-Roberts (essentially) shows that in a massive RQFT if the observables commute at space-like distances, then the fields of the corresponding QFT either commute or anticommute at space-like distances in $d = 3+1$. Moreover the spin-statistics theorem proves that fields with integer spin are bosons and those with half-integer spin are fermions. Hence we know that in the massive case fields with integer spin commute and those with half-integer spin anticommute at space-like distances.
Therefore we need to impose such (anti)commutation relations for our relativistic fields at space-like separated coordinates. Since $a(p)\,|0\rangle = 0$, the state obtained applying $\psi^\dagger(x)$ to $|0\rangle$ is the same if one add an additional contribution $\sim a$ to $\psi^\dagger(x)$. Let's define

$$\phi(x) = \psi(x) + \psi^\dagger(x) \tag{2.35}$$

where the presence of both ψ and $\psi^\dagger$ is reminiscent of the fact that the dispersion relation $p_\mu p^\mu = m^2$ has two solutions $\pm\sqrt{\boldsymbol{p}^2+m^2}$, one to one associated to ψ and $\psi^\dagger$. Then $\phi\,|0\rangle = \psi^\dagger\,|0\rangle$, ϕ (smeared) is a self-adjoint field operator and, as we desired,

$$[\phi(x),\phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \tag{2.36}$$

or more precisely

$$[\phi(f),\phi(g)] = 0 \quad \text{if} \quad \operatorname{supp} f \mathsf{X} \operatorname{supp} g \tag{2.37}$$

However, it is clear that even if $\phi(x)$ is "localized in x" (so that it cannot affect points space like separated) or better $\phi(f)$ is "localized in $\operatorname{supp} f$", nevertheless the state $\phi(x)\,|0\rangle$ is not localized in x and $\phi(f)\,|0\rangle$ is not localized in $\operatorname{supp} f$. In fact the *two point correlation function* for a real field reads

$$\begin{aligned}\langle 0|\,\phi(x)\phi(y)\,|0\rangle = \langle 0|\,\psi(x)\psi^\dagger(y)\,|0\rangle = \langle 0|\,[\psi(x)\psi^\dagger(y)]\,|0\rangle = \\ = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2\sqrt{\boldsymbol{p}^2+m^2}}e^{ip\cdot(x-y)} \sim \frac{m^{1/2}}{|\boldsymbol{x}-\boldsymbol{y}|^{3/2}}e^{-m|\boldsymbol{x}-\boldsymbol{y}|} \neq 0\end{aligned} \tag{2.38}$$

where in the second line $p^0 := \sqrt{\boldsymbol{p}^2 + m^2}$ and the approximation holds for $(x-y)^2 \ll -1$.[II]
Differently from NRQFT case, where

$$\langle 0|\, \psi(\boldsymbol{x})\psi^\dagger(\boldsymbol{y})\, |0\rangle = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.44}$$

holds, in RQFT $a^\dagger(p)$ applied at the vacuum creates a one-particle state with 4-momentum p^μ, but $\phi(x)$ applied to the vacuum does not create a particle localized in x^μ (i.e. with "wave function" given by a δ localized in x^μ).
Now it's very clear that $\hat{x}^\mu$ is not a good observable in RQFT, whereas $\hat{p}^\mu$ is a good observable in the one particle Hilbert state $\mathcal{H}_1$. It is indeed possible to create a particle with well defined momentum p^μ (in an improper state) but is impossible to create a particle with well defined position x^μ.
The physical underlying reason as we already mentioned is that the measure of $\hat{x}^\mu$ would produce a diverging fluctuation in p due to Heisenberg principle $\Delta\hat{x}^\mu\Delta\hat{p}^\mu \gtrsim \hbar$, allowing the production of particle-hole pairs. Thus the "space-time coordinate x" looses its meaning, since it is impossible to understand to which particle it refers.
Finally notice that from its definition $\phi(x)$ satisfies the homogeneous Klein-Gordon equation

$$(\Box + m^2)\phi(x) := \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + m^2\right)\phi(x) = 0 \tag{2.45}$$

If we replace c with the phonon velocity this is the same equation appearing in Condensed Matter for optical phonon at small (quasi)-momentum (in the lattice p^μ is periodic), showing a first example that some excitations in material exhibits "relativistic features" in some regime of parameters.

2.2 Characteristic features of Fock space

[NO98]; [BR97, Section 5.2]

The aim of this section is to understand how CCR and CAR affect the properties of the Fock space we built up. We have seen that the creation and annihilation operators for bosons and fermions satisfy the CCR and the CAR respectively

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij} \tag{2.46}$$

Since such relations are purely algebraic, they can be thought as the caracterizing rules for an algebra (i.e. a vector space where we defined the multiplication) endowed with an involution denoted by † (which

[II] Let's see how the result is obtained. First replace $(x-y) \mapsto x$ and then set $\boldsymbol{p}\cdot\boldsymbol{x} = |\boldsymbol{p}||\boldsymbol{x}|\cos\theta =: |\boldsymbol{p}||\boldsymbol{x}|u$ (here $u := \cos\theta$). Then

$$\int \frac{\mathrm{d}^3p}{\sqrt{\boldsymbol{p}^2+m^2}} e^{ip^0x^0} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \sim \int \frac{|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|\,\mathrm{d}u\,\mathrm{d}\phi}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} \sim \int_0^\infty \frac{|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} \int_{-1}^1 \mathrm{d}u\, e^{-i|\boldsymbol{p}||\boldsymbol{x}|u} =$$
$$= \int_0^\infty \frac{|\boldsymbol{p}|^2\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{ip^0x^0} \frac{1}{-i|\boldsymbol{p}||\boldsymbol{x}|}\left(e^{-i|\boldsymbol{p}||\boldsymbol{x}|} - e^{i|\boldsymbol{p}||\boldsymbol{x}|}\right) = \frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^\infty \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i(p^0x^0+|\boldsymbol{p}||\boldsymbol{x}|)} \tag{2.39}$$

Due to $x^2 \ll -1$, i.e. $|\boldsymbol{x}| \gg x^0$ and $|\boldsymbol{x}| \gg 1$, we have that

$$\frac{\partial}{\partial|\boldsymbol{p}|}(p^0x^0 + |\boldsymbol{p}||\boldsymbol{x}|) = \frac{|\boldsymbol{p}|x^0}{p^0} + |\boldsymbol{x}| \approx |\boldsymbol{x}| \tag{2.40}$$

hence the contribution of p^0x^0 to the variation of the total phase respect to $|\boldsymbol{p}|$ is negligible, and it will give just a small correction to the final result

$$\frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^\infty \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i(p^0x^0+|\boldsymbol{p}||\boldsymbol{x}|)} \approx \frac{1}{i|\boldsymbol{x}|}\int_{-\infty}^\infty \frac{|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|}{\sqrt{|\boldsymbol{p}|^2+m^2}} e^{i|\boldsymbol{p}||\boldsymbol{x}|} = \frac{1}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}|\boldsymbol{p}|\, \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2+m^2}} \tag{2.41}$$

Substituting $p \mapsto m\sinh(t)$, $\mathrm{d}p = \sqrt{|\boldsymbol{p}|^2+m^2}\mathrm{d}t$, we get

$$\frac{1}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}|\boldsymbol{p}|\, \frac{|\boldsymbol{p}|\sin(|\boldsymbol{p}||\boldsymbol{x}|)}{\sqrt{|\boldsymbol{p}|^2+m^2}} = \frac{m}{i|\boldsymbol{x}|}\int_0^\infty \mathrm{d}t\, \sinh(t)\sin(m|\boldsymbol{x}|\sinh(t)) = \frac{m}{i|\boldsymbol{x}|}\mathrm{K}_1(m|\boldsymbol{x}|) \tag{2.42}$$

where $\mathrm{K}_n(x)$ is the modified Bessel function, which satisfies

$$\mathrm{K}_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \quad \text{for} \quad x \gg n \tag{2.43}$$

promote the algebra to a *-algebra), generated by $\{a_i\}_{i\in I}$. From this point of view a and $a^\dagger$ are then just representations of this algebra as operators acting on $\mathscr{F}^\pm$ ($\mathscr{F}^\pm$ are representation spaces of the algebra). If the set I is finite, i.e. the algebra is finitely generated, then the *von Neumann uniqueness theorem* ensures that all the representation of CCR and CAR are unitarily equivalent, hence all possible representations live in the same abstract Hilbert space.
For instance this occur in QM for an N particles system in $\mathbb{R}^d$ (finite degrees of freedom (d.o.f.)), where we can put $a_i = \frac{1}{\sqrt{2}}(q_i + p_i)$, $i = 1, \dots, d$, with representation space $L^2((\mathbb{R}^d)^N, \mathrm{d}^d x_1 \dots \mathrm{d}^d x_N)$.
This is not true anymore for infinitely generated algebras (infinite d.o.f.). Roughly speaking, a unitary operator mapping one representation of the algebra into another one can be thought in zero dimensions as $e^{iN} \in U(1)$ for finite N but as $N \to \infty$ it vanishes. Infinitely generated CCR and CAR algebras have infinite inequivalent representations acting in completely disjoint Hilbert spaces. Moreover each of these possible representations describes completely different physics.
The point is now to understand what actually characterizes a specific $\mathscr{F}$, since CCR and CAR are the same also for infinitely many different Hilbert spaces. The answer is given by the number operator $\hat{N}$: indeed it is a well defined observable if and only if the elements of $\mathscr{F}$ are created by a specific representation of $a_i^\dagger$, or in other words if the excitations created out from the vacuum by the $a_i^\dagger$ are the ones that can be counted (even if infinitely many) by a specific $\hat{N}$.
Mathematically, $\hat{N} := \sum_{i\in I} a_i^\dagger a_i$ is well defined in $\mathcal{H}$ if exists a domain D dense in $\mathcal{H}$ containing $|0\rangle$ in which $\hat{N}$ is self-adjoint. This also implies that its spectrum is $\sigma(\hat{N}) = \mathbb{N}$ by the standard argument seen for the harmonic oscillator in QM.
Representations of CCR or CAR on which $\hat{N}$ is not well defined are called *non-Fock representation*. We now wonder whether is possible or not to create non-Fock representations. A simple example of this is given by *Bogoliubov transformations*: starting from given a_i, $a_i^\dagger$, we define

$$\begin{aligned} a_i' &:= \alpha_i a_i + \beta_i a_i^\dagger \\ a_i'^\dagger &:= \alpha_i^* a_i^\dagger + \beta_i^* a_i \end{aligned} \tag{2.47}$$

for $\alpha_i, \beta_i \in \mathbb{C}$. If $\{a_i\}_i$ generates a representation of CCR then

$$[a_i', a_j'^\dagger] = (|\alpha_i|^2 - |\beta_i|^2)\delta_{ij} \tag{2.48}$$

from which we see that $\{a_i'\}_i$ gives a new representation of CCR provided that $|\alpha_i|^2 - |\beta_i|^2 = 1$. Conversely if $\{a_i\}_i$ generates a representation of CAR then

$$\{a_i', a_j'^\dagger\} = (|\alpha_i|^2 + |\beta_i|^2)\delta_{ij} \tag{2.49}$$

from which we see that $\{a_i'\}_i$ gives a new representation of CAR provided that $|\alpha_i|^2 + |\beta_i|^2 = 1$. Let $|0\rangle$ be the vacuum in the a_i representation, then

$$\langle 0| \hat{N}' |0\rangle = \sum_{i\in I} \langle 0| (\alpha_i^* a_i^\dagger + \beta_i^* a_i)(\alpha_i a_i + \beta_i a_i^\dagger) |0\rangle = \sum_{i\in I} |\beta_i|^2 \tag{2.50}$$

If $\sum_i |\beta_i|^2 < \infty$ then such transformation is allowed and since it is unitary then the Fock space of a_i is a Fock space for a_i' too (even if states, including the vacuum, may not be the same). Otherwise, for $\sum_i |\beta_i|^2 = \infty$, the vacuum $|0\rangle$ in the a_i representation is not even an element of $D(\hat{N}') := \{ |\psi\rangle \text{ s.t. } ||\hat{N}' |\psi\rangle|| < \infty\}$, nor is any excitation created from $|0\rangle$ using $a_i^\dagger$.[III] Therefore if for $\sum_i |\beta_i|^2 = \infty$ the Fock space for $\{a_i\}_{i\in I}$ is not the right space for the excitations described by a_i'. Actually we proved something more, we proved that the Fock space for a_i and a_i' are completely disjoint since no state of one of the two spaces is contained into the other one.
This actually happens in practice also in some simple situations such as for *(spin 1/2) Fermi gas*.[IV] Let's consider N fermions in a finite volume V and zero temperature $T = 0$. In such conditions the density of

[III] Indeed using CCR and CAR one obtains

$$\hat{N}' a_i^\dagger = \alpha_i a_i'^\dagger \pm \beta_i^* a_i' + a_i^\dagger \hat{N}' \tag{2.51}$$

hence for any polynomial in $a_i^\dagger$ eventually $\hat{N}'$ hit the vacuum producing a divergence.
[IV] This example is presented in [Str85, Part B, sec. 1.1].

particles in the ground state $|\psi_0\rangle$ is

$$n(\boldsymbol{k}) = \begin{cases} 2 & \text{for} \quad |\boldsymbol{k}| \leq k_F \\ 0 & \text{for} \quad |\boldsymbol{k}| > k_F \end{cases} \tag{2.52}$$

where we have two particles for each momentum state due the spin $1/2$ and to the Fermi statistics. Here k_F is the Fermi momentum. Clearly when $|\boldsymbol{k}| \leq k_F$ the ground state $|\psi_0\rangle$ corresponding to such configuration $n(\boldsymbol{k})$ is not a vacuum for the annihilation operators $a(\boldsymbol{k}, s)$ because $a(\boldsymbol{k}, s)\,|\psi_0\rangle \neq 0$, since $a(\boldsymbol{k}, s)$ just annihilates a particle which certainly exists in the whole ensemble where all configurations for $|\boldsymbol{k}| \leq k_F$ are occupied.
Nevertheless if one defines the Bogoliubov transformation

$$a'(\boldsymbol{k}, s) = \alpha(\boldsymbol{k}, s)a(\boldsymbol{k}, s) + \beta(-\boldsymbol{k}, -s)a^\dagger(-\boldsymbol{k}, -s) \tag{2.53}$$

with

$$\begin{cases} \alpha(\boldsymbol{k}, s) = 1 & \beta(-\boldsymbol{k}, -s) = 0 \qquad \text{for} \quad |\boldsymbol{k}| > k_F \\ \alpha(\boldsymbol{k}, s) = 0 & \beta(-\boldsymbol{k}, -s) = 1 \qquad \text{for} \quad |\boldsymbol{k}| \leq k_F \end{cases} \tag{2.54}$$

then the ground state is a Fock vacuum for $a'(\boldsymbol{k}, s)$:

$$a'(\boldsymbol{k}, s)\,|\psi_0\rangle = 0 \tag{2.55}$$

According to the previous argument we know that $|\psi_0\rangle \in \mathscr{F}$ (for $a(\boldsymbol{k}, s)$) if and only if $\sum_{\boldsymbol{k},s} |\beta(\boldsymbol{k}, s)|^2 < \infty$. This is clearly true if $N < \infty$, but it is false in the limit $N \to \infty$, $V \to \infty$, N/V fixed. Therefore, in the thermodynamic limit the Hilbert space of a quantum Fermi gas (the one built with $a'(\boldsymbol{k}, s)$) is not equivalent to the Fock space of the non-relativistic fermions (the one built with $a(\boldsymbol{k}, s)$).
Another example of non-Fock space is provided by the radiation emitted by a charged particle changing its momentum $\boldsymbol{p} \mapsto \boldsymbol{p'}$. A simple qualitative argument goes as follows: the energy emitted by radiation goes like

$$\mathcal{E} = \int \mathrm{d}^3k\, |\mathbf{E}_{\mathrm{rad}}(\boldsymbol{k})|^2 \tag{2.56}$$

and we know that is is a finite number, moreover since photons asymptotically (where they are free) have energy $\omega(\boldsymbol{k}) = c|\boldsymbol{k}|$, then the number of emitted photons is

$$N = \int \mathrm{d}^3k \frac{|\mathbf{E}_{\mathrm{rad}}(\boldsymbol{k})|^2}{c|\boldsymbol{k}|} \tag{2.57}$$

The problem is that the electric field of the radiation behaves in the infrared ($|\boldsymbol{x}| \to \infty$) as $\mathbf{E}_{\mathrm{rad}}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-2}$ implying $\mathbf{E}(\boldsymbol{k}) \sim |\boldsymbol{k}|^{-1}$ as $|\boldsymbol{k}| \to 0$. Hence

$$N \sim \int \mathrm{d}^3k\, \frac{1}{|\boldsymbol{k}|} \left(\frac{1}{|\boldsymbol{k}|}\right)^2 = \infty \tag{2.58}$$

therefore the Hilbert space of photons at the end of the process is not the Fock space $\mathscr{F}_{\mathrm{in}}$ of the initial photons.

2.3 Interacting fields

[NO98]; [GR96, Chapter 8, 9]; [Kle15, Section 1.9, Chapter 10]; [BS80, Chapter 4]

Up to now we have considered only "free" fields whose Hilbert space is a Fock space, however to obtain physical informations we need interactions. Now, starting from the case of particle physics (zero temperature $T = 0$, zero density $n = 0$, relativistic case), and then moving to the condensed matter case, we will introduce the perturbative calculation of the interactions. We are mainly interested in the understanding of the problems that arise in the perturbative approach and in general in the need of non-perturbative techniques.
In particle physics most of the informations are extracted from scattering experiments. As a very simple model we can consider the ϕ^4 model, which in its complex version describe the (low energy of the) Higgs

field with only quartic self-interactions taken into account, in the unbroken symmetry phase (which we presume was present in the early universe).
To be concrete we consider a real field $\phi(x)$ with canonical lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \text{and} \quad \mathcal{L}_I \quad = -\frac{\lambda}{4}\phi^4 \tag{2.59}$$

The corresponding Hamiltonian is

$$H = H_0 + H_I = \int \mathrm{d}^3\boldsymbol{x}\ (\mathcal{H}_0(\boldsymbol{x}) + \mathcal{H}_I(\boldsymbol{x})) \tag{2.60}$$

with

$$\mathcal{H}_0(\boldsymbol{x}) = \frac{\pi^2}{2}(\boldsymbol{x}) + \frac{(\boldsymbol{\nabla}\phi)^2(\boldsymbol{x})}{2} + \frac{m^2}{2}\phi^2(\boldsymbol{x}) \quad \text{and} \quad \mathcal{H}_I(\boldsymbol{x}) = \frac{\lambda}{4}\phi^4(\boldsymbol{x}) \tag{2.61}$$

where the canonical momentum is $\pi(x) = \dot{\phi}(x)$ and the canonical quantization would give

$$[\hat{\phi}(\boldsymbol{x},t), \dot{\hat{\phi}}(\boldsymbol{y},t)] = i\hbar\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.62}$$

Now one may wonder in which Hilbert space the field $\hat{\phi}$ is defined as an operator-valued distribution, and if such space is the same as the Fock space of the free scalar field. Such question is actually very non-trivial. To gain the maximum support from our knowledge of the free field let us first suppose that we switch-off the interaction term H_I for large $|t|$, i.e. we replace λ by a C^∞ function of t, $\lambda_\varepsilon(t)$, such that

$$\lambda_\varepsilon(t) = \begin{cases} \lambda & |t| \leq \varepsilon^{-1} \\ 0 & |t| \to \infty \end{cases} \tag{2.63}$$

and we denote the new interacting Hamiltonian with H^I_ε. Then we will take the limit $\varepsilon \to 0$ to restore the physical situation. For any finite $\varepsilon > 0$ in the limit $t \to -\infty$ the field $\hat{\phi}(\boldsymbol{x},t)$ tends to the free field $\hat{\phi}_{\text{in}}$, defined in the Fock space $\mathscr{F}_{\text{in}}$ and in the limit $t \to +\infty$ to the free field $\hat{\phi}_{\text{out}}$ defined in $\mathscr{F}_{\text{out}}$.
Notice that in general (as we will comment later on) in order to have a well defined expression we need to cutoff also in the spatial direction, so that ε become an infrared regulator of spacetime, and possibly we need to introduce also ultraviolet counterterms.
Since we *assumed* that $\hat{\phi}$ (using the Heisenberg picture) satisfies the CCR eq. (2.62) (but this assumption is not guaranteed) and by definition CCR are obeyed also by $\hat{\phi}_{\text{in}}$ and $\hat{\phi}_{\text{out}}$ since these are free fields, then it exists a unitary one-parameter group $U^I_\varepsilon(t)$ such that

$$\begin{aligned} U^{I\dagger}_\varepsilon(t)\hat{\phi}_{\text{in}}(\boldsymbol{x},t)U^I_\varepsilon(t) &= \hat{\phi}(\boldsymbol{x},t) \\ U^{I\dagger}_\varepsilon(t)\hat{\pi}_{\text{in}}(\boldsymbol{x},t)U^I_\varepsilon(t) &= \hat{\pi}(\boldsymbol{x},t) \end{aligned} \tag{2.64}$$

If this is true then

$$\begin{aligned} U^I_\varepsilon(t) &\xrightarrow[t\to-\infty]{} \mathbb{1} \\ U^I_\varepsilon(t) &\xrightarrow[t\to+\infty]{} S_\varepsilon \end{aligned} \tag{2.65}$$

where (since U^I_ε is unitary) S_ε is a unitary operator called *Scattering matrix (with cutoff ε)* such that

$$S^\dagger_\varepsilon\hat{\phi}_{\text{in}}S_\varepsilon = \hat{\phi}_{\text{out}} \tag{2.66}$$

Notice that by consistency, if $U(t) = \exp\{-\frac{itH}{\hbar}\}$ denotes the unitary evolution in Heisenberg picture of $\hat{\phi}$ using the complete Hamiltonian, then eq. (2.64) implies that

$$U^{I\dagger}_\varepsilon(t)e^{+\frac{itH_0}{\hbar}}\hat{\phi}_{\text{in}}(\boldsymbol{x},0)e^{-\frac{itH_0}{\hbar}}U^I_\varepsilon(t) = U^\dagger(t)\hat{\phi}(\boldsymbol{x},0)U(t) \tag{2.67}$$

must hold, so that

$$U^I_\varepsilon(t) = e^{\frac{itH_0}{\hbar}}U(t) \tag{2.68}$$

This provides the evolution operator in the *interaction picture* for the states, while as we know the fields in such picture evolve according to Heisenberg picture with the free Hamiltonian, so that

$$\langle\chi|\,\hat{\phi}(\boldsymbol{x},t)\,|\psi\rangle = \underbrace{\langle\chi|\,U_\varepsilon^{I\dagger}(t)}_{\substack{\text{Sch. evol. of } |\chi\rangle \\ \text{using } U_\varepsilon^I}} \underbrace{e^{\frac{itH_0}{\hbar}}\,\hat{\phi}_{\text{in}}(\boldsymbol{x},0)e^{-\frac{itH_0}{\hbar}}}_{\substack{\text{Heis. evol. of } \hat{\phi} \\ \text{using } H_0.}} \underbrace{U_\varepsilon^I(t)\,|\psi\rangle}_{\substack{\text{Sch. evol. of } |\psi\rangle \\ \text{using } U_\varepsilon^I}} \tag{2.69}$$

In order to determine $U_\varepsilon^I(t)$, we can differentiate respect to t in eq. (2.68) and we get

$$\begin{aligned}\frac{\mathrm{d}U_\varepsilon^I(t)}{\mathrm{d}t} &= \frac{i}{\hbar}H_0 e^{\frac{itH_0}{\hbar}}U(t) - \frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}(H_0+H_\varepsilon^I)U(t) \\ &= -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H_\varepsilon^I U(t) = -\frac{i}{\hbar}e^{\frac{itH_0}{\hbar}}H_\varepsilon^I e^{-\frac{itH_0}{\hbar}}U_\varepsilon^I(t) = -\frac{i}{\hbar}H_\varepsilon^I(t)U_\varepsilon^I(t)\end{aligned} \tag{2.70}$$

and using the boundary condition $U_\varepsilon^I(-\infty)=\mathbb{1}$ we finally get the integral equation

$$U_\varepsilon^I(t) = \mathbb{1} - \frac{i}{\hbar}\int_{-\infty}^{t} H_\varepsilon^I(t')U_\varepsilon^I(t')\mathrm{d}t' \tag{2.71}$$

By successively re-inserting the l.h.s. of eq. (2.71) we get

$$U_\varepsilon^I(t) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{t}\mathrm{d}t_1\cdots\int_{-\infty}^{t}\mathrm{d}t_n\,\mathrm{T}[H_\varepsilon^I(t_1)\dots H_\varepsilon^I(t_n)] =: \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_{-\infty}^{t}H_\varepsilon^I(t')\mathrm{d}t'\right] \tag{2.72}$$

where $\mathrm{T}[A_1(t_1)\cdots A_n(t_n)]$ is the *time-ordering* defined by

$$\mathrm{T}[A_1(t_1)\cdots A_n(t_n)] = \sum_{\pi\in\Sigma_n}\Theta(t_{\pi(1)},\dots,t_{\pi(n)})\epsilon^{\sigma(\pi)}A_{\pi(1)}(t_{\pi(1)})\cdots A_{\pi(n)}(t_{\pi(n)}) \tag{2.73}$$

with $\varepsilon=\pm1$ according to the statistics of the operators and

$$\Theta(t_{\pi(1)},\dots,t_{\pi(n)}) = \begin{cases}1 & \text{if} \quad t_{\pi(1)}\geq t_{\pi(2)}\geq\cdots\geq t_{\pi(n)} \\ 0 & \text{otherwise}\end{cases} \tag{2.74}$$

By construction $\mathscr{F}_{\text{in}},\mathscr{F}_{\text{out}}\subset\mathcal{H}$ (the space of $\hat{\phi}$) and if $|\text{in}\rangle\in\mathscr{F}_{\text{in}}$ is a state of free particles prepared at $t=-\infty$ and $|\text{out}'\rangle\in\mathscr{F}_{\text{out}}$ a state of free particles found at $t=+\infty$, then the probability to get this transition is given, according to the rules of QM, by

$$|\,\langle\text{in}|\text{out}'\rangle\,|^2 = |\langle\text{in}|\,S_\varepsilon\,|\text{in}'\rangle|^2 \tag{2.75}$$

Hence the scattering process can be analyzed in terms of the matrix elements of S_ε.
These matrix elements are in turn related to the *correlation functions or Green functions* of $\hat{\phi}$

$$G^{(n)}(x_1,\dots,x_n) := \langle 0|\,\mathrm{T}[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)]\,|0\rangle \tag{2.76}$$

by the *LSZ (Lehmann-Symanzik-Zimmermann) formula*: if the in-state is given by particles with momenta $q_1,\dots,q_m$ and the out state by particles with momenta $p_1,\dots,p_n$, then

$$\begin{aligned}&\langle q_1,\dots,q_m\,\text{in}|p_1,\dots,p_n\,\text{out}\rangle = \\ &\quad= i^{m+n}\int\mathrm{d}^4x_1\dots\mathrm{d}^4x_m\int\mathrm{d}^4y_1\dots\mathrm{d}^4y_n\times \\ &\qquad\times e^{-i(q_1x_1+\dots+q_mx_m)}e^{i(p_1y_1+\dots+p_ny_n)}\times \\ &\qquad\times(\Box_{x_1}+m^2)\cdots(\Box_{x_m}+m^2)(\Box_{y_1}+m^2)\cdots(\Box_{y_n}+m^2)\times \\ &\qquad\times\langle 0|\,\mathrm{T}[\hat{\phi}(y_1)\cdots\hat{\phi}(y_n)\hat{\phi}(x_1)\cdots\hat{\phi}(x_m)]\,|0\rangle+ \\ &\qquad+\text{disconnected terms without interactions not contributing to the cross section}\end{aligned} \tag{2.77}$$

The perturbative approach tries to compute the correlation functions $\langle 0|\,\mathrm{T}[\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)]\,|0\rangle$ of interacting Heisenberg fields in terms of the free in-fields $\hat{\phi}_{\text{in}}(x)$. Using the expression of $U_\varepsilon^I(t_1,t_2) :=$

$U_\varepsilon^I(t_1)U_\varepsilon^{I\,-1}(t_2)$ one can prove the following identity, called *Gell-Mann - Low formula*. Let $|0\rangle$ be the vacuum state in $\mathcal{H}$, then

$$\langle 0|\,\mathrm{T}[\hat\phi(x_1)\dots\hat\phi(x_n)]\,|0\rangle = \frac{\langle 0_{\rm in}|\,\mathrm{T}[\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)\,e^{-\frac{i}{\hbar}\int \mathrm{d}t\,H_\varepsilon^I(t)}]\,|0_{\rm in}\rangle}{\langle 0_{\rm in}|\,\mathrm{T}[\,e^{-\frac{i}{\hbar}\int \mathrm{d}t\,H_\varepsilon^I(t)}]\,|0_{\rm in}\rangle} \tag{2.78}$$

where integrations in t go from $-\infty$ to $+\infty$. This formula connects interacting Heisenberg fields $\hat\phi$ with in-fields $\hat\phi_{\rm in}$ assuming that we have a finite infrared cutoff ε in time.
We now give a sketch of the proof of this result.[V] Assume $t \gg t_1 > t_2 > \ldots > t_n$, then using unitarity

$$U_\varepsilon^{I\dagger}(t_1) = U_\varepsilon^{I\,-1}(t_1) = U_\varepsilon^{I\,-1}(t)U_\varepsilon^I(t,t_1) = U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1) \quad , \quad U_\varepsilon^I(t_n) = U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t) \tag{2.79}$$

and

$$\begin{aligned}
\langle 0|\,\mathrm{T}[\hat\phi(x_1)\cdots\hat\phi(x_n)]\,|0\rangle &= \langle 0|\,\hat\phi(x_1)\cdots\hat\phi(x_n)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t_1)\hat\phi_{\rm in}(x_1)U_\varepsilon^I(t_1)\cdots U_\varepsilon^{I\dagger}(t_n)\hat\phi_{\rm in}(x_n)U_\varepsilon^I(t_n)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t)U_\varepsilon^I(t,t_1)\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)U_\varepsilon^I(t_n,-t)U_\varepsilon^I(-t)\,|0\rangle \\
&= \langle 0|\,U_\varepsilon^{I\dagger}(t)\mathrm{T}[\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\,H_\varepsilon^I(t)}]U_\varepsilon^I(-t)\,|0\rangle
\end{aligned} \tag{2.80}$$

where in the last step we moved on the right all the unitary operators inside the time ordering (since they are inside the time ordering the result is the same). Now since

$$U_\varepsilon^I(t)\,|0\rangle \xrightarrow[t\to-\infty]{} |0_{\rm in}\rangle \quad , \quad U_\varepsilon^I(t)\,|0\rangle \xrightarrow[t\to+\infty]{} |0_{\rm out}\rangle \tag{2.81}$$

we have

$$\langle 0|\,\mathrm{T}[\hat\phi(x_1)\cdots\hat\phi(x_n)]\,|0\rangle = \langle 0_{\rm out}|\,\mathrm{T}[\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)e^{-\frac{i}{\hbar}\int_{-t}^{t}\mathrm{d}t\,H_\varepsilon^I(t)}]\,|0_{\rm in}\rangle \tag{2.82}$$

If we assume that the vacuum is non degenerate, we expect that under adiabatic evolution (i.e. changing the value of ε) the initial free state cannot evolve in another state, hence

$$|0_{\rm out}\rangle = e^{iL}\,|0_{\rm in}\rangle \tag{2.83}$$

for some operator L. Now $\langle 0_{\rm in}|0_{\rm out}\rangle = e^{iL}$ therefore

$$\langle 0_{\rm out}| \,= \langle 0_{\rm in}|\,e^{-iL} = \frac{\langle 0_{\rm in}|}{\langle 0_{\rm in}|0_{\rm out}\rangle} = \frac{\langle 0_{\rm in}|}{\langle 0_{\rm in}|\,S_\varepsilon\,|0_{\rm in}\rangle} \tag{2.84}$$

and taking $t\to+\infty$ we finally get eq. (2.78).
In general is very hard to evaluate expressions like eq. (2.78). In order to make this effort, the perturbative approach consists in replacing

$$\begin{gathered}
\langle 0_{\rm in}|\,\mathrm{T}[\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\,H_\varepsilon^I(t')\right)^\ell]\,|0_{\rm in}\rangle \\
\downarrow \\
\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\langle 0_{\rm in}|\,\mathrm{T}[\hat\phi_{\rm in}(x_1)\cdots\hat\phi_{\rm in}(x_n)\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t'\,H_\varepsilon^I(t')\right)^\ell]\,|0_{\rm in}\rangle
\end{gathered} \tag{2.85}$$

i.e. moving the infinite series outside the expectation value. The terms inside the sum can now be computed in terms of in-fields using Wick theorem or equivalently Feynman diagrams.
Resumming our steps, we expressed the scattering amplitudes in terms of correlators of interacting fields using LSZ formula, then Gell-Mann - Low formula allows us to express such correlators in terms of correlators of in-fields (free). Finally the perturbative approach prescripts the extraction of the infinite series outside the expectation value, in such a way that we can obtain our result computing expectation values of free fields through Feynman diagrams. We'll see in the next chapter how dangerous is the perturbative approach prescription.

[V] Reference: https://authors.library.caltech.edu/60474/1/PhysRev.84.350.pdf

2.4 Condensed matter systems

[NO98], [Kle15, Sections 1.16, 2.17–2.19, 10.9]; [Fjæ13]

Let us turn to condensed matter systems. A source of information on the physical properties (typically transport properties) are the correlation functions of local observables, such as spin, density, currents, etc; in particular two-points functions. Hence in this case physical informations are much more involved than just scattering amplitudes in particle physics. These two-points functions appear naturally in particular when one studies the *linear response*, that is the response to a test (i.e. infinitesimal) perturbation.
Let us consider a system with Hamiltonian H (which might include the classical chemical potential term μN), we want to see the effect on the mean value of an observable O_1 or a field (in general not self-adjoint) of a test perturbation typically generated by another observable O_2 in the form

$$V^\varepsilon(t) := \xi_\varepsilon(t) O_2^H(t) \tag{2.86}$$

where $\xi_\varepsilon(t)$ is a coupling vanishing as $|t| \to \infty$ similarly to $\lambda_\varepsilon(t)$ previously considered and $O_2^H(t) = e^{itH} O_2 e^{-itH}$. Notice that we do not assume H to be free as in the scattering case.
Let $\{|n\rangle\}$ be the set of eigenfunctions of H ($H|n\rangle = \mathcal{E}_n |n\rangle$) generating a Dirac completeness in $\mathcal{H}$ and consider the first order modification induced by the perturbation in

$$\langle n| O_1^H(t) |n\rangle = \langle n| e^{itH} O_1 e^{-itH} |n\rangle \tag{2.87}$$

The time evolution of O_1 in the presence of the perturbation is

$$O_1^{H+V_\varepsilon}(t) = U^{H+V_\varepsilon\,\dagger}(t) O_1 U^{H+V_\varepsilon}(t) \tag{2.88}$$

We want to rewrite this in terms of the evolution generated by H in an "interaction picture", i.e.

$$O_1^{H+V_\varepsilon}(t) = U_\varepsilon^{V\,\dagger}(t) O_1^H(t) U_\varepsilon^V(t) \tag{2.89}$$

where $U_\varepsilon^V(t)$ gives the contribution of the interaction $V_\varepsilon(t)$ to the "free" evolution described by $O_1^H(t)$.
The situation in this respect is similar to the one considered before in RQFT, just replacing of H_I^ε by V_ε we get that

$$U_\varepsilon^V(t) = \mathrm{T}[e^{-i\int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t'}] \tag{2.90}$$

We calculate the response to the perturbation through a variation, using the perturbative ansatz eq. (2.89) together with eq. (2.90) and then taking only first order contribution of V_ε in the perturbative expansion:

$$\begin{aligned}
\delta\langle n| O_1^{H+V_\varepsilon}(t) |n\rangle &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon'}\langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle\big|_{\varepsilon'=0} = \lim_{\varepsilon'\to 0} \frac{\langle n| O_1^{H+\varepsilon' V_\varepsilon}(t) |n\rangle - \langle n| O_1^H(t) |n\rangle}{\varepsilon'} = \\
&= \lim_{\varepsilon'\to 0} \frac{1}{\varepsilon'}\langle n| \left[\left(1 + i\varepsilon' \int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t' + O(\varepsilon')^2\right) O_1^H(t) \left(1 - i\varepsilon' \int_{-\infty}^t V_\varepsilon(t')\mathrm{d}t' + O(\varepsilon')^2\right) - O_1^H(t)\right] |n\rangle = \\
&= i\int_{-\infty}^t \mathrm{d}t'\, \xi_\varepsilon(t') \langle n| [O_2^H(t'), O_1^H(t)] |n\rangle = -i\int_{-\infty}^{+\infty} \mathrm{d}t'\, \xi_\varepsilon(t') \theta(t-t') \langle n| [O_1^H(t), O_2^H(t')] |n\rangle
\end{aligned} \tag{2.91}$$

One can now just sum over $\{|n\rangle\}$ to perform the thermal expectation value ($\mathcal{H}$ is the multiparticle Hilbert space of the system[VI])

$$\langle(\bullet)\rangle_{\beta:=\frac{1}{k_b T}} = \frac{\mathrm{Tr}_{\mathcal{H}}(\bullet) e^{-\beta H}}{\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}} = \frac{\sum_n \langle n| (\bullet) |n\rangle e^{-\beta \mathcal{E}_n}}{\sum_n e^{-\beta \mathcal{E}_n}} \tag{2.92}$$

obtaining the expectation value at a given temperature T

$$\delta\langle O_1^{H+V_\varepsilon}(t)\rangle_\beta = \int \mathrm{d}t'\, \xi_\varepsilon(t') \left(-i\theta(t-t')\langle [O_1^H(t), O_2^H(t')]\rangle_\beta\right) = \int_{-\infty}^{+\infty} \mathrm{d}t'\, \xi_\varepsilon(t') G_{\mathrm{ret}}^{O_1 O_2}(t,t') \tag{2.93}$$

where

$$G_{\mathrm{ret}}^{O_1 O_2}(t,t') := -i\theta(t-t')\langle [O_1^H(t), O_2^H(t')]\rangle_\beta \tag{2.94}$$

[VI] For $\varepsilon > 0$ such space can be regarded as a Fock space, even though in the thermodynamic limit is not ensured that it is still a Fock space.

is called *retarded correlation function*. Also in this case, similarly to what happens in RQFT, the experimental data are obtained from particular correlation functions.
If O_1, O_2 and ξ_ε depend also on space coordinates then eq. (2.93) becomes:

$$\begin{aligned}\delta\langle O_1^{H+V_\varepsilon}(\boldsymbol{x},t)\rangle_T &= \int \mathrm{d}^3x'\,\mathrm{d}t'\,\xi_\varepsilon(\boldsymbol{x'},t')(-i\theta(t-t')\langle[O_1^H(\boldsymbol{x},t),O_2^H(\boldsymbol{x'},t')]\rangle_\beta) = \\ &= \int \mathrm{d}^3x'\,\mathrm{d}t'\,\xi_\varepsilon(\boldsymbol{x'},t')\,G_{\mathrm{ret}}^{O_1O_2}(\boldsymbol{x},t,\boldsymbol{x'},t')\end{aligned} \tag{2.95}$$

where

$$G_{\mathrm{ret}}^{O_1O_2}(\boldsymbol{x},t,\boldsymbol{x'},t') := -i\theta(t-t')\langle[O_1^H(\boldsymbol{x},t),O_2^H(\boldsymbol{x'},t')]\rangle_\beta \tag{2.96}$$

Assuming translational invariance, i.e. $G_{\mathrm{ret}}^{O_1O_2}$ depends only on $t-t'$, and the space difference $\boldsymbol{x}-\boldsymbol{x'}$, we can easily perform the Fourier transform of eq. (2.95), since it corresponds to a convolution:

$$\delta\langle O_1^{H+V_\varepsilon}(\boldsymbol{q},\omega)\rangle_\beta = \tilde{\xi}_\varepsilon(\boldsymbol{q},\omega)\,\tilde{G}_{\mathrm{ret}}^{O_1O_2}(\boldsymbol{q},\omega) \tag{2.97}$$

The retarded correlators are typically directly connected to experiments. For example, suppose to measure the magnetization of a spin system, with spin $\boldsymbol{S}(\boldsymbol{x})$, in presence of a test magnetic field $\boldsymbol{B}(\boldsymbol{x},t)$. The coupling between the magnetic field and the spin is given by

$$\int \mathrm{d}^3x\,\boldsymbol{B}(\boldsymbol{x},t)\cdot\boldsymbol{S}(\boldsymbol{x}) \tag{2.98}$$

The linear response to the perturbation is determined by

$$-i\theta(t_1-t_2)\langle[\boldsymbol{S}^H(\boldsymbol{x_1},t_1),\boldsymbol{S}^H(\boldsymbol{x_2},t_2)]\rangle_\beta \tag{2.99}$$

whose Fourier transform is precisely the dynamic magnetic susceptibility $\chi_s(\boldsymbol{q},\omega)$ measurable by neutrons. Analogously, charged particles are coupled to the electromagnetic field (in the gauge $A_0=0$) by

$$\int \mathrm{d}^3x\,\boldsymbol{A}(\boldsymbol{x},t)\cdot\boldsymbol{j}(\boldsymbol{x}) \tag{2.100}$$

where in the free case the current $\boldsymbol{j}(\boldsymbol{x})$ is given by

$$\boldsymbol{j}(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\frac{\overleftrightarrow{\boldsymbol{\nabla}}}{2mi}\psi(\boldsymbol{x}) := \psi^\dagger(\boldsymbol{x})\left(\frac{\boldsymbol{\nabla}}{2mi}\psi(\boldsymbol{x})\right) - \left(\frac{\boldsymbol{\nabla}}{2mi}\psi^\dagger(\boldsymbol{x})\right)\psi(\boldsymbol{x}) \tag{2.101}$$

and the *conductivity* $\sigma_{\alpha\beta}$ is directly related to the Fourier transform of

$$-i\theta(t_1-t_2)\langle[\boldsymbol{j}_\alpha^H(\boldsymbol{x_1},t_1),\boldsymbol{j}_\beta^H(\boldsymbol{x_2},t_2)]\rangle_\beta \tag{2.102}$$

Notice that since the Hamiltonian should be a scalar, we get that if O_1 and O_2 are fermionic fields then $\xi_\varepsilon(t)$ must be an anticommuting function so that

$$O_1^H\xi_\varepsilon O_2^H - \xi_\varepsilon O_2^H O_1^H = -\xi_\varepsilon(O_1^H O_2^H + O_2^H O_1^H) \tag{2.103}$$

and in fact the retarded correlation function for fermionic fields O_1 and O_2 is

$$G_{\mathrm{ret}}^{O_1O_2} = i\theta(t_1-t_2)\langle\{O_1^H(t_1),O_2^H(t_2)\}\rangle_\beta \tag{2.104}$$

For example the intensity of response in metals to high frequency photons (if the direction of the photon is fixed then the determination of such intensity is called *Angle-Resolved Photoemission Spectroscopy* (*ARPES*)) is related to the imaginary part of the Fourier transform of

$$i\theta(t_1-t_2)\langle\{\psi^\dagger(\boldsymbol{x_1},t_1),\psi(\boldsymbol{x_2},t_2)\}\rangle_\beta \tag{2.105}$$

Matsubara formalism

If H is not free the next question is how to compute the retarded correlation functions. Let $H = H_0 + H_I$, with H_0 "free" (typically this means that contains only the quadratic terms in the fields, with no mixed components). For $T = 0$ one just have to replace

$$\langle(\bullet)\rangle_\beta \to \langle 0|\,(\bullet)\,|0\rangle \tag{2.106}$$

and the Gell-Mann Low formula applies as before with adiabatic switching as in the relativistic case and the perturbative approach is completely analogous.
For $T > 0$ the situation is more complicated because H_I would appear in two places: in the time evolution eq. (2.90) as in $T = 0$ but also in the Boltzmann weight $e^{-\beta(H_0+H_I)}$. This makes the standard perturbative treatment inapplicable, since we need to disentangle a perturbation in these two places. The way to solve this issue is due to Matsubara, and for this reason is called the *Matsubara formalism*. It consists in defining new correlators for which an analogue of the Gell-Mann Low formula applies, and then relate these new correlators to the retarded correlators by analytic continuation. Let's see how this works.
We define an "evolution" of the operators by a new parameter τ:

$$O^H(\tau) = e^{\tau H} O e^{-\tau H} \quad \text{with} \quad 0 \le \tau \le \beta \tag{2.107}$$

and then instead of computing correlation functions at ordinary time t, we compute them using the modified evolution and the parameters τ_i:

$$\langle O_1^H(\tau_1)\cdots O_n^H(\tau_n)\rangle_\beta \tag{2.108}$$

Suppose that $H = H_0 + H_I$ (notice that the cutoff ε is not required here since the domain of τ is already finite, $\tau \in [0, \beta]$) and write again the evolution separating the contributions of H_0 and H_I as in the interaction picture

$$O^H(\tau) = U^{I\,\dagger}(\tau) O^{H_0}(\tau) U^I(\tau) \tag{2.109}$$

getting as before

$$U^I(\tau) = e^{\tau H_0} e^{-\tau H} \quad \text{with} \quad U^I(0) = \mathbb{1} \tag{2.110}$$

and again

$$U^I(\tau) = \mathrm{T}_\tau[e^{-\int_0^\tau \mathrm{d}\tau'\, H_I(\tau')}] \tag{2.111}$$

Note that equations (2.109), (2.110) and (2.111) are the analogues of equations (2.64), (2.68) and (2.72) respectively (and can be obtained with the same procedure).
We define, in analogy to eq. (2.76), the *Matsubara correlator / Green function* by

$$\begin{aligned} G_M^{O_1\cdots O_n}(\tau_1,\ldots,\tau_n) &:= -\langle \mathrm{T}_\tau[O_1^H(\tau_1)\cdots O_n^H(\tau_n)]\rangle_\beta \\ &\overset{(2.92)}{=} -\frac{1}{Z}\,\mathrm{Tr}_{\mathcal{H}}\left(\mathrm{T}_\tau[O_1^H(\tau_1)\cdots O_n^H(\tau_n)]e^{-\beta H}\right) \end{aligned} \tag{2.112}$$

where Z is the *partition function* defined by

$$Z := \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} \tag{2.113}$$

with the trace summing over all the multiparticle states of the system.
Unlike for the case of retarded Green functions, for Matsubara Green functions an analogue of the Gell-Mann Low formula applies

$$\langle \mathrm{T}_\tau[O_1^H(\tau_1)\cdots O_n^H(\tau_n)]\rangle_\beta = \frac{\langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1)\cdots O_n^{H_0}(\tau_n) e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}]\rangle_\beta^0}{\langle \mathrm{T}_\tau[e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}]\rangle_\beta^0} \tag{2.114}$$

where $\langle\ \rangle_\beta^0$ is the thermal average computed using H_0 as Hamiltonian. In comparison with the Gell-Mann Low formula for particle physics, in this case the vacuum expectation values are replaced by the Boltzmann-weighted thermal traces, and the vacuum expectation value of the S-matrix operator $U_\varepsilon^I(+\infty)$

in the denominator is replaced by the Boltzmann-weighted trace of the interaction operator $U^I(\beta)$ along the euclidean time axis τ. Moreover, the integration over τ goes up to β, since $0 \leq \tau \leq \beta$.
Finally, the perturbative treatment prescripts

$$\begin{gathered}\langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1)\dots O_n^{H_0}(\tau_n)e^{-\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')}]\rangle_\beta^0 \\ \downarrow \\ \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\langle \mathrm{T}_\tau[O_1^{H_0}(\tau_1)\dots O_n^{H_0}(\tau_n)\Big(\int_0^\beta \mathrm{d}\tau'\, H_I(\tau')\Big)^\ell]\rangle_\beta^0\end{gathered} \tag{2.115}$$

This leads to a series of thermally averaged products of many fields which evolve according to the free field equations. Therefore Wick's theorem can be applied and we obtain an expansion of the Matsubara Green function completely analogous to that used in the (relativistic) field theoretic setting. The only difference is the finite-time interaction. Now everything is again expressed in terms of the free fields and can be computed using Feynman diagrams.
To be precise in most cases one also need to use an infrared cutoff ε, which in this case (the domain of τ is finite) is required only in the space coordinates, and will be removed at the end of the calculation.

KMS condition and Matsubara frequencies

According to the definition eq. (2.112), the 2-points Matsubara correlator (the only one related to the computation of retarded correlators in perturbation theory) reads:

$$\begin{aligned}G_M^{O_1O_2}(\tau_1,\tau_2) &= -\frac{1}{Z}\big\{\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}e^{\tau_1 H}O_1e^{-\tau_1 H}e^{\tau_2 H}O_2e^{-\tau_2 H}]\theta(\tau_1-\tau_2) \\ &\qquad \pm \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}e^{\tau_2 H}O_2e^{-\tau_2 H}e^{\tau_1 H}O_1e^{-\tau_1 H}]\theta(\tau_2-\tau_1)\big\} = \\ &= -\frac{1}{Z}\big\{\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_1e^{-(\tau_1-\tau_2)H}O_2e^{(\tau_1-\tau_2)H}]\theta(\tau_1-\tau_2) \\ &\qquad \pm \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_2e^{-(\tau_2-\tau_1)H}O_1e^{(\tau_2-\tau_1)H}]\theta(\tau_2-\tau_1)\big\}\end{aligned} \tag{2.116}$$

where in the second step we used the cyclicity of the trace and the choice of the sign depends on the commutation relation between O_1^H and O_2^H (+ if they commute or − if they anticommute). This computation shows that the two points correlator is just a function of the difference $\tau := \tau_1 - \tau_2$.

$$G_M^{O_1O_2}(\tau) = -\frac{1}{Z}\big\{\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_1e^{-\tau H}O_2e^{\tau H}]\theta(\tau) \pm \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_2e^{\tau H}O_1e^{-\tau H}]\theta(-\tau)\big\} \tag{2.117}$$

By construction $0 \leq \tau_i \leq \beta$, $i = 1, 2$, hence we get that $\tau \in [-\beta, \beta]$. Notice that for $\tau \in [-\beta, 0]$, $\tau + \beta \geq 0$ and since

$$\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_2e^{\tau H}O_1e^{-\tau H}] = \mathrm{Tr}_{\mathcal{H}}[O_2e^{(\tau+\beta)H}e^{-\beta H}O_1e^{-(\tau+\beta)H}] = \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}O_1e^{-(\tau+\beta)H}O_2e^{(\tau+\beta)H}] \tag{2.118}$$

we have $G_M^{O_1O_2}(\tau) = \pm\, G_M^{O_1O_2}(\tau+\beta)$ and we can regard $G_M^{O_1O_2}(\tau)$ as an (anti)periodic function of period β defined for any $\beta \in \mathbb{R}$.
In general traces can be infinite or the limit $\varepsilon \to 0$ of the cutoff might not exists, making the perturbative approach not working (notice that an infinite trace is the thermal-equivalent of a divergent vacuum expectation value of the zero-temperature case). Nevertheless the periodicity of the Matsubara correlator is a non-perturbative result which hold anyway, even if the definition of the trace lose its meaning. Such relation is called *Kubo-Martin-Schwinger (KMS) condition*:

$$G_M(\tau) = \pm\, G_M(\tau+\beta) \tag{2.119}$$

which is independent on the choice of ε and works also for $\varepsilon \to 0$.
Due to periodicity, $G_M(\tau)$ can be represented in terms of Fourier coefficients

$$G_M(\tau) = \frac{1}{\beta}\sum_{n\in\mathbb{Z}} e^{-i\omega_n\tau}G_M(\omega_n) \tag{2.120}$$

with

$$\omega_n = \begin{cases} \dfrac{2n\pi}{\beta} & \text{if} \quad G_M(\tau) = +G_M(\tau+\beta) \\ \dfrac{(2n+1)\pi}{\beta} & \text{if} \quad G_M(\tau) = -G_M(\tau+\beta) \end{cases} \tag{2.121}$$

called *Matsubara frequencies* and

$$G_M(\omega_n) = \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n \tau} G_M(\tau) \tag{2.122}$$

Notice that as $T \to 0$ Matsubara coefficients become continuous and eq. (2.120) become a Fourier transformation.

Lehmann spectral representation of the correlators ($T > 0$)

Now the question is how this thermal averaged operator can be related to the retarded correlators needed to compute the linear response of our system.
The Wick rotation of the t-axis in eq. (2.94) exactly coincides with the axis on which τ is defined, and the Wick rotation send the energies in the field theoretic Green functions eq. (2.94) to the axis along which Matsubara frequencies are situated. The retarded Green functions are related to the imaginary-time Matsubara Green functions by an analytical continuation.
To see this relation we introduce a Dirac completeness of eigenstates $\{|n\rangle\}$ of H inside the coefficients of the Fourier series eq. (2.122) (we change the label for the Fourier coefficients from n to s) using eq. (2.117):

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z}\int_0^\beta \mathrm{d}\tau\, e^{i\omega_s\tau}\, \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H} O_1 e^{-\tau H} O_2 e^{\tau H}] \\ &= -\frac{1}{Z}\int_0^\beta \mathrm{d}\tau \sum_{m,n} e^{i\omega_s\tau} e^{-\beta\mathcal{E}_n} e^{\tau(\mathcal{E}_n - \mathcal{E}_m)} \langle n|\, O_1\, |m\rangle\langle m|\, O_2\, |n\rangle \end{aligned} \tag{2.123}$$

Using $\int_0^\beta \mathrm{d}\tau\, e^{\alpha\tau} = \frac{1}{\alpha}(e^{\alpha\beta} - 1)$ we can integrate:

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z}\sum_{m,n} e^{-\beta\mathcal{E}_n} \frac{e^{\beta(i\omega_s + \mathcal{E}_n - \mathcal{E}_m)} - 1}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n|\, O_1\, |m\rangle\langle m|\, O_2\, |n\rangle \\ &= -\frac{1}{Z}\sum_{m,n} \frac{e^{\beta(i\omega_s - \mathcal{E}_m)} - e^{-\beta\mathcal{E}_n}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n|\, O_1\, |m\rangle\langle m|\, O_2\, |n\rangle \end{aligned} \tag{2.124}$$

Finally using the explicit values of Matsubara frequencies eq. (2.121), one gets

$$\begin{aligned} G_M(\omega_s) &= -\frac{1}{Z}\sum_{m,n} \frac{\pm e^{-\beta\mathcal{E}_m} - e^{-\beta\mathcal{E}_n}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n|\, O_1\, |m\rangle\langle m|\, O_2\, |n\rangle = \\ &= \frac{1}{Z}\sum_{m,n} \frac{e^{-\beta\mathcal{E}_n} \mp e^{-\beta\mathcal{E}_m}}{i\omega_s + \mathcal{E}_n - \mathcal{E}_m} \langle n|\, O_1\, |m\rangle\langle m|\, O_2\, |n\rangle \quad \text{for} \quad G_M(\tau) = \pm G_M(\tau+\beta) \end{aligned} \tag{2.125}$$

Such expression for the Green function is called *Lehmann representation* (of G_M, in this case). Notice that in this representation the Fourier series coefficients can be computed in terms of the eigenvalues of H and the matrix elements of O_1 and O_2 only (for this reason such representation is also called "spectral" representation). Notice that these coefficients have poles along the imaginary axis.
Let's try to find the Lehmann representation of the retarded Green function we introduced before. We use instead of eq. (2.94) or its fermionic analogue the synthetic notation

$$G_{\mathrm{ret}}^{O_1 O_2}(t,t') := -i\theta(t-t')\langle [O_1^H(t), O_2^H(t')]_\mp \rangle_\beta \tag{2.126}$$

taking the commutator or the anticommutator depending on the label after the square bracket. Rewriting eq. (2.126) introducing a Dirac completeness as in eq. (2.123) we get

$$\begin{aligned} G_{\text{ret}}(t_1,t_2) &= -i\theta(t_1-t_2)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}[e^{it_1H}O_1e^{-it_1H}, e^{it_2H}O_2e^{-it_2H}]_{\mp}] \\ &= -\frac{i}{Z}\theta(t_1-t_2)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}(e^{it_1H}O_1e^{-it_1H}e^{it_2H}O_2e^{-it_2H} \mp e^{it_2H}O_2e^{-it_2H}e^{it_1H}O_1e^{-it_1H})] \\ &= -\frac{i}{Z}\theta(t)\,\mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}e^{itH}O_1e^{-itH}O_2 \mp e^{itH}O_1e^{-\beta H}e^{-itH}O_2] \\ &= -\frac{i}{Z}\theta(t)\sum_{m,n}(e^{-\beta\mathcal{E}_n} \mp e^{-\beta\mathcal{E}_m})e^{it(\mathcal{E}_n-\mathcal{E}_m)}\langle n|\,O_1\,|m\rangle\langle m|\,O_2\,|n\rangle \end{aligned} \tag{2.127}$$

where $t := t_1 - t_2$ and in the third step we used the cyclicity of the trace. Using the Fourier Transform of the following functions (the limit $\delta \to 0^+$ is understood)

$$\begin{aligned} \mathcal{F}[\theta(t)](\omega) = \frac{i}{\omega+i\delta} \quad , \quad \mathcal{F}[e^{i\alpha t}](\omega) = \delta(\omega+\alpha) \\ \mathcal{F}[f(t)g(t)](\omega) = \int \mathrm{d}\omega'\,\tilde{f}(\omega-\omega')\tilde{g}(\omega') \end{aligned} \tag{2.128}$$

one get the Fourier transform of $G_{\text{ret}}(t)$:

$$\begin{aligned} G_{\text{ret}}(\omega) &= -\frac{i}{Z}\int \mathrm{d}\omega'\,\frac{i}{\omega-\omega'+i\delta}\sum_{m,n}(e^{-\beta\mathcal{E}_n} \mp e^{-\beta\mathcal{E}_m})\delta(\omega'+\mathcal{E}_n-\mathcal{E}_m)\langle n|\,O_1\,|m\rangle\langle m|\,O_2\,|n\rangle \\ &= \frac{1}{Z}\sum_{m,n}\frac{e^{-\beta\mathcal{E}_n} \mp e^{-\beta\mathcal{E}_m}}{\omega+\mathcal{E}_n-\mathcal{E}_m+i\delta}\langle n|\,O_1\,|m\rangle\langle m|\,O_2\,|n\rangle \end{aligned} \tag{2.129}$$

which is the Lehmann representation of G_{ret}. Notice that the prescription $i\delta$ is needed to move slightly the poles of the functions from the real axis, in such a way that the Wick rotation of the function is allowed.
Is now clear comparing eq. (2.125) and eq. (2.129) that the retarded correlator is just the analytical continuation of the Matsubara correlator

$$G_{\text{ret}}(\omega) = G_M(i\omega_s \to \omega + i\delta) \tag{2.130}$$

Can be proved[VII] using Carleman's theorem that such relation is well-defined and can be used to reconstruct uniquely $G_{\text{ret}}(\omega)$ by analytical continuation of $G_M(\omega_n)$. The only requirement on the Matsubara Green function is that it satisfy the KMS condition eq. (2.119). Therefore it is possible to compute G_M as in the high energy case and then obtain G_{ret} from analytical continuation.

2.5 Path integral formalism

[NO98], [Zha99], [Kle15, Chapter 14]

In this section we just sketch in a "heuristic way", without details, how the path-integral is defined, with some more details in the non relativistic setting and $T > 0$.
The key ingredients of Feynman path-integrals both in real and imaginary time are

(1) a Dirac completeness like $\int \mathrm{d}x\ |x\rangle\langle x| = \mathbb{1}$, appropriately generalized for fields;

(2) the representation of the kernel of infinitesimal time evolution in terms of the action.

[VII] Original proof by Baym and Mermin (1960): https://doi.org/10.1063/1.1703704, useful reference: Appendix A in https://fks.sk/~bzduso/physics/master/thesis.pdf.

Path integral for bosonic fields

For bosonic fields the completeness employs the *coherent states*. Recall that for the harmonic QM oscillator the coherent states $|z\rangle$, $z \in \mathbb{C}$, are of the form (with a annihilation operator)

$$a\,|z\rangle = z\,|z\rangle \quad , \quad |z\rangle = e^{za^\dagger}\,|0\rangle \tag{2.131}$$

One easily verifies that

$$\langle z|\,a^\dagger = z^*\langle z| \quad , \quad \langle z|z'\rangle = e^{z^*z'} \quad , \quad \int_{\mathbb{C}} \frac{\mathrm{d}z\,\mathrm{d}z^*}{2\pi i}\,|z\rangle\langle z|\,e^{-|z|^2} = \int_{\mathbb{R}^2} \frac{\mathrm{d}\,\mathrm{Re}(z)\,\mathrm{d}\,\mathrm{Im}(z)}{\pi}\,|z\rangle\langle z|\,e^{-|z|^2} = \mathbb{1} \tag{2.132}$$

The concept of coherent states easily generalizes to fields, just replacing the single annihilation operator in QM with a set of annihilation operators $\{a_\alpha\}_{\alpha\in A}$. Indeed a generic field can be written as[VIII]

$$\hat{\phi}(\boldsymbol{x}) = \sum_{\alpha\in A} \phi_\alpha(\boldsymbol{x}) a_\alpha \tag{2.133}$$

The set $\{a_\alpha\}_{\alpha\in A}$ is commuting, hence all the operators admits a common eigenbasis made of eigenvectors in the form $|\phi\rangle := |\{\phi_\alpha\}_{\alpha\in A}\rangle$, such that for any $\alpha \in A$

$$a_\alpha\,|\phi\rangle = \phi_\alpha\,|\phi\rangle \quad \text{with eigenvalues} \quad \phi_\alpha \in \mathbb{C} \tag{2.134}$$

Moreover, using commutation relations for ladder operators, this implies

$$|\phi\rangle = e^{\sum_\alpha \phi_\alpha a_\alpha^\dagger}\,|0\rangle \quad \text{and} \quad \langle\phi|\phi'\rangle = e^{\sum_\alpha \phi_\alpha^*\phi'_\alpha} \tag{2.135}$$

It is clear that vectors $\{\,|\phi\rangle\}$ are eigenvectors also for the field operator:[IX]

$$\hat{\phi}(\boldsymbol{x})\,|\phi\rangle = \sum_{\alpha\in A} \phi_\alpha(\boldsymbol{x})\phi_\alpha\,|\phi\rangle = \phi(\boldsymbol{x})\,|\phi\rangle \tag{2.136}$$

with eigenvalue $\phi(\boldsymbol{x}) := \sum_{\alpha\in A}\phi_\alpha(\boldsymbol{x})\phi_\alpha$ (is a function). In particular, for a RQFT we have the operator

$$\hat{\phi}(\boldsymbol{x}) = \sum_\alpha \phi_\alpha(\boldsymbol{x})a_\alpha + \phi_\alpha^*(\boldsymbol{x})a_\alpha^\dagger \tag{2.137}$$

and correspondingly we have the eigenvalues

$$\phi(\boldsymbol{x}) = \sum_\alpha \phi_\alpha(\boldsymbol{x})\phi_\alpha + \phi_\alpha^*(\boldsymbol{x})\phi_\alpha^* \tag{2.138}$$

Notice that up to now we have properly generalized eq. (2.131) in the case of fields.
Let's see how the last identity in eq. (2.132) generalizes in this setting. The following Dirac completeness can be proved:[X]

$$\int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi}\,|\phi\rangle\langle\phi| \;:=\; \int \prod_{\alpha\in A} \frac{\mathrm{d}\phi_\alpha \mathrm{d}\phi_\alpha^*}{2\pi i} e^{-\sum_\alpha \phi_\alpha^*\phi_\alpha}\,|\phi\rangle\langle\phi| \;=\mathbb{1} \tag{2.139}$$

(from now on we will often suppress the label $\alpha \in A$).
For each operator A and a basis $\{\,|n\rangle\}$ satisfying Dirac completeness $\sum_n |n\rangle\langle n| = \mathbb{1}$ we have

$$\begin{aligned} \mathrm{Tr}\,A &= \sum_n \langle n|\,A\,|n\rangle = \sum_n \langle n|\,A \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi}\,|\phi\rangle\,\langle\phi|n\rangle \\ &= \sum_n \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi}\,\langle\phi|n\rangle\,\langle n|\,A\,|\phi\rangle \\ &= \int \frac{\mathrm{d}\phi\,\mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi}\langle\phi|\,A\,|\phi\rangle \end{aligned} \tag{2.140}$$

[VIII] We consider the field at $t = 0$ in analogy with QM, the evolution can be recovered according to Heisenberg picture.
[IX] Notice the difference between ϕ_α and $\phi_\alpha(\boldsymbol{x})$: the former is the eigenvalue of a_α on $|\phi\rangle$, whereas the latter comes from the decomposition eq. (2.133).
[X] Here ϕ should be regarded as a vector in $\mathbb{C}^{|A|}$ whose components are $\{\phi_\alpha\}_{\alpha\in A}$ with scalar product $\phi^*\phi' = \sum_\alpha \phi_\alpha^*\phi'_\alpha$.

In the Gell-Mann - Low formula or in the Matsubara formalism (to which we now address our effort) beside the fields at fixed time $\hat{\phi}(\boldsymbol{x})$ there are also the time-evolution operators. Let us consider an infinitesimal one in the non-relativistic case: for H the Hamiltonian in terms of free fields, $\Delta\tau$ infinitesimal "imaginary time" interval, the infinitesimal time-evolution operator is

$$e^{-\Delta\tau H(a^\dagger_\alpha, a_\alpha)} \tag{2.141}$$

We rewrite it in "normal order" $: \ :$ form, i.e. with all the creation operators on the left of the annihilation ones:

$$e^{-\Delta\tau H(a^\dagger_\alpha, a_\alpha)} =: e^{-\Delta\tau H(a^\dagger_\alpha, a_\alpha)} : \ + O((\Delta\tau)^2) \tag{2.142}$$

Let us now study the partition function of the system. It reads

$$Z := \operatorname{Tr} e^{-\beta H} = \int \frac{\mathrm{d}\phi\, \mathrm{d}\phi^*}{2\pi i} e^{-\phi^*\phi} \langle\phi| \, e^{-\beta H} \, |\phi\rangle \tag{2.143}$$

We now split the interval $[0, \beta]$ into N small intervals of length $\beta/N =: \Delta\tau$ and at the "imaginary times" multiples of $\Delta\tau$ we insert the Dirac completeness eq. (2.139). Finally we take the limit $N \to \infty$ so that the intervals become arbitrarily small. We get[XI]

$$Z = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j \, \mathrm{d}\phi_j^*}{2\pi i} e^{-\phi_j^* \phi_j} \langle\phi_j| \, e^{-\Delta\tau H} \, |\phi_{j-1}\rangle \quad \text{where} \quad \phi_0 := \phi_N \tag{2.144}$$

For each $|\phi_j\rangle$, $j = 1, \dots, N$, the integral runs over all possible states of the completeness eq. (2.139). Due to the limit $N \to \infty$ and eq. (2.142) we have $\Delta\tau \approx 0$ and then

$$\begin{aligned} &\langle\phi_j| \, e^{-\Delta\tau H(a^\dagger_\alpha, a_\alpha)} \, |\phi_{j-1}\rangle \simeq \langle\phi_j| : e^{-\Delta\tau H(a^\dagger_\alpha, a_\alpha)} : \, |\phi_{j-1}\rangle = \\ &\quad = e^{-\Delta\tau H(\phi_j^*, \phi_{j-1})} \langle\phi_j|\phi_{j-1}\rangle = e^{-\Delta\tau H(\phi_j^*, \phi_{j-1})} e^{\sum_\alpha \phi^*_{\alpha\, j} \phi_{\alpha\, j-1}} \end{aligned} \tag{2.145}$$

Putting things together we have

$$Z = \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j \, \mathrm{d}\phi_j^*}{2\pi i} e^{-S_N(\phi^*, \phi)} \tag{2.146}$$

with

$$S_N(\phi^*, \phi) := \Delta\tau \left(\sum_{j=1}^{N} \sum_\alpha \phi^*_{\alpha\, j} \frac{\phi_{\alpha\, j} - \phi_{\alpha\, j-1}}{\Delta\tau} + H(\phi_j^*, \phi_{j-1}) \right) \quad \text{where} \quad \phi_0 := \phi_N \tag{2.147}$$

We now define the *path integral measure* $\mathcal{D}\phi(\tau)$ such that

$$\int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau) \, \dots := \lim_{N\to\infty} \int \prod_{j=1}^{N} \frac{\mathrm{d}\phi_j \, \mathrm{d}\phi_j^*}{2\pi i} \, \dots \tag{2.148}$$

and taking the limit on S_N as a Riemann integral, the label j is replaced by a continuous variable τ:

$$S_N \xrightarrow[N\to\infty]{} \int_0^\beta \mathrm{d}\tau \left(\phi^*(\tau) \frac{\partial}{\partial\tau} \phi(\tau) + H(\phi^*(\tau), \phi(\tau)) \right) =: S(\phi^*, \phi) \quad \text{where} \quad \phi(0) = \phi(\beta) \tag{2.149}$$

In this way we obtained the *Euclidean action* S with time τ of period β, since $\phi_0 := \phi_N$ has been replaced by the condition $\phi(0) = \phi(\beta)$. Hence for the partition function we get

$$Z = \int_{\phi(0)=\phi(\beta)} \mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau) \, e^{-S(\phi^*, \phi)} \tag{2.150}$$

[XI] Here ϕ_j are vectors as in the footnote X.

We can interpret $e^{-S(\phi^*,\phi)}$ as a Boltzmann weight for field configurations. In general expression eq. (2.150) has not a rigorous mathematical definition. One exception is provided by free fields, for which one can give a rigorous meaning to the condensed expression

$$\mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\, e^{-S(\phi^*,\phi)} \tag{2.151}$$

as a Gaussian measure (notice that for free fields the action is quadratic), and then the whole integral is well defined.
The original non-commutativity of the field operators $\hat{\phi}$ is replaced here by the integration over all possible "alternatives" of the function fields ϕ, without the classical constraint of the equation of motion.
It worth to notice that in general the expansion in annihilation operators eq. (2.133) is performed in momentum space, hence the labels $\{\alpha \in A\}$ are the momentum vectors $\boldsymbol{p} \in \mathbb{R}^3$. Since each element $\phi(\tau)$ should be meant as a collection $\{\phi_\alpha(\tau)\}_{\alpha\in A} = \{\phi_{\boldsymbol{p}}(\tau)\}_{\boldsymbol{p}\in\mathbb{R}^3}$, then we can define a function $\phi(\tau, \boldsymbol{p}) := \phi_{\boldsymbol{p}}(\tau)$. Similarly, the expansion of $\hat{\phi}$ can be done in the standard space of elements $\{\boldsymbol{x} \in \mathbb{R}^3\}$, defining in this way the functions $\phi(\tau, \boldsymbol{x})$. Therefore the path-integral measure can be thought as an integration over all possible functions defined on $\mathbb{R}^4$, periodic in τ, and the summation $\sum_\alpha$ can always be replaced by $\int \mathrm{d}^3x$ or $\int \mathrm{d}^3p$.

Path integral for fermionic fields

If we have fermions the above procedure needs a little modification since now annihilation states satisfy a different algebra

$$\{a_\alpha, a_{\alpha'}\} = 0 \tag{2.152}$$

so that for each common eigenstate $|\phi\rangle$ of the annihilation operators we have

$$0 = \{a_\alpha, a_{\alpha'}\}\,|\phi\rangle = \{\phi_\alpha, \phi_{\alpha'}\}\,|\phi\rangle \tag{2.153}$$

hence the eigenvalues $\phi_\alpha, \phi_{\alpha'}$ of the annihilation operators cannot be complex numbers, and we need to introduce some *Grassmann variables*, i.e. anticommuting objects such that

$$\xi_\alpha\xi_{\alpha'} + \xi_{\alpha'}\xi_\alpha = 0 \tag{2.154}$$

A strategy similar to the one previously sketched replacing bosonic operators $\hat{\phi}$ by complex functions ϕ can be adapted, replacing fermionic operators $\hat{\phi}$ by Grassmann functions ξ. For instance the trace of an operator now reads (omitting indices $\alpha \in A$)

$$\mathrm{Tr}\, A = \int \mathrm{d}\xi^*\mathrm{d}\xi\, e^{\sum_\alpha \xi^*_\alpha\xi_\alpha}\langle -\xi|\, A\,|\xi\rangle \tag{2.155}$$

so that

$$Z = \lim_{N\to\infty}\int \prod_{j=1}^{N} \mathrm{d}\xi^*_j\mathrm{d}\xi_j\, e^{-S_N(\xi^*,\xi)} =: \int_{\xi(0)=-\xi(\beta)} \mathcal{D}\xi^*(\tau)\mathcal{D}\xi(\tau)\, e^{-S(\xi^*,\xi)} \tag{2.156}$$

with the *Euclidean action* defined in terms of Grassmann fields

$$S(\xi^*,\xi) := \int_0^\beta \left(\xi^*(\tau)\frac{\partial}{\partial\tau}\xi(\tau) + H(\xi^*(\tau),\xi(\tau)\right) \tag{2.157}$$

Again one can give a rigorous meaning to the above formulas for free fields by proving that $\mathcal{D}\xi^*(\tau)\mathcal{D}\xi(\tau)\, e^{-S(\xi^*,\xi)}$ is a "Grassmann Gaussian integral".
In the following we denote by ϕ both the complex fields in the bosonic case and the Grassmann fields in the fermionic case.

Computation of Matsubara Green functions

As an example, let's see how to compute Matsubara Green functions using the path integral formalism in the Fock space in the finite volume case. Since we are working in the Fock space we can use for

the free field the ladder operators $\{a_\alpha, a_\alpha^\dagger\}_{\alpha\in A}$, and using the Matsubara formalism we can introduce a dependence on τ for these operators:

$$a_\alpha(\tau) = e^{\tau H} a_\alpha e^{-\tau H} \tag{2.158}$$

Then according to eq. (2.112) in Fock space the Matsubara Green functions are defined (up to a sign) by

$$\begin{aligned}\langle \mathrm{T}_\tau[a_{\alpha_1}(\tau_1)\cdots a_{\alpha_n}(\tau_n)a^\dagger_{\alpha_{n+1}}(\tau_{n+1})\cdots a^\dagger_{\alpha_{2n}}(\tau_{2n})]\rangle_\beta :=& \\ := \frac{\mathrm{Tr}_{\mathscr{F}^\pm}\left(e^{-\beta H}\mathrm{T}_\tau[a_{\alpha_1}(\tau_1)\cdots a_{\alpha_n}(\tau_n)a^\dagger_{\alpha_{n+1}}(\tau_{n+1})\cdots a^\dagger_{\alpha_{2n}}(\tau_{2n})]\right)}{\mathrm{Tr}_{\mathscr{F}^\pm} e^{-\beta H}}&\end{aligned} \tag{2.159}$$

where we consider $\mathscr{F}^+$ for bosonic fields and $\mathscr{F}^-$ for fermionic fields. This formula holds provided that $Z = \mathrm{Tr}_{\mathscr{F}^\pm} e^{-\beta H}$ is finite. The requirement of finite volume makes this assumption more easily satisfied, but obviously there may be problems with the thermodynamic limit.
The time ordering product can be removed by means of a permutation π that puts the imaginary times in chronological order, $\tau_{\pi(i)} > \tau_{\pi(i+1)}$, then using notation

$$\tilde{a}_{\alpha_i}(\tau_i) = \begin{cases} a_{\alpha_i}(\tau_i) & \text{if} \quad i \le n \\ a^\dagger_{\alpha_i}(\tau_i) & \text{if} \quad i > n \end{cases} \tag{2.160}$$

then

$$\begin{aligned}&\langle \mathrm{T}_\tau[a_{\alpha_1}(\tau_1)\cdots a_{\alpha_n}(\tau_n)a^\dagger_{\alpha_{n+1}}(\tau_{n+1})\cdots a^\dagger_{\alpha_{2n}}(\tau_{2n})]\rangle_\beta = \\ &= \frac{1}{Z}\epsilon^{\sigma(\pi)}\,\mathrm{Tr}\left(e^{-\beta H}\tilde{a}_{\alpha_{\pi(1)}}(\tau_{\pi(1)})\cdots\tilde{a}_{\alpha_{\pi(2n)}}(\tau_{\pi(2n)})\right) \\ &= \frac{1}{Z}\epsilon^{\sigma(\pi)}\int_{\phi(0)=\pm\phi(\beta)}\mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\,e^{-S(\phi^*,\phi)}\tilde{\phi}_{\alpha_{\pi(1)}}(\tau_{\pi(1)})\cdots\tilde{\phi}_{\alpha_{\pi(2n)}}(\tau_{\pi(2n)}) \\ &= \frac{1}{Z}\int_{\phi(0)=\pm\phi(\beta)}\mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\,e^{-S(\phi^*,\phi)}\tilde{\phi}_{\alpha_1}(\tau_1)\cdots\tilde{\phi}_{\alpha_{2n}}(\tau_{2n}) \\ &= \frac{1}{Z}\int_{\phi(0)=\pm\phi(\beta)}\mathcal{D}\phi(\tau)\mathcal{D}\phi^*(\tau)\,e^{-S(\phi^*,\phi)}\phi_{\alpha_1}(\tau_1)\cdots\phi_{\alpha_n}(\tau_n)\phi^*_{\alpha_{n+1}}(\tau_{n+1})\cdots\phi^*_{\alpha_{2n}}(\tau_{2n})\end{aligned} \tag{2.161}$$

where in the last step we put the operators in the original order, killing in this way the factor $\epsilon^{\sigma(\pi)}$. We saw in this way that the path integral is very useful to avoid problems related to the time ordering inside the Gell-Mann Low formula.

Relativistic case

Completely analogous formulas holds in the relativistic case, where $S(\phi^*, \phi)$ is replaced by the Euclidean relativistic action. For instance for the massive bosonic field it is given by

$$S(\phi^*, \phi) = \frac{1}{\hbar}\int \mathrm{d}^{d+1}x\,\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi^* + m^2\phi^*\phi)(x,\tau) \tag{2.162}$$

Chapter 3

The need of a non-perturbative approach

We saw in the previous chapter that all the interesting physical quantities can be computed, both in high-energy physics and in condensed matter physics, using the perturbative approach. In this chapter we'll see why what we said doesn't really work so good, and in some instances we need a non-perturbative approach. We already anticipated some problems in the introduction, now we'll go more in details.

3.1 The asymptotic series

Divergent quantities

Let's start from the simple example provided by a massive theory with quartic coupling in $d = 0$. Consider the following quantity, which naturally emerges in perturbative computations:

$$\frac{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} e^{-\lambda x^4}}{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \quad \text{for} \quad \lambda > 0 \quad , \quad \mathrm{Re}\,\alpha > 0 \tag{3.1}$$

Such quantity is clearly smaller than 1, since the numerator contains the integral of a function which is everywhere smaller than the function in the integral present in the denominator. Nevertheless if one applies the perturbative prescription, the previous quantity is substituted by

$$\frac{\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} (-\lambda x^4)^n}{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \tag{3.2}$$

But if one tries to compute explicitly the coefficient of the series

$$\frac{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} (\lambda x^4)^n}{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} = \lambda^n \frac{(4n)!}{(2n)!} \frac{1}{2^{2n+1}} \frac{1}{\alpha^{2n}} \geq \frac{\lambda^n}{2} \frac{((2n)!)^2}{(2n)!} \frac{1}{(2\alpha)^{2n}} = \frac{\lambda^n}{2} \frac{(2n)!}{(2\alpha)^{2n}} \geq \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2} \right)^n (n!)^2 \tag{3.3}$$

one gets that the series eq. (3.2) is absolutely divergent

$$\sum_{n=0}^{\infty} \left| \frac{(-\lambda)^n}{n!} \frac{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2} x^{4n}}{\int \mathrm{d}x\, e^{-\frac{\alpha}{2}x^2}} \right| \geq \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{\lambda}{(2\alpha)^2} \right)^n n! = +\infty \tag{3.4}$$

In this case the perturbative prescription didn't worked, but can be proved that the same issue appears also in higher dimension. Actually this problem is more general, indeed all the perturbation series needed to compute the correlator using Gell-Mann Low formula are divergent.
Nevertheless, it is well known that usually the perturbative approach works (for instance in QED it works very well, and allows to obtain extremely precise predictions) hence we wonder if it is somehow possible to obtain the right expression of the original correlator using the coefficients of the perturbative series. In other words we want to know if, although divergent, could the perturbative series at least determine uniquely the corresponding correlation function.

The asymptotic series

One says that a series $\sum_{n=0}^{\infty} a_n \lambda^n$, $\lambda > 0$, is *asymptotic to a function* $f(\lambda)$ if

$$\lim_{\lambda \to 0^+} \frac{\left| f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n \right|}{\lambda^N} = 0 \quad \text{for all} \quad N > 0 \tag{3.5}$$

This means that the absolute difference between $f(\lambda)$ and the truncated series at order N is $O(\lambda^{N+1})$, so that for $\lambda \ll 1$ small enough we can make this difference as mild as we prefer.
Apparently with the increase of the order of perturbation the truncated series approximate better and better $f(\lambda)$. This naive idea is obviously wrong, due to the fact that increasing the perturbation order one should also decrease the value of λ. Indeed if one fixes the value of λ and increases the perturbation order N, the absolute difference between $f(\lambda)$ and the truncated series initially decreases, but eventually it reaches its minimum and starts to increase again, diverging as $N \to \infty$. At higher perturbative orders one should take values of λ smaller and smaller to make the correction negligible. The fact that we can take $\lambda \ll 1$ to make the difference negligible doesn't solve our problem, since in physical applications the value of λ is given by $\hbar$, and decreasing its value is meaningless, since it is fixed experimentally.

Non-analytical contributions

The second problem is that any series asymptotic to some function is also asymptotic to infinitely many other functions. For instance, if a series is asymptotic to some $f(\lambda)$ it is also asymptotic to $f(\lambda) + e^{-\frac{1}{\lambda \alpha}}$, since

$$\lim_{\lambda \to 0^+} \frac{e^{-\frac{1}{\lambda \alpha}}}{\lambda^N} = 0 \quad \text{for any} \quad N \in \mathbb{N} \quad \text{and} \quad a > 0 \tag{3.6}$$

This is due to the fact that the series expansion of $e^{-\frac{1}{\lambda \alpha}}$ for $\lambda \ll 1$ has all the coefficients equal to zero. Unfortunately terms in the form $e^{-\frac{1}{\lambda \alpha}}$ are exactly the contributions that arises if there are non-trivial topological configurations of the fields in the correlation function.
This makes completely impossible any unique reconstruction of the original function starting from its asymptotic series, since in any case the reconstructed function would be defined up to non-analytical terms which didn't appear in its asymptotic series.

Is the perturbative series asymptotic to the correlation function?

Even if it's impossible to reconstruct the correlator from its asymptotic series, at least it has been shown that in many QFT where the ultraviolet renormalization does not involve coupling constants (the so called *super-renormalizable theories*) the renormalized perturbation series is asymptotic to the non-perturbatively defined correlation function (e.g. QED for $d < 3 + 1$ and ϕ^4 in $d < 3 + 1$).
Nevertheless if our theory is *renormalizable* (hence not super-renormalizable) in the only case we have almost completely rigorous control of the RQFT, i.e. ϕ^4 in $d = 3 + 1$, it has been proved that the renormalized perturbation series is not asymptotic to the non-perturbative correlator. Even if we don't have a rigorous proof, this seems the case also for QED in $d = 3 + 1$.

Borel resummation

In some cases, called *Borel resummable theories*, even if the perturbative series is divergent it is possible to resum it via *Borel resummation* obtaining a finite result.
The idea is to introduce the identity

$$1 = \frac{1}{n!}\int_0^\infty x^n e^{-x}\mathrm{d}x \tag{3.7}$$

inside the perturbative series

$$\sum_{n=0}^{\infty} a_n \lambda^n = \int_0^\infty \sum_{n=0}^{\infty} \frac{a_n}{n!}(x\lambda)^n e^{-x}\mathrm{d}x \tag{3.8}$$

In some instances, the introduction of the factor $\frac{1}{n!}$ inside the perturbative coefficients makes the new series converge: $\sum_n \frac{1}{n!}a_n(x\lambda)^n$ might converge even if $\sum_n a_n\lambda^n$ does not. If the new coefficients are smooth enough to make the integral, we are then able to obtain the resummed series.
The first problem is that the resummed series is still not sensible to non-analytical terms (they cannot be reconstructed just resumming the analytical contributions), moreover a-posteriori one has to check that the resummed series is asymptotic to the non-perturbative result, since the right convergence is not ensured in general.

Resurgence

We just mention that there is a very recent technique called *resurgence* which applies to Quantum Mechanics and allows to obtain non-perturbative results just using perturbative techniques. Is still unknown whether such technique can be implemented or not also in QFT.

3.2 The Källen-Lehmann representation and the Lehmann representation

[GR96, Section 9.3]

The second problem of the perturbative approach is related to the limit $\varepsilon \to 0$ of the IR cutoff of the relation

$$U_\varepsilon^{I\dagger}(t)\hat{\phi}_{\mathrm{in}}(\boldsymbol{x},t)U_\varepsilon^I(t) = \hat{\phi}(\boldsymbol{x},t) \tag{3.9}$$

Indeed the limit $\varepsilon \to 0$ should be taken in a way compatible with physical properties of the interacting theory, in particular translational invariance and, in high energy physics, Poincaré invariance.
In order to exploit the problem we first need to introduce the spectral representation of the Green functions (the same we introduced for Matsubara and retarded correlators, but from a more general and deep point of view).

Källen-Lehmann representation (relativistic case)

If $\mathcal{H}$ is the Hilbert space of the interacting theory, in order to have Poincaré invariance, some properties has to be satisfied: in particular we need in $\mathcal{H}$

(1) A unitary representation of the covering of the restricted Poincaré group $\tilde{\mathscr{P}}_+^\uparrow$. Let's denote by $\hat{P}^\mu$ the corresponding generators of space-time translations.

(2) A vacuum vector $|0\rangle$ invariant under the representation of spacetime translations $U(a)$, $a \in \mathbb{R}^{d+1}$ $(d = 3)$ (invariance is due to homogeneity of spacetime).

(3) The spectrum of $\hat{P}^\mu$ is contained in the forward light cone, $\sigma(\hat{P}^\mu) \subseteq V_0^+$.

(4) The operator $\hat{\phi}$ should transform in a irreducible representation of $\tilde{\mathscr{P}}_+^\uparrow$ under $U(a)$. In particular assuming that $\hat{\phi}$ is scalar we have

$$U(a)\hat{\phi}(x)U(a)^\dagger = \hat{\phi}(x-a) \tag{3.10}$$

From (1) we get that exists a Dirac completeness $|\alpha\rangle$ of (generalized) eigenvectors of $\hat{P}^\mu$ with eigenvalues p_α (for simplicity we write $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$ also if $\hat{P}^\mu$ has continuum spectrum).
Consider the 2-points function of a scalar RQFT

$$\begin{aligned}\langle 0|\,\hat{\phi}(x)\hat{\phi}(y)\,|0\rangle &= \sum_\alpha \langle 0|\,\hat{\phi}(x)\,|\alpha\rangle\langle\alpha|\,\hat{\phi}(y)\,|\alpha\rangle \overset{(4)}{=} \sum_\alpha \langle 0|\,U(x)\hat{\phi}(0)U^\dagger(x)\,|\alpha\rangle\langle\alpha|\,U(y)\hat{\phi}(0)U^\dagger(y)\,|0\rangle \\ &= \sum_\alpha e^{-ip_\alpha(x-y)}|\langle\alpha|\,\hat{\phi}(0)\,|0\rangle|^2 \overset{d=3+1}{=} \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)}\end{aligned} \tag{3.11}$$

where in the last step we used the identity $1 = \int \mathrm{d}^4q\,\delta(q-p_\alpha)$ and we defined the Fourier transform (up to a factor 2π) of the 2-point function

$$\rho_+(q) := (2\pi)^3 \sum_\alpha \delta(q-p_\alpha)|\langle 0|\,\hat{\phi}(0)\,|\alpha\rangle|^2 \tag{3.12}$$

which has some interesting properties:

(a) $\rho_+(q) \geq 0$;

(b) $\rho_+(q) = 0$ if $q \notin \overline{V}_0^+$, thanks to (3), the bar over the forward light cone indicate its closure;

(c) $\rho_+(\Lambda q) = \rho_+(q)$, for $\Lambda \in \mathscr{L}_+^\uparrow$, thanks to (4).

Therefore from all these conditions we get that the most general form for ρ_+ is[I]

$$\rho_+(q) = \sigma(q^2)\theta(q^0) + \text{const}\cdot\delta(q) \tag{3.13}$$

Notice that property (a), $\rho_+(q) \geq 0$, implies that $\sigma(q^2)$ is a semi-definite positive function, i.e. $\sigma(q^2) \geq 0$ for any value of $q \in \overline{V}_0^+$. Moreover from (b) we see that $\sigma(q^2) = 0$ if $q^2 < 0$.
Let's introduce the *spectral function*[II]

$$\rho(q) := \rho_+(q) - \rho_+(-q) \tag{3.15}$$

Using expression eq. (3.13) we get

$$\rho(q) = \rho_+(q) - \rho_+(-q) = \sigma(q^2)\theta(q^0) + \cancel{c\,\delta(q)} - \sigma(q^2)\theta(-q^0) - \cancel{c\,\delta(-q)} = \mathrm{sign}(q^0)\sigma(q^2) \tag{3.16}$$

Due to eq. (3.11) the spectral function ρ is the Fourier transform (up to a factor 2π) of the correlator $\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle$, indeed using eq. (3.11) we get

$$\begin{aligned}\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle &= \langle 0|\,\hat{\phi}(x)\hat{\phi}(y)\,|0\rangle - \langle 0|\,\hat{\phi}(y)\hat{\phi}(x)\,|0\rangle \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(y-x)} \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(q)e^{-iq(x-y)} - \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho_+(-q)e^{-iq(x-y)} \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho(q)e^{-iq(x-y)}\end{aligned} \tag{3.17}$$

Representation of correlation functions given by eq. (3.11) and eq. (3.17) are called *Källen-Lehmann representations* since they was firstly derived by these two physicist.[III] They are also known as *spectral representations*, since ρ_+, ρ and σ can be expressed in terms of the matrix elements of $\hat{\phi}(0)$.

[I] The delta function is introduced to take into account possible non vanishing vacuum expectation values.

[II] Sometimes $\rho(q)$ is defined with an additional factor 2π in such a way that it is exactly the Fourier transform of $\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle$. In some other cases it is defined (equivalently to eq. (3.14)) as 2 times the immaginary part of the retarded correlator, using the relations

$$\mathcal{F}(-i\theta(t))(\omega) = \frac{1}{\omega+i\delta} \quad \text{and} \quad \mathrm{Im}\,\frac{1}{\omega+i\delta} = \pi\delta(\omega) \tag{3.14}$$

where $\omega = q_0$.

[III] Källen: https://doi.org/10.5169%2Fseals-112316, Lehmann: https://doi.org/10.1007%2Fbf02783624.

If $\hat{\phi}$ obeys the CCR $[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)] = i\delta(\boldsymbol{x}-\boldsymbol{y})$ then ρ satisfies the *sum sule*

$$\int_{-\infty}^{+\infty} \mathrm{d}q^0\, q^0 \rho(q) = 1 \tag{3.18}$$

Indeed consider the identity

$$\begin{aligned}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} &= \delta(\boldsymbol{x}-\boldsymbol{y}) = -i\langle 0|\,[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)]\,|0\rangle = -i\partial_{y^0}\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle\big|_{\substack{x_0=t\\y_0=t}} = \\ &= -i\partial_{y^0}\int \frac{\mathrm{d}^4 q}{(2\pi)^3}\,\rho(q)e^{-iq(x-y)}\big|_{\substack{x_0=t\\y_0=t}} = \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, q^0\rho(q)e^{-iq(x-y)}\big|_{\substack{x_0=t\\y_0=t}} = \\ &= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\, q^0\rho(q)e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\end{aligned} \tag{3.19}$$

then comparing the left and the right side of the previous identity one gets exactly eq. (3.18). Moreover, using eq. (3.16) one can rewrite

$$1 = \int_{-\infty}^{+\infty} \mathrm{d}q^0\, q^0\rho(q) = \int_{-\infty}^{+\infty} \mathrm{d}q^0\, |q^0|\sigma(q^2) = 2\int_0^{+\infty} \mathrm{d}q^0\, q^0\sigma(q^2) = \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2) \tag{3.20}$$

where using $m^2 = q^2 = (q^0)^2 - \boldsymbol{q}^2$ and the fact that in our integration $\boldsymbol{q}$ is fixed, we applied the change of variable $q^0 \mapsto m^2$, $2q^0\mathrm{d}q^0 \mapsto \mathrm{d}m^2$.
Recalling that $\sigma(q^2) \geq 0$, we have that $q^0\rho(q) = |q^0|\sigma(q^2) \geq 0$ too, hence using eq. (3.18) and eq. (3.20) we get that both

$$A(q) := q^0\rho(q) \overset{(3.16)}{=} |q^0|\,\sigma(q^2) \tag{3.21}$$

and $\sigma(m^2)$ are probability densities. In particular can be proved that $\sigma(m^2)$ is the probability density to find in the state $\int e^{iqx}\hat{\phi}(x)\,|0\rangle$ a state of mass m^2, while $A(q)$ is the probability density to find in the same state $\int e^{iqx}\hat{\phi}(x)\,|0\rangle$ a state of energy q^0 for fixed value $\boldsymbol{q}$.
We just mention that for fermions the same sum rule eq. (3.18) holds equivalently provided that we have the CAR $\{\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)\} = i\delta(\boldsymbol{x}-\boldsymbol{y})$.

Now, suppose that we do not know if our interacting field satisfies CCR (hence eq. (3.18) does not apply to it), although we are sure that the associated free theory does. Denoting by

$$\Delta_+(x-y;m) := \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\delta(q^2-m^2)\theta(q^0)e^{-iq(x-y)} \tag{3.22}$$

the 2-points function of the free scalar field of mass m, then inserting inside eq. (3.11) the equality $\int_0^{+\infty}\mathrm{d}m^2\,\delta(q^2-m^2) = 1$ we get

$$\begin{aligned}\langle 0|\,\phi(x)\phi(y)\,|0\rangle &= \int \frac{\mathrm{d}^4 q}{(2\pi)^3}\sigma(q^2)\theta(q^0)e^{-iq(x-y)} + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2 \\ &= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\int \frac{\mathrm{d}^4 q}{(2\pi)^3}\delta(q^2-m^2)\theta(q^0)e^{-iq(x-y)} + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2 \\ &= \int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)\Delta_+(x-y;m) + |\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2\end{aligned} \tag{3.23}$$

where $|\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2$ is the constant which comes from eq. (3.12) when evaluated at $q=0$ due to the term $\delta(q)$ in eq. (3.13), and can be formally removed just shifting $\hat{\phi}(x)$ (although physically the non-vanishing of $|\langle 0|\,\hat{\phi}(0)\,|0\rangle|^2$ is a non-trivial issue signalling a symmetry breaking). This means that even in the interacting theory we can write the 2-points function as an integral of free 2-points functions of the free fields with varying mass weighted by a suitable measure $\mathrm{d}m^2\,\sigma(m^2)$. This fact is a purely relativistic effect, and has no analogue for NRQFT, it is due to Lorentz invariance, and it is no more true if we break Lorentz symmetry.

One of the consequences is that, since the 2-point function of the interacting theory can be written in terms of weighted two point functions of free theories, which satisfies CCR, also the interacting theory should satisfy CCR, up to some constant factor c which can be reabsorbed in the field if it is finite:

$$\begin{aligned}\langle 0|\,[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)]\,|0\rangle &= \partial_{y^0}\langle 0|\,[\hat{\phi}(x),\hat{\phi}(y)]\,|0\rangle\big|_{\substack{x_0=t\\y_0=t}}\\ &= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\,\partial_{y^0}\big(\Delta_+(x-y;m)-\Delta_+(y-x;m)\big)\big|_{\substack{x_0=t\\y_0=t}}\\ &= \int_0^{+\infty} \mathrm{d}m^2\,\sigma(m^2)\,i\delta(\boldsymbol{x}-\boldsymbol{y}) = c\,i\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned} \tag{3.24}$$

with $c=\int_0^{+\infty}\mathrm{d}m^2\,\sigma(m^2)$ strictly positive.

Lehmann representation (non-relativistic case)

Lehmann applied the previous procedure also to the non-relativistic case. Let's see how to generalize the previous representation without assuming Lorentz invariance.
In this case in the Hilbert space $\mathcal{H}$ of our theory we need:

(1') A unitary representation U of spacetime translations (one can also add rotational invariance).

(2') The field $\hat{\phi}$ should transform under an irreducible representation of translations under U.[IV] We assume that $\hat{\phi}$ transforms as a scalar.

As in the relativistic case, (1') implies that exists a Dirac completeness for the generators $\hat{\boldsymbol{P}}$, H of $U(a)$ with eigenvalues p_α. Applying the same strategy as in the relativistic case (the symbol $\pm$ in $\langle\bullet\rangle_{\beta\,\pm}$ indicates whether the operators are bosons or fermions when we compute the average at finite temperature T, while $[\bullet,\bullet]_+$ is the anticommutator and $[\bullet,\bullet]_-$ is the commutator)

$$\begin{aligned}\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_\beta &= \frac{\mathrm{Tr}\Big[e^{-\beta H}[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\Big]}{\mathrm{Tr}\big[e^{-\beta H}\big]}\\ &= \frac{1}{Z}\sum_{\alpha,\alpha'}\big(\langle\alpha'|\,e^{-\beta H}\hat{\phi}(x)\,|\alpha\rangle\langle\alpha|\,\hat{\phi}^\dagger(y)\,|\alpha'\rangle \pm \langle\alpha|\,e^{-\beta H}\hat{\phi}^\dagger(y)\,|\alpha'\rangle\langle\alpha'|\,\hat{\phi}(x)\,|\alpha\rangle\big)\\ &= \frac{1}{Z}\sum_{\alpha,\alpha'}\big(e^{-\beta\mathcal{E}_{\alpha'}}\pm e^{-\beta\mathcal{E}_\alpha}\big)e^{-i(p_\alpha-p_{\alpha'})(x-y)}|\langle\alpha'|\,\hat{\phi}(0)\,|\alpha\rangle|^2\\ &\overset{d=3}{=} \int\frac{\mathrm{d}^4q}{(2\pi)^3}\,\rho^\pm(q)e^{-q(x-y)}\end{aligned} \tag{3.25}$$

with *spectral function*

$$\rho^\pm(q) = \frac{(2\pi)^3}{Z}\sum_{\alpha,\alpha'}\big(e^{-\beta\mathcal{E}_{\alpha'}}\pm e^{-\beta\mathcal{E}_\alpha}\big)\delta(q-p_\alpha+p_{\alpha'})|\langle\alpha'|\,\hat{\phi}(0)\,|\alpha\rangle|^2 \tag{3.26}$$

Eq. (3.25) is called *Lehmann representation* of $\langle[\hat{\phi}(x),\hat{\phi}^\dagger(y)]_\pm\rangle_{\beta\,\pm}$.
For fermions, provided that CAR $\{\hat{\phi}(\boldsymbol{x},t),\hat{\phi}^\dagger(\boldsymbol{y},t)\}=i\delta(\boldsymbol{x}-\boldsymbol{y})$ is satisfied[V] and $T=0$, we get that $A(\omega,\boldsymbol{q}):=\rho^+(q)$ (without q^0 in the non relativistic case) is a probability density, and it describes the probability density to find the state $\int e^{-iqx}\hat{\phi}(x)\,|0\rangle$ in a state of energy ω for fixed $\boldsymbol{q}$.

The constraint $0\le Z(\boldsymbol{q})\le 1$

In general in a QFT (relativistic or not) if there is a particle (or a quasi-particle) excitation with dispersion $\omega=\omega(\boldsymbol{q})$ (in the relativistic case $\omega=\sqrt{m^2+\boldsymbol{q}^2}$) (we assume $\langle 0|\,\hat{\phi}(0)\,|0\rangle=0$, otherwise we can shift the

[IV] For lattice theories U is restricted to discrete spatial lattice translations and typically the time lattice translations are represented contractively, i.e. through terms of the form e^{-tH} (continuum version of the transfer matrix).
[V] Notice that in the non-relativistic case there is no time derivative in the CAR.

field) then in $A(\omega, \boldsymbol{q})$ there should be a term

$$\begin{aligned} &Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q})) \quad \text{in the non-relativistic case} \\ &Z\delta(q^2 - m^2)\theta(q^0) \quad \text{in the relativistic case} \end{aligned} \tag{3.27}$$

Notice that in the relativistic case there cannot be any dependence on $\boldsymbol{q}$ in Z due to Lorentz invariance. Assuming CCR, consider the non-relativistic case[VI] and suppose that we have only one kind of particle excitation in the system, since $A(\omega, \boldsymbol{q})$ describes the probability density to measure energy ω in the state $\int e^{-iqx}\hat{\phi}(x)\,|0\rangle$ for fixed $\boldsymbol{q}$ we have that in general

$$A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q})\delta(\omega - \omega(\boldsymbol{q})) + A_{\text{inc}}(\omega, \boldsymbol{q}) \tag{3.28}$$

where A_{inc} (inc=incoherent) is a regular (differently from δ) "featureless" positive function describing contributions of multi-particle states in the two points function (we still allow our particle to generate some other particles which eventually recombine in the original one due to energy uncertainty, if there are interactions in our theory). These give a (small) non-zero probability to find a particle in an energy state different from $\omega(\boldsymbol{q})$.

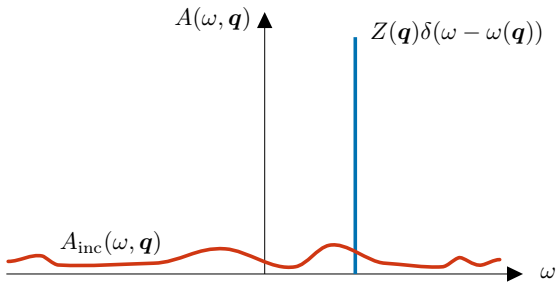


Figure 3.1: One-dimensional pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a stable particle.

Just notice that since $A(\omega, \boldsymbol{q})$ has an explicitly physical meaning as probability density, the plot shown in Fig. 3.1 can be obtained directly in experiments, up to experimental uncertainties which turn the δ-function into a smooth but very localized function.
From the sum rule we get

$$1 = \int \mathrm{d}\omega\, A(\omega, \boldsymbol{q}) = Z(\boldsymbol{q}) + \underbrace{\int \mathrm{d}\omega\, A_{\text{inc}}(\omega, \boldsymbol{q})}_{\geq 0} \tag{3.29}$$

which implies

$$0 \leq Z(\boldsymbol{q}) \leq 1 \tag{3.30}$$

Notice that in the case $Z(\boldsymbol{q}) = 1$ the theory is free, indeed for a free particle $A(\omega, \boldsymbol{q}) = \delta(\omega - \omega(\boldsymbol{q}))$. Our problem is that, as we'll see, it is very difficult to satisfy $0 \leq Z < 1$ in an interacting situation.
We can weaken a little the above requirement of a stable particle excitation: let's see what happen if we do not have a stable particle but a long-life resonance (i.e. a particle with a lifetime large respect to the typical time of the system). The δ-function in eq. (3.27) can be rewritten as

$$\delta(\omega - \omega(\boldsymbol{q})) \quad \mapsto \quad \frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{\omega - \omega(\boldsymbol{q}) + i\delta} \quad \text{where the limit } \delta \to 0^+ \text{ is understood} \tag{3.31}$$

and this implies that the retarded correlator has a pole at $\omega = \omega(\boldsymbol{q}) - i\delta$. Notice that δ essentially correspond to the inverse lifetime τ^{-1} of the particle, hence the limit $\delta \to 0^+$ correspond to the infinite lifetime of the particle.
A resonance is described through a Lorentzian distribution, by replacing the above pole by a complex pole at $\omega = f(\omega, \boldsymbol{q})$ where f is a complex function with $\mathrm{Im}\, f(\omega, \boldsymbol{q}) < 0$ (in such a way that $A(\omega, \boldsymbol{q}) \geq 0$ is satisfied). Then

$$\mathrm{Im}\,\frac{1}{\omega - f(\omega, \boldsymbol{q})} = \frac{-\,\mathrm{Im}\, f(\omega, \boldsymbol{q})}{(\omega - \mathrm{Re}\, f(\omega, \boldsymbol{q}))^2 + (\mathrm{Im}\, f(\omega, \boldsymbol{q}))^2} \tag{3.32}$$

[VI] The relativistic one is obtained imposing $q^2 = m^2$.

where $\omega = \operatorname{Re} f(\omega, \boldsymbol{q})$ describes the dispersion relation of the unstable particle and $-\operatorname{Im} f(\omega, \boldsymbol{q})$ its inverse lifetime.
If we take $\operatorname{Im} f(\omega, \boldsymbol{q}) \to 0^-$ then $\operatorname{Im} \frac{1}{\pi} \frac{1}{\omega - f(\omega, \boldsymbol{q})} \to \delta(\omega - f(\omega, \boldsymbol{q}))$ with dispersion relation $\omega(\boldsymbol{q})$ solution of the equation $\omega - \operatorname{Re} f(\omega, \boldsymbol{q}) = 0$ and we recover the case of a stable particle:

$$\delta(\omega - f(\omega, \boldsymbol{q})) = \frac{\delta(\omega - \omega(\boldsymbol{q}))}{\left| \frac{\partial f}{\partial \omega}(\omega, \boldsymbol{q}) \right| \Big|_{\omega = \omega(\boldsymbol{q})}} = Z(\boldsymbol{q}) \delta(\omega - \omega(\boldsymbol{q})) \tag{3.33}$$

This means that in the case of the resonance we just broaden the δ-function in the spectral function, as shown in fig. 3.2.

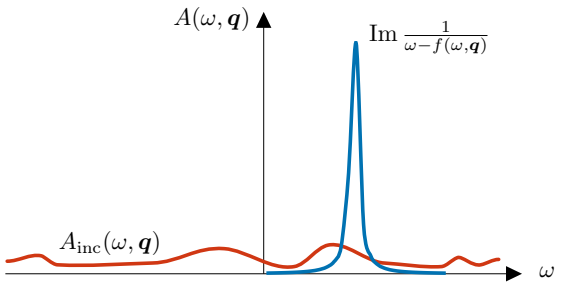


Figure 3.2: One-dimensionally pictorially representation of the function $A(\omega, \boldsymbol{q})$ for a unstable particle. The width of the blue peak is proportional to the inverse lifetime τ^{-1} of the particle.

Again, provided CCR or CAR (depending on the physical situation) hold, we again have the contstraint

$$0 \leq Z(\boldsymbol{q}) \leq 1 \tag{3.34}$$

hence the constraint applies even if we consider unstable particles.

3.3 The incompatibility between CCR and U^I

Let's see what have to do the previous spectral representations with the inconsistency of the perturbation theory as $\varepsilon \to 0$, first considering the relativistic case.
Let's inspect the relation between Z and asymptotic fields. With the introduction of the infrared cutoff ε (at least in t) we got that the free field is related to the interacting field by a unitary operator

$$\hat{\phi}(\boldsymbol{x}, t) = U_\varepsilon^{I\dagger}(t) \hat{\phi}_{\text{in}}(\boldsymbol{x}, t) U_\varepsilon^I(t) \tag{3.35}$$

such that

$$\hat{\phi}(\boldsymbol{x}, t) \xrightarrow[t \to -\infty]{} \hat{\phi}_{\text{in}}(\boldsymbol{x}, t) \quad \text{and} \quad U_\varepsilon^I(t) \xrightarrow[t \to -\infty]{} \mathbb{1} \tag{3.36}$$

Let's see if such unitary relation between free and interacting field is possible preserving translational invariance.
Consider a system with only one species of particle created by the real field $\hat{\phi}$, since $\langle 0 | \hat{\phi}_{\text{in}} \hat{\phi}_{\text{in}} | 0 \rangle$ has exactly a pole for $q^2 = m^2$ we can identify $\langle 0 | \hat{\phi}_{\text{in}} \hat{\phi}_{\text{in}} | 0 \rangle$ as responsible for the contribution

$$Z \delta(q^2 - m^2) \theta(q^0) \tag{3.37}$$

in the spectral function $A(q)$ of $\hat{\phi}$, eq. (3.28). We want that $\hat{\phi}$ describes an interacting field, so we assume $Z \neq 1$ (otherwise $\hat{\phi}$ is free), conversely for $\hat{\phi}_{\text{in}}$ we already know that eq. (3.37) is satisfied for $Z = 1$. Therefore it is impossible to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} \hat{\phi}_{\text{in}}$, one can at best hope to satisfy $\hat{\phi} \xrightarrow[t \to -\infty]{} Z^{1/2} \hat{\phi}_{\text{in}}$, i.e.

$$\hat{\phi}(\boldsymbol{x}, t) = Z^{1/2} U_\varepsilon^{I\dagger}(t) \hat{\phi}_{\text{in}}(\boldsymbol{x}, t) U_\varepsilon^I(t) \tag{3.38}$$

which means that $\hat{\phi}_{\text{in}}$ is related by a unitary transformation to the renormalized field $Z^{-1/2} \hat{\phi}$.

But if $\hat{\phi}_{\text{in}}$ and $\hat{\phi}$ are canonical, i.e. satisfy CCR, then eq. (3.38) must be wrong, indeed from eq. (3.38) we get

$$[\hat{\phi}(\boldsymbol{x},t),\dot{\hat{\phi}}(\boldsymbol{y},t)] \xrightarrow[t\to-\infty]{} Z[\hat{\phi}_{\text{in}}(\boldsymbol{x},t),\dot{\hat{\phi}}_{\text{in}}(\boldsymbol{y},t)] \tag{3.39}$$

but then using CCR for $\hat{\phi}_{\text{in}}$ and $\hat{\phi}$ we get

$$\begin{aligned}\delta(\boldsymbol{x}-\boldsymbol{y}) &\overset{\text{CCR}}{=} \lim_{x^0\to-\infty}\langle 0|\,[\hat{\phi}(\boldsymbol{x},x^0),\dot{\hat{\phi}}(\boldsymbol{y},y^0)]\,|0\rangle\big|_{x^0=y^0} = \\ &= Z\lim_{x^0\to-\infty}\langle 0|\,[\hat{\phi}_{\text{in}}(\boldsymbol{x},x^0),\dot{\hat{\phi}}_{\text{in}}(\boldsymbol{y},y^0)]\,|0\rangle\big|_{x^0=y^0} \overset{\text{CCR}}{=} Z\delta(\boldsymbol{x}-\boldsymbol{y})\end{aligned} \tag{3.40}$$

which give $Z=1$, but this is wrong since we assumed that $\hat{\phi}$ is interacting.

Notice that the above argument about the inconsistency between CCR and eq. (3.38) requires that both x^0 and y^0 are sent to $-\infty$ simultaneously in eq. (3.39). Conversely it is still possible that $\hat{\phi}$ approaches $Z^{1/2}\hat{\phi}_{\text{in}}$ as $t\to-\infty$ in the weak sense $\hat{\phi} \xrightarrow[t\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, i.e. in matrix elements: taken a sequence A_n of operators the *weak limit*

$$A_n \xrightarrow[n\to+\infty]{} A \tag{3.41}$$

means that

$$|\langle\psi,(A_n-A)\phi\rangle| \xrightarrow[n\to+\infty]{} 0 \tag{3.42}$$

Indeed taken $A_n \rightharpoonup A$ and $B_n \rightharpoonup B$ is not ensured that $A_nB_n \rightharpoonup AB$, therefore (with smearing) $\hat{\phi}(x) \xrightarrow[x^0\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$ and $\dot{\hat{\phi}}(y) \xrightarrow[y^0\to-\infty]{} Z^{1/2}\dot{\hat{\phi}}_{\text{in}}(y)$ does not imply $[\hat{\phi}(x),\dot{\hat{\phi}}(y)] \xrightarrow[x^0,y^0\to-\infty]{} Z[\hat{\phi}_{\text{in}}(x),\dot{\hat{\phi}}_{\text{in}}(y)]$ and eq. (3.39) does not hold anymore. Therefore if $\hat{\phi}(x) \xrightarrow[x^0\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}(x)$ we don't have any inconsistency between CCR and eq. (3.38).[VII]
Notice that LSZ formula uses precisely such weak limit for $t\to-\infty$, so there is no inconstistence with it.[VIII]
The serious problem arises when in the perturbative approach is used the Gell-Mann Low formula, which requires the existence of a unitary operator $U^I=\lim_{\varepsilon\to0^+}U^I_\varepsilon$ interpolating between $\hat{\phi}$ and $\hat{\phi}_{\text{in}}$, or at least $Z\hat{\phi}_{\text{in}}$, but if $\hat{\phi}$ obeys CCR and translation invariance is recovered (i.e. we take the limit $\varepsilon\to0^+$) we get back to the difficulty discussed above.

3.4 The Haag's theorem

[EF05, Section 3]; [SW00, Section 4.5]; [Str13, Pages 39-40, 95–96]

The important argument from Haag[IX] proves that the situation does not improve even if we do not require $\hat{\phi}$ to be canonical. Except very special non-relativistic cases, the Hilbert space of the free field is disjoint from the Hilbert space of the interacting field (i.e. doesn't exist any unitary operator which maps the states of one Hilbert space into the states of the other one).

Theorem 3.1 (Haag, [EF05, Page 8])**.** *Assume that the unique translationally invariant state of an interacting scalar QFT with Hilbert space $\mathcal{H}$ and field $\hat{\phi}\in\mathcal{H}$ is the vacuum $|0\rangle\in\mathcal{H}$* [X] *(for lattice theories only discrete translations are considered). Suppose that the Hamiltonian of the theory can be written as $H=H_0+H_I$, with H_0 free Hamiltonian.*
Then there is no state $|0_F\rangle\in\mathcal{H}$ such that $H_0|0_F\rangle=0$ and $H|0_F\rangle\neq0$ i.e. if the interacting Hamiltonian H does not annihilate the vacuum of the free theory $|0_F\rangle$ then $|0_F\rangle$ is not an element of $\mathcal{H}$.

[VII] More about this fact in [GR96, Section 9.2].
[VIII] The derivation of the LSZ formula for the scalar theory can be found in [GR96, Section 9.4].
[IX] Original paper: [Haa55].
[X] It is reasonable to assume that the only translationally invariant state should be the vacuum, as the presence of any particle broke translational invariance. This is ensured if the underlying free theory has a mass gap, i.e. all particles of our theory are massive. In general these assumptions does not hold if the free theory has massless particles. See [EF05, Section 3] for further comments about this fact.

Proof. Let $U(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, be the unitary representation of translations in $\mathcal{H}$. Assume that the Fock vacuum $|0_F\rangle$ of the free theory with Hamiltonian H_0 is an element of $\mathcal{H}$, i.e. $|0_F\rangle \in \mathcal{H}$ and $H_0 |0_F\rangle = 0$. Denote by $a_F(\boldsymbol{q})$ the annihilation operator of H_0 ($F =$ free), then

$$U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) U(\boldsymbol{x}) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} a_F(\boldsymbol{q}) \tag{3.43}$$

therefore

$$0 = U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) |0_F\rangle = U^\dagger(\boldsymbol{x}) a_F(\boldsymbol{q}) U(\boldsymbol{x}) U^\dagger(\boldsymbol{x}) |0_F\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} a_F(\boldsymbol{q}) U^\dagger(\boldsymbol{x}) |0_F\rangle \tag{3.44}$$

and this means that $a_F(\boldsymbol{q})$ annihilates $U^\dagger(\boldsymbol{x}) |0_F\rangle$, hence $U^\dagger(\boldsymbol{x}) |0_F\rangle$ is translationally invariant. By uniqueness of the translational invariant state $|0_F\rangle = c\,|0\rangle$ for some constant c (actually if our states are normalized c is just a phase). Then

$$0 = H |0\rangle = (H_0 + H_I) c |0_F\rangle = c\, H_I |0_F\rangle \tag{3.45}$$

which give us $H_I |0_F\rangle = 0$ and $H |0_F\rangle = 0$. □

Under some hypothesis, this first version of Haag's theorem proves that the vacuum of the free theory cannot be an element of the Hilbert space of the interacting theory. The relative simplicity of this theorem is purchased at the expense of generality since it appeals to the fact that typical Hamiltonians for interacting fields do not annihilate $|0_F\rangle$, $H |0_F\rangle \neq 0$, (*excitations of the vacuum*).[XI]
It was the ambition of the second version of his theorem to dispense with any reference to the particular form of the interacting field Hamiltonian. However, Haag's attempted proof of such general version was found "inconclusive". The gap was filled by a pair of theorems presented by Hall and Wightman,[XII] which together constitutes the so called *Haag-Hall-Wightman (HHW) theorem*, which applies to any pair of neutral scalar field.

Theorem 3.2 (HHW Part 1, [HW57, Section 3, Thm. 1], [SW00, Thm. 4.14])**.** *Consider two QFTs with scalar fields $\hat{\phi}_1$, $\hat{\phi}_2$ and Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and assume (with $j = 1, 2$)) that*

(i) the Euclidean group $\mathbb{E}_d$ (roto-traslations) is represented unitarily by U_1 and U_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively:

$$U_j(\boldsymbol{a}, \boldsymbol{R}) \hat{\phi}_j(\boldsymbol{x}, t) U_j^\dagger(\boldsymbol{a}, \boldsymbol{R}) = \hat{\phi}_j(\boldsymbol{R}\,\boldsymbol{x} + \boldsymbol{a}, t) \tag{3.47}$$

(ii) the representation of the fields is irreducible, i.e. if an operator $\hat{A}$ defined in $\mathcal{H}_j$ commutes with all the fields $\{\hat{\phi}_j\}$ then it is a multiple of the identity, $\hat{A} = a\,\mathbb{1}$ for $a \in \mathbb{C}$;

(iii) the fields are related at some time t by a unitary transformation $V(t)$

$$\hat{\phi}_2(\boldsymbol{x}, t) = V(t) \hat{\phi}_1(\boldsymbol{x}, t) V^\dagger(t) \tag{3.48}$$

Then

$$U_2(\boldsymbol{a}, \boldsymbol{R}) = V(t) U_1(\boldsymbol{a}, \boldsymbol{R}) V^\dagger(t) \tag{3.49}$$

Further, if there are unique states $|0_1\rangle$ and $|0_2\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a}, \boldsymbol{R}) |0_j\rangle = |0_j\rangle$ for $j = 1, 2$, then

$$V(t) |0_1\rangle = c\, |0_2\rangle \quad \textit{with} \quad c \in \mathbb{C} \tag{3.50}$$

In particular if $|0_j\rangle$, $j = 1, 2$ are normalized to 1 then $|c| = 1$.

Notice that this part of the theorem holds also in the non-relativistic case. The operator V is meant to be the unitary operator interpolating between the free and the interacting fields in the Gell-Mann Low formula.

[XI] As an illustration, consider the $\hat{\phi}^4$ scalar field in $\mathbb{R}^{3+1}$ with (formal) Hamiltonian

$$H = H_0 + \lambda \int \mathrm{d}^3 x \, : \hat{\phi}^4 : - C \tag{3.46}$$

where C is a constant c-number chosen to give vanishing ground state energy, and $:\ :$ indicates the Wick normal product. This H will not annihilate the bare vacuum $|0_F\rangle$ since the factor that follows the coupling constat λ contains a term with a product of four creation operator $\sim (a^\dagger)^4$ (and there is only one term of this form, so it will not be canceled by another term).

[XII] Original paper: [HW57]

Proof. Consider the operator $U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)$, this commutes with all $\hat{\phi}_1 \in \mathcal{H}_1$, indeed:

$$\begin{aligned}
&U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x},t)V^\dagger(t)U_2^\dagger(\boldsymbol{a},\boldsymbol{R})V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_2(\boldsymbol{x},t)U_2^\dagger(\boldsymbol{a},\boldsymbol{R})V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)\hat{\phi}_2(\boldsymbol{R}\,\boldsymbol{x}+\boldsymbol{a},t)V(t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_1(\boldsymbol{R}\,\boldsymbol{x}+\boldsymbol{a},t)U_1(\boldsymbol{a},\boldsymbol{R}) = \\
&= U_1^\dagger(\boldsymbol{a},\boldsymbol{R})U_1(\boldsymbol{a},\boldsymbol{R})\hat{\phi}_1(\boldsymbol{x},t)U_1(\boldsymbol{a},\boldsymbol{R})U_1^\dagger(\boldsymbol{a},\boldsymbol{R}) = \\
&= \hat{\phi}_1(\boldsymbol{x},t)
\end{aligned} \tag{3.51}$$

which implies

$$U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\hat{\phi}_1(\boldsymbol{x},t) = \hat{\phi}_1(\boldsymbol{x},t)U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t) \tag{3.52}$$

Due to the irreducibility of the representation we get that

$$U_1^\dagger(\boldsymbol{a},\boldsymbol{R})V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t) = \omega(\boldsymbol{a},\boldsymbol{R})\mathbb{1} \tag{3.53}$$

for some complex function $\omega(\boldsymbol{a},\boldsymbol{R})$ which depend on the element $(\boldsymbol{a},\boldsymbol{R}) \in \mathbb{E}_d$ and then

$$\omega(\boldsymbol{a},\boldsymbol{R})U_1(\boldsymbol{a},\boldsymbol{R}) = V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t) \tag{3.54}$$

Notice that in order to preserve unitarity we must have $|\omega(\boldsymbol{a},\boldsymbol{R})| = 1$ for all $(\boldsymbol{a},\boldsymbol{R}) \in \mathbb{E}_d$. By definition eq. (3.53) $\omega(\boldsymbol{a},\boldsymbol{R})$ provides a continuous one-dimensional representation (on $\mathbb{C}$) of a non-compact group ($\mathbb{E}_d$), and it is a general mathematical result that the unique representation with these properties is the trivial one; hence one may take $\omega(\boldsymbol{a},\boldsymbol{R}) = 1$.
If there are unique states $|0_j\rangle$ invariant under $\mathbb{E}_d$, $U_j(\boldsymbol{a},\boldsymbol{R})\,|0_j\rangle = |0_j\rangle$, then from $U_1(\boldsymbol{a},\boldsymbol{R})\,|0_1\rangle = |0_1\rangle$ one gets using eq. (3.54)

$$V^\dagger(t)U_2(\boldsymbol{a},\boldsymbol{R})V(t)\,|0_1\rangle = |0_1\rangle \tag{3.55}$$

and

$$U_2(\boldsymbol{a},\boldsymbol{R})V(t)\,|0_1\rangle = V(t)\,|0_1\rangle \tag{3.56}$$

but the only state in $\mathcal{H}_2$ invariant under $U_2(\boldsymbol{a},\boldsymbol{R})$ is $|0_2\rangle$, therefore we must have

$$V(t)\,|0_1\rangle = c\,|0_2\rangle \tag{3.57}$$

for some constant $c \in \mathbb{C}$, which should be a phase, $|c| = 1$, if $|0_1\rangle$ and $|0_2\rangle$ are normalized to 1. □

Corollary 3.2.1. *For any pair of theories satisfying the hypothesis of theorem 3.2 the equal time correlation functions of $\hat{\phi}_1$ and $\hat{\phi}_2$ coincides*

$$\langle 0_1|\,\hat{\phi}_1(\boldsymbol{x}_1,t)\cdots\hat{\phi}_1(\boldsymbol{x}_n,t)\,|0_1\rangle = \langle 0_2|\,\hat{\phi}_2(\boldsymbol{x}_1,t)\cdots\hat{\phi}_2(\boldsymbol{x}_n,t)\,|0_2\rangle \tag{3.58}$$

Proof. The statement directly follows from eq. (3.48) and eq. (3.50). □

In principle an operator V satisfying the hypothesis of Gell-Mann Low formula can still exists, and corollary 3.2.1 still admits different dynamics for $\hat{\phi}_1$ and $\hat{\phi}_2$ since it applies only to equal time correlators. As we consider correlators at different times one should use time evolution operators involving the Hamiltonian, and no useful conclusion can be stated at this point if there is no vacuum polarization. We need one more result, which however works only in the relativistic case:

Theorem 3.3 (HHW Part 2, [HW57, Section 3, Thm. 2], [SW00, Thm. 4.16])**.** *Under the hypothesis of theorem 3.2 assume further that the (covering of the) restricted Poincaré group $\mathscr{P}_+^\uparrow$ is represented unitarily by U_1 and U_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, i.e.*

$$U_j(\Lambda,a)\hat{\phi}_j(x)U_j^\dagger(\Lambda,a) = \hat{\phi}_j(\Lambda\,x+a) \tag{3.59}$$

Then

$$\langle 0_1|\,\hat{\phi}_1(x)\hat{\phi}_1(y)\,|0_1\rangle = \langle 0_2|\,\hat{\phi}_2(x)\hat{\phi}_2(y)\,|0_2\rangle \tag{3.60}$$

and if $\hat{\phi}_1$ is a free field of mass $m > 0$, then $\hat{\phi}_2$ is a free field of mass $m > 0$ too.

Proof. The proof is much more involved than the previous one, since it requires a preliminary result of R. Jost and B. Schroer, which can be found in [SW00, Thm. 4.15]. We just sketch the argument for the proof: by Poincaré covariance through a boost transformation we can bring any two points in the space-time to equal time, in such a way that corollary 3.2.1 applies. Then the correlator we obtained can be extended to the initial points by analiticity, preserving the value of Z and the position of the pole in the Källen-Lehmann representation. This is enough to prove the statement of the theorem. □

The immediate consequence of the latter is that, if the hypothesis holds, any interacting theory related to a free one by a unitary transformation is actually free. The conclusion that all the fields are free is unacceptable because the intention is to represent an interacting field, therefore at least one of the assumptions of the theorem must be dropped. An obvious candidate is the assumption that at some t there exists a unitary transformation $V(t)$ relating the fields. Therefore the Heisenberg dynamics of the interacting field does not exist in the Hilbert space of the free field, but requires a different Hilbert space, inequivalent to the free one.
The second part of the HHW theorem applies only to relativistic QFT, nevertheless the unitary operator U^I interpolating between free and interacting fields does not exists also in most of the non-relativistic QFT, except for very special cases. Therefore almost always the perturbative approach is not mathematically consistent.

3.5 Ultraviolet singularities and canonical quantization

[Str13, Section 2.4]

Consider a RQFT such that $Z = 0$ (recall that Z is the constant connecting the renormalized and the free field, $\phi \xrightarrow[t\to-\infty]{} Z^{1/2}\hat{\phi}_{\text{in}}$, $\hat{\phi}_{\text{ren}} := Z^{-1/2}\hat{\phi} \xrightarrow[t\to-\infty]{} \hat{\phi}_{\text{in}}$) as in the case of perturbation theory, where one usually gets

$$Z = 1 - \lambda\infty \sim \frac{1}{1+\lambda\infty} = 0 \tag{3.61}$$

where ∞ simply denotes some divergent quantity which has been removed through the renormalization procedure. Then at fixed time the field itself does not exist as an operator. In order to have a well defined operator, one needs to smear $\hat{\phi}(x)$ not only in space

$$\hat{\phi}(x^0, f) = \int \mathrm{d}^3x\, f(\boldsymbol{x})\hat{\phi}(x^0, \boldsymbol{x}) \tag{3.62}$$

but also in time

$$\hat{\phi}(g) = \int \mathrm{d}^4x\, g(x)\hat{\phi}(x) \tag{3.63}$$

Indeed suppose that $\hat{\phi}$ is the Heisenberg interacting field, and suppose that it is well defined at fixed time t, then according to Källen-Lehmann representation (recall eq. (3.24)) we get

$$\langle 0|\,[\hat{\phi}(t,\boldsymbol{x}), \dot{\hat{\phi}}(t,\boldsymbol{y})]\,|0\rangle = c\, i\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{3.64}$$

for a constant c, which may diverge and in general is strictly positive. Therefore the "renormalized" field $\hat{\phi}_{\text{ren}} = Z^{-1/2}\hat{\phi}$ which approaches weakly $\hat{\phi}_{\text{in}}$ as $t \to -\infty$, satisfies

$$\langle 0|\,[\hat{\phi}_{\text{ren}}(t,\boldsymbol{x}), \dot{\hat{\phi}}_{\text{ren}}(t,\boldsymbol{y})]\,|0\rangle = c\frac{i}{Z}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{3.65}$$

Clearly if $Z = 0$ this expression is ill-defined (recall that usually $c > 0$), but no assumption has been made except that $\hat{\phi}$ is well defined at fixed time, hence if $Z = 0$ this assumption should be wrong, i.e. the renormalized field cannot be defined at fixed time. A Schrödinger picture requires that we have a definite state at a fixed time, hence we also proved that it is impossible to defined a picture of dynamics using Schrödinger idea if $Z = 0$.

Chapter 4

The reconstruction theory

Beside the literature present at the beginning of each section, have a look at [Kaz99] for this chapter and the following one.

4.1 How to avoid Haag's theorem

[Str13, Section 2.5]

Let's see how to avoid the problems raised by the Haag theorem. The idea is to start by introducing IR (volume) and UV cutoffs (the UV one is not needed for lattice models) so that the cutoff model describes a system of finite degrees of freedom, so that the interaction picture is still correct, thanks to von Neumann uniqueness theorem, which guarantees that free and interacting fields are unitarily equivalent and defined in the same Hilbert space. Moreover one can also require that the fields satisfy CCR (or CAR) since no problem arises in this framework.
Consider a $T = 0$ RQFT (the restriction $T = 0$ is not really needed, but simplify the treatment) and let $\hat{\phi}_0(\boldsymbol{x}) := \hat{\phi}(t = 0, \boldsymbol{x})$ be the $t = 0$ field, then the cutoff field at arbitrary time is given by

$$\hat{\phi}^{\Lambda}_{\#}(t, \boldsymbol{x}) = e^{itH_{\#}} \hat{\phi}_0(\boldsymbol{x}) e^{-itH_{\#}} \tag{4.1}$$

where # denotes either the free ($\# = 0$) of the interacting ($\# = I$) theory and Λ denotes the introduced cutoffs. The consistency of the previous time evolution is ensured by the von Newman theorem, assuming that the free and the interacting fields coincides at the initial time $t = 0$. Let $|0^{\Lambda}_I\rangle$ be the vacuum of the interacting theory, which is still in the Fock space of the free field.
We construct the following functions (actually they are distributions, but with abuse of terminology, we call "functions" also these expectation values)

$$W^{\Lambda}_n(x_1, \dots, x_n) := \langle 0^{\Lambda}_I | \, \hat{\phi}^{\Lambda}_I(x_1) \cdots \hat{\phi}^{\Lambda}_I(x_n) \, |0^{\Lambda}_I\rangle \tag{4.2}$$

We stress the fact that since translational invariance (and thus also Poincaré invariance) is broken by the IR cutoff, Haag's theorem does not apply.
Then we identify the necessary counterterms to be added to the Hamiltonian (or to the Lagrangian if the path integral approach is used) and the field renormalization Z_{Λ} (called *wave-function renormalization constant*) leading to "*renormalized fields*", for which the correlation functions have a well-defined limit when IR an UV cutoffs are removed from outside[I] of the correlation function, i.e. such that the following limit

$$W_n(x_1, \dots, x_n) := \lim_{\substack{\Lambda_{\mathrm{UV}} \to \infty \\ \Lambda_{\mathrm{IR}} \to 0}} Z^{-n}_{\Lambda_{\mathrm{UV}}} \langle 0^{\Lambda}_I | \, \hat{\phi}^{\Lambda}_I(x_1) \cdots \hat{\phi}^{\Lambda}_I(x_n) \, |0^{\Lambda}_I\rangle \tag{4.3}$$

exists in a suitable space of functions or distributions, e.g. $\mathcal{S}'(\mathbb{R}^d)$. This is possible since the previous limit is a limit of functions, hence much weaker than the operator limit (for the fields) which cannot exists due to Haag's theorem.

[I] It is important to notice that the limit should be removed from outside, otherwise Haag's theorem apply and we are stuck.

The crucial fact is that if the limiting functions satisfy certain properties we can reconstruct quantum fields acting in an appropriate Hilbert space, such that their expectation values are precisely the functions we started from. The reconstructed field $\hat{\phi}$ now has no reasons to be equivalent to a free field, nor the Hilbert space of $\hat{\phi}$ has reason to be the same Fock space we started from (actually in general it could even not be a Fock space), hence the reconstruction is not in conflict with Haag's argument.

4.2 Wightman's reconstruction theorem

[SW00, Chapter 3]; [Jos65, Chapter 3]; [Str13, Chapter 3]; [Str93, Sections 1.3-1.4]

As anticipated, provided some properties for the limiting functions

$$W_n(x_1,\dots,x_n) := \lim_{\substack{\Lambda_{\rm UV}\to\infty \\ \Lambda_{\rm IR}\to 0}} Z_{\Lambda_{\rm UV}}^{-n} W_n^{\Lambda}(x_1,\dots,x_n) \tag{4.4}$$

are satisfied, these functions themselves determine states and fields of the interacting theory, avoiding Haag's theorem. In this section we first describe how this reconstruction works for a RQFT at $T=0$, then we'll sketch how to adapt the formalism to more general situations.
We give the statement and the complete proof of the *reconstruction theorem* for a scalar field at $T=0$ for a relativistic theory.

Theorem 4.1 (Wightman's reconstruction theorem, [Wig56], [SW00, Thm. 3.7], [Jos65, Section 3.4, Thm. 1])**.** *Let $\{W_n\}_{n=0}^{\infty}$ be a sequence of (tempered) distributions, $W_n \in \mathcal{S}'(\mathbb{R}^{(d+1)n})$ for all $n>0$, $W_0 := 1$. Suppose that they satisfy the following properties:*

*(1) (*Positive Definiteness*) For any finite sequence of N test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1,x_2),\dots)$, $f_n \in \mathcal{S}(\mathbb{R}^{(d+1)n})$, the following inequality holds:*

$$\sum_{j,k=0}^{N} \int \mathrm{d}x_1\dots\mathrm{d}x_j\mathrm{d}y_1\dots\mathrm{d}y_k\, f_j^*(x_1,\dots,x_j) W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k) f_k(y_1,\dots,y_k) \geq 0 \tag{4.5}$$

or in more compact notation $(\underline{f}, W\underline{f}) \geq 0$.[II]

*(2) (*Poincaré covariance*) For any $(a,\Lambda) \in \mathscr{P}_+^{\uparrow}$, $\Lambda \in \mathscr{L}_+^{\uparrow}$ and $a \in \mathbb{R}^{d+1}$,*

$$W_n(x_1,\dots,x_n) = W_n(\Lambda x_1 + a,\dots,\Lambda x_n + a) \tag{4.6}$$

for all $n \in \mathbb{N}$.

*(3) (*Spectral condition*) For each W_n, $n>0$, exists $\widetilde{W}_n$ such that*

$$W_n(x_1,\dots,x_n) = \int \mathrm{d}p_1\dots\mathrm{d}p_{n-1}\, \widetilde{W}_{n-1}(p_1,\dots,p_{n-1})\, e^{i(x_n-x_{n-1})p_{n-1}}\cdots e^{i(x_2-x_1)p_1} \tag{4.7}$$

with $\widetilde{W}_{n-1}(p_1,\dots,p_{n-1}) = 0$ if $p_j \notin \overline{V}^+$ for some j, $1 \leq j \leq n-1$.

*(4) (*Local commutativity condition / Symmetry*) If x_i and x_{i+1} are space-like separated for some i, that is $(x_i - x_{i+1})^2 < 0$, then for all $n > i$*

$$W_n(x_1,\dots,x_i,x_{i+1},\dots,x_n) = W_n(x_1,\dots,x_{i+1},x_i,\dots,x_n) \tag{4.8}$$

*(5) (*Cluster property*) For any $n \geq 2$, $j<n$,*

$$\lim_{\substack{a\to\infty \\ a^2<0}} W_n(x_1,\dots x_j, x_{j+1}+a,\dots,x_n+a) = W_j(x_1,\dots,x_j) W_{n-j}(x_{j+1},\dots,x_n) \tag{4.9}$$

Then exist

[II] Notice that the proof of this property is not easy, since one should also prove the existence of the limit $W_n^{\Lambda} \to W_n$.

(1') *a separable Hilbert space* $\mathcal{H}$*,*

(2') *a continuous unitary representation* $U(a,\Lambda)$ *of* $\mathscr{P}_+^\uparrow$ [III] *in* $\mathcal{H}$*,*

(3') *a translationally invariant state* $|\Omega\rangle \in \mathcal{H}$*, called* vacuum*,*

(4') *an operator-valued distribution* $\hat\phi(x)$ *such that* $\hat\phi(f) := \int \mathrm{d}x\, f(x)\hat\phi(x)$ *is an operator with domain* D *dense in* $\mathcal{H}$ *for any test function* f*; moreover* $|\Omega\rangle \in D$*, and denoting by* $\mathcal{P}(\hat\phi(f))$ *the set of polynomials of* $\hat\phi(f)$*,*[IV] *the subspace of* $\mathcal{H}$ *generated by* $\mathcal{P}(\hat\phi(f))\,|\Omega\rangle$ *is dense in* $\mathcal{H}$

and the following properties hold:

(5') *the field* $\hat\phi(x)$ *transforms covariantly under* $\mathscr{P}_+^\uparrow$*:*

$$U(a,\Lambda)\hat\phi(x)U^\dagger(a,\Lambda) = \hat\phi(\Lambda x + a) \tag{4.10}$$

(6') *let* $\sigma(\hat P^\mu)$ *be the spectrum of the generator* $\hat P^\mu$ *of the subgroup of translations, then* $\sigma(\hat P^\mu) \subseteq \overline{V}^+$*, i.e.* $\sigma(\hat P^\mu \hat P_\mu) \geq 0$ *and* $\sigma(\hat P^0) \geq 0$*;*

(7') *(*Locality*) fields in space-like separated regions commute*

$$[\hat\phi(x), \hat\phi(y)] = 0 \quad \textit{if} \quad (x-y)^2 < 0 \tag{4.11}$$

or more precisely, in terms of smeared fields f, g*,*

$$[\hat\phi(f), \hat\phi(g)] = 0 \quad \textit{if} \quad \operatorname{supp} f \mathbin{\mathsf{X}} \operatorname{supp} g \tag{4.12}$$

where X *denotes that the two regions are space-like separated,*

(8') *(*Uniqueness of the vacuum*) The vector* Ω *is the unique vector in* $\mathcal{H}$ *translationally invariant;*

(9') *for all* $n \in \mathbb{N}$

$$\langle\Omega|\,\hat\phi(x_1)\cdots\hat\phi(x_n)\,|\Omega\rangle = W_n(x_1,\dots,x_n) \tag{4.13}$$

Proof. (*sketch*)
(1') Consider the vector space $\underline{\mathcal{S}}$ of sequences of test functions $\underline{f} = (f_0, f_1(x_1), f_2(x_1,x_2), \dots)$, $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{(d+1)k})$, with finite number of non-vanishing elements.[V] Addition and multiplication by complex scalars are definite by

$$\begin{gathered}(f_0, f_1, \dots) + (g_0, g_1, \dots) = (f_0+g_0, f_1+g_1, \dots) \\ \alpha(f_0, f_1, \dots) = (\alpha f_1, \alpha f_2, \dots)\end{gathered} \tag{4.14}$$

Next we introduce a scalar product defined on pairs of vectors of $\underline{\mathcal{S}}$

$$(\underline{f}, \underline{g}) := \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \dots \mathrm{d}x_j \mathrm{d}y_1 \dots \mathrm{d}y_k f_j^*(x_1,\dots,x_j) W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k) g_k(y_1,\dots,y_k) \tag{4.15}$$

or using the previous compact notation $(\underline{f}, \underline{g}) := (\underline{f}, W\underline{g})$. Recall $W_0 = 1$ by definition. The linearity in $\underline{g}$ and anti-linearity in $\underline{f}$ are evident from the definition, furthermore from *(1)* we get that the norm $\|\underline{f}\|^2 = (\underline{f}, \underline{f})$ is semi-definite positive, i.e. $\|\underline{f}\| \geq 0$ for any $\underline{f}$. Therefore the operation $(-,-)$ is a well defined inner product in $\underline{\mathcal{S}}$, but $\underline{\mathcal{S}}$ is not a pre-Hilbert space yet since it may contain vectors of zero norm. In order to get a pre-Hilbert, note that the set

$$\underline{\mathcal{S}_0} := \{\underline{f} \in \underline{\mathcal{S}} \text{ s.t. } \|\underline{f}\| = 0\} \subseteq \underline{\mathcal{S}} \tag{4.16}$$

[III] In the case of fields with half-integer spin $\mathscr{P}_+^\uparrow$ is replaced by $\tilde{\mathscr{P}}_+^\uparrow$.
[IV] It is important to notice that $\mathcal{P}(\hat\phi(f))$ denotes polynomials of fields smeared (in general) with different test functions, not only one test function f, for instance $\hat\phi(f_1)\hat\phi(f_2)\dots\hat\phi(f_n) \in \mathcal{P}(\hat\phi(f))$.
[V] Equivalently, for each $\underline{f} \in \underline{\mathcal{S}}$ exists $N \in \mathbb{N}$ such that $f_k = 0$ for all $k \geq N$.

is an isotropic subspace of $\underline{\mathcal{S}}$, that is, a subspace in which each vector is orthogonal to every other vector, indeed by the Schwartz inequality (which is valid as long as the scalar product is non-negative)

$$|(\underline{f},\underline{g})| \leq \|\underline{f}\|\,\|\underline{g}\| \tag{4.17}$$

thus, if $\underline{f}$ and $\underline{g}$ are of zero norm, then $\underline{f}$ is orthogonal to $\underline{g}$ and $\alpha\underline{f}+\beta\underline{g}$ is of zero norm. We now form equivalence classes of sequences $\underline{f}$, two sequences being equivalent if they differ by a sequence of zero norm. These equivalence classes form in a natural way a vector space, denoted by $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$, on which the scalar product induced by $\underline{\mathcal{S}}$ is well-defined and positive definite, since $\|[\underline{f}]\| = 0$ implies $[\underline{f}] = [\underline{0}]$. Therefore $\underline{\mathcal{S}}$ is a pre-Hilbert space. A generic element of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ is of the form

$$\underline{\mathcal{S}}/\underline{\mathcal{S}_0} \ni [\underline{f}] = \{\underline{g} \in \underline{\mathcal{S}} \mid \underline{f}-\underline{g} \in \underline{\mathcal{S}_0}\} \tag{4.18}$$

Then we can finally define the Hilbert space $\mathcal{H}$ by completion[VI] of $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ respect to the induced norm:

$$\mathcal{H} := \overline{\underline{\mathcal{S}}/\underline{\mathcal{S}_0}} \tag{4.19}$$

that is, let $\mathfrak{h}$ be the space of Cauchy sequences $F = \{[\underline{f_1}],[\underline{f_2}],\ldots\}$ with scalar product

$$(F,G) := \lim_{n\to\infty}([\underline{f_n}],[\underline{g_n}]) \tag{4.20}$$

and $\mathfrak{h}_0$ its subspace given by vectors of zero norm, then $\mathcal{H} := \mathfrak{h}/\mathfrak{h}_0$. In the following we'll denote by $|\underline{f}_1,\underline{f}_2,\ldots\rangle$ the elements of $\mathcal{H}$ (i.e. as elements of $\mathfrak{h}$) rather then $[([\underline{f}_1],[\underline{f}_2],\ldots)]$, we give as understood the fact that we refer to the associated equivalence class. Moreover, to simplify the notation we denote by $|\underline{f}\rangle$ the element $[([\underline{f}],[\underline{f}],\ldots)]$.
Being $\mathcal{S}$ separable so is $\underline{\mathcal{S}}$, and since the latter is dense in $\mathcal{H}$ also $\mathcal{H}$ is separable.

(2') We define the linear transformation $U(a,\Lambda)$ in $\underline{\mathcal{S}}$ by

$$U(a,\Lambda)(f_0,f_1,f_2,\ldots) := (f_0,\{a,\Lambda\}f_1,\{a,\Lambda\}f_2,\ldots) \tag{4.21}$$

where

$$\{a,\Lambda\}f_k(x_1,\ldots,x_k) := f_k(\Lambda^{-1}(x_1-a),\ldots,\Lambda^{-1}(x_k-a)) \tag{4.22}$$

The operator $U(a,\Lambda)$ leaves the scalar product of $\underline{\mathcal{S}}$ invariant by virtue of *(2)*:

$$\begin{aligned}
&(U(a,\Lambda)\underline{f},U(a,\Lambda)\underline{g}) = \\
&= \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\ldots\mathrm{d}x_j\mathrm{d}y_1\ldots\mathrm{d}y_k f_j^*(\Lambda^{-1}(x_1-a),\ldots,\Lambda^{-1}(x_j-a))\times \\
&\qquad\qquad \times W_{j+k}(x_1,\ldots,x_j,y_1,\ldots,y_k)g_k(\Lambda^{-1}(y_1-a),\ldots,\Lambda^{-1}(y_k-a)) \\
&= \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\ldots\mathrm{d}x_j\mathrm{d}y_1\ldots\mathrm{d}y_k f_j^*(x_1,\ldots,x_j)W_{j+k}(\Lambda x_1+a,\ldots,\Lambda y_k+a)g_k(y_1,\ldots,y_k) \\
&\overset{(2)}{=} \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\ldots\mathrm{d}x_j\mathrm{d}y_1\ldots\mathrm{d}y_k f_j^*(x_1,\ldots,x_j)W_{j+k}(x_1,\ldots,x_j,y_1,\ldots,y_k)g_k(y_1,\ldots,y_k) \\
&= (\underline{f},\underline{g})
\end{aligned} \tag{4.23}$$

thus $U(a,\Lambda)$ leaves invariant also $\underline{\mathcal{S}_0}$, since $\|\underline{f}\| = 0$ implies $\|U(a,\Lambda)\underline{f}\| = 0$. We have to check that $U(a,\Lambda)$ is actually a mapping of equivalence classes in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. But due to the invariance of the scalar product if $[\underline{f}] = [\underline{g}]$ then $[U(a,\Lambda)\underline{f}] = [U(a,\Lambda)\underline{g}]$, hence $U(a,\Lambda)$ is well defined also in $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$. Being bounded it extends by continuity to $\mathcal{H}$, preserving the scalar product in $\mathcal{H}$, and since it admits the inverse it is unitary. It is easy to prove that $U(a,\Lambda)$ is a unitary representation of $\mathscr{P}_+^{\uparrow}$ on $\mathcal{H}$.

(3') Notice that

$$U(a,\Lambda)(1,0,\ldots,0,\ldots) = (1,0,\ldots,0,\ldots) =: \underline{\mathbb{1}} \tag{4.24}$$

[VI] The completion procedure is described in [SW00, pages 121-122].

hence $\underline{\mathbb{1}} \in \underline{\mathcal{S}}$ is translational invariant in $\underline{\mathcal{S}}$ and the same holds for $[\,\underline{\mathbb{1}}\,]$ in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. Finally, the vacuum in $\mathcal{H}$ is given by the Cauchy sequence $|\Omega\rangle := |\underline{\mathbb{1}}\rangle$.

(4') We introduce in $\underline{\mathcal{S}}$ a linear operator $\hat{\phi}(h)$ for each test function $h \in \mathcal{S}(\mathbb{R}^{d+1})$ by the equation

$$\hat{\phi}(h)\underline{f} := (0, hf_0, h \otimes f_1, h \otimes f_2, \ldots) \tag{4.25}$$

or in compact notation $\hat{\phi}(h)\underline{f} := h \times \underline{f}$, where

$$(h \otimes f_k)(x_1, \ldots, x_{k+1}) := h(x_1) f_k(x_2, x_3, \ldots, x_{k+1}) \tag{4.26}$$

is clearly a test function. As functionals of h the matrix elements $(\underline{f}, \hat{\phi}(h)\underline{g})$ are tempered distributions since they are finite sums of W's, and furthermore

$$(\underline{f}, \hat{\phi}(h)\underline{g}) = (\hat{\phi}(h^*)\underline{f}, \underline{g}) \tag{4.27}$$

We have to check that $\hat{\phi}(h)$ is a mapping of equivalence classes in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. That $\|\underline{f}\| = 0$ implies $\|\hat{\phi}(h)\underline{f}\| = 0$ follows from previous definition and Schwartz inequality:

$$(\hat{\phi}(h)\underline{f}, \hat{\phi}(h)\underline{f}) = (\underline{f}, \hat{\phi}(h^*)\hat{\phi}(h), \underline{f}) \leq \|\underline{f}\|\, \|\hat{\phi}(h^*)\hat{\phi}(h)\underline{f}\| = 0 \tag{4.28}$$

if $\|\underline{f}\| = 0$. Therefore $\hat{\phi}(h)$ is well-defined also in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, $\hat{\phi}(h)[\,\underline{f}\,] := [\,\hat{\phi}(h)\underline{f}\,]$. Hence $\hat{\phi}$ is defined in a dense subset D of $\mathcal{H}$, $D \cong \underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. In particular, being $|\Omega\rangle \in D$, this defines $\hat{\phi}(h)$ on $\mathcal{P}(\hat{\phi}(h))\,|\Omega\rangle$. Notice that

$$\hat{\phi}(h_1) \cdots \hat{\phi}(h_n)\underline{\mathbb{1}} = \hat{\phi}(h_1) \cdots \hat{\phi}(h_{n-1})(0, h_n, 0, 0, \ldots) = (\underbrace{0, \ldots, 0}_{n}, h_1 \otimes h_2 \otimes \ldots \otimes h_n, 0, 0, \ldots) \tag{4.29}$$

hence $\mathcal{P}(\hat{\phi}(h))\,\underline{\mathbb{1}} \cong \underline{\mathcal{S}}$ and then $\mathcal{P}(\hat{\phi}(h))\,|\Omega\rangle \cong D$ is dense in $\mathcal{H}$.

(9') The vacuum expectation value in $\mathcal{H}$ reads

$$\begin{aligned}
\langle\Omega|\,\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\,|\Omega\rangle &= ([\,\underline{\mathbb{1}}\,], \hat{\phi}(h_1)\ldots\hat{\phi}(h_n)[\,\underline{\mathbb{1}}\,]) \qquad \text{scalar product in } \underline{\mathcal{S}}\,/\underline{\mathcal{S}_0} \\
&= (\underline{\mathbb{1}}, \hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\underline{\mathbb{1}}) \qquad \text{scalar product in } \underline{\mathcal{S}} \\
&= \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, W_n(x_1, \ldots, x_n) h_1(x_1) \ldots h_n(x_n) \\
&= W_n(h_1, \ldots, h_n)
\end{aligned} \tag{4.30}$$

where in the first step we computed the scalar product in $\mathcal{H}$ using eq. (4.20), in the second step we chosed the representative $\underline{\mathbb{1}}$ for the equivalence class $[\,\underline{\mathbb{1}}\,]$ and then we computed the scalar product in $\underline{\mathcal{S}}$ according to the definition eq. (4.15) using eq. (4.29) to compute $\hat{\phi}(h_1)\cdots\hat{\phi}(h_n)\underline{\mathbb{1}}$. Then we noted that the result we obtained coincide with the smearing of the Wightman function. Finally eq. (4.13) follows immediately from $\langle\Omega|\,\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\,|\Omega\rangle = W_n(h_1, \ldots, h_n)$.

(5') That $\hat{\phi}(h)$ satisfies the transformation law

$$U(a, \Lambda)\hat{\phi}(h)U^\dagger(a, \Lambda) = \hat{\phi}(\{a, \Lambda\}h) \tag{4.31}$$

can be easy proved in $\underline{\mathcal{S}}$:

$$\begin{aligned}
U(a, \Lambda)\hat{\phi}(h)(f_0, f_1, \ldots) &= U(a, \Lambda)(0, hf_0, h \otimes f_1, \ldots) \\
&= (0, \{a, \Lambda\}hf_0, \{a, \Lambda\}h \otimes \{a, \Lambda\}f_1, \ldots) \\
&= \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda)(f_0, f_1, \ldots)
\end{aligned} \tag{4.32}$$

then since the previous computation holds for each $\underline{f} \in \underline{\mathcal{S}}$ and both $\hat{\phi}(h)$ and $U(a, \Lambda)$ are mapping of equivalence classes then

$$U(a, \Lambda)\hat{\phi}(h)[\,\underline{f}\,] = \hat{\phi}(\{a, \Lambda\}h)U(a, \Lambda)[\,\underline{f}\,] \tag{4.33}$$

and finally we can extend it by continuity to $\mathcal{H}$, since $\underline{\mathcal{S}}/\underline{\mathcal{S}_0}$ is dense in it:

$$U(a,\Lambda)\hat{\phi}(h)\,|\underline{f}_1,\underline{f}_2,\ldots\rangle = \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda)\,|\underline{f}_1,\underline{f}_2,\ldots\rangle \tag{4.34}$$

Since this holds for any element of $\mathcal{H}$ we get

$$U(a,\Lambda)\hat{\phi}(h) = \hat{\phi}(\{a,\Lambda\}h)U(a,\Lambda) \quad \Rightarrow \quad \hat{\phi}(\{a,\Lambda\}h) = U(a,\Lambda)\hat{\phi}(h)U^{-1}(a,\Lambda) \tag{4.35}$$

and we are done, since "unsmearing" the fields we get

$$\int \mathrm{d}x\, h(\Lambda^{-1}(x-a))\hat{\phi}(x) = U(a,\Lambda)\int \mathrm{d}x\, h(x)\hat{\phi}(x)U^{-1}(a,\Lambda) \tag{4.36}$$

and by a change of variable on the r.h.s.:

$$\int \mathrm{d}x\, h(x)\hat{\phi}(\Lambda x+a) = \int \mathrm{d}x\, h(x)U(a,\Lambda)\hat{\phi}(x)U^{-1}(a,\Lambda) \tag{4.37}$$

(6') Consider the unitary group of translations $\{U(a)\} := \{U(a,\mathbb{1})\}$. Stone's theorem[VII] apply to such operators (the group can be decomposed in $(d+1)$ one-parameter unitary groups), then exists a unique operator $\hat{P} : D_P \to \mathcal{H}$, self-adjoint in the dense domain D_P, such that $U(a) = e^{ia\hat{P}}$ for any $a \in \mathbb{R}^{d+1}$. Moreover thanks to *(5')* we have

$$\hat{\phi}(x) = e^{ix\hat{P}}\hat{\phi}(0)e^{-ix\hat{P}} \tag{4.38}$$

Then, using *(9')* together with *(3')* we have

$$\begin{aligned}
&W_n(x_1,\ldots,x_n) = \\
&\quad = \langle\Omega|\,\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\,|\Omega\rangle \\
&\quad = \langle\Omega|\,\hat{\phi}(0)e^{i\hat{P}(x_2-x_1)}\hat{\phi}(0)\ldots\hat{\phi}(0)e^{i\hat{P}(x_n-x_{n-1})}\hat{\phi}(0)\,|\Omega\rangle \\
&\quad = \int \mathrm{d}p_1\ldots\mathrm{d}p_{n-1}\,\langle\Omega|\,\hat{\phi}(0)e^{i\hat{P}(x_2-x_1)}\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\ldots\hat{\phi}(0)e^{i\hat{P}(x_n-x_{n-1})}\,|p_{n-1}\rangle\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle \\
&\quad = \int \mathrm{d}p_1\ldots\mathrm{d}p_{n-1}\,e^{ip_1(x_2-x_1)}\ldots e^{ip_{n-1}(x_n-x_{n-1})}\langle\Omega|\,\hat{\phi}(0)\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\,|p_2\rangle\ldots\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle
\end{aligned} \tag{4.39}$$

where $\{\,|p_i\rangle\}$ denotes an orthonormal basis of generalized eigenvectors of $\hat{P}$, with generalized eigenvalues $\{p_i\}$. Comparing this expression with eq. (4.7) we get

$$\widetilde{W}_{n-1}(p_1,\ldots,p_{n-1}) = \langle\Omega|\,\hat{\phi}(0)\,|p_1\rangle\langle p_1|\,\hat{\phi}(0)\,|p_2\rangle\ldots\langle p_{n-1}|\,\hat{\phi}(0)\,|\Omega\rangle \tag{4.40}$$

Then the claim follows using property *(3)*. Indeed, since

$$|p_i\rangle\langle p_i| = \int_{\sigma(\hat{P})} \mathrm{d}^4p\,\delta(p-p_i)\,|p\rangle\langle p| = \delta(\hat{P}-p_i) \tag{4.41}$$

then from *(3)* we get

$$\delta(\hat{P}-p_i) = 0 \quad \text{if} \quad p_i \notin \overline{V}_+ \tag{4.42}$$

implying that $\sigma(\hat{P}) \subseteq \overline{V}_+$.

(7') Take $f,g \in \mathcal{S}(\mathbb{R}^{d+1})$ space-like separated, that is $\operatorname{supp} f \mathbin{\mathsf{X}} \operatorname{supp} g$, and a sequence of test functions $\{h_i\}_{i=0}^{\infty}$. Then from *(4)* we know that, for each value of n,

$$W_{n+2}(f,g,h_1,\ldots,h_n) = W_{n+2}(g,f,h_1,\ldots,h_n) \tag{4.43}$$

and *(9')* tells us that

$$\langle\Omega|\,\hat{\phi}(f)\hat{\phi}(g)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\,|\Omega\rangle = \langle\Omega|\,\hat{\phi}(g)\hat{\phi}(f)\hat{\phi}(h_1)\ldots\hat{\phi}(h_n)\,|\Omega\rangle \tag{4.44}$$

[VII] https://doi.org/10.2307%2F1968538

Then, by linearity

$$\langle\Omega|\,[\hat{\phi}(f),\hat{\phi}(g)]\hat{\phi}(h_1)\dots\hat{\phi}(h_n)\,|\Omega\rangle = 0 \tag{4.45}$$

An analogous procedure together with eq. (4.27) lead us also to

$$\big\langle[\hat{\phi}(f),\hat{\phi}(g)]\hat{\phi}(h_1)\dots\hat{\phi}(h_n)\Omega\big|[\hat{\phi}(f),\hat{\phi}(g)]\hat{\phi}(h_1)\dots\hat{\phi}(h_n)\Omega\big\rangle = \|[\hat{\phi}(f),\hat{\phi}(g)]\hat{\phi}(h_1)\dots\hat{\phi}(h_n)\,|\Omega\rangle\|^2 = 0 \tag{4.46}$$

Since the scalar product is non-degenerate, this implies that $[\hat{\phi}(f),\hat{\phi}(g)]\hat{\phi}(h_1)\dots\hat{\phi}(h_n)\,|\Omega\rangle = 0$ is the zero vector in $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$. The only way to satisfy this for any choice of n and any set of smearing test functions $\{h_i\}_{i=1}^{\infty}$ is that in $\mathcal{P}(\hat{\phi}(h))\Omega$ we have

$$[\hat{\phi}(f),\hat{\phi}(g)] = 0 \tag{4.47}$$

meaning that the operator $[\hat{\phi}(f),\hat{\phi}(g)]$ maps all the elements of $\mathcal{P}(\hat{\phi}(h))\Omega$ into the zero vector. But then the claim extends to the whole Hilbert space $\mathcal{H}$ since $\mathcal{P}(\hat{\phi}(h))\Omega$ is dense in $\mathcal{H}$.

(8') Assume that, beside $|\Omega\rangle = |\underline{\mathbb{1}}\rangle$ there is another translationally invariant state Ω'. Without loss of generality we can orthonormalize $|\Omega'\rangle$ respect to $|\Omega\rangle$, i.e. we take $\langle\Omega'|\Omega\rangle = 0$ and $\langle\Omega'|\Omega'\rangle = 1$. If $|\Omega'\rangle = |\underline{f}\rangle$, which means that $|\Omega'\rangle \in \underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, then we would have an immediate contradiction because

$$\begin{aligned}
1 = \langle\Omega'|\Omega'\rangle &\overset{(a)}{=} \lim_{\substack{a\to\infty\\ a^2<0}} \langle\Omega'|U(a)\Omega'\rangle = \lim_{\substack{a\to\infty\\ a^2<0}} ((f_0,f_1,\dots),(f_0,\{a,\mathbb{1}\}f_1,\{a,\mathbb{1}\}f_2,\dots)) \\
&\overset{(4.15)}{=} \lim_{\substack{a\to\infty\\ a^2<0}} \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\dots\mathrm{d}x_j\mathrm{d}y_1\dots\mathrm{d}y_k\, f_j^*(x_1,\dots,x_j)W_{j+k}(x_1,\dots,x_j,y_1,\dots,y_k)f_k(y_1-a,\dots,y_k-a) \\
&= \lim_{\substack{a\to\infty\\ a^2<0}} \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\dots\mathrm{d}x_j\mathrm{d}y_1\dots\mathrm{d}y_k\, f_j^*(x_1,\dots,x_j)W_{j+k}(x_1,\dots,x_j,y_1+a,\dots,y_k+a)f_k(y_1,\dots,y_k) \\
&\overset{(5)}{=} \sum_{j,k=0}^{\infty}\int \mathrm{d}x_1\dots\mathrm{d}x_j\mathrm{d}y_1\dots\mathrm{d}y_k\, f_j^*(x_1,\dots,x_j)W_j(x_1,\dots,x_j)W_k(y_1,\dots,y_k)f_k(y_1,\dots,y_k) \\
&= \Big(\sum_{j=0}^{\infty}\int \mathrm{d}x_1\dots\mathrm{d}x_j\, f_j^*(x_1,\dots,x_j)W_j(x_1,\dots,x_j)\Big)\Big(\sum_{k=0}^{\infty}\int \mathrm{d}y_1\dots\mathrm{d}y_k\, W_k(y_1,\dots,y_k)f_k(y_1,\dots,y_k)\Big) \\
&= \langle\Omega'|\Omega\rangle\langle\Omega|\Omega'\rangle = 0
\end{aligned} \tag{4.48}$$

where in (a) we used the translational invariance of Ω'. In general Ω' is not an element of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$, but because the latter is dense then Ω' can be approximated by elements of $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ by arbitrary accuracy, then the contradiction easily extends for $\Omega' \in \mathcal{H}$, see [SW00, page 124] for the details. □

The most important consequence of the theorem is that in order to exhibit a relativistic quantum field theory model, it is enough to give a set of Wightman functions satisfying properties *(1)-(5)*. As we'll discuss in the next section, these properties provide a non-perturbative substitute for canonical quantization, since allow to construct a quantum field theory without use the canonical quantization procedure and CCR (or CAR) which are inconsistent in the interacting case due to Haag's theorem.
Properties *(1')* - *(8')* are called *Wightman axioms*[VIII], and define axiomatically a RQFT at $T = 0$ for a real scalar field. Provided appropriated versions of properties *(1)-(5)*, more general forms of the reconstruction theorem allow to reconstruct also theories with complex fields and more general spins, satisfying appropriate Wightman axioms.
Notice that property *(9')* tells us that the vacuum expectation values of the reconstructed theory are exactly the Wightman function we started from.
Conversely, if a theory satisfy Wightman axioms then its vacuum expectation values satisfy the properties required in the reconstruction theorem.[IX] In particular, if $\hat{\phi}_F$ is a free theory defined in a Fock space, then the vacuum expectation values $\langle 0|\,\hat{\phi}_F(x_1)\dots\hat{\phi}_F(x_n)\,|0\rangle$ satisfy properties *(1)-(5)* and the reconstructed theory coincide with the initial one, $\hat{\phi}_{\text{rec}} = \hat{\phi}_F$. On the other side, a reconstructed interacting theory is defined in a Hilbert space disjoint from the Fock space of the free theory, due to Haag's theorem.

[VIII] These axioms are discussed in [SW00, Section 3.1] and [Jos65, Section 3.2].
[IX] See [SW00, Theorems 3.1-3.4] and [Jos65, Section 3.3].

Can also be proved that the reconstructed theory is unique, i.e. any other theory satisfying properties *(1') - (9')* (in particular with same vacuum expectation values $\{W_n\}_{n=0}^{\infty}$) is unitarily equivalent to the one reconstructed in the proof of the theorem.[X]
Finding a set of Wightman functions satisfying properties *(1)-(5)* turned out to be a very hard problem, apart from the non-interacting case, also because it is difficult to satisfy the positivity condition, which has a non-linear structure, in contrast to the other properties, which have a linear structure. We will see in the following how this problem has been solved by computing Wightman functions in the Euclidean space, where properties *(1)-(5)* take a simpler form.

In the proof of the theorem we constructed the Hilbert space of the theory as (the completion of) finite sequences of test functions. In order to give a better interpretation of the Hilbert space, notice that if we define

$$\hat{\phi}(\underline{f}) := f_0 + \int \mathrm{d}x_1\, \hat{\phi}(x_1) f_1(x_1) + \int \mathrm{d}x_1 \mathrm{d}x_2\, \hat{\phi}(x_1)\hat{\phi}(x_2) f_2(x_1, x_2) + \dots \tag{4.49}$$

such that

$$\hat{\phi}(\underline{f})\underline{g} := (f_0 g_0, f_0 g_1 + f_1 g_0, f_0 g_2 + f_1 \otimes g_1 + f_2 g_0, \dots) \quad , \quad \hat{\phi}(\underline{f})[\,\underline{g}\,] := [\,\hat{\phi}(\underline{f})\underline{g}\,] \tag{4.50}$$

and in particular

$$\hat{\phi}(\underline{f})\underline{\mathbb{1}} = (f_0, f_1, f_2, \dots) = \underline{f} \quad , \quad \hat{\phi}(\underline{f})[\,\underline{\mathbb{1}}\,] = [\,\underline{f}\,] \tag{4.51}$$

Then the scalar product in $\underline{\mathcal{S}}$ and $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ can be rewritten as[XI]

$$(\underline{f}, \underline{g}) = \langle \hat{\phi}(\underline{f})\underline{\mathbb{1}}, \hat{\phi}(\underline{g})\underline{\mathbb{1}} \rangle \quad , \quad ([\,\underline{f}\,], [\,\underline{g}\,]) = \langle \hat{\phi}(\underline{f})[\,\underline{\mathbb{1}}\,], \hat{\phi}(\underline{g})[\,\underline{\mathbb{1}}\,] \rangle \tag{4.52}$$

Hence we can identify by unitary equivalence (i.e. up to representations) $\underline{\mathcal{S}}\,/\underline{\mathcal{S}_0}$ with $\{\hat{\phi}(\underline{f})[\,\underline{\mathbb{1}}\,] \mid \underline{f} \in \underline{\mathcal{S}}\}$ by

$$\begin{aligned} U \,:\, \underline{\mathcal{S}}\,/\underline{\mathcal{S}_0} &\to \{\hat{\phi}(\underline{f})[\,\underline{\mathbb{1}}\,] \mid \underline{f} \in \underline{\mathcal{S}}\} \\ [\,\underline{f}\,] &\mapsto \hat{\phi}(\underline{f})[\,\underline{\mathbb{1}}\,] \end{aligned} \tag{4.53}$$

since the scalar products defined in the two spaces are the same. This easily extends to the whole $\mathcal{H}$ due to the density of $\mathcal{P}(\hat{\phi}(f))\,|\Omega\rangle$ in $\mathcal{H}$. Then we are done: in the Hilbert space one can forget the structure underlying $\underline{f}$ and $\underline{\mathbb{1}}$ and define the Hilbert space by applying the operator $\hat{\phi}(\underline{f})$ on the vacuum state $|\Omega\rangle$, as usual.

4.3 Additional remarks

[Str13, Sections 3.4, 4.1–4.4]; [Jos65, Chapters 4, 5]; [SW00, Chapter 4]

By exploiting the analyticity properties of the Wightman's functions, general results have been derived. In this section we will focus on some of the main results which have been derived in the specific case of relativistic QFT's.

Spin-statistics theorem

The connection between spin and statistics, which is at the basis of Pauli principle (namely, in the alternative of (anti)commutation relations at space-like separated points, fields carrying half-integer/integer spin must anti-commute/commute), have been first derived by Pauli for free theories in Fock space. It's generalization to interacting theories holds in the Wightman axiomatic formalism, and is a consequence of Lorentz covariance, as it has been proved in *spin-statistics theorem*:

[X] Actually this claim is usually included in the statement of the reconstruction theorem, see [SW00, Thm. 3.7] and [Jos65, Section 3.4, Thm. 1].

[XI] The bilinear $(-,-)$ denotes as usual the scalar product in $\underline{\mathcal{S}}$, whereas $\langle -,- \rangle$ denotes the scalar product in $\{\hat{\phi}(\underline{f})\underline{\mathbb{1}} \mid \underline{f} \in \underline{\mathcal{S}}\}$. Analogous notations hold for the quotient sets.

Theorem 4.2 (Spin-statistics, [Str13, Section 4.2], [SW00, Section 4.4], [Jos65, Section 5.3]). *Let $\psi_{\alpha,\dot{\beta}}$ be a spinor field*[XII] *transforming as $\mathcal{D}^{j/2,k/2}$, and therefore carrying a integer/half-integer spin, corresponding to $j+k$ =even/odd, then the wrong connection between spin and statistics, i.e., for integer/half-integer spin the field anti-commutes/commutes at space-like separations, implies that $\psi\,|\Omega\rangle = 0$ and, if locality holds, $\psi = 0$.*

PCT theorem

The (up to now experimentally established) *PCT symmetry* also follows from Wightman functions properties, being in particular related to local commutativity:

Theorem 4.3 (PCT, [Str13, Section 4.3-4.4], [SW00, Section 4.3], [Jos65, Section 5.2]). *If a spinor field satisfies covariance, spectral condition, and locality, then the corresponding Wightman functions are PCT symmetric.*

Reeh-Schlieder theorem

Wightman's functions analyticity can be proved[XIII] and imply that the knowledge of the Wightman functions of the fields localized in a given open set $\mathcal{O} \subset \mathbb{R}^4$, however small, completely determines the theory. In fact, if $\mathcal{P}(\mathcal{O})$ denotes the polynomial algebra of the fields smeared with test functions with support in $\mathcal{O}$, one has

Theorem 4.4 (Reeh-Schlieder, [SW00, Section 4.2], [Jos65, Section 4.8], [Str13, Section 3.4.4], [Haa96, Section II.5.3]). *For any open set $\mathcal{O} \subset \mathbb{R}^4$, the vacuum is cyclic with respect to the algebra $\mathcal{P}(\mathcal{O})$, i.e.*

$$\{\mathcal{P}(\hat{\phi}(f)) \text{ s.t. } \mathrm{supp}(f) \subseteq \mathcal{O}\}\,|\Omega\rangle \tag{4.54}$$

is dense in $\mathcal{H}$. Moreover, if $A \in \mathcal{P}(\mathcal{O})$ and $A\,|\Omega\rangle = 0$ then $A = 0$.

This means that the information contained in fields localized in any, arbitrarily small, open subset of $\mathbb{R}^4$, can be used to reconstruct the entire RQFT.

Non-commutativity of the algebra of interacting fields

As we already pointed out equal-time canonical (anti-)commutation relations cannot be used to quantize field equations for interacting theories. A natural question is whether and where a quantization condition is contained in the general requirements of Wightman's axioms, or equivalently in the properties of Wightman functions. To this purpose, one may show that, as a consequence of the spectral condition, the field algebra of the reconstructed theory cannot be commutative because in that case $W_2(x-y)$ would be constant, hence properties *(1)-(5)* imply quantization. Since can be proved[XIV] that field obeying free fields equations satisfy either CCR of CAR, Wightman's properties qualify as the way of defining quantum fields by providing the correct substitute of canonical quantization in the interacting case.

4.4 Haag-Ruelle scattering theory (massive case) and LSZ reduction formula

[Rue62]; [Hep65]; [Haa96, Chapter II.4]; [Jos65, Chapters 6]; [Str13, Sections 6.1-6.2]

Another important consequence of properties *(1)-(5)* for relativistic QFT's is the existence of asymptotic fields which provide a non-perturbative definition of the S-matrix, which is unitary if asymptotic completeness holds. Also the LSZ asymptotic condition and the LSZ reduction formula can be derived from the same properties, as proved by Hepp; furthermore, one can prove dispersion relations for scattering amplitudes, yielding experimentally measurable relations.

[XII] Here we consider a spinor field since it can describe the most general case, i.e. both integer and half integer fields. We will do the same also for the spin-statistics theorem.
[XIII] [Str13, Sections 3.4.1-3.4.3], [SW00, Chapter 4]
[XIV] [Str13, Section 4.1]

One should notice that the axioms of Wightman do not introduce the notion of a particle. The particle notion is, however, of crucial importance if any connection of the theory with experimental physics is to be achieved. The principal observable of elementary particle physics, the S-matrix, depends essentially on the possibility of defining asymptotic incoming and outgoing states, which themselves can be described by free particles and their corresponding free fields.
It is highly gratifying that, under natural conditions, a Wightman field theory allows the precise definition of such asymptotic fields. This was proved by Ruelle, who applied Haag's idea of construct asymptotic ingoing and outgoing states as strong limits in Hilbert space, if a certain "space-like asymptotic condition" is verified by the vacuum expectation values of products of field operators.
In his theory, Ruelle had to make some natural additional assumptions which go beyond the axioms of Wightman. This is not astonishing, it is easily seen that in a theory which allows a complete particle interpretation the spectrum of the energy-momentum vector must have special properties.

We shall not present the general theory in this section. The special case to which we shall restrict ourselves, however, shows most of the interesting aspects of the general case. Let's assume that our theory has a *mass gap*. This means that the spectrum of the generators of the spacetime translations, as determined by the Wightman functions and the reconstruction theorem, satisfies the following:
Mass gap condition: Above the isolated point $p = 0$ (corresponding to the vacuum state) there is a gap up to an isolated hyperboloid $p^2 = m^2$ denoted by V_m^+ ($V_m^+ \subset \sigma(\hat{P}^\mu) \subseteq \overline{V}^+$), with finite multiplicity (one-particle state), followed by a continuous spectrum starting at $p^2 = (2m)^2$ (two-particles states).
It suffices that such a mass gap condition holds in the subspace characterized by the conserved quantum numbers of the asymptotic states under investigation. Notice that such condition cannot be satisfied if theories admits massless particles, for instance it does not apply to QED.

Let us restrict to the one-body problem, i.e. the description of the asymptotic state for a one particle system. The existence of free asymptotic fields of mass m should be related to the existence of an isolated hyperboloid $p^2 = m^2$ in the two-point spectral function of the field $\hat{\phi}$.
Let $\mathcal{H}_m$ be the improper[XV] subspace of $\mathcal{H}$, which is eigenspace of the operator $\hat{P}^\mu \hat{P}_\mu$ associated to the eigenvalue m^2. Suppose that our (possibly renormalized, if needed) field $\hat{\phi}$ when applied to the vacuum has a non-empty intersection with $\mathcal{H}_m$, that is $\hat{\phi}(x)\,|\Omega\rangle \not\perp \mathcal{H}_m$. Let $h(x)$ be a test function whose Fourier transform satisfies $\operatorname{supp} \tilde{h}(p) \cap \sigma(\hat{P}^\mu) \subseteq V_m^+$, or equivalently, its support in momentum space does not intersect the continuum region above V_m^+ of $\overline{V}^+$ (the one associated to multi-particles states).
Let's define the operator

$$\hat{\phi}_m(h) := \int \mathrm{d}p\, \tilde{h}(p) \tilde{\hat{\phi}}(p) \tag{4.55}$$

and consider the two-points function

$$\langle \Omega |\, \hat{\phi}_m(h) \hat{\phi}_m(h')\, |\Omega\rangle \tag{4.56}$$

where h' satisfies same properties as h. Such two-point function in the Källen-Lehmann representation contains only an isolated term like $\delta(p^2 - m^2)\theta(p^0)$, since the localization of h on the one-particle hyperboloid acts as a cutoff for the continuum part in the spectral representation of the correlator.
The field $\hat{\phi}_m(h)$ is not a free field, since in general $\operatorname{supp} h \not\subset \sigma(\hat{P}^\mu)$ and $(p^2 - m^2)\tilde{\hat{\phi}}_m(p) \neq 0$, however $(p^2 - m^2)\tilde{\hat{\phi}}_m(p)\Omega = 0$, hence we selected at the field level a neighbourhood of the mass shell.

Let's see how to construct a state for more than one body. Suppose that we have n test functions, $\{h_i\}_{i=1}^n$, $\operatorname{supp} \tilde{h}_i(p) \cap \sigma(\hat{P}^\mu) \subseteq V_m^+$, such that their supports in terms of velocities are disjoint, that is

$$\frac{\boldsymbol{p}_i}{p_i^0} \neq \frac{\boldsymbol{p}_j}{p_j^0} \quad \text{for any} \quad p_i \in \operatorname{supp} \tilde{h}_i \quad , \quad p_j \in \operatorname{supp} \tilde{h}_j \quad , \quad i \neq j \tag{4.57}$$

Due to condition (4.57), for very large values of t the operators

$$e^{-itH} \hat{\phi}_m(h_i) e^{itH} \quad , \quad i = 1, \dots, n \tag{4.58}$$

[XV] It is an improper subspace since it is made of generalized vectors, in the sense of Dirac's formalism.

have their supports separated by large distances, hence in the asymptotic limit $t \to \pm\infty$ they are free (since interactions decay exponentially for $m > 0$, see [Str13, Section 6.2.2]). Then to have fields defined for all times we can "bring them back" using free evolution, i.e. using the standard dispersion relation $\sqrt{\boldsymbol{p}^2 + m^2}$:

$$\hat{\phi}_m(h,t) := \int \mathrm{d}p\, e^{it\sqrt{\boldsymbol{p}^2+m^2}} \tilde{h}(p) e^{-itH} \tilde{\hat{\phi}}_m(p) e^{itH} \tag{4.59}$$

Such field behaves for $t \to \pm\infty$ as a free field.
The final result is provided by the following theorem

Theorem 4.5 (Haag-Ruelle, [Rue62, Section 4]; [Str13, Theorem 6.2.1], [Haa96, Theorem II.4.2.1], [Jos65, Theorems 1, 2 section 6.3])**.** *The following asymptotic limits*

$$\begin{aligned} \lim_{t\to-\infty} \hat{\phi}_m(h_1,t)\cdots\hat{\phi}_m(h_n,t)\,|\Omega\rangle =: |h_1,\ldots,h_n\rangle_{\mathrm{in}} \\ \lim_{t\to+\infty} \hat{\phi}_m(h_1,t)\cdots\hat{\phi}_m(h_n,t)\,|\Omega\rangle =: |h_1,\ldots,h_n\rangle_{\mathrm{out}} \end{aligned} \tag{4.60}$$

exist in the strong sense and define asymptotic in/out scattering states. The closure of the space of scattering states $\mathcal{H}_{\mathrm{in/out}}$ is contained in $\mathcal{H}$. In $\mathcal{H}_{\mathrm{in/out}}$ are defined the operators $\hat{\phi}_{\mathrm{in/out}}$ such that

$$\begin{aligned} \hat{\phi}_{\mathrm{in}}(h)\,|h_1,\ldots,h_n\rangle &= \lim_{t\to-\infty} \hat{\phi}_m(h,t)\,|h_1,\ldots,h_n\rangle_{\mathrm{in}} \\ \hat{\phi}_{\mathrm{out}}(h)\,|h_1,\ldots,h_n\rangle &= \lim_{t\to+\infty} \hat{\phi}_m(h,t)\,|h_1,\ldots,h_n\rangle_{\mathrm{out}} \end{aligned} \tag{4.61}$$

Let's see what happens if we abandon the restriction on the support of the test function, i.e. if we define

$$\hat{\phi}(f,t) = \int \mathrm{d}p\, \tilde{f}(p) e^{itH} \tilde{\hat{\phi}}(p) e^{-itH} e^{it\sqrt{\boldsymbol{p}^2+m^2}} \tag{4.62}$$

for some (generic) test function f. Hepp proved that it admits a weak limit $t \to \pm\infty$ on scattering states, and can be used as asymptotic fields in the LSZ formula

Theorem 4.6 (Hepp, [Hep65])**.** *The following weak limits exist:*

$$\begin{aligned} \lim_{t\to-\infty} {}_{\mathrm{in}}\langle h_1,\ldots,h_n|\hat{\phi}(f,t)\,|h'_1,\ldots,h'_m\rangle_{\mathrm{in}} &= {}_{\mathrm{in}}\langle h_1,\ldots,h_n|\hat{\phi}_{\mathrm{in}}(f)\,|h'_1,\ldots,h'_m\rangle_{\mathrm{in}} \\ \lim_{t\to+\infty} {}_{\mathrm{out}}\langle h_1,\ldots,h_n|\hat{\phi}(f,t)\,|h'_1,\ldots,h'_m\rangle_{\mathrm{out}} &= {}_{\mathrm{out}}\langle h_1,\ldots,h_n|\hat{\phi}_{\mathrm{out}}(f)\,|h'_1,\ldots,h'_m\rangle_{\mathrm{out}} \end{aligned} \tag{4.63}$$

and define scattering fields $\hat{\phi}_{\mathrm{in/out}}$. These fields can be used inside the LSZ reduction formula, hence the reduction formalism works even in this axiomatic framework starting from Wightman functions.

Hence all the formalism of asymptotic states and fields is mathematically well defined within Wightman theory.

Chapter 5

Reconstruction in the Euclidean spacetime

[SW00, Pages 185-191], [Sum16], [Wig76], [Str13, Section 5.1], [GJ87, Section 6.1, Chapter 19]

We now have to deal with the construction of the Wightman functions or their non relativistic counterpart, which as we have seen are the cornerstone of our formalism. It turns out that the easiest way to construct them is the following

(1) go back to the Gell-Man - Low formula and introduce an IR cutoff (and in the relativistic case also an UV one) so that we avoid Haag's no-go theorem;

(2) rewrite such formula using the path-integral formalism as described in Sec. 2.5 (when the cutoffs are introduced, we'll denote the action by S_Λ), in particular

   (a) for a relativistic theory the path integral is defined in an Euclidean space-time,

   (b) for a non relativistic theory the path integral is defined in an Euclidean space-time for an "imaginary" time;

(3) remove the cutoffs from outside the path integral, we assume that this is possible, otherwise the procedure fails;

(4) provided some properties hold for the correlation functions, analytically continue them to the appropriate space, in particular

   (a) for a relativistic theory, the correlation functions in the Euclidean space are called *Schwinger functions*, denoted by $\{S_n\}$

$$S_n(x_1,\dots,x_n) = \lim_{\substack{\Lambda_{\mathrm{UV}}\to\infty \\ \Lambda_{\mathrm{IR}}\to 0}} \frac{\int \mathcal{D}\phi\mathcal{D}\phi^*\, e^{-S_\Lambda(\phi^*,\phi)}\phi(x_1)\dots\phi(x_n)}{\int \mathcal{D}\phi\mathcal{D}\phi^*\, e^{-S_\Lambda(\phi^*,\phi)}} \tag{5.1}$$

   they determine Wightman function by analytic continuation to the Minkowski space if a set of properties, called *Osterwalder-Schrader (OS) axioms*, are satisfied,

   (b) for a non relativistic theory with $T > 0$ the Matsubara Green functions we obtained for an "imaginary" time can be analytically continued to the real time provided that other suitable axioms are satisfied, defining in this way the retarded functions for the interacting theory.

We first discuss the relativistic case and later we make some comments on the non relativistic $T > 0$ and the lattice QFT cases.
Let's heuristically motivate why it is easier to work in the Euclidean space rather than in the Minkowski one. In the Euclidean space the exponential inside the path integral has a real argument, e^{-S}, whereas in the Minkowski one the argument is imaginary: e^{iS}. In the Euclidean space one can use more powerful techniques such as saddle point method, moreover in general the path integral measure can be defined

with a real exponential, while the definition in the Minkowski space is much more difficult and up to now it is not really understood if the path integral can be defined in a rigorous way for an infinite-dimensional configuration space as for a field theory. Without enter in the details, notice for instance that $\int e^{-x^2} \mathrm{d}x$ clearly converges, whereas $\int e^{ix^2} \mathrm{d}x$ should be regularized. For the same reason, in the non relativistic case is better to work with imaginary time rather than with the real one.

5.1 Osterwalder-Schrader reconstruction

[OS73], [OS75], [Ost73], [Str13, Chapter 5], [Str93, Chapter 3]

We now list the properties of Schwinger functions needed to determine Wightman functions of an interacting (scalar) RQFT. Such properties are called *Osterwalder-Schrader (OS) axioms*:

(0) (*Regularity*) The $\{S_n(x_1, \dots, x_n)\}_{n=0}^{\infty}$, $x_i \in \mathbb{R}^{d+1}$ with Euclidean metric, are distributions in $\mathcal{S}'(\mathbb{R}^{n(d+1)})$ (or more generally spaces of distributions admitting Fourier transform) and "should not grow too fast in n" (we omit the details about this requirement).

(1) (*Euclidean covariance*) They transform covariantly under the Euclidean $d+1$-dimensional group $\mathbb{E}^{d+1}$. In particular for a scalar field ϕ, $(a, R) \in \mathbb{E}^{d+1}$, we have

$$S_n(Rx_1 + a, \dots, Rx_n + a) = S_n(x_1, \dots, x_n) \tag{5.2}$$

(2) (*Osterwalder-Schrader positivity / Reflection positivity*) Let r denote the reflection of a point of $\mathbb{R}^{d+1}$ with respect to the time-zero d-dimensional space, $rx = r(\tau, \boldsymbol{x}) := (-\tau, \boldsymbol{x})$. Then for each finite sequence $\underline{f}$ of test functions with support in the upper half-space in the Euclidean time direction (i.e. the support of the test functions is completely contained in the $\tau > 0$ region) the following inequality holds:

$$\sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \dots \mathrm{d}x_j \mathrm{d}y_1 \dots \mathrm{d}y_k \, f_j^*(rx_1, \dots, rx_j) S_{j+k}(x_1, \dots, x_j, y_1, \dots, y_k) f_k(y_1, \dots, y_k) \geq 0 \tag{5.3}$$

(3) (*Symmetry*) Schwinger functions of bosonic (fermionic) fields are symmetric (antisymmetric) under any exchange of the arguments. For a scalar bosonic theory we have $S_n(x_1, \dots, x_n) = S_n(x_{\pi(1)}, \dots, x_{\pi(n)})$ for any permutation π.

(4) (*Cluster property*) Schwinger functions asymptotically factorize when two sets of the arguments are taken apart to ∞:

$$\lim_{a \to \infty} S_n(x_1 + a, \dots, x_j + a, x_{j+1}, \dots, x_n) = S_j(x_1, \dots, x_j) S_{n-j}(x_{j+1}, \dots, x_n) \tag{5.4}$$

Notice that it is much more easy to verify this axioms (beside OS positivity) rather than properties of Wightman functions. For instance in this case there is no spectral condition, since it is very cleverly hidden inside OS positivity.
To Schwinger functions satisfying OS axioms a reconstruction theorem can be applied which is a variant of Wightman reconstruction theorem. Then Schwinger functions are directly related to Wightman functions (whose existence is now guaranteed by such theorem) by analytical continuation[I] in the time variable.

Theorem 5.1 (Osterwalder-Schrader)**.** *Assume that the Schwinger functions $\{S_n(x_1, \dots, x_n)\}_{n=0}^{\infty}$ satisfy the OS axioms. Then they uniquely determine*

(1') a Hilbert space of states $\mathcal{H}$;

(2') a continuous representation $U(a, \Lambda)$ of $\tilde{\mathscr{P}}_+^{\uparrow}$ on $\mathcal{H}$;

(3') a unique vector $|\Omega\rangle \in \mathcal{H}$ invariant under U;

[I] Notice that in order to make an analytical continuation from imaginary time to real time one should prove that there is no obstruction in the complex plane such as poles or cuts.

(4') *a field operator (in the bosonic case)* $\hat{\phi}(\boldsymbol{x})$ *such that, for* $t_1 < t_2 < \ldots < t_{n-1} < t_n$,

$$\begin{aligned} S_n(x_1, \ldots, x_n) &= \langle \Omega | \Big(\prod_{j=1}^{n-1} \hat{\phi}(\boldsymbol{x}_j) e^{-(t_{j+1}-t_j)H} \Big) \hat{\phi}(\boldsymbol{x}_n) \, | \Omega \rangle \\ &= \langle \Omega | \, e^{-t_1 H} \hat{\phi}(\boldsymbol{x}_1) e^{t_1 H} e^{-t_2 H} \hat{\phi}(\boldsymbol{x}_2) e^{t_2 H} \cdots e^{-t_n H} \hat{\phi}(\boldsymbol{x}_n) e^{t_n H} \, | \Omega \rangle \end{aligned} \tag{5.5}$$

where H *is the generator of* $U(a^0)$ *and* $H \geq 0$.[II]

Furthermore such $S_n(x_1, \ldots, x_n)$ *are the restriction of functions* $W_n((z_1, \boldsymbol{x}_1), \ldots, (z_n, \boldsymbol{x}_n))$, *analytic in* $\{z_i\}_{i=1}^n$ *on the domain* $\operatorname{Im} z_i < \operatorname{Im} z_{i+1}$, *to the region* $z_j = it_j$, *for* $j = 1, \ldots, n$. *Formally*[III]

$$S_n(x_1, \ldots, x_n) = W_n((it_1, \boldsymbol{x}_1), \ldots, (it_n, \boldsymbol{x}_n)) \quad \textit{where} \quad x_j = (t_j, \boldsymbol{x}_j) \tag{5.6}$$

The functions $W_n(x_1, \ldots, x_n)$ *are the Wightman functions of a RQFT. In other words the Wightman functions are obtained from the Schwinger functions by analytical continuation in time.*
Properties (5')-(9') in Wightman reconstruction theorem are satisfied.
The Hilbert space $\mathcal{H}$, *the representation* U, *the vacuum* $|\Omega\rangle$ *and the field* $\hat{\phi}$ *constructed in the theorem are the same obtained by Wightman reconstruction starting from the Wightman functions* $\{W_n\}$ *obtained by analytical continuation.*

Notice that the fact that we reconstruct the theory directly from the Euclidean space, without using Wightman functions, turns out to be crucial in different situations, for instance OS reconstruction theorem apply to the quantization of solitons, whereas Wightman reconstruction theorem doesn't.
Moreover we already know that the path integral can be used to build Schwinger functions for the theory regularized by cutoffs (since they are simply vacuum expectation values and we can apply previous formulas, which certainly give well-defined results thanks to the cutoffs we introduced), whereas it is not well defined in the Minkowski space, where Wightman functions are defined.
After the construction of Schwinger functions using the path integral for the regularized theory, one has to check that after the removal of the cutoffs the OS axioms are satisfied.

Proof. (*sketch*) *(1')* Let $\underline{\mathcal{S}}_+$ be the space of finite sequences $\underline{f}$ of test functions with support in the positive Euclidean time half-space. Define an inner product by

$$(\underline{f}, \underline{g}) := \sum_{j,k=0}^{\infty} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \mathrm{d}y_1 \ldots \mathrm{d}y_k \, f_j^*(rx_1, \ldots, rx_j) S_{j+k}(x_1, \ldots, x_j, y_1, \ldots, y_k) g_k(y_1, \ldots, y_k) \tag{5.7}$$

which is semi-definite positive by OS positivity. Denoting by θ the convolution, called *OS reflection*, given by the reflection respect to $\tau = 0$ hyperplane together with the complex conjugation, we can rewrite eq. (5.7) in compact notation by $\langle f \theta g \rangle$, where the average is computed using the Schwinger functions.
Then the construction of $\mathcal{H}$ is performed like in the Wightman reconstruction theorem:

$$\mathcal{H} := \underline{\mathcal{S}}_+ / \mathcal{N} \quad \text{where} \quad \mathcal{N} = \{ \underline{f} \in \underline{\mathcal{S}}_+ \mid \|\underline{f}\| = 0 \} \tag{5.8}$$

We denote by $|\underline{f}\rangle$ the vectors in $\mathcal{H}$ corresponding to $\underline{f}$.
(2') Given a sequence $\underline{f}$ let $\underline{f}_t$ be the sequence with all arguments translated in the Euclidean time by $t > 0$ and $\underline{f}_{\boldsymbol{a},R}$ the sequence with arguments rotated by R and then translated spatially by $\boldsymbol{a}$. Set also the compact notation $\underline{f}_{\boldsymbol{a}} := \underline{f}_{\boldsymbol{a},\mathbb{1}}$.
Spatial roto-translations are well-defined by the same arguments that we used for $U(a, \Lambda)$ in the Wightman reconstruction theorem, with a unitary operator $U(\boldsymbol{a}, R)$ such that:

$$(\underline{f}, \underline{g}_{\boldsymbol{a},R}) = (\underline{f}_{(\boldsymbol{a},R)^{-1}}, \underline{g}) = (\underline{f}, U(\boldsymbol{a}, R)\underline{g}) \quad \text{and} \quad (\underline{f}, \underline{g}_{\boldsymbol{a}}) = (\underline{f}_{-\boldsymbol{a}}, \underline{g}) = (\underline{f}, e^{i\hat{\boldsymbol{P}}\boldsymbol{a}}\underline{g}) \tag{5.9}$$

[II] H is semidefinite positive, so it is reasonable that Schwinger functions can be analytically continued to the real time obtaining Wightman functions, since this provides the right sign inside the time evolution exponentials.
[III] Notice that the relation between Wightman functions and Schwinger functions is very similar to the relation between retarded correlators and Matsubara functions.

Different arguments are required for time translations, since restricting ourself to $\tau > 0$ we broke time translational invariance. We want to prove that

$$(\underline{f}, \underline{g}_t) = (\underline{f}_t, \underline{g}) = (\underline{f}, e^{-tH}\underline{g}) \quad \text{with} \quad H \geq 0 \tag{5.10}$$

The continuity in t of $(\underline{f}, \underline{g}_t)$ is obvious, and

$$(\underline{f}, \underline{g}_t) = (\underline{f}_t, \underline{g}) \tag{5.11}$$

follows directly from the definition eq. (5.7) using eq. (5.2). Let's define the operator $P_t : \underline{g} \mapsto \underline{g}_t$, then it is symmetric thanks to eq. (5.11). Moreover

$$\begin{aligned} \|P_t\underline{g}\|^2 = \|\underline{g}_t\|^2 = |(\underline{g}_t, \underline{g}_t)| = |(\underline{g}, \underline{g}_{2t})| &\leq |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}_{2t}, \underline{g}_{2t})|^{1/2} = |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}, \underline{g}_{4t})|^{1/2} \\ &\leq |(\underline{g}, \underline{g})|^{1/2} |(\underline{g}, \underline{g})|^{1/4} |(\underline{g}_{4t}, \underline{g}_{4t})|^{1/4} \\ &\leq \dots \\ &\leq |(\underline{g}, \underline{g})|^{\sum_{n=1}^{N} \frac{1}{2^n}} |(\underline{g}_{2^N t}, \underline{g}_{2^N t})|^{1/2^N} \\ &\leq |(\underline{g}, \underline{g})|^{\sum_{n=1}^{\infty} \frac{1}{2^n}} |(\underline{g}_{\infty}, \underline{g}_{\infty})|^{0} \\ &= |(\underline{g}, \underline{g})| = \|\underline{g}\|^2 \end{aligned} \tag{5.12}$$

where inequalities are due to Schwartz. Due to continuity and $\|P_t\underline{g}\|^2 \leq \|\underline{g}\|^2$ an analogue of Stone's theorem proves the existence of $H \geq 0$ such that $P_t = e^{-tH}$. In this way we proved eq. (5.10).
Moreover since H is positive, this suggest us that analytical continuation of e^{-tH} to e^{-itH} may be possible.
For boosts the situation is much more complicated and it will not be discussed here. We just say that they can be built starting from representations of spatial roto-translations and time translations.
(3') The vacuum $|\Omega\rangle$ is then given (up to quotienting and completion) by $\underline{\mathbb{1}} = (1, 0, \dots, 0, \dots)$. The proof of uniqueness is omitted.
(4') We just sketch how to construct the field $\hat{\phi}$, without proving the well-definiteness and its properties. Define the domain

$$D_+ = \bigcup_{\varepsilon > 0} e^{-\varepsilon H} \underline{\mathcal{S}}_+ / \mathcal{N} \tag{5.13}$$

given by sequences shifted in the positive time direction by arbitrary $\varepsilon > 0$. In such domain we can define the field at time $t = 0$, $\hat{\phi}_0$, as the quadratic form[IV] in $D_+ \times D_+$.[V] In particular, let $\underline{f}, \underline{h} \in D_+$ and $g \in \mathcal{S}(\mathbb{R}^d)$, then the field at $t = 0$ is defined by

$$\langle \underline{f} | \, \hat{\phi}_0(g) \, | \underline{h} \rangle := \big(\underline{f}, (g(\boldsymbol{x}) \otimes \delta(\tau)) \times \underline{h} \big) \tag{5.14}$$

where

$$(g \otimes \delta) \times \underline{h} = (0, (g \otimes \delta) h_0, (g \otimes \delta) h_1, \dots) \tag{5.15}$$

[IV] A *quadratic form* is a quantity defined in matrix elements only when both the vectors are in a specified dense domain. This provides a generalization of the concept of self-adjoint operator in some cases in which an operator would be unbounded.
Indeed, given an operator O and its domain $D(O)$, if $|\psi\rangle \in D(O)$ then $O|\psi\rangle \in \mathcal{H}$ and $\langle\phi| O |\psi\rangle$ is well defined for any $|\phi\rangle \in \mathcal{H}$. Instead, a quadratic form Q has domain $D(Q)$ such that $\langle\phi| Q |\psi\rangle$ is well defined for $|\phi\rangle, |\psi\rangle \in D(Q)$. Comparing the two definitions, the latter is weaker that the former, since in general $Q|\psi\rangle$ is not defined for $|\psi\rangle \in D(Q)$.
For instance, consider the formal map $Q : g(x) \mapsto \delta(x) g(x)$ with the scalar product $(f, Qg) = \int \mathrm{d}x f(x) \delta(x) g(x)$. The multiplication by $\delta(x)$ is not an operator, hence Q can not be defined as an operator, but Q can be defined as a quadratic form. Indeed one should require that the function $g(x)$ is not singular in $x = 0$, but this is not enough, one should also require that the same property is satisfied also by $f(x)$.
This problems can be avoided when one consider a system in QM with potential $V(x) = \delta(x)$. Indeed if we consider the operator $-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ associated to the Schrödinger equation this is well-defined since the ground state of the wave function has a jump in the first derivative, so that the double derivative produces a delta function which cancels with the one produced by the potential.
Reference: [RS80, Section 8.6].

[V] We are forced to use a quadratic form since in general it is not guaranteed that the field exists as an operator at a fixed time. For instance we have seen that this it is not possible if the wave function renormalization is vanishing: $Z = 0$.

This is well-defined since all sequences in D_+ are moved away from the $\tau = 0$ axis, and then the previous object is always finite provided that both $\underline{f}, \underline{g} \in D_+$. Notice that the r.h.s. of eq. (5.14) is exactly the generalization of smeared field

$$\hat{\phi}(g) = \int \mathrm{d}\boldsymbol{x}\, \hat{\phi}(\boldsymbol{x}, 0) g(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\tau\, \hat{\phi}(\boldsymbol{x}, \tau) g(\boldsymbol{x}) \delta(\tau) \tag{5.16}$$

where we defined it as a quadratic function (i.e. using matrix elements) in an appropriate domain such that the delta function gives always a finite contribution.
Notice that for $\tau = 0$ the field is identical to its analytic continuation to the real time, hence $\hat{\phi}_0$ provides a well defined field for $t = 0$ (as quadratic form) where t is the real time. We can then generalize the definition to arbitrary real time just using the unitary time evolution defined in Minkowski, and this can be done in such a way that the final field turns out to be an operator rather than just a quadratic form, just by appropriately smearing in time the resulting field. In particular, let $f \in \mathcal{S}(\mathbb{R}^{d+1})$, $f(\boldsymbol{x}, t) = f_t(\boldsymbol{x})$, be a test function in Minkowski, then the following is a well-defined operator in a dense domain:

$$\hat{\phi}(f) = \int_{-\infty}^{+\infty} \mathrm{d}t\, e^{itH} \int \mathrm{d}\boldsymbol{x}\, \hat{\phi}_0(\boldsymbol{x}) f_t(\boldsymbol{x}) e^{-itH} \tag{5.17}$$

Indeed for any fixed value of t, the function $f_t(\boldsymbol{x})$ is an element of $\mathcal{S}(\mathbb{R}^d)$ such as g in eq. (5.14), such that $\int \mathrm{d}\boldsymbol{x}\, \hat{\phi}_0(\boldsymbol{x}) f_t(\boldsymbol{x})$ is a well defined quadratic form. The function $f(\boldsymbol{x}, t)$ should be seen as a set of functions of $\boldsymbol{x}$ labelled by t, and then smearing in t we promote the quadratic form to an operator defined in a dense domain.
(Locality) We just make a comment about the proof of the locality for the field we just constructed. The proof can be done in analogy to what we have done for the Wightman reconstruction theorem.
Let's consider two points x, y space-like separated, it always exists a boost putting them at equal time and by time translational invariance we can put both of them at $t = 0$. We denote the resulting space-like component by $\boldsymbol{x}'$ and $\boldsymbol{y}'$ respectively, $\boldsymbol{x}' \neq \boldsymbol{y}'$.
Now consider $\underline{f}, \underline{g} \in D_+$ and construct the following matrix elements

$$\langle \underline{f}|\, \hat{\phi}(\boldsymbol{x}', 0) e^{-\varepsilon H} \phi(\boldsymbol{y}', 0)\, |\underline{g}\rangle \quad \text{and} \quad \langle \underline{f}|\, \hat{\phi}(\boldsymbol{y}', 0) e^{-\varepsilon H} \phi(\boldsymbol{x}', 0)\, |\underline{g}\rangle \tag{5.18}$$

for some $\varepsilon > 0$. If one considers the matrix element on the l.h.s., the exponential $e^{-\varepsilon H}$ raised slightly the argument of $\phi(\boldsymbol{y}', 0)\, |\underline{g}\rangle$ above the $t = 0$ hyperplane. Looking at this matrix element from the right to the left, $\phi(\boldsymbol{y}', 0)\, |\underline{g}\rangle$ is well defined since $\underline{g} \in D_+$, but then also $e^{-\varepsilon H} \phi(\boldsymbol{y}', 0)\, |\underline{g}\rangle$ is an element of D_+ due to the effect of $e^{-\varepsilon H}$, and this finally imply that such matrix element is well defined since both the vectors on which the quadratic form $\hat{\phi}(\boldsymbol{x}', 0)$ acts on are elements of D_+. Similarly, also the matrix elements on the r.h.s. is well defined, just inverting the role of $\boldsymbol{x}'$ and $\boldsymbol{y}'$.
But then

$$\lim_{\varepsilon \to 0} \langle \underline{f}|\, \hat{\phi}(\boldsymbol{x}', 0) e^{-\varepsilon H} \phi(\boldsymbol{y}', 0)\, |\underline{g}\rangle = \lim_{\varepsilon \to 0} \langle \underline{f}|\, \hat{\phi}(\boldsymbol{y}', 0) e^{-\varepsilon H} \phi(\boldsymbol{x}', 0)\, |\underline{g}\rangle \tag{5.19}$$

due to symmetry of the Schwinger functions (we are still in the Euclidean space), which implies that, on matrix elements,

$$[\hat{\phi}(\boldsymbol{x}', 0), \hat{\phi}(\boldsymbol{y}', 0)] = 0 \tag{5.20}$$

Now using again boosts (and Lorentz invariance) this vanishing condition apply also for the initial space-like separated points, and locality is proven.
The proof of all the other points of the theorem (e.g. uniqueness of the vacuum and analytical continuation of Schwinger functions) are omitted. □

5.2 Generalizations of OS reconstruction theorem

[Sum16], [GJ87, Section 6.1]

There are several variants of the OS reconstruction theorem, e.g. applied to lattice theories, to fields not supported on points (like Wilson loops in gauge theories, which in abelian case take the form $\exp\left(-\oint_{\mathcal{C}} A\right)$, to finite temperature and also to soliton fields. We'll see in the following some examples. All these variants are based on the same idea, just changing appropriately the OS axioms depending on the situation.
For this reason, let us now pinpoint where the properties of OS axioms have been used:

(1) the OS positivity was used to construct $\mathcal{H}$ and Ω;

(2) (i) time translational invariance was used to construct the time evolution e^{-tH} and the field operator $\hat{\phi}(x)$;

(ii) space roto-translational invariance was used to construct $U(\boldsymbol{a}, R)$;

(iii) Euclidean symmetry was used to construct the full representation of the Poincaré group and, combined with OS positivity, to prove that $\sigma(\hat{P}^\mu) \subseteq \overline{V}^+$;

(3) cluster property was used to ensure the uniqueness of Ω as translational invariant state;

(4) symmetry was used to prove the locality of $\hat{\phi}$.

5.2.1 Lattice and gauge theories

[Frö80, Section 4], [OS78], [Sei82], [Str93, Section 6.4]

In view of the previous remark it is not difficult to understand the variants of OS reconstruction for lattice theories involving also non-local fields, such as the Wilson loop in gauge theories. For abelian theories with gauge potential A_μ the Wilson loop is defined as follows. Let $\mathcal{C}$ be a loop, then the *Wilson loop* is defined by[VI]

$$W(\mathcal{C}) = e^{i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu} \tag{5.21}$$

One can prove that the the collection of $W(\mathcal{C})$ for all loops $\mathcal{C}$ completely determines a gauge theory without matter fields. Therefore Wilson loops are those objects that contain all the information about such gauge theory in a gauge invariant way.
In the non-abelian case we need to take also a path ordering $\mathrm{P}(-)$, which is just the time-ordering for the "time" parametrizing the loop, combined with the trace of the elements of the Lie group[VII]

$$W(\mathcal{C}) = \mathrm{Tr}\,\mathrm{P}\left(e^{i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu}\right) \tag{5.22}$$

Consider a lattice whose sites are labelled by i, j, then for each pair of nearest neighbouring sites (nearest neighbouring sites are denoted by $\langle i, j\rangle$) we can define (for the abelian case)

$$U_{\langle i,j\rangle} := \mathrm{P}\left(e^{i \int_i^j A_\mu \mathrm{d}x^\mu}\right) \tag{5.23}$$

Then given a loop in the lattice let's define

$$W(\mathcal{C}) := \prod_{\langle i,j\rangle \in \mathcal{C}} U_{\langle i,j\rangle} \tag{5.24}$$

such that this object is exactly the lattice regularization of eq. (5.21).
We can now take

$$\langle W(\mathcal{C}_1) \cdots W(\mathcal{C}_n)\rangle \tag{5.25}$$

as the correlator for the lattice theory.
Using such regularization it has been computed (using computers) some very interesting results for QED and QCD, which gave very good approximations of experimental results.

OS axioms for lattice theories

Let $\{O_\alpha(\mathcal{C}_i)\}_{\alpha \in A}$ be lattice fields, each of them with support on a compact connected set of cells $\{\mathcal{C}_i\}_{i=1}^n$ of the lattice (the supports $\mathcal{C}_i$ can be points, lines, surfaces, etc, in the case of Wilson loops the support is given by a closed line). Let's define the correlation functions as expectation values of these fields

$$S_n(\alpha_1, \mathcal{C}_1, \ldots, \alpha_n, \mathcal{C}_n) := \langle O_{\alpha_1}(\mathcal{C}_1) \cdots O_{\alpha_n}(\mathcal{C}_n)\rangle \tag{5.26}$$

[VI] Recall that in QM not only the field strength is an observable. Due to Aharonov-Bohm effect also objects of the form eq. (5.21) are observables.

[VII] In the non-Abelian case A_μ are matrices and so is $e^{i \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu}$. So the trace is defined as usual.

Notice that in general is not required that all the $\{\mathcal{C}_i\}$ are of the same type, for instance we can have both lines and surfaces. Let $F(\{O_{\alpha_i}(\mathcal{C}_i)\}) \in \mathcal{F}$ where $\mathcal{F}$ is the space of functions in these lattice fields with supports $\{\mathcal{C}_i\}$ contained in the positive "Euclidean-time" half-lattice. Let θ denote the OS reflection, i.e. the reflection with respect to the euclidean time 0 space together with (for bosonic fields) complex conjugation. Moreover, set the lattice spacing to 1.

Theorem 5.2 (OS reconstruction theorem for lattice theories)**.** *Given a set of correlation functions* $\{S_n(\alpha_1, \mathcal{C}_1, \dots, \alpha_n, \mathcal{C}_n)\}_{n=0}^{\infty}$ *satisfying*

(1) lattice translational invariance,

(2) OS positivity.

Then one can reconstruct from $\{S_n\}$

(1') a separable Hilbert space $\mathcal{H}$*,*

(2') a vector $|\Omega\rangle \in \mathcal{H}$*, the vacuum,*

(3') a self adjoint transfer matrix $T = e^{-H}$ *and unitary space translation operators* U_μ*,* $\mu = 1, \dots, d$*, acting on* $\mathcal{H}$ [VIII] *such that* $T|\Omega\rangle = |\Omega\rangle$*,* $U_\mu|\Omega\rangle = |\Omega\rangle$*,*

Moreover if cluster property holds, i.e.

$$\begin{aligned}\lim_{a\to\infty} S_n(\alpha_1, \mathcal{C}_1 + a, \dots, \alpha_j, \mathcal{C}_j + a, \alpha_{j+1}, \mathcal{C}_{j+1}, \dots, \alpha_n, \mathcal{C}_n) = \\ = S_j(\alpha_1, \mathcal{C}_1, \dots, \alpha_j, \mathcal{C}_j) S_{n-j}(\alpha_{j+1}, \mathcal{C}_{j+1}, \dots, \alpha_n, \mathcal{C}_n)\end{aligned} \tag{5.27}$$

then

(4') $|\Omega\rangle$ is the unique translationally (both in space and time directions) invariant vector in $\mathcal{H}$,

(5') there is a set of vectors in $\mathcal{H}$, $\{|\alpha_1, \mathcal{C}_1, \dots, \alpha_n, \mathcal{C}_n\rangle\}$ such that the set of their linear combinations is dense in $\mathcal{H}$ and their scalar product is defined analogously to eq. (5.7):

$$\langle \alpha_1, \mathcal{C}_1, \dots, \alpha_m, \mathcal{C}_m | \alpha'_1, \mathcal{C}'_1, \dots, \alpha_n, \mathcal{C}'_n \rangle = \langle O_{\alpha'_1}(\mathcal{C}'_1) \cdots O_{\alpha'_n}(\mathcal{C}'_n) \theta[O_{\alpha_1}(\mathcal{C}_1) \cdots O_{\alpha_m}(\mathcal{C}_m)] \rangle \tag{5.28}$$

(6') there are some "field operators" $\hat{O}_\alpha(\mathcal{C})$, with $\mathcal{C}$ contained in the strip $[0, \tau]$, $\tau \in \mathbb{N} \setminus \{0\}$, such that

$$\hat{O}_\alpha(\mathcal{C}) T(\tau) |\alpha_1, \mathcal{C}_1, \dots, \alpha_n, \mathcal{C}_n\rangle = |\alpha, \mathcal{C}, \alpha_1, \mathcal{C}_1 + \tau, \dots, \alpha_n, \mathcal{C}_n + \tau\rangle \tag{5.29}$$

with $T(\tau) := T^\tau$ representing a time translation by τ time units.

5.2.2 Reconstruction at finite temperature

[BF02], [Frö75]

Suppose that we have a set of Matsubara functions S_n depending on creation and annihilation operators $a^{\#}_{\alpha_i}$, where $^{\#}$ denotes either annihilation a_{α_i} or creation $a^\dagger_{\alpha_i}$ operators, and on euclidean times τ_i, time ordered.
These functions are constructed starting from some expectation values of ladder operators at finite volume V, where we have finite degrees of freedom, and for ordered time parameters $\tau_1 < \tau_2 < \dots$

$$\langle a^{\#}_{\alpha_1}(\tau_1) \cdots a^{\#}_{\alpha_n}(\tau_n) \rangle^\beta_V \tag{5.30}$$

and then taking them thermodynamic limit

$$S^\beta_n(a^{\#}_{\alpha_1}, \tau_1, \dots a^{\#}_{\alpha_n}, \tau_n) := \lim_{V\to\infty} \langle a^{\#}_{\alpha_1}(\tau_1) \cdots a^{\#}_{\alpha_n}(\tau_n) \rangle^\beta_V \tag{5.31}$$

Indeed in general functions S^β_n cannot be defined as expectation values of operators defined in the infinite volume limit since such operators cannot exists due to Haag's theorem.
Due to usual boundary conditions operators a_{α_i} should be thought as operators defined on a circle of circumference β in euclidean time.

[VIII] The matrix T describes the time evolution by a unit of time, while U_μ describe a space translation o f a unit of space in the direction given by μ.

Theorem 5.3 (OS reconstruction theorem for finite temperature)**.** *Suppose that functions* $\{S_n^\beta\}_{n=0}^\infty$ *satisfy the following properties* ($\tau_1 < \tau_2 < \ldots$)*:*

(1) Regularity *conditions in* a_{α_i} *and* $\{\tau_i\}$*.*

(2) Time translational invariance

$$S_n^\beta(a_{\alpha_1}^\#, \tau_1, \ldots a_{\alpha_n}^\#, \tau_n) = S_n^\beta(a_{\alpha_1}^\#, \tau_1 + \tau, \ldots a_{\alpha_n}^\#, \tau_n + \tau) \tag{5.32}$$

for arbitrary $\tau \in \mathbb{R}$*.*

(3) KMS condition

$$S_n^\beta(a_{\alpha_1}^\#, \tau_1, \ldots a_{\alpha_n}^\#, \tau_n) = S_n^\beta(a_{\alpha_{j+1}}^\#, \tau_{j+1}, \ldots, a_{\alpha_n}^\#, \tau_n, a_{\alpha_1}^\#, \tau_1 + \beta, \ldots, a_{\alpha_j}^\#, \tau_j + \beta) \tag{5.33}$$

for any $j = 1, \ldots, n-1$*.*

(4) OS positivity*: Consider all polynomials in* $\{a_i^\#(\tau_i)\}$ *variables,* $\mathcal{P}(\{a_i^\#(\tau_i)\})$*, such that the time variables are ordered and bounded by* $\beta/2$*, that is*

$$\tau_1 < \tau_2 < \ldots < \tau_n < \beta/2 \tag{5.34}$$

For such polynomial we define the convolution θ *by*[IX]

$$\theta\mathcal{P}(\{a_i^\#(\tau_i)\}) = \mathcal{P}(\{a_i^\#(\beta - \tau_i)\})^\dagger \tag{5.35}$$

with again $\beta - \tau_i$ *ordered. Then the OS positivity is the requirement*

$$\big\langle \mathcal{P}(\{a_i^\#(\tau_i)\})\theta\mathcal{P}(\{a_j^\#(\tau_j)\})\big\rangle \geq 0 \tag{5.36}$$

for any couple of these polynomials, when the expectation value $\langle\ \rangle$ *is written in terms of* $\{S_n^\beta\}$*.*

Then the $\{S_n^\beta\}$ *uniquely determine*

(1') *a separable Hilbert space* $\mathcal{H}_\beta$*,*

(2') *a vector* $\Omega_\beta \in \mathcal{H}_\beta$*,*

(3') *a continuous unitary one-parameter group* $\{e^{itL}, t \in \mathbb{R}\}$ *(*t *is now the real time, not the Euclidean one) leaving* Ω_β *invariant,*

(4') *a representation in terms of operators in* $\mathcal{H}$ *of the algebra of* $a_{\alpha_i}^\#$*,* $\Pi(a_{\alpha_i}^\#)$*, such that the functions* S_n^β *are obtained from the real-time "Wightman" functions*

$$W_n(a_{\alpha_1}^\#, t_1, \ldots, a_{\alpha_n}^\#, t_n) = \Big\langle \prod_{j=1}^n e^{it_j L}\Pi(a_{\alpha_i}^\#)e^{-it_j L}\Omega_\beta \Big| \Omega_\beta \Big\rangle \tag{5.37}$$

(the expression in the r.h.s. denotes the scalar product between elements of $\mathcal{H}_\beta$*) by analytic continuation in the time variables* $t_1, \ldots, t_n$ *to the domain* $(z_1, \ldots, z_n)$*,* $z_i \in \mathbb{C}$*, with* $\operatorname{Im} z_1 < \operatorname{Im} z_i < \operatorname{Im} z_{i+1} < \operatorname{Im} z_n < \operatorname{Im} z_1 + \beta$*, and then restricting* $(z_1, \ldots, z_n)$ *to* $(i\tau_1, \ldots, i\tau_n)$ *with* $0 < \tau_i < \tau_{i+1} < \beta$*.*

From the real-time "Wightman" functions one can of course construct the retarded Green functions just using $\theta(t)$ and (anti)commutators.

[IX] The definition of such convolution is just what we expect adapting the definition of θ that we gave for $T = 0$. First it reflects the time variables in the circle of circumference β respect to the line passing through $\tau = 0$ and $\tau = \beta/2$. Then takes the adjoint of the resulting polynomial, such that the order of $a^\#$ is inverted and each $a^\#$ is substituted with the corresponding adjoint. In some sense this is the appropriate adjustment of the reflection $\tau \to -\tau$.

A comment on the Hilbert space $\mathcal{H}_\beta$

For finite volume $\mathcal{H}_\beta$ constructed in the proof of theorem 5.3 is the Hilbert space of operators A such that $\operatorname{Tr} A^\dagger A < \infty$ with inner product and vacuum defined by

$$\langle A|B\rangle := \operatorname{Tr} A^\dagger B \quad \text{and} \quad |\Omega_\beta\rangle := \frac{|e^{-\beta H/2}\rangle}{\sqrt{\operatorname{Tr} e^{-\beta H}}} \tag{5.38}$$

where the denominator in the definition of $|\Omega_\beta\rangle$ appropriately normalizes the state to 1.
Heuristically, the reason beside this strange structure of the Hilbert space is due to the fact that in finite temperature we are not considering an isolated system anymore, but a system in a thermal bath. Such thermal bath is an infinite volume system, and now we have to take care of the fluctuations coming from it. These fluctuations induces an infinite renormalization of the energy, so if one proceeds as before defining the Hamiltonian starting from the temporal evolution would obtain a divergent result in the infinite volume limit. Therefore one should define a new operator H in such a way that fluctuations coming from the thermal bath are killed. This turns out to be impossible using a Fock space, whereas is possible if one uses such strange Hilbert space.
This is also related to the fact that in statistical mechanics the Hamiltonian of a system in a thermal bath cannot be positive definite, so one has to consider the Liouvillian L instead of the Hamiltonian H. In this case these objects are just related by a renormalization constant.
More formally, in $\mathcal{H}_\beta$ at finite volume we have two commuting representation of observables $A(\{a_\alpha^\#\})$:

$$\Pi(A)\,|B\rangle = |AB\rangle \quad \text{and} \quad \Pi_r(A)\,|B\rangle = |BA^\dagger\rangle \tag{5.39}$$

where the first one is the one used in the statement of the theorem. It turns out that the generator of time evolution is

$$L = \Pi(H) - \Pi_r(H) \tag{5.40}$$

The subtraction of $\Pi_r(H)$, since Π_r and Π commutes, does not change the equations of motion of the observable in Π. Indeed $\Pi_r(H)$ acts as a constant, and cancel the infinities appearing in H in the infinite volume limit.
Let us see the relation with the standard formalism with the Boltzmann weight $e^{-\beta H}$. For a finite system at finite T we have two equivalent descriptions: in the Fock space $\mathscr{F}$ the equilibrium state is a mixed state of the form

$$\rho = \frac{e^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \tag{5.41}$$

whereas in the Hilbert space $\mathcal{H}_\beta$ defined above the equilibrium state Ω_β is a vector of the form

$$|\sqrt{\rho}\rangle = \frac{|e^{-\beta H/2}\rangle}{\sqrt{\operatorname{Tr} e^{-\beta H}}} \tag{5.42}$$

(instead of a mixed state) which is a eigenvector of $L = \Pi(H) - \Pi_r(H)$ of zero eigenvalue, indeed using eq. (5.39) ad eq. (5.40) one gets

$$L\,|\sqrt{\rho}\rangle = \frac{|He^{-\beta H/2}\rangle - |e^{-\beta H/2}H^\dagger\rangle}{\sqrt{\operatorname{Tr} e^{-\beta H/2}}} \overset{H=H^\dagger}{=} 0 \tag{5.43}$$

since the two vectors in the numerator cancel each other.
Notice that as $V \to \infty$ the expectation value of H in ρ diverges

$$\frac{\operatorname{Tr} He^{-\beta H}}{\operatorname{Tr} e^{-\beta H}} \to \infty \tag{5.44}$$

since as we said before in the thermodynamic limit $V \to \infty$ the Hamiltonian H should take into account the infinite fluctuations coming from the thermal bath. Conversely (the symbol $(\bullet)$ should be substituted by some operator)

$$\langle \Omega_\beta|\,(\bullet)\,|\Omega_\beta\rangle_V = \frac{\operatorname{Tr}_V (\bullet) e^{-\beta H}}{\operatorname{Tr}_V e^{-\beta H}} \tag{5.45}$$

provides an expectation value, with respect to the vector $|\Omega_\beta\rangle \in \mathcal{H}_\beta$, well defined even in the infinite volume limit (for operators in an appropriate domain, from which H is excluded due to eq. (5.44)) which still satisfies $L\,|\Omega_\beta\rangle = 0$ (thanks to eq. (5.43)) and the KMS condition.

Summary

The procedure to construct retarded Green functions is the following: first one constructs the Fock space for the theory at finite temperature appropriately regularized (lattice and finite volume regularization), then one computes the Matsubara Green functions in the Fock space using the path integral method and assumes that the thermodynamic limit exists. If Matsubara functions in the thermodynamic limit exist and satisfy the axioms then one can reconstruct completely the theory even in the thermodynamic limit. Obviously the Hilbert space we constructed has nothing to do with the original Fock space, and in particular is not the generalization of the original Fock space but instead is the generalization of the finite volume Hilbert space $\mathcal{H}_\beta$, which is a space of operators. By analytical continuation one defines an analogue of Wightman functions and combining them with Heaviside functions and commutators one finally gets the desired retarded Green functions.

Part II

Topological structures in QFT

Chapter 6

Quantum solitons

6.1 Introduction to topological objects

The first appearance of a classical *soliton* was in a report by J. Scott Russel in 1842, he was in a boat in a channel and noticed "a solitary elevation, a rounded, smooth and well-defined heap of water which continued in course along the channel apparently without change of form (i.e. without dispersion/dissipation) or diminution of speed (i.e. constant velocity)".
The term "soliton" (which comes from "solitary wave") was much later introduced to characterize solutions of wave equations that do not dispense and preserve their form during the propagation. Hence in a sense a classical soliton behaves as a particle in spite of being a wave.
The stationary nature of these "wave" solitons is due to some conservation law. If this conservation law is of topological origin (i.e. the conserved quantity is not related to the (first) Noether theorem and the conservation holds without using the equations of motion) these solitons are called *topological*.
The original concept of topologically protected "wave" solitons has been later extended, in particular by high-energy physicists, and today we define a classical topological soliton as a topologically stable, finite energy solution of the Hamiltonian equations of motion of a classical field theory.
Since already classically solitons behaves as particles, one can naturally expect, correctly, that their quantized version provides a quantum particle, but as we will see later it is a peculiar one.

6.2 Spontaneously broken symmetry

[Str85, Chapter C.1], [Str12], [Shi12, Chapter 1]

Before entering in the discussion of solitons, a brief formal discussion of the spontaneously broken symmetry (SSB) phenomenon is required.
We already know that the *observables* of a system are the quantities of the physical system that we can measure and the algebra that they generate is called the *observable algebra* $\mathcal{A}$. The *states* contain the information on the system, if the information is maximal they are called *pure*, if it is not maximal are said *mixed*.
By definition pure states can be obtained one from the other by operations physically performed on the system (including limiting procedure, such as the infinite volume limit or the thermodynamic limit, but excluding transformations which require to cross would-be states of infinite energy). Both in the classical and quantum settings the mixed states can be viewed as complex combinations of pure states and a state is pure if it cannot be written as convex combination of the pure states. For equilibrium states at finite temperature by analogy we introduce the concept of *pure phase*: an equilibrium state is a pure phase if it cannot be written as a convex combination of other equilibrium states.

For instance consider the Ising model: take a finite lattice of volume V, where to each cell i is associated a classical spin σ_i with possible values ± 1. Suppose that we imposed vanishing boundary conditions, i.e. outside the volume we do not have any spin (the system is confined), and try to take the infinite volume limit. For spatial dimension $d = 2$ and temperature $T < T_c$ where T_c is the *critical temperature*

the system has two possible ground states: for the first one $\langle\sigma_i\rangle > 0$ for all the cells, for the second one $\langle\sigma_i\rangle < 0$. Obviously, the equilibrium state for the system in such condition is described by $\langle\sigma_i\rangle = 0$ for all the cells, and one can show that it is not a pure phase as it can be described as a convex combination of the two ground states, which in turn are the pure phases.

We can now distinguish different kinds of symmetries. An *algebraic symmetry* is an invertible algebra homomorphism $\mathcal{A} \to \mathcal{A}$ (i.e. a map preserving the algebraic relations, e.g. equations of motion are preserved in form and also (in quantum setting) the commutation relations between observables). A *physical symmetry* is an algebraic symmetry together with an invertible map $\mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is the space of pure states, which preserves the expectation values:

$$\begin{array}{rl} \alpha : \mathcal{A} \longrightarrow \mathcal{A} \\ A \longmapsto A' \end{array} \quad \text{and} \quad \begin{array}{rl} \tilde{\alpha} : \mathcal{S} \longrightarrow \mathcal{S} \\ \Sigma \longmapsto \Sigma' \end{array} \tag{6.1}$$

such that

$$\langle A\rangle_\Sigma = \langle A'\rangle_{\Sigma'} \tag{6.2}$$

A *dynamical symmetry* is either an algebraic or a physical symmetry leaving the Hamiltonian invariant

$$H \longmapsto H' = H \tag{6.3}$$

A dynamical symmetry is said *spontaneously broken* if it cannot be realized as a physical symmetry, i.e. is an algebraic symmetry leaving H invariant but it does no exists a map in the space of pure states of the system preserving the expectation values.

Naively it seems impossible to define a spontaneously broken symmetry, as for each map $\alpha : \mathcal{A} \to \mathcal{A}$ always exists a map $\tilde{\alpha} : \mathcal{S} \to \mathcal{S}$ defined by

$$\langle \alpha^{-1}(A)\rangle_\Sigma = \langle A\rangle_{\tilde{\alpha}(\Sigma)} \tag{6.4}$$

which clearly implies

$$\langle A\rangle_\Sigma = \langle \alpha(A)\rangle_{\tilde{\alpha}(\Sigma)} = \langle A'\rangle_{\Sigma'} \tag{6.5}$$

What goes wrong is that such map $\tilde{\alpha} : \Sigma \mapsto \Sigma'$ in principle may send the space $\mathcal{S}$ not in itself, i.e. $\tilde{\alpha}(\Sigma) \notin \mathcal{S}$, but in another space of states not realizable by acting on $\mathcal{S}$ with physical operations.

Take for instance the Heisenberg model of a ferromagnet in three dimension, where the spin is classically described by a unit vector in a lattice. Let's then consider the thermodynamic limit. It is well known that such system has a critical temperature T_c, such that for $T < T_c$ the ground states have an expectation value $\langle \boldsymbol{S}_i\rangle \neq 0$. For $T = 0$ in the ground state all the spin are aligned in the same direction. The Hamiltonian for this system is

$$H = J \sum_{\langle i,j\rangle} (\boldsymbol{S}_i - \boldsymbol{S}_j)^2 \tag{6.6}$$

which is clearly invariant under rotations. Consider two ground states at $T = 0$ which differs for the orientation of the spins, and let's see if these could be related by some allowed transformation or not, if not we would have a spontaneous symmetry breaking. We are allowed to do only physical (in particular, local) transformations, at least by some limiting procedure. Nevertheless we already took the thermodynamic limit, therefore we are not allowed to move all the spins (including those at infinity) at the same time with a single transformation. We are forced to use some limiting procedure, for instance we can consider neighbourhoods of circles of radius R and smoothly rotate the spins in these regions, while we increase R from 0 to ∞. But as R approaches ∞ there are infinitely many spins rotated simultaneously, hence due to the non trivial interacting energy between spins, one can expect that such procedure requires an infinite amount of energy. If the explicit computation of the energy actually give us an infinite energy cost, then spontaneous symmetry breaking occured.
Notice that in general one cannot guess whether we have SSB or not, a computation of the energy (or some other relevant physical quantity) is needed.

Unfortunately sometimes one refers to SSB in the case in which the Hamiltonian is invariant under a symmetry α, but each of its pure equilibrium states are not invariant under $\tilde{\alpha}$, hence restricting the non-invariance to equilibrium states.
For field theories the two definitions are basically equivalent from the practical point of view but for systems with finite degrees of freedom they are not. For instance consider a horizontal plane where gravitational force applies in the vertical direction, and put a point particle at rest on the plane, then certainly this describes an equilibrium state. The Hamiltonian is invariant under horizontal space translations, nevertheless the equilibrium state is not, hence according to the latter definition the symmetry is spontaneously broken. On the other side, certainly this is not a SSB according to the first definition, because we can easily move to another equilibrium state by some physically allowed translation, since all the intermediate states are pure states of the system.

Chapter 7

Kinks

[Shi12, Chapter 2], [Gou+12], [Raj82, Chapters 2, 5, 8]

The fist (topological) soliton that we consider is the *kink* in ϕ_2^4, i.e. the relativistic ϕ^4 theory in 1+1 dimensions in the spontaneously broken phase.

7.1 Classical treatment

The classical Hamiltonian we consider is on the real line $\mathbb{R}$, labelled by x, and if the system is described by a real scalar field $\phi(x,t)$ with conjugate momentum $\pi(x,t)$ it reads

$$H = \int \mathrm{d}x\, \mathcal{H}(\pi,\phi) = \int \mathrm{d}x\, \left[\frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \frac{g^2}{4}\big(\phi^2 - v^2\big)^2\right] \tag{7.1}$$

with $v, g \in \mathbb{R}$. The potential appearing here is the field analogue of the double well with potential $\frac{g^2}{4}(x^2 - v^2)^2$. The equations of motion in Hamiltonian formalism are

$$\begin{aligned} \dot\phi &= \frac{\delta H}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi} = \pi \\ \dot\pi &= \frac{\delta H}{\delta \phi} = \frac{\partial \mathcal{H}}{\partial \phi} = -\frac{\partial^2}{\partial x^2}\phi + g^2\big(\phi^2 - v^2\big)\phi \end{aligned} \tag{7.2}$$

where for the second equation we used an appropriate integration by parts in H. Equilibrium states are given by the equilibrium condition $\dot\phi = \dot\pi = 0$, which implies (for each $x \in \mathbb{R}$)

$$\pi = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2}\phi = g^2\big(\phi^2 - v^2\big)\phi \tag{7.3}$$

Clearly, from the structure of the Hamiltonian, finite energy solutions must satisfy

$$\pi \to 0 \quad \text{and} \quad \phi \to \pm v \quad \text{for} \quad x \to \pm\infty \tag{7.4}$$

as otherwise H is divergent. Indeed all the terms inside H are strictly positive, hence both the term depending on π and the one depending on ϕ should vanish identically at infinite. The absolute minima of H are given by the two configurations

$$\phi_+^0 = +v \quad , \quad \pi^0 = 0 \quad \text{and} \quad \phi_-^0 = -v \quad , \quad \pi^0 = 0 \tag{7.5}$$

In such cases H vanishes, hence both these solutions are called *vacuum*. The expectation values of the field at the minima are

$$\langle\phi\rangle_{\phi_\pm^0,\pi^0} = \pm v \tag{7.6}$$

The Hamiltonian is invariant under the symmetry α

$$\alpha : (\phi,\pi) \mapsto (-\phi,-\pi) \tag{7.7}$$

We see that in order to satisfy

$$\langle\phi\rangle_\Sigma = \langle\alpha(\phi)\rangle_{\tilde{\alpha}(\Sigma)} \tag{7.8}$$

for $\Sigma = (\phi^0_\pm, \pi^0)$ we should have

$$\pm v = \langle -\phi\rangle_{\tilde{\alpha}(\phi^0_\pm, \pi^0)} \tag{7.9}$$

which is satisfied for $\tilde{\alpha}(\phi^0_\pm, \pi^0) = (\phi^0_\mp, \pi^0)$. However it is impossible to reach continuously ϕ^0_- from ϕ^0_+ and vice versa, since the set of boundary conditions at infinity, $\{+v, -v\}$, is not connected and to reach one minimum from the other with (the limit of) local transformations we need to cross infinite energy states. Therefore symmetry eq. (7.7) is spontaneously broken, and the two minima that we found live in disjoint spaces of states related by a parity transformation.

The kink solution

However there are two local minima in the space of finite-energy solutions: we can find them with the following trick. For $\pi = \dot{\phi} = 0$ the equation of motion reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi = g^2(\phi^2 - v^2)\phi \tag{7.10}$$

where now ϕ depends only on x. If we think about x as a time coordinate and ϕ as the position q of a particle, then eq. (7.10) become a Newton equation with potential $V'(q) = -\frac{g^2}{4}(q^2 - v^2)^2$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi = g^2(\phi^2 - v^2)\phi \quad \leftrightarrow \quad \ddot{q} = -\frac{\mathrm{d}V'}{\mathrm{d}q} \tag{7.11}$$

Due to energy conservation we get

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 - \frac{g^2}{4}(\phi^2 - v^2)^2 = \text{const (in } x) \overset{(7.4)}{=} 0 \quad \leftrightarrow \quad \frac{1}{2}\dot{q}^2 + V'(q) = \text{const (in } t) \tag{7.12}$$

Hence we whould solve the differential equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \mp\frac{g}{\sqrt{2}}(\phi^2 - v^2) \tag{7.13}$$

These are standard first order equations trivially solved by

$$\int \mathrm{d}x = \mp\frac{\sqrt{2}}{g}\int \frac{\mathrm{d}\phi}{\phi^2 - v^2} \tag{7.14}$$

which have solutions

$$x = \pm\frac{\sqrt{2}}{gv}\,\mathrm{arctanh}\left(\frac{\phi}{v}\right) + x_0 \tag{7.15}$$

i.e.

$$\phi^S_\pm(x) = \pm v\tanh\left(\frac{gv}{\sqrt{2}}(x - x_0)\right) \tag{7.16}$$

where we denoted the two solutions by S, which means "soliton". In fig. 7.1 one can find a plot of the function $\phi^S_+(x)$. The value x_0 is called *collective coordinate* or *modulus* of the soliton, called (classical) *kink* ($\phi^S_+(x)$) or (classical) *anti-kink* ($\phi^S_-(x)$).
We see that ϕ^S_+, ϕ^S_-, ϕ^0_+ and ϕ^0_- cannot be deformed one into other by operations physically implementable, since the boundary conditions at ∞ are disjoint. This statement can be translated for the solitons in the conservation of a charge

$$Q = \int \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\phi = \phi(+\infty) - \phi(-\infty) \tag{7.17}$$

which is topological (only the behaviour of ϕ at infinity is relevant). The corresponding conserved current is

$$J_\mu(x) = \epsilon_{\mu\nu}\partial^\nu\phi \tag{7.18}$$

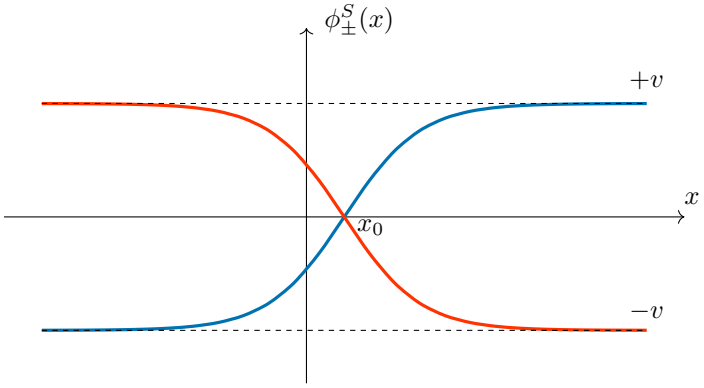


Figure 7.1: Shape of the solutions $\phi^S_+(x)$ (blue) and $\phi^S_-(x)$ (red).

with $\mu, \nu = 0, 1$ where $\mu = 0$ is the time t component and $\mu = 1$ is the spatial x component, moreover ϵ is the Levi-Civita tensor. We can easily see that J_0 actually coincides with the conserved density associated to (7.17):

$$\int \mathrm{d}x\, J_0 = \int \mathrm{d}x\, \epsilon_{0\nu}\partial^\nu \phi = \int \mathrm{d}x\, \partial^1 \phi = \int \mathrm{d}x\, \frac{\partial}{\partial x}\phi = Q \tag{7.19}$$

while J_1 is the unique component of the vector current. The current J_μ is automatically conserved without using the equations of motion, since the conservation directly follows from the contraction of a symmetric tensor with an antisymmetric one

$$\partial^\mu J_\mu = \epsilon_{\mu\nu}\partial^\mu\partial^\nu\phi = \epsilon_{[\mu\nu]}\partial^{\{\mu}\partial^{\nu\}}\phi = 0 \tag{7.20}$$

therefore the kink in this model is a topological soliton.

The Bogomol'nyi bound

There is a more instructive way to prove that $\phi^S_\pm$ are the local minima at $(\mp v, \pm v)$ boundary conditions (b.c.). Let's write the potential $V(\phi) := \frac{g^2}{4}(\phi^2 - v^2)^2$ in terms of a function $\mathcal{W}(\phi)$ called *superpotential* (since it is largely used in supersymmetric theories)

$$V(\phi) = \frac{1}{2}\left(\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi}\right)^2 \tag{7.21}$$

which implies that $\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi} = \frac{g}{\sqrt{2}}(\phi^2 - v^2)$ and then

$$\mathcal{W} = \frac{g}{\sqrt{2}}\left(\frac{1}{3}\phi^3 - v^2\phi\right) \tag{7.22}$$

The Hamiltonian can be rewritten for $\pi = 0$ as

$$H = \int \mathrm{d}x \left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \pm \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi}\right)^2 \mp \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi}\right] \quad \text{for b.c. } (\mp v, \pm v) \tag{7.23}$$

Now notice that the last term gives

$$\int \mathrm{d}x\, \frac{\mathrm{d}\phi}{\mathrm{d}x}\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi} = \mathcal{W}(\phi(+\infty)) - \mathcal{W}(\phi(-\infty)) = \pm\Delta\mathcal{W} \quad \text{for b.c. } (\mp v, \pm v) \tag{7.24}$$

with

$$\Delta\mathcal{W} := \mathcal{W}(v) - \mathcal{W}(-v) = -\frac{4}{3\sqrt{2}}gv^3 < 0 \tag{7.25}$$

hence we get that

$$H \geq -\Delta\mathcal{W} > 0 \tag{7.26}$$

for both the possible boundary conditions. This lower bound is called *Bogomol'nyi bound* (as we will see has several generalizations). This bound is saturated, hence the equality in eq. (7.26) holds, only if the term inside square brackets in eq. (7.23) vanishes, i.e. if the following first order equation is satisfied

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \mp \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}\phi} \quad \text{for b.c. } (\mp v, \pm v) \tag{7.27}$$

which coincides exactly with eq. (7.13). Hence the solutions $\phi^S_\pm$ are local minima for which the energy coincides with the Bogomol'nyi bound, notice also that such solutions are the ones with minimal energy given the b.c. $(\mp v, \pm v)$.

Mass and position of the kink

The energy density associated to $\phi^S_\pm(x_0)$, depending on the modulus x_0, is given by the Hamiltonian density evaluated for the field $\phi^S_\pm(x_0)$ (it obviously depends on the spatial coordinate x and on the choice of x_0):

$$\mathcal{E}^{\mathrm{cl}} = \frac{1}{2}\left(\frac{\mathrm{d}\phi^S_\pm}{\mathrm{d}x}\right)^2 + \frac{g^2}{4}\left(\phi^{S\,2}_\pm - v^2\right)^2 \overset{(7.13)}{=} 2\cdot\frac{1}{2}\left(\frac{\mathrm{d}\phi^S_\pm}{\mathrm{d}x}\right)^2 = \left(\frac{\mathrm{d}\phi^S_\pm}{\mathrm{d}x}\right)^2 \tag{7.28}$$

In fig. 7.2 one can see the shape of $\mathcal{E}(x)$ associated to the field of fig. 7.1. It is clearly localized near x_0. Hence the soliton exhibits an energy density profile as a " dump" around x_0, thus behaving "like a particle" localized around x_0. The integral

$$M^{\mathrm{cl}}_S := \int \mathrm{d}x\, \mathcal{E}^{\mathrm{cl}}(x) = -\Delta\mathcal{W} = \frac{4}{3\sqrt{2}} g v^3 \tag{7.29}$$

can be interpreted as the "classical" mass of the soliton, i.e. the energy of the soliton in its rest frame.

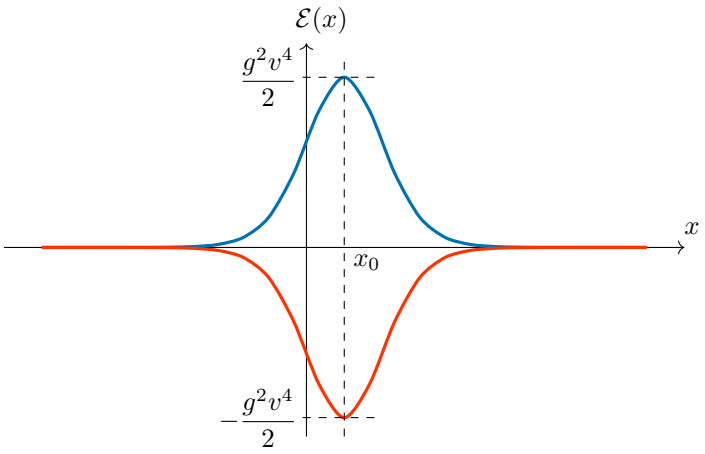


Figure 7.2: Shape of the energy density for the fields $\phi^S_+(x)$ (blue) and $\phi^S_-(x)$ (red) plotted in fig. 7.1 (the x-axis has not been rescaled).

Since (looking at the Hamiltonian) the theory is Lorentz invariant (on 1+1 dimension) one can "change the reference frame" and make the soliton move at a constant velocity v:

$$\phi^S_\pm(x, x_0, \mathrm{v}) = \pm v \tanh\left(\frac{gv}{\sqrt{2}}\,\frac{x - x_0 - \mathrm{v}t}{\sqrt{1-\mathrm{v}^2}}\right) \tag{7.30}$$

Then the kink behaves like a particle with equation of motion $x = x_0 + \mathrm{v}t$.

7.2 Semi-classical treatment

Up to now we just described the soliton at classical level. Let us turn to the quantum world. We first give a standard semi-classical (heuristic) Hamiltonian treatment, then in section 7.4 we'll make some comments on a rigorous approach in the spirit of the reconstruction theorem previously discussed.

Semi-classical description of fluctuations around the vacuum

Let us start from the quantization of fluctuations around one vacuum. To quantize perturbatively fluctuations around $\phi^0_+ = v$ we rewrite the quantum field as

$$\hat{\phi}(x) = v + \hat{\chi}(x) \tag{7.31}$$

Formally (in principle regularizations are needed) inserting the previous expansion in the Hamiltonian we get

$$\begin{aligned} H &= \int \mathrm{d}x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\frac{\mathrm{d}\hat{\chi}}{\mathrm{d}x}\right)^2 + \frac{g^2}{4}\left(2v\hat{\chi} + \hat{\chi}^2\right)^2 \right] \\ &= \int \mathrm{d}x \left[\frac{1}{2}\hat{\pi} + \frac{1}{2}\left(\frac{\mathrm{d}\hat{\chi}}{\mathrm{d}x}\right)^2 + \frac{2(gv)^2}{2}\hat{\chi}^2 + \mathcal{P}(\hat{\chi}) \right] \end{aligned} \tag{7.32}$$

where $\mathcal{P}(\hat{\chi})$ is a polynomial in $\hat{\chi}$ of order higher than 2. Integrating by part we get

$$H_{\leq 2} = \int \mathrm{d}x \left[\frac{1}{2}\hat{\pi} + \frac{1}{2}\hat{\chi}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(gv)^2\right)\hat{\chi} \right] \tag{7.33}$$

($H_{\leq 2}$ is obtained from H neglecting $\mathcal{P}(\hat{\chi})$). One can then expand $\hat{\chi}$ in terms of the complete set of solutions $\{\chi_k(x) = e^{ikx}\}$ of the differential equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2(gv)^2\right)\chi_k = \omega_k^2\,\chi_k \tag{7.34}$$

with eigenvalues

$$\omega_k^2 = k^2 + 2(gv)^2 \tag{7.35}$$

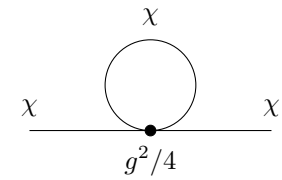
so that one can write

$$\hat{\chi}(x) = \sum_k \hat{a}_k\,\chi_k(x) \tag{7.36}$$

From the quadratic terms of H we see that $\hat{\chi}$ is a massive field with (bare) mass $m = \sqrt{2}gv$, hence we recover the usual dispersion relation $\omega_k = \sqrt{k^2 + m^2}$. Self-interaction are given by $\mathcal{P}(\hat{\chi})$.
Perturbatively we impose the CCR on $\hat{\chi}$ and $\hat{\pi} = \dot{\hat{\chi}}$

$$[\hat{\chi}(x), \hat{\pi}(y)] = [\hat{\chi}(x), \dot{\hat{\chi}}(y)] = i\hbar\,\delta(x-y) \tag{7.37}$$

and in order to make sense to the previous discussion we insert a normal ordering in the Hamiltonian, so that the free H_2 Hamiltonian has zero vacuum energy (otherwise it would be divergent) and then proceed in the usual way.
Clearly in perturbation theory the bare mass of $\hat{\chi}$ is renormalized by the polynomial interaction, and the relevant term, coming from $:\hat{\chi}^4:$, is the counterterm corresponding to the following Feynman graph

$(-1)\cdot$ 

$$= -4\cdot 3\cdot \frac{g^2}{4}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\hbar}{p^2+m^2} = -\frac{3g^2}{4\pi}\int_0^\infty \mathrm{d}p \frac{\mathrm{d}}{\mathrm{d}p}\log\left(p^2+m^2\right) \tag{7.38}$$

where $4\cdot 3$ is a combinatorial factor[I], and from now on in this section we set $\hbar = 1$. Introducing a cutoff $\Lambda \gg m$ we get renormalized mass

$$m_R^2 = m^2 - \frac{3g^2}{2\pi}\log\left(\frac{\Lambda^2}{m^2}\right) \tag{7.39}$$

[I]Notice that the coupling $\frac{g^2}{4}$ already contains the factor 1/4! associated to the possible permutations of the legs of the vertex. This can be easily understood by comparing the interaction potential of the theory with the potential of $\lambda\phi^4$ theory. Since there is only one vertex in the graph no symmetry factor associated to the exchange of the vertices is needed. Hence we just have to take care of the $4\cdot 3$ possible ways to connect the external fields to the vertex.

where we multiplied the correction coming from (7.38) by 2 in order to take into account the factor $1/2$ in the mass term of the Hamiltonian. The quantity m_R^2 is exactly the square of the mass that one detects performing an experimental measure of the mass of the fluctuation χ around the vacuum.

Semi-classical description of fluctuations around soliton solution

Let us now turn to the quantization around the soliton solutions $\phi_\pm^S$, consider for instance ϕ_+^S. Since this is a local minima, we can still expand around such solution very similarly to what we did for the fluctuations around the vacuum. First notice that, expressed in terms of the bare mass $m = \sqrt{2}gv$, the classical mass of the kink eq. (7.29) reads

$$M_S^{\rm cl} = \frac{m^3}{3g^2} \sim \frac{1}{g^2} \tag{7.40}$$

hence it cannot be recovered by perturbations around $g = 0$, since the perturbative expansion in g^{-2} is not well defined for $g \approx 0$. The behaviour $\sim g^{-2}$ of the energy or the action is typical of the soliton solutions.
A naive approach to quantization would be to write

$$\hat\phi(x,t) = \phi_+^S(x) + \hat\chi(x,t) \tag{7.41}$$

Inserting in the Hamiltonian we get

$$\begin{aligned} H = \int \mathrm{d}x \Bigg[&\frac{1}{2}\dot\phi_+^{S\,2} + \dot\phi_+^S\dot{\hat\chi} + \frac{1}{2}\dot{\hat\chi}^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi_+^S}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}\phi_+^S}{\mathrm{d}x}\frac{\mathrm{d}\hat\chi}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}\hat\chi}{\mathrm{d}x}\right)^2 + \\ &+ \frac{g^2}{4}\left[(\phi_+^{S\,2} - v^2)^2 + (4\phi_+^{S\,3} - 4v^2\phi_+^S)\hat\chi + (6\phi_+^{S\,2} - 2v^2)\hat\chi^2 + 4\phi_+^S\hat\chi^3 + \hat\chi^4\right]\Bigg] \end{aligned} \tag{7.42}$$

By integrating by parts with boundary conditions $\hat\chi(\pm\infty) = 0$ (otherwise the field would have infinite energy) we get

$$\begin{aligned} H = \int \mathrm{d}x \Bigg[&\frac{1}{2}\dot\phi_+^{S\,2} + \dot\phi_+^S\dot{\hat\chi} + \frac{1}{2}\dot{\hat\chi}^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi_+^S}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}^2\phi_+^S}{\mathrm{d}x^2}\hat\chi - \frac{1}{2}\hat\chi\frac{\mathrm{d}^2\hat\chi}{\mathrm{d}x^2} + \\ &+ \frac{g^2}{4}\left[(\phi_+^{S\,2} - v^2)^2 + (4\phi_+^{S\,3} - 4v^2\phi_+^S)\hat\chi + (6\phi_+^{S\,2} - 2v^2)\hat\chi^2 + 4\phi_+^S\hat\chi^3 + \hat\chi^4\right]\Bigg] \end{aligned} \tag{7.43}$$

The terms linear in $\hat\chi$ vanish due to the equation of motion of ϕ_+^S

$$\left(\frac{\mathrm{d}^2\phi_+^S}{\mathrm{d}x^2} - g^2(\phi_+^{S\,3} - v^2\phi_+^S)\right)\hat\chi = 0 \tag{7.44}$$

and combining with (7.28) we get

$$\begin{aligned} H &= M_S^{\rm cl} + \int \mathrm{d}x \left[\frac{1}{2}\dot\phi_+^{S\,2} + \dot\phi_+^S\dot{\hat\chi} + \frac{1}{2}\dot{\hat\chi}^2 - \frac{1}{2}\hat\chi\frac{\mathrm{d}^2\hat\chi}{\mathrm{d}x^2} + \frac{g^2}{4}\left[(6\phi_+^{S\,2} - 2v^2)\hat\chi^2 + 4\phi_+^S\hat\chi^3 + \hat\chi^4\right]\right] \\ &= M_S^{\rm cl} + \int \mathrm{d}x \left[\frac{1}{2}\dot\phi_+^{S\,2} + \dot\phi_+^S\dot{\hat\chi} + \frac{1}{2}\dot{\hat\chi}^2 + \frac{1}{2}\hat\chi L_2\hat\chi + \frac{g^2}{4}\left(4\phi_+^S\hat\chi^3 + \hat\chi^4\right)\right] \end{aligned} \tag{7.45}$$

with

$$L_2 = \frac{g^2}{2}(6\phi_+^{S\,2} - 2v^2) - \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{m^2}{2}\left(3\tanh^2\frac{mx}{2} - 1\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{7.46}$$

Up to now we kept terms proportional to $\dot\phi_+^S$ because in the following we will introduce a time dependence in ϕ_+^S. However at this stage the field is constant, hence setting $\phi_+^S = 0$ the terms quadratic in $\hat\chi$ give

$$H_2 = \int \mathrm{d}x \left(\frac{1}{2}\dot{\hat\chi}^2 + \frac{1}{2}\hat\chi L_2\hat\chi\right) \tag{7.47}$$

and the remaining terms are third and quartic interactions in $\hat{\chi}$.
Now the problem is similar to the free one, except for the fact that the operator associated to the harmonic oscillator has been replaced by L_2. We have to solve the "Schrödinger equation"

$$L_2\chi_n(x) = \omega_n^2\chi_n(x) \tag{7.48}$$

where n and χ_n play the previous roles of k and e^{ikx} respectively. Then we can expand

$$\hat{\chi}(x,t) = \sum_{n\in\mathbb{Z}} \hat{a}_n(t)\chi_n(x) \tag{7.49}$$

We can rewrite the Hamiltonian as

$$H_2 = \sum_{n\in\mathbb{Z}} \left(\frac{\dot{\hat{a}}_n^2}{2} + \frac{\omega_n^2}{2}\hat{a}_n^2 \right) \tag{7.50}$$

i.e. naively as a sum of uncoupled oscillators, as in the free case.

This would actually be true if $\omega_n > 0$, but one can prove that there is one vanishing eigenvalue ω_0, it is a "zero mode" and must be treated separately because the fluctuations along the direction of this mode are not small.[II] Such zero mode is associated to fluctuations of the modulus of the kink solution: changing the value of x_0 the solution has the same energy as the original one, hence this do not contribute to the Hamiltonian, moreover there is no limit of this kind of fluctuations, as all possible values of x_0 give kink solutions with the same energy.
The occurrence of the zero mode can be understood by a general argument. Expansion eq. (7.41) depends on the arbitrary choice of x_0, hence fixing the solution the translational invariance has been explicitly broken in the space of equilibrium states.[III] Such invariance is recovered by taking into account all possible values of x_0, and the standard approach, called *adiabatic*, to recover the invariance is to convert the position x_0 of the soliton into a quantum dynamical variable $\hat{x}_0(t)$ (a quantum position variable cannot assume a sharp value, hence there is no breaking of translational invariance) and to assume that the only dependence on time of the kink solution enter through its modulus[IV]

$$\phi_+^S(x) \longmapsto \phi_+^S(x - \hat{x}_0(t)) \quad \text{and} \quad \chi_n(x) \longmapsto \chi_n(x - \hat{x}_0(t)) \tag{7.51}$$

This is in agreement with the previous interpretation of the classical kink as a localized particle, indeed moving to a quantum mechanical description a semi-classical[V] particle should only depend on a quantum operator describing its position.
Expanding $\hat{\phi}$ in $\hat{x}_0$ we are able to recover the quantum mechanics of the kink. The same adiabatic strategy applies in general for the description of the solitons, but we will discuss it in details only in this instance. The time derivative of $\hat{\phi}$ is given by (notice that using the adiabatic approach we eliminated the zero mode ω_0)[VI]

$$\begin{aligned}
\dot{\hat{\phi}}(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_+^S(x-\hat{x}_0(t)) + \hat{\chi}(x-\hat{x}_0(t),t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\phi_+^S(x-\hat{x}_0(t)) + \sum_{n\in\mathbb{Z}^*}\hat{a}_n(t)\chi_n(x-\hat{x}_0(t))\right) \\
&= \left[-\frac{\mathrm{d}\phi_+^S(x-\hat{x}_0(t))}{\mathrm{d}x} - \sum_{n\in\mathbb{Z}^*}\hat{a}_n(t)\frac{\mathrm{d}\chi_n(x-\hat{x}_0(t))}{\mathrm{d}x}\right]\dot{\hat{x}}_0(t) + \sum_{n\in\mathbb{Z}^*}\dot{\hat{a}}_n(t)\chi_n(x-\hat{x}_0(t)) \\
&= \left[-\frac{\mathrm{d}\phi_+^S(x-\hat{x}_0(t))}{\mathrm{d}x} - \frac{\mathrm{d}\hat{\chi}(x-\hat{x}_0(t),t)}{\mathrm{d}x}\right]\dot{\hat{x}}_0(t) + \sum_{n\in\mathbb{Z}^*}\dot{\hat{a}}_n(t)\chi_n(x-\hat{x}_0(t)) \\
&= -\frac{\mathrm{d}\hat{\phi}(x,t)}{\mathrm{d}x}\dot{\hat{x}}_0(t) + \sum_{n\in\mathbb{Z}^*}\dot{\hat{a}}_n(t)\chi_n(x-\hat{x}_0(t))
\end{aligned} \tag{7.52}$$

[II] Fluctuations associated to $\omega_n = 0$ are not limited by any physical argument, as the kinetic energy is identically vanishing for this mode. Even worse, $\omega_n < 0$ would imply that the system is unstable, as fluctuations naturally become arbitrarily large. Luckily, the case $\omega_n < 0$ does not occur in the treatment of the kink.
[III] This is very similar to the previous example of a classical particle in a flat plane, subject to a gravitational force.
[IV] Adiabaticity means "very slow changes" of the system, in this way translations of the modulus are slow enough to leave invariant the shape of the field during the variation of x_0. This motivates the dependence on time of the kink only through $x_0(t)$.
[V] I.e. without any other quantum numbers.
[VI] Let $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ be the set of non-zero integers.

The Hamiltonian around the soliton solution up to quadratic terms is

$$\begin{aligned} H_{\leq 2} &\overset{(7.45)}{=} M_S^{\mathrm{cl}} + \int \mathrm{d}x \left[\frac{1}{2} {\dot{\phi}_+^S}^2 + \dot{\phi}_+^S \dot{\hat{\chi}} + \frac{1}{2} \dot{\hat{\chi}}^2 + \frac{1}{2} \hat{\chi} L_2 \hat{\chi} \right] \\ &\overset{(7.52)}{=} M_S^{\mathrm{cl}} + \int \mathrm{d}x \left[\left(\frac{\mathrm{d}\phi_+^S}{\mathrm{d}x} \right)^2 \dot{\hat{x}}_0^2 - \frac{\mathrm{d}\phi_+^S}{\mathrm{d}x} \dot{\hat{x}}_0 \dot{\hat{\chi}} + \frac{1}{2} \dot{\hat{\chi}}^2 + \frac{1}{2} \hat{\chi} L_2 \hat{\chi} \right] \end{aligned} \tag{7.53}$$

As a consequence of the adiabaticity assumption[VII] the effect of the time dependence of the modulus and the presence of the fluctuations should be considered separately in the Hamiltonian, i.e. the term $\dot{\hat{x}}_0 \dot{\hat{\chi}}$ gives higher order corrections which we can neglect. From another point of view, typically the Fourier transform of $\dot{\hat{x}}_0$ should be characterized by low frequencies, whereas typically fluctuations highly oscillate, hence the support in the Fourier space of $\dot{\hat{\chi}}$ is typically charaterized by high frequencies. Since Fourier components of different wavelengths are orthogonal we get $\int \mathrm{d}x\, \dot{\hat{x}}_0 \dot{\hat{\chi}} \approx 0$. For the same reason, we can neglect $\frac{\mathrm{d}\chi}{\mathrm{d}x} \dot{\hat{x}}_0$ in (7.52). Using eq. (7.28) we finally get

$$\begin{aligned} H_{\leq 2} &= M_S^{\mathrm{cl}} + \frac{1}{2} M_S^{\mathrm{cl}}\, \dot{\hat{x}}_0^2 + \int \mathrm{d}x \left[\frac{1}{2} \dot{\hat{\chi}}^2 + \frac{1}{2} \hat{\chi} L_2 \hat{\chi} \right] \\ &= M_S^{\mathrm{cl}} + \frac{1}{2} M_S^{\mathrm{cl}}\, \dot{\hat{x}}_0^2 + \sum_{n \in \mathbb{Z}^*} \left(\frac{\dot{\hat{a}}_n^2}{2} + \frac{\omega_n^2}{2} \hat{a}_n^2 \right) \\ &= M_S^{\mathrm{cl}} + \frac{\hat{p}_0^2}{2 M_S^{\mathrm{cl}}} + \sum_{n \in \mathbb{Z}^*} \left(\frac{\dot{\hat{a}}_n^2}{2} + \frac{\omega_n^2}{2} \hat{a}_n^2 \right) \end{aligned} \tag{7.54}$$

where we introduced the momentum $\hat{p}_0 := M_S^{\mathrm{cl}} \dot{\hat{x}}_0$. In this expression we have three contribution: the classical mass of the soliton, a term corresponding to the kinetic energy of the soliton as a quantum particle, and a set of oscillators describing the effect of fluctuations around the classical solution. Recall that we made an expansion in $\dot{\hat{x}}_0$ in a relativistic framework, hence we can reconstruct the complete energy of the soliton meant as a quantum particle (no fluctuations):

$$M_S^{\mathrm{cl}} + \frac{\hat{p}_0^2}{2 M_S^{\mathrm{cl}}} \quad \longrightarrow \quad \sqrt{{M_S^{\mathrm{cl}}}^2 + \hat{p}_0^2} \tag{7.55}$$

(we won't use the complete energy since we are working at quadratic order in the operators). Notice that there is no potential for $\hat{x}_0$, because consistently with translational invariance H cannot depend on $\hat{x}_0$ itself but only on $\hat{p}_0$. Quantization is finally archived imposing commutation relations

$$[\hat{x}_0, \hat{p}_0] = i\hbar \quad \text{and} \quad [\hat{a}_n, \dot{\hat{a}}_m] = i\hbar \delta_{nm} \tag{7.56}$$

Using eq. (7.54) one can approximatively compute the renormalization of the mass M_S^{cl} due to the quantum modes. Since non-zero modes are oscillatory modes we know that

$$\sigma \left(\sum_{n \in \mathbb{Z}^*} \frac{\dot{\hat{a}}_n^2}{2} + \frac{\omega_n^2}{2} \hat{a}_n^2 \right) = \sum_{n \in \mathbb{Z}^*} \left(N_n + \frac{1}{2} \right) \omega_n \quad \text{with} \quad N_n \in \mathbb{N} \tag{7.57}$$

Hence in its ground state ($N_n = 0$, $p_0 = 0$) the energy of the kink becomes

$$M_S^{\mathrm{cl}} + \sum_{n \in \mathbb{Z}^*} \frac{\omega_n^2}{2} \tag{7.58}$$

However we should remember that in RQFT the Hamiltonian is defined so that the vacuum is annihilated by H, hence we have to subtract to the previous expression the vacuum expectation value of the "unrenormalized" Hamiltonian $H_{\leq 2}$. Therefore, the energy of the ground state for the Hamiltonian is

$$M_S^{\mathrm{q}} = M_S^{\mathrm{cl}} + \sum_{n \in \mathbb{Z}^*} \left(\frac{\omega_n}{2} - \frac{\omega_{\mathrm{vac},n}}{2} \right) \tag{7.59}$$

[VII] Recall that from the practical point of view, adiabaticity consists in assuming that our "particle" moves very slowly, in such a way that the motion do not affect the shape of the field.

where $\omega_{\text{vac},n}$ are the frequencies of the oscillatory modes in the vacuum (which in the previous subsection we denoted simply by ω_n, a symbol here used for the oscillator modes in the presence of the kink). In the thermodynamic limit $L \to \infty$ these frequencies are labelled by a continuum index p and since we are in a relativistic context the on-shell condition implies $\omega_n \to \omega(p) = \sqrt{p^2+m^2}$. To get a well defined subtraction in M_S^{q} we first put the system in a region of finite length L, and then we take the limit $L \to \infty$. One can fix the boundary conditions in some different ways provided that in the thermodynamic limit they coincide with the previous ones. We use the (Dirichlet) vanishing boundary conditions at the boundary of L, such that

$$\omega_{\text{vac},n} = \sqrt{p_n^2+m^2} \quad \text{with} \quad p_n L = \pi n \tag{7.60}$$

For the modes around the kink solution, apart from the eigenvalue 0, there is another isolated eigenvalue $\omega_1 = \frac{\sqrt{3}}{2}m$, while all the other eigenvalues lie above m^2. In the thermodynamic limit the eigenvalues larger than m^2 are given by a continuum $\omega(\tilde{p}) = \sqrt{\tilde{p}^2+m^2}$ corresponding to the asymptotic behaviour of the eigenfunctions of the operator L_2

$$\begin{aligned} \chi_{\tilde{p}}(x) &\xrightarrow[x\to+\infty]{} e^{\pm i\tilde{p}x} \\ &\xrightarrow[x\to-\infty]{} e^{\pm i(\tilde{p}x+\delta(\tilde{p}))} \end{aligned} \tag{7.61}$$

where $e^{i\delta(\tilde{p})}$ is a correction called *phase shift* (as it gives an phase shift to the asymptotic behaviour) which always comes out for non-trivial scatterings and is given by

$$e^{i\delta(\tilde{p})} = e^{i\left[2\arctan\frac{\tilde{p}}{m}+2\arctan\frac{2\tilde{p}}{m}\right]} \tag{7.62}$$

For the system at finite length with vanishing boundary conditions the phase shift is given by

$$\tilde{p}_n L - \delta(\tilde{p}_n) = \pi n \quad \Rightarrow \quad \tilde{p}_n = p_n + \frac{\delta(\tilde{p}_n)}{L} \overset{L\gg 1}{\approx} p_n + \frac{\delta(p_n)}{L} \tag{7.63}$$

Then eq. (7.59) reads (discarding the finite difference due to the eigenvalue ω_0 and ω_1, as we are interested in the divergent terms)

$$M_S^{\text{q}} - M_S^{\text{cl}} \approx \sum_{n\in\mathbb{Z}^*} \left(\frac{\sqrt{\tilde{p}_n^2+m^2}}{2} - \frac{\sqrt{p_n^2+m^2}}{2} \right) \tag{7.64}$$

Going to the thermodynamic limit and using $n = p_n L/\pi$ we can replace, according to Riemann integration[VIII]

$$\sum_{n\in\mathbb{Z}^*} f(p_n) \quad \longmapsto \quad \int_{-\infty}^{\infty} \frac{\mathrm{d}p\, L}{\pi} f(p) \tag{7.65}$$

and then we get

$$\begin{aligned} M_S^{\text{q}} - M_S^{\text{cl}} &\approx \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p\, L}{\pi} \left(\sqrt{\left(p+\frac{\delta(p)}{L}\right)^2+m^2} - \sqrt{p^2+m^2} \right) \overset{\delta(p)\ll L}{\approx} \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{p\,\delta(p)}{\sqrt{p^2+m^2}} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\,\delta(p)\,\frac{\mathrm{d}\sqrt{p^2+m^2}}{\mathrm{d}p} = \int_0^{\infty} \frac{\mathrm{d}p}{\pi}\,\delta(p)\,\frac{\mathrm{d}\sqrt{p^2+m^2}}{\mathrm{d}p} \end{aligned} \tag{7.66}$$

where in the third step we expanded around $\delta(p)/L \approx 0$ and in the last we used $\delta(p) = -\delta(-p)$. Integrating by parts with boundary conditions $\delta(0) = \delta(\infty) = 0$ (directly follows from eq. (7.62)) we finally get

$$\begin{aligned} M_S^{\text{q}} - M_S^{\text{cl}} &\approx -\int_0^{\infty} \frac{\mathrm{d}p}{\pi}\sqrt{p^2+m^2}\,\frac{\mathrm{d}\delta(p)}{\mathrm{d}p} \overset{(7.62)}{=} -\int_0^{\infty} \frac{\mathrm{d}p}{\pi}\sqrt{p^2+m^2}\left(\frac{2m}{p^2+m^2}+\frac{4m}{4p^2+m^2}\right) \\ &= -\frac{2m}{\pi}\int_0^{\infty} \mathrm{d}y\,\sqrt{y^2+1}\left(\frac{1}{y^2+1}+\frac{2}{4y^2+1}\right) \qquad \text{with} \quad y = \frac{p}{m} \end{aligned} \tag{7.67}$$

[VIII] The value $p = 0$ has zero measure, hence we can include it in the integral.

The last integral is clearly logarithmically divergent. Introducing a cutoff Λ the divergent term is indeed (the previous integral is finite for small p, hence we consider only the high energy part)

$$M_S^{\mathrm{q}} - M_S^{\mathrm{cl}} \approx -\frac{3m}{\pi}\int^{\Lambda/m}\frac{\mathrm{d}y}{y} = -\frac{3m}{\pi}\log\frac{\Lambda}{m} = -\frac{3m}{2\pi}\log\frac{\Lambda^2}{m^2} \tag{7.68}$$

This divergence arises since we considered the unrenormalized mass rather than the renormalized one, indeed in our discussion so far we did not take into account higher order interactions. We should take into account divergences due to third and quartic coupling, as we have done for the fluctuations around the vacuum.
The one-loop correction δm to the bare mass, given by eq. (7.39) and due to higher order interactions, implies the following correction to the kink energy density:

$$\delta\mathcal{E} = \frac{1}{2}\delta m^2(\phi^2_{\mathrm{kink}} - \phi^2_{\mathrm{vac}}) = \frac{1}{2}\delta m^2 v^2\left(\tanh^2\frac{gv(x-x_0)}{\sqrt{2}} - 1\right) \tag{7.69}$$

which, by integration, give the one-loop correction to the mass of the kink

$$\begin{aligned}\delta M_S &= \frac{1}{2}\delta m^2\frac{m^2}{2g^2}\int_{-\infty}^{+\infty}\mathrm{d}x\,\left(\tanh^2\frac{m}{2}(x-x_0) - 1\right)\\ &= -\delta m^2\,\frac{m}{2g^2}\int_{-\infty}^{+\infty}\mathrm{d}y\,\frac{\mathrm{d}}{\mathrm{d}y}\tanh y \qquad \text{with} \quad y = \frac{1}{2}m(x-x_0)\\ &= -\delta m^2\,\frac{m}{g^2} \overset{(7.39)}{=} \frac{3m}{2\pi}\log\frac{\Lambda^2}{m^2}\end{aligned} \tag{7.70}$$

which is exactly the factor needed to cancel the divergence in the bare mass of χ in eq. (7.68). So, at least at first order, (the divergent part of) the correction due to the kinetic energy of fluctuations (eq. (7.68)) cancels exactly (the divergent part of) the correction coming from the renormalization of the bare mass (eq. (7.70)).

7.3 Applications

Before leaving the semi-classical treatment of the kink let us comment some physical applications.

High-energy physics

Let's consider the same theory in $D = 3+1$ dimension. Then adding two dimensions, the "soliton" center x_0 instead of a point become a plane (along y and z directions), and the kink become a *domain wall*, as it separates the space in two domains. Clearly in the thermodynamic limit the energy of such object is infinite, but the energy per unit area A

$$T = \frac{H}{A} \tag{7.71}$$

is still finite and is called *wall tension*. Notice that the energy H is not well defined as it is divergent, but still we can define T and then take its thermodynamic limit, which is well-defined. The world with a domain wall is disjoint from the world without a domain wall, hence we have a spontaneous breaking of translational invariance, except for some special instances, for example in the case of space-time with a particular topology (e.g. if the space-time is a torus, then H and A are finite and the wall just "open" the torus in a cylinder) or in the case where the wall is a purely quantum effect and has a "virtual area", i.e. it exists only for finite time.
In $D > 2$ the quantum version of x_0, $\hat{x}_0(t)$, becomes a function of the coordinate of the wall, for example in $D = 4$ we have $\hat{x}_0(t, y, z)$. A calculation completely analogous to the previous one gives the following Hamiltonian (at quadratic order) for the kink

$$H_{\leq 2} = \int \mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\,\frac{1}{2}\left[\hat{p}_0(t,y,z)^2 + (\partial_y\hat{x}_0(t,y,z))^2 + (\partial_z\hat{x}_0(t,y,z))^2\right] \tag{7.72}$$

A complete treatment including higher order terms in the Lagrangian formalism give the so called *Nambu-Goto* or *Polyakov actions*. Moreover, if a domain world is coupled to gravity, it "antigravitates", that is, it is repulsive for both other walls or for massive particles.

Solid state physics

Kinks appear in many materials, especially in those which can be described as one-dimensional chains. A famous one is the polyacetylene. This is a linear polymer made of a sequence of units of $(C_2H_2)_n$ with two minimal energy configurations, shown in fig. 7.3, which can be obtained from each other by reflection, i.e. by a $\mathbb{Z}_2$ symmetry.

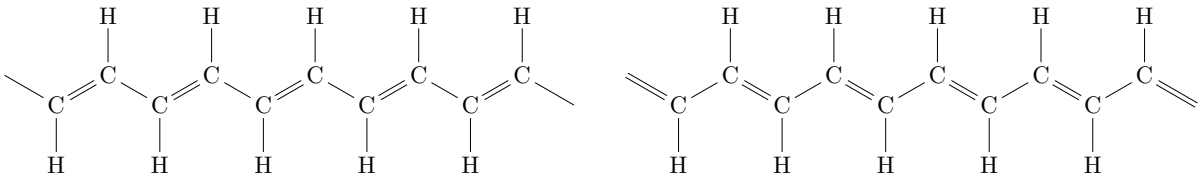


Figure 7.3: Two minimal energy configurations of *trans*-polyacetylene.

A kink interpolates between the two structures and in terms of the displacement of the double bond it has exactly the form of the kink of ϕ^4 in the lattice renormalization. An example of such situation is shown in fig. 7.4.

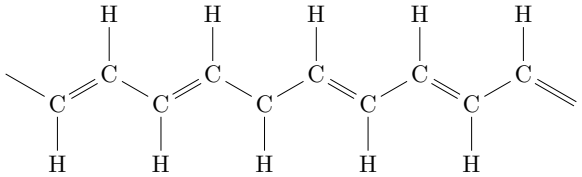


Figure 7.4: A state of *trans*-polyacetylene chain with one of the double bonds converted into a single bond form a domain wall.

There are a lot of one-dimensional systems with n ground states related by $\mathbb{Z}_n$ symmetry, spontaneously broken, which exhibits kinks interpolating between these ground states.

7.4 Quantum field theory treatment

[FM88], [FM90], [Mar88], see also [Mar86], [Mar87], [FM87]

Up to now we have discussed classical and quantum mechanical treatment of kinks. Let's now really consider a QFT treatment.

The space of configurations of the path integral

We start by asking how relevant is the classical solution of a field theory for the understanding of the corresponding QFT behaviour. In the path-integral approach the configurations of the field $\phi(x)$ are weighted by $e^{-\frac{1}{\hbar}S(\phi)}$, where $S(\phi)$ is the classical action, and the classical equations of motion can be derived as minima of $S(\phi)$.
The naive answer to the above question is that classical solutions give the dominant contribution to the path integral at least in the semi-classical limit of small $\hbar$ or more in general if the action contribution $e^{-S(\phi)}$ dominates over the measure $\mathcal{D}\phi$, e.g. when one can rewrite the action in the form $\beta S(\phi)$ with β sufficiently large.[IX] In our ϕ^4 model this is achieved rescaling

$$\phi \to g\phi \tag{7.73}$$

so that

$$S(\phi) \to \frac{1}{g^2}\int \mathrm{d}^2x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + \frac{1}{4}\left(\phi - (vg)^2\right)^2\right) \tag{7.74}$$

[IX] This is often the case for supersymmetric theories, at least in low dimensions.

hence for small g^2, keeping vg fixed, the exponential weight is dominant in the path integral.

However, if there is no UV cutoff, this naive reasoning is wrong, because the typical configurations of ϕ are extremely singular, whereas solutions of equations of motion are typically very regular. If we consider just the quadratic term of the action, denoted by $Q(\phi)$, there is a simple way to understand the typical regularity of ϕ. Let formally

$$C(x-y) \equiv \langle \phi(x)\phi(y) \rangle_2 := \frac{\int \mathcal{D}\phi \, e^{-Q(\phi)} \phi(x)\phi(y)}{\int \mathcal{D}\phi \, e^{-Q(\phi)}} \tag{7.75}$$

usually called *covariance*, then assuming translational invariance we can perform the Fourier transform of C, $\tilde{C}(k)$, and then there is a theorem proving that typical configurations of ϕ satisfy with probability 1 the estimate

$$|\phi(x) - \phi(y)| < \text{const} \cdot |x-y|^\alpha \tag{7.76}$$

for every α such that

$$\int \mathrm{d}^D k \, \tilde{C}(k) \, k^{2\alpha} < \infty \tag{7.77}$$

Conversely, eq. (7.76) is almost never satisfied if eq. (7.77) does not hold.
For $D = 1$ (i.e. $\phi(x) = q(t)$, we are in the quantum mechanical case) with

$$\tilde{C}(k_0) = \frac{1}{k_0^2 + m^2} \tag{7.78}$$

which is the case for the harmonic oscillator, we have

$$\int \mathrm{d}k_0 \, \frac{1}{k_0^2 + m^2} \, k_0^{2\alpha} < \infty \quad \text{only if} \quad \alpha < \frac{1}{2} \tag{7.79}$$

hence $q(t)$ is continuous in t (continuity is ensured by $\alpha > 0$) but it is not differentiable (differentiability requires $\alpha > 1$).[X]
For arbitrary D and

$$\tilde{C}(k) = \frac{1}{k^2 + m^2} \tag{7.80}$$

condition eq. (7.77) is satisfied provided that $D + 2\alpha < 2$ holds. This means that $\phi(x)$ is not even continuous for $D \geq 2$. For this reason our initial claim is wrong, and classical solutions are not the dominant contributions to the path integral. Notice that the situation become worse and worse as we increase D, since greater values of D implies smaller values of α. The reason behind this is that higher is the dimension, stronger are the ultraviolet effects, as one can easily see from dimensional analysis.

Typical configurations of ϕ are distributions. In a sense we should have expected this, since quantum fields are described by operator-valued distributions in the operator formalism. It turns out that configurations with finite action are of measure 0 in the space of all field configurations over which the path integral is performed.
Summarizing, for a theory where the ultraviolet cutoff is pushed to infinity and the translational invariance holds, we can apply the previous theorem, which tells us exactly what happens when the path integral is Gaussian, i.e. when the action si quadratic, described by Q. In particular, we know that typical configurations are extremely singular: they are mostly continuous only in $D = 1$, whereas in all greater dimensions they are almost always not differentiable. As we will see, this suggest which configurations dominate the path integral, and the description of such configuration will lead us to the notion of *defect*, which is the cornerstone for the description of solitons in QFT.

[X] This may be interpreted as follows: since at any time the particle can be found at an arbitrary position, the probability that the sequence of positions follows a smooth path is practically vanishing.

The semi-classical approximation

The reason why classical field solutions are relevant, is that they can be used as a starting point for the following approximation. In analogy to the procedure we followed before, we expand the field ϕ around a minima ϕ_{cl} of the action, $\phi \to \phi_{\text{cl}} + \chi$, as

$$S(\phi) = S(\phi_{\text{cl}}) + Q(\chi) + R(\chi) \tag{7.81}$$

where $Q(\chi)$ contains quadratic (and possibly linear) terms in χ and $R(\chi)$ is the reminder (contains all higher order terms in χ). The semi-classical approximation is based on the idea that the contribution of $R(\chi)$ is typically small respect to the one of $S(\phi_{\text{cl}}) + Q(\chi)$, hence configurations which differ significantly from the classical solutions are suppressed by the quadratic term $Q(\chi)$ and in the path integration the only configurations whose contributions sum up coherently in a non-zero result are those which are "close" to the classical one. Therefore, the main contribution to the path integral is given by those configuration localized around the classical solution, in the configurations space. Such situation is represented in fig. 7.5.

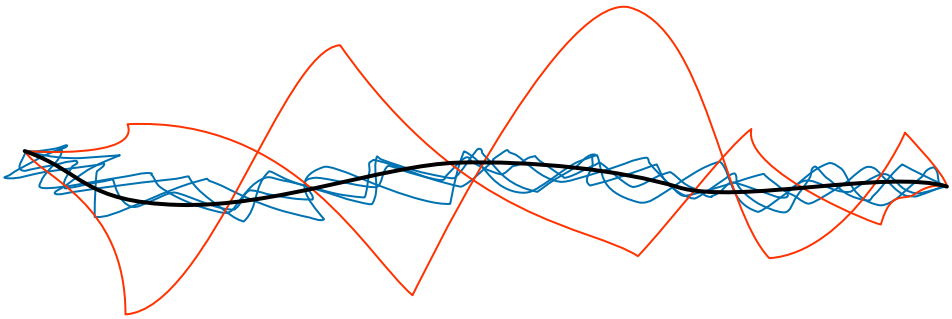


Figure 7.5: The black line corresponds to the classical solution, ϕ_{cl}. The blue lines describes the configurations which sum up coherently in the path integral, whereas red lines represent those configurations whose contributions to the path integral is suppressed by $Q(\chi)$ since they strongly deviate from the classical solution.

Notice that if such approximation holds, the soliton can be analized as a (extended) quantum mechanical particle localized around the classical solution, as we will see. However, in order to give a complete QFT description, we should improve our description, in such a way that we are able to describe creation and annihilation of solitons.

Notice that some of the dominant classical configurations (blue lines in fig. 7.5) will not be associated to a specific solution of the equations of motion, because in different regions of the Euclidean space-time the field may be close to different local minima of the action, and we need to interpolate among them. For example, in our ϕ^4 model the field ϕ in some regions may be close to the absolute minima ϕ_+, and in others to the kink solution ϕ^S.
In order to take into account this issue, we wish to find a set of classical configurations $\{\phi_i\}_{i\in I}$ and a partition $\{\mathcal{C}_i\}_{i\in I}$ of the space $\mathcal{C}$ of all configurations of ϕ,[XI] with $\phi_i \in \mathcal{C}_i$, such that

(0) for each $\phi \in \mathcal{C}$ and $i \in I$, expanding $\phi = \phi_i + \chi_i$ we can rewrite

$$S(\phi) = S(\phi_i) + Q_i(\chi_i) + R_i(\chi_i) \tag{7.82}$$

with $Q_i(\chi_i)$ containing terms up to the second order in χ_i and $R_i(\chi_i)$ containing terms of higher order;

(1) if $\phi_i + \chi \in \mathcal{C}_i$ then $|e^{-R_i(\chi)} - 1| \ll 1$;

(2) if $\phi_i + \chi \notin \mathcal{C}_i$ then $e^{-Q_i(\chi)} \ll 1$ (i.e. ϕ strongly deviate from ϕ_i);

(3) $\bigcup_i \mathcal{C}_i = \mathcal{C}$ is the configuration space of ϕ;

[XI] The space $\mathcal{C}$ is very singular, for example in ϕ_2^4 it may be $\mathcal{S}'(\mathbb{R}^2)$.

(4) $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$.

One should think at $\{\phi_i\}_{i\in I}$ as the union of all the classical solutions of equations of motion (vacuum configurations and local minima of the action) together with a set of "inequivalent" configurations which interpolate between different classical solutions. The meaning of "inequivalent" is specified by the above conditions. For instance, there may be a configuration ϕ_i which coincide with ϕ^S_+ far to the left of x_0, coincide with ϕ^S_- far to the right of x_0 and near x_0 behaves approximately as a classical kink. Then using (1) and (2) we see that $R_i(\phi - \phi_i)$ is very close to zero if ϕ is close to ϕ_i, and $e^{Q_i(\phi-\phi_i)}$ is almost zero if ϕ strongly deviates from ϕ_i. The five properties together ensures that there are no other configurations $\phi_j \notin \mathcal{C}_i$ which are a slight deformation of ϕ_i.
According to (3) and (4) we can define a partition of unity

$$1_{\mathcal{C}} = \sum_i X_i(\phi) \quad \text{with} \quad X_i(\phi) = \begin{cases} 1 & \text{if} \quad \phi \in \mathcal{C}_i \\ 0 & \text{if} \quad \phi \notin \mathcal{C}_i \end{cases} \tag{7.83}$$

which can be inserted in the path integral obtaining an "improved semi-classical approximation":[XII]

$$\begin{aligned} \int_{\mathcal{C}} \mathcal{D}\phi\, e^{-S(\phi)} F(\phi) &= \sum_{i\in I} \int_{\mathcal{C}} \mathcal{D}\phi\, e^{-S(\phi)} F(\phi) X_i(\phi) \\ &= \sum_{i\in I} \int_{\mathcal{C}_i} \mathcal{D}\phi\, e^{-S(\phi)} F(\phi) \\ &= \sum_{i\in I} \int_{\mathcal{C}_i-\phi_i} \mathcal{D}\chi_i\, e^{-S(\phi_i)-Q_i(\chi_i)-R_i(\chi_i)} F(\phi_i+\chi_i) \\ &\overset{(1)}{\approx} \sum_{i\in I} e^{-S(\phi_i)} \int_{\mathcal{C}_i-\phi_i} \mathcal{D}\chi_i\, e^{-Q_i(\chi_i)} F(\phi_i+\chi_i) \\ &\overset{(2)}{\approx} \sum_{i\in I} e^{-S(\phi_i)} \int_{\mathcal{C}-\phi_i} \mathcal{D}\chi_i\, e^{-Q_i(\chi_i)} F(\phi_i+\chi_i) \\ &= \sum_{i\in I} e^{-S(\phi_i)} \int_{\mathcal{C}} \mathcal{D}\chi_i\, e^{-Q_i(\chi_i)} F(\phi_i+\chi_i) \end{aligned} \tag{7.84}$$

Hence such approximation considers fluctuations around various possible local minima of the action, neglecting higher order contributions coming from the various $R_i(\chi_i)$'s.

Defects in QFT

For a fixed configuration ϕ let Γ be a region in the Euclidean space-time where the ϕ deviates strongly from the global minimum of the action (i.e. from the vacuum) but it is close to a local minimum (usually corresponding to a classical solution of the equations of motion). If Γ is connected it is said to be the support of a *defect*.[XIII] If the Euclidean space-time minus the support of the defect Γ is contractible to the Euclidean space-time minus a n-dimensional manifold, then we call the defect a n-dimensional defect. In particular we can have a point defect, a line defect, etc. In mathematical language, Γ is a point/line/surface/$\dots$/n-dimensional defect if its support is the tubular neighbourhood of a point/line/surface/$\dots$/n-dimensional manifold.
In high energy language, sometimes one also uses the word *p-brane* for a $(p+1)$-dimensional defect, although often this terminology is deserved to cases in which the defect is strictly $(p+1)$-dimensional (without contracting its complement).
Notice that free theories has only one minimum, therefore in order to have defects we should consider interacting theories.

[XII] Here we denote by $\int_{\mathcal{C}_i-\phi} \mathcal{D}\chi$ the integration over those configurations χ such that $\phi + \chi \in \mathcal{C}_i$. The last step follows easily since $\mathcal{C}$ contains all the possible configurations, hence $\mathcal{C} - \phi = \mathcal{C}$ for any ϕ.
[XIII] The notion of defect comes from the study of crystals: a defect is an irregularity of the structure. Such irregularities usually create new energy levels, which have higher energy with respect to the ground state. Similarly, in QFT the notion of defect has been adapted, describing local minima of the action, disjointed from the global ones (the vacua).

For our ϕ_2^4 theory the kink is a line defect. Indeed Γ appears in the partition function when the field ϕ changes sign around a bounded region. The line where ϕ vanishes is called the *locus of the defect* and is clearly a closed line, unless it reaches the boundaries of the spacetime. For the classical kink discussed before in the quantum mechanical treatment, the locus of the defect is time independent and is parametrized by the modulus of the kink, therefore it reaches infinities in the time direction and it is not closed. However, for the moment let's restrict ourselves on defects whose loci have compact support, an example is shown in fig. 7.6. A typical configuration of ϕ far away from the locus of the defect should be close to one of the two global minima $\phi_\pm$. Instead near the region $\phi = 0$ typical configurations should be close to a kink, in order to maintain $S(\phi)$ close to a (local) minimum.

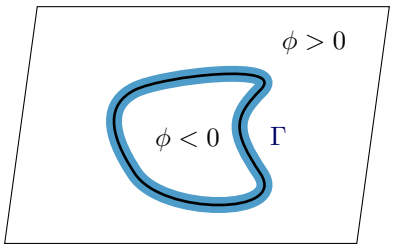


Figure 7.6: The defect Γ is represented by the blue thick line. The black line correspond to the locus of the defect, i.e. where $\phi = 0$. In the white regions the configuration is close to the vacuum, $|\phi| \approx v$.

If the vacuum expectation value is positive, $\langle\phi\rangle > 0$, which can be achieved by imposing ϕ_+ boundary conditions at ∞, the situation is the one represented in fig. 7.7. Notice that for such boundary conditions it is impossible to have defects reaching ∞.

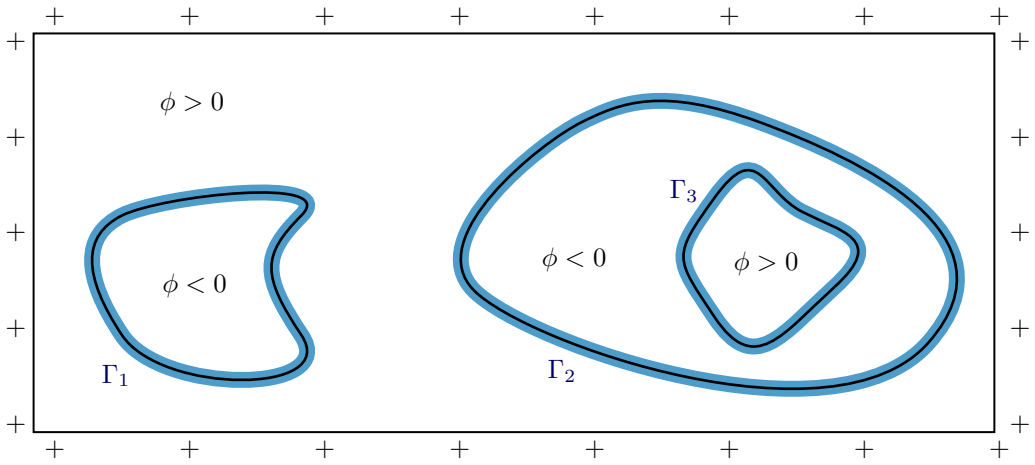


Figure 7.7: Support of defects (blue lines) and locus of such defects (black lines) in presence of $\langle\phi\rangle > 0$ boundary conditions, in the finite volume regularization.

In the lattice approximation (replacing ϕ^4 by the Ising model) such defects are called *Peierls contours*.

Let us consider again eq. (7.84). Notice that to each $\mathcal{C}_i$ we can associate the defects with support[XIV] $\Gamma_{i,1}, \dots, \Gamma_{i,n}$ appearing in the configurations $\phi \in \mathcal{C}_i$. Then, we can consider

$$\int_{\mathcal{C}_i - \phi_i} \mathcal{D}\chi_i \, e^{-S(\phi_i) - Q_i(\chi_i)} \tag{7.85}$$

as a Boltzmann weight of the *gas of defects* $\{\Gamma_{i,1}, \dots, \Gamma_{i,N}\}$ appearing in the configurations of $\mathcal{C}_i$. From

[XIV] For simplicity we will now denote also the defects by their support, assuming that no ambiguity arises from this simplification.

such Boltzmann weight, we can define the action $S(\Gamma_{i,1}, \dots, \Gamma_{i,N})$ associated to the gas

$$e^{-S(\Gamma_{i,1}, \dots, \Gamma_{i,N})} = \int_{\mathcal{C}_i - \phi_i} \mathcal{D}\chi_i \, e^{-S(\phi_i) - Q_i(\chi_i)} \tag{7.86}$$

Notice that the same defect Γ can be associated to different sets $\mathcal{C}_j$, as in general classical configurations may interpolate between several local minima of the action, but each gas of defects $\{\Gamma_1, \dots, \Gamma_N\}$ is uniquely associated to some $\mathcal{C}_i$, for the same reasons which lead us to eq. (7.84).
We can now define the interaction between defects in the gas $\{\Gamma_1, \dots, \Gamma_N\}$ as

$$V(\Gamma_1, \dots, \Gamma_N) := S(\Gamma_1, \dots, \Gamma_N) - \sum_{k=1}^{N} S(\Gamma_k) \tag{7.87}$$

One should notice that if the defects are separated enough from each others, so that no configuration with non-zero measure interpolate between them, then $V(\Gamma_1, \dots, \Gamma_N) = 0$ and the defects do not interact.
Finally, neglecting contributions coming from fluctuations around the vacuum, we get that in the "improved semi-classical approximation" (s.c.) the partition function of the system can be expressed as

$$\begin{aligned} Z_{\text{s.c.}} := \int_{\mathcal{C}} \mathcal{D}\phi \, e^{-S(\phi)} &\approx \sum_{i \in I} e^{-S(\Gamma_{i,1}, \dots, \Gamma_{i,N})} \\ &= \sum_{i \in I} \underbrace{\prod_{k=1}^{N} e^{-S(\Gamma_{i,k})}}_{\substack{\text{Non-interacting} \\ \text{gas of defects}}} \underbrace{e^{-V(\Gamma_{i,1}, \dots, \Gamma_{i,N})}}_{\substack{\text{Interactions} \\ \text{among defects}}} \end{aligned} \tag{7.88}$$

or equivalently, setting formally $e^{S(\Gamma_1, \dots, \Gamma_N)} = 0$ if $\{\Gamma_1, \dots, \Gamma_N\}$ does not correspond to any $\mathcal{C}_i$,

$$Z_{\text{s.c.}} \approx \sum_{N=0}^{\infty} \sum_{\{\Gamma_1, \dots, \Gamma_N\}} e^{-S(\Gamma_1, \dots, \Gamma_N)} \tag{7.89}$$

where the sum over $\{\Gamma_1, \dots, \Gamma_N\}$ considers all possible combinations of N defects. From eq. (7.89) we see that $Z_{\text{s.c.}}$ can be interpreted as a partition function of the interacting gas of defects, which by construction have closed locus.

In the lattice line defects are close contours in the dual lattice, between sites in which the field ϕ has opposite sign, as shown in fig. 7.8 and fig. 7.9.

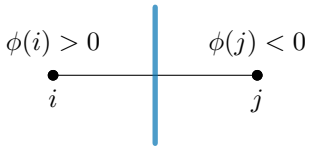
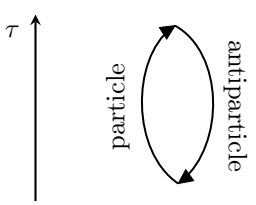


Figure 7.8: The elementary component (blue line) of a defect. Given a link in the lattice between two sites where ϕ has opposite sign, the defect is made by considering the orthogonal link in the dual lattice.

Particle-antiparticle virtual pairs

Since the support of line defects locally looks like the world line of a particle, closed loops naturally correspond to particle-antiparticle virtual pairs.



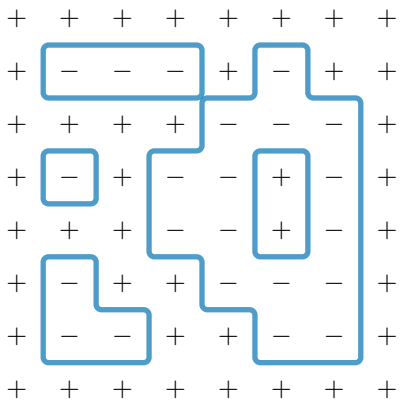


Figure 7.9: The sites of the lattice are denoted by + and − depending on whether the field has positive or negative value on them. The blue lines represent defects which are supported in the dual lattice and are constructed according to fig. 7.8.

A natural question is then how to construct quantum field operators creating and annihilating particles giving rise to the defects, as quantum particles. We first show that the Euclidean two-points function of a free massive scalar field[XV]

$$\langle \phi(x)\phi(y)\rangle = (-\Delta + m^2)^{-1}(x,y) \tag{7.90}$$

can be written as a "sum" of open paths (worldlines) with boundaries x, y with suitable Boltzmann weights, whereas the partition function of the same field can be written as a "sum" of closed paths (defects) with the same Boltzmann weights (but with different coefficients). Actually, our description will be heuristic (with ill-defined measure in the path integral) but the same task can be done similarly in a rigorous way, using the Wiener measure.

In some sense we can go from partition functions to correlation functions "opening" the closed paths of the partition function in such a way that their boundaries correspond to the position of the insertion of the fields in the correlator in the Euclidean space-time.

Let us consider a massive scalar free field with two points function eq. (7.90). From the reconstruction theorem, if $\hat{\phi}(\boldsymbol{x})$ is the time-0 quantum field "operator", $\hat{\phi}(x^0, \boldsymbol{x}) = e^{x^0 H}\hat{\phi}(\boldsymbol{x})e^{-x^0 H}$ is the field operator, H is the Hamiltonian and $|\Omega\rangle$ is the vacuum, then for $x^0 < y^0$, then

$$\langle \phi(x)\phi(y)\rangle = \langle \Omega | \hat{\phi}(x^0, \boldsymbol{x})\hat{\phi}(y^0, \boldsymbol{y}) | \Omega \rangle \tag{7.91}$$

We can also write[XVI]

$$(-\Delta + m^2)^{-1}(x,y) = \int_0^\infty \mathrm{d}s \left(e^{-s(-\Delta+m^2)}\right)(x,y) = \int_0^\infty \mathrm{d}s\, e^{-sm^2} e^{s\Delta}(x,y) \tag{7.93}$$

where $e^{s\Delta}(x,y)$ is the Euclidean analogue of the kernel of the (real) time evolution e^{-itH} for $H = \Delta$

$$\left(e^{-itH}\psi\right)(x) = \int \mathrm{d}y \left(e^{-itH}\right)(x,y)\psi(y) \tag{7.94}$$

We know that the above kernel can be represented in terms of a path integral

$$\left(e^{-is\Delta}\right)(x,y) = \int\limits_{\substack{q(0)=x \\ q(s)=y}} \mathcal{D}q(t)\, e^{\,i\int_0^s \mathrm{d}t\, \dot{q}^2(t)/4} \tag{7.95}$$

[XV] The object $(-\Delta + m^2)^{-1}(x,y)$ is the kernel of the operator $(-\Delta + m^2)^{-1}$. If you are not familiar with the notion of the *kernel of an integral operator* look at https://encyclopediaofmath.org/wiki/Integral_operator and https://encyclopediaofmath.org/wiki/Kernel_of_an_integral_operator.

[XVI] Let us write a function of a definite positive operator A as the linear combination $f(A) = \sum_n f(\lambda_n)\,|\lambda_n\rangle\langle\lambda_n|$, where $\{\lambda_n\}$ is the set of positive eigenvalues of A with eigenvectors $|\lambda_n\rangle$. For $a > 0$ we have

$$\int_0^\infty \mathrm{d}s\, e^{-as} = \frac{1}{a} \quad \Rightarrow \quad A^{-1} = \sum_n \frac{1}{\lambda_n}|\lambda_n\rangle\langle\lambda_n| = \int_0^\infty \mathrm{d}s \sum_n e^{-\lambda_n s}|\lambda_n\rangle\langle\lambda_n| = \int_0^\infty \mathrm{d}s\, e^{-sA} \tag{7.92}$$

which implies

$$\left(e^{s\Delta}\right)(x,y) = \int\limits_{\substack{q(0)=x\\ q(s)=y}} \mathcal{D}q(t)\, e^{-\int_0^s \mathrm{d}t\, \dot{q}^2(t)/4} \tag{7.96}$$

so that finally

$$\langle\phi(x)\phi(y)\rangle = \int_0^\infty \mathrm{d}s \int\limits_{\substack{q(0)=x\\ q(s)=y}} \mathcal{D}q(t)\, \underbrace{e^{-sm^2 - \int_0^s \mathrm{d}t\, \dot{q}^2(t)/4}}_{\text{Boltzmann weight}} \tag{7.97}$$

that is, we expressed the two points function as a weighted "sum" over trajectories with boundary points x and y.
Analogously, for the partition function[XVII]

$$\begin{aligned}
Z &= \left(\det(-\Delta + m^2)\right)^{-1} \\
&= \exp\left[\mathrm{Tr}\log\left(-\Delta + m^2\right)^{-1}\right] \\
&= \exp\left[\mathrm{Tr}\int_0^\infty \frac{\mathrm{d}s}{s}\, e^{-sm^2} e^{s\Delta}\right] \\
&= \exp\left[\int \mathrm{d}^{d+1}x \int_0^\infty \frac{\mathrm{d}s}{s}\, e^{-sm^2} e^{s\Delta}(x,x)\right] \\
&= \exp\left[\int \mathrm{d}^{d+1}x \int_0^\infty \frac{\mathrm{d}s}{s}\, e^{-sm^2} \int\limits_{q(0)=q(s)=x} \mathcal{D}q(t)\, e^{-\int_0^s \mathrm{d}t\, \dot{q}(t)/4}\right] \\
&= \sum_{n=0}^\infty \frac{1}{n!}\left[\int \mathrm{d}^{d+1}x \int_0^\infty \frac{\mathrm{d}s}{s} \int\limits_{q(0)=q(s)=x} \mathcal{D}q(t)\, \underbrace{e^{-sm^2 - \int_0^s \mathrm{d}t\, \dot{q}(t)/4}}_{\text{Boltzmann weight}}\right]^n
\end{aligned} \tag{7.99}$$

where in the second line we used the identity $(\det A)^{-1} = \exp(\mathrm{Tr}\log A)$ and in the fourth line we used (A operator in a space with orthonormal basis $\{|\phi_n\rangle\}$)

$$\begin{aligned}
\mathrm{Tr}\, A &= \sum_n \langle\phi_n| A |\phi_n\rangle = \sum_n \langle\phi_n| \overbrace{\int \mathrm{d}x\, |x\rangle\langle x|}^{\mathbb{1}} A \overbrace{\int \mathrm{d}y\, |y\rangle\langle y|}^{\mathbb{1}} \phi_n\rangle = \int \mathrm{d}x\mathrm{d}y\, \langle x| A |y\rangle \sum_n \langle y|\phi_n\rangle\langle\phi_n|x\rangle \\
&= \int \mathrm{d}x\mathrm{d}y\, \langle x| A |y\rangle \langle y|x\rangle = \int \mathrm{d}x\, \langle x| A |x\rangle = \int \mathrm{d}x\, A(x,x)
\end{aligned} \tag{7.100}$$

Hence Z can be written as a weighted "sum" over closed paths, with the same weights as for the 2-points function.

Knowing the structure of the Boltzmann weight in the partition function, the basic idea is to recover the two points function "opening' the paths used in Z in such a way that boundaries of the resulting lines coincides with points in the two points function.
Let us now turn to the description of kinks. As we have seen the partition function of ϕ_2^4 in the broken symmetry phase can be written as a sum over closed line defects "associated to kinks", whose locus is the region of spacetime where ϕ changes sign. This suggests that the Green function of a quantum field kink operator can be constructed if we are able to introduce open line defects whose loci have boundaries at the points of the insertion of the kink operator.

[XVII] Analogously to what we have done in eq. (7.92), for A definite positive, $a > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sa} = -\int_0^\infty e^{-sa} = -\frac{1}{a} \quad \Rightarrow \quad \int_0^\infty \frac{\mathrm{d}s}{s} e^{-sa} = -\log a + c \quad \Rightarrow \quad \log A = -\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sA} - c \tag{7.98}$$

where c is an (infinite) constant which can be reabsorbed in the partition function.

As in the previous example with the ϕ operator, the crucial point is that, although the correlation function is written in terms of (also[XVIII]) open line defects, it should depend only on the boundary points of the locus of these defects.

Construction of the 2-points correlation functions for the kink

In the following we will sketch the procedure to "open" the defects for the kinks in ϕ_2^4 in the case of the 2-points function. In general the complete procedure is not so easy.
Let x_1, x_2 be the Euclidean spacetime points of the kink operator insertion, i.e. the points of the kink correlation function. Let γ be a path with boundary points x_1 and x_2. Define the "parallel transporter" of γ along an arbitrary path ω supported in $\mathbb{R}^2 \setminus \{x_1, x_2\}$ by

$$U(\omega|\gamma) := (-1)^{I(\omega,\gamma)} \tag{7.101}$$

where

$$I(\omega,\gamma) := \text{number of intersections between } \omega \text{ and } \gamma \tag{7.102}$$

This allow us to define a covariant derivative of ϕ: let e_μ be the unit vector in the $\mu = 0, 1$ directions, then the covariant derivative is defined by

$$\nabla_\mu^\gamma \phi(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\phi\left(x + \frac{\varepsilon}{2}e_\mu\right) - U\left(\langle x - \frac{\varepsilon}{2}e_\mu, x + \frac{\varepsilon}{2}e_\mu\rangle|\gamma\right)\phi\left(x - \frac{\varepsilon}{2}e_\mu\right)\right] \tag{7.103}$$

where $\langle p, q\rangle$ denotes the straight line from p to q. Look at fig. 7.10 for a representation of such construction.

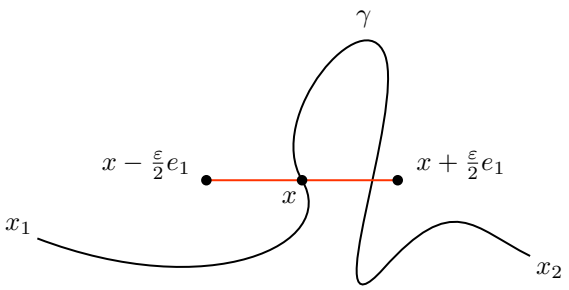


Figure 7.10: Construction of the covariant derivative $\nabla_\mu^\gamma \phi(x)$ in eq. (7.103), where the black line is the path γ and the red one is the line $\langle x - \frac{\varepsilon}{2}e_\mu, x + \frac{\varepsilon}{2}e_\mu\rangle$. In this case $I\left(\langle x - \frac{\varepsilon}{2}e_\mu, x + \frac{\varepsilon}{2}e_\mu\rangle, \gamma\right) = 2$. The field ϕ computed at $x - \frac{\varepsilon}{2}e_1$ is parallel transported using eq. (7.101) along the red line to the final point $x + \frac{\varepsilon}{2}e_1$ and then compared with ϕ evaluated at the same point. The resulting infinitesimal variation determines the value of $\nabla_\mu^\gamma \phi(x)$.

From fig. 7.10 one can see that if $x \in \gamma$ but γ does not intersect itself in x, then taking ε arbitrarily small eventually $I\left(\langle x - \frac{\varepsilon}{2}e_\mu, x + \frac{\varepsilon}{2}e_\mu\rangle, \gamma\right) = 1$. If ϕ is regular then

$$\lim_{\varepsilon\to 0} \frac{\phi\left(x + \frac{\varepsilon}{2}e_\mu\right) + \phi\left(x - \frac{\varepsilon}{2}e_\mu\right)}{\varepsilon} \sim \lim_{\varepsilon\to 0} \frac{2\phi(x)}{\varepsilon} \tag{7.104}$$

and the existence of $\nabla_\mu^\gamma \phi(x)$ implies that ϕ vanishes on x, as otherwise $\lim_{\varepsilon\to 0} \frac{2\phi(x)}{\varepsilon}$ diverges. Hence, the introduction of the parallel transporter $U(\omega|\gamma)$ creates the locus of an open line defect on the regular curve γ. In the following we always assume that γ does not intersect itself (which is the case we are interested in, since we want to build open defects). Notice that this is not restrictive, indeed any loop in γ describes a closed defect, hence we can decompose γ as the union of open and closed defects.
For $x \notin \gamma$ we have $\lim_{\varepsilon\to 0} I\left(\langle x - \frac{\varepsilon}{2}e_\mu, x + \frac{\varepsilon}{2}e_\mu\rangle, \gamma\right) = 0$ and the covariant derivative coincides with the ordinary one: $\nabla_\mu^\gamma \phi(x) = \partial_\mu \phi(x)$.

[XVIII] In the previous case we have seen that the correlation function can be written in terms of open line defects, but this is not completely true, in general different defects should be taken into account.

We define the modified action introducing the parallel transporter in the original one, so that we have an open defects with locus γ:

$$S_\gamma(\phi) = \int \mathrm{d}^2x \, |\nabla^\gamma \phi|^2 + \frac{g^2}{4}(\phi^2 - v^2)^2 \tag{7.105}$$

Then the two-points function reads[XIX]

$$\langle s(x)s(y)\rangle := \left(\frac{\int \mathcal{D}\phi \, e^{-S_\gamma(\phi)}}{\int \mathcal{D}\phi \, e^{-S(\phi)}} \right)_{\text{ren}} \tag{7.106}$$

where γ is a regular curve connecting x and y and $(\dots)_{\text{ren}}$ denotes a suitable renormalization needed to remove the cutoffs we introduced to avoid Haag theorem. For higher (even) number of insertion points different curves connecting pairs of points should be considered, and an appropriate covariant derivative should be used.[XX]
We can represent $\langle s(x)s(y)\rangle$ in terms of defects, such representation in fact contains an open line defect whose locus has boundaries x and y. Typical configurations of the defects in the numerator of eq. (7.106) would be of the form shown in fig. 7.11.

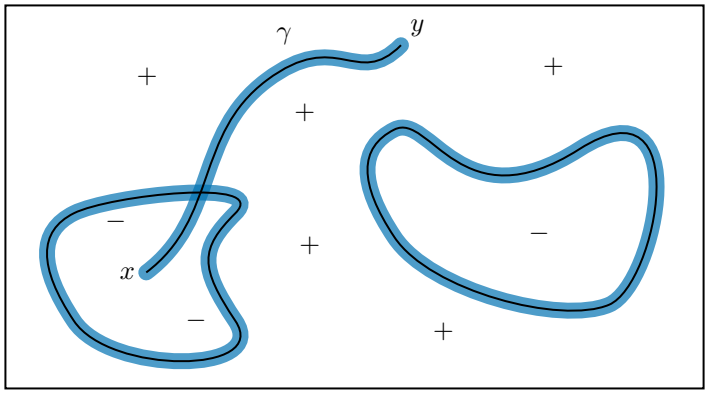


Figure 7.11: Open and closed defects with their loci in the spacetime. The open defect is generated by the curve γ, connecting x and y.

A crucial point is to prove that $\langle s(x)s(y)\rangle$ in eq. (7.106) depends only on x and y and not on γ itself. This is a consequence of the $\mathbb{Z}_2$ gauge invariance, where the $\mathbb{Z}_2$-gauge transformations are defined as follows. Given a partition of $\mathbb{R}^2$ into two disjoint subsets B and B^C define the gauge transformation

$$\big(\xi_B\phi\big)(x) = \begin{cases} -\phi(x) & \text{if} \quad x \in B \\ +\phi(x) & \text{if} \quad x \in B^C \end{cases} \qquad \text{and} \quad \xi_B\gamma = \gamma \cup \partial B \tag{7.107}$$

where ∂B is the boundary of the region B, moreover since ∂B is necessarily closed, then $\partial(\xi_B\gamma) = \partial\gamma$. Notice that if the intersection of γ and ∂B is non-empty, then we should neglect it in $\xi_B\gamma$ since the curves in eq. (7.107) are not oriented, similarly to what happens in the case of $\mathbb{Z}_2$-valued[XXI] simplicial homology.
Then is simple to prove that

$$\nabla^{\xi_B\gamma}(\xi_B\phi) = \xi_B(\nabla^\gamma\phi) \tag{7.108}$$

and

$$(\xi_B\phi)^{2n} = \phi^{2n} \tag{7.109}$$

[XIX] Using the terminology of fibre bundle theory, the integration in the numerator $\int \mathcal{D}\phi \, e^{-S_\gamma(\phi)}$ is performed over distributions of a $\mathbb{Z}_2$ bundle over $\mathbb{R}^2 \setminus \{x, y\}$ with holonomy -1 around x and y.
[XX] It is not needed to consider different possible connections between points, as in the path integral all the possible choices are taken into account considering also all possible closed defects.
[XXI] Meaning that the overlapping of two simplices, instead of being counted twice, is neglected, since $[2] = [0]$ in $\mathbb{Z}_2$.

therefore

$$S^{\xi_B\gamma}(\xi_B\phi) = S^{\gamma}(\phi) \tag{7.110}$$

Furthermore, we clearly have

$$\mathcal{D}\phi = \mathcal{D}(\xi_B\phi) \tag{7.111}$$

provided that B is compact. This implies that the correlator eq. (7.106) is invariant under ξ_B.
Since we can deform arbitrarily γ acting on it with several $\mathbb{Z}_2$-transformations with different choices of B, provided that x and y are fixed (indeed $\partial(\xi_B\gamma) = \partial\gamma$ implies that x and y cannot change under ξ_B), then the invariance of S^γ and $\mathcal{D}\phi$ under ξ_B implies that eq. (7.106) depends only on x and y and not on γ, as desired.
If different open defects are present (several insertion points in the correlator) then the interaction between these defects depend only on the insertion points, and not on how close are the curves chosen to build the covariant derivative.

The reconstruction of the kink operator

One can prove that a natural generalization of the above construction of the correlator for an arbitrary even number of points (correlators of an odd number of points are set to zero) satisfy the OS axioms, hence we can reconstruct a quantum kink field operator $\hat{s}(x)$ which creates and destroy kinks, such that

$$\langle\Omega|\,\hat{s}(\boldsymbol{x})e^{-(x^0-y^0)H}\hat{s}(\boldsymbol{y})\,|\Omega\rangle = \langle s(x)s(y)\rangle \tag{7.112}$$

One can prove (rigorously in the lattice approximation) that $\hat{s}(x)$ couples to the vacuum $|\Omega\rangle$ to a one-particle state, so that $\hat{s}(x)$ really creates and annihilate kink particles, which for this reason should be included in scattering states.

The insertion of the meson field - The dual algebra

Notice that up to now we only considered the QFT description of the soliton, but in our theory we can also describe the creation and the annihilation of particles associated to the original scalar field describing the mesons. The QFT description of such particles is just the usual ϕ^4 theory, and we can describe the associated operators without any difficulty computing the correlation functions in the Euclidean spacetime satisfying OS axioms. However describing a theory involving both kinks and mesons is more involved.
First of all, notice that the field $\phi(x)$ is not $\mathbb{Z}_2$-gauge invariant, however we can define a new field

$$\phi_\gamma(x) := \phi(x)\,U(\omega_x|\gamma) \tag{7.113}$$

where ω_x is a straight line at constant Euclidean time from x to $+\infty$, as shown in fig. 7.12, which is $\mathbb{Z}_2$-gauge invariant.

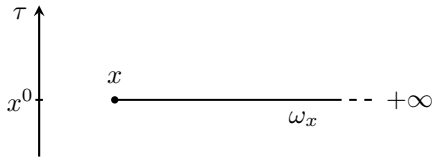


Figure 7.12: Representation of the line ω_x.

With the insertion of $\phi_\gamma(x)$ the correlation functions become

$$\langle s(x)s(y)\phi(z)\phi(w)\rangle = \frac{\int \mathcal{D}\phi\, e^{-S_\gamma(\phi)}\phi_\gamma(z)\phi_\gamma(w)}{\int \mathcal{D}\phi\, e^{-S_\gamma(\phi)}} \tag{7.114}$$

with γ line from x to y, and satisfy OS axioms except for the symmetry property between kink and meson insertions, which means that the quantum field $\hat{\phi}(x)$ and $\hat{s}(z)$ are not relatively local.

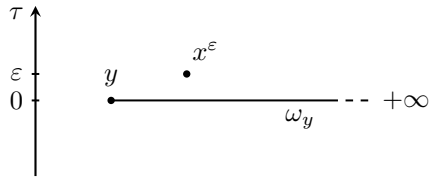
However it is easy to see that if we consider the insertion of the kink at $x^{\pm\varepsilon} = (x^0 = \pm\varepsilon, x^1)$ and the insertion of the meson at $y = (y^0 = 0, y^1)$ and we denote by[XXII] $S^{2n,m}(x_1, \dots, x_{2n}; y_1, \dots, y_m)$ the Schwinger functions for $2n$ kinks and m mesons we get

$$\lim_{\varepsilon\to 0} S^{2n,m}(\dots, x^\varepsilon; y, \dots) = \lim_{\varepsilon\to 0}(-1)^{\theta(x^1-y^1)} S^{2n,m}(\dots, x^{-\varepsilon}; y, \dots) \tag{7.115}$$

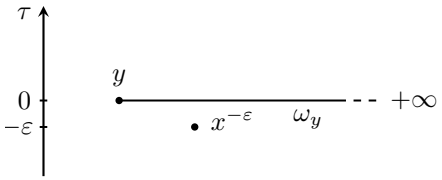
so that the reconstructed fields satisfy a new commutation relation called *dual algebra*[XXIII] (writing x^1 instead of $(0, x^1)$, same for y_1)

$$\hat{s}(x^1)\hat{\phi}(y^1) = (-1)^{\theta(x^1-y^1)}\hat{\phi}(y^1)\hat{s}(x^1) \tag{7.116}$$

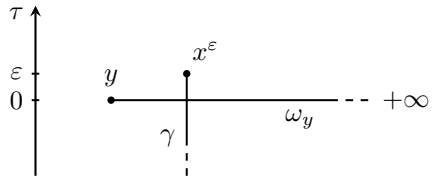
where θ is the Heaviside step function, proving that among fields of a relativistic QFT are possible also such "strange" commutation relations in presence of solitons.
Let us motivate eq. (7.115) at least graphically. We start from the case $x^1 > y^1$. If we insert a kink in x^ε and a meson at y, the situation is the following



whereas, inserting the kink in $x^{-\varepsilon}$ we get



Now we have to fix the second boundary point defining the open defect of the kink, and suppose without loss of generality that such point has time coordinate strictly smaller than $-\varepsilon$. Then the we get



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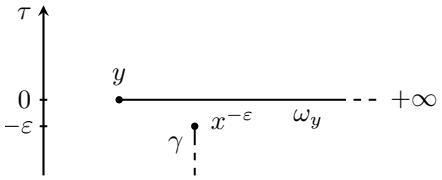
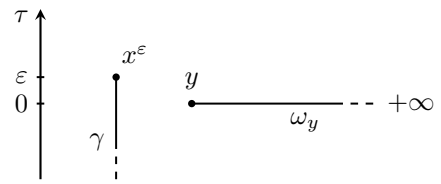


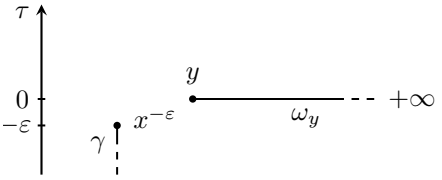
so that in the first case $U(\omega_y|\gamma) = -1$ whereas in the second one $U(\omega_y|\gamma) = 1$. In terms of the field ϕ_γ we get $\phi_\gamma(y) = -\phi(y)$ if the boundary point of γ is x^ε, and $\phi_\gamma(y) = \phi(y)$ if the boundary point is $x^{-\varepsilon}$. Hence in the limit $\varepsilon \to 0$ we get eq. (7.115) for $x^1 > y^1$ (consider for instance the simplest case of eq. (7.114)).

[XXII] The first $2n$ variables denote the boundaries of n curves $\gamma_1, \dots, \gamma_n$, describing the $2n$ insertion points of the kinks, and the last m variables denotes the insertion points of the m mesons.
[XXIII] Such new algebra appeared for the first time in [Frö76].

Instead, in the case $x^1 < y^1$, the situation in the following:



and



In this case $U(\omega_y|\gamma) = U(\omega_y|\gamma) = 1$, so $\phi_\gamma(y) = \phi(y)$ for both x^ε and $x^{-\varepsilon}$ and we get eq. (7.115) for $x^1 < y^1$.

Notice that eq. (7.116) give a new type of commutation relations, different respect to both bosonic and fermionic ones, which depend on the space coordinates. Such statistics is the *braid statistic* in 1 dimension, it differs from that in 2 dimensions since there the phase introduced in the commutation relation depends on whether the particles are interchanged in the clockwise or in the anticlockwise direction, whereas in 1 dimension it depends on whether the order of the quantum fields agrees with the order of their position in $\mathbb{R}$. It worth to highlight the fact that the existence of both these statistics depends on the possibility of give a well-define ordering of the points of our spacetime. For instance in $D = 3 + 1$ such statistics cannot exists, at least for massive particles. The factor $(-1)^{\theta(x^1-y^1)}$ in eq. (7.116) provides the R-matrix representation of an oriented exchange, a generator of the braid group.

The fermionic excitations

Let us define a new field

$$\hat{\psi}(x) := \hat{\phi}(x)\hat{s}(x) \tag{7.117}$$

then we have (writing x^1 instead of $(0, x^1)$, same for y_1)

$$\begin{aligned}
\hat{\psi}(x^1)\hat{\psi}(y^1) &= \hat{\phi}(x^1)\hat{s}(x^1)\hat{\phi}(y^1)\hat{s}(y^1) \\
&= (-1)^{\theta(x^1-y^1)}\hat{\phi}(x^1)\hat{\phi}(y^1)\hat{s}(x^1)\hat{s}(y^1) \\
&= (-1)^{\theta(x^1-y^1)}\hat{\phi}(y^1)\hat{\phi}(x^1)\hat{s}(y^1)\hat{s}(x^1) \\
&= (-1)^{\theta(x^1-y^1)}(-1)^{\theta(y^1-x^1)}\hat{\phi}(y^1)\hat{s}(y^1)\hat{\phi}(x^1)\hat{s}(x^1) \\
&= -\hat{\psi}(y^1)\hat{\psi}(x^1)
\end{aligned} \tag{7.118}$$

hence $\hat{\psi}$ describes a fermionic field, and the theory admits fermionic excitations.
The operator $\hat{\psi}$ is the analogue of the Onsager fermion field at the critical point in the Ising model, with the difference that the Onsager field is free whereas $\hat{\psi}$ describes a strongly interacting field.

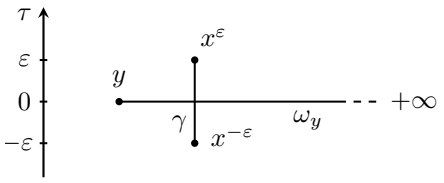
The reconstructed Hilbert space

We now discuss which is the relation between the kink operator and the classical kink solution. Let us compute the expectation value of $\hat{\phi}(y^1)$ in presence of the kink operator, in particular we compute (we

are still writing x^1 instead of $(0, x^1)$, same for y_1)[XXIV]

$$\tilde{\phi}_S(x^1, y^1) := \frac{\langle e^{-\varepsilon H}\hat{s}(x^1)\Omega|\,\hat{\phi}(y^1)\,|e^{-\varepsilon H}\hat{s}(x^1)\Omega\rangle}{\langle e^{-\varepsilon H}\hat{s}(x^1)\Omega|e^{-\varepsilon H}\hat{s}(x^1)\Omega\rangle} \tag{7.119}$$

The numerator in the previous expression can be represented by



so that $\tilde{\phi}_S$ must have a zero at x^1, since at the defect γ the field changes sign. Moreover, if we impose positive boundaries conditions for the field: $\langle\Omega|\,\hat{\phi}\,|\Omega\rangle = \phi_+$, then

$$\phi_\gamma(y^1) = \phi(y^1)U(\omega_y|\gamma) \to \pm\phi(y^1) \quad \text{for} \quad y^1 \to \pm\infty \tag{7.120}$$

since ω_y either does not intersect γ or intersect it certainly. Then by cluster property we have

$$\langle e^{-\varepsilon H}\hat{s}(x^1)\Omega|\,\hat{\phi}(y^1 \to \pm\infty)\,|e^{-\varepsilon H}\hat{s}(x^1)\Omega\rangle \quad \to \quad \phi_\pm\,\langle e^{-\varepsilon H}\hat{s}(x^1)\Omega\big|e^{-\varepsilon H}\hat{s}(x^1)\Omega\rangle \tag{7.121}$$

so that $\tilde{\phi}_S(x^1, y^1)$ defined in eq. (7.119) behaves topologically (that is, at boundaries) as the classical kink with coordinate y^1 and modulus x^1, which we denoted by $\phi_S(y^1 - x^1)$:

$$\tilde{\phi}_S(x^1, y^1) \simeq \phi_S(y^1 - x^1) \tag{7.122}$$

Moreover both the objects vanish at $y^1 = x^1$.

With positive boundaries conditions the reconstruction theorem proves that the Hilbert space can be separated in a vacuum sector and a kink sector

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_S \quad \text{with} \quad |\Omega\rangle \in \mathcal{H}_+ \quad \text{and} \quad \hat{s}(x)\,|\Omega\rangle \in \mathcal{H}_S \tag{7.123}$$

The direct sum (hence the non-interference between elements of the two spaces) is due to the fact that there are no inner products between states with an even and an odd number of solitons, because the correlation function with an odd number of kink insertion vanish. Moreover clearly $\hat{s}$ maps vectors of $\mathcal{H}_+$ in vectors of $\mathcal{H}_S$, $\hat{s}(x) : \mathcal{H}_+ \to \mathcal{H}_S$. The splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_S$ can be equivalently stated saying that $\mathcal{H}_S$ is a *superselection sector*.
The fact that a correlator with an odd number of insertion points vanishes can be motivated as follows: we know that a kink is described by a open line defect, hence is associated to two points. The only configuration of the defects which allows for an odd number of insertion points is the one where one (or more) open defect γ connects a point z in the spacetime to the boundary ∞. However, since the kink carries a mass, the energy required to move the kink from z to ∞ is proportional to $m_{\text{kink}} \cdot \text{length}(\gamma) = \infty$.

In the "unbroken" phase with $\mathbb{Z}_2$-global symmetry, it turns out that $\mathcal{H}$ cannot be separated in the direct sum of two subspaces. Let us show intuitively why. If the $\mathbb{Z}_2$ symmetry is restored, then all typical configurations are closed to the vacuum ϕ_0, which now vanishes.[XXV] Therefore it is much easier to find configurations where ϕ vanishes. Hence there are much more defects respect to the broken-phase case, and typically the net of defects (the collection of regions where ϕ vanishes) reaches the boundary ∞ of the spacetime. If this is the case, then an additional line defect γ from z to ∞ cannot affect the correlation function. But then by the cluster property the kink position sent to infinity is incompatible with $\langle\Omega|\,\hat{s}(x \to \infty)\,|\Omega\rangle = 0$. Hence the Hilbert space $\mathcal{H}$ has no superselection kink sectors, $\mathcal{H} \equiv \mathcal{H}_0$, and the phase in which the net of defects reaches the infinity corresponds to the unbroken $\mathbb{Z}_2$-symmetry.

[XXIV] The factor $e^{-\varepsilon H}$ comes from the proof the the OS reconstruction theorem, in particular is required to have a well defined expression since $\hat{\phi}$ is defined in the domain eq. (5.13).
[XXV] Analogously, in the Ising model the $\mathbb{Z}_2$ symmetry is restored in the paramagnetic phase, where $\langle S_i\rangle = 0$.

The configuration, which appears in the unbroken phase, in which the defects almost covers the whole space time, is said *condensation of kinks*.

By reconstruction theorem we know that in $\mathcal{H}$ there is a unitary representation of the Poincaré group, which in turn implies that we have unitary representation of the Poincaré group also in both $\mathcal{H}_+$ and $\mathcal{H}_S$. Recall that at classical level the modulus of the kink breaks translational invariance, which at the level of quantum mechanics is recovered by promoting the modulus to a quantum coordinate. In the QFT framework, instead, translational invariance is recovered directly in the proof of the OS reconstruction theorem. By uniqueness of the vacuum, there are no translational invariant vector states in $\mathcal{H}_S$ (as the only one is the vacuum, which is an element of $\mathcal{H}_+$).

Chapter 8

Vortices

The second quantum soliton that we consider is the vortex in $2+1$ dimensions in a model[I] that in high-energy is known as *Abelian Higgs model* and in its version in $3+1$ dimensions was the first model proposed to make massive gauge fields in a gauge theory without losing gauge-invariance.
In its non-relativistic version in condensed matter is described by the *Landau-Ginzburg model* and in 3 space dimensions it was proposed as a phenomenological model for superconductors.
Our discussion will be performed in the Lagrangian formalism, starting from the classical model.

8.1 Classical treatment

[Shi12, Chapter 3], [FM89]

The classical Lagrangian

The field content is made of a complex scalar field ϕ (whose complex conjugate is denoted by ϕ^*) and a $U(1)$ gauge field A_μ. The classical relativistic Lagrangian is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + |D^\mu\phi|^2 - \lambda(|\phi|^2 - v^2)^2 \tag{8.1}$$

where e is the electric charge, the covariant derivative is defined by

$$D_\mu\phi := (\partial_\mu - in_eA_\mu)\phi \tag{8.2}$$

and n_e is the electric charge of ϕ in units of e.
The non-relativistic Euclidean version replaces $|D^0\phi|^2$ by a first order term

$$|D^0\phi|^2 \quad \to \quad \phi^*(\partial_0 - in_eA_0)\phi \tag{8.3}$$

For the model of superconductivity ϕ is a field representing the large distance behaviour of the Cooper pairs generated by phonon attraction and $n_e \equiv 2$. The vortices that will be discussed later in fact really appear in nature.

Application of vortices

A lattice configuration of such vortices, called *Abrikosov*[II] *vortices*, is the equilibrium state of a class of superconductors in the presence of a magnetic field, orthogonal to the surface of the superconductors, whose direction will be denoted by z. The z-dependence is then trivial, and in the gauge $A_z = 0$ the $3+1$ model reduces to a $2+1$ model.
Notice that a typical characteristics of superconductors is the expulsion of the magnetic field (*Meissner effect*), but there are two behaviours of superconducting materials in this respect, called *type I* and *type*

[I] In this chapter $x \equiv (x^0, x^1, x^2) \equiv (x^0, \boldsymbol{x})$.
[II] Nobel prize in the 2003.

II. In type I the magnetic flux is completely expelled from the bulk of the material, whereas in type II it penetrates in the superconductor in tubes whose two-dimensional cross section are the vortices, as shown in fig. 8.1. Each of these tubes contains a flux $\frac{h}{e}$ and if they are sufficiently many[III] at equilibrium they are arranged in a triangular lattice, the *Abrikosov lattice*.

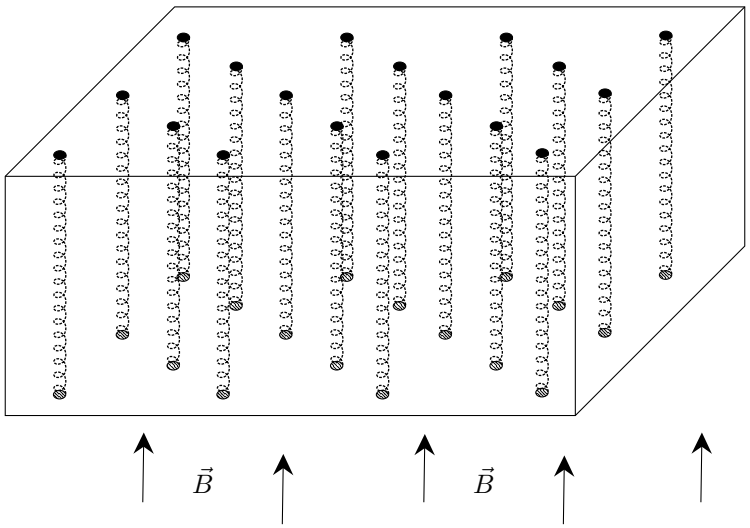


Figure 8.1: Vortices created by the magnetic field in a type II superconductor.

Gauge symmetry, energy density and Higgs mechanism

The model is invariant under the $U(1)$ gauge transformation[IV]

$$\begin{cases} \phi(x) & \to & e^{i\beta(x)}\phi(x) \\ A_\mu(x) & \to & A_\mu(x) + \dfrac{1}{in_e} e^{-i\beta(x)}\partial_\mu e^{i\beta(x)} = A_\mu(x) + \dfrac{1}{n_e}\partial_\mu\beta(x) \end{cases} \tag{8.5}$$

Let us first consider static configurations (no dependence on x^0) in the *temporal* gauge $A_0 = 0$, so that there is no difference between relativistic and non-relativistic models. The energy is given by

$$\int \mathrm{d}^2x\, \mathcal{E}\big(\boldsymbol{A}(\boldsymbol{x}), \phi(\boldsymbol{x})\big) = \int \mathrm{d}^2x \left[\frac{1}{4e^2}F_{ij}^2 + |D_i\phi|^2 + \lambda\big(|\phi|^2 - v^2\big)^2\right] \tag{8.6}$$

and it has global minima at

$$\phi(\boldsymbol{x}) = ve^{i\theta} \quad \text{for} \quad \theta \in [0, 2\pi) \quad , \quad A_i(\boldsymbol{x}) = 0 \tag{8.7}$$

However by gauge invariance this configuration is physically equivalent to

$$\phi(\boldsymbol{x}) = ve^{i[\theta+\beta(\boldsymbol{x})]} \quad , \quad A_\mu(\boldsymbol{x}) = \frac{1}{in_e} e^{-i\beta(\boldsymbol{x})}\partial_\mu e^{i\beta(\boldsymbol{x})} \tag{8.8}$$

with $\beta(\boldsymbol{x})$ globally defined of compact support (indeed it cannot act on the boundary of the spacetime, otherwise it changes the boundary conditions).

The existence of degenerate global minima (up to gauge equivalence) labelled by θ suggests that the global $U(1)$ symmetry is spontaneously broken. Indeed the degeneracy of the vacuum cannot be regarded

[III] In order to increase the number of tubes one can increase the strength of the magnetic field.

[IV] Notice that the r.h.s. of

$$e^{-i\beta(x)}\partial_\mu e^{i\beta(x)} = i\partial_\mu\beta(x) \tag{8.4}$$

is not really well defined, since $\beta(x)$ may be defined with "jumps" of 2π (more formally, we have $\beta : \mathbb{R}^2 \to \mathbb{R}/2\pi\mathbb{Z}$) hence its derivative is not completely well defined on $\mathbb{R}$. Instead, $e^{i\beta(x)}$ is a smooth function, hence its derivative is well defined and smooth as well. However we'll sometime use such notation since it is more compact.

as a manifestation of the (local) gauge symmetry breaking of the theory, since gauge symmetries cannot act at boundaries of the spacetime, whereas the action of $e^{i\theta}$ extends also at the infinity. The presence of spontaneously symmetry breaking of the theory can be shown perturbatively (as in the SM), non-perturbatively in some specific gauge (e.g. in the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$, for which a change of boundary conditions at ∞ is impossible), and using a non-local order parameter (which turns out to have a non zero expectation value independent from the gauge choice).
Due to the *Anderson-Higgs mechanism* (in high energy physics also called *BEH*, due to Brout-Englert-Higgs[V]), let $v=\langle\phi\rangle$ be the (real) expectation value of ϕ in the broken symmetry phase, the gauge field A_μ acquire a mass gap, whose inverse in condensed matter (in the Landau-Ginzburg model) is called *penetration depth*, given in the quadratic approximation by

$$m_A=\sqrt{2}vn_e e \tag{8.9}$$

and also, writing

$$\phi(x)=(v+\chi(x))\,e^{i\theta(x)}\quad\text{with}\quad\chi(x),\theta(x)\text{ real fields} \tag{8.10}$$

the Higgs field $\chi(x)$ acquires a mass gap , whose inverse in condensed matter is called *coherence length*, which in the quadratic approximation is given by

$$m_H=2\sqrt{\lambda}v \tag{8.11}$$

This can be easily proved: inserting the expansion eq. (8.10) in the Lagrangian eq. (8.1) we get

$$\begin{aligned}\mathcal{L}&=-\frac{1}{4e^2}F_{\mu\nu}^2+|(\partial_\mu-in_eA_\mu)[(v+\chi)\,e^{i\theta}]|^2-\lambda(|\,(v+\chi)\,e^{i\theta}|^2-v^2)^2\\&=-\frac{1}{4e^2}F_{\mu\nu}^2+|\partial_\mu\chi+i(v+\chi)(\partial_\mu\theta-n_eA_\mu)|^2-\lambda(2v\chi+\chi^2)^2\end{aligned} \tag{8.12}$$

Defining $\tilde{A}_\mu(x):=A_\mu(x)-\frac{1}{n_e}\partial_\mu\theta(x)$ we finally get

$$\begin{aligned}\mathcal{L}&=-\frac{1}{4e^2}\tilde{F}_{\mu\nu}^2+|\partial_\mu\chi-in_e(v+\chi)\tilde{A}_\mu|^2-\lambda(2v\chi+\chi^2)^2\\&=-\frac{1}{4e^2}\tilde{F}_{\mu\nu}^2+(\partial_\mu\chi)^2+n_e^2v^2\tilde{A}_\mu^2+2n_e^2v\tilde{A}_\mu^2\chi-4\lambda v^2\chi^2-4\lambda v\chi^3-\lambda\chi^4\\&=\underbrace{-\frac{1}{4e^2}\tilde{F}_{\mu\nu}^2+\frac{1}{2e^2}m_A^2\tilde{A}_\mu^2+(\partial_\mu\chi)^2-\frac{1}{2}m_H^2\chi^2}_{\text{kinetic term}}\underbrace{+2n_e^2v\tilde{A}_\mu^2\chi-4\lambda v\chi^3-\lambda\chi^4}_{\text{interaction term}}\end{aligned} \tag{8.13}$$

At this point one can change the notation $\tilde{A}\mapsto A$ and then the masses of A_μ and the χ are exactly m_A and m_H, as we claimed above. Notice that in order to get rid of the field θ in eq. (8.13) we had to absorb it in the gauge potential, the new gauge choice for the potential is usually referred to as *unitary gauge*. The strange factor e^{-2} in $\frac{1}{2e^2}m_A^2\tilde{A}_\mu^2$ should be inserted for dimensional reasons.

Finite energy solutions - Vortices

Let us now impose the finiteness of the energy. The last term in eq. (8.6) forces

$$|\phi|\xrightarrow[|\boldsymbol{x}|\to\infty]{}ve^{if(\alpha)} \tag{8.14}$$

where α is the angle in polar coordinates of $\boldsymbol{x}$ at ∞, $\boldsymbol{x}=|\boldsymbol{x}|e^{i\alpha(\boldsymbol{x})}$, and f is a real function satisfying $e^{if(2\pi)}=e^{if(0)}$. The first term in eq. (8.6) implies that $F_{ij}=0$ at ∞, i.e. A_i is a pure gauge at the boundaries. The second term in eq. (8.6) implies that

$$|e^{if(\alpha)}\partial_je^{if(\alpha)}-in_eA_j|\xrightarrow[|\boldsymbol{x}|\to\infty]{}0 \tag{8.15}$$

so A_j, asymptotically, is the pure gauge

$$A_j\xrightarrow[|\boldsymbol{x}|\to\infty]{}\frac{1}{in_e}e^{-if(\alpha)}\partial_je^{if(\alpha)}=\frac{1}{in_e}\partial_j\log e^{if(\alpha)} \tag{8.16}$$

[V] Nobel prizes in the 2013.

As a consequence of eq. (8.14) we have, asymptotically, the following map

$$e(\boldsymbol{x}) := \frac{\phi(\boldsymbol{x})}{|\phi(\boldsymbol{x})|} \quad : \quad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ \alpha & \longmapsto & e^{if(\alpha)} \end{array} \tag{8.17}$$

which, if continuous, define a homotopy class in $\pi_1(S^1) \cong \mathbb{Z}$.[VI] These maps are classified by the number of times that $e^{if(\alpha)}$ covers the circle labelled by α. For example, if $f(\alpha) = n\alpha$, the homotopy class of $e^{if(\alpha)}$ is $[e^{if(\alpha)}] = n \in \pi_1(S^1)$. In fact, using continuous gauge transformations one can always put $f(\alpha)$ in the form $n\alpha$ for some $n \in \mathbb{Z}$. Let's prove this. From eq. (8.16) we have that

$$\begin{aligned} \lim_{R\to\infty} \oint_{|\boldsymbol{x}|=R} A_j \, \mathrm{d}x^j &= \int_0^{2\pi} \mathrm{d}\alpha \, \frac{1}{in_e} \frac{\mathrm{d}x^j}{\mathrm{d}\alpha} \partial_j \log e^{if(\alpha)} \\ &= \frac{1}{in_e} \int_0^{2\pi} \mathrm{d}\alpha \, \frac{\mathrm{d}}{\mathrm{d}\alpha} \log e^{if(\alpha)} \\ &= \frac{1}{n_e}\big(f(2\pi) - f(0)\big) = 2\pi \frac{n}{n_e} \end{aligned} \tag{8.18}$$

hence we can compute n starting from A_j. If $\beta(x)$ labels a gauge transformation globally defined in $\mathbb{R}^2$ and its derivative is continuous (i.e. $\beta \in \mathcal{C}^1$) then by Stokes theorem

$$\oint_{|\boldsymbol{x}|=R} \partial_j \beta(x) \mathrm{d}x^j = 0 \tag{8.19}$$

hence, for a given $f(\alpha)$ and the corresponding n, we can perform a gauge transformation of A_j with $\beta(x) \in \mathcal{C}^1$ such that

$$\beta(x) \xrightarrow[|\boldsymbol{x}|\to\infty]{} -f(\alpha) + n\alpha \tag{8.20}$$

so that $f(\alpha)$ is replaced by $n\alpha$ in eq. (8.16). Such gauge transformation, even if it acts also at boundaries, do not change the boundary conditions, thanks to eq. (8.19), hence it is allowed in our gauge theory.

Vorticity and magnetic field

Using Stokes theorem one can also derive

$$\lim_{R\to\infty} \oint_{|\boldsymbol{x}|=R} A_j \frac{\mathrm{d}x^j}{2\pi} = \int_{\mathbb{R}^2} \frac{\epsilon_{ij} F^{ij}}{2\pi} \mathrm{d}^2x = \int_{\mathbb{R}^2} \frac{B}{2\pi} \mathrm{d}^2x \tag{8.21}$$

where $B(x)$ is the magnetic field associated to F^{ij} (pointing along the third spatial direction). Hence n can also be interpreted as the magnetic flux, and rescaling A_μ to give the standard form of the Maxwell free action $-\frac{1}{4}F_{\mu\nu}^2$, we get

$$\int B \, \mathrm{d}^2x = n \frac{2\pi}{e} \hbar = n \frac{h}{e} \tag{8.22}$$

where we restored $\hbar$. Notice that $\frac{h}{e}$ is the unit of quantum flux. The number n is also called *vorticity* and its topological origin is at the basis of the stability of the vortices that we will now discuss in details. The configurations minimizing the energy for $n = \pm 1$ are called *vortex* and *anti-vortex* respectively. Plotting the complex field ϕ as a vector, after the gauge transformation eq. (8.20) we get fig. 8.2, which shows clearly the analogy between our discussion and well known vortices in water.

[VI] The homotopy class $\pi_1(S^1)$ is the set of equivalent maps from S^1 to S^1, where the equivalence is defined as follows: two maps are said equivalent if it is possible to deform one map into the other continuously, or better, if exists a continuous map interpolating between the two initial ones. The set $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ since to each equivalence class in $\pi_1(S^1)$ can be associated the number of windings of the map around the second circle when the argument of the map goes from 0 to 2π once.

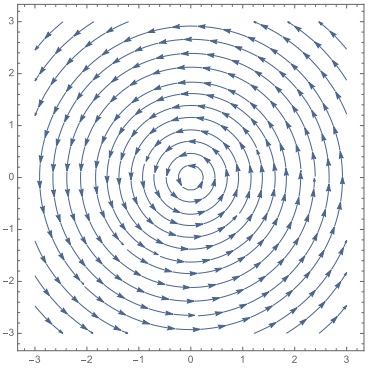


Figure 8.2: Plot of the field ϕ, with $\mathrm{Re}\,\phi$ along x and $\mathrm{Im}\,\phi$ along y, in the case $n = 1$.

Localization of vortices

According to previous considerations, we want to see how the vortices are localized (which is a property required for solitons), at least qualitatively. Let us consider the case $n = 1$, $f(\alpha) = \alpha$, and in the usual notation for polar coordinates we set $r \equiv |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2}$ and $\varphi \equiv \alpha$. Using $\varphi = \arctan \frac{x^2}{x^1}$ we get the following asymptotic behaviours

$$\begin{aligned} A_i(\boldsymbol{x}) &= \frac{1}{n_e}\partial_i\varphi = -\frac{1}{n_e}\epsilon_{ij}\frac{x^j}{r^2} \\ \phi(\boldsymbol{x}) &= ve^{i\varphi} = v\frac{x^1 + ix^2}{r} \end{aligned} \quad \text{for} \quad |\boldsymbol{x}| \to \infty \tag{8.23}$$

We can improve our approximation in finite regions of the spacetime introducing some corrections to our fields, in such a way that we can really obtain a solution of the equations of motion, in terms of some functions $g_A(r)$ and $g_H(r)$:[VII]

$$\begin{aligned} A_i(\boldsymbol{x}) &= \frac{1}{n_e}\partial_i\varphi(1 - g_A(r)) = -\frac{1}{n_e}\epsilon_{ij}\frac{x^j}{r^2}(1 - g_A(r)) \\ \phi(\boldsymbol{x}) &= ve^{i\varphi}(1 - g_H(r)) = v\frac{x^1 + ix^2}{r}(1 - g_H(r)) \end{aligned} \tag{8.24}$$

We claim that a characteristic feature of vortex configurations is that F_{ij} is significantly different from 0 in a region of radius $O(m_A^{-1})$ from the center of the vortex and $\frac{|\phi|}{v}$ is significantly different from 1 in a region of radius $O(m_H^{-1})$, or more precisely

$$\begin{aligned} g_A(r) &\sim e^{-m_A r} \\ g_H(r) &\sim e^{-m_H r} \end{aligned} \quad \text{for} \quad |\boldsymbol{x}| \to \infty \tag{8.25}$$

We will show this with some simplifications. Continuity at 0 implies that

$$g_A(0) = 1 = g_H(0) \tag{8.26}$$

so that ϕ vanishes at the center of the vortex

$$\phi(0) = 0 \tag{8.27}$$

In polar coordinates we can rewrite the first of eq. (8.24) as

$$A_r = 0 \quad \text{and} \quad A_\varphi = \frac{1}{n_e}(1 - g_A(r)) \tag{8.28}$$

Since

$$F_{r\varphi} = \partial_r A_\varphi - \partial_\varphi A_r \tag{8.29}$$

[VII] Such functions depends only on r in such a way that we preserve the behaviour of the vortex.

we get (recall that in polar coordinates the metric which we use to raise indices is not the flat one)

$$\partial^r F_{r\varphi} = \partial_r \frac{1}{r} \partial_r A_\varphi \tag{8.30}$$

so that the equation of motion for A_φ become

$$\partial_r \frac{1}{r} \partial_r A_\varphi = 2i \frac{n_e e^2}{r} \left(\phi^* \partial_\varphi \phi - i n_e \phi^* A_\varphi \phi \right) \tag{8.31}$$

Introducing our ansatz eq. (8.24) we get

$$r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} g_A(r) - 2 n_e^2 e^2 v^2 (1 - g_H(r))^2 g_A(r) = 0 \tag{8.32}$$

and linearizing at large r (hence $g_H \sim 0 \sim g_A$) we get

$$r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} g_A(r) - m_A^2 g_A(r) = 0 \tag{8.33}$$

Putting $g_A(r) = r^\gamma \tilde{g}_A(r)$ we have

$$(2\gamma - 1) r^{\gamma - 1} \frac{\mathrm{d}\tilde{g}_A(r)}{\mathrm{d}r} + r^\gamma \left[\frac{\mathrm{d}^2 \tilde{g}_A(r)}{\mathrm{d}r^2} - m_A^2 \tilde{g}_A(r) \right] = 0 \tag{8.34}$$

We see that the piece in the square brackets gives the exponential behaviour, and in order to compensate $r^{\gamma-1}$ and r^γ we necessarily have $\gamma = \frac{1}{2}$, so that the first factor cancels. Finally we have

$$g_A(r) \sim \sqrt{r} e^{-m_A r} \tag{8.35}$$

hence $g_A(r)$ decays exponentially with the penetration depth m_A^{-1}.
Doing the same for the ϕ, we have the following equation of motion:

$$\Delta_A \phi + m_e^2 e^2 \phi (|\phi|^2 - v^2) = 0 \tag{8.36}$$

where the covariant Laplacian Δ_A (recall that $A_r = 0$) can be written in polar coordinate as

$$\Delta_A = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \varphi} - i A_\varphi \right)^2 \tag{8.37}$$

so that introducing the ansatz eq. (8.24) we get

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} g_H(r) + \frac{1}{r^2} g_A^2(r)(1 - g_H(r)) + 2\lambda v^2 (1 - g_H(r))(-2 g_H(r) + g_H^2(r)) = 0 \tag{8.38}$$

Again linearizing at large r, we can neglect non-linear terms in g_A and g_H, so that the equation of motion become

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} g_H(r) - m_H^2 g_H(r) = 0 \tag{8.39}$$

and, inserting $g_H(r) = r^\gamma \tilde{g}_H(r)$, we finally get the behaviour of g_H:

$$g_H(r) \sim \frac{1}{\sqrt{r}} e^{-m_H r} \tag{8.40}$$

Hence outside of a compact region of radius $O(m_A^{-1}, m_H^{-1})$ around the center of the vortex both the magnetic field associated to A_μ and the scalar field ϕ approach the vacuum, and therefore the vortex behaves as an "extended particle" in the region $O(m_A^{-1}, m_H^{-1})$.

Interactions among vortices - BPS limit

In general these vortices interact among each other, but for $m_H = m_A$, i.e. $\lambda = \frac{1}{2}n_e^2e^2$ a clever inequality (again of the Bogomol'nyi type, sometimes called *Bogomol'nyi-Prasad-Sommerfield (BPS)* in this context) shows that they are free, non-interacting.
To prove such inequality, let us start by rewriting the energy density as

$$\begin{aligned}\int \mathrm{d}^2x\,\mathcal{E}(\boldsymbol{A}(\boldsymbol{x}),\phi(\boldsymbol{x})) = \int \mathrm{d}^2x \left[\frac{1}{2}\left(\frac{B}{e}+n_e e(|\phi|^2-v^2)\right)^2 + |(D_1+iD_2)\phi|^2 \right.\\ \left. + \left(\lambda - \frac{n_e^2e^2}{2}\right)(|\phi|^2-v^2)^2 + n_eeBv^2 - i\partial_j(\epsilon_{ij}\phi^* D_i\phi)\right]\end{aligned} \tag{8.41}$$

Indeed using

$$\begin{aligned}|(D_1+iD_2)\phi|^2 &= |D_i\phi|^2 + i\epsilon_{ij}\phi^* D_jD_i\phi + i\partial_j(\epsilon_{ij}\phi^*D_i\phi)\\ &= |D_i\phi|^2 - n_eeB|\phi|^2 + i\partial_j(\epsilon_{ij}\phi^*D_i\phi)\\ &= |D_i\phi|^2 - n_eeB(|\phi|^2-v^2) + i\partial_j(\epsilon\phi^*D_i\phi) - n_eeBv^2\end{aligned} \tag{8.42}$$

we get from eq. (8.41)

$$\begin{aligned}\int \mathrm{d}^2x\,\mathcal{E} &= \int \mathrm{d}^2x \left[\frac{1}{2e^2}B^2 + \frac{n_e^2e^2}{2}(|\phi|^2-v^2)^2 + Bn_e(|\phi|^2-v^2) + |D_i\phi|^2 - n_eeB(|\phi|^2-v^2)\right.\\ &\quad \left. + i\partial_j(\epsilon\phi^*D_i\phi) - n_eeBv^2 + n_eeBv^2 - i\partial_j(\epsilon_{ij}\phi^*D_i\phi) + \left(\lambda - \frac{n_e^2e^2}{2}\right)(|\phi|^2-v^2)^2\right] \qquad (8.43)\\ &= \int \mathrm{d}^2x \left[\frac{1}{2e^2}B^2 + |D_i\phi|^2 + \lambda(|\phi|^2-v^2)^2\right]\end{aligned}$$

hence eq. (8.41) really coincides with eq. (8.6). Notice that

$$\int_{\mathbb{R}^2} \mathrm{d}^2x\,\partial_j(\epsilon_{ij}\phi^*D_i\phi) = \int_{S^1_\infty} \epsilon_{ij}\phi^*D_i\phi\,\mathrm{d}x^j = 0 \tag{8.44}$$

hence

$$\int \mathrm{d}^2x\,\mathcal{E} = \int \mathrm{d}^2x \left[\frac{1}{2}\left(\frac{B}{e}+n_e e(|\phi|^2-v^2)\right)^2 + |(D_1+iD_2)\phi|^2 + \left(\lambda - \frac{n_e^2e^2}{2}\right)(|\phi|^2-v^2)^2 + n_eeBv^2\right] \tag{8.45}$$

and

$$\text{if} \quad \lambda \geq \frac{(n_ee)^2}{2} \quad \text{then} \quad \int \mathrm{d}^2x\,\mathcal{E} \geq n_eev^2 \int \mathrm{d}^2x\,B \tag{8.46}$$

since the other terms in $\mathcal{E}$ are strictly positive.

In particular for the *BPS limit* or *Bogomol'nyi limit*, where $\lambda = \frac{1}{2}(n_ee)^2$ (i.e. $m_A = m_H$) and

$$\begin{cases}\frac{1}{e}|B| + n_e e(|\phi|^2-v^2) = 0\\ (D_1+iD_2)\phi = 0\end{cases} \tag{8.47}$$

then the energy is proportional to the vorticity

$$\int \mathrm{d}^2x\,\mathcal{E} = n_eev^2 \int \mathrm{d}^2x\,B = 2\pi nv^2 \tag{8.48}$$

where we used the previous identity

$$\int \mathrm{d}^2x\,B = \frac{2\pi n}{n_ee} \tag{8.49}$$

Since $\int \mathrm{d}^2x\,\mathcal{E} \sim n$ this means that vortices are free in the BPS limit. The configurations saturating the *BPS bound* (in this case $m_A = m_H$) are called *BPS states* and even in higher dimensions, where instead of vortices forming world-lines we have higher dimensional vortex defects, they play an important role

in supersymmetric theories, since the preserve some supersymmetry generators and guarantees a special stability of the system (for instance, in our case, BPS states are non-interacting).

From eq. (8.46) we see that for $\lambda > \frac{1}{2}(n_e e)^2$ the energy density is greater than in the non-interacting case, hence vortices repels each others, whereas for $\lambda < \frac{1}{2}(n_e e)^2$ the energy density is smaller and vortices attract each others.
For the BPS solution

$$|B| = n_e e^2(v^2 - |\phi|^2) \tag{8.50}$$

while for $\lambda < \frac{1}{2}(n_e e)^2$ it turns out that

$$|B| \leq c(v^2 - |\phi|^2) \tag{8.51}$$

for some constant c. If vortices attracts each other, the value of $|B|$ increases and at some point we have a violation of eq. (8.51), which implies that vortices cannot exists. Therefore for $\lambda > \frac{1}{2}(n_e e)^2$ we have superconductors of type II, while for $\lambda < \frac{1}{2}(n_e e)^2$ we have superconductors of type I.

Sketch of semi-classical treatment

As for the position x_0 of the kink, the two coordinates of the center of the vortex are the moduli of the vortex, and break the translational invariance of the theory. A semi-classical quantization of the vortex can be obtained by promoting those coordinates to be quantum operators $\hat{\boldsymbol{x}}_0(t) := (\hat{x}_0^1(t), \hat{x}_0^2(t))$ depending on t, restoring the translational invariance.

8.2 Quantum field theory treatment

[FM89], [FM87]

Let us now outline briefly the construction of the quantum vortex operator, using the path-integral formalism, analogously to the construction of the kink operator.

Closed defects

In 3 Euclidean dimensions the vortex is a line defect D. In the path integral formulation of the partition function D appears where $|\phi| = 0$, which is the *locus of the defect*.
In the partition function, for a typical configuration far away from the locus of a defect we have $|\phi| \sim v$ and $A_\mu \sim \partial_\mu \theta$, which implies that $F_{\mu\nu}$ vanishes, hence such configurations are close to the vacuum configurations. Instead, near the locus of a defect, the configurations should be close to a vortex solution in order to maintain the action close to a minimum.
As for the kink, in fig. 8.3 we pictorically describe a typical field configuration of the improved semi-classical approximation for the partition function.

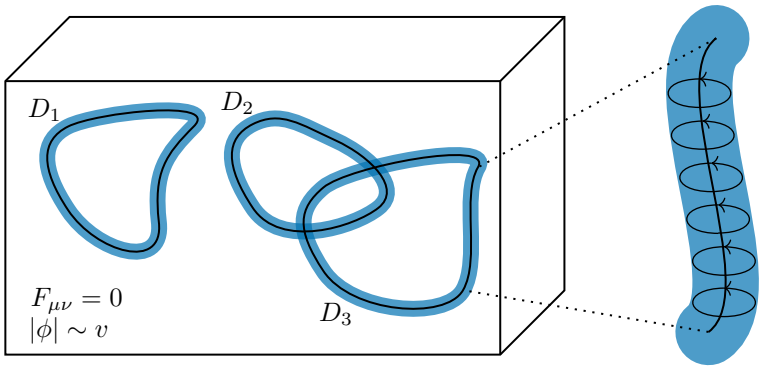


Figure 8.3: Defects (blue closed lines) and their loci (black closed lines). On the right side of the picture, is represented in more detail the vortex associated to the defect D_3.

In order to understand how to "open" these line defects, let's first consider the restriction of the field strength F of the vortex to a 2 dimensional disk crossing transversally the locus of the defect, such that the boundary of the disk identify the region where the configuration approximately approaches the vacuum one, as represented in fig. 8.4.

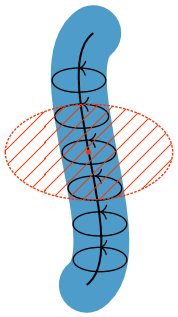


Figure 8.4: Disk crossing the locus of the defect, whose boundary identify the region where the configuration approaches the vacuum.

We then identify the points of the boundary to one-point, obtaining a closed 2-dimensional surface S^2 (topologically but with flat metric), as represented in fig. 8.5 (but for a plane, i.e. for an infinite radius of the disk, and for a sphere not only topological, but metric).

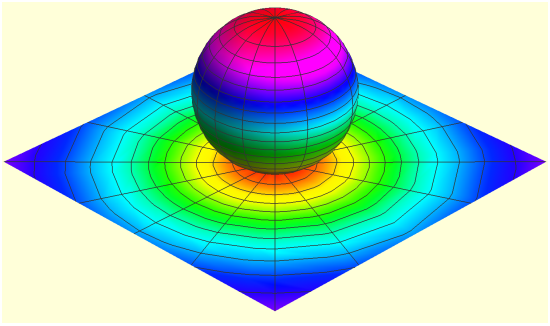


Figure 8.5: Compactification of the plane by identification of the boundaries. Source: https://demonstrations.wolfram.com/TheRiemannSphereAsAStereographicProjection/

According to eq. (8.21) and eq. (8.22), integrating over such sphere (using spatial indices along the disk), we get

$$\frac{1}{2\pi}\int_{S^2} F_{ij}\mathrm{d}S^{ij} = n = \text{vorticity carried by the defect } D \tag{8.52}$$

where we set $n_e e = 1$ in order to get rid of the coupling constant between ϕ and A. Let's see how A_i behaves on such sphere. Using definition eq. (8.2) and writing $\phi = |\phi|e^{i\theta}$, we get

$$\begin{aligned} D_i\phi &= e^{i\theta}\partial_i|\phi| + |\phi|e^{i\theta}(e^{-i\theta}\partial_i e^{i\theta} - iA_i) \\ (D_i\phi)^* &= e^{-i\theta}\partial_i|\phi| + |\phi|e^{-i\theta}(e^{i\theta}\partial_i e^{-i\theta} + iA_i) \end{aligned} \tag{8.53}$$

and then

$$A_i = \underbrace{\frac{i}{2}\frac{e^{-i\theta}D_i\phi - e^{i\theta}(D_i\phi)^*}{|\phi|}}_{\text{regular term } a_i} + \underbrace{\frac{i}{2}(e^{i\theta}\partial_i e^{-i\theta} - e^{-i\theta}\partial_i e^{i\theta})}_{\text{singular term}} \tag{8.54}$$

where the first term is gauge-invariant, hence globally defined, and the curl of the second term is non-vanishing if and only if such term is singular (which may happen for $|\phi| = 0$, since θ is ill defined at this point). Recall that $\theta(x)$ is an angle around the center of the vortex, where $|\phi| = 0$ due to eq. (8.27). Without loss of generality we can assume that the center of the vortex coincide with the origin of the

space. In the sense of distributions, the spatial components of the field strength of the singular term above is[VIII]

$$\epsilon^{ij}\partial_i\left(\frac{i}{2}(e^{i\theta}\partial_j e^{-i\theta}-e^{-i\theta}\partial_j e^{i\theta})\right)=\frac{1}{i}\epsilon^{ij}\partial_i\partial_j\log e^{i\theta}=2\pi\,\delta_{|\phi|^{-1}(0)}(x) \tag{8.55}$$

since the dependence of θ on the space coordinates goes with $\arctan\frac{y}{x}$. Hence such field strength is non zero (actually, singular) only in the center of the vortex. Therefore the spatial part of the field strength of A_μ is

$$F_{ij}(x)=\partial_{[i}a_{j]}(x)+2\pi\,\delta_{|\phi|^{-1}(0)}(x)\epsilon_{ij} \tag{8.56}$$

where a_ν denotes the regular term of eq. (8.54).
We see then that one can identify the locus of the defect in $F_{\mu\nu}$ in terms of a "singular current" (i.e. "δ-like") having the support where $|\phi|$ vanishes, which is necessarily closed because the boundary condition at infinity is $|\phi|=v$.

Open defects

If we want to open a defect and x is a boundary of the locus of such defect, we need to construct a 2-tensor $F^x_{\mu\nu}$ such that for a metric (i.e. curved embedded in $\mathbb{R}^3$) 2-sphere S^2_x centered at x and of arbitrarily small radius we have[IX]

$$\int_{S^2_x}F^x_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu=2\pi n \tag{8.57}$$

as required by eq. (8.52). Then if B^3_x is the ball centered in x whose boundary is S^2_x, by Gauss theorem we have

$$\int_{S^2_x}F^x_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu=\int_{B^3_x}\epsilon^{\mu\nu\rho}\partial_\mu F^x_{\nu\rho}\mathrm{d}^3y=2\pi n \tag{8.58}$$

and this should hold for any arbitrarily small radius, which implies

$$\epsilon^{\mu\nu\rho}\partial_\mu F^x_{\nu\rho}(y)=2\pi n\,\delta(x-y) \tag{8.59}$$

or equivalently, using $\epsilon^{\mu\nu\rho}F^x_{\nu\rho}=B^\mu_x$ where B^μ_x is the magnetic field centered at x,

$$\big(\nabla\cdot B_x\big)(y)=2\pi n\,\delta(x-y) \tag{8.60}$$

This is precisely the magnetic field of a *monopole* in $\mathbb{R}^3$, i.e. the analogue of a point-like electric charge q which obeys

$$\big(\nabla\cdot E_x\big)(y)=q\,\delta(x-y) \tag{8.61}$$

with the magnetic field replacing the electric field.
Therefore we can build open defects constructing line defects with monopoles at the boundaries. With respect to fig. 8.4, this is equivalent to the introduction of two points at the boundaries of the line, describing classical monopoles, as represented in fig. 8.6. Comparing eq. (8.52) and eq. (8.60) we see that such monopoles "provide the right vorticity" to the defect.

Figure 8.6: Representation of an open defect, whose boundaries are described by monopoles (red circles).

[VIII] The distribution $\delta_{|\phi|^{-1}(0)}(x)$ is made of δ-distributions centered at points of $|\phi|^{-1}(0)$.
[IX] This is the same of eq. (8.52), assuming that there is no component along x^0 in $S^2_{\boldsymbol{x}}$.

Correlators

Similarly to what we have done for the kink, where we modified the action introducing a new parallel transporter to construct the open defects, now we should insert classical monopole-antimonopole pairs at the points where we create and annihilate the quantum vortices of our theory. Correlators of the quantum vortex field will have insertion points at the monopoles positions. As usual, our construction can be made rigorous in the lattice approximation.

Let $F^{xy}_{\mu\nu}$ be the field strength of a monopole at y ($n = 1$) and a antimonopole ($n = -1$) at x and let A^{xy}_{μ} be the corresponding gauge potential. A possible choice for $F^{xy}_{\mu\nu}$ and A^{xy}_{μ} is the following. Let γ_{xy} be a curve with boundary x and y in $\mathbb{R}^3$ oriented from y to x and define the singular current[X]

$$j^{xy}_{\mu\nu}(z) := \int_{\gamma_{xy}} \mathrm{d}w^{\alpha}\, \epsilon_{\alpha\mu\nu}\delta(z-w) \tag{8.62}$$

and take

$$\begin{aligned} F^{xy}_{\mu\nu}(z) &:= -2\pi \int \mathrm{d}^3 w\, \partial^{\alpha}\Delta^{-1}(z-w)\, \frac{1}{2}\partial_{[\alpha} j^{xy}_{\mu\nu]}(w) \\ &= 2\pi \int \mathrm{d}^3 w\, \partial^{\alpha}\Delta^{-1}(z-w)\epsilon_{\alpha\mu\nu}[\delta(w-x) - \delta(w-y)] \end{aligned} \tag{8.63}$$

where $\Delta^{-1}(x)$ is the kernel of the inverse of the Laplacian in 3 dimensions, and in the second line we used Gauss theorem to evaluate the integral at the boundaries of γ_{xy} (the change of sign is due to the orientation of γ_{xy}). For the gauge potential we can choose

$$A^{xy}_{\mu}(z) := 2\pi \int \mathrm{d}^3 w\, \partial^{\alpha}\Delta^{-1}(z-w) j^{xy}_{\alpha\mu}(w) \tag{8.64}$$

Let us check that A^{xy}_{μ} and $F^{xy}_{\nu\rho}$ really describe monopoles at the boundaries of the vortex, i.e. satisfy eq. (8.59), which now reads

$$\epsilon^{\mu\nu\rho}\partial_{\mu} F^{xy}_{\nu\rho}(z) = 2\pi\big(\delta(z-x) - \delta(z-y)\big) \tag{8.65}$$

[X] This is the *Poincaré dual* of γ_{xy}.

According to our definitions we have[XI]

$$
\begin{aligned}
&\epsilon^{\mu\nu\rho}\partial_\mu F^{xy}_{\nu\rho}(z) = \\
&\quad = -2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \partial^\alpha\Delta^{-1}(z-w)\,\frac{1}{2}\partial_{[\alpha}j^{xy}_{\nu\rho]}(w) \\
&\quad = -2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \partial^\alpha\Delta^{-1}(z-w)\,\big(\partial_\alpha j^{xy}_{\nu\rho}(w) + \partial_\nu j^{xy}_{\rho\alpha}(w) + \partial_\rho j^{xy}_{\alpha\nu}(w)\big) \\
&\quad = -2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \big(\partial_\alpha\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\nu\rho}(w) + \partial_\nu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\rho\alpha}(w) + \partial_\rho\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\alpha\nu}(w)\big) \\
&\quad = -2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \big(\partial_\alpha\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\nu\rho}(w) + \partial_\nu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\rho\alpha}(w) - \partial_\nu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\alpha\rho}(w)\big) \\
&\quad = -2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \big(\partial_\alpha\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\nu\rho}(w) + 2\partial_\nu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\rho\alpha}(w)\big) \\
&\quad = 2\pi\epsilon^{\mu\nu\rho}\partial_\mu \int \mathrm{d}^3w\, \delta(z-w) j^{xy}_{\nu\rho}(w) \\
&\quad = 2\pi\epsilon^{\mu\nu\rho}\partial_\mu\, j^{xy}_{\nu\rho}(z) \\
&\quad = 2\pi\,\partial_\mu \int_{\gamma_{xy}} \mathrm{d}w^\alpha \overbrace{\epsilon^{\mu\nu\rho}\epsilon_{\alpha\nu\rho}}^{\delta^\mu_\alpha}\delta(z-w) \\
&\quad = 2\pi \int_{\gamma_{xy}} \mathrm{d}w^\alpha\, \partial_\alpha\delta(z-w) \\
&\quad = 2\pi\,\big(\delta(z-x) - \delta(z-y)\big)
\end{aligned}
\tag{8.67}
$$

as we claimed. Now, let's derive the relation between $F^{xy}_{\mu\nu}$ and A^{xy}_μ, which should coincide with eq. (8.56). One can write

$$
\begin{aligned}
F^{xy}_{\mu\nu}(z) &= -2\pi \int \mathrm{d}^3w\, \partial^\alpha\Delta^{-1}(z-w)\,\frac{1}{2}\partial_{[\alpha}j^{xy}_{\mu\nu]}(w) \\
&= -2\pi \int \mathrm{d}^3w\, \big(\partial_\alpha\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\mu\nu}(w) + \partial_\mu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\nu\alpha}(w) + \partial_\nu\partial^\alpha\Delta^{-1}(z-w) j^{xy}_{\alpha\mu}(w)\big) \\
&= 2\pi\, j^{xy}_{\mu\nu}(z) + \partial_\mu A^{xy}_\nu(z) - \partial_\nu A^{xy}_\mu(z) \\
&= 2\pi\, j^{xy}_{\mu\nu}(z) + \partial_{[\mu}A^{xy}_{\nu]}(z)
\end{aligned}
\tag{8.68}
$$

Hence the field strength of A_μ is equal to the usual curl of the gauge potential, $2\,\partial_{[\mu}A^{xy}_{\nu]}(z)$, up to a singular current $2\pi\, j^{xy}_{\mu\nu}(z)$. Such current is called *Dirac string*, and was introduced by Dirac, in order to write the field strength of a monopole.

We can prove that $F^{xy}_{\mu\nu}$ is independent on the choice of γ^{xy}, i.e. of the Dirac string, it depends only on the boundary points. Therefore the Dirac string is an unphysical quantity. Indeed if we choose another curve γ'^{xy} with the same boundaries as γ^{xy}, the difference $\gamma'^{xy} - \gamma^{xy}$ has no boundary, hence

$$
\epsilon^{\mu\nu\rho}\partial_\mu(j'^{xy}_{\nu\rho} - j^{xy}_{\nu\rho}) = 0 \quad \Rightarrow \quad j'^{xy}_{\nu\rho} - j^{xy}_{\nu\rho} = \partial_\nu\xi_\rho - \partial_\rho\xi_\nu \tag{8.69}
$$

[XI] In the third identity we used integration by parts together with

$$
\int \mathrm{d}^3w\, \Delta^{-1}(z-w)\partial^{(w)}_\alpha j(w) = -\int \mathrm{d}^3w\, \partial^{(w)}_\alpha\Delta^{-1}(z-w) j(w) = \int \mathrm{d}^3w\, \partial^{(z)}_\alpha\Delta^{-1}(z-w) j(w) \tag{8.66}
$$

In the fourth we exchanged the labels $\nu \leftrightarrow \rho$ in the third term and we used $\epsilon^{\mu\nu\rho} = -\epsilon^{\mu\rho\nu}$. In the fifth we used $j^{xy}_{\mu\nu} = -j^{xy}_{\nu\mu}$. In the sixth we used $\partial^\alpha\partial_\alpha = -\Delta$ (according to our mostly minus signature), so that $\partial_\alpha\partial^\alpha\Delta^{-1}(z-w) = -\delta(z-w)$, and $\epsilon^{\mu\nu\rho}\partial_\mu\partial_\nu = 0$.

with ξ_ν and integer current. But then

$$\begin{aligned}
&\partial_{[\mu}A'^{xy}_{\nu]}(z) - \partial_{[\mu}A^{xy}_{\nu]}(z) \\
&\quad = 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\partial^\alpha\Delta^{-1}(z-w)(j'^{xy}_{\nu]\alpha}(w) - j^{xy}_{\nu]\alpha}(w)) \\
&\quad = 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\partial^\alpha\Delta^{-1}(z-w)\partial_{\nu]}\xi_\alpha(w) - 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\partial^\alpha\Delta^{-1}(z-w)\partial_\alpha\xi_{\nu]}(w) \\
&\quad = 2\pi\int \mathrm{d}^3w\,\partial_{[\mu}\partial^\alpha\Delta^{-1}(z-w)\partial_{\nu]}\xi_\alpha(w) + 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\partial^\alpha\Delta^{-1}(z-w)\cancel{\partial_{[\mu}\partial_{\nu]}}\xi_\alpha(w) - \\
&\qquad - 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\partial_\alpha\partial^\alpha\Delta^{-1}(z-w)\xi_{\nu]}(w) \\
&\quad = 2\pi\int \mathrm{d}^3w\,\cancel{\partial_{[\mu}\partial_{\nu]}}\partial^\alpha\Delta^{-1}(z-w)\xi_\alpha(w) + 2\pi\,\partial_{[\mu}\int \mathrm{d}^3w\,\delta(z-w)\xi_{\nu]}(w) \\
&\quad = 2\pi\,\partial_{[\mu}\xi_{\nu]}(z) \\
&\quad = 2\pi\big(j'^{xy}_{\mu\nu}(z) - j^{xy}_{\mu\nu}(z)\big)
\end{aligned} \tag{8.70}$$

where in the second line α is not included in the antisymmetrization of the indices. Hence using eq. (8.68) we see from eq. (8.68) that $F^{xy}_{\mu\nu}$ is unchanged under the replacement $\gamma^{xy} \mapsto \gamma'^{xy}$.

From eq. (8.68) we see also that $F^{xy}_{\mu\nu}$ has the same structure of the field strength of closed defects, eq. (8.56), when restricted to the transverse plane. The difference is that now $j_{\mu\nu}$ has a boundary: the location of the monopole-antimonopole pairs, since eq. (8.65) is satisfied.

Now we can come back to our vortex correlation function. We define the Euclidean modified Lagrangian (for $n_e = 1$) by including terms associated to the monopoles

$$\mathcal{L}^{xy} = \frac{1}{4e^2}(F_{\mu\nu} + F^{xy}_{\mu\nu})^2 + |(\partial_\mu - i(A_\mu + A^{xy}_\mu))\phi|^2 + \lambda(|\phi|^2 - v^2)^2 + \begin{array}{c}\text{possible counterterms}\\ \text{and gauge fixing terms}\end{array} \tag{8.71}$$

and, in the non-relativistic case case, we substitute

$$|(\partial_0 - i(A_0 + A^{xy}_0))\phi|^2 \quad \to \quad \phi^*(\partial_0 - i(A_0 + A^{xy}_0))\phi \tag{8.72}$$

We denote by S^{xy} the action associated to $\mathcal{L}^{xy}$. Notice that

$$\int \mathrm{d}^3z\,F^{\mu\nu}F^{xy}_{\mu\nu} = \int \mathrm{d}^3z\,\partial^{[\mu}A^{\nu]}F^{xy}_{\mu\nu} = \int \mathrm{d}^3z\,\partial^\mu A^\nu F^{xy}_{\mu\nu} = -\int \mathrm{d}^3z\,A^\nu\partial^\mu F^{xy}_{\mu\nu} \overset{(8.63)}{=} 0 \tag{8.73}$$

where we used $\partial^\mu F^{xy}_{\mu\nu} = 0$, which follows directly from the definition eq. (8.63), and

$$\begin{aligned}
\int \mathrm{d}^3z\,(F^{xy}_{\mu\nu})^2 &= (2\pi)^2\int \mathrm{d}^3z\,\mathrm{d}^3w\,\mathrm{d}^3\tilde{w}\,\partial^\alpha\Delta^{-1}(z-w)\epsilon_{\alpha\mu\nu}[\delta(w-x) - \delta(w-y)]\times \\
&\qquad \times \partial_\beta\Delta^{-1}(z-w)\epsilon^{\beta\mu\nu}[\delta(\tilde{w}-x) - \delta(\tilde{w}-y)] \\
&= (2\pi)^2\int \mathrm{d}^3z\,\mathrm{d}^3w\,\mathrm{d}^3\tilde{w}\,\partial^\alpha\Delta^{-1}(z-w)[\delta(w-x) - \delta(w-y)]\times \\
&\qquad \times \partial_\alpha\Delta^{-1}(z-\tilde{w})[\delta(\tilde{w}-x) - \delta(\tilde{w}-y)] \\
&= (2\pi)^2\int \mathrm{d}^3z\,\mathrm{d}^3w\,\Delta^{-1}(z-w)[\delta(w-x) - \delta(w-y)][\delta(z-x) - \delta(z-y)] \\
&= (2\pi)^2\int \mathrm{d}^3z\,\big[\Delta^{-1}(z-x) - \Delta^{-1}(z-y)\big][\delta(z-x) - \delta(z-y)]
\end{aligned} \tag{8.74}$$

where from the second to the third line we integrated by parts in z and then we used $\partial^\alpha\partial_\alpha\Delta^{-1}(z-\tilde{w}) = -\delta(z-\tilde{w})$. The last line is clearly divergent since both $\int \mathrm{d}^3z\,\Delta^{-1}(z-x)\delta(z-x)$ and $\int \mathrm{d}^3z\,\Delta^{-1}(z-y)\delta(z-y)$ are divergent quantities. However, (once integrated over z) such divergent quantities do not depend on the spacetime points, hence can be eliminated with an appropriate counternterm.

Finally the 2-point correlation function of the vortex in the path integral formalism is given by

$$\langle v_{+1}(x)v_{-1}(y)\rangle := \left[\frac{\int \mathcal{D}\phi\mathcal{D}A\, e^{-S^{xy}}}{\int \mathcal{D}\phi\mathcal{D}A\, e^{-S}}\right]_{\text{ren}} \tag{8.75}$$

where $[\dots]_{\text{ren}}$ denote the previous renormalization and $v_n(x)$ denotes the vortex insertion at x with the monopole at x providing the vorticity n of the vortex.
One can generalize this construction to an arbitrary number of vortex insertions, provided that the total charge of the monopoles is zero. We can also include insertion of ordinary gauge invariant field, such as the electromagnetic field strength of the Wilson loop. However such insertion need a modification with respect to the insertion in the vacuum, as it happened in the kinks case. For example, in order to introduce the x, y vortex insertion in a electromagnetic field correlator we should replace the electromagnetic field strength by

$$F_{\mu\nu} \quad \to \quad F_{\mu\nu} + F^{xy}_{\mu\nu} \tag{8.76}$$

or in the case of a Wilson loop $W_\alpha(\mathcal{C})$ with support on $\mathcal{C}$, we should replace

$$W_\alpha(\mathcal{C}) := e^{i\alpha \oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu} \quad \to \quad W_\alpha(\Sigma) := e^{i\alpha \left[\oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu + \int_\Sigma F^{xy}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right]} \tag{8.77}$$

where Σ is any surface such that $\partial\Sigma = \mathcal{C}$.

The reconstructed QFT

These correlators satisfy OS axioms and the reconstructed vortex field operator $\hat{v}_n(x)$ satisfy

$$\langle v_{+1}(x)v_{-1}(y)\rangle = \langle\Omega|\, \hat{v}_{-1}(\boldsymbol{y})\, e^{-(x^0-y^0)H}\, \hat{v}_{+1}(\boldsymbol{x})\, |\Omega\rangle \tag{8.78}$$

The Wilson loop operator $\hat{W}_\alpha(\Sigma)$ reconstructed from $W_\alpha(\Sigma)$ can be used to define the topological charge $\hat{Q}$ measuring the vorticity n of the vortex. Indeed we can define $\hat{Q}$ by

$$e^{i\alpha 2\pi\hat{Q}} = \lim_{R\to\infty} \frac{\hat{W}_\alpha(\mathcal{C}_R)}{\langle\Omega|\, \hat{W}_\alpha(\mathcal{C}_R)\, |\Omega\rangle} \tag{8.79}$$

where $\mathcal{C}_R$ is a circle of radius R and time coordinate 0, and the limit is taken in the weak topology. Let Σ_R be a flat surface bounded by $\mathcal{C}_R$ and r by the reflection respect to the Euclidean time 0 plane. Then for $x^0 > 0, y^0 > 0$ we have

$$\begin{aligned}
\langle\hat{v}_1(y)\Omega|\, e^{i\alpha 2\pi\hat{Q}}\, |\hat{v}_1(x)\Omega\rangle &= \lim_{R\to\infty} \frac{\langle v_{-1}(ry) \exp\left[i\alpha\left(\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu + \int_\Sigma F^{x(ry)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)\right] v_{+1}(x)\rangle}{\langle \exp\left[i\alpha\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu\right]\rangle} \\
&= \lim_{R\to\infty} \frac{\langle v_{-1}(ry) \exp\left[i\alpha\left(\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu + 2\pi\int_\Sigma j^{x(ry)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\right)\right] v_{+1}(x)\rangle}{\langle \exp\left[i\alpha\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu\right]\rangle} \\
&= \exp\left[2\pi i\alpha \underbrace{\int_\Sigma j^{x(ry)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}_{n}\right] \frac{\cancel{\langle \exp\left[i\alpha\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu\right]\rangle}\langle v_{-1}(ry)v_{+1}(x)\rangle}{\cancel{\langle \exp\left[i\alpha\oint_{\mathcal{C}_R} A_\mu \mathrm{d}x^\mu\right]\rangle}} \\
&= e^{i\alpha 2\pi n}\, \langle\hat{v}_1(y)\Omega|\hat{v}_1(x)\Omega\rangle
\end{aligned} \tag{8.80}$$

where from the first line to the second one we used eq. (8.68) and we changed variable in the path integral of the numerator from A to $\tilde{A} = A + A^{x(ry)}$, and from the second line to the third we used cluster property, as represented in fig. 8.7. As it is clear from the figure, $\int_\Sigma j^{x(ry)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ equals the number of intersection between the curve $\gamma^{x(ry)}$ and Σ_R counted with signs depending on the orientation (as expected in our case $n = 1$).

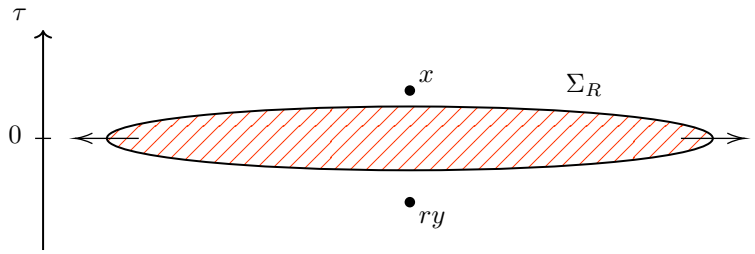


Figure 8.7: Representation of the limit in eq. (8.80) in one dimension less. Arrows represent the limit $R \to \infty$. The ellipse is Σ_R and is completely contained in the $\tau = 0$ plane. As $R \to \infty$ it is clear that any line from ry to x intersects Σ_R an odd number of times with pairs of opposite orientation.

If the total sum of topological charges (vorticity) of the vortices in a correlator is non-vanishing, e.g. $n \in \mathbb{Z}^*$, we can define the correlator with a compensating charge $-n$ and send it to ∞, as in fig. 8.8. Since the mass of the vortex is finite, the contribution in the action of the defect associated to the compensating charge is infinite. This implies that all correlation functions with vanishing total charge are 0, or in other words there are no coherent superposition of states with different total vorticity. Since these correlators can be interpreted as scalar products between states with quantum vortices in the reconstructed Hilbert space, this means that the Hilbert space obtained by OS reconstruction is splitted in superselection sectors, $\mathcal{H}_n$, labelled by the total vorticity charge $n \in \mathbb{Z}$:

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \tag{8.81}$$

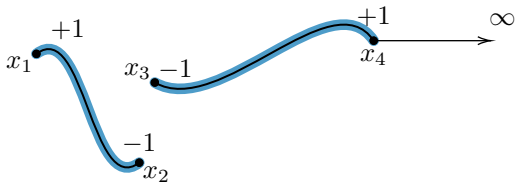


Figure 8.8: Representation of a system with total vorticity -1 (the charge in x_4 cannot be reached by any measure.

One can prove (rigorously in the lattice approixmation) that actually $\hat{v}(x)$ couple the vacuum to a one particle state, hence $\hat{v}(x)$ create and annihilate vortex quantum particles. So there are really particles corresponding to vortices that we have seen, which can be created by scattering. They give a contribution in the spectrum of the Hamiltonian and in the non-relativistic framework they appear as "quasi-particles". In fact in superconductors the presence of these particles is evident because they modify the spectral weight and possibly the dispersion relation of the other particles excitations.

Notice that all the above discussion was in absence of external magnetic field. If an external uniform magnetic field is introduced[XII] then the globally neutral gas of vortices discussed above turns into a gas with total vortex charge per unit area comparable with the uniform magnetic flux through that area.

If we have no symmetry breaking of the global $U(1)$, both in the relativistic and the non-relativistic case, at the minima we have $|\phi| \sim 0$ instead of $|\phi| \sim v$, and we have a condensation of defects reaching the infinity, making the additional defect from x_3 to ∞ in fig. 8.8 invisible. So the Hilbert space of states cannot be splitted into superselection sectors

$$\mathcal{H} = \mathcal{H}_0 \tag{8.82}$$

and the presence of vortices in the partition function is completely irrelevant.

[XII] Of curse in the thermodynamic limit it would need an infinite energy.

Chapter 9

Monopoles

The last solitons we will take into account are the monopoles, in particular we will consider general monopoles and t'Hooft-Polyakov monopoles. As usual we start from them classical theory.

9.1 Introduction

Let us first consider classical electrodynamics in empty space, Maxwell's equations can be written as (we set $c = 1$)

$$\begin{cases} -\dfrac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{B} = 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \end{cases} \quad \leftrightarrow \quad \partial_\mu F^{\mu\nu} = 0$$
$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \end{cases} \quad \leftrightarrow \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma=0} \quad \leftrightarrow \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{9.1}$$

where $\tilde{F}$ is the dual of F: $\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. This system of equations remains unchanged under the replacement

$$\boldsymbol{E} \mapsto \boldsymbol{B} \quad , \quad \boldsymbol{B} \mapsto -\boldsymbol{E} \tag{9.2}$$

or equivalently

$$F_{\mu\nu} \mapsto \tilde{F}_{\mu\nu} \quad , \quad \tilde{F}_{\mu\nu} \mapsto -F_{\mu\nu} \tag{9.3}$$

Such transformation is called *electromagnetic duality*. However, when we add electric sources via a current j_e^μ the invariance under duality is broken, because Maxwell's equations become

$$\begin{cases} \partial_\mu F^{\mu\nu} = j_e^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{cases} \tag{9.4}$$

and clearly they are no more invariant under eq. (9.3).

In 1931 Dirac had the idea to recover the duality invariance introducing also a "magnetic current" j_m^μ

$$\begin{cases} \partial_\mu F^{\mu\nu} = j_e^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} = j_m^\nu \end{cases} \tag{9.5}$$

so that eq. (9.3) still holds provided that under the transformation we exchange the currents $j_e^\nu \leftrightarrow j_m^\nu$. However the introduction of the magnetic current raises a problem, since j_m violates the Bianchi identity

$$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0 \tag{9.6}$$

that guarantees the global existence of the gauge potential A_μ,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{9.7}$$

In fact, assume for simplicity the temporal gauge $A^0 = 0$ and the global existence of $\boldsymbol{A}$. Consider then the zero-component of the second of eq. (9.5), that is $j_m^0 = \partial_\mu \tilde{F}^{\mu 0} = \boldsymbol{\nabla} \cdot \boldsymbol{B}$, and integrate it at a fixed time over a ball B^3 containing a magnetic charge. Then we get

$$Q_m = \int_{B^3} \mathrm{d}^3x\, j_m^0 = \int_{B^3} \mathrm{d}^3x\, \boldsymbol{\nabla} \cdot \boldsymbol{B} = \oint_{\partial B^3 = S^2} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_{S^2} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S^2 = \emptyset} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = 0 \tag{9.8}$$

hence the magnetic charge associated to j_m should vanish if the gauge potential exists globally. This problem can be avoided in two ways:

- In the *Wu-Yang approach* we can define A^μ only on patches (open sets) $\{U_\alpha\}_{\alpha \in A}$ covering S^2, then in $U_\alpha \cap U_\beta$ we have $F_\alpha^{\mu\nu} = F_\beta^{\mu\nu}$, $\alpha, \beta \in A$, but $A_\alpha^\mu = A_\beta^\mu + \partial^\mu \lambda_{\alpha\beta}$ for some gauge transformation $\delta A = \partial^\mu \lambda_{\alpha\beta}$. In this way $F^{\mu\nu}$ is still defined globally even if A^μ isn't, but this is enough in the classical description of physics.
- The other alternative is the one introduced by Dirac, that is the *Dirac string* $j^{\mu\nu}$, introduced in the previous chapter for the quantization of vortices, such that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + j^{\mu\nu} \tag{9.9}$$

The Wu-Yang approach

Let's make some comments about the Wu-Yang approach (as the Dirac solution has already been discussed in the last chapter). On the sphere S^2 centered on the position of the point-like magnetic charge (*monopole*) we introduce spherical coordinates[I] (r, θ, φ) and define two patches whose union covers S^2:

$$U_1 = S^2 \setminus \{\theta = \pi\} \quad , \quad U_2 = S^2 \setminus \{\theta = 0\} \tag{9.11}$$

On U_1 we define

$$\boldsymbol{A}_1 = \frac{Q_m}{4\pi} \frac{\cos\theta - 1}{r \sin\theta} \boldsymbol{e}_\varphi \tag{9.12}$$

where Q_m is the magnetic charge and $\boldsymbol{e}_\varphi$ is the unit vector along φ: in Cartesian coordinates $\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$. It is clear that A_1 is well defined on S^2 except for $\theta = \pi$. Analogously on U_2 we define

$$\boldsymbol{A}_2 = \frac{Q_m}{4\pi} \frac{\cos\theta + 1}{r \sin\theta} \boldsymbol{e}_\varphi \tag{9.13}$$

Let us check that in the intersection of U_1 and U_2 the gauge potential A_1 and A_2 differ only by a gauge transformation. Writing

$$\boldsymbol{A} = A_r \boldsymbol{e}_r + A_\theta \boldsymbol{e}_\theta + A_\varphi \boldsymbol{e}_\varphi \tag{9.14}$$

we find

$$(\boldsymbol{A}_1 - \boldsymbol{A}_2)\big|_{U_1 \cap U_2} = \frac{Q_m}{4\pi} \left[\frac{\cos\theta + 1}{r \sin\theta} - \frac{\cos\theta - 1}{r \sin\theta} \right] \boldsymbol{e}_\varphi = \frac{Q_m}{2\pi} \frac{1}{r \sin\theta} \boldsymbol{e}_\varphi = \frac{1}{2\pi i} e^{-iQ_m\varphi} \boldsymbol{\nabla} e^{iQ_m\varphi} \tag{9.15}$$

where in the last line we used eq. (9.10). We can also write the last result as "$\boldsymbol{\nabla} \frac{Q_m}{2\pi} \varphi$", even if such notation is not completely correct since φ is defined up to jumps of 2π. If we compute the magnetic field we get, using $\boldsymbol{\nabla} \times \boldsymbol{\nabla} = 0$,

$$\boldsymbol{B}_1 = \boldsymbol{\nabla} \times \boldsymbol{A}_1 = \boldsymbol{\nabla} \times \boldsymbol{A}_2 = \boldsymbol{B}_2 \tag{9.16}$$

[I] Some relations for spherical coordinates which will be used in the following:

$$\begin{aligned} \mathrm{d}\boldsymbol{x} &= \frac{\partial \boldsymbol{x}}{\partial r}\mathrm{d}r + \frac{\partial \boldsymbol{x}}{\partial \theta}\mathrm{d}\theta + \frac{\partial \boldsymbol{x}}{\partial \varphi}\mathrm{d}\varphi = \left|\frac{\partial \boldsymbol{x}}{\partial r}\right| \mathrm{d}r \boldsymbol{e}_r + \left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right| \mathrm{d}\theta \boldsymbol{e}_\theta + \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right| \mathrm{d}\varphi \boldsymbol{e}_\varphi = \mathrm{d}r \boldsymbol{e}_r + r\mathrm{d}\theta \boldsymbol{e}_\theta + r\sin\theta \mathrm{d}\varphi \boldsymbol{e}_\varphi \\ \boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{x} &= \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{\partial f}{\partial \varphi}\mathrm{d}\varphi = \frac{\partial f}{\partial r}\boldsymbol{e}_r \cdot \mathrm{d}\boldsymbol{x} + \frac{\partial f}{\partial \theta}\frac{1}{r}\boldsymbol{e}_\theta \cdot \mathrm{d}\boldsymbol{x} + \frac{\partial f}{\partial \varphi}\frac{1}{r\sin\theta}\boldsymbol{e}_\varphi \cdot \mathrm{d}\boldsymbol{x} \\ \boldsymbol{\nabla} &= \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \end{aligned} \tag{9.10}$$

so $\boldsymbol{B}$ is globally well defined on S^2. Setting[II] $\boldsymbol{a}_j := 2\pi \boldsymbol{A}_j$ we see that $\boldsymbol{a}_1$ is related to $\boldsymbol{a}_2$ in $U_1 \cap U_2$ by a well defined $U(1)$ gauge transformation

$$e^{-iQ_m\varphi}\frac{\boldsymbol{\nabla}}{i}e^{iQ_m\varphi} \tag{9.17}$$

with parameter $\lambda_{ij} = Q_m\varphi$. Then also the field strength F_{ij} is globally well defined on S^2 and setting $F_{ij} = \frac{1}{2\pi} f_{ij}$ (f_{ij} turns out to be the curvature of a $U(1)$ connection) we get

$$\int_{S^2} F_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2\pi}\int_{S^2} f_{ij}\mathrm{d}x^i\mathrm{d}x^j = Q_m \tag{9.18}$$

which is also called the *first Chern number*. We may also write

$$f_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{Q_m}{2}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \frac{Q_m}{2}\frac{1}{r^3}\epsilon_{ijk}x^i\mathrm{d}x^j\mathrm{d}x^k \tag{9.19}$$

which implies

$$f_{ij} = \frac{Q_m}{2}\frac{1}{r^3}\epsilon_{ijk}x^k \quad \Rightarrow \quad f_{ij}^2 = \frac{Q_m^2}{4}\frac{1}{r^6}\epsilon_{ijk}\epsilon^{ijl}x^k x_l = \frac{Q_m^2}{4}\frac{1}{r^4} \tag{9.20}$$

Topological constraints and problems in the quantization

By choosing U_1 as the upper semisphere S^2_+ and U_2 as the lower semisphere S^2_-, then $U_1 \cap U_2$ is just the circle in the $z = 0$ plane and

$$\begin{aligned}\int_{S^2} F_{ij}\mathrm{d}x^i\mathrm{d}x^j &= \int_{S^2_+} F_{ij}\mathrm{d}x^i\mathrm{d}x^j + \int_{S^2_-} F_{ij}\mathrm{d}x^i\mathrm{d}x^j = \int_{S^2_+} \boldsymbol{\nabla}\times\boldsymbol{A}_1\cdot\mathrm{d}\boldsymbol{\Sigma} + \int_{S^2_-}\boldsymbol{\nabla}\times\boldsymbol{A}_2\cdot\mathrm{d}\boldsymbol{\Sigma} \\ &= \oint_{S^1}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\mathrm{d}\boldsymbol{x} = \oint_{S^1} e^{-iQ_m\varphi}\frac{\boldsymbol{\nabla}}{2\pi i}e^{iQ_m\varphi}\mathrm{d}\boldsymbol{x} = Q_m\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi} = Q_m\end{aligned} \tag{9.21}$$

Hence Q_m have a new interpretation, namely it counts how many times the gauge transformation $e^{iQ_m\varphi}$ goes around the circle S^1 as φ goes from 0 to 2π (recall that an element of $U(1)$, the gauge group, can be identified with an element of S^1). In other words, $e^{iQ_m\varphi} \in \pi_1(S^1) \cong \mathbb{Z}$, where π_1 is again the first homotopy group. The fact that such $U(1)$ map cannot be deformed to a constant guarantees the stability of the monopole.

Up to now the monopole was considered at a fixed time. If we consider a $3+1$ dimensional theory one can quantize the theory, first considering a static monopole with 3 moduli corresponding to the 3 coordinates of the center of the monopole, so that monopole worldlines in a $3+1$ QFT produce line defects. However, in the case of monopoles we have a qualitative difference with respect to kinks and vortices: the energy of the monopole is UV divergent, indeed

$$\int_{\mathbb{R}^3}\mathrm{d}^3x\,\mathcal{E} \sim \int_{\mathbb{R}^3}\mathrm{d}^3x\,F_{ij}^2 \overset{(9.20)}{\sim} \int_0^\infty \mathrm{d}r\,\frac{1}{r^2} := \lim_{R\to 0}\int_R^\infty\frac{\mathrm{d}r}{r^2} = \lim_{R\to 0}\frac{1}{R} = +\infty \tag{9.22}$$

where $B^3(R)$ denotes a 3-ball of radius R centered on the monopole. Since at quantum level the UV divergence of the energy implies divergence in the semi-classical approximation, this suggest that monopoles in the electrodynamical setting are only defined with a UV cutoff, e.g. on a lattice, as we will discuss later on.

Let us show where this singular behaviour of the monopole comes from. One can view this as a consequence of the impossibility of deforming the $U(1)$ transformation $e^{iQ_m\varphi}$ to a constant over the circle S^1 of arbitrarily small radius, which means that the charge Q_m (which due to eq. (9.21) gives such constraint on the possible deformations) is concentrated in a single point, the center of the monopole. Indeed, the same kind of divergence also appears in the classical description of the electric field, and is due to the localization of the charge of the electron in a point. In the case of the electric field the problem is solved in QFT by replacing the electron with a quantum field, that is an operator valued distribution, which makes sense only when smeared with a test function. The problem here is that in the case of the monopole

[II] Notice that $\boldsymbol{a}$ is the gauge potential with the normalization used in the previous chapter.

even the semi-classical approximation is inconsistent, hence we cannot solve this issue as in the case of the electric field.

Consider our $U(1)$ group as a subgroup of $SU(2)$, and recall that a 4π rotation in $SU(2)$ can be deformed to the identity, since $SU(2)$ is a double cover of $SO(3)$. Suppose that we have a $SU(2)$ gauge theory, instead of the $U(1)$ discussed up to now, in which we are able to construct a solution of the previous equations of motion that behaves like a $U(1)$ monopole of charge 2 at large distances from its center but near the center can explore the entire structure of $SU(2)$.
In particular, from eq. (9.22) this means that at high energies (small r) the whole structure of the $SU(2)$ group is evident, whereas at low energies (large r) only the subgroup $U(1)$ is manifest, that is the $SU(2)$ group structure is "spontaneously broken" at low energies, and only the $U(1)$ generator remains unbroken. Then the previous argument implying infinite energy would not be valid anymore, since a charge 2 in $U(1)$ is the same as the zero charge in $SU(2)$, so the topological charge in the central point turns out to vanish even if asymptotically we have the structure of a charge 2 monopole. This is the basic idea which leads to the description of the t'Hooft-Polyakov monopole.

9.2 Classical treatment of the t'Hooft-Polyakov monopole

[Shi12, Chapter 4]

The *t'Hooft-Polyakov monopole* was first introduced in the *Georgi-Glashow model*, which was one of the first attempts to get a unified theory of weak and electromagnetic interactions (now proved to be wrong, but so suggestive that Glashow shared the Nobel prize with the inventors of the Standard Model, Weinberg and Salam). The basic underlying idea was that the $U(1)$ gauge symmetry of QED was just a subgroup of a larger gauge symmetry, $SO(3)$, "spontaneously broken"[III] to $U(1)$. By the Anderson-Higgs mechanism the $U(1)$ component of the gauge field create massless photons, whereas the other 2 components, corresponding to the spontaneously broken symmetry, create massive vector mesons $W^{\pm}_{\mu}$.[IV]

Let's start the description of the Georgi-Glashow model from the fields of the theory. The $SO(3)$ gauge field is described by

$$A := A^a_\mu \frac{\tau^a}{2} \quad \text{for} \quad a = 1,2,3 \quad \text{and} \quad \tau^a \text{ Pauli matrices} \tag{9.23}$$

and its field strength has components

$$G^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \tag{9.24}$$

The model contains also a 3-components real scalar field

$$\phi := \phi^a \frac{\tau^a}{2} \tag{9.25}$$

and its $SO(3)$ covariant derivative is given by

$$D_\mu \phi^a := \partial_\mu \phi^a + \epsilon^{abc} A^b_\mu \phi^c \tag{9.26}$$

The Euclidean Lagrangian in $3+1$ dimensions, a kind of non-Abelian generalization of the $2+1$ Higgs model discussed for vortices, is given by

$$\mathcal{L} = \frac{1}{4g^2}(G^a_{\mu\nu})^2 + \frac{1}{2}(D_\mu \phi^a)^2 + \lambda\big((\phi^a)^2 - v^2\big) \tag{9.27}$$

From the action we can derive the following equations of motion

$$\begin{cases} D_\mu G^{\mu\nu\, a} = -g^2 \epsilon^{abc} \phi^b D^\nu \phi^c \\ D_{[\mu} G^a_{\nu\rho]} = 0 \\ D_\mu D^\mu \phi^a = 4\lambda \phi^a \big((\phi^b)^2 - v^2\big) \end{cases} \tag{9.28}$$

[III] Here the concept of "spontaneous symmetry breaking" apply either perturbatively, in some gauges, or using a non local order parameter, not as in section 6.2.

[IV] The Z^0 massive uncharged meson was missing: it was introduced in the Standard Model to describe neutral currents, by replacing $SU(3)$ with $SU(2) \times U(1)$.

where the first and the second equations corresponds to the first and the second of eq. (9.4) respectively, and the third correspond to the equation of motion of the scalar field.
The action is invariant under the following gauge transformations: for $g(x) \in SU(2)$[V]

$$\begin{cases} A_\mu & \mapsto & g^{-1}A_\mu g + g^{-1}\partial_\mu g \\ \phi & \mapsto & g^{-1}\phi g \end{cases} \tag{9.29}$$

Symmetry breaking boundary conditions

In order to choose a vacuum configuration, let's impose $SO(3)$ breaking boundary conditions, e.g.

$$\phi^a(\infty) = v\delta^{a3} \quad \leftrightarrow \quad \phi(\infty) = v\frac{\tau^3}{2} \tag{9.30}$$

The direction of the vector $\phi^a(\infty)$ in the $SO(3)$ (inducing the $SO(3)$ breaking) can be chosen arbitrarily, but when once it still leaves a $U(1)$ subgroup of $SO(3)$ unbroken, corresponding to rotations around the chosen axis, e.g.

$$e^{i\alpha\frac{\tau^3}{2}} \underbrace{v\frac{\tau^3}{2}}_{\phi(\infty)} e^{-i\alpha\frac{\tau_3}{2}} = \underbrace{v\frac{\tau^3}{2}}_{\phi(\infty)} \tag{9.31}$$

At least perturbatively one often perform a gauge transformation (*unitary gauge*) reducing $\phi = v\frac{\tau^3}{2}$ everywhere. Then A^3_μ remains gapless and we identify it with the "photon field", whereas

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\frac{1}{g}(A^1_\mu \pm iA^2_\mu) \tag{9.32}$$

are the massive vector meson fields.

Vacuum sector

The vacua of the theory are given by the vanishing of all squares in the energy density $\mathcal{E}$. Define the electric and the magnetic field by $G^{a\,0i} =: E^{a\,i}$ and $G^a_{ij} =: -\frac{1}{2}\epsilon_{ijk}B^{a\,k}$ respectively, then for a static configuration, in the gauge $A^a_0 = 0$,

$$\int \mathrm{d}^3\boldsymbol{x}\,\mathcal{E} = \int \mathrm{d}^3\boldsymbol{x}\left[\frac{1}{2}\left(\frac{(B^{ai})^2}{g^2} + (D_i\phi^a)^2\right) + \lambda\big((\phi^a)^2 - v^2\big)^2\right] \tag{9.33}$$

The global minimum of the energy, for the given boundary conditions, is given by

$$\begin{cases} B^a = 0 & \Rightarrow & A^a_i = g^{-1}\partial_i g \\ \phi = v\dfrac{\tau^3}{2} & & \\ D_i\phi^a = 0 & \Rightarrow & A^a_i = 0 \end{cases} \tag{9.34}$$

where we used the first and the second conditions to obtain the third.

Monopoles

Let's consider the monopole as a static field configuration. First, in order to find the monopole, we want to find the correct "magnetic field" of the model. The "$SO(3)$-magnetic field" B^a_i is not gauge invariant, hence it is unphysical. But its projections on ϕ is gauge invariant, so a natural choice for the "magnetic field" is

$$B^a_i\frac{\phi^a}{|\phi|} = \frac{1}{2}\epsilon_{ijk}G^a_{jk}\frac{\phi^a}{|\phi|} \tag{9.35}$$

and its "magnetic charge" in units of g can be defined as

$$Q_m = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R}\mathrm{d}\Sigma^i\,B^a_i\frac{\phi^a}{|\phi|} \tag{9.36}$$

[V] Actually the gauge transformations of eq. (9.29) leave the center of $SU(2)$ invariant, hence they behaves as $SO(3)$ transformations.

We want to find some finite energy configuration with $Q_m \neq 0$. From the finiteness of the energy eq. (9.33) as $r \to \infty$ we should have[VI] $(\phi^a(\infty))^2 = v^2$ hence at boundaries (described by a sphere S^2_∞ of infinite radius) ϕ takes values in a 2-sphere S^2_ϕ of radius $|v|$. The continuous maps between these spheres are labelled by an integer N corresponding to an element of the homotopy group $\pi_2(S^2) \cong \mathbb{Z}$, identifying how many times ϕ sweeps S^2_ϕ when $\boldsymbol{x}$ sweeps S^2_∞.
Let us consider the case $N = 1$, this clearly occours if at ∞ we have

$$\phi^a \underset{r\to\infty}{\sim} v\frac{x^a}{r} =: vn^a \tag{9.37}$$

We see that the group index "a" is referred to as a group index but also as a spatial index, hence gets "entangled" with a coordinate index of $\boldsymbol{x}$ (this is possible since in both cases it runs over 3 components), whereas this would be impossible in $U(1)$). Furthermore finiteness of the energy imposes also that as $r \to \infty$ the derivative $D_i\phi^a$ decays faster than $r^{-3/2}$. For eq. (9.37) this means that

$$\partial_i\phi^a = \partial_i\left(\frac{vx^a}{r}\right) = \frac{v}{r}(\delta^{ai} - n^a n^i) \underset{r\to\infty}{\sim} \frac{1}{r} \tag{9.38}$$

which is not enough to satisfy $D_i\phi^a \sim r^{-3/2}$. Hence we must choose A^b_i so that

$$D_i\phi^a = \partial_i\phi^a + \epsilon^{abc}A^b_i\phi^c \underset{r\to\infty}{\sim} O(r^{-3/2}) \tag{9.39}$$

This requires

$$A^a_i \underset{r\to\infty}{\sim} \epsilon_{aij}\frac{n^j}{r} \tag{9.40}$$

in fact for such choice of A^a_i[VII]

$$\epsilon^{abc}A^b_i\phi^c = \epsilon^{abc}\epsilon^{bij}\frac{n_j}{r}\frac{x^c}{r}v = \frac{v}{r}(-\delta^{ai} + n^a n^i) \tag{9.43}$$

Again, in eq. (9.40) group and spatial indices are "entangled".
We just obtained that in order to have a finite energy configuration with Q_m non-zero and $N = 1$, eq. (9.37) and eq. (9.40) should hold:

$$\phi^a \underset{r\to\infty}{\sim} v\frac{x^a}{r} = vn^a \quad \text{and} \quad A^a_i \underset{r\to\infty}{\sim} \epsilon^{aij}\frac{x_j}{r^2} = \epsilon^{aij}\frac{n_j}{r} \tag{9.44}$$

Let us compute Q_m for such configuration using eq. (9.36). First notice that

$$\begin{aligned}\frac{\phi^a}{|\phi|}B^a_i = \frac{x^a}{r}B^a_i = \frac{x^a}{r}\left[-\frac{1}{2}\epsilon_{ijk}(\partial_j A^a_k - \partial_k A^a_j + \epsilon^{abc}A^b_j A^c_k)\right] \\ = \frac{x^a}{r}\left[-\epsilon_{ijk}\delta^m_j\frac{\epsilon^{akm}}{r^2} - \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\epsilon^{bjm}\frac{x_m}{r^2}\epsilon^{ckn}r^2\right] = \frac{x_i}{r^3}\end{aligned} \tag{9.45}$$

and then

$$Q_m = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R} \mathrm{d}\Sigma^i\, B^a_i\frac{\phi^a}{|\phi|} = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R}\mathrm{d}\Sigma^i\,\frac{x_i}{r^3} = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R} r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{x^i}{r}\frac{x_i}{r^3} = \frac{4\pi}{g} \tag{9.46}$$

and this is exactly the expected magnetic charge for a $N = 1$ monopole, hence fields as in eq. (9.44) are very good candidates for the asymptotic description of the monopole.

[VI] The boundary condition eq. (9.30) was imposed in the vacuum configuration.
[VII] The following identity is needed:

$$\epsilon^{ijk}\epsilon_{ij'k'} = \delta^j_{j'}\delta^k_{k'} - \delta^j_{k'}\delta^k_{j'} \tag{9.41}$$

We will also use

$$\epsilon^{ijk}\epsilon_{ijk'} = \delta^k_{k'} \tag{9.42}$$

In order to find the description of the fields associated to the monopole at finite distances, we introduce the following ansatz, analogous to the one used for the vortices,

$$A_i^a = \epsilon^{aij}\frac{x_j}{r^2}(1-g_A(r)) \quad , \quad \phi^a = v\frac{x^a}{r}(1-g_H(r)) \tag{9.47}$$

with g_A and g_H functions vanishing for $r\to\infty$. The requirement that the energy eq. (9.33) is finite as $r\to 0$ gives

$$r^2B^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \quad \Rightarrow \quad r^2\frac{1}{r^4}(1-g_A)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \tag{9.48}$$

and

$$r^2(D_i\phi)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \quad \Rightarrow \quad r^2\frac{1}{r^2}(1-g_H)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \tag{9.49}$$

where we used $B\sim\frac{1}{r^2}(1-g_A)$ and $D_i\phi\sim\frac{1}{r}(1-g_H)$. Requiring that B and ϕ are regular, we obtain

$$r^2B^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad r^2\frac{1}{r^4}(1-g_A)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad 1-g_A \underset{r\to 0}{\sim} r \tag{9.50}$$

and simultaneously

$$r^2(D_i\phi)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad r^2\frac{1}{r^2}(1-g_H)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad 1-g_H \underset{r\to 0}{\sim} 1 \tag{9.51}$$

Hence for $1-g_A = O(r)$ and $1-g_H = O(1)$ the solution has finite energy.
These conditions give a different result for the magnetic charge density respect to the case of the Dirac monopole, indeed

$$\lim_{R\to 0}\frac{1}{g}\int_{S^2_R} \mathrm{d}\Sigma^i\, B_i^a\frac{\phi^a}{|\phi|} = 0 \tag{9.52}$$

so the "magnetic charge density" is not concentrated in a point like in the Dirac monopole, and this allows the finiteness of the energy. Indeed eq. (9.47) give really a static solution of the Georgi-Glashow equations of motion eq. (9.28) for $1-g_A = O(r)$ and $1-g_H = O(1)$, and such solution is the (static) t'Hooft-Polyakov monopole.

The $U(1)$ symmetry of the monopole

Recall that a monopole, according to Dirac's description, should be related to a $U(1)$ symmetry. t'Hooft proved that one can define, starting from the non-Abelian field strength $G^a_{\mu\nu}$ of the Georgi-Glashow model, an Abelian $U(1)$ gauge invariant field strength whose singularity is exactly the one of the Dirac monopole. In this way it is possible to see the real monopole structure of the t'Hooft-Polyakov monopole, namely the one corresponding to the Dirac monopole. The correct $U(1)$ "magnetic field" found by t'Hooft is obtained by adding to $B_i^a\frac{\phi^a}{|\phi|}$ a contribution vanishing at ∞, still gauge invariant, so that instead of the regular structure defined up to now we have the singular structure of the Dirac monopole.
In order to simplify the notation, let $e^a := \frac{\phi^a}{|\phi|}$ be the unit vector defined on the points where $|\phi|\neq 0$, i.e. outside the center ($r=0$) of the monopole. The $U(1)$ gauge "magnetic field strength", $SO(3)$ invariant, proposed by t'Hooft is

$$F^{U(1)}_{\mu\nu} := e^a\left[G^a_{\mu\nu} - \epsilon^{abc}D_\mu e^b D_\nu e^c\right] \tag{9.53}$$

Indeed, for $|\phi|\neq 0$ (brackets are omitted when a derivative acts only on the first element on its side, e.g. $\partial_\mu e^b\partial_\nu e^c = (\partial_\mu e^b)(\partial_\nu e^c)$)

$$\begin{aligned} F^{U(1)}_{\mu\nu} &= e^a\big[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu - \epsilon^{abc}\partial_\mu e^b\partial_\nu e^c - \epsilon^{abc}\epsilon^{blm}A^l_\mu e^m\partial_\nu e^c - \\ &\quad - \epsilon^{abc}\partial_\mu e^b\epsilon^{crs}A^r_\nu e^s - \epsilon^{abc}\epsilon^{blm}A^l_\mu e^m\epsilon^{crs}A^r_\nu e^s\big] \\ &= e^a\big[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu - \epsilon^{abc}\partial_\mu e^b\partial_\nu e^c + A^a_\mu e^c\cancel{\partial_\nu e^e} - A^c_\mu e^a\partial_\nu e^c - \\ &\quad - \cancel{\partial_\mu e^b e^b}A^a_\nu + \partial_\mu e^b A^b_\nu e^a - \epsilon^{blm}A^l_\mu A^a_\nu e^m e^c + \epsilon^{blm}A^l_\mu A^b_\nu e^m e^a\big] \\ &= \partial_\mu(e^aA^a_\nu) - \partial_\nu(e^aA^a_\mu) - \epsilon^{abc}e^a\partial_\mu e^b\partial_\nu e^c \end{aligned} \tag{9.54}$$

where cancellations are due to $e^c\partial_\nu e^c = \frac{1}{2}\partial_\nu(e^c e^c) = \frac{1}{2}\partial_\nu(1) = 0$. Notice that $e^a A^a_\mu$ is the projection of A_μ along ϕ, and $F^{U(1)}_{\mu\nu}$ is defined everywhere, beside in the center of the monopole where $|\phi| = 0$. For the monopole solution eq. (9.47) we get

$$e^a = \frac{x^a}{r} \quad \text{and} \quad A^a_i e^a = \epsilon^{aij}\frac{x^a}{r}\frac{x_j}{r^2}(1 - g_A(r)) = 0 \tag{9.55}$$

hence

$$\begin{aligned} F^{U(1)}_{ij} &= -\epsilon^{abc} e^a \partial_\mu e^b \partial_\nu e^c \\ &= -\epsilon^{abc}\frac{x^a}{r}\partial_i\frac{x^b}{r}\partial_j\frac{x^c}{r} \\ &= -\epsilon^{abc}\frac{x^a}{r^3}\partial_i x^b \partial_j x^c \end{aligned} \tag{9.56}$$

where in the third line we used the antisymmetry of ϵ^{abc}. This result is, up to an overall constant factor, the field strength of the Dirac monopole, eq. (9.20). Hence $F^{U(1)}_{ij}$ has the same singularity at $r = 0$ as in the case of the Dirac monopole, but we obtained it from a finite energy configuration of the Georgi-Glashow model, so at least formally it can be quantized in the continuum.

Mass of the monopole

Let's compute the mass of the classical monopole. This is particularly simple in the case $\lambda = 0$ (but still using $\phi^a(\infty) = v\delta^{a3}$ boundary conditions for the vacuum, eq. (9.30)), called *BPS limit*. In this case, in analogy to the Bogomol'nyi treatment of vortices eq. (8.41), one can rewrite

$$\int \mathrm{d}^3x\, \mathcal{E} = \int \mathrm{d}^3x \left[\frac{1}{2g^2}(B^a_i - gD_i\phi^a)^2 + \frac{1}{g}B^a_i D_i\phi^a\right] \tag{9.57}$$

The second term, using the equation of motion $0 = D_{[i}G_{jk]} = D_i B^a_i$, can be rewritten as

$$\int \mathrm{d}^3x\, \frac{1}{g}B^a_i D_i\phi^a = \frac{1}{g}\int \mathrm{d}^3x\, \partial_i(B^a_i\phi^a) - \frac{1}{g}\int \mathrm{d}^3x\, (D_i B^a_i)\phi^a = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R} \mathrm{d}\Sigma^i\, B^a_i\phi^a = vQ_m \tag{9.58}$$

Hence if the monopole satisfies $B^a_i = gD_i\phi^a$ as local minimum of the energy, we have

$$\int \mathrm{d}^3x\, \mathcal{E} = M_m = vQ_m \tag{9.59}$$

This also implies that the t'Hooft-Polyakov monopole in the BPS limit is free and very heavy if g is small. Even if $W^\pm$ mesons described by the Georgi-Glashow model are quite different respect those described by the Standard Model, we may assume that the mass of $W^\pm$ is the same in the two models, and in this case the expected mass of the monopole would be $M_m \sim 10\mathrm{TeV}/c^2$ (about the maximum energy reached in the LHC). It has not be found (yet).

9.3 Quantum mechanical treatment of t'Hooft-Polyakov monopole

[Shi12, Section 15.5]

In the case of the t'Hooft-Polyakov monopole it is important to make some more comments about its quantum mechanical version, since up to now there is no agreed and well defined QFT description of it. The quantum mechanical version of the t'Hooft-Polyakov monopole is obtained promoting the moduli of the the classical solution, corresponding to symmetries broken by the specific choice of the monopole solution, to quantum mechanical (time-dependent) variables. Three moduli correspond to the position of the center of the monopole $\boldsymbol{x}_0$ (up to now $\boldsymbol{x}_0 = \mathbf{0}$). There is however a fourth modulus, corresponding to the fact that the "global"[VIII] $U(1)$-electromagnetic symmetry is unbroken, but a fixed monopole solution

[VIII] Meaning that the symmetry is still described by gauge transformations, but whose group parameters are constant over the whole spacetime.

breaks it. In fact let $e_m(\infty)$ denote the asymptotic behaviour of the normalized scalar field of the monopole. A $U(1)$ transformation of the form $e^{i\alpha e_m(x)}$, where α is constant, leaves unchanged the boundary condition of ϕ_m (the monopole configuration of the scalar field) but modifies the gauge potential by

$$A_i \;\mapsto\; A_i^{(\alpha)} := e^{-i\alpha e_m} A_i e^{i\alpha e_m} + \frac{1}{i} e^{-i\alpha e_m} \partial_j e^{i\alpha e_m} \tag{9.60}$$

We should consider α as a new modulus associated to the specific monopole solution. Hence in the quantum mechanical treatment 4 time dependent variables $\hat{\boldsymbol{x}}_0(t)$ and $\hat{\alpha}(t)$ are introduced, together with their conjugated momenta $\hat{\boldsymbol{p}}_0(t)$ and $\hat{p}_\alpha(t)$, and then the quantum mechanical Hamiltonian of the monopole in the quadratic approximation reads

$$H_m = M_m + \frac{\hat{\boldsymbol{p}}_0(t)}{2M_m} + m_W^2 \frac{\hat{p}_\alpha^2(t)}{2M_m} \tag{9.61}$$

where the factor m_W is needed since α is an angle, hence $\frac{1}{2M_m}\hat{p}_\alpha^2$ has not the right dimension in the Hamiltonian. Since $\hat{\alpha}$ is periodic, then $\hat{p}_\alpha$ has discrete spectrum: $\sigma(\hat{p}_\alpha) \subseteq \mathbb{Z}$ (for $\hbar = 1$). To understand the meaning of these eigenvalues in the simplest case of a BPS solution, notice that

$$\hat{p}_\alpha = \frac{M_m}{m_W^2} \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \tag{9.62}$$

where m_W^2 again comes from a dimensional argument, and define the gauge invariant $SO(3)$ electric field for the monopole solution evaluated in the gauge $A_0^a = 0$ as

$$E_i = E_i^a \frac{\phi^a}{|\phi|} = \mathrm{Tr}(E_i e_m) \quad \text{with} \quad E_i = \partial_0 A_i^{(\alpha)} \tag{9.63}$$

whose quantum mechanical version is

$$\hat{E}_i = \mathrm{Tr}\left(\partial_0 \hat{A}_i^{(\alpha)} e_m\right) = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \mathrm{Tr}\left(\partial_\alpha \hat{A}_i^{(\alpha)} e_m\right) \overset{(9.60)}{=} \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \mathrm{Tr}\left(D_i \hat{e}_m \, e_m\right) \tag{9.64}$$

At large distances, since $|\phi| = v$, we can replace e_m by $\frac{\phi}{v}$, so that asymptotically

$$\hat{E}_i = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \frac{1}{v} \mathrm{Tr}(D_i \phi \, e_m) \tag{9.65}$$

Then using the BPS equation $B_i^a = D_i \phi^a$ in this limit one can write

$$\hat{E}_i = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \frac{1}{vg} \mathrm{Tr}\left(B_i e_m\right) = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \frac{1}{m_W} \mathrm{Tr}\left(B_i e_m\right) = \frac{m_W}{M_m} \hat{p}_\alpha \mathrm{Tr}\left(B_i e_m\right) \tag{9.66}$$

The electric charge in units of g for the eigenvalue $k \in \mathbb{Z}$ of $\hat{p}_\alpha$ is then given by

$$Q_e = \lim_{R\to\infty} \frac{1}{g} \int_{S_R^2} \mathrm{d}\Sigma^i \, \hat{E}_i = \lim_{R\to\infty} \frac{1}{g} \int_{S_R^2} \mathrm{d}\Sigma^i \, \frac{m_W}{M_m} k \, \mathrm{Tr}\left(B_i e_m\right) = \frac{1}{g} m_W k \frac{Q_m}{M_m} = k \tag{9.67}$$

where we used $M_m = v Q_m$ and $m_W = vg$. Hence if $k \neq 0$ then both Q_m and Q_e are non-vanishing. These solutions, where both electric and magnetic charge are not zero, are called *dyons*. By consistency, they should appear in the Georgi-Glashow model (meaning that they cannot be removed from the spectrum of the Hamiltonian).
Hence, while the first three moduli $\boldsymbol{x}_0$ become just the position variables of a quantum mechanical particle, the modulus connected with the symmetry which has been broken by a specific choice of a modulus solution in the internal space is just the information that the spectrum of the theory contains particles with both electric and magnetic charges different from 0.

9.4 Quantum field theory treatment of Dirac monopoles

Unfortunately there is no well understood quantum field theory of the 't Hooft-Polyakov monopoles, so in order to discuss a quantum field theory of monopoles we turn again to Dirac monopoles. As previously

discussed an UV cutoff is needed for the description of such monopoles, and it is understood in the following of the chapter. However, formally we will write formulas in the continuum notation.
Again we construct monopole quantum field operators from Euclidean monopole correlators using OS reconstruction. To construct monopole correlation functions it is easier to exploit the duality invariance discussed at the beginning of this chapter: we first construct the correlation functions of charged fields in a scalar electrodynamic without monopoles and then by duality we obtain the Dirac monopole correlators (at least in a theory without electric charges).

Strocchi's theorem

[Str13, Chapter 7]

The electric charge in scalar electrodynamic appears as the generator of a global subgroup of a local gauge invariance, this implies an important difference with respect to the topological charges previously discussed. Up to now, starting from the correlation functions of the (Euclidean) fields, via reconstruction theorems, we have constructed a Hilbert space of states which was naturally endowed with a positive definite scalar product. However for gauge theories, both in the relativistic (such as QED) and non-relativistic cases, the situation is not so simple. The key issue is that a local gauge invariance implies a Gauss' law.
Let us first consider the problem classically. The Noether's theorem for a global (gauge) invariance yields a conserved current j_μ using equations of motions (of both the gauge field and the charged field). However, if we mimic its standard proof in the case of local gauge invariance we get the conservation of the current without using the equations of motions of the charged field and the corresponding charge can be determined from measurements at infinity, hence the charge is not local.

To simplify the discussion we consider the Abelian case, a scalar electrodynamic with charged field ϕ, but the non Abelian case can be treated similarly. In order to simplify the notation we assume unit charge. From the gauge invariance of the Lagrangian density

$$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*, A_\nu, \partial_\mu A_\nu) \tag{9.68}$$

for an infinitesimal gauge transformation parametrized by $\Lambda(x)$

$$\delta\phi(x) = i\Lambda(x)\phi(x) \quad , \quad \delta\phi^*(x) = -i\Lambda(x)\phi^*(x) \quad , \quad \delta A_\mu(x) = -i\partial_\mu\Lambda(x) \tag{9.69}$$

we get

$$\begin{aligned} 0 = \delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial_\mu\delta\phi^* + \frac{\delta\mathcal{L}}{\delta A^\nu}\delta A^\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\delta A_\nu \\ &= i\left[\frac{\delta\mathcal{L}}{\delta\phi}\phi - \frac{\delta\mathcal{L}}{\delta\phi^*}\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial_\mu\phi^*\right]\Lambda + i\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} - \frac{\delta\mathcal{L}}{\delta A_\mu}\right]\partial_\mu\Lambda - i\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\partial_\nu\Lambda \end{aligned} \tag{9.70}$$

for $\Lambda(x)$ arbitrary. In the case of global symmetry Λ is independent on x, and using equations of motion of ϕ and ϕ^*

$$\frac{\delta\mathcal{L}}{\delta\phi} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \quad \text{and} \quad \frac{\delta\mathcal{L}}{\delta\phi^*} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} \tag{9.71}$$

we get

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^*\right] = 0 \tag{9.72}$$

so that the Noether current

$$j^\mu := \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^* \tag{9.73}$$

is conserved: $\partial_\mu j^\mu = 0$.

But if $\Lambda(x)$ is arbitrary (depending on x), also coefficients of $\partial_\mu\Lambda$ and $\partial_\mu\partial_\nu\Lambda$ in eq. (9.70) should vanish, hence we have also

$$j^\mu - \frac{\delta\mathcal{L}}{\delta A_\mu} = 0 \quad \text{and} \quad \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\partial_\nu\Lambda = 0 \tag{9.74}$$

and the latter is satisfied only if

$$G^{\mu\nu} := -\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} \tag{9.75}$$

is antisymmetric, $G^{\mu\nu} = G^{[\mu\nu]}$. But then from the first of eq. (9.74) and the equation of motion of A we have

$$j^\nu = \partial_\mu G^{\mu\nu} \tag{9.76}$$

and due to the antisymmetry of $G^{\mu\nu}$

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu G^{\mu\nu} = 0 \tag{9.77}$$

hence j^ν is automatically conserved without using equations of motion of ϕ and ϕ^*. Moreover, defining the charge

$$Q := \int \mathrm{d}^d x \, j^0 = \int \mathrm{d}^d x \, \partial_i G^{i0} = \lim_{R\to\infty} \int_{S_R^{d-1}} \mathrm{d}\Sigma_i \, G^{i0} \tag{9.78}$$

associated to j^ν, we see that such global charge does not depend on the local behaviour of the density j^0 but only upon the behaviour at infinity of the "electromagnetic" field strength $G^{\mu\nu}$.

Since property eq. (9.76) is a key consequence of the local gauge invariance we must impose it also at quantum level at least in the weakest possible sense, as we are going to see.
Denote by $\hat{j}_\mu$ and $\hat{G}_{\mu\nu}$ the field operator corresponding to classical j_μ and $G_{\mu\nu}$, and by $|\psi\rangle$ and $|\chi\rangle$ the vectors in the space $\mathcal{V}$ of states generated by quantum fields of a quantum gauge theory. According to the previous discussion we require that at least in a subspace[IX] $\mathcal{V}_{\text{phys}} \subseteq \mathcal{V}$ of physical states the matrix elements satisfy

$$\langle\psi| \, (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle = 0 \quad \text{for} \quad |\psi\rangle, \, |\chi\rangle \in \mathcal{V}_{\text{phys}} \tag{9.79}$$

and

$$\text{if} \quad |\chi\rangle \in \mathcal{V}_{\text{phys}} \quad \text{then} \quad (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle \in \mathcal{V}_{\text{phys}} \tag{9.80}$$

for a smearing test function f needed in the continuum. Then if $\mathcal{V}$ has a semi-definite positive inner product, we get the stronger statement

$$(\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle = 0 \quad \text{for all} \quad |\chi\rangle \in \mathcal{V}_{\text{phys}} \tag{9.81}$$

In fact, for $|\psi\rangle \in \mathcal{V}$ and $|\chi\rangle \in \mathcal{V}_{\text{phys}}$, using Schwartz inequality

$$0 \leq |\langle\psi| \, (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle|^2 \leq \langle\psi|\psi\rangle \underbrace{\langle\chi| \, (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})^\dagger(f)}_{\in \mathcal{V}_{\text{phys}}} \underbrace{(\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu}) \, |\chi\rangle}_{\in \mathcal{V}_{\text{phys}}} = 0 \tag{9.82}$$

hence eq. (9.81), i.e. the Gauss' law as an operator identity, automatically holds in the subspace of physical states.

Let's consider the transformation of the field $\hat{\phi}$, charged with respect to the global gauge group, i.e. (modulo smearings)

$$0 \neq \delta\hat{\phi}(y) = [\hat{Q}, \hat{\phi}(y)] = \lim_{R\to\infty} \int_{B_R^d} \mathrm{d}^d x [\hat{j}^0(x), \hat{\phi}(y)] \tag{9.83}$$

If $\mathcal{V}$ has a semi-definite inner product and $\hat{\phi}(y)$ is local, or at least localized in a bounded region, then for $|\psi\rangle, \, |\chi\rangle \in \mathcal{V}_{\text{phys}}$ we have a contraddiction

$$0 \neq \langle\psi| \, \delta\hat{\phi}(y) \, |\chi\rangle = \langle\psi| \lim_{R\to\infty} \int_{B_R^d} \mathrm{d}^d x \, [\hat{j}^0(x), \hat{\phi}(y)] \, |\chi\rangle \overset{(9.81)}{=} \langle\psi| \lim_{R\to\infty} \int_{S_R^{d-1}} \mathrm{d}\Sigma_i \, [\hat{G}^{i0}(x), \hat{\phi}(y)] \, |\chi\rangle = 0 \tag{9.84}$$

where the vanishing of the commutator is a consequence of locality.
Hence positivity of $\mathcal{V}$ and locality (in a bounded region) of charged fields are incompatible in a gauge theory. This result in known as *Strocchi's theorem*, and can be rephrased as the impossibility in a gauge theory of having both positivity of the Hilbert space and locality of the charged field.

[IX] Due to the gauge invariance there are unphysical states, which are elements of $\mathcal{V}$ but not of $\mathcal{V}_{\text{phys}}$.

At least formally (neglecting UV problems) the quantity

$$\hat{Q}(\Lambda) := \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})(\hat{j}^0(\boldsymbol{x}) - \partial_i \hat{G}^{i0}(\boldsymbol{x})) \tag{9.85}$$

generates the time-independent gauge transformations of parameters $\Lambda(\boldsymbol{x})$. In fact defining

$$\delta\hat{\phi}(\boldsymbol{y}) := [\hat{Q}(\Lambda), \hat{\phi}(\boldsymbol{y})] = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})[\hat{j}^0(\boldsymbol{x}) - \partial_i \hat{G}^{i0}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] \tag{9.86}$$

and using

$$\hat{j}^0 \overset{(9.73)}{=} (\hat{\phi}^*\partial_0\hat{\phi} - \hat{\phi}\partial_0\hat{\phi}^*) \quad \text{and} \quad [\partial_0\hat{\phi}^*(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] = -i\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{9.87}$$

we get

$$\delta\hat{\phi}(\boldsymbol{y}) = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})\hat{\phi}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}) = i\Lambda(\boldsymbol{y})\hat{\phi}(\boldsymbol{y}) \tag{9.88}$$

which coincides with eq. (9.69). Similarly, for the gauge field

$$\delta\hat{A}_j(\boldsymbol{y}) := [\hat{Q}(\Lambda), \hat{A}_j(\boldsymbol{y})] = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})[-\partial_i\hat{G}^{i0}(\boldsymbol{x}), \hat{A}_j(\boldsymbol{y})] = i\int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})\partial_i\delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ij} = -i\partial_j\Lambda(\boldsymbol{y}) \tag{9.89}$$

This suggest that if in the reconstruction theorem we use only fields invariant under local gauge transformations ($\delta\hat{\phi} = \delta\hat{A}_j = 0$) vanishing at infinity, then $\hat{Q}(\Lambda)|_{\mathcal{V}_{\text{phys}}} = 0$ and from eq. (9.85) we get that the Gauss' law is automatically implemented in the reconstructed space of states. If OS positivity holds this must be a Hilbert space, but then charged field operators are non-local, reaching ∞ even at fixed time. This is not the standard approach based on covariant quantization, gauge-fixing and locality, indeed for the consistency of the monopole, in particular for the unphysical nature of the Dirac string, preserving the gauge invariance appears to be crucial and this approach based on positivity but not locality is the most natural for charged fields.
Therefore we try to get the monopole quantum field starting from a charged non-local quantum field, easier to define, in the electromagnetic dual theory obtained via OS reconstruction from its correlators. Then we apply to these correlators the duality transformation. Let us show how to do this.

Duality transformation in the path integral

Notice that in the path integral formalism there is a simple way to derive the duality, starting from the free theory.
Consider the free photon partition function in the Lorenz gauge[X]

$$\begin{aligned}
Z_A &= \int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A_{\nu]})^2}\delta(\partial^\mu A_\mu) = \\
&= \text{const}\cdot\int \mathcal{D}A^T_\mu\, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A^T_{\nu]})^2} = \\
&= \text{const}\cdot\int \mathcal{D}A^T_\mu \int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\, e^{i\int B_{\mu\nu}\partial^{[\mu}A^{\nu]}_T} = \\
&= \text{const}\cdot\int \mathcal{D}A^T_\mu \int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\, e^{-i\int(\partial^{[\mu}B_{\mu\nu})A^{\nu]}_T} = \\
&= \text{const}\cdot\int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\,\delta(\partial^\mu B_{\mu\nu})
\end{aligned} \tag{9.90}$$

where in the first step we splitted $A_\mu = A^L_\mu + A^T_\mu$ with $\partial^\mu A^T_\mu = 0$ and $\partial^\mu A_\mu = \partial^\mu A^L_\mu$, in the second step we performed a Gaussian transformation with antisymmetric tensor[XI] $B_{\mu\nu}$ (can be checked by integrating over $B_{\mu\nu}$ from the second to the first line), in the third step we integrated by parts in the second exponential, and in the last step we integrated over A^T_μ.

[X] In the following for simplicity we often don't write explicitly the measure d^4x in the integrals.
[XI] Notice that the tensor $B_{\mu\nu}$ plays the role of a Lagrange multiplier.

The constraint $\partial^\mu B_{\mu\nu} = 0$ can be solved writing $B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \tilde{A}^\sigma$, for some gauge potential $\tilde{A}^\sigma$, with possible gauge-fixing $\partial_\mu \tilde{A}^\mu = 0$ which fixes the correspondence $B_{\mu\nu} \leftrightarrow \tilde{A}_\mu$. In this way the partition function become

$$\begin{aligned} Z_A &= \text{const} \cdot \int \mathcal{D}\tilde{A}_\mu \, e^{-\frac{e^2}{2}\int (\epsilon_{\mu\nu\rho\sigma}\partial^\rho \tilde{A}^\sigma)^2} \, \delta(\partial^\mu \tilde{A}_\mu) \\ &= \text{const} \cdot \int \mathcal{D}\tilde{A}_\mu \, e^{-\frac{e^2}{2}\int (\partial^{[\rho}\tilde{A}^{\sigma]})^2} \, \delta(\partial^\mu \tilde{A}_\mu) \end{aligned} \tag{9.91}$$

Varying with respect to $B_{\mu\nu}$ in the mixed $B_{\mu\nu}$-A_μ action one finds, using saddle point approximation (which works since the action is quadratic),

$$\frac{i}{e}\partial_{[\mu}A_{\nu]} = eB_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \tilde{A}^\sigma \tag{9.92}$$

(notice that in Minkowski space the first i is missing). We can now construct field strengths of A and $\tilde{A}$ as follows

$$F^{\mu\nu} = \partial^{[\mu}A^{\nu]} \quad \text{and} \quad \tilde{F}^{\mu\nu} = \partial^{[\mu}\tilde{A}^{\nu]} \tag{9.93}$$

which satisfy equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} = 0 \quad &\overset{(9.92)}{\longleftrightarrow} \quad \epsilon_{\mu\nu\rho\sigma}\partial^\nu \tilde{F}^{\rho\sigma} = 0 \\ \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \quad &\overset{(9.92)}{\longleftrightarrow} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{aligned} \tag{9.94}$$

thus recovering the electromagnetic duality for the free theory. Thus we see how in the free theory the duality arises from a clever change of variables, performing a Gaussian (Fourier) transformation together with the introduction of $B_{\mu\nu}$ and the solution of the constraint on $B_{\mu\nu}$ by means of the field $\tilde{A}_\mu$. Such procedure is general and can be applied to more general situations.

The closed defects

Let us now see how by duality we obtain a theory of electrodynamic with monopoles. We start from the Stückelberg model (for high-energy physics) or the gauged XY model (for condensed matter physics). Under duality monopoles will correspond to the charges of these models. Fields of the model are the angular field $\theta(x)$ and the Abelian gauge field $A_\mu(x)$. The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2e^2}(\partial_{[\mu}A_{\nu]})^2 + \frac{\lambda}{2}(-ie^{i\theta}\partial_\mu e^{i\theta} - A_\mu)^2 \tag{9.95}$$

One can write (*Hodge decomposition*)

$$\frac{1}{i}e^{-i\theta}\partial_\mu e^{i\theta} = \partial_\mu \Lambda + 2\pi n_\mu \tag{9.96}$$

where Λ is real and globally defined and $2\pi n_\mu$ is a singular current corresponding to jumps $0 \to 2\pi$ in $\theta(x)$, so that $n_\mu(x) \in \mathbb{Z}$. The "continuity up to jumps of 2π" of θ implies[XII] that the singular component n_μ is non-vanishing only along some lines, therefore it can be regarded as a singular integer valued line-current, that is, a singular integer valued current supported on lines.
Let's state the Poisson resummation formula: for an angle φ we have the Fourier transform

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(\varphi + 2\pi n)^2} = \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2\lambda}j^2} e^{i\varphi j} \tag{9.97}$$

We can apply its x-dependent version to the Lagrangian ($\varphi \mapsto \partial_\mu\Lambda - A_\mu$)

$$\sum_{n_\mu} e^{-\frac{\lambda}{2}\int(\partial_\mu\Lambda + 2\pi n_\mu - A_\mu)^2} = \sum_{j^\mu} e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{i\int(\partial_\mu\Lambda - A_\mu)j^\mu} \tag{9.98}$$

[XII] Let's justify this claim. The function θ can be regarded as a continuous function with values in $\mathbb{R}$ rather than in $[0, 2\pi)$. Then $2\pi n_\mu$ is non-vanishing only in those points where $\theta = 2\pi n$ for some $n \in \mathbb{Z}$.

where now $n_\mu(x)$ and $j^\mu(x)$ are integer valued line-currents. In particular j^μ is the current of charged particles in the Stückelberg model. Integration over the scalar field $\Lambda(x)$ gives

$$\int \mathcal{D}\Lambda\, e^{i\int \mathrm{d}^4x\,(\partial_\mu\Lambda)j^\mu} = \prod_{x\in\mathbb{R}^4} \delta(\partial^\mu j_\mu(x)) =: \delta(\partial^\mu j_\mu) \tag{9.99}$$

and $\partial_\mu j^\mu = 0$ implies that the support of the current j_μ is made of closed lines. We see that the term $\exp\big(-i\int \mathrm{d}^4x\, A_\mu j^\mu\big)$ describes the world line of a charged particle-antiparticle pair. The closed lines in the support of j_μ then correspond to boundaries of surfaces. Let's denote by $S^{\rho\sigma}(x)$ the integer current with support on one of these surfaces for fixed j_μ, so that

$$j_\mu(x) = \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(x) \tag{9.100}$$

(of course there are many possible choices for the surface, but all these possible choices are equivalent).

The partition function of the Stückelberg model become

$$\begin{aligned}
Z_\lambda &= \int \mathcal{D}A_\mu \int \mathcal{D}\Lambda \sum_{n_\mu} e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} e^{-\frac{\lambda}{2}\int(\partial_\mu\Lambda - A_\mu + 2\pi n_\mu)^2}\delta(\partial^\mu A_\mu) \\
&= \int \mathcal{D}A^T_\mu \int \mathcal{D}\Lambda \sum_{n_\mu} e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} e^{-\frac{\lambda}{2}\int(\partial_\mu\Lambda - A_\mu + 2\pi n_\mu)^2} \\
&\overset{(9.98)}{=} \int \mathcal{D}A^T_\mu \int \mathcal{D}\Lambda \sum_{j_\mu} e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A^T_{\nu]})^2} e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{i\int(-A^T_\mu + \partial_\mu\Lambda)j^\mu} \\
&\overset{\substack{(9.99)\\(9.100)}}{=} \int \mathcal{D}A^T_\mu \sum_{S^{\rho\sigma}} e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A^T_{\nu]})^2} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu_T\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}} \\
&= \int \mathcal{D}A_\mu \sum_{S^{\rho\sigma}} e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}}\delta(\partial^\mu A_\mu)
\end{aligned} \tag{9.101}$$

Applying then the duality procedure as in eq. (9.90) we have

$$Z_\lambda = \int \mathcal{D}A^T_\mu \int \mathcal{D}B_{\mu\nu} \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int B^2_{\mu\nu}} e^{i\int B_{\mu\nu}\partial^\mu A^\nu_T} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu_T\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}} \tag{9.102}$$

and then integrating out A^T_μ and taking the limit $\lambda\to\infty$ we get

$$Z_\infty = \int \mathcal{D}B_{\mu\nu} \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int B^2_{\mu\nu}} \delta(\partial^\nu B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}) \tag{9.103}$$

The constraint on $B_{\mu\nu}$ can be solved introducing a gauge field $\tilde{A}_\mu$ such that

$$B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma \tag{9.104}$$

so that

$$\begin{aligned}
Z_\infty &= \int \mathcal{D}\tilde{A}^\mu \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int(\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\tilde{A}^\sigma + S^{\rho\sigma}))^2}\delta(\partial^\mu\tilde{A}_\mu) \\
&= \int \mathcal{D}\tilde{A}_\mu \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int(\partial^{[\rho}\tilde{A}^{\sigma]} + S^{\rho\sigma})^2}\delta(\partial^\mu\tilde{A}_\mu)
\end{aligned} \tag{9.105}$$

If we define the field strength

$$\tilde{\mathcal{F}}^{\rho\sigma} = \partial^{[\rho}\tilde{A}^{\sigma]} + S^{\rho\sigma} \tag{9.106}$$

we see that it satisfies

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu\tilde{\mathcal{F}}^{\rho\sigma} = \cancel{\epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^{[\rho}\tilde{A}^{\sigma]}} + \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma} \overset{(9.100)}{=} j_\mu \tag{9.107}$$

This proves that j^μ are monopole currents with respect to $\tilde{A}^\mu$, i.e. the worldlines of monopole-antimonopole pairs. Accordingly $S^{\rho\sigma}$ should describe the 2-dimensional worldsheet spanned by the Dirac string between the monopole and the antimonopole of the pair. The arbitrariness of the choice of $S^{\rho\sigma}$ reflects the arbitrariness of the Dirac string. Hence monopole currents are the locus of the defects in the electrodynamics of the monopoles.

Open defects and correlators

Let us see how to "open" such line defects using duality. A charged gauge-invariant non-local field in the Stückelberg model can be constructed using a recipe suggested by Dirac in the operator setting (for QED), adapted to the path-integral formalism.
The charged field of the Stückelberg model (given by quantization of the complex field ϕ with $|\phi| = 1$) is $e^{i\hat{\theta}(\boldsymbol{x})}$, clearly not-gauge invariant under the gauge transformation

$$e^{i\hat{\theta}(\boldsymbol{x})} \quad \to \quad e^{i(\hat{\theta}(\boldsymbol{x}) + e\Lambda(\boldsymbol{x}))} \tag{9.108}$$

Dirac suggested[XIII] to make it gauge invariant multiplying it by

$$e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \tag{9.109}$$

where $\hat{A}^i$ is the gauge potential of the quantum "photon" field and $E_i^{\boldsymbol{x}}$ denote the (classical) electric field generated by a unit charge at the position $\boldsymbol{x}$, so that by the classical Gauss' law

$$\partial^i E_i^{\boldsymbol{x}}(\boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{9.110}$$

In fact, under a gauge transformation $\Lambda(\boldsymbol{x})$ vanishing at ∞

$$\begin{aligned} e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \quad \to \quad & e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})(\hat{A}^i(\boldsymbol{y}) + \partial_i\Lambda(\boldsymbol{y}))} = \\ & = e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}_i(\boldsymbol{y})} e^{-ie\int \mathrm{d}^3\boldsymbol{y}\, \partial^i E_i^{\boldsymbol{x}}(\boldsymbol{y})\Lambda(\boldsymbol{y})} = \\ & = e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}_i(\boldsymbol{y})} e^{-ie\Lambda(\boldsymbol{x})} \end{aligned} \tag{9.111}$$

so that

$$e^{i\hat{\theta}(\boldsymbol{x})} e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \tag{9.112}$$

is gauge invariant but still charged, hence it is physical and should be added to the Hilbert space of the theory due to completeness. The second term in eq. (9.112) describes the Coulomb field always attached to a charged particle.
It is easy to define the corresponding Euclidean field (now in $\mathbb{R}^4$). Define

$$E_i^x(y) = \delta(x^0 - y^0) E_i^{\boldsymbol{x}}(\boldsymbol{y}) \quad , \quad E_o^x(y) = 0 \tag{9.113}$$

then the charged non-local Euclidean field is given by

$$e^{i\theta(x)} e^{ie\int \mathrm{d}^4 y\, E_\mu^x(y) A^\mu(y)} \tag{9.114}$$

The Euclidean correlator of such fields naturally satisfy the OS positivity, hence using the OS reconstruction it is possible to obtain a charged quantum field operator (in the lattice).
However we are interested in the quantum field operator for the monopole, so let's apply the duality transformation to

$$\left\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4 z\, E_\mu^x(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4 w\, E_\mu^y(w) A^\mu(w)} \right\rangle_\lambda \tag{9.115}$$

where $\langle \quad \rangle_\lambda$ denotes the expectation value in the model with the fixed value λ. To apply the duality transformation we need to re-express the θ-dependence appearing in the expectation value in terms of $\frac{1}{i} e^{-i\theta}\partial_\mu e^{i\theta} = \partial_\mu\Lambda + 2\pi n_\mu$. This can be achieved introducing a current j_μ^{xy} satisfying

$$\partial^\mu j_\mu^{xy}(w) = \delta(y - w) - \delta(x - w) \tag{9.116}$$

using this current we can write

$$\begin{aligned} e^{\int \mathrm{d}^4 w\, j_\mu^{xy}(w) e^{-i\theta(w)}\partial^\mu e^{i\theta(w)}} & = e^{\int \mathrm{d}^4 w\, j_\mu^{xy}(w)\partial_\mu \log e^{i\theta(w)}} = e^{-\int \mathrm{d}^4 w\, \partial^\mu j_\mu^{xy}(w) \log e^{i\theta(w)}} \\ & = e^{-\log e^{i\theta(y)} + \log e^{i\theta(x)}} = e^{i(\theta(x) - \theta(y))} \end{aligned} \tag{9.117}$$

[XIII] The idea of Dirac, qualitatively, it that an electron, even asymptotically, carries with it also its Coulomb field, hence when an electron is created its Coulomb field should be created as well. Applying this idea to the scalar field (rather than to the electron field) we get the procedure we are describing in the following.

so our 2-point function eq. (9.115) according to the decomposition eq. (9.96) becomes

$$\begin{aligned}\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4 z\, E^x_\mu(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4 w\, E^y_\mu(w) A^\mu(w)}\rangle_\lambda = \\ = \langle e^{ie\int \mathrm{d}^4 w\, (E^x_\mu(w) - E^y_\mu(w)) A^\mu(w)} e^{i\int \mathrm{d}^4 w\, j^{xy}_\mu(w)(\partial^\mu \Lambda + 2\pi n^\mu)(w)}\rangle_\lambda\end{aligned} \tag{9.118}$$

Note that

$$e^{i\int \mathrm{d}^4 w\, j^{xy}_\mu(w) 2\pi n^\mu(w)} = 1 \tag{9.119}$$

since both j^{xy}_μ and n_μ are integer valued, hence we get

$$\begin{aligned}&\langle e^{i\theta(x)} e^{ie\int E^x_\mu A^\mu} e^{-i\theta(y)} e^{-ie\int E^y_\mu A^\mu}\rangle_\lambda = \\ &= \frac{1}{Z_\lambda}\int \mathcal{D}A_\mu \int \mathcal{D}\Lambda \sum_{n_\mu} e^{ie\int (E^x_\mu - E^y_\mu) A^\mu} e^{i\int j^{xy}_\mu \partial^\mu \Lambda} e^{-\frac{1}{2e^2}\int (\partial_{[\mu} A_{\nu]})^2} e^{-\frac{\lambda}{2}\int (\partial_\mu \Lambda - A_\mu + 2\pi n_\mu)^2} \delta(\partial^\mu A_\mu) \\ &\overset{(9.98)}{=} \frac{1}{Z_\lambda}\int \mathcal{D}A_\mu \int \mathcal{D}\Lambda \sum_{j_\mu} e^{ie\int (E^x_\mu - E^y_\mu) A^\mu} e^{i\int j^{xy}_\mu \partial^\mu \Lambda} e^{-\frac{1}{2e^2}\int (\partial_{[\mu} A_{\nu]})^2} e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{i\int (\partial_\mu \Lambda - A_\mu) j^\mu} \delta(\partial^\mu A_\mu) \\ &\overset{(9.99)}{=} \frac{1}{Z_\lambda}\int \mathcal{D}A_\mu \sum_{j_\mu} e^{ie\int (E^x_\mu - E^y_\mu) A^\mu} e^{-\frac{1}{2e^2}\int (\partial_{[\mu} A_{\nu]})^2} e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{-i\int A_\mu j^\mu} \delta(\partial^\mu j_\mu + \partial^\mu j^{xy}_\mu)\delta(\partial^\mu A_\mu)\end{aligned} \tag{9.120}$$

hence $\partial^\mu j_\mu(w) = -\partial^\mu j^{xy}_\mu(w) = \delta(w-x) - \delta(w-y)$, or equivalently $\partial^\mu j_\mu = \delta_x - \delta_y$, where we defined $\delta_x(w) := \delta(w-x)$. Finally in the limit $\lambda \to \infty$, which we have seen is dual to the electrodynamic with monopoles, the correlation function becomes

$$\begin{aligned}&\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4 z\, E^x_\mu(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4 w\, E^y_\mu(w) A^\mu(w)}\rangle_\infty = \\ &= \frac{1}{Z_\infty}\int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int (\partial_{[\mu} A_{\nu]})^2} \delta(\partial^\mu A_\mu) \sum_{\substack{j_\mu \\ \partial^\mu j_\mu = \delta_x - \delta_y}} e^{i\int A_\mu(-j_\mu + E^x_\mu - E^y_\mu)} = \\ &= \frac{1}{Z_\infty}\int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int (\partial_{[\mu} A_{\nu]})^2} \delta(\partial^\mu A_\mu) \sum_{\substack{j_\mu \\ \partial^\mu j_\mu = 0}} e^{i\int A_\mu(-j_\mu + j^{xy}_\mu + E^x_\mu - E^y_\mu)}\end{aligned} \tag{9.121}$$

The situation is represented in fig. 9.1.

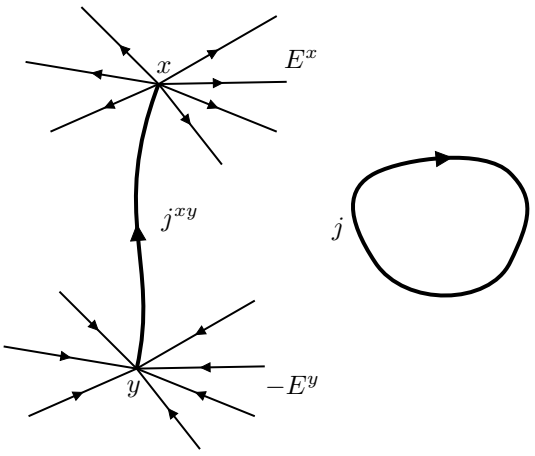


Figure 9.1: Representation of currents and fields in eq. (9.121) in one dimension less.

Since $\partial^\mu(j^{xy}_\mu + E^x_\mu - E^y_\mu) = 0$, it exists a tensor $S^{\rho\sigma}(z|j^{xy}, E^x, E^y)$ such that

$$-(j^{xy}_\mu + E^x_\mu - E^y_\mu)(z) = \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(z|j^{xy}, E^x, E^y) \tag{9.122}$$

Using the Hodge decomposition one can give a more explicit expression for this tensor

$$S^{\rho\sigma}(z|j^{xy},E^x,E^y)=\int \mathrm{d}w\,\epsilon^{\rho\sigma\alpha\beta}\partial_\alpha\Delta^{-1}(z-w)(j^{xy}+E^x-E^y)_\beta \tag{9.123}$$

in fact, using $\epsilon^{\rho\sigma\alpha\beta}\epsilon_{\mu\nu\rho\sigma}=\delta^\alpha_\mu\delta^\beta_\nu-\delta^\alpha_\nu\delta^\beta_\mu$, we get

$$\begin{aligned}\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(z|j^{xy},E^x,E^y)&=\int \mathrm{d}^4w\,\epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}\partial^\nu\partial_\alpha\Delta^{-1}(z-w)(j^{xy}+E^x-E^y)_\beta(w)\\&=\int \mathrm{d}^4w\,\partial_\mu\Delta^{-1}(z-w)\partial^\nu(j^{xy}+E^x-E^y)_\nu(w)-\\&\quad-\int \mathrm{d}^4w\,\underbrace{\partial_\nu\partial^\nu\Delta^{-1}(z-w)}_{\delta(z-w)}(j^{xy}+E^x-E^y)_\mu(w)\\&=-(j^{xy}+E^x-E^y)_\mu(z)\end{aligned} \tag{9.124}$$

Since j_μ appearing in eq. (9.121) could be rewritten as in the partition function using $j_\mu=\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}$ one has

$$\begin{aligned}&\langle e^{i\theta(x)}e^{ie\int \mathrm{d}^4z\,E^x_\mu(z)A^\mu(z)}e^{-i\theta(y)}e^{-ie\int \mathrm{d}^4w\,E^y_\mu(w)A^\mu(w)}\rangle_\infty=\\&=\frac{1}{Z_\infty}\int\mathcal{D}A_\mu\,e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A_{\nu]})^2}\delta(\partial^\mu A_\mu)\sum_{S^{\rho\sigma}}e^{-i\int A_\mu\epsilon_{\mu\nu\rho\sigma}\partial^\nu(S^{\rho\sigma}+S^{\rho\sigma}(j^{xy},E^x,E^y))}\\&=\frac{1}{Z_\infty}\int\mathcal{D}A^T_\mu\int\mathcal{D}B_{\mu\nu}\sum_{S^{\rho\sigma}}e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}e^{i\int A^\mu_T\partial^\nu(B_{\mu\nu}-\epsilon_{\mu\nu\rho\sigma}(S^{\rho\sigma}+S^{\rho\sigma}(j^{xy}E^x,E^y)))}\\&=\frac{1}{Z_\infty}\int\mathcal{D}B_{\mu\nu}\sum_{S^{\rho\sigma}}e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\delta(\partial^\nu(B_{\mu\nu}-\epsilon_{\mu\nu\rho\sigma}(S^{\rho\sigma}+S^{\rho\sigma}(j^{xy}E^x,E^y))))\end{aligned} \tag{9.125}$$

where in the third step we applied duality as in eq. (9.90). We should solve the constraint which arises integrating out A^T_μ: for a gauge field $\tilde{A}^\mu$ we get

$$B_{\mu\nu}-\epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}+\epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}(j^{xy},E^x,E^y)=\epsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma \tag{9.126}$$

Then we finally get (recall eq. (9.105))

$$\begin{aligned}&\langle e^{i\theta(x)}e^{ie\int \mathrm{d}^4z\,E^x_\mu(z)A^\mu(z)}e^{-i\theta(y)}e^{-ie\int \mathrm{d}^4w\,E^y_\mu(w)A^\mu(w)}\rangle_\infty=\\&=\frac{\displaystyle\int\mathcal{D}\tilde{A}_\mu\,\delta(\partial^\mu\tilde{A}_\mu)\sum_{S_{\mu\nu}}e^{-\frac{e^2}{2}\int(\partial_{[\mu}\tilde{A}_{\nu]}+S_{\mu\nu}+S_{\mu\nu}(j^{xy},E^x,E^y))^2}}{\displaystyle\int\mathcal{D}\tilde{A}_\mu\,\delta(\partial^\mu\tilde{A}_\mu)\sum_{S_{\mu\nu}}e^{-\frac{e^2}{2}\int(\partial_{[\mu}\tilde{A}_{\nu]}+S_{\mu\nu})^2}}\\&=:\langle M(x)M^\dagger(y)\rangle\end{aligned} \tag{9.127}$$

which is the correlation function of the monopole in the dual theory of electrodynamic with monopoles. Notice that the additional "monopole current" introduced in the correlator is

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_{\rho\sigma}(j^{xy},E^x,E^y)=j^{xy}+E^x-E^y \tag{9.128}$$

hence the "classical electric fields" E^x and E^y in the dual theory play the role of the classical magnetic fields of the monopole, which is expected on the basis of duality since in the Stückelberg model E^x and E^y were just the classical electric field describing the Coulomb interaction associated to the charged field.

Reconstruction of the theory

Since the charged correlator of the Stückelberg model satisfy OS positivity, in spite of the appearance the same must be true for the monopole correlator. Therefore we can apply OS reconstruction theorem

and construct a non-local monopole field operator $\hat{M}(\boldsymbol{x})$ (it is supported in the entire $t = 0$ 3-plane due to the non-locality of E^x and E^y), so that in the electrodynamic with monopoles the correlators and the field operators are related through

$$\langle M(x)M(y)\rangle = \langle 0|\, \hat{M}(\boldsymbol{x}) e^{-H(x^0 - y^0)} \hat{M}^\dagger(\boldsymbol{y})\, |0\rangle \tag{9.129}$$

In fig. 9.2 one clearly sees the difference in locality between vortices and monopoles as consequence of the gauge invariance resulting in a Coulomb interaction for charged particles.

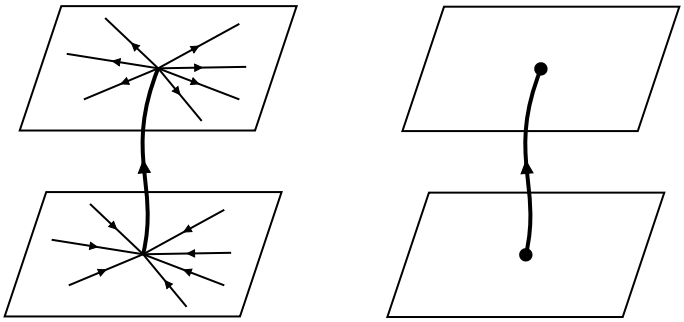


Figure 9.2: Representation in one dimension less of the typical structure of "monopole currents" and "vortex currents" in 2-point correlators

One can almost rigorously prove that for e^2 large enough (strong coupling limit) the monopole quantum field operator $\hat{M}(\boldsymbol{x})$ couple the vacuum $|0\rangle$ to a 1-particle state together with an infrared Coulomb tail (called *one-infraparticle state*[XIV]), hence $\hat{M}(\boldsymbol{x})$ creates and annihilates monopole infra-particles.
Furthermore the mass of the monopole turns out to be $O(e^2)$, hence, similarly to the case of vortices, monopole correlators with non-vanishing total monopole charge vanish, so that the Hilbert space of states of electrodynamic with monopoles for large e^2 develops monopole superselection sectors $\mathcal{H}_q$ labelled by the total magnetic charge. Therefore the total Hilbert space of the theory is given by

$$\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q \tag{9.130}$$

with

$$\hat{M}^\dagger \,:\, \mathcal{H}_q \to \mathcal{H}_{q+1} \tag{9.131}$$

Conversely to the case $e^2 \gg 1$, where opening a closed defect "costs a lot", for e^2 small enough (weak coupling limit) the monopole "condense", i.e. there is a huge amount of open line defects in the partition function, and such defects "hides" the monopoles, so that $\mathcal{H} = \mathcal{H}_0$.

9.5 Spin ice

[CMS08]

Although Dirac monopoles have not been found as elementary particles, and in fact they suffer of an UV problem, in 2008 a kind of exotic magnets called *spin ice* have been studied, in which there seems to appear excitations similar to Dirac monopoles.[XV] The name comes from the fact that their structure is reminiscent of that of the ice.
The structure appearing in the portion of oxygen and hydrogen in ice is tetrahedral, as represented in fig. 9.3.

[XIV] One cannot think about such state as a pole of the Green function, since the mass hyperboloid associated to the monopole interacts with the rest for the froward light cone, due to the presence of the field E^x, so that the pole of the Green function is blurred by the photon. This cannot be proved perturbatively, but using the non-perturbative Bloch-Nordsieck treatment.
[XV] Original work: [CMS08].

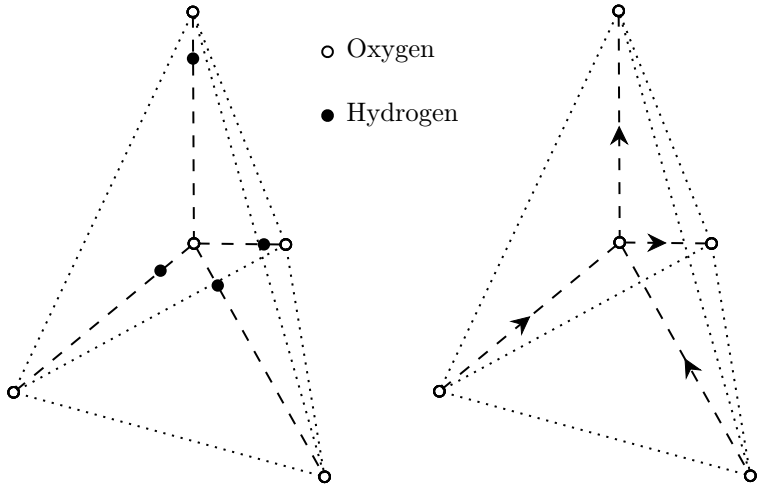


Figure 9.3: Tetrahedral structure of the ice: one oxygen is connected to other oxygens along the direction of a tetrahedron, and in between two oxygens there is an hydrogen. However of these four hydrogens contained in the tetrahedron two are nearer to the central oxygen and their bounds are covariant, and two are more far, and their bounds are of hydrogen type. In the second picture we represented the positions of the hydrogens by means of displacement vectors, pointing in the direction of the displacement of the hydrogen atom with respect to the center of the bound. Hence in a tetrahedron two displacement vectors are inword and two are outword.

In spin ice materials the hydrogen is replaced is replaced by rare-earth ions and the displacement vector is replaced by a spin vector, each spin vector has only one possible direction and two possible orientations, i.e. Ising-like spins, along the bound.
The Hamiltonian can be taken as

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i^{\hat{z}_i} \cdot \boldsymbol{S}_j^{\hat{z}_j} + D \sum_{i,j} \left[\frac{\boldsymbol{S}_i^{\hat{z}_i} \cdot \boldsymbol{S}_j^{\hat{z}_j}}{r_{ij}^3} - 3 \frac{(\boldsymbol{S}_i^{\hat{z}_i} \cdot \boldsymbol{r}_{ij})(\boldsymbol{S}^{\hat{z}_j} \cdot \boldsymbol{r}_{ij})}{r_{ij}^5} \right] \tag{9.132}$$

where $\boldsymbol{S}_i^{\hat{z}_i}$ is the spin along the $\hat{z}_i$ radial direction from the center of the tetrahedron towards its vertex labelled by the site i, $\boldsymbol{r}_{ij}$ is the vector connecting the site i and the site j. Furthermore the orientation of the spins in the first term are favouring the combination 2-in/2-out for each tetrahedron, as this is the minimal energy configuration. Indeed, since the angle θ between bounds give $\cos\theta = -\frac{1}{3}$, we get the following energy contributions for the short range interaction:

$$\begin{cases} \mathcal{E}(2 \text{ in, } 2 \text{ out}) = -2\dfrac{J}{3}S^2 \\ \mathcal{E}(1 \text{ in, } 3 \text{ out}) = \mathcal{E}(3 \text{ in, } 1 \text{ out}) = 0\dfrac{J}{3}S^2 \\ \mathcal{E}(0 \text{ in, } 4 \text{ out}) = \mathcal{E}(4 \text{ in, } 0 \text{ out}) = 2\dfrac{J}{3}S^2 \end{cases} \tag{9.133}$$

Notice that the minimal energy configuration 2-in/2-out is highly degenerate. The long-range part of the Hamiltonian is reminiscent of the dipolar interaction. Indeed the potential of a dipole $\boldsymbol{p} = q\boldsymbol{d}$ is

$$\phi(\boldsymbol{x}) = \frac{\boldsymbol{p} \cdot \boldsymbol{x}}{|\boldsymbol{x}|^3} \tag{9.134}$$

where q is the charge and $\boldsymbol{d}$ is the vector connecting the two charges of the dipole, and at large distances the potential energy of two interacting dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in electrodynamics is given up to a sign by the scalar product between one of the dipoles and the electric field generated by the other one. Therefore if we call $\boldsymbol{r}$ the vector from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ and $r := |\boldsymbol{r}|$ then the energy of the dipolar interaction reads

$$\mathcal{E} = \boldsymbol{p}_1 \cdot \boldsymbol{\nabla}\phi_2(\boldsymbol{r}) = \boldsymbol{p}_1 \cdot \left(\frac{\boldsymbol{p}_2}{r^3} - 3\frac{(\boldsymbol{r} \cdot \boldsymbol{p}_2)\boldsymbol{x}}{r^5} \right) = \frac{\boldsymbol{p}_i \cdot \boldsymbol{p}_2}{r^3} - 3\frac{(\boldsymbol{r} \cdot \boldsymbol{p}_1)(\boldsymbol{r} \cdot \boldsymbol{p}_2)}{r^5} \tag{9.135}$$

Therefore we can replace the spins in the spin ice by dipoles "at the end of the spin vectors", so that the + charge corresponds to the tail of the spin vector and the − charge to its head, and such charges interact via Coulomb potential. It is natural to think of these charges as magnetic charges since the spin is naturally related to magnetism. In this way at the center of each tetrahedron the nearer charges of the 4 dipoles accumulate. More precisely, if the site center of the tetrahedron is labelled by i we have for its charge

$$Q_i = q_{i1} + q_{i2} + q_{i3} + q_{i4} \tag{9.136}$$

where $q_{i\ell}$ is the charge of the ℓ-th dipole which is closer to the site i. Charges of the centers interact via Coulomb interaction

$$V = \frac{Q_i Q_j}{r_{ij}} \tag{9.137}$$

Imposing an onsite energy $v_0 \sum_i Q_i^2$, so that $Q_i = 0$ on the ground state, enforces the ice rule, that is the configuration 2 in/2 out for each tetrahedron, as in the case of fig. 9.4.

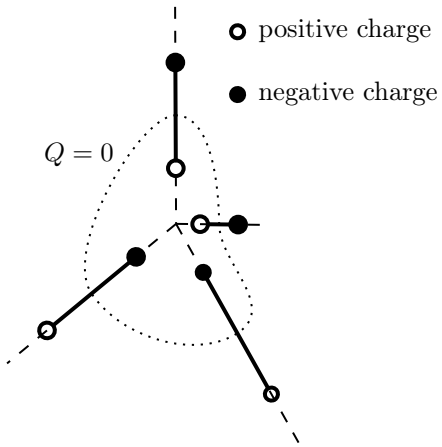


Figure 9.4: The tetrahedral cell of spin ice in the "dipole" representation, in the minimal energy configuration (2 in-2 out). Notice that in the region close to the center of the tetrahedron the total charge is $Q = 1 + 1 - 1 - 1 = 0$.

If we flip a spin, in one of the two tetrahedrons connected by that spin the total charge become 2, and in the other one become -2. The defects in the partition function of the model correspond to chains of spin flips. If a site is intermediate between two spin flips cleverly chosen, then two opposite charges of that site in the dipole picture have changed their signs, so we still have $Q = 0$ in that site. However at the ends of a chain of spin flips we have 3 charges of the same sign and one with the opposite one, so that $Q \neq 0$. Since these charges interact according to the Coulomb interaction eq. (9.137), they can be interpreted as monopoles. Therefore at each end of a chain of spin flips we have a "magnetic monopole" if $Q > 0$ of a "magnetic antimonopole" if $Q < 0$. The string of spin flips is nothing but a "Dirac string". Such situation is represented in fig. 9.5

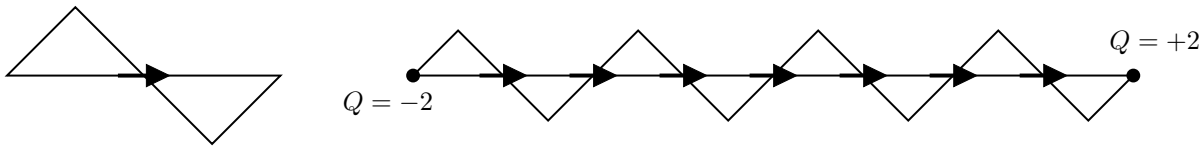


Figure 9.5: On the left, the two triangles represent two close tetrahedrons, and the arrow represent a flip of the spin shared between them. On the right, we have the representation of a chain of spin flips, chosen so that it represent a "Dirac string" whose boundaries constitute a "monopole-antimonopole pair".

If we introduce the time dimension these configurations describes precisely worldlines of virtual monopole-antimonopole pairs with a Dirac string between them spanning a surface bounded by that worldline.

9.6 Condensation of defects

We close the discussion about solitons with a final remark about phase transitions. We have seen that quantum solitons correspond in their Euclidean description to line-defects. We also remarked that it may happen that line-defects reach ∞ with finite probability, i.e. they give raise to the phenomenon called *condensation of defects*. When condensation of defects occurs the soliton sectors disappear. The transition from a situation where configurations allowed are only with finite defects ("dilute gas of defects") to a situation where also infinitely extended defects appear corresponds to a *phase transition*.

As we said, defects can have arbitrary dimension[XVI] k. We now show heuristically that for $k > 0$ condensation of defects is a general mechanism of phase transition. Suppose for simplicity that only one species of defects appears in the model under consideration, but the extension to multiple species is straightforward.
For $k > 0$ the mean action $\bar{S}$ of a k-dimensional defect D^k is typically proportional to the volume of the locus of D^k, i.e.

$$\bar{S}(D^k) \sim c_k \beta |D^k| \tag{9.138}$$

where β is the relevant coupling constant of the model, $c_k > 0$ is a suitable constant and $|D^k|$ is the volume of the locus of the defect suitably discretized.
The number of such defects containing a fixed point, say the origin, and with fixed discretized volume $|D^k|$, is bounded by $e^{d_k |D^k|}$, for some constant $d_k > 0$. To get a hint of this bound consider the simplest case of a line defect in dimension $d = 2$. Consider a kink containing the origin, since we are considering a discretized space, it is trivial to notice that the first "piece" of the kink can be chosen is $2d$ ways, since there are $2d$ possible locus around the origin which can be connected with the origin itself. Then we may add another "piece" of kink in $2d - 1$ ways, as we cannot "go back", and the same is true for all other "pieces",[XVII] obtaining in this way a line defect of dimension $L = |D^1|$ after $L - 1$ steps. Hence there are at most

$$2d(2d-1)^{L-1} \leq e^{L \log 2d} \tag{9.139}$$

line defects of length L containing the origin, and this proves that for $k = 1$ we have $d_1 = \log 2d$.
As discussed for the kink in ϕ^4_2, one can write in general the partition function of the model with k-defects, denoted D from now on for simplicity, as the partition function of an interacting gas of defects. Assuming that defects are weakly interacting we get from eq. (7.89) that

$$Z \approx \sum_{N=0}^{\infty} \sum_{\{D_1, \ldots, D_N\}} \prod_{i=1}^{N} e^{-\bar{S}(D_i)} \tag{9.140}$$

Let's now consider the contribution to the partition function of the defects of volume $|D|$ containing the origin. By the esitmate above such contribution goes like

$$e^{-\bar{S}(D)} e^{d_k |D|} \approx e^{(d_k - c_k \beta)|D|} \tag{9.141}$$

We immediately see that for $|D| \to \infty$

$$e^{(d_k - c_k \beta)|D|} \xrightarrow[|D| \to \infty]{} \begin{cases} 0 & \text{if} \quad d_k - c_k \beta < 0 \\ \infty & \text{if} \quad d_k - c_k \beta > 0 \end{cases} \tag{9.142}$$

The first case implies that there are no defects of infinite volume containing the origin (their Boltzmann weight vanish), the second case instead suggests that there are defects containing the origin reaching infinity with finite probability and that there is a critical value β_c for β where the transition occurs.
One can heuristically interpret in this way the transition from the symmetry breaking phase to the unbroken phase in the case of kinks, from superconductors with massive photon to the usual Coulomb phase with massless photons in the case of vortices, and so on.
Although this mechanism of phase transition is quite general, specific for the line defects is the related appearance of soliton sectors and usually of quantum soliton particles in the phase where the gas of defects is dilute, i.e. where there are no infinite-long defects.

[XVI] Recall that the dimensin of the defect is the dimension of its locus, where the singularity appears.
[XVII] Actually, in some cases it may happen that we have less than $2d - 1$ possible choices, as otherwise we get a closed defect, but we ignore this issue as we are looking for an higher bound to the number of possible defects.

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