Chapter 1

Monopoles

Qui non ho alcuna bibliografia, se mi suggerisce qualche titolo lo aggiungo volentieri

The last solitons we will take into account are the monopoles, in particular we will consider general monopoles and t'Hooft-Polyakov monopoles. As usual we start from its classical theory.

1.1 Introduction

Let us first consider classical electrodynamics in empty space, Maxwell's equations can be written as (we set $c = 1$)

$$\begin{aligned} &\begin{cases} -\dfrac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{B} = 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \end{cases} \qquad \leftrightarrow \qquad \partial_\mu F^{\mu\nu} = 0 \\ &\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \end{cases} \qquad \leftrightarrow \qquad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma=0} \qquad \leftrightarrow \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{aligned} \tag{1.1}$$

where $\tilde{F}$ is the dual of F: $\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. This system of equations remains unchanged under the replacement

$$\boldsymbol{E} \mapsto \boldsymbol{B} \quad , \quad \boldsymbol{B} \mapsto -\boldsymbol{E} \tag{1.2}$$

or equivalently

$$F_{\mu\nu} \mapsto \tilde{F}_{\mu\nu} \quad , \quad \tilde{F}_{\mu\nu} \mapsto -F_{\mu\nu} \tag{1.3}$$

Such transformation is called *electromagnetic duality*. However, when we add electric sources via a current j_e^μ the invariance under duality is broken, because Maxwell's equations become

$$\begin{cases} \partial_\mu F^{\mu\nu} = j_e^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{cases} \tag{1.4}$$

and clearly the first is no more invariant under eq. (1.3).

In 1931 Dirac had the idea to recover the duality invariance introducing also a "magnetic current", j_m^μ, so that

$$\begin{cases} \partial_\mu F^{\mu\nu} = j_e^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} = j_m^\nu \end{cases} \tag{1.5}$$

so that eq. (1.3) still holds provided that under the transformation we exchange the currents $j_e^\nu \leftrightarrow j_m^\nu$. However the introduction of the magnetic current raises a problem, since j_m violates the Bianchi identity

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \tag{1.6}$$

that guarantees the global existence of the gauge potential A_μ,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.7}$$

In fact, assume for simplicity the temporal gauge $A^0 = 0$ and the global existence of $\boldsymbol{A}$. Consider then the zero-component of the second of eq. (1.5), that is $j^0_m = \partial_\mu \tilde{F}^{\mu 0} = \boldsymbol{\nabla} \cdot \boldsymbol{B}$, and integrate it at a fixed time over a ball B^3 containing a magnetic charge. Then we get

$$Q_m = \int_{B^3} \mathrm{d}^3x\, j^0_m = \int_{B^3} \mathrm{d}^3x\, \boldsymbol{\nabla} \cdot \boldsymbol{B} = \oint_{\partial B^3 = S^2} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_{S^2} \boldsymbol{\nabla} \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S^2 = \emptyset} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = 0 \tag{1.8}$$

hence the magnetic charge associated to j_m should vanish if the gauge potential exists globally. This problem can be avoided in two ways:

- In the *Wu-Yang approach* we can define A^μ only on patches (open sets) $\{U_\alpha\}_{\alpha \in A}$ covering S^2, then in $U_\alpha \cap U_\beta$ we have $F^{\mu\nu}_\alpha = F^{\mu\nu}_\beta$, $\alpha, \beta \in A$, but $A^\mu_\alpha = A^\mu_\beta + \partial^\mu \lambda_{\alpha\beta}$ for some gauge transformation $\delta A = \partial^\mu \lambda_{\alpha\beta}$. In this way $F^{\mu\nu}$ is still defined globally even if A^μ isn't, but this is enough in the classical description of physics.
- The other alternative is the one introduced by Dirac, that is the *Dirac string* $j^{\mu\nu}$, introduced in the previous chapter for the quantization of vortices, such that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + j^{\mu\nu} \tag{1.9}$$

Nelle sue note, a pag 241-242, c'era un commento sulla stringa di Dirac che non ha fatto a lezione. Per ora non l'ho riportato in queste note.

The Wu-Yang approach

Let's make some comment about the Wu-Yang approach (as the Dirac solution has already been discussed in the last chapter). On the sphere S^2 centered on the position of the point-like magnetic charge (*monopole*) we introduce spherical coordinates[I] (r, θ, φ) and define two patches whose union covers S^2:

$$U_1 = S^2 \setminus \{\theta = \pi\} \quad , \quad U_2 = S^2 \setminus \{\theta = 0\} \tag{1.11}$$

On U_1 we define

$$\boldsymbol{A}_1 = \frac{Q_m}{4\pi} \frac{\cos\theta - 1}{r \sin\theta} \boldsymbol{e}_\phi \tag{1.12}$$

where Q_m is the magnetic charge and $\boldsymbol{e}_\phi$ is the unit vector along φ: in Cartesian coordinates $\boldsymbol{e}_\phi = (-\sin\varphi, \cos\varphi, 0)$. I is clear that A_1 is well defined on S^2 except for $\theta = \pi$. Analogously on U_2 we define

$$\boldsymbol{A}_2 = \frac{Q_m}{4\pi} \frac{\cos\theta + 1}{r \sin\theta} \boldsymbol{e}_\varphi \tag{1.13}$$

Let us check that in the intersection of U_1 and U_2 the gauge potential A_1 and A_2 differ only by a gauge transformation. Writing

$$\boldsymbol{A} = A_r \boldsymbol{e}_r + A_\theta \boldsymbol{e}_\theta + A_\varphi \boldsymbol{e}_\varphi \tag{1.14}$$

we find

$$(\boldsymbol{A}_1 - \boldsymbol{A}_2)\big|_{U_1 \cap U_2} = \frac{Q_m}{2\pi} \left[\frac{\cos\theta + 1}{r \sin\theta} - \frac{\cos\theta - 1}{r \sin\theta} \right] \boldsymbol{e}_\varphi = \frac{Q_m}{2\pi} \frac{1}{r \sin\theta} \boldsymbol{e}_\varphi = \frac{1}{2\pi i} e^{-iQ_m\varphi} \boldsymbol{\nabla} e^{iQ_m\varphi} \tag{1.15}$$

where in the last line we used that φ is periodic of period 2π and $Q_m \in \mathbb{Z}$. In a not very formal notation we can also write the last result as "$\boldsymbol{\nabla} \frac{Q_m}{2\pi} \varphi$", so that it is clear that we have a gauge transformation with $\lambda_{12} = \frac{Q_m}{2\pi} \varphi$. If we compute the magnetic field we get, using $\boldsymbol{\nabla} \times \boldsymbol{\nabla} = 0$,

Non è corretto matematicamente perché φ è definita a meno di salti di 2π?

$$\boldsymbol{B}_1 = \boldsymbol{\nabla} \times \boldsymbol{A}_1 = \boldsymbol{\nabla} \times \boldsymbol{A}_2 = \boldsymbol{B}_2 \tag{1.16}$$

[I] Some relations for spherical coordinates which will be used in the following:

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} &= \frac{\partial \boldsymbol{x}}{\partial r}\mathrm{d}r + \frac{\partial \boldsymbol{x}}{\partial \theta}\mathrm{d}\theta + \frac{\partial \boldsymbol{x}}{\partial \varphi}\mathrm{d}\varphi = \left|\frac{\partial \boldsymbol{x}}{\partial r}\right| \mathrm{d}r \boldsymbol{e}_r + \left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right| \mathrm{d}\theta \boldsymbol{e}_\theta + \left|\frac{\partial \boldsymbol{x}}{\partial \varphi}\right| \mathrm{d}\varphi \boldsymbol{e}_\varphi = \mathrm{d}r \boldsymbol{e}_r + r\mathrm{d}\theta \boldsymbol{e}_\theta + r\sin\theta \mathrm{d}\varphi \boldsymbol{e}_\varphi \\
\boldsymbol{\nabla} f \cdot \mathrm{d}\boldsymbol{x} &= \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{\partial f}{\partial \varphi}\mathrm{d}\varphi = \frac{\partial f}{\partial r} \boldsymbol{e}_r \cdot \mathrm{d}\boldsymbol{x} + \frac{\partial f}{\partial \theta}\frac{1}{r} \boldsymbol{e}_\theta \cdot \mathrm{d}\boldsymbol{x} + \frac{\partial f}{\partial \varphi}\frac{1}{r \sin\theta} \boldsymbol{e}_\varphi \cdot \mathrm{d}\boldsymbol{x} \\
\boldsymbol{\nabla} &= \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}
\end{aligned} \tag{1.10}$$

so $\boldsymbol{B}$ is well defined on S^2.
Notice that by setting[II] $\boldsymbol{A} = \frac{1}{2\pi}\boldsymbol{a}$, $\boldsymbol{a}_1$ differs from $\boldsymbol{a}_2$ in $U_1 \cap U_2$ by a well defined $U(1)$ gauge transformation

$$e^{-iQ_m\varphi}\frac{\boldsymbol{\nabla}}{i}e^{iQ_m\varphi} \tag{1.17}$$

so that setting $F_{ij} = \frac{1}{2\pi}f_{ij}$ (f_{ij} turns out to be the curvature of a $U(1)$ connection) and

$$\int_{S^2} F_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2\pi}\int f_{ij}\mathrm{d}x^i\mathrm{d}x^j = Q_m \tag{1.18}$$

is called the *first Chern number*. We may also write

$$f_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{Q_m}{2}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \frac{Q_m}{2}\frac{1}{r^3}\epsilon_{ijk}x^i\mathrm{d}x^j\mathrm{d}x^k \tag{1.19}$$

Topological constraints and problems in the quantization

Notice that by choosing U_1 as the upper semisphere S^2_+ and U_2 as the lower semisphere S^2_-, then $U_1 \cap U_2$ is just the circle in the $z = 0$ plane and

$$\begin{aligned}\int_{S^2} F_{ij}\mathrm{d}x^i\mathrm{d}x^j &= \int_{S^2_+} F_{ij}\mathrm{d}x^i\mathrm{d}x^j + \int_{S^2_-} F_{ij}\mathrm{d}x^i\mathrm{d}x^j = \int_{S^2_+} \boldsymbol{\nabla}\times\boldsymbol{A}_1\cdot\mathrm{d}\boldsymbol{\Sigma} + \int_{S^2_-} \boldsymbol{\nabla}\times\boldsymbol{A}_1\cdot\mathrm{d}\boldsymbol{\Sigma} \\ &= \oint_{S^1}(\boldsymbol{A}_1 - \boldsymbol{A}_2)\mathrm{d}\boldsymbol{x} = \oint_{S^1} e^{-iQ_m\varphi}\frac{\boldsymbol{\nabla}}{2\pi i}e^{iQ_m\varphi}\mathrm{d}\boldsymbol{x} = Q_m\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi} = Q_m\end{aligned} \tag{1.20}$$

Hence Q_m have a new interpretation, namely it counts how many times the gauge transformation $e^{iQ_m\varphi}$ goes around the circle S^1 as φ goes from 0 to 2π (recall that an element of $U(1)$, the gauge group, can be identified with an element of S^1). In other words, $e^{iQ_m\varphi} \in \pi_1(S^1) \simeq \mathbb{Z}$, where π_1 is again the first homotopy group. The fact that such $U(1)$ map cannot be deformed to a constant guarantees the stability of the monopole.

Up to now the monopole was considered at a fixed time. If we consider a $3+1$ dimensional theory one can quantize the theory, first considering a static monopole with 3 moduli corresponding to the 3 coordinates of the center of the monopole, so that the monopole worldline in a $3+1$ QFT produce line defects. However, in the case of monopoles we have a qualitative difference with respect to kinks and vortices: if we compute the energy of the monopole we find that it is UV divergent:

$$\int \mathrm{d}^3x\,\mathcal{E} \sim \int_{\mathbb{R}^3}\mathrm{d}^3x\,F_{ij}^2 \overset{(1.19)}{\sim} \int_0^\infty r^2\mathrm{d}r\,\frac{1}{r^4} \tag{1.21}$$

so that for a 3-ball $B^3(R)$ centered on the monopole of radius R we have

$$\lim_{R\to 0}\int_{\mathbb{R}^3\setminus B^3(R)}\mathrm{d}^3x\,F_{ij}^2 \sim \lim_{R\to 0}\int_R^\infty \frac{\mathrm{d}r}{r^2} = +\infty \tag{1.22}$$

even if we excluded the singularity $r = 0$ of $\frac{1}{r^2}$ from the integration domain.
Since at quantum level this implies divergence in the semi-classical approximation, this suggest that monopoles in the electrodynamic setting are only defined with a UV cutoff, e.g. on a lattice, as we will discuss later on.

Let us show where this singular behaviour of the monopole comes from. One can view this as a consequence of the impossibility of deforming the $U(1)$ transformation $e^{iQ_m\varphi}$ to a constant over the circle S^1 of arbitrarily small radius, which means that the charge Q_m (which due to eq. (1.20) gives such constraint on the possible deformations) is concentrated in a single point. Indeed, the same kind of divergence appears also in the classical description of the electric field, and is due to the fact that the charge of the electron is concentrated in a point. In the case of the electric field the problem is solved in QFT

[II] Notice that $\boldsymbol{a}$ is the gauge potential with the normalization used in the previous chapter.

by replacing the electron to a quantum field, that is an operator valued distribution, which makes sense only when smeared with a test function. The problem here is that in the case of the monopole even the semi-classical approximation is inconsistent, hence we cannot solve this issue as in the case of the electric field.

Let us suppose that our $U(1)$ group is embedded in $SU(2)$, then a 4π rotation in $SU(2)$ can be deformed to the identity, since $SU(2)$ is a double cover of $SO(3)$. So suppose that we have a $SU(2)$ gauge theory, instead of the $U(1)$ discussed up to now, in which we are able to construct a solution of the previous equations of motion that behaves like a $U(1)$ monopole of charge[III] 2 at large distances from its center but near the center can explore the entire structure of $SU(2)$.

Dal nostro ragionamento mi verrebbe più spontaneo pensare che la carica del monopolo debba essere $Q_m \in \mathbb{Z}_2$, visto che dopo 4π siamo nell'origine.

Then the previous argument implying infinite energy would not be valid anymore, and this is the basic idea which leads to the t'Hooft-Polyakov monopole.

1.2 Classical treatment of the t'Hooft-Polyakov monopole

[**Shifman:2012**]

The *t'Hooft-Polyakov monopole* was first introduced in the *Georgi-Glashow model*, which was one of the first attempts to get a unified theory of weak and electromagnetic interactions (now proved to be wrong, but so suggestive that Glashow shared the Nobel prize with the inventors of the Standard Model, Weinberg and Salam). The basic underlying idea was that the $U(1)$ gauge symmetry of QED was just a subgroup of a larger gauge symmetry, $SO(3)$, "spontaneously broken"[IV] to $U(1)$. By the Anderson-Higgs mechanism the $U(1)$ component of the gauge field create massless photons, whereas the other 2 components, corresponding to the spontaneously broken symmetry, create massive vector mesons $W_\mu^\pm$.[V]

Let's start the description of the Georgi-Glashow model from the fields of the theory. The $SO(3)$ gauge field is described by

$$A \equiv A_\mu^a \frac{\tau^a}{2} \quad \text{for} \quad a = 1, 2, 3 \quad \text{and} \quad \tau^a \text{ Pauli matrices} \tag{1.23}$$

and its field strength has components

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \tag{1.24}$$

The model contains also a 3-components real scalar field

$$\phi \equiv \phi^a \frac{\tau^a}{2} \tag{1.25}$$

and its $SO(3)$ covariant derivative is given by

$$D_\mu \phi^a := \partial_\mu \phi^a + \epsilon^{abc} A_\mu^b \phi^c \tag{1.26}$$

The Euclidean Lagrangian in $3+1$ dimensions, a kind of non-Abelian generalization of the $2+1$ Higgs model discussed for vortices, is given by

$$\mathcal{L} = \frac{1}{4g^2}(G_{\mu\nu}^a)^2 + \frac{1}{2}(D_\mu \phi^a)^2 + \lambda\big((\phi^a)^2 - v^2\big) \tag{1.27}$$

with $SO(3)$ breaking boundary conditions, e.g.

$$\phi^a(\infty) = v\delta^{a3} \quad \leftrightarrow \quad \phi(\infty) = v\frac{\tau^3}{2} \tag{1.28}$$

[III] As we need a rotation of 4π to reach the identity in $SU(2)$.
[IV] Here the concept of "spontaneous symmetry breaking" apply either perturbatively, in some gauges, or using a non local order parameter, not as in section **??**.
[V] The Z^0 massive uncharged meson was missing: it was introduced in the Standard Model to describe neutral currents, by replacing $SU(3)$ with $SU(2) \times U(1)$.

The direction of the vector $\phi^a(\infty)$ in the $SO(3)$ (inducing the $SO(3)$ breaking) can be chosen arbitrarily, but when fixed it still leaves a $U(1)$ subgroup of $SO(3)$, corresponding to rotations around the chosen axis, unbroken:

$$e^{i\alpha\frac{\tau^3}{2}}\underbrace{v\frac{\tau^3}{2}}_{\phi(\infty)}e^{-i\alpha\frac{\tau_3}{2}} = \underbrace{v\frac{\tau^3}{2}}_{\phi(\infty)} \tag{1.29}$$

From the action we can derive the following equations of motion

$$\begin{cases} D_\mu G^{\mu\nu\,a} = -g^2\epsilon^{abc}\phi^b D^\nu\phi^c \\ D_{[\mu}G^a_{\nu\rho]} = 0 \\ (D_\mu D^\mu\phi^a) = 4\lambda\phi^a\big((\phi^b)^2 - v^2\big) \end{cases} \tag{1.30}$$

where the first and the second equations corresponds to the first and the second of eq. (1.4) respectively, and the third correspond to the equation of motion of the scalar field.
The action is invariant under the gauge transformations: for $g(x) \in SU(2)$[VI]

$$\begin{cases} A_\mu & \mapsto & g^{-1}A_\mu g + g^{-1}\partial_\mu g \\ \phi & \mapsto & g^{-1}\phi g \end{cases} \tag{1.31}$$

At least perturbatively one often perform a gauge transformation (*unitary gauge*) reducing $\phi = v\frac{\tau^3}{2}$ everywhere. Then A^3_μ remains gapless and we identify it with the “photon field”, whereas

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\frac{1}{g}(A^1_\mu \pm iA_\mu)^2 \tag{1.32}$$

are the massive vector meson fields.

Vacuum sector

The vacua of the theory are found by putting to zero all squares in the energy density $\mathcal{E}$. Define $G^{a\,0i} := E^{a\,i}$ and $G^a_{ij} := -\frac{1}{2}\epsilon_{ijk}B^{ak}$, then for a static configuration, with $A^a_0 = 0$,

$$\int \mathrm{d}^3x\,\mathcal{E} = \int \mathrm{d}^3x\,\frac{1}{2}\left[\frac{(B^{ai})^2}{g^2} + (D_i\phi^a)^2\right] + \lambda\big((\phi^a)^2 - v^2\big)^2 \tag{1.33}$$

The global minimum of the energy, for the given boundary conditions, is given by

$$\begin{cases} B^a = 0 \quad \Rightarrow \quad A^a_i = g^{-1}\partial_i g \\ \phi = v\dfrac{\tau^3}{2} \\ D_i\phi^a = 0 \quad \Rightarrow \quad A^a_i = 0 \end{cases} \tag{1.34}$$

where we used the first and the second conditions to obtain the third.

Monopoles

Let’s consider the monopole as a static field configuration. First, in order to find the monopole, we want to find the correct “magnetic field” of the model. The “$SO(3)$-magnetic field” B^a_i is not gauge invariant, hence unphysical. But its projections on ϕ is gauge invariant, so a natural choice for the “magnetic field” is

$$B^a_i\frac{\phi^a}{|\phi|} = \frac{1}{2}\epsilon_{ijk}G^a_{jk}\frac{\phi^a}{|\phi|} \tag{1.35}$$

and its “magnetic charge” in units of g can be defined as

$$Q_m = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R}\mathrm{d}\Sigma^i\,B^a_i\frac{\phi^a}{|\phi|} \tag{1.36}$$

We want to find some finite energy configuration with $Q_m \neq 0$. From the finiteness of the energy eq. (1.33) we have that as $r \to \infty$ we should have[VII] $(\phi^a(\infty))^2 = v^2$ hence ϕ on the sphere at ∞, S^2_∞, takes values

[VI] Actually the gauge transformations eq. (1.31) leaves the center of $SU(2)$ invariant, hence they behaves as $SO(3)$ transformations.

[VII] The boundary condition eq. (1.28) was imposed in the vacuum configuration.

in a 2-sphere S^2_ϕ of radius $|v|$.
The continuous maps between these spheres are labelled by an integer n corresponding to an element of the homotopy group $\pi_2(S^2) \simeq \mathbb{Z}$, identifying how many times ϕ sweeps S^2_ϕ when $\boldsymbol{x}$ sweeps S^2_∞.
Let us consider the case $N = 1$, this clearly occours if at ∞ we have

$$\phi^a \underset{r\to\infty}{\sim} v\frac{x^a}{r} =: vn^a \tag{1.37}$$

We see that the group index "a" is referred to both as group index and as spatial index, hence gets "entangled" with a coordinate index of $\boldsymbol{x}$ (this is possible since in both cases it runs over $\mathbb{R}^3$, conversely this would be impossible in $U(1)$).
Furthermore finiteness of the energy imposes also that as $r \to \infty$ the derivative $D_i\phi^a$ decays faster than $r^{-3/2}$. For eq. (1.37) this means that

$$\partial_i\phi^a = \partial_i\left(\frac{vx^a}{r}\right) = \frac{v}{r}(\delta^{ai} - n^a n^i) \underset{r\to\infty}{\sim} \frac{1}{r} \tag{1.38}$$

which is not enough to satisfy $D_i\phi^a \sim r^{-3/2}$. Hence we must choose A^b_i so that

$$D_i\phi^a = \partial_i\phi^a + \epsilon^{abc}A^b_i\phi^c \underset{r\to\infty}{\sim} O(r^{-3/2}) \tag{1.39}$$

This requires

$$A^a_i \underset{r\to\infty}{\sim} \epsilon_{aij}\frac{n^j}{r} \tag{1.40}$$

in fact for such choice of A^a_i [VIII]

$$\epsilon^{abc}A^b_i\phi^c = \epsilon^{abc}\epsilon^{bij}\frac{n_j}{r}\frac{x^c}{r}v = \frac{v}{r}(-\delta^{ai} + n^a n^i) \tag{1.43}$$

Again, in eq. (1.40) group and spatial indices are "entangled".

We obtained that in order to have a finite energy configuration with Q_m such that $N = 1$, eq. (1.37) and eq. (1.40) should hold:

$$\phi^a \sim v\frac{x^a}{r} = vn^a \quad \text{and} \quad A^a_i \underset{r\to\infty}{\sim} \epsilon^{aij}\frac{x_j}{r^2} = \epsilon^{aij}\frac{n_j}{r} \tag{1.44}$$

Let us compute Q_m for such configuration using eq. (1.36). First notice that

$$\begin{aligned}\frac{\phi^a}{|\phi|}B^a_i &= \frac{x^a}{r}B^a_i = \frac{x^a}{r}\left[-\frac{1}{2}\epsilon_{ijk}(\partial_j A^a_k - \partial_k A^a_j + \epsilon^{abc}A^b_j A^c_k)\right] \\ &= \frac{x^a}{r}\left[-\epsilon_{ijk}\delta^m_j\frac{\epsilon^{akm}}{r^2} - \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\epsilon^{bjm}\frac{x_m}{r^2}\epsilon^{ckn}r^2\right] = \frac{x_i}{r^3}\end{aligned} \tag{1.45}$$

and then

$$Q_m = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R}\mathrm{d}\Sigma^i\, B^a_i\frac{\phi^a}{|\phi|} = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R}\mathrm{d}\Sigma^i\,\frac{x_i}{r^3} = \lim_{R\to\infty}\frac{1}{g}\int_{S^2_R} r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{x^i}{r}\frac{x_i}{r^3} = \frac{4\pi}{g} \tag{1.46}$$

and this is exactly the expected magnetic charge for a $N = 1$ monopole, hence fields as in eq. (1.44) are very good candidates for the asymptotic description of the monopole.

Non mi è ben chiaro come mai questa sia la forma corretta della carica del monopolo.

In order to find the description of the fields associated to the monopole at finite distances, we introduce the following ansatz, analogous to the one used for the vortices,

$$A^a_i = \epsilon^{aij}\frac{x_j}{r^2}(1 - g_A(r)) \quad , \quad \phi^a = v\frac{x^a}{r}(1 - g_H(r)) \tag{1.47}$$

[VIII] The following identity is needed:

$$\epsilon^{ijk}\epsilon_{ij'k'} = \delta^j_{j'}\delta^k_{k'} - \delta^j_{k'}\delta^k_{j'} \tag{1.41}$$

We will also use

$$\epsilon^{ijk}\epsilon_{ijk'} = \delta^k_{k'} \tag{1.42}$$

with g_A and g_H functions vanishing for $r \to \infty$. The requirement that the energy eq. (1.33) is finite as $r \to 0$ gives

$$r^2 B^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \quad \Rightarrow \quad r^2 \frac{1}{r^4}(1-g_A)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \tag{1.48}$$

and

$$r^2 (D_i\phi)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \quad \Rightarrow \quad r^2 \frac{1}{r^2}(1-g_H)^2 \underset{r\to 0}{\sim} \frac{1}{r^{1-\varepsilon}} \tag{1.49}$$

where we used $B \sim \frac{1}{r^2}(1-g_A)$ and $D_i\phi \sim \frac{1}{r}(1-g_H)$. Requiring that B and ϕ are regular, we obtain

$$r^2 B^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad r^2 \frac{1}{r^4}(1-g_A)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad 1-g_A \underset{r\to 0}{\sim} r \tag{1.50}$$

and simultaneously

$$r^2 (D_i\phi)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad r^2 \frac{1}{r^2}(1-g_H)^2 \underset{r\to 0}{\sim} 1 \quad \Rightarrow \quad 1-g_H \underset{r\to 0}{\sim} 1 \tag{1.51}$$

Hence for $1-g_A = O(r)$ and $1-g_H = O(1)$ the solution has finite energy.
These conditions give a different result for the magnetic charge density respect to the case of the Dirac monopole, eq. (1.22), indeed

$$\lim_{R\to 0} \frac{1}{g} \int_{S^2_R} \mathrm{d}\Sigma^i \, B^a_i \frac{\phi^a}{|\phi|} = 0 \tag{1.52}$$

so the "magnetic charge density" is not concentrated in a point like in the Dirac monopole, and this allows the finiteness of the energy. Indeed eq. (1.47) give really a static solution of the Georgi-Glashow equations of motion eq. (1.30) for $1-g_A = O(r)$ and $1-g_H = O(1)$, and such solution is the (static) t'Hooft-Polyakov monopole.

Recall that a monopole, according to Dirac's description, should be related to a $U(1)$ symmetry. t'Hooft proved that one can define, starting from the non-Abelian field strength $G^a_{\mu\nu}$ of the Georgi-Glashow model, an Abelian $U(1)$ gauge invariant field strength whose singularity is exactly the one of the Dirac monopole. In this way it is possible to see the real monopole structure of the t'Hooft-Polyakov monopole, namely the one corresponding to the Dirac monopole. The correct $U(1)$ "magnetic field" found by t'Hooft is obtained by adding to $B^a_i \frac{\phi^a}{|\phi|}$ a contribution vanishing at ∞, still gauge invariant, so that instead of the regular structure defined up to now we have the singular structure of the Dirac monopole.
In order to simplify the notation, let's $e^a := \frac{\phi^a}{|\phi|}$ defined on the points where $|\phi| \neq 0$, i.e. outside the center ($r = 0$) of the monopole. The $U(1)$ gauge "magnetic field strength", $SO(3)$ invariant, proposed by t'Hooft is

$$F^{U(1)}_{\mu\nu} := e^a \left[G^a_{\mu\nu} - \epsilon^{abc} D_\mu e^b D_\nu e^c\right] \tag{1.53}$$

Indeed, for $|\phi| \neq 0$ (brackets are omitted, however the derivatives act only on the element on their side, e.g. $\partial_\mu e^b \partial_\nu e^c = (\partial_\mu e^b)(\partial_\nu e^c)$)

$$\begin{aligned}
F^{U(1)}_{\mu\nu} &= e^a \big[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu - \epsilon^{abc}\partial_\mu e^b \partial_\nu e^c - \epsilon^{abc}\epsilon^{blm} A^l_\mu e^m \partial_\nu e^c - \\
&\quad - \epsilon^{abc}\partial_\mu e^b \epsilon^{crs} A^r_\nu e^s - \epsilon^{abc}\epsilon^{blm} A^l_\mu e^m \epsilon^{crs} A^r_\nu e^s\big] \\
&= e^a \big[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu - \epsilon^{abc}\partial_\mu e^b \partial_\nu e^c + A^a_\mu \cancel{e^c \partial_\nu e^c} - A^c_\mu e^a \partial_\nu e^c - \\
&\quad - \cancel{\partial_\mu e^b e^b} A^a_\nu + \partial_\mu e^b A^b_\nu e^a - \epsilon^{blm} A^l_\mu A^a_\nu e^m e^c + \epsilon^{blm} A^l_\mu A^b_\nu e^m e^a\big] \\
&= \partial_\mu (e^a A^a_\nu) - \partial_\nu (e^a A^a_\mu) - \epsilon^{abc} e^a \partial_\mu e^b \partial_\nu e^c
\end{aligned} \tag{1.54}$$

notice that $e^a A^a_\mu$ is the projection of A_μ along ϕ, and $F^{U(1)}_{\mu\nu}$ is defined everywhere, beside in the center of the monopole where $|\phi| = 0$. Using eq. (1.47) we see that for the monopole solution

$$e^a = \frac{x^a}{r} \quad \text{and} \quad A^a_i e^a = \cancel{\epsilon^{aij} \frac{x^a}{r} \frac{x_j}{r^2}} (1 - g_A(r)) = 0 \tag{1.55}$$

hence, for the monopole solution

$$
\begin{aligned}
F_{ij}^{U(1)} &= -\epsilon^{abc} e^a \partial_\mu e^b \partial_\nu e^c \\
&= -\epsilon^{abc} \frac{x^a}{r} \partial_i \frac{x^b}{r} \partial_j \frac{x^c}{r} \\
&= -\epsilon^{abc} \frac{x^a}{r^3} \partial_i x^b \partial_j x^c
\end{aligned}
\tag{1.56}
$$

where in the third line we used the antisymmetry of ϵ^{abc}. The result is up to an overall constant factor the field strength of the Dirac monopole, eq. (1.19). Hence $F_{ij}^{U(1)}$ has the same singularity at $r = 0$ as in the case of the Dirac monopole, but we obtained it from a finite energy configuration starting from the Georgi-Glashow model.

Mass of the monopole

Let's compute the mass of the classical monopole. This is particularly simple in the case $\lambda = 0$ (but still $\phi^a(\infty) = v\delta^{a3}$ boundary conditions in the vacuum, eq. (1.28)), called *BPS limit*. In this case, in analogy to the Bogomol'nyi treatment of vortices, one can rewrite

$$
\int \mathrm{d}^3x\, \mathcal{E} = \int \mathrm{d}^3x\, \frac{1}{2g^2}(B_i^a - gD_i\phi^a)^2 + \frac{1}{g} B_i^a D_i \phi^a \tag{1.57}
$$

The second term, using the equation of motion $0 = D_{[i}G_{jk]} = D_i B_i^a$, can be rewritten as

$$
\int \mathrm{d}^3x\, \frac{1}{g} B_i^a D_i \phi^a = \frac{1}{g}\int \mathrm{d}^3x\, \partial_i (B_i^a \phi^a) - \frac{1}{g}\int \mathrm{d}^3x\, (D_i B_i^a)\phi^a = \lim_{R\to\infty} \frac{1}{g}\int_{S_R^2} \mathrm{d}\Sigma^i\, B_i^a \phi^a = vQ_m \tag{1.58}
$$

Hence if the monopole satisfies $B_i^a = D_i\phi^a$ as local minimum of the energy, we have

$$
\int \mathrm{d}^3x\, \mathcal{E} = M_m = vQ_m \tag{1.59}
$$

This also implies that the t'Hooft-Polyakov monopole in the BPS limit is free and very heavy if g is small. Even if $W^\pm$ mesons described by the Georgi-Glashow model are quite different respect those described by the Standard Model, we may assume that the mass of $W^\pm$ is the same in the two models, and in this case the expected mass of the monopole would be $M_m \sim 10\mathrm{TeV}/c^2$ (about the maximum energy reached by the LHC). It has not be found (yet).

1.3 Quantum mechanical treatment of t'Hooft-Polyakov monopole

In the case of the t'Hooft-Polyakov monopole it is important to make some more comments about its quantum mechanical version, since up to now there is no agreed and well defined QFT description of it. The quantum mechanical version of the t'Hooft Polyakov monopole is obtained promoting the moduli of the the classical solution, corresponding to symmetries broken by the specific choice of the monopole solution, to quantum mechanical (time-dependent) variables. Three moduli correspond to the position of the center of the monopole $\boldsymbol{x}_0$ (up to now $\boldsymbol{x}_0 = \boldsymbol{0}$). There is however a fourth modulus, corresponding to the fact that the "global"[IX] $U(1)$-electromagnetic symmetry is unbroken, but a fixed monopole solution breaks it. In fact let $e_m(\infty)$ denote the asymptotic behaviour of the normalized scalar field of the monopole. A $U(1)$ transformation of the form $e^{i\alpha e_m(x)}$, where α is constant, leaves the boundary condition of ϕ_m (the monopole configuration of the scalar field) unchanged but modifies the gauge potential by

$$
A_i \;\mapsto\; A_i^{(\alpha)} = e^{-i\alpha e_m} A_i e^{i\alpha e_m} + \frac{1}{i} e^{-i\alpha e_m}\partial_j e^{i\alpha e_m} \tag{1.60}
$$

We should consider α as a new modulus associated to the specific monopole solution. Hence in the quantum mechanical treatment we should introduce 4 time dependent variables $\hat{\boldsymbol{x}}_0(t)$ and $\hat{\alpha}(t)$, together

[IX] Meaning that the symmetry is still described by gauge transformations, but whose group parameter are constant over the whole spacetime.

with their conjugated momenta $\hat{\boldsymbol{p}}_0(t)$ and $\hat{p}_\alpha(t)$, and then the quantum mechanical hamiltonian of the monopole in the quadratic approximation is

$$H_m = M_m + \frac{\hat{\boldsymbol{p}}_0(t)}{2M_m} + m_W^2 \frac{\hat{p}_\alpha^2(t)}{2M_m} \tag{1.61}$$

where the factor m_W is needed since α is an angle, hence $\frac{1}{2M_m}\hat{p}_\alpha^2$ has not the right dimension in the Hamiltonian. Since $\hat{\alpha}$ is periodic, then $\hat{p}_\alpha$ has discrete spectrum: $\sigma(\hat{p}_\alpha) \subset \mathbb{Z}$ (for $\hbar = 1$).
To understand the meaning of these eigenvalues in the simplest case of a BPS solution, notice that

$$\hat{p}_\alpha = \frac{M_m}{m_W^2}\frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t} \tag{1.62}$$

where m_W^2 again comes from a power counting argument, and define the gauge invariant $SO(3)$ electric field for the monopole solution evaluated in the gauge $A_0^a = 0$ as

$$E_i = E_i^a \frac{\phi^a}{|\phi|} = \mathrm{Tr}(E_i e_m) \quad \text{with} \quad E_i = \partial_0 A_i^{(\alpha)} \tag{1.63}$$

whose quantum mechanical version is

$$\hat{E}_i = \mathrm{Tr}\left(\partial_0 \hat{A}_i^{(\alpha)} e_m\right) = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t}\mathrm{Tr}\left(\partial_\alpha \hat{A}_i^{(\alpha)} e_m\right) \overset{(1.60)}{=} \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t}\mathrm{Tr}\left(D_i \hat{e}_m\, e_m\right) \tag{1.64}$$

At large distances, since $|\phi| = v$, we can replace e_m by $\frac{\phi}{v}$ allowing to write the behaviour of the electric field as

$$\hat{E}_i = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t}\frac{1}{v}\mathrm{Tr}(D_i\phi\, e_m) \tag{1.65}$$

Then using the BPS equation $B_i^a = D_i\phi^a$ in this limit one can write

$$\hat{E}_i = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t}\frac{1}{vg}\mathrm{Tr}\left(B_i e_m\right) = \frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}t}\frac{1}{m_W}\mathrm{Tr}\left(B_i e_m\right) = \frac{m_W}{M_m}\hat{p}_\alpha\,\mathrm{Tr}\left(B_i e_m\right) \tag{1.66}$$

The electric charge in units of g for the eigenvalue $k \in \mathbb{Z}$ of $\hat{p}_\alpha$ is then given by

$$Q_e = \lim_{R\to\infty}\frac{1}{g}\int_{S_R^2}\mathrm{d}\Sigma^i\,\hat{E}_i = \lim_{R\to\infty}\frac{1}{g}\int_{S_R^2}\mathrm{d}\Sigma^i\,\frac{m_W}{M_m}k\,\mathrm{Tr}\left(B_i e_m\right) = \frac{1}{g}m_W k\frac{Q_m}{M_m} = k \tag{1.67}$$

where we used $M_m = vQ_m$ and $m_W = vg$. Hence if $k \neq 0$ then both Q_m and Q_E are non-vanishing. These solutions, where both electric and magnetic charge are not zero, are called *dyons*. By consistency, they should appear in the Georgi-Glashow model (meaning that they cannot be removed from the spectrum of the Hamiltonian).
Hence, while the first three moduli $\boldsymbol{x}_0$ become just the position variables of a quantum mechanical particle, the modulus connected with the symmetry which has been broken by a specific choice of a modulus solution in the internal space is just the information that the spectrum of the theory contains particles with both electric and magnetic charges different from 0.

1.4 Quantum field theory treatment of Dirac monopoles

Unfortunately there is no well understood quantum field theory of the 't Hooft-Polyakov monopoles, so in order to discuss a quantum field theory of monopoles we turn again to Dirac monopoles. As previously discussed an UV cutoff is needed for the description of such monopoles, and it is understood in the following of the chapter. However, formally we will write formulas in the continuum notation.
Again we construct monopole quantum field operators from Euclidean monopole correlators using OS reconstruction. To construct monopole correlation functions it is easier to exploit the duality invariance discussed at the beginning of this chapter: we first construct the correlation functions of charged fields in a scalar electrodynamic without monopoles and then by duality we obtain the Dirac monopole correlators (at least in a theory without electric charges).

Strocchi's theorem

The electric charge in the first model appears as the generator of a global subgroup of a local gauge invariance, this implies an important difference with respect to the topological charges previously discussed. Up to now starting from the fields of the correlation functions via reconstruction theorems, we have constructed a Hilbert space of states which was naturally endowed with a positive definite scalar product, in case via reconstruction. However for gauge theories, both relativistic (such as QED) and non-relativistic cases, the situation is not so simple. The key issue is that a local gauge invariance implies a Gauss law.
Let us first consider the problem classically. The Noether's theorem for a global (gauge) invariance yields a conserved current j_μ, using equations of motions (of both the gauge field and the charged field). However, if we mimic its standard proof in the case of local gauge invariance we get the conservation of the current without using the equations of motions of the charged field and the corresponding charge can be determined from measurements at infinity.

To simplify the discussion we consider the Abelian case, a scalar electrodynamic in the charged field ϕ, but the non Abelian case can be treated similarly. In order to simplify the notation we assume unit charge.
From the gauge invariance of the Lagrangian density

$$\mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*, A_\nu, \partial_\mu A_\nu) \tag{1.68}$$

for an infinitesimal gauge transformation parametrized by $\Lambda(x)$ such that

$$\delta\phi(x) = i\Lambda(x)\phi(x) \quad , \quad \delta\phi^*(x) = -i\Lambda(x)\phi^*(x) \quad , \quad \delta A_\mu(x) = -i\partial_\mu\Lambda(x) \tag{1.69}$$

we get

$$\begin{aligned} 0 &= \left[\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial_\mu\delta\phi^* + \frac{\delta\mathcal{L}}{\delta A^\nu}\delta A^\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\delta A_\nu\right] \\ &= i\left[\frac{\delta\mathcal{L}}{\delta\phi}\phi - \frac{\delta\mathcal{L}}{\delta\phi^*}\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\partial_\mu\phi^*\right]\Lambda + \\ &\quad + i\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} - \frac{\delta\mathcal{L}}{\delta A_\mu}\right]\partial_\mu\Lambda - i\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\partial_\nu\Lambda \end{aligned} \tag{1.70}$$

for $\Lambda(x)$ arbitrary. In the case of global symmetry $\Lambda(x)$ is independent on x, and using equations of motion of ϕ and ϕ^*

$$\frac{\delta\mathcal{L}}{\delta\phi} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \quad \text{and} \quad \frac{\delta\mathcal{L}}{\delta\phi^*} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} \tag{1.71}$$

we get

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^*\right] = 0 \tag{1.72}$$

so that the Noether current

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^* \tag{1.73}$$

is conserved: $\partial_\mu j^\mu = 0$.

But if Λ is arbitrary, also coefficients of $\partial_\mu\Lambda$ and $\partial_\mu\partial_\nu\Lambda$ in eq. (1.70) should vanish, hence we should have also

$$j^\mu - \frac{\delta\mathcal{L}}{\delta A_\mu} = 0 \quad \text{and} \quad \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\partial_\mu\partial_\nu\Lambda = 0 \tag{1.74}$$

and the latter is satisfied only if

$$G^{\mu\nu} := -\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} \tag{1.75}$$

is antisymmetric, $G^{\mu\nu} = G^{[\mu\nu]}$. But then from the first of eq. (1.74) and the equation of motion of A we have

$$j^\nu = \partial_\mu G^{\mu\nu} \tag{1.76}$$

and by antisymmetry

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu G^{\mu\nu} = 0 \tag{1.77}$$

hence j^ν is automatically conserved without using equations of motion of ϕ and ϕ^*. Moreover, defining the charge

$$Q := \int \mathrm{d}^d x \, j^0 = \int \mathrm{d}^d x \, \partial_i G^{i0} = \lim_{R\to\infty} \int_{S_R^{d-1}} \mathrm{d}\Sigma_i \, G^{i0} \tag{1.78}$$

associated to j^ν, we see that such global charge does not depend on the local behaviour of the density j^0 but only upon the behaviour at infinity of the "electromagnetic" field strength $G^{\mu\nu}$.

Since property eq. (1.76) is a key consequence of the local gauge invariance we must impose it also at quantum level at least in the weakest possible sense, as we are going to see.
Denote by $\hat{j}_\mu$ and $\hat{G}_{\mu\nu}$ the field operator corresponding to classical j_μ and $G_{\mu\nu}$, and by $|\psi\rangle$, $|\chi\rangle$ vectors in the space $\mathcal{V}$ of vector states generated by quantum fields of a quantum gauge theory. According to the previous discussion we require that at least for a subspace[X] $\mathcal{V}_{\mathrm{phys}} \subseteq \mathcal{V}$ of physical states the matrix elements satisfy

$$\langle\psi| \, (\hat{j}^\nu - \partial_\mu G^{\mu\nu})(f) \, |\chi\rangle = 0 \quad \text{for} \quad |\psi\rangle, |\chi\rangle \in \mathcal{V}_{\mathrm{phys}} \tag{1.79}$$

and

$$\text{if} \quad |\chi\rangle \in \mathcal{V}_{\mathrm{phys}} \quad \text{then} \quad (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle \in \mathcal{V}_{\mathrm{phys}} \tag{1.80}$$

for f smearing test function needed in the continuum. Then if $\mathcal{V}$ has a semi-definite positive inner product, this implies the stronger statement

$$(\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})(f) \, |\chi\rangle = 0 \quad \text{for all} \quad |\chi\rangle \in \mathcal{V}_{\mathrm{phys}} \tag{1.81}$$

In fact, for $|\psi\rangle \in \mathcal{V}$ and $|\chi\rangle \in \mathcal{V}_{\mathrm{phys}}$, using Schwartz inequality

$$0 \le |\langle\psi| \, (\hat{j}^\nu - \partial_\mu G^{\mu\nu})(f) \, |\chi\rangle|^2 \le \langle\psi|\psi\rangle \, \underbrace{\langle\chi| \, (\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu})^\dagger(f)}_{\in\mathcal{V}_{\mathrm{phys}}} \underbrace{(\hat{j}^\nu - \partial_\mu \hat{G}^{\mu\nu}) \, |\chi\rangle}_{\in\mathcal{V}_{\mathrm{phys}}} = 0 \tag{1.82}$$

hence eq. (1.81), i.e. the Gauss law as an operator identity, automatically holds.
Let's consider the transformation of the field $\hat{\phi}$, charged with respect to the global gauge group, i.e. (modulo smearings)

$$[Q, \hat{\phi}(y)] = \lim_{R\to\infty} \int_{B_R^d} \mathrm{d}^d x [\hat{j}^0(x), \hat{\phi}(y)] = \delta\hat{\phi}(y) \neq 0 \tag{1.83}$$

If $\mathcal{V}$ has a semi-definite inner product and $\hat{\phi}(y)$ is local, or at least localized in a bounded region, then for $|\psi\rangle$, $|\chi\rangle \in \mathcal{V}_{\mathrm{phys}}$ we have

$$\langle\psi| \lim_{R\to\infty} \int_{B_R^d} \mathrm{d}^d x \, [\hat{j}^0(x), \hat{\phi}(y)] \, |\chi\rangle = \langle\psi| \, \delta\hat{\phi}(y) \, |\chi\rangle \neq 0 \tag{1.84}$$

but also

$$\langle\psi| \lim_{R\to\infty} \int_{B_R^d} \mathrm{d}^d x \, [\hat{j}^0(x), \hat{\phi}(y)] \, |\chi\rangle = \langle\psi| \lim_{R\to\infty} \int_{S_R^{d-1}} \mathrm{d}\Sigma_i \, [G^{i0}(\boldsymbol{x}), \hat{\phi}(y)] \, |\chi\rangle = 0 \tag{1.85}$$

where the vanishing of the commutator is a consequence of locality, contraddicting $\langle\psi| \, \delta\hat{\phi} \, |\chi\rangle \neq 0$.
Hence positivity of $\mathcal{V}$ and locality (in a bounded region) of charged fields are incompatible in a gauge theory. This result in known as *Strocchi's theorem*, and can be rephrased as the impossibility in a gauge theory of having both positivity of the Hilbert space and locality of the charged field.

At least formally (neglecting UV problems) the quantity

$$Q(\Lambda) := \int \mathrm{d}\boldsymbol{x} \, \Lambda(\boldsymbol{x}) (\hat{j}^0(\boldsymbol{x}) - \partial_i G^{i0}(\boldsymbol{x})) \tag{1.86}$$

[X]Due to the gauge invariance there are unphysical states, which are elements of $\mathcal{V}$ but not of $\mathcal{V}_{\mathrm{phys}}$.

generates the time-independent gauge transformations of parameters $\Lambda(\boldsymbol{x})$. In fact defining

$$\delta\hat{\phi}(\boldsymbol{y}) := [Q(\lambda), \hat{\phi}(\boldsymbol{y})] = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})[\hat{j}^0(\boldsymbol{x}) - \partial_i G^{i0}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] \tag{1.87}$$

and using

$$\hat{j}^0 = (\hat{\phi}^*\partial_0\hat{\phi} - \hat{\phi}\partial_0\hat{\phi}^*) \quad \text{and} \quad [\partial_0\hat{\phi}^*(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] = -i\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.88}$$

we get

$$\delta\hat{\phi}(\boldsymbol{y}) = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})\hat{\phi}(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y}) = i\Lambda(\boldsymbol{y})\hat{\phi}(\boldsymbol{y}) \tag{1.89}$$

Similarly, for the gauge field

$$\delta A_j(\boldsymbol{y}) := [Q(\Lambda), \Lambda_j(\boldsymbol{y})] = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x})[-\partial_i F^{i0}(\boldsymbol{x}), A_j(\boldsymbol{y})] = \int \mathrm{d}^3\boldsymbol{x}\, \Lambda(\boldsymbol{x}) i\partial_i\delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ij} = -i\partial_j\Lambda(\boldsymbol{y}) \tag{1.90}$$

This suggest that if in the reconstruction theorem we use only fields invariant under local gauge transformations vanishing at infinity, in the reconstructed space of states the Gauss law is automatically implemented. If OS positivity holds this must be a Hilbert space, but then charged field operators are non-local, reaching ∞ even at fixed time.
This is not the standard approach based on covariant quantization, gauge-fixing and locality, but for the consistency of the monopole, in particular for the unphysical nature of the Dirac string, preserving the gauge invariance appears to be crucial and this approach based on positivity but not locality is the most natural for charged fields.
Therefore we try to get the monopole quantum field starting from a charged non-local quantum field, easier to define, in the dual theory obtained via OS reconstruction from its correlators. Then we apply to these correlators the duality transformation. Let us show how to do this.

Duality transformation in the path integral

Notice that in the path integral formalism there is a simple way to derive the duality, starting from the free theory.
Consider the free photon partition function in the Lorenz gauge (with simplified notation for integrals over x)

$$\begin{aligned} Z_A &= \int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A_{\nu]})^2}\delta(\partial^\mu A_\mu) = \\ &= \text{const}\cdot\int \mathcal{D}A_\mu^T\, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A^T_{\nu]})^2} = \\ &= \text{const}\cdot\int \mathcal{D}A_\mu^T \int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\, e^{i\int B_{\mu\nu}\partial^{[\mu}A_T^{\nu]}} = \\ &= \text{const}\cdot\int \mathcal{D}A_\mu^T \int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\, e^{-i\int(\partial^{[\mu}B_{\mu\nu})A_T^{\nu]}} = \\ &= \text{const}\cdot\int \mathcal{D}B_{\mu\nu}\, e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\,\delta(\partial^\mu B_{\mu\nu}) \end{aligned} \tag{1.91}$$

where in the first step we splitted $A_\mu = A_\mu^L + A_\mu^T$ with $\partial^\mu A_\mu^T = 0$ and $\partial^\mu A_\mu = \partial^\mu A_\mu^L$, in the second step we performed a Gaussian transformation with antisymmetric tensor[XI] $B_{\mu\nu}$ (can be checked by integrating over $B_{\mu\nu}$ from the second to the first line), in the third step we integrated by parts in the second exponential, and in the last step we integrated over A_μ^T.
The constraint $\partial^\mu B_{\mu\nu} = 0$ can be solved writing $B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma$, for some gauge potential $\tilde{A}^\sigma$, with possible gauge-fixing $\partial_\mu\tilde{A}^\mu = 0$ which fixes the correspondence $B_{\mu\nu} \leftrightarrow \tilde{A}_\mu$. In this way the partition function become

$$\begin{aligned} Z_A &= \text{const}\cdot\int \mathcal{D}\tilde{A}_\mu\, e^{-\frac{e^2}{2}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma)^2}\,\delta(\partial^\mu\tilde{A}_\mu) \\ &= \text{const}\cdot\int \mathcal{D}\tilde{A}_\mu\, e^{-\frac{e^2}{2}\int(\partial^{[\rho}\tilde{A}^{\sigma]})^2}\,\delta(\partial^\mu\tilde{A}_\mu) \end{aligned} \tag{1.92}$$

[XI] Notice that the tensor $B_{\mu\nu}$ plays the role of a Lagrange multiplier.

Varying with respect to $B_{\mu\nu}$ in the mixed $B_{\mu\nu}$-A_μ action one finds, using saddle point approximation (which works since the action is quadratic),

$$\frac{i}{e}\partial^{[\mu}A^{\nu]} = eB_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \tilde{A}^\sigma \tag{1.93}$$

(notice that in Minkowski space the first i is missing). We can now construct field strengths of A and $\tilde{A}$

$$F^{\mu\nu} = \partial^{[\mu}A^{\nu]} \quad \text{and} \quad \tilde{F}^{\mu\nu} = \partial^{[\mu}\tilde{A}^{\nu]} \tag{1.94}$$

which satisfy equations of motion

$$\begin{aligned} \partial_\mu F^{\mu\nu} = 0 \quad &\overset{(1.93)}{\longleftrightarrow} \quad \epsilon_{\mu\nu\rho\sigma}\partial^\nu \tilde{F}^{\rho\sigma} = 0 \\ \epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0 \quad &\overset{(1.93)}{\longleftrightarrow} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \end{aligned} \tag{1.95}$$

thus recovering the duality for the free theory. Thus we see how in the free theory the duality arises from a clever change of variables, performing a Gaussian (Fourier) transformation together with the introduction of $B_{\mu\nu}$ and the solution of the constraint on $B_{\mu\nu}$ by means of the field $\tilde{A}_\mu$. Such procedure is general and can be applied to more general situations.

The closed defects

Let us now see how by duality we obtain an electrodynamic with monopoles. We start from the Stückelberg model (for high-energy physics) or the gauged XY model (for condensed matter physics). Monopoles will correspond to the charges of these models. The fields of the model are the angular field $\theta(x)$ and the abelian gauge field $A_\mu(x)$. The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2e^2}(\partial_{[\mu}A_{\nu]})^2 + \frac{\lambda}{2}(-ie^{i\theta}\partial_\mu e^{i\theta} - A_\mu)^2 \tag{1.96}$$

One can write (*Hodge decomposition*)

$$\frac{1}{i}e^{-i\theta}\partial_\mu e^{i\theta} = \partial_\mu \Lambda + 2\pi n_\mu \tag{1.97}$$

where Λ is real and globally defined and $2\pi n_\mu$ is a singular current corresponding to the jump $0 \to 2\pi$ in $\theta(x)$, so that $n_\mu(x) \in \mathbb{Z}$.
Let's state the Poisson resummation formula: for and angle φ we have the Fourier transform

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\lambda}{2}(\varphi+2\pi n)^2} = \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2\lambda}j^2}\, e^{i\varphi j} \tag{1.98}$$

We can apply its x-dependent version to the Lagrangian:

$$\sum_n e^{-\frac{\lambda}{2}\int(\partial_\mu\Lambda + A_\mu + 2\pi n_\mu)^2} = \sum_j e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{i\int(\partial_\mu\Lambda - A_\mu)j^\mu} \tag{1.99}$$

where now $n_\mu(x)$ and $j^\mu(x)$ are integer valued line-currents (i.e. supported on lines). In particular j^μ is the current of charged particles in the Stückelberg model. Integrating over the scalar field $\Lambda(x)$ gives

$$\int \mathcal{D}\Lambda\, e^{i\int \mathrm{d}^4x\,(\partial_\mu\Lambda)j^\mu} = \prod_{x\in\mathbb{R}^4} \delta(\partial^\mu j_\mu) = \delta(\partial^\mu j_\mu) \tag{1.100}$$

and $\partial_\mu j^\mu = 0$ implies that the support of the current j_μ is made of closed lines. We then see that the term $\exp\left(-i\int \mathrm{d}^4x\, A_\mu j^\mu\right)$ describes the world line of charged particle-antiparticle pairs. The closed lines in the support of j_μ then correspond to boundaries of surfaces. Let's denote by $S^{\rho\sigma}(x)$ the integer current with support on one of these surfaces for fixed j_μ, so that

$$j_\mu = \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(x) \tag{1.101}$$

(of course there are many possible choices for the surface, but all these possible choices are equivalent). According to the summation formula eq. (1.98) the partition function of the Stückelberg model then become

$$
\begin{aligned}
Z_\lambda &= \int \mathcal{D}A_\mu \sum_{n_\mu} e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A_{\nu]})^2} e^{-\frac{\lambda}{2}\int(\partial_\mu\Lambda - A_\mu + 2\pi n_\mu)^2}\delta(\partial^\mu A_\mu) \\
&= \int \mathcal{D}A^T_\mu \int \mathcal{D}\Lambda \sum_{j_\mu} e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A^T_{\nu]})^2} e^{-\frac{1}{2\lambda}\int j_\mu^2} e^{i\int(-A^T_\mu + \partial_\mu\Lambda)j^\mu} \\
&= \int \mathcal{D}A^T_\mu \, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A^T_{\nu]})^2} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu_T \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}} \\
&= \int \mathcal{D}A_\mu \, e^{-\frac{1}{2e^2}\int(\partial_{[\mu}A_{\nu]})^2} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}}\delta(\partial^\mu A_\mu)
\end{aligned}
\tag{1.102}
$$

Applying then the duality procedure as in eq. (1.91) we have

$$
Z_\lambda = \int \mathcal{D}A^T_\mu \int \mathcal{D}B_{\mu\nu} \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int B^2_{\mu\nu}} e^{i\int B_{\mu\nu}\partial^\mu A^\nu_T} e^{-\frac{1}{2\lambda}\int(\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})^2} e^{-i\int A^\mu_T\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}} \tag{1.103}
$$

and then integrating out A^T_μ and taking the limit $\lambda \to \infty$ we get

$$
Z_\infty = \int \mathcal{D}B_{\mu\nu} \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int B^2_{\mu\nu}} \delta(\partial^\nu B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma})) \tag{1.104}
$$

The constraint on $B_{\mu\nu}$ can be solved introducing a gauge field $\tilde{A}_\mu$ such that

$$
B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho\tilde{A}^\sigma \tag{1.105}
$$

so that

$$
\begin{aligned}
Z_\infty &= \int \mathcal{D}\tilde{A}^\mu \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int(\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\tilde{A}^\sigma + S^{\rho\sigma}))^2}\delta(\partial^\mu\tilde{A}_\mu) \\
&= \int \mathcal{D}\tilde{A}_\mu \sum_{S^{\rho\sigma}} e^{-\frac{e^2}{2}\int(\partial^{[\rho}\tilde{A}^{\sigma]} + S^{\rho\sigma})^2}\delta(\partial^\mu\tilde{A}_\mu)
\end{aligned}
\tag{1.106}
$$

If we define the field strength

$$
\tilde{\mathcal{F}}^{\rho\sigma} = \partial^{[\rho}\tilde{A}^{\sigma]} + S^{\rho\sigma} \tag{1.107}
$$

we see that it satisfies

$$
\epsilon_{\mu\nu\rho\sigma}\partial^\nu\tilde{\mathcal{F}}^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\partial^\nu\partial^{[\rho}\tilde{A}^{\sigma]} + \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma} = j_\mu \tag{1.108}
$$

This proves that j^μ are monopole currents with respect to $\tilde{A}^\mu$, i.e. the worldlines of monopole-antimonopole pairs. Accordingly $S^{\rho\sigma}$ should describe the 2-dimensional worldsheet spanned by the Dirac string between the monopole and the antimonopole of the pair. The arbitrariness of the choice of $S^{\rho\sigma}$ reflects the arbitrariness of the Dirac string. Hence the monopole current are the locus of the defects of the electrodynamics of the monopoles.

Open defects and correlators

Let us see how to "open" such line defects using duality. A charged gauge-invariant non-local field in the Stückelberg model can be constructed using a recipe suggested by Dirac in the operator setting (for QED), adapted to the path-integral formalism.
The charged field of the Stückelberg model (given by quantization of the complex field ϕ with $|\phi| = 1$) is $e^{i\hat{\theta}(\boldsymbol{x})}$, clearly not-gauge invariant under the gauge transformation (if the field has unit charge $e = 1$)

$$
e^{i\hat{\theta}(\boldsymbol{x})} \quad \to \quad e^{i(\hat{\theta}(\boldsymbol{x}) + e\Lambda(\boldsymbol{x}))} \tag{1.109}
$$

Dirac suggested[XII] to make it gauge invariant by multiplying it by

$$
e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E^{\boldsymbol{x}}_i(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \tag{1.110}
$$

[XII] The idea of Dirac, qualitatively, it that an electron, even asymptotically, carries with it also its Coulomb field, hence when an electron is created its Coulomb field should be created as well. Applying this idea to the scalar field (rather than to the electron field) we get the procedure we are describing in the following.

where $\hat{A}^i$ is the gauge potential of the quantum "photon" field and $E_i^{\boldsymbol{x}}$ denote the (classical) electric field generated by a unit charge at the position $\boldsymbol{x}$, so that by the classical Gauss law

$$\partial^i E_i^{\boldsymbol{x}}(\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.111}$$

In fact, under a gauge transformation $\Lambda(\boldsymbol{x})$ vanishing at ∞

$$\begin{aligned} e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \quad \to \quad & e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})(\hat{A}^i(\boldsymbol{y})+\partial_i\Lambda(\boldsymbol{y}))} = \\ & = e^{ie\int \mathrm{d}^3\boldsymbol{y}\, e_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}_i(\boldsymbol{y})} e^{-ie\int \partial^u E_i^{\boldsymbol{x}}(\boldsymbol{y})\Lambda(\boldsymbol{y})} = \\ & = e^{ie\int \mathrm{d}^3\boldsymbol{y}\, e_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}_i(\boldsymbol{y})} e^{-ie\Lambda(\boldsymbol{x})} \end{aligned} \tag{1.112}$$

so that

$$e^{i\hat{\theta}(\boldsymbol{x})} e^{ie\int \mathrm{d}^3\boldsymbol{y}\, E_i^{\boldsymbol{x}}(\boldsymbol{y})\hat{A}^i(\boldsymbol{y})} \tag{1.113}$$

is gauge invariant but still charged, hence it is physical and should be added to the Hilbert space of the theory due to completeness. The second term in Dirac recipe describe the Coulomb field always attached to a charged particle.
It is easy to define the corresponding Euclidean field (now in $\mathbb{R}^4$). Define

$$E_i^x(y) = \delta(x^0 - y^0) E_i^{\boldsymbol{x}}(\boldsymbol{y}) \quad , \quad E_o^x(y) = 0 \tag{1.114}$$

then the charged non-local Euclidean field is given by

$$e^{i\theta(x)} e^{ie\int \mathrm{d}^4y\, E_\mu^x(y)A^\mu(y)} \tag{1.115}$$

Then the Euclidean correlator of such fields naturally satisfy the OS positivity, hence using the OS reconstruction it is possible to obtain a charged quantum field operator (in the lattice).
However we are interested in the quantum field operator for the monopole, so let's apply the duality transformation

$$\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4z\, E_\mu^x(z)A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4w\, E_\mu^y(w)A^\mu(w)} \rangle_\lambda \tag{1.116}$$

where $\langle \quad \rangle_\lambda$ denotes the expectation value in the model with the fixed value λ. To apply the duality transfomration we need to reexpress the θ-dependence appearing in the exprectation value in terms of $-ie^{-i\theta}\partial_\mu e^{i\theta} = \partial_\mu\Lambda + 2\pi n_\mu$. This can be achieved introducing a current j_μ^{xy} satisfying

$$\partial^\mu j_\mu^{xy}(w) = \delta(y-w) - \delta(x-w) \tag{1.117}$$

Then

$$e^{i(\theta(x)-\theta(y))} = e^{\int \mathrm{d}^4w\, j_\mu^{xy}(w) e^{-i\theta(w)}\partial_\mu e^{i\theta(w)}} \tag{1.118}$$

in fact

$$e^{\int \mathrm{d}^4w\, j_\mu^{xy}(w) e^{-i\theta(w)}\partial_\mu e^{i\theta(w)}} = e^{\int \mathrm{d}^4w\, j_\mu^{xy}(w)\partial_\mu \log e^{i\theta(w)}} = e^{-\int \mathrm{d}^4w\, \partial^\mu j_\mu^{xy}(w)\log e^{i\theta(w)}} = e^{-\log e^{i\theta(y)} + \log e^{i\theta(x)}} \tag{1.119}$$

Then our 2-point function eq. (1.116) becomes

$$\begin{aligned} & \langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4z\, E_\mu^x(z)A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4w\, E_\mu^y(w)A^\mu(w)} \rangle_\lambda = \\ & = \langle e^{ie\int \mathrm{d}^4w\, (E_\mu^x(w) - E_\mu^y(w))A_\mu(w)} e^{i\int \mathrm{d}^4w\, j_\mu^{xy}(w)(\partial_\mu\Lambda + 2\pi n_\mu)(w)} \rangle_\lambda \end{aligned} \tag{1.120}$$

Note that

$$e^{i\int \mathrm{d}^4w\, j_\mu^{xy}(w) 2\pi n_\mu(w)} = 1 \tag{1.121}$$

since both j_μ^{xy} and n_μ are integer valued. Then the integration over Λ in the partition function gives

$$\int \mathcal{D}\Lambda\, e^{i\int j_\mu(\partial_\mu\Lambda + 2\pi n_\mu - A_\mu)} e^{i\int j_\mu^{xy}\partial_\mu\Lambda} = e^{-i\int j_\mu A_\mu}\delta(\partial^\mu j_\mu + \partial^\mu j_\mu^{xy}) \tag{1.122}$$

hence $\partial^\mu j_\mu(w) = -\partial^\mu j^{xy}_\mu(w) = \delta(w-x) - \delta(w-y)$, or equivalently $\partial^\mu j_\mu = \delta_x - \delta_y$, where we defined $\delta_x(w) := \delta(w-x)$. Finally in the limit $\lambda \to \infty$, which we have seen is dual to the electrodynamic with a monopole, the correlation function becomes

$$
\begin{aligned}
&\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4 z\, E^x_\mu(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4 w\, E^y_\mu(w) A^\mu(w)} \rangle_\infty = \\
&\quad = \frac{1}{Z}\int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} \delta(\partial^\mu A_\mu) \sum_{j_\mu\,:\,\partial^\mu j_\mu = \delta_x - \delta_y} e^{i\int A_\mu(-j_\mu + E^x_\mu - E^y_\mu)} = \\
&\quad = \frac{1}{Z}\int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} \delta(\partial^\mu A_\mu) \sum_{j_\mu\,:\,\partial^\mu j_\mu = 0} e^{i\int A_\mu(-j_\mu + j^{xy}_\mu + E^x_\mu - E^y_\mu)}
\end{aligned}
\tag{1.123}
$$

The situation is represented in fig. 1.1.

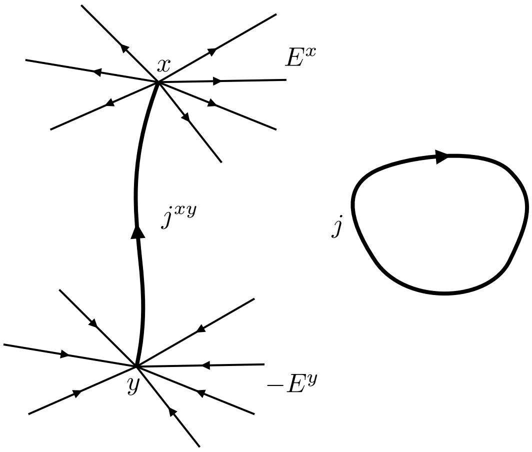

Figure 1.1: One dimensionless representation of currents and fields in eq. (1.123).

Since $\partial^\mu(j^{xy}_\mu + E^x_\mu - E^y_\mu) = 0$, it exists a tensor

$$
S^{\rho\sigma}(z|j^{xy}, E^x, E^y) \quad \text{such that} \quad -(j^{xy}_\mu + E^x_\mu - E^y_\mu)(z) = \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(z|j^{xy}, E^x, E^y) \tag{1.124}
$$

Using the Hodge decomposition one can give a more explicit expression for this tensor

$$
S^{\rho\sigma}(z|j^{xy}, E^x, E^y) = \int \mathrm{d}w\, \epsilon^{\rho\sigma\alpha\beta}\partial_\alpha \Delta^{-1}(z-w)(j^{xy} + E^x - E^y)_\beta \tag{1.125}
$$

in fact, using $\epsilon^{\rho\sigma\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$, we get

$$
\begin{aligned}
\epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}(z|j^{xy}, E^x, E^y) &= \int \mathrm{d}^4 w\, \epsilon_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\alpha\beta}\partial^\nu\partial_\alpha \Delta^{-1}(z-w)(j^{xy} + E^x - E^y)_\beta(w) \\
&= \int \mathrm{d}^4 w\, \partial_\mu \Delta^{-1}(z-w)\cancel{\partial^\beta(j^{xy} + E^x - E^y)_\beta(w)} - \\
&\quad - \int \mathrm{d}^4 w\, \underbrace{\partial_\nu\partial^\nu \Delta^{-1}(z-w)}_{\delta(z-w)}(j^{xy} + E^x - E^y)_\mu(w) \\
&= -(j^{xy} + E^x - E^y)_\mu(z)
\end{aligned}
\tag{1.126}
$$

Since j_μ appearing in eq. (1.123) could be rewritten as in the partition function using $j_\mu = \epsilon_{\mu\nu\rho\sigma}\partial^\nu S^{\rho\sigma}$ one can rewrite

$$
\begin{aligned}
&\langle e^{i\theta(x)} e^{ie\int \mathrm{d}^4 z\, E^x_\mu(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int \mathrm{d}^4 w\, E^y_\mu(w) A^\mu(w)} \rangle_\infty = \\
&\quad = \frac{1}{Z}\int \mathcal{D}A_\mu\, e^{-\frac{1}{2e^2}\int (\partial_{[\mu}A_{\nu]})^2} \delta(\partial^\mu A_\mu) \sum_{S^{\rho\sigma}} e^{-i\int A_\mu \epsilon_{\mu\nu\rho\sigma}\partial^\nu (S^{\rho\sigma} + S^{\rho\sigma}(j^{xy}, E^x, E^y))} \\
&\quad = \frac{1}{Z}\int \mathcal{D}A^T_\mu \int \mathcal{D}B_{\mu\nu} e^{-\frac{e^2}{2}\int B^2_{\mu\nu}}\, e^{i\int A^\mu_T \partial^\nu (B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma}(S^{\rho\sigma} + S^{\rho\sigma}(j^{xy} E^x, E^y)))}
\end{aligned}
\tag{1.127}
$$

where in the third step we applied duality. We should solve the constraint which arises integrating out A^T_μ: for a gauge field $\tilde{A}^\mu$ we get

$$B_{\mu\nu} - \epsilon_{\mu\nu\rho\sigma} S^{\rho\sigma} + S^{\rho\sigma}(j^{xy}, E^x, E^y) = \epsilon_{\mu\nu\rho\sigma} \partial^\rho \tilde{A}^\sigma \tag{1.128}$$

and then we finally get

$$\begin{aligned} &\langle e^{i\theta(x)} e^{ie\int d^4z\, E^x_\mu(z) A^\mu(z)} e^{-i\theta(y)} e^{-ie\int d^4w\, E^y_\mu(w) A^\mu(w)} \rangle_\infty = \\ &= \frac{\displaystyle\int \mathcal{D}\tilde{A}_\mu\, \delta(\partial^\mu \tilde{A}_\mu) \sum_{S^{\mu\nu}} e^{-\frac{i}{2}\int (\partial_{[\mu}\tilde{A}_{\nu]} + S_{\mu\nu} + S_{\mu\nu}(j^{xy}, E^x, E^y))^2}}{\displaystyle\int \mathcal{D}\tilde{A}_\mu\, \delta(\partial^\mu \tilde{A}_\mu) e^{-\frac{i}{2}\int (\partial_{[\mu}\tilde{A}_{\nu]} + S_{\mu\nu})^2}} \\ &=: \langle M(x) M^\dagger(y) \rangle \end{aligned} \tag{1.129}$$

as the correlation function of the monopole in the dual theory of electrodynamic with monopoles. Notice that the additional "monopole current"introduced in the correlation is

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu S_{\rho\sigma}(j^{xy}, E^x, E^y) = j^{xy} + E^x - E^y \tag{1.130}$$

hence here the "classical electric fields" E^x and E^y in the dual theory play the role of the classical magnetic field of the monopole, which is expected on the basis of duality since in the Stückelberg model E^x and E^y were just the classical electric field describing the Coulomb interaction associated to the charged field.

Reconstruction of the theory

Since the charged correlator of the Stückelberg model satisfy OS positivity, in spite of the appearance the same must be true for the monopole correlator. Therefore we can apply OS reconstruction theorem and construct a non-local monopole field operator $\hat{M}(\boldsymbol{x})$ (it is supported in the $x^0 = 0$ plane due to the non-locality of E^x and E^y), so that in the electrodynamic with monopoles

$$\langle M(x) M(y) \rangle = \langle 0 | \hat{M}(\boldsymbol{x}) e^{-H(x^0 - y^0)} \hat{M}^\dagger(\boldsymbol{y}) | 0 \rangle \tag{1.131}$$

In fig. 1.2 one clearly sees the difference in locality between vortices and monopoles as consequence of the gauge invariance resulting in a Coulomb interaction for charged particles.

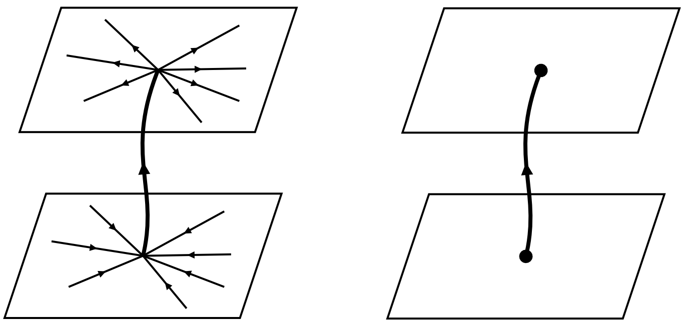

Figure 1.2: One dimensionless representation of the typical structure of "monopole currents" and "vortex currents" in 2-point correlators

One can almost rigorously prove that for e^2 large enough (strong coupling limit) the monopole quantum field operator $\hat{M}(\boldsymbol{x})$ couple the vacuum $|0\rangle$ to a 1-particle state together with an infrared Coulomb tail (called *one-infraparticle state*[XIII]), hence $\hat{M}(\boldsymbol{x})$ creates and annihilates monopole infra-particles.
Furthermore the mass of the monopole turns out to be $O(e^2)$, hence, similarly to the case of vortices, monopole correlators with non-vanishing total monopole charge vanish, so that the Hilbert space of states

[XIII] One cannot think about such state as a pole of the Green function, since the mass hyperboloid associated to the monopole interacts with the rest for the froward light cone, due to the presence of the field E^x, so that the pole of the Green function is blurred by the photon. This cannot be proved perturbatively, but using the non-perturbative Bloch-Nordick treatment.

of electrodynamic with monopoles for large e^2 develops monopole superselection setor $\mathcal{H}_q$ labelled by the total magnetic charge. Therefore the total Hilbert space of the theory is given by

$$\mathcal{H} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q \tag{1.132}$$

with

$$\hat{M}^\dagger : \mathcal{H}_1 \to \mathcal{H}_{q+1} \tag{1.133}$$

Conversely to the case $e^2 \gg 1$, where opening a closed defect "costs a lot", for e^2 small enough (weak coupling limit) the monopole "condense", i.e. there is a huge amount of open line defects in the partition function, and such defects "hides" the monopoles, so that $\mathcal{H} = \mathcal{H}_0$.

1.5 "Dirac monopoles" in spin ice

[**Castelnovo:2008aa**]

Although Dirac monopoles have not been found as elementary particles, and in fact they suffer of an UV problem, in 2008 a kind of exotic magnets called *spin ice* have been studied, in which there seems to appear excitations similar to Dirac monopoles.[XIV] The name comes from the fact that their structure is reminiscent of that of the ice.
The structure appearing in the portion of oxygen and hydrogen in ice is tetrahedral, as represented in fig. 1.3.

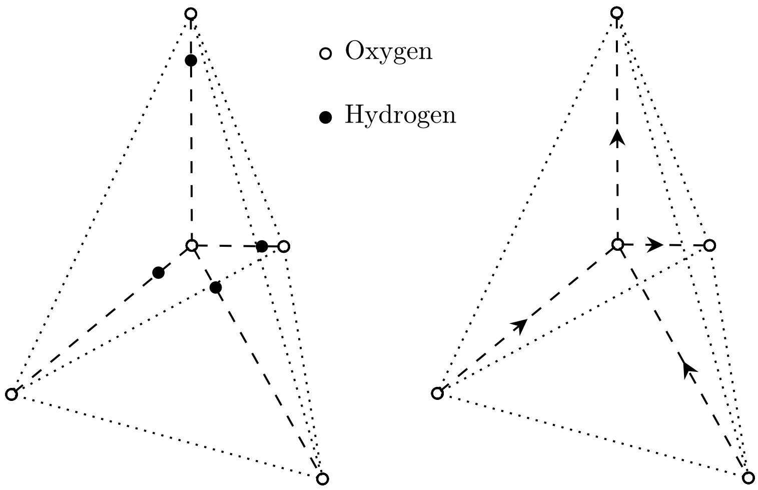

Figure 1.3: Tetrahedral structure of the ice: one oxygen is connected to other oxygens along the direction of a tetrahedron, and in between two oxygens there is an hydrogen. However of these four hydrogens contained in the tetrahedron two are nearer to the central oxygen and their bound is covariant, and two are more far, and their bound is of hydrogen type. In the second picture we represented the positions of the hydrogens by means of displacement vectors, pointing in the direction of the displacement of the hydrogen atom with respect to the center of the bound. Hence in a tetrahedron two displacement vectors are inword and two are outword.

In spin ice materials the hydrogen is replaced is replaced by rare-earth ions and the displacement vector is replaced by a spin vector, each spin vector has only one possible direction and two possible orientations, i.e. Ising-like spins, along the bound.
The Hamiltonian can be taken as

$$H = -J\sum_{\langle i,j\rangle} \boldsymbol{S}_i^{\hat{z}_i}\cdot\boldsymbol{S}_j^{\hat{z}_j} + D\sum_{i,j}\left[\frac{\boldsymbol{S}_i^{\hat{z}_i}\cdot\boldsymbol{S}_j^{\hat{z}_j}}{r_{ij}^3} - 3\frac{(\boldsymbol{S}_i^{\hat{z}_i}\cdot\boldsymbol{r}_{ij})(\boldsymbol{S}^{\hat{z}_j}\cdot\boldsymbol{r}_{ij})}{r_{ij}^5}\right] \tag{1.134}$$

[XIV] Original work: [**Castelnovo:2008aa**].

where $\boldsymbol{S}_i^{\hat{z}_i}$ is the spin along the $\hat{z}_i$ direction in the radial direction from the center of the tetrahedron towards its vertex labelled by the site i, $\boldsymbol{r}_{ij}$ is the vector connecting the site i and the site j. Furthermore the orientation of the spins in the first term are favouring the combination 2-in/2-out for each tetrahedron, as this is the minimal energy configuration. Indeed, since the angle between bounds give $\cos\theta = \frac{1}{3}$, we get the following energy contributions for the short range interaction:

$$\begin{cases} \mathcal{E}(2 \text{ in}, 2 \text{ out}) = -2\dfrac{J}{3}S^2 \\ \mathcal{E}(1 \text{ in}, 3 \text{ out}) = \mathcal{E}(3 \text{ in}, 1 \text{ out}) = 0\dfrac{J}{3}S^2 \\ \mathcal{E}(0 \text{ in}, 4 \text{ out}) = \mathcal{E}(4 \text{ in}, 0 \text{ out}) = 2\dfrac{J}{3}S^2 \end{cases} \tag{1.135}$$

Notice that the minimal energy configuration of such a system is highly degenerate. The long-range part of the Hamiltonian is reminiscent of the dipolar interaction: the potential of a dipole $\boldsymbol{p} = q\boldsymbol{d}$ is

$$\phi(\boldsymbol{x}) = \frac{\boldsymbol{p}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^3} \tag{1.136}$$

where q is the charge and $\boldsymbol{d}$ is the vector connecting the two charges of the dipole. At large distances the potential energy of two interacting dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in electrodynamics is given up to a sign by the scalar product of a dipole for the electric field generated by the other.
Therefore for $\boldsymbol{r}$ the vector from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ the energy reads

$$\mathcal{E} = \boldsymbol{p}_1 \cdot \boldsymbol{\nabla}\phi_2(\boldsymbol{r}) = \boldsymbol{p}_1 \cdot \left(\frac{\boldsymbol{p}_2}{r^3} - 3\frac{(\boldsymbol{r}\cdot\boldsymbol{p}_2)\boldsymbol{x}}{r^5}\right) = \frac{\boldsymbol{p}_i\cdot\boldsymbol{p}_2}{r^3} - 3\frac{(\boldsymbol{r}\cdot\boldsymbol{p}_1)(\boldsymbol{r}\cdot\boldsymbol{p}_2)}{r^5} \tag{1.137}$$

Then we can replace the spins in the spin ice by dipoles “at the end of the spin vectors", so that the $+$ charge corresponds to the tail and the $-$ charge to the head of the vector, and such charges interact via a Coulomb potential. It is natural to think of these charges as magnetic ones since the spin is naturally related to magnetism. In this way at the center of each tetrahedron the nearer charges of the 4 dipoles accumulate. More precisely, if the site center of the tetrahedron is labelled by i we have for its charge

$$Q_i = q_{i1} + q_{i2} + q_{i3} + q_{i4} \tag{1.138}$$

where $q_{i\ell}$ is the charge of the ℓ-dipole which is closer to the site i. Charges of the centers interact via Coulomb interaction

$$V = \frac{Q_i Q_j}{r_{ij}} \tag{1.139}$$

Imposing an onsite energy $v_0 \sum_i Q_i^2$, so that $Q_i = 0$ on the ground state, enforces the ice rule, that is the configuration 2 in/2 out for each tetrahedron, as in the case of fig. 1.4.
If we split a spin, in one of the two tetrahedron connected by the spin the total charge become 2, and in the other become -2. The defect in the partition function of the model correspond to a chain of spin flips. If a site is intermediate between two spin flips cleverly chosen, then two charges of opposite charge in the dipole picture have changed their sign, so still we have $Q = 0$ in this site. However at the ends of a chain of spin flips we have 3 charges of the same sign and one with the opposite one, so that $Q \neq 0$. Since these charges interact according to the Coulomb interaction eq. (1.139), they can be interpreted as monopoles. Therefore at each end of a chain of spin flips we have a “magnetic monopole” if $Q > 0$ of a “magnetic antimonopole” if $Q < 0$. The string of spin flips is nothing but a “Dirac string”. Such situation is represented in fig. 1.5
If we introduce the time dimension these configurations describes precisely worldlines of virtual monopole-antimonopole pairs with a Dirac string between them spanning a surface bounded by that worldline.

1.6 Condensation of defects

We close the discussion about solitons with a final remark about phase transitions. We have seen that quantum solitons correspond in their Euclidean description to line-defects. We also remarked that it may happen that line-defects reach ∞ with finite probability, i.e. give raise to the phenomenon called

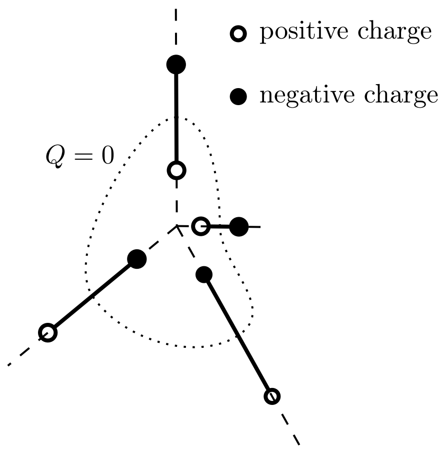

Figure 1.4: The tetrahedral cell of spin ice in the "dipole" representation, in the minimal energy configuration (2 in-2out). Notice that in the region close to the center of the tetrahedron the total charge is $Q = 1 + 1 - 1 - 1 = 0$.

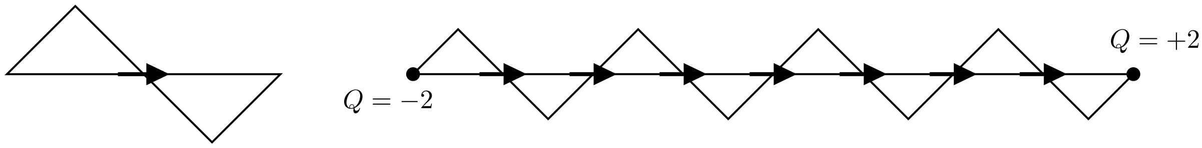

Figure 1.5: On the left, the two triangles represent two close tetrahedrons, and the arrow represent a spin flip in the spin shared between them. On the right, we have the representation of a chain of spin flips, chosen so that it represent a "Dirac string" whose boundaries constitute a "monopole-antimonopole pair".

condensation of defects. When condensation of defects occurs the soliton sectors disappear. The transition from a situation where configurations allowed are only with finite defects ("dilute gas of defects") to a situation where also infinitely extended defects appear corresponds to a phase transition.

As we said, defects can have arbitrary dimension[XV] k. We now show heuristically that for $k > 0$ condensation of defects is a general mechanism of phase transition. Suppose for simplicity that only one species of defects appear in the model under consideration, but the extension to multiple species is straightforward.
For $k > 0$ the mean action $\bar{S}$ of a k-dimensional defect, denoted by D^k, is typically proportional to the volume of its locus, i.e.

$$\bar{S}(D^k) \sim c_k \beta |D^k| \tag{1.140}$$

where β is the relevant coupling constant of the model, $c_k > 0$ is a suitable constant and $|D^k|$ is the volume of the locus of the defect suitably discretized.
The number of such defects which contain a fixed point, say the origin, and with fixed discretized volume $|D^k|$, is bounded by $e^{d_k |D^k|}$, for some constant $d_k > 0$. To get a hint of this bound consider the simplest case of a line defect in dimension $d = 2$. Consider a kink containing the origin, since we are considering a discretized space, it is trivial to notice that the first "piece" of the kink can be chosen is $2d$ ways, since there are $2d$ possible locus around the origin which can be connected with it. Then we may add another "piece" of kink in $2d - 1$ ways, as we cannot "go back", and the same is true for all other "pieces",[XVI] obtaining in this way a line defect of dimension $L = |D^1|$ after $L - 1$ steps. Hence if the length of the defect we consider is L there are at most

$$2d(2d-1)^{L-1} \leq e^{L \log 2d} \tag{1.141}$$

[XV] Recall that the dimensin of the defect is the dimension of its locus, where the singularity appears.
[XVI] Actually, in some cases it may happen that we have less than $2d - 1$ possible choices, as otherwise we get a closed defect, but we ignore this issue as we are looking for an higher bound to the number of possible defects.

line defects of such length containing the origin.
As discussed for the kink in ϕ^4_2, one can write in general the partition function of the model with k-defects, denoted D from now on for simplicity, as the partition function of an interacting gas of defects. Assuming they are weakly interacting

$$Z \approx \sum_{N=0}^{\infty} \sum_{\{D_1,\ldots,D_N\}} \prod_{i=1}^{N} e^{-\bar{S}(D_i)} \tag{1.142}$$

Let's now consider the contribution of the defects of volume $|D|$ containing the origin. By the esitmate above it goes like

$$e^{-\bar{S}(D)} e^{d_k|D|} \approx e^{(d_k - c_k\beta)|D|} \tag{1.143}$$

We immediately see that for $|D| \to \infty$

$$e^{(d_k - c_k\beta)|D|} \xrightarrow[|D|\to\infty]{} \begin{cases} 0 & \text{if} \quad d_k - c_k\beta < 0 \\ \infty & \text{if} \quad d_k - c_k\beta > 0 \end{cases} \tag{1.144}$$

The first case implies that there are no defects of infinite volume containing the origin, the second instead suggests that there are defects containing the origin reaching infinity with finite probability and that there is a critical value β_c for β where the transition occur.
One can heuristically interpret in this way the transition from the symmetry breaking phase to the unbroken phase in the case of kinks, from superconductors with massive photon to the usual Coulomb phase with massless photons in the case of vortices, and so on.
Although this mechanism of phase transition is quite general, specific for the line defects is the related appearance of soliton sectors and usually of quantum soliton particles in the phase where the gas of defects is dilute, i.e. where there are no infinite-long defects.