Notes on Macroeconomic Model with Habit Formation

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1 Framework

1. The Core Question and Strategy The model aims to explain:

Why did the correlation between bond and stock returns switch from positive (pre-2000) to negative (post-2000)?

The strategy is to build a model that:

- Generates **time-varying risk premia** (key for explaining asset return comovements)
- Links these risk premia to **macroeconomic fundamentals** (output, inflation, interest rates)
- Is grounded in **microeconomic utility theory**, so that all equations are internally consistent

2. Step 1 Preferences and Asset Pricing Kernel To generate time-varying risk premia, the model uses external **habit formation** in preferences:

$$U(C_t, H_t) = \frac{(C_t - H_t)^{1-\gamma} - 1}{1 - \gamma}$$

This defines the **stochastic discount factor** (SDF) used to price all assets:

$$M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)} = \beta \cdot \left(\frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\gamma}$$

This SDF has two key properties:

- It creates a natural link between consumption dynamics and asset pricing
- It embeds time-varying **risk aversion** via the surplus consumption ratio S_t, which fluctuates with economic conditions

So: the pricing of all bonds and stocks will reflect fluctuations in S_t, which in turn responds to macroeconomic states.

3. Step 2 Connecting the Macro Side: Euler Equation for Output We now want to link macroeconomic variables (which are observable) to consumption (which is central in pricing). To do this, the model assumes:

$$x_t = \hat{c}_t = c_t - (1-\phi)\sum_{i=0}^{\infty}\phi^i c_{t-1-i}$$

That is, the output gap x_t is detrended log consumption.

This substitution allows us to rewrite the Euler equation in terms of x_t, making it usable in macro models and estimation.

From this, we derive:

$$x_t = f_x \mathbb{E}_t x_{t+1} + \rho_x x_{t-1} - \psi r_t$$

This is a standard forward-looking New Keynesian Euler equation. Importantly, it is **not imposed**, but derived from preferences and equilibrium. It links real interest rates to expected output growth.

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4. Step 3 Completing the Macro Block: Inflation and Rates To fully capture macroeconomic evolution, we must model how:

- Inflation π_t
- Short rate i_t

respond to shocks and to each other. These are essential for pricing nominal bonds.

The authors assume inflation and interest rates have a **common unit root** component π_t^*, and define gaps as:

$$\hat{\pi}_t = \pi_t - \pi_t^*, \quad \hat{i}_t = i_t - \pi_t^*$$

Then the system is:

$$\begin{aligned}
\hat{\pi}_t &= b_{\pi x} x_{t-1} + b_{\pi\pi}\hat{\pi}_{t-1} + b_{\pi i}\hat{i}_{t-1} + v_{\pi,t} \\
\hat{i}_t &= b_{ix} x_{t-1} + b_{i\pi}\hat{\pi}_{t-1} + b_{ii}\hat{i}_{t-1} + v_{i,t} \\
\pi_t^* &= \pi_{t-1}^* + v_t^*
\end{aligned}$$

These equations do two things:

- They allow for rich, persistent inflation dynamics (which are important for bond pricing)
- They close the macro block so the model can be solved for the joint evolution of x_t, π_t, i_t

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5. Step 4 Recursive Asset Pricing Given the stochastic discount factor M_{t+1}, we can price any asset:

$$P_t = \mathbb{E}_t[M_{t+1}D_{t+1} + M_{t+1}P_{t+1}]$$

Applied recursively:

Bonds: Nominal bond prices satisfy:

$$P^{\$}_{n,t} = \mathbb{E}_t[M_{t+1}e^{-\pi_{t+1}}P^{\$}_{n-1,t+1}]$$

Stocks: Stocks are modeled as claims on aggregate consumption via:

$$R^{\delta}_{t+1} = \frac{1}{\delta}R^{c}_{t+1} - \frac{1-\delta}{\delta}e^{r_t}$$

These recursions determine the price-dividend ratios and expected returns, including risk premia.

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6. Step 5 Linking Macro Shocks to Risk Premia This is the heart of the model.

Equity:

$$\mathbb{E}_t\left[r^{\text{stock}}_{t+1} - r_t\right] \propto \gamma(1+\lambda(s_t))\sigma_x^2$$

Bonds:

$$\mathbb{E}_t\left[r^{\text{bond}}_{t+1} - r_t\right] \propto \gamma(1+\lambda(s_t))\text{Cov}_t(x_{t+1}, -i_{t+1} - \pi_{t+1})$$

The **key insight** is that:

- Both premia rise in recessions (s_t low, $\lambda(s_t)$ high)
- Bond risk premia change sign depending on how inflation covaries with the business cycle

This mechanism explains the shift in bond-stock correlation.

7. Step 6 Empirical Implementation To take the model to data:

- Estimate structural break in inflation-output dynamics (around 2001)
- Estimate macro parameters in each period using simulated method of moments
- Compute bond-stock return correlations from the model and compare to data

The model reproduces:

- positive bond-stock correlation before 2001
- negative correlation after 2001

This validates the theoretical channel.

Conclusion The entire framework connects:

Preferences → SDF → Euler Equation+Inflation System → Asset Prices → Risk Premia and Comovement

Each formula is used to logically construct a system where:

Observed changes in macro dynamics $\Rightarrow$ Changes in risk premia $\Rightarrow$ Changes in return correlations

Let me know when you'd like to dive into derivations, calibration, or specific asset pricing components.

2 Calibration

Where Calibration with ϕ, θ_1, θ_2 Enters the Model These parameters ϕ, θ_1, and θ_2 enter the model at the core intersection between macroeconomic dynamics and asset pricing: the derivation of the **Euler equation for the output gap**.

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1. Why These Parameters Matter The key macroeconomic relation derived from preferences is the loglinearized Euler equation:

$$x_t = f_x \mathbb{E}_t x_{t+1} + \rho_x x_{t-1} - \psi r_t$$

This equation relates:

- Expected future output gap: $\mathbb{E}_t x_{t+1}$
- Lagged output gap: x_{t-1}
- Real interest rate: r_t

The coefficients f_x, ρ_x, and ψ are *not* estimated, but are *calibrated* from deeper preference and consumption-dynamics parameters:

$$f_x = \frac{1}{\phi - \theta_1}, \quad \rho_x = \frac{\theta_2}{\phi - \theta_1}, \quad \psi = \frac{1}{\gamma(\phi - \theta_1)}$$

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2. Role of Each Parameter

- ϕ: the smoothing parameter that defines how consumption is detrended to form the output gap:
$$x_t = c_t - (1-\phi)\sum_{i=0}^{\infty} \phi^i c_{t-1-i}$$
It determines how strongly x_t co-moves with consumption.

- θ_1, θ_2: parameters from the surplus consumption ratio dynamics:
$$s_{t+1} = (1-\theta_0)\bar{s} + \theta_0 s_t + \theta_1 x_t + \theta_2 x_{t-1} + \lambda(s_t)\varepsilon_{c,t+1}$$
These control how current and lagged output affect the evolution of risk aversion.
  - θ_1: affects forward-looking weight in Euler equation
  - θ_2: adds a backward-looking component to match empirical hump-shaped impulse responses

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3. Why They Are Calibrated These parameters are **not freely estimated**, for several reasons:

- They define structural relationships between consumption, output, and preferences.
- Choosing them arbitrarily could make the model inconsistent with asset pricing theory.
- Calibration ensures that the Euler equation (used to solve the macro block) is fully consistent with the surplus consumption dynamics and asset pricing kernel.

4. Specific Calibration Choices in the Paper

$$\phi = 0.93, \quad \theta_1 = -0.05, \quad \theta_2 = 0.02$$

- $\phi = 0.93$: chosen to match the empirical correlation between detrended consumption and the output gap
- θ_1, θ_2: chosen via a grid search to best match impulse responses in the data (though still viewed as part of the calibration block)

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Conclusion The calibration with ϕ, θ_1, and θ_2 defines how consumption dynamics enter the output Euler equation. This links the micro-founded habit preferences to the macroeconomic system and ensures that the real interest rate affects the output gap in a way consistent with both asset pricing theory and macroeconomic data. Their role is to shape:

$$x_t = f_x \mathbb{E}_t x_{t+1} + \rho_x x_{t-1} - \psi r_t$$

and thereby influence both macro impulse responses and the pricing of bonds and equities.