The Stochastic Knapsack Problem

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***Abstract*—Considered is a knapsack with integer volume F and which is capable of holding K different classes of objects. An object from class k has integer volume b_k, $k = 1, \cdots, K$. Objects arrive randomly to the knapsack; interarrivals are exponential with mean depending on the state of the system. The sojourn time of an object has a general class-dependent distribution. An object in the knapsack from class k accrues revenue at a rate r_k. The problem is to find a control policy in order to accept/reject the arriving objects as a function of the current state in order to maximize the average revenue. Optimization is carried out over the class of coordinate convex policies. For the case of $K = 2$, we show for a wide range of parameters that the optimal control is of the threshold type. In the case of Poisson arrivals and of knapsack and object volumes being integer multiples of each other, it is shown that the optimal policy is always of the double-threshold type. An $O(F)$ algorithm to determine the revenue of threshold policies is also given. For the general case of K classes, we consider the problem of finding the optimal static control where for each class a portion of the knapsack is dedicated. An efficient finite-stage dynamic programming algorithm for locating the optimal static control is presented. Furthermore, variants of the optimal static control which allow some sharing among classes are also discussed.**

I. Introduction

The classical knapsack problem is to pack a knapsack of integer volume F with objects from K different classes in order to maximize profit. Let an object from class k, $k = 1, \cdots, K$, consume b_k integer units of the knapsack and produce a profit r_k. In the case when the knapsack volume F is an integer multiple of the object volumes b_k, $k = 1, \cdots, K$, then the problem has a simple solution: fill the knapsack entirely with objects from the class k that has the highest profit to volume ratio r_k/b_k. If the knapsack volume ratio is not an integer multiple of the object volumes, then the problem can still be solved in $O(FK)$ time with dynamic programming.

In this paper, we suppose that the objects arrive to and depart from the knapsack at random times. No assumption is made on the sojourn distribution of an object and it is permitted to be class-dependent. It is assumed that the K classes of objects arrive according to independent birth processes. In particular, interarrival times from any given class are assumed to be exponentially distributed with mean depending on the current number of objects of that class in the knapsack. An arriving object bypasses the knapsack if insufficient volume is present. Since objects remain in the knapsack for a finite time in this stochastic model, we shall refer to r_k as the revenue rate rather than the profit of class k. The optimization problem considered is to accept/block arriving objects as a function of the current system state (i.e., the current number of objects of each class in the knapsack) in order to maximize the long-run average revenue.

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The stochastic knapsack model is motivated by the problem of accepting and blocking calls to a circuit-switched telecommunication system which supports a variety of traffic types (e.g., voice, video, facsimile, etc.), each of which having different bandwidth requirements and holding-time distributions (see [14] [12] [8] [13]). By modeling the communication channel as the knapsack, the traffic types as the object classes, and the bandwidth requirements as the object volumes, the problem of optimally accepting calls in order to maximize average revenue is equivalent to the stochastic knapsack problem. Other applications of the stochastic knapsack model include: parallel processing where jobs require a varying number of processors as a function of their class; sharing memory where jobs from different classes require different amounts of memory.

Note that the classical knapsack problem can be viewed as an extended case of our stochastic model by setting the arrival rates for each of the classes equal to infinity. In this case, a complete partitioning (CP) policy would be optimal where the knapsack is partitioned and each object class has exclusive use of its dedicated portion of the knapsack volume. At the other extreme, where the arrival rates are all "small," we would expect complete sharing (CS) to be optimal where an object is always offered access whenever sufficient volume is available.

In a related paper, Ross and Tsang [13] posed the access control problem as a Markov decision process (MDP) and developed algorithms to locate optimal policies for several objective criteria (see also Gopal and Stern [8] and Tijms [16], [17]). With a careful choice of state descriptor and MDP algorithm, it was shown in [13] that the optimal policy could be found in reasonable CPU time even if the available volume F and the number of classes K is large (in fact, it was shown that optimal policies could also be found for circuit-switched *networks* with a small number of origin–destination pairs). It was also shown in [13] that the performance can often be significantly improved by exercising a control, i.e., the average revenue of the optimal policy can be significantly better than that of complete sharing (see also [12]). However, the optimal policy is in general complicated, and can therefore be difficult to implement due to its storage requirement and table look-up time. We are therefore motivated to search for high-performing policies which have a simple structure. This is done by limiting our consideration to coordinate convex policies (defined in Section II), which is a rich class of policies, but which may exclude the optimal. An advantage of the coordinate convex policies is that they give rise to product-form solutions for the associated equilibrium state probabilities, from which all of the performance measures of interest can be determined. However, the product form involves a normalization constant, which may be infeasible to directly evaluate for many problems of practical interest.

For the case of two object classes with nonincreasing arrival rates (which includes both Erlang and Engset arrivals), we first consider the problem of finding the optimal coordinate convex policy. For a wide class of parameters (involving arrival, service and revenue rates as well as the volume requirements) the optimal policy is of the threshold type, i.e., one class is always accepted and the other is accepted up to a threshold. We then consider the same problem under the assumption that the arrivals are Poisson (Erlang model) and

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that knapsack volume and the object volumes are integer multiples of each other. In this case, it is shown that the optimal policy is always of a double-threshold form. We also prove that the optimal threshold levels can only occur at the points where the amount of idling knapsack volume is an integer multiple of the volume ratios. Therefore, in order to find the optimal policy it is only necessary to evaluate the average revenue over a relatively small number of threshold policies. An $O(F)$ algorithm is given to recursively determine the revenue for the threshold policies.

The above structural results generalize those obtained by Foschini, Gopinath, and Hayes [6], who considered the special case of each class having the same volume requirement. They also assumed that the arrival rates are generated by a finite population model (i.e., the Engset case). Under these conditions and with a general objective function, they showed that the optimal policy is of the threshold type. Their proofs involved extensive algebraic manipulations, whereas our proofs require probabilistic and sample-path arguements.

We have not been able to establish structural properties for the case of more than two object classes (i.e., $K \geq 3$). We therefore consider the problem of finding an optimal CP policy for the general case of K classes. This is equivalent to finding an optimal fixed allocation of the knapsack in order to maximize the revenue. Such a policy would in general be suboptimal over the class of coordinate convex policies. However, preliminary computational testing has shown that it performs well in many regions of the parameter space; in particular, it performs well if the input traffic rates are high for all classes or just those classes that have the largest values of r_k/b_k (as one would expect since in heavy traffic the problem resembles the classical static knapsack model). Moreover, the optimal CP policy is easy to implement and can be efficiently determined via finite-stage dynamic programming.

II. The Optimal Access Problem

The system state is represented by a vector $\boldsymbol{n} = (n_1, n_2, \cdots, n_K)$ where n_k is the number of class k objects in system. When the system is in state $\boldsymbol{n} = (n_1, n_2, \cdots, n_K)$, a class k object arrives with rate $\lambda_k(n_k)$. Thus, for each class the interarrival times are exponentially distributed with mean depending on the number of objects of that class currently in the system. In the case $\lambda_k(n_k) = \lambda_k$, the model is said to be Erlang. If $\lambda_k(n_k) = (N_k - n_k)\lambda_k$, with N_k representing the population size of class k objects, the model is said to be Engset. The sojourn time for a class k object is permitted to have an arbitrary distribution with mean $1/\mu_k$. A class k object requires b_k volume units (servers) where b_k is a positive integer. Denote $\boldsymbol{b} = (b_1, b_2, \cdots, b_K)$ for the volume requirement vector. Also denote $\boldsymbol{e}_k$ as the K dimensional vector of all zeros except for a one in the kth component. The set of all possible states is given by

$$\Omega_{CS} := \{\boldsymbol{n} : \boldsymbol{n} \cdot \boldsymbol{b} \leq F\}.$$

A policy determines for each system state and for each class of arriving object whether the arriving object will be accepted (see [13]). In this paper, we shall limit our discussion to coordinate convex policies as first defined by Aein and Kosovych [1].

Definition 1: A policy is said to be coordinate convex if there exists a subset $\Omega \subseteq \Omega_{CS}$ such that 1) $\boldsymbol{n} \in \Omega$ and $n_k > 0$ imply $\boldsymbol{n} - \boldsymbol{e}_k \in \Omega$ for $k = 1, \cdots, K$, 2) for any $\boldsymbol{n} \in \Omega_{CS}$ and $k = 1, \cdots, K$, a class k object is accepted when in state $\boldsymbol{n}$ if and only if $\boldsymbol{n} + \boldsymbol{e}_k \in \Omega$.

There is a 1–1 correspondence between coordinate convex policies and subsets $\Omega \subseteq \Omega_{CS}$ satisfying 1) of Definition 1. Henceforth, a coordinate convex policy shall be defined by the subset Ω with which it corresponds. Thus, the CS policy is denoted as Ω_{CS}. Note that the CP policies are also coordinate convex.

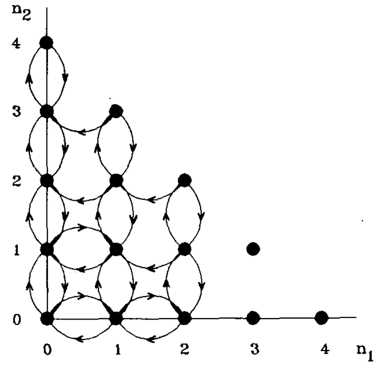

Fig. 1. State transition diagram for a policy which is not coordinate convex.

As an example of a policy which is not coordinate convex, consider a system with $K = 2$, $F = 4$ and $b_1 = b_2 = 1$. Let π be a policy that always accepts a type-2 object (as long as the knapsack is not full) and accepts a type-1 object whenever the total number of objects (type-1 and type-2 combined) in the system does not exceed 1. The state transition diagram for this policy is given in Fig. 1. Note that the state (1, 1) is accessible from (2, 1), but that (2, 1) is not accessible from (1, 1). Thus, π is not coordinate convex.

The class of coordinate convex policies is particularly appealing since they lead to a product-form solution.

Proposition 1: For a give coordinate convex policy Ω, denote $P_\Omega(\boldsymbol{n})$ for the equilibrium probability of being in state $\boldsymbol{n}$. Then

$$P_\Omega(\boldsymbol{n}) = G^{-1}(\Omega) \prod_{k=1}^{K} q_k(n_k)$$

where

$$q_k(n_k) := \frac{\prod_{j=0}^{n_k-1} \lambda_k(j)}{n_k! \mu_k^{n_k}}$$

and

$$G(\Omega) := \sum_{\boldsymbol{n} \in \Omega} \prod_{k=1}^{K} q_k(n_k). \tag{1}$$

With exponential service times, the product form solution of Proposition 1 follows from the well-known properties of truncated reversible processes (see Corollary 1.10 of Kelly [11]). The insensitivity result is easily obtained by modifying the arguments of Kaufman [10] in order to allow for state-dependent arrival rates (see also Burman *et al.* [2]). Henceforth, the term "policy" will mean "coordinate convex policy".

Let $r_k > 0$, $k = 1, \cdots, K$, be the rate of revenue generated by a class k object. Then the long-run average revenue for a policy Ω is given by

$$J(\Omega) := \sum_{\boldsymbol{n} \in \Omega} (\boldsymbol{n} \cdot \boldsymbol{r}) P_\Omega(\boldsymbol{n}). \tag{2}$$

A policy Ω^* is said to be *optimal* if it maximizes $J(\Omega)$ over the class of all coordinate convex policies. Note that if $r_k = b_k$, $k = 1, \cdots, K$, then $J(\Omega)$ is the average knapsack utilization generated by policy Ω.

III. The Structure of the Optimal Policy

Throughout this and the next section we shall suppose that $K = 2$. Note that the maximum number of class k objects, $k = 1, 2$, that can be in the system is given by $\lfloor F/b_k \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. We shall also assume that $\lambda_k(\cdot)$ is nonincreasing for $k = 1, 2$, i.e., $\lambda_k(0) \geq \lambda_k(1) \geq \cdots \geq \lambda_k(\lfloor F/b_k \rfloor)$ for $k = 1, 2$. This last assumption holds for both the Erlang and Engset models. Without loss of generality, we suppose that $b_2 \geq b_1$. Denote $B = \lceil b_2/b_1 \rceil$ for the volume ratio where $\lceil x \rceil$ is the smallest integer greater than or equal to x. Also denote $R = r_2/r_1$ for the revenue ratio.

Definition 2: A policy Ω is said to be threshold type-k ($k = 1, 2$) if for some $l_k = 0, 1, \cdots, \lfloor F/b_k \rfloor$,

$$\Omega = \{(n_1, n_2) \in \Omega_{CS} : n_k \leq l_k\}.$$

A policy Ω is said to be double-threshold if for some $l_1 = 0, \cdots, \lfloor F/b_1 \rfloor$ and for some $l_2 = 0, \cdots, \lfloor F/b_2 \rfloor$,

$$\Omega = \{(n_1, n_2) \in \Omega_{CS} : n_1 \leq l_1,\ n_2 \leq l_2\}.$$

A threshold type-1 policy limits the number of class-1 objects permitted in the system and always accepts a class-2 object when sufficient volume exists (see Fig. 2). Similarly, a threshold type-2 policy limits the number of class-2 objects permitted in the system and always accepts a class-1 object when sufficient volume exists (see Fig. 3). Note that the class of double-threshold policies includes the classes of threshold type-k policies, $k = 1, 2$. Furthermore, a policy Ω is both threshold type-1 and threshold type-2 if and only if $\Omega = \Omega_{CS}$.

Before stating the main result of this section we need to introduce some additional notation. For $k = 1, 2$ and for all nonnegative integers a, b with $b \geq a$, let $X_k(a, b)$ be a random variable taking values in $\{a, a+1, \cdots, b\}$ with probability mass function

$$P(X_k(a, b) = j) = \frac{q_k(j)}{\sum_{j=a}^{b} q_k(j)} \quad a \leq j \leq b.$$

Note that the random variable $X_k(a, b)$ corresponds to the equilibrium state of a birth-death process with birth rates $\lambda(j)$, $j \geq 0$, given by

$$\lambda(j) = \begin{cases} \lambda_k(j) & j < b \\ 0 & j = b \end{cases},$$

and death rates $\mu(j)$, $j \geq 1$, given by

$$\mu(j) = \begin{cases} j\mu_k & j > a \\ 0 & j = a \end{cases}.$$

Denote $x_k(a, b) = E[X_k(a, b)]$. Further define $l(j) = \lfloor F - jb_2/b_1 \rfloor$, which is equal to the maximum number of class-1 objects that can be present in the system when j class-2 objects are present (see Fig. 4). Set $l(\lfloor F/b_2 \rfloor + 1) + 1 := 0$.

The first main result of this section is as follows.

Theorem 1: Suppose Ω^* is optimal. 1) If $R > x_1(0, B)$, then Ω^* is threshold type-1. Moreover, the threshold must be equal to $l(j)$ for some $j = 0, 1, \cdots, \lfloor F/b_2 \rfloor$. 2) If $R < L$ where

$$L := \frac{\min\left\{x_1[l(j)+1, l(j-1)] - x_1[l(j+1)+1, l(j)] : j = 1, \cdots, \left\lfloor \frac{F}{b_2} \right\rfloor\right\}}{x_2(0, 1)},$$

then Ω^* is threshold type-2. 3) If $x_1(0, B) < R < L$, then $\Omega^* = \Omega_{CS}$.

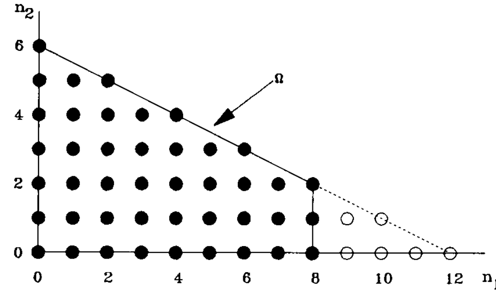

Fig. 2. Type-1 threshold policy: $F = 12$, $b_1 = 1$, $b_2 = 2$, and $l_1 = 8$.

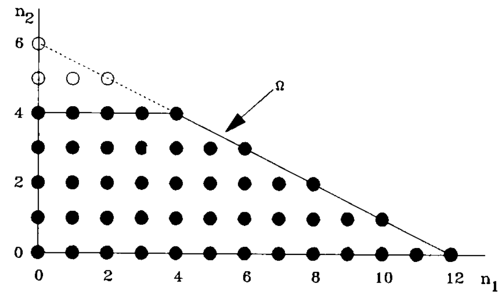

Fig. 3. Type-2 threshold policy: $F = 12$, $b_1 = 1$, $b_2 = 2$, and $l_2 = 4$.

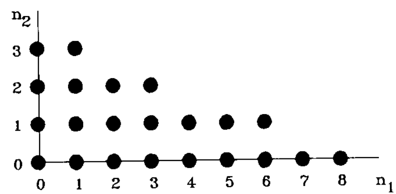

Fig. 4. An example of the geometry of CS: $F = 17$, $b_1 = 2$, $b_2 = 5$; $l(0) = 8$, $l(1) = 6$, $l(2) = 3$, $l(3) = 1$.

The proof of Theorem 1 is the subject of the next subsection. Note that statement 3) immediately follows from 1) and 2) combined. Let $\rho_k := \lambda_k(0)/\mu_k$, $k = 1, 2$. The following weaker result should shed some insight on Theorem 1.

Corollary 1: Suppose that Ω^* is optimal. If $R \geq \min(\rho_1, B)$, then Ω^* is threshold type-1. If $R \leq 1/\min(\rho_2, 1)$, then Ω^* is threshold type-2. If $\min(\rho_1, B) \leq R \leq 1/\min(\rho_2, 1)$, then $\Omega^* = \Omega_{CS}$.

Proof: For Erlang arrivals it is well known that $x_k(0, \infty) = \rho_k$, $k = 1, 2$. But it is easily shown (and intuitively obvious) that the expected number of objects in a system with arrivals at constant rate $\lambda_k(0)$ is at least equal to the expected number of objects in a system with (state-dependent) arrival rates less than or equal to $\lambda_k(0)$. Thus, $x_k(0, \infty) \leq \rho_k$ always holds under the basic assumption that the arrival rates are nonincreasing; clearly then, $x_k(0, a) \leq \min(\rho_k, a)$ for all $a \geq 0$. Combining this with Theorem 1 completes the proof. □

It follows from Corollary 1 that if the traffic is sufficiently light (more precisely, if $\rho_1 \leq R$ and $\rho_2 \leq 1/R$) then the optimal policy is Ω_{CS}. It also follows that if b_2/b_1 is an integer and if the optimization criterion is to maximize average utilization (i.e., $r_1 = b_1$, $r_2 = b_2$), then the optimal policy is threshold type-1. Finally, if $b_1 = 1$ and $b_2 = 1$ (the case of [6]), then it follows from Corollary 1 that the optimal policy is threshold type-1 [resp., threshold type-2] if $r_2 \geq r_1$ [resp., if $r_1 \geq r_2$].

A. Proof of Theorem 1

Definition 3: Let Ω be a policy and $(\alpha, \beta) \in \Omega_{CS} - \Omega$. The tuple (α, β) is a corner point for Ω if $\alpha \geq 1$, $\beta \geq 1$, $(\alpha - 1, \beta) \in \Omega$ and $(\alpha, \beta - 1) \in \Omega$. The tuple (α, β) is a type-1 corner point for Ω if $\beta \geq 1$, $(\alpha, \beta - 1) \in \Omega$ and either $\alpha = 0$ or $(\alpha - 1, \beta) \in \Omega$. The tuple (α, β) is a type-2 corner point for Ω if $\alpha \geq 1$, $(\alpha - 1, \beta) \in \Omega$ and either $\beta = 0$ or $(\alpha, \beta - 1) \in \Omega$.

Throughout our discussion we will need to make use of various geometrical properties of coordinate convex policies. The analytical verifications of these facts are tedious and uninteresting, and will not generally be given. We instead suggest that the reader verify these results visually with the aid of Fig. 4. The first of these geometrical properties is as follows.

Lemma 1: A policy is double-threshold if and only if it has no corner points. A policy is a type-k threshold if and only if it has no type-k corner points ($k = 1, 2$).

Lemma 1 outlines a procedure for establishing the threshold properties. Show that the various corner points are not possible for an optimal policy. This shall be done with the aid of the following definition and lemma, which are due to Foschini, Gopinath, and Hayes [6]. In order to state the definition and lemma, we shall need to extend the definition of $G(S)$ and $J(S)$ for all subsets $S \subseteq \Omega_{CS}$ via transformations (1) and (2), respectively. Of course, if S does not satisfy condition 1) of Definition 1, then S is not a policy. It is straightforward to establish from (2) that if $S = \{a, a + 1, \cdots, b\} \times \{c, c + 1, \cdots, d\}$, then $J(S) = r_1x_1(a, b) + r_2x_2(c, d)$. Also denote

$$H(S) := \sum_{n \in S} (n \cdot r) \prod_{k=1}^{K} q_k(n_k),$$

so that $J(S) = H(S)/G(S)$ for all $S \subseteq \Omega_{CS}$.

Definition 4: A nonempty subset S of Ω_{CS} is incrementally admissible (IA) with respect to a policy Ω if $\Omega \cap S = \phi$ and $\Omega \cup S$ is also a policy. A subset S of a policy Ω is incrementally removable (IR) with respect to Ω if $\Omega - S$ is also a policy.

Lemma 2: Let Ω^* be an optimal policy. Then

1) if S is IA with respect to Ω^*, then $J(S) \leq J(\Omega^*)$;

2) if S is IR with respect to Ω^*, then $J(S) \geq J(\Omega^*)$.

Proof: Suppose S is IA with respect to Ω^*. From (1) and optimality of Ω^*, we have

$$\frac{H(\Omega^*)+H(S)}{G(\Omega^*)+G(S)} = J(\Omega^* \cup S) \leq J(\Omega^*) = \frac{H(\Omega^*)}{G(\Omega^*)},$$

which in turn implies $J(S) = H(S)/G(S) \leq H(\Omega^*)/G(\Omega^*) = J(\Omega^*)$. The proof of 2) is similar. □

The basic idea behind the proof of Theorem 1 is to show that the existence of corner points leads to IA and IR sets which contradict Lemma 2. Before proving Theorem 1, we need to establish two additional lemmas.

Lemma 3: For any nonnegative integers a, b, c, d, e, f with $b \geq a$, $d \geq c$, $f \geq e$, $b + d \geq f$, $a + c \geq e$, we have

$$x_k(a, b) + x_k(c, d) \geq x_k(e, f) \quad k = 1, 2.$$

Proof: Fix k and let $Q_1(t)$, $Q_2(t)$, and $Q(t)$ be independent birth-death processes corresponding to $X_k(a, b)$, $X_k(c, d)$, and $X_k(e, f)$, respectively. Let $Z(t) = Q_1(t) + Q_2(t)$, which moves up and down by steps of one, but which is not (strictly speaking) a birth-death process. Note that when $Z(t) = j$, the process $Z(t)$ moves down at rate no greater than $\gamma(j)$ where

$$\gamma(j) = \begin{cases} 0 & j \leq a+c \\ j\mu_k & j > a+c \end{cases};$$

and due to the fact that $\lambda_k(\cdot)$ is nonincreasing, $Z(t)$ moves up at rate no less than $\nu(j)$ where

$$\nu(j) = \begin{cases} \lambda_k(j) & j < b+d \\ 0 & j \geq b+d \end{cases}.$$

Also note that when $Q(t) = j$, then $Q(t)$ moves down at rate no less than $\gamma(j)$ and moves up at rate no greater than $\nu(j)$. These facts combined with Theorem 5 of Smith and Whitt [15] complete the proof. □

Lemma 4: Suppose Ω^* is optimal. If (α, β) is a type-1 corner point for Ω^*, then for some $j = 2, 3, \cdots, \lfloor F/b_2 \rfloor + 1$ and $i = 1, 2, \cdots, j - 1$ we have

$$\alpha = l(j) + 1 \tag{3}$$

and

$$x_1[l(j)+1, l(i-1)] - x_1[l(j)+1, l(i)] \geq R. \tag{4}$$

If (α, β) is a type-2 corner point for Ω^*, then for some $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor$ we have

$$\alpha = l(j) + 1 \tag{5}$$

and

$$Rx_2(0, 1) \geq x_1[l(j)+1, l(j-1)] - x_1[l(j+1)+1, l(j)]. \tag{6}$$

Proof: We begin by establishing (5) and then (3). Suppose that (α, β) is a type-2 corner point for Ω^* and that $\alpha \neq l(j) + 1$ for all $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor$. In this case, there is a nonnegative integer n such that $S^- = \{(\alpha - 1, \beta + i): i = 0, \cdots, n\}$ is IR with respect to Ω^* and $S^+ = \{(\alpha, \beta + i): i = 0, \cdots, n\}$ is IA with respect to Ω^* (see Fig. 4). But $J(S^-) = r_1(\alpha - 1) + r_2x_2(\beta, \beta + n) < r_1\alpha + r_2x_2(\beta, \beta + n) = J(S^+)$, which contradicts Lemma 2. Thus, (5) holds for some $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor$. Now suppose that (α, β) is a type-1 corner point for Ω^*. Then either (α, β) is a corner point (and hence, a type-2 corner point) or $\alpha = 0 = l(\lfloor F/b_2 \rfloor + 1) + 1$. Thus, from (5), we have $\alpha = l(j) + 1$ for some $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor + 1$. But if $\alpha = l(1) + 1$, then $\beta = 0$, so that (α, β) is not a type-1 corner point. Hence, (3) holds true for some $j = 2, 3, \cdots, \lfloor F/b_2 \rfloor + 1$.

We now establish (4). Suppose that (α, β) is a type-1 corner point, so that $\alpha = l(j) + 1$ for some $j = 2, 3, \cdots, \lfloor F/b_2 \rfloor + 1$. Then for some $i = 1, 2, \cdots, j - 1$ the set $S_1^+ = \{l(j) + 1, \cdots, l(i)\} \times \{\beta\}$ is IA with respect to Ω^* and for some ξ, $l(i) \leq \xi \leq l(i - 1)$, the set $S_1^- = \{l(j) + 1, \cdots, \xi\} \times \{\beta - 1\}$ is IR with respect to Ω^* (see Fig. 4). We have $J(S_1^-) = r_1x_1[l(j) + 1, \xi] + r_2(\beta - 1)$ and $J(S_1^+) = r_1x_1[l(j) + 1, l(i)] + r_2\beta$. Combining these two equalities with Lemma 2 gives

$$x_1[l(j)+1, \xi] - x_1[l(j)+1, l(i)] \geq R,$$

from which (4) directly follows.

We now establish (6). Suppose that (α, β) is a type-2 corner point, so that $\alpha = l(j) + 1$ for some $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor$. In this case there exists integers m, n with $0 \leq n - m \leq 1$ such that $S_2^- = \{(l(j + 1) + 1, \cdots, l(j)\} \times \{\beta, \cdots, \beta + n\}$ is IR with respect to Ω^* and $S_2^+ = \{l(j) + 1, \cdots, l(j - 1)\} \times \{\beta, \cdots, \beta + m\}$ is IA with respect to Ω^*. We have $J(S_2^-) = r_1x_1[l(j + 1) + 1, l(j)] + r_2x_2(\beta, \beta + n)$ and

$J(S_2^+) = r_1x_1[l(j)+1, l(j-1)] + r_2x_2(\beta, \beta+m)$. Combining these two equalities with Lemma 2 gives

$$R\{x_2(\beta, \beta+n)-x_2(\beta, \beta+m)\} \geq x_1[l(j)+1, l(j-1)] - x_1[l(j+1)+1, l(j)],$$

which when combined with Lemma 3 gives (6). □

Proof of Theorem 1: Let Ω^* be optimal and $R > x_1(0, B)$. Suppose (α, β) is a type-1 corner point for Ω^*. Let i, j be as in Lemma 4. Combining (4) with Lemma 3 gives

$$x_1[0, l(i-1)-l(i)] \geq R.$$

But $B \geq l(i-1) - l(i)$, which when combined with the above gives $x_1(0, B) \geq R$, a contradiction. Thus, Ω^* does not have a type-1 corner point and, in the light of Lemma 1, we may conclude that Ω^* is a type-1 threshold policy. Now let the threshold for Ω^* be l_1. Then either $l_1 = \lfloor F/b_1 \rfloor := l(0)$ or $(l_1 + 1, 0)$ is a type-2 corner point for Ω^*. In the latter case, Lemma 4 implies that $l_1 = l(j)$ for some $j = 1, 2, \cdots, \lfloor F/b_2 \rfloor$. Hence, statement 1) is proved.

Statement 2) is proved in a similar manner with (6) of Lemma 4. □

IV. Simplified Model Assumption

Theorem 1 characterizes the optimal policy for a wide range of system parameters. The only case that remains unspoken for is $L \leq R \leq x_1(0, B)$, which shall be studied under the following.

Simplified Model Assumption: 1) $b_1 = 1$ and F is an integer multiple of $b_2 = B$. 2) For each $k = 1, 2$, we have $\lambda_k(j) = \lambda_k$ for $j = 0, \cdots, F/b_k - 1$, i.e., arrivals follow the Erlang model.

Under the simplified model assumption, L becomes

$$L = \frac{\min\left\{x_1[qB+1, (q+1)B] - x_1[(q-1)B+1, qB] : q=0, \cdots, \frac{F}{B}-1\right\}}{x_2(0, 1)} \quad (7)$$

where we have redefined $-B + 1 := 0$. Our second main result makes use of the simplified model assumption in order to characterize the structure of the optimal policy over the entire parameter range.

Theorem 2: Suppose the simplified model assumption holds and Ω^* is optimal. Then Ω^* is double-threshold.

The proof of Theorem 2 (and Theorem 3 below) is delayed to the next section. Note that Theorem 2 does not exclude the possibility that the optimal policy is either threshold type-1 or threshold type-2. Theorem 1 can be combined with Theorem 2 to help determine this more detailed structure, if it is present. It turns out that the more detailed structure is always present when $B = 2$:

Theorem 3: Suppose the simplified model assumption holds, $B = 2$ and Ω^* is optimal. Then Ω^* is either threshold type-1 or threshold type-2. Moreover, either 1) $R > L$, in which case Ω^* is threshold type-1, or 2) $x_1(0, B) \leq R \leq L$, in which case $\Omega^* = \Omega_{CS}$, or 3) $R < x_1(0, B)$, in which case Ω^* is threshold type-2.

The above case $B = 2$ turns out to be somewhat unusual; for $B \geq 3$ it is possible that the optimal policy is neither threshold type-1 nor threshold type-2 (although by Theorem 2 it is guaranteed to be at least double-threshold). Indeed, for the case of $F = 9$, $B = 3$, $R = 2.82$, $\rho_1 = 18$, and $\rho_2 = 1470$, Ω^* is a double-threshold policy ($l_1 = 3$, $l_2 = 2$) with $J(\Omega^*) = 8.461835$. For the same parameters, the best single-threshold policy Ω is a type-1 threshold ($l_1 = 3$) with $J(\Omega) = 8.461288$. However, preliminary computational experience has indicated that single-threshold suboptimality only occurs in a small region of the parameter space; moreover, in this region, the revenue of the best single-threshold policy is only slightly less than that of the optimal double-threshold policy.

A. Proof of Theorems 2 and 3

Under the simplified model assumption, Lemma 4 directly translates to the following.

Lemma 5: Suppose Ω^* is optimal. If (α, β) is a type-1 corner point for Ω^*, then for some $q = -1, 0, \cdots, F/B - 2$ and $p = q+1, \cdots, F/B - 1$ we have

$$\alpha = qB + 1 \quad (8)$$

and

$$x_1[qB+1, (p+1)B] - x_1[qB+1, pB] \geq R. \quad (9)$$

If (α, β) is a type-2 corner point for Ω^*, then for some $q = 0, 1, \cdots, F/B - 1$ we have

$$\alpha = qB + 1 \quad (10)$$

and

$$R > x_1[qB+1, (q+1)B] - x_1[(q-1)B+1, qB]. \quad (11)$$

In order to prove Theorem 2, we need the following technical, but crucial result.

Lemma 6: For all $q = 0, 1, \cdots$ and $p = q+1, q+2, \cdots$, we have

$$x_1[(q-1)B+1, qB] + x_1[qB+1, (p+1)B] \leq x_1[qB+1, (q+1)B] + x_1[qB+1, pB].$$

Proof: The case $q = 0$ follows immediately from Lemma 3. Henceforth, suppose $q \geq 1$ and $p > q$. Also, in order to simplify notation, set $\lambda = \lambda_1$ and $\mu = \mu_1$.

Let $Q_1(t)$, $Q_2(t)$, $Q_3(t)$, and $Q_4(t)$ be birth-death processes associated with the equilibrium random variables $X_1[(q-1)B+1, qB]$, $X_1[qB+1, (p+1)B]$, $X_1[qB+1, (q+1)B]$ and $X_1[qB+1, pB]$, respectively. For $i = 1, \cdots, 4$, denote a_i and b_i for the lowest and highest value, respectively, that $Q_i(t)$ can take on (e.g., $a_1 = (q-1)B+1$ and $b_1 = qB$). We construct a common probability space to support the four birth-death processes so that for all $t \geq 0$

$$Q_i(t) - (b_i - b_1) \leq Q_1(t) \quad i = 2, 3, 4 \quad (12)$$

$$Q_3(t) \leq Q_2(t) \quad (13)$$

$$Q_1(t) + Q_2(t) \leq Q_3(t) + Q_4(t) \quad (14)$$

over the entire sample space. The result would then directly follow from (14).

The probability space is constructed as follows. Define an underlying Poisson process generating event epochs at rate $\gamma = \lambda + 4(p+1)B\mu$. Whenever this Poisson process "fires," one of the following events occurs:

a) Fictitious event with probability

$$1 - \frac{\lambda + \mu[Q_4(t) + Q_3(t) - Q_1(t)]}{\gamma},$$

b) arrival event with probability λ/γ,

c) type-(0, 1, 1, 0) departure event with probability

$$\frac{\mu[Q_3(t) - Q_1(t)]}{\gamma},$$

d) type-(0, 1, 0, 1) departure event with probability

$$\frac{\mu[Q_2(t)-Q_3(t)]}{\gamma},$$

e) type-(0, 0, 0, 1) departure event with probability

$$\frac{\mu[Q_3(t)+Q_4(t)-Q_1(t)-Q_2(t)]}{\gamma},$$

f) type-(1, 1, 1, 1) departure event with probability $\mu Q_1(t)/\gamma$.

The four birth–death processes are defined on the probability space as follows. Suppose the underlying Poisson process fires at time $t = s$. If the fictitious event a) occurs, then there is no change in the processes, i.e., $Q_i(s) = Q_i(s^-)$ for $i = 1, \cdots, 4$. If the arrival event b) occurs, then for $i = 1, \cdots, 4$, the process $Q_i(t)$ will increase at time $t = s$ if and only if $Q_i(s^-) < b_i$. If a type-$(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ departure event occurs, then for $i = 1, \cdots, 4$ the process $Q_i(t)$ will decrease at time $t = s$ if and only if $\Delta_i = 1$ and $Q_i(s^-) > a_i$. Also define $Q_i(0) = b_i$ for $i = 1, \cdots, 4$. Note that for each $i = 1, \cdots, 4$, the process $Q_i(t)$ has been constructed so that at time t a birth occurs with rate $\lambda \cdot 1[Q_i(t) < b_i]$ and a death occurs with rate $\mu Q_i(t) \cdot 1[Q_i(t) > a_i]$ where $1[\cdot]$ is the indicator function. Thus, for each $i = 1, \cdots, 4$, $Q_i(t)$ is a birth–death process yielding the proper equilibrium distribution.

We now establish (12)–(14) by induction. Fix an outcome in the sample space. Clearly, (12)–(14) hold true at time $t = 0$. Suppose that

$$Q_i(s^-)-(b_i-b_1) \le Q_1(s^-) \quad i=2, 3, 4 \tag{15}$$

$$Q_3(s^-) \le Q_2(s^-) \tag{16}$$

$$Q_1(s^-)+Q_2(s^-) \le Q_3(s^-)+Q_4(s^-) \tag{17}$$

and that the underlying Poisson process fires at time $t = s$. It suffices to show that (12)–(14) remain true at time $t = s$.

Proof of (12) at $t = s$: For the fictitious event a), there is no change in $Q_i(t)$, $i = 1, \cdots, 4$, at time $t = s$; thus, (15) implies that (12) holds true at $t = s$. Suppose the arrival event b) occurs. Except in the case $Q_1(s^-) = b_1$ and $Q_i(s^-) < b_i$, if $Q_i(t)$ increases at $t = s$, then so will $Q_1(t)$ at $t = s$, in which case (15) would imply that (12) holds true at $t = s$. On the other hand, if $Q_1(s^-) = b_1$ and $Q_i(s^-) < b_i$, then $Q_1(s^-) > Q_i(s^-) - (b_i - b_1)$; hence, $Q_1(s) = Q_1(s^-) \ge Q_i(s^-) - (b_i - b_1) + 1 = Q_i(s) - (b_i - b_1)$, so that (12) again holds true at $t = s$.

Suppose that one of the departure events c)–f) occurs. Note that for events c)–e), a death cannot occur in $Q_1(t)$ at time $t = s$, so that (12) holds true by (15). In the case of event f), unless $Q_1(s^-) > a_1$ and $Q_i(s^-) = a_i$, if $Q_1(t)$ decreases at $t = s$, then so will $Q_i(t)$ at $t = s$, in which case (15) would imply that (12) remains true at $t = s$. On the other hand, if $Q_1(s^-) > a_1$ and $Q_i(s^-) = a_i$, then $Q_i(s^-) - (b_i - b_1) = a_i - (b_i - b_1) \le a_1 < Q_1(s^-)$; hence, $Q_1(s) = Q_1(s^-) - 1 \ge Q_i(s^-) - (b_i - b_1) = Q_1(s) - (b_i - b_1)$, so that (15) remains true at $t = s$.

Proof of (13) at $t = s$: For the fictitious event a), there is no change in the processes, so that (13) remains true at $t = s$. Suppose that the arrival event b) occurs. Except in the case of $Q_2(s^-) = b_2$ and $Q_3(s^-) < b_3$, if $Q_3(t)$ increases at $t = s$, then so will $Q_2(t)$ at time $t = s$, in which case (16) implies that (13) holds true at $t = s$. If $Q_2(s^-) = b_2$ and $Q_3(s^-) < b_3$, then $Q_2(s^-) > Q_3(s^-)$, implying that (13) holds true at $t = s$.

Suppose that one of the departure events c)–f) occurs. Under event e), there is no decrease in $Q_2(t)$ at $t = s$, so that (16) implies (13) at $t = s$. Suppose that event c) or f) occurs. Except in the case $Q_2(s^-) > a_2$ and $Q_3(s^-) = a_3$, if there is a decrease in $Q_2(t)$ at $t = s$, then there is a decrease in $Q_3(t)$ at $t = s$, in which case (13) holds true at $t = s$ by (16). If $Q_2(s^-) > a_2$ and $Q_3(s^-) = a_3$, then $Q_3(s^-) < Q_2(s^-)$, which implies that (13) remains true at $t = s$. Finally, suppose event d) occurs. In order for this event to occur with positive probability, we need $Q_2(s^-) > Q_3(s^-)$, which implies that (13) holds true at $t = s$.

Proof of (14) at $t = s$: For the fictitious event a), there is no change in the processes, so that (14) remains true at $t = s$. Suppose that the arrival event b) occurs. If $Q_3(s^-) = b_3$ and $Q_4(s^-) = b_4$, then $Q_3(s) + Q_4(s) = b_3 + b_4 \ge Q_1(s) + Q_2(s)$, so that (14) holds true at $t = s$. If either $Q_3(s^-) = b_3$ or $Q_4(s^-) = b_4$, but not both, then (15) implies that $Q_1(s^-) = b_1$, so that $Q_3(t) + Q_4(t)$ increases by one at $t = s$, whereas $Q_1(t) + Q_2(t)$ increases by at most one at $t = s$; combining this with (17) gives (14) at $t = s$. Finally, if $Q_3(s^-) < b_3$ and $Q_4(s^-) < b_4$, then $Q_3(t) + Q_4(t)$ will increase by two at $t = s$; combining this with (17) gives (14) at $t = s$.

Suppose that one of the departure events occurs. If $Q_2(s^-) = a_2$, then $Q_1(s^-) + Q_2(s^-) \le b_1 + a_2 < a_3 + a_4$; thus, $Q_1(s) + Q_2(s) \le Q_1(s^-) + Q_2(s^-) < a_3 + a_4 \le Q_3(s) + Q_4(s)$, so that (14) holds true at $t = s$. Henceforth, assume $Q_2(s^-) > a_2$.

If either event c) or d) occurs, then $Q_1(t) + Q_2(t)$ will decrease by one at $t = s$, whereas $Q_3(t) + Q_4(t)$ will decrease by at most one at $t = s$; combining this with (17) gives (14) at $t = s$.

In order for event (e) to occur with positive probability at $t = s$ we need $Q_1(s^-) + Q_2(s^-) < Q_3(s^-) + Q_4(s^-)$. Combining this with the fact that $Q_3(t) + Q_4(t)$ decreases by at most one at $t = s$ under event e) gives (14) at $t = s$.

Finally, suppose that event f) occurs. If $Q_1(s^-) > a_1$, then from $Q_2(s^-)$ and (17) it follows that (14) holds true at $t = s$. On the other hand, if $Q_1(s^-) = a_1$, then (15) implies that $Q_3(s^-) = a_3$; combining this with $Q_2(s^-) > a_2$ and (17) gives (14) at $t = s$. □

Proof of Theorem 2: Suppose Ω^* is optimal and (α, β) is a corner point for Ω^*. From Lemma 5, we have $\alpha = qB + 1$ for some $q = 0, 1, \cdots, F/B - 2$. Combining (9) with (11) gives

$$x_1[qB+1, (p+1)B]+x_1[(q-1)B+1, qB] > x_1[qB+1, (q+1)B]+x_1[qB+1, pB]$$

for some $p = q + 1, \cdots, F/B - 1$. But this contradicts Lemma 6, so that Ω^* does not have any corner points. Combining this with Lemma 1 completes the proof. □

Proof of Theorem 3: The result would follow immediately from Theorem 1 if we can show $x_1(0, 2) \le L$, which is equivalent to showing for all $q = 0, 1, \cdots, F/B - 1$ that

$$x_1(0, 2)+x_1(2q-1, 2q) \le x_1(2q+1, 2q+2). \tag{18}$$

The remainder of the proof is in the spirit of the proof of Lemma 3. When there are $j = 2q + 1$ objects in the system, the arrival rate corresponding to both the left and right side of (18) is λ_1. When there are $j = 2q + 2$ objects in the system, the departure rates corresponding to both the left and the right sides of (18) is $(2q + 2)\mu_1$. Since the system for the right side of (18) holds either $2q + 1$ or $2q + 2$ objects, whereas the system for the left of (18) can hold as few as $2q - 1$ objects and no more than $2q + 2$ objects, it follows from the above statements and Theorem 5 of [15] that (18) holds true. □

B. An Efficient Algorithm for the Average Revenue

In this section, an algorithm is given that recursively determines the average revenue for (potentially optimal) type-1 threshold policies. For notational convenience, the algorithm is presented under the simplified model assumption; it is easily extended to the general case.

Suppose that $R \geq x_1(0, B)$. By Theorem 1, the optimal policy must take the form

$$\Omega_q = \bigcup_{p=0}^{q} S_p,$$

for some $q = 0, \cdots, F/B$ where

$$S_p = \{(p-1)B+1, \cdots, pB\} \times \left\{0, \cdots, \frac{F}{B} - p\right\}.$$

Define for $k = 1, 2$

$$g_k(a, b) := \sum_{j=a}^{b} q_k(j)$$

$$h_k(a, b) := \sum_{j=a}^{b} j q_k(j),$$

so that $x_k(a, b) = h_k(a, b)/g_k(a, b)$. The following result follows directly form the definitions of $G(S)$ and $H(S)$, $S \subseteq \Omega_{CS}$.

Theorem 4: $G(\Omega_0) = G(S_0)$ and $H(\Omega_0) = H(S_0)$. For $q = 1, \cdots, F/B$, we have

$$G(\Omega_q) = G(\Omega_{q-1}) + G(S_q) \tag{19}$$

$$H(\Omega_q) = H(\Omega_{q-1}) + H(S_q). \tag{20}$$

Moreover, for $q = 0, \cdots, F/B$,

$$G(S_q) = g_1[(q-1)B+1, qB] \cdot g_2\left[0, \frac{F}{B} - q\right]. \tag{21}$$

and

$$H(S_q) = r_1 h_1[(q-1)B+1, qB] \cdot g_2\left[0, \frac{F}{B} - q\right] + r_2 h_2\left[0, \frac{F}{B} - q\right] \cdot g_1[(q-1)B+1, qB]. \tag{22}$$

Theorem 4 specifies a recursive algorithm yielding $G(\Omega_q)$ and $H(\Omega_q)$ [and hence $J(\Omega_q) = H(\Omega_q)/G(\Omega_q)$] for all $q = 0, \cdots, F/B$. The threshold policy Ω_q, $q = 0, \cdots, F/B$, giving the largest value $J(\Omega_q)$ is optimal. In order to carry out this algorithm, it is best to first determine $g_2(0, q)$ and $h_2(0, q)$, $q = 0, \cdots, F/B$, recursively, which can be done in $O(F/B)$ time. The quantities $g_1[(q-1)B+1, qB]$, $h_1[(q-1)B + 1, qB]$, $q = 1, \cdots, F/B$, can all together be determined in $O(F)$ time. Finally, (19)–(22) would then be employed to determine $J(\Omega_q)$, $q = 0, \cdots, F/B$, which requires $O(F/B)$ time. Thus, the overall complexity of the algorithm is $O(F)$.

In an analogous manner, a recursive algorithm can also be developed to determine the revenue of the type-2 threshold policies. Furthermore, the above algorithm can easily be modified to determine the average revenue for an arbitrary coordinate-convex policy (even with the simplified model assumption relaxed). Algorithms are given in [18] in order to determine the performance of threshold policies for $K \geq 3$.

V. The Optimal CP Policy

In this section, for the general case of K classes, the problem of finding an optimal CP policy is addressed. Note that a CP policy allocates a fixed amount of the knapsack to each class. Denote s_k, $k = 1, \cdots, K$, for the number of volume units allocated to class k objects. Any allocation $\boldsymbol{s} = (s_1, \cdots, s_K)$ must satisfy $s_1 + s_2 + \cdots + s_K = F$. A given allocation $\boldsymbol{s} = (s_1, \cdots, s_K)$ decouples the knapsack into K smaller knapsacks where knapsack k has s_k volume units fully dedicated to class k objects. If s_k volume units are allocated to class k objects, then the average revenue generated by class k objects is given by

$$R_k(s_k) = r_k x_k\left(0, \left\lfloor \frac{s_k}{b_k} \right\rfloor\right).$$

Thus, the optimal CP policy is given by the solution of the following resource allocation problem:

$$\max\ J(\boldsymbol{s}) = R_1(s_1) + R_2(s_2) + \cdots + R_K(s_K)$$
$$\text{s.t. } s_1 + s_2 + \cdots + s_K = F$$
$$0 \leq s_k \leq F,\ s_k \text{ integer},\ k = 1, \cdots, K.$$

An optimal solution can be obtained via dynamic programming. Let $f_k(y)$ be the maximum revenue generated by classes 1 through k with y units of volume available. The corresponding dynamic programming equations are

$$f_1(y) = R_1(y),\ 0 \leq y \leq F$$

$$f_k(y) = \max_{0 \leq s \leq y} \{R_k(s) + f_{k-1}(y-s)\}, \quad 0 \leq y \leq F,\ k = 2, \cdots, K.$$

The above recursive equations have complexity $O(F^2K)$. It gives as a byproduct the optimal CP policy $\boldsymbol{s} = (s_1, \cdots, s_K)$. Reaching can also be employed to exploit the special structure of $R_k(\cdot)$, $k = 1, \cdots, K$, yielding a faster algorithm [4]. If $R_k(\cdot)$ is concave for all $k = 1, \cdots, K$, then more efficient marginal analysis algorithms can be employed (e.g., see [4], [7], [3], [9]). However, $R_k(\cdot)$ is only concave when $b_k = 1$.

It is interesting to note that the above technique also applies when volume is allocated to *groups* of object classes, rather than to individual object classes. Suppose that the object classes are organized into fixed groups $A_1, A_2, \cdots, A_q$ where $\cup_{p=1}^{q} A_p = \{1, \cdots, K\}$. (See Kraimeche and Schwartz [12] for a discussion on how the groups should be organized.) An allocation would then be given by $\boldsymbol{s} = (s_1, s_2, \cdots, s_q)$ where s_p volume units are allocated to group p. Within each of the groups, one of the following policies would be employed (depending on the designer's willingness to tradeoff performance for ease of implementation).

1) The CS policy.
2) The optimal coordinate convex policy.
3) The optimal policy determined from a Markov decision algorithm (see [13]).

For any of the above policies, the optimal revenue $R_p(s_p)$ would first be determined for $s_p = 0, \cdots, F$ and $p = 1, \cdots, q$. Then the optimal allocation $\boldsymbol{s} = (s_1, \cdots, s_q)$ would then be determined via the above dynamic programming equations. If the CS policy is applied for each group, then the revenue $R_p(s_p)$, for $s_p = 0, \cdots, F$, $p = 1, \cdots, q$, could be rapidly obtained via Kaufman's one-dimensional recursive algorithm [10]. If the optimal coordinate convex policy is employed for groups with two classes, an optimal threshold policy could then be applied to these groups. This grouping/partitioning scheme should offer the designer a variety of performance and implementation scenarios.

VI. Conclusion

For the case $K = 2$, we have shown for a wide class of parameters that the optimal policy has a single-threshold. If the simplified model assumption is not satisfied, the structure of the optimal policy for the remaining parameters ($L \leq R \leq x_1[0, B]$) remains an open question and merits further research. If the simplified model assumption is satisfied, then the optimal policy always has (at least) a double-threshold structure. The revenue associated with any threshold policy can be efficiently calculated via the recursive technique of Section IV-B.

The case $K \geq 3$ remains very much an open question. The complicated geometry makes this class of problems very challenging. Foschini and Gopinath [5] have obtained some results for $K = 3$ in a related (queueing) problem with $b_k = 1$, $k = 1, 2, 3$.

It would also be interesting to compare the performance of the best coordinate convex policy with that of the absolutely best policy (in the Markov decision sense). This would involve an extensive numerical study. Preliminary findings have found that the optimal coordinate convex policy gives close to optimal performance.

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