ROBUST PORTFOLIO OPTIMIZATION: A STUDY OF BSE30 AND BSE100

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by

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to the

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CERTIFICATE

This is to certify that the work contained in this project report entitled "**Robust Portfolio Optimization: A Study of BSE30 and BSE100**" submitted by **Mohammed Bilal Girach (Roll No. 150123024)** and **Shashank Oberoi (Roll No. 150123047)** to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of **Bachelor of Technology** in Mathematics and Computing has been carried out by them under my supervision.

It is also certified that, along with literature survey, a few new results are established using computational implementations carried out by the students under the project.

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November 2018	Project Supervisor

ABSTRACT

We begin with a discussion on the classical Markowitz portfolio optimization, its drawbacks and consequent motivation of the alternate approach of robust portfolio optimization. This is followed by presenting several robust optimization models. Using uncertainty sets, we then present computational results with BSE30 and BSE100 followed by a simulation study using true mean and covariance of asset returns. We undertake a comparison of performance of the robust optimization approaches as compared to Markowitz optimization. We finally discuss the advantages of the robust optimization from the standpoint of number of stocks, number of samples and types of data.

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Chapter 1

Introduction

Investment in an individual security always has an associated risk, which can be minimized through diversification, a process involving investment in a portfolio consisting of several securities. For optimal allocation of weights in a diversified portfolio, one of the well-established methods is the classical mean-variance portfolio optimization introduced by Markowitz [31, 32]. Mean and covariance matrix of returns of securities are used as the measures for giving a quantitative sense to the return and the risk, respectively, of the portfolio. Despite being considered as the most basic theoretical framework in the field of portfolio optimization, there are several drawbacks associated with incorporating the Markowitz model in a practical setup.

Theoretically, Markowitz based portfolio optimization can result in assigning extreme weights to the securities comprising the portfolio. However, investment in securities can not be made in such extreme positions like large short positions if one takes active trading into account. Such kind of scenarios can be avoided by introducing appropriate constraints on the weights. Black and Litterman [12] argued that there is an added disadvantage since there are high chances of the optimal portfolio lying in the neighborhood of the imposed constraints. Thus, imposition of constraints leads to strong dependence of the constructed portfolio upon the constraints. For example, disallowing short sales often results in assigning zero weights to many securities and largely positive weights to the securities having small market capitalization.

One of the most major limitations of the mean-variance model is the sensitivity of the optimal portfolios to the errors in the estimation of return and risk parameters. These parameters are estimated using sample mean and sample covariance matrix, which are maximum likelihood estimates (MLEs) (calculated using historical data) under the assumption that the asset returns are normally distributed. According to DeMiguel and Nogales [18], since the efficiency of MLEs is extremely sensitive to deviations of the distribution of asset returns from the assumed normal distribution, it results in the optimal portfolios being vulnerable to the errors in estimation of input parameters. Additionally, the historical data neglects various other market factors and is not an accurate representation for estimates of future returns. Taking into account the above reasons, Michaud [33] argued that the mean-variance analysis tends to maximize the impact of estimation errors associated with the return and the risk parameters for the securities. As a result, Markowitz portfolio optimization often overweighs (underweighs) the securities having higher (lower) expected return, lower (higher) variance of returns and negative (positive) correlation between their returns. Labelling the model as "estimation-error maximizers", he stated that it often leads to financially counter-intuitive portfolios, which, in some cases, perform worse than the equal-weighted portfolio. Broadie [13] investigated the error maximization property of mean-variance analysis. Accordingly, he conducted a simulation based study to compare the estimated efficient frontier with the actual frontier computed using true parameter values. He observed that points on the estimated efficient frontier show superior performance as compared to the corresponding points on the actual frontier. He supported his argument of over-estimation of expected returns of optimal portfolios through his simulated results of obtaining the estimated frontier lying above the actual frontier. Additionally, he pointed out that non-stationarity in the data of returns can further increase the errors in computing the efficient frontier. Chopra and Ziemba [16] performed the sensitivity analysis of performance of optimal portfolios by studying the relative effect of estimation errors in means, variances and covariances of security returns, taking the investors' risk tolerance into consideration as well. They observed that at a high risk tolerance (to be defined in later chapters) of around fifty, cash equivalent loss for estimation errors in means is about eleven times greater than that for errors in variances or covariances. Accordingly, they pointed out that if the investors have superior estimates for means of security returns, they should prefer using them over the sample means calculated from historical data. Best and Grauer [9, 10] also arrived at similar conclusions by studying the sensitivity of weights of optimal portfolios with respect to changes in estimated means of returns on individual securities. Further, on imposition of no short selling constraint on the securities, they observed that a small change in estimated mean return of an individual security can assign zero weights to almost half the securities comprising the portfolio, which is counter-intuitive.

The discussed literature arrives at a common conclusion that the optimal portfolios are extremely sensitive towards the estimated values of input parameters, particularly expected returns of individual securities. In order to address this issue, there has been significant progress in recent years in the area of robust portfolio optimization. Several methods have been proposed in this area. **We are particularly interested in the approaches falling in the category related to enhancing robustness by optimizing the portfolio performance in worst-case scenarios.**

Significant efforts have been made towards formulating these kinds of approaches from Markowitz based mean-variance analysis. The robust optimization approach incorporates uncertainty in the input parameters directly into the optimization problem. Tütüncü and Koenig [40] described uncertainty, using an uncertainty set that includes almost all possible realizations of the uncertain input parameters. Accordingly, they formulated the problem of robust portfolio optimization by optimizing the portfolio performance under the worst possible realizations of the uncertain input parameters. They conducted numerous experiments applying the robust allocation methods to the market data and concluded that robust optimization can be considered as a viable asset allocation alternative for conservative investors. According to Ceria and Stubbs [15], the standard approach of robust optimization is too conservative. They argued that it is too pessimistic to adjust the return estimate of each asset downwards. Accordingly, they introduced new variants of robust optimization, taking into account the estimation errors in input parameters while formulating the optimization problem. They observed that the constructed robust portfolios perform superior in comparison to those constructed using mean-variance analysis in most of the cases but not in each month with certainty. Utilizing the standard framework of robust optimization, Scherer [38] showed that robust methods are equivalent to Bayesian shrinkage estimators and do not lead to significant change in the efficient set. Constructing an example, he showed that robust portfolio underperforms out of the sample in comparison to Markowitz portfolio, especially in the case of low risk aversion and high uncertainty aversion. He also argued that performance of robust portfolio is dependent upon the consistency between uncertainty aversion and risk aversion which is quite complicated. Santos [37] performed similar experiments to compare two types of robust approaches, namely, the standard robust optimization discussed in Scherer's work [38] and zero net alpha-adjusted robust optimization proposed by Ceria and Stubbs [15], with the traditional optimization methods. The empirical results indicated better performance of robust approaches in comparison to the portfolios constructed using mean-variance analysis in the case of simulated data unlike in the case of real market data.

Chapter 2

Robust Portfolio Optimization Models using Uncertainty Sets

All the real world optimizing problems inevitably have uncertain parameters embedded in them. In order to tackle such problems, a framework called "Stochastic Programming" [11] is used, which can model such problems having uncertain parameters. These models take the probability distributions of the underlying data into consideration. To improve the stability of the solutions, robust methods such as re-sampling techniques, robust estimators and Bayesian approaches were developed. One of the approaches is **robust optimization**, which is used when the parameters are known to lie in a certain range. In this chapter, we discuss some robust models with worst-case optimization approaches for a given objective function within a predefined "uncertainty" set.

The concept of uncertainty sets was introduced by Soyster [39], where he uses a different definition for defining a feasible region of a convex programming problem. In this definition, the convex inequalities are replaced by convex sets with a condition that the finite sum of convex sets again should be within another convex set. In another way, he defines a new linear programming problem (LPP) with uncertain truth value, but it is bound to lie within a defined convex set. Later, El Ghaoui and Lebret [20] extend these uncertainty sets to define a robust formulation while tackling the least-squares problem having uncertain parameters, but they are bounded matrices. In their work, they describe the problem of finding a worst-case residual and refer the solution as a robust least-squares solution. Furthermore, they show that it can be computed via semi-definite or second order cone programming. El Ghaoui, Oustry, and Lebret [21] further study how to integrate bounded uncertain parameters in semidefinite programming. They introduce robust-formulations for semidefinite programming and provide sufficient conditions to guarantee the existence of such robust solutions. Ben-Tal and Nemirovski [6] mainly focus on the uncertainty related with *hard* constraints and which are *ought* to be satisfied, irrespective of the representation of the data. They suggest a methodology where they replace an actual uncertain linear programming problem by its robust counterpart. They show that the robust counterpart of an LPP with the ellipsoidal uncertainty set is computationally attractive, as it reduces to a polynomial time solvable conic quadratic program. Additionally, they use interior points methods [7] to compute the solutions efficiently. Along the same lines, Goldfarb and Iyengar [23], focus on the robust convex quadratically constrained programs which are a subclass of the robust convex programs of Ben-Tal and Nemirovski [6]. They mainly work on finding uncertainty sets which structures this subclass of programs as second-order cone programs.

In its early phases, the major directions of research were to introduce robust formulations and to build uncertainty sets for robust counterparts of the LPP as they are computationally attractive. Once the basic framework of robust optimization was established, it is now applied across various domains such as learning, statistics, finance and numerous areas of engineering.

2.1 Uncertainty Sets

The determination of the structure of the uncertainty sets, so as to obtain computationally tractable solutions has been a key step in robust optimization. In the real world, even the distribution of asset returns has an uncertainty associated. In order to address this issue, a most frequently used technique is to find an estimate of the uncertain parameter and to define a geometrical bound around it. Empirically, historical data is used to compute the estimates of these uncertain parameters. For a given optimization problem, determining the geometry of the uncertainty set is a difficult task. In this section, we discuss a couple of popular types of uncertainty sets which are used in portfolio optimization. Accordingly, we first introduce the common notations.

1. N: Number of assets.
2. $\mathbf{x}$: Weight vector for a portfolio.
3. $\boldsymbol{\mu}$: Vector for expected return.
4. Σ: Covariance matrix for asset returns.
5. λ: Risk aversion.
6. $\mathcal{U}_{\mu,\Sigma}$: General uncertainty set with μ and Σ as uncertain parameters.
7. $\mathbf{1}$: Unity vector of length N.

The classical Markowitz model formulation with no short selling constraint is given by the following problem:

$$\max_{\mathbf{x}} \left\{\boldsymbol{\mu}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x}\right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{2.1}$$

Most of the robust models deal with optimizing a given objective function with a predefined "uncertainty set" for obtaining computationally tractable solutions. For any general uncertainty set $\mathcal{U}_{\mu,\Sigma}$, the worst case classical Markowitz model formulation [27, 25] with no short selling constraint is given as:

$$\max_{\mathbf{x}} \left\{ \min_{(\boldsymbol{\mu},\boldsymbol{\Sigma}) \in \mathcal{U}_{\mu,\Sigma}} \boldsymbol{\mu}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} \right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0, \tag{2.2}$$

In literature, there are many extensions of uncertainty sets varying from simple polytopes to statistically derived conic-representable sets. A *polytopic* [19] uncertainty set which resembles a "box", it is defined as

$$U_\delta(\hat{\mathbf{a}}) = \{\mathbf{a} : |a_i - \hat{a_i}| \leq \delta_i, i = 1, 2, 3, ..., N\}, \tag{2.3}$$

where $\mathbf{a} = (a_i, a_2, ..., a_N)$ is a vector of values of uncertain parameters of dimension N and $\hat{\mathbf{a}} = (\hat{a_1}, \hat{a_2}, ..., \hat{a_N})$ is generally the estimate for $\mathbf{a}$.

In order to capture more information from the data, the consideration of the second moment gives rise to another class of uncertainty sets, namely, ellipsoidal uncertainty sets. One of the most popular way of defining them [19] is

$$U(\hat{\mathbf{a}}) = \left\{\mathbf{a} : \mathbf{a} = \hat{\mathbf{a}} + \mathbf{P}^{1/2}\mathbf{u}, ||\mathbf{u}|| \leq 1\right\}, \tag{2.4}$$

where the choice of $\mathbf{P}$ is driven by the optimization problem. The main motivation behind the use of such kind of sets is that they come up naturally when one tries to estimate uncertain parameters using techniques like regression. Additionally, these sets take probabilistic properties into account. We further discuss how to model the uncertainties for some of the financial indicators.

2.1.1 Uncertainty in Expected Returns

Recently, many attempts have been made to model the uncertainty in the expected returns because of several reasons. When compared with variances and covariances, it is known that the effect on the performance of portfolio due to the estimation error is more in case of expected returns. Though it is unlikely that the future returns of the assets are equal to the estimated value of expected return, one can foresee that they will be within a certain range of the estimated return. Accordingly, one can define uncertainty sets in such a way so that expected values lie inside the geometric bound around the estimated value, say $\hat{\boldsymbol{\mu}}$.

In a simple scenario, one can define possible intervals for the expected returns of each individual asset by using box uncertainty set. Mathematically, it can be expressed as [27]:

$$U_{\boldsymbol{\delta}}(\hat{\boldsymbol{\mu}}) = \left\{\boldsymbol{\mu} : |\mu_i - \hat{\mu}_i| \leq \delta_i, i = 1, 2, 3, ..., N\right\}, \tag{2.5}$$

where N represents the number of stocks and δ_i represents the value which determines the confidence interval region for individual assets. Clearly from the above expression, for the asset i, the estimated error has an upper bound limit of δ_i. On incorporating the box uncertainty set (2.5), the max-min robust formulation (2.2) reduces to the following maximization problem:

$$\max_{\mathbf{x}} \quad \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} - \boldsymbol{\delta}^\top |\mathbf{x}| \quad \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0, \tag{2.6}$$

The most popular choice is to use ellipsoidal uncertainty set, as it takes the second moments into account. Uncertainty in expected return using ellipsoidal uncertainty set is expressed as [27] :

$$U_\delta(\hat{\boldsymbol{\mu}}) = \left\{ \boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \Sigma_{\boldsymbol{\mu}}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \delta^2 \right\}, \tag{2.7}$$

where Σ_μ is a variance-covariance matrix of the estimation error of expected returns of the assets. The max-min robust formulation (2.2) in conjunction with the ellipsoidal uncertainty set (2.7) results in the following maximization problem:

$$\max_{\mathbf{x}} \left\{ \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} - \delta \sqrt{\mathbf{x}^\top \Sigma_{\boldsymbol{\mu}} \mathbf{x}} \right\} \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{2.8}$$

While dealing with the box uncertainty, it is assumed that the returns follow normal distribution as it eases the task of computing the desired confidence intervals for each individual asset. We define δ_i for $100(1-\alpha)\%$ confidence level as follows:

$$\delta_i = \sigma_i \, z_{\frac{\alpha}{2}} \, n^{-\frac{1}{2}} \tag{2.9}$$

where $z_{\frac{\alpha}{2}}$ represents the inverse of standard normal distribution, σ_i is the standard deviation of returns of asset i and n is the number of observations of returns for asset i.

For the same reason, if the uncertainty set follows ellipsoid model, the underlying distribution is assumed to be tracing a χ^2 distribution with the number of assets being the degrees of freedom (df). Accordingly, for $100(1-\alpha)\%$ confidence level, δ is defined in the following manner

$$\delta^2 = \chi^2_N(\alpha) \tag{2.10}$$

where $\chi^2_N(\alpha)$ is the inverse of a chi square distribution with N degrees of freedom.

2.1.2 Separable Uncertainty Set

As mentioned earlier, portfolio performance is more sensitive towards estimation error in mean returns of assets in comparison to variances and covariances of asset returns. This is one of the major reasons behind research works laying less emphasis upon the uncertainty set for covariance matrix of asset returns. The robust approaches discussed in previous section model only the expected returns using uncertainty sets. Hence, in order to also encapsulate the uncertainty in the covariances, box uncertainty set for the covariance matrix of returns is defined on similar lines as that for expected returns. Lower bound $\underline{\Sigma}_{ij}$ and upper bound $\overline{\Sigma}_{ij}$ can be specified for each entry Σ_{ij} of the covariance matrix. Using this methodology, the constructed box uncertainty set for covariance matrix is expressed in the following form [40] :

$$U_\Sigma = \{\Sigma : \underline{\Sigma} \leq \Sigma \leq \overline{\Sigma},\ \Sigma \succeq 0\}. \tag{2.11}$$

In the above equation, the condition $\Sigma \succeq 0$ implies that Σ is a symmetric positive semidefinite matrix. This condition is necessary in most of the robust optimization approaches, particularly those involving Markowitz model as the basic theoretical framework.

Tütüncü and Koenig [40] discuss a method to solve the robust formulation of Markowitz optimization problem having non-negativity constraints upon the weights of assets. They define the uncertainty set for covariance matrix as per above equation and uncertainty set for expected returns as $U_{\boldsymbol{\mu}} = \{\boldsymbol{\mu} : \underline{\boldsymbol{\mu}} \leq \boldsymbol{\mu} \leq \overline{\boldsymbol{\mu}}\}$, where $\underline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\mu}}$ represent lower and upper bounds on mean return vector $\boldsymbol{\mu}$ respectively. Consequently, the max-min robust formulation (2.2) can be formulated as the following maximization problem:

$$\max_{\mathbf{x}} \left\{ \underline{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \overline{\Sigma} \mathbf{x} \right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{2.12}$$

Above approach involves the use of "separable" uncertainty sets, which implies, the uncertainty sets for mean returns and covariance matrix are defined independent of each other.

2.1.3 Joint Uncertainty Set

There are certain drawbacks associated with separable uncertainty sets. Lu [30] argues that such kind of uncertainty sets don't take the knowledge of actual confidence level into consideration. Secondly, separable uncertainty sets don't incorporate the joint behavior of mean returns and covariance matrix. As a result, these uncertainty sets are completely or partially similar to box uncertainty sets. This is one of the major reasons behind robust portfolios being conservative or highly non-diversified as observed in numerous computations. In order to address these drawbacks, Lu proposes a "joint uncertainty set". This uncertainty set is constructed as per desired confidence level using a statistical procedure that takes the factor model [24] for asset returns into consideration.

Delage and Ye [17] define a joint uncertainty set that takes into consideration the uncertainty in distribution of asset returns as well as moments (mean returns and covariance matrix of returns). The proposed uncertainty set having confidence parameters, $\gamma_1 \geq 0$ and $\gamma_2 \geq 1$, is given by:

$$\begin{aligned}
&(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \ \hat{\Sigma}^{-1} \ (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \gamma_1, \\
&E[(\mathbf{r} - \hat{\boldsymbol{\mu}})(\mathbf{r} - \hat{\boldsymbol{\mu}})^\top] \leq \gamma_2 \hat{\Sigma}.
\end{aligned} \tag{2.13}$$

In the above equation, $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$ represent the estimates of mean return vector and covariance matrix of asset returns respectively, and $\mathbf{r}$ is the random return vector. Using this uncertainty set, they formulate the portfolio optimization problem as a Distributionally Robust Stochastic Program (DRSR). Accordingly, they demonstrate that the problem is computationally tractable by solving it as a semidefinite program.

Chapter 3

Computational Results

We analyze the performance of robust portfolio optimization methods discussed in the preceding chapter vis-à-vis the Markowitz model in a practical setup involving domestic market data and simulated data. The analysis is performed under two scenarios, namely, the number of stocks N being 31 and the number of stocks N being 98. This is done in order to observe the effect of increase in number of stocks on the performance of robust methods with respect to the Markowitz model. These numbers were chosen since they represent the number of stocks in S&P BSE 30 and S&P BSE 100 indices, respectively.

For the first scenario, we use the daily log-returns based on daily adjusted close price of the 31 stocks comprising BSE 30 (data source: Yahoo Finance [3]). Accordingly, we have considered the period from December 18, 2017 to September 30, 2018 (both inclusive) comprising of a total of 194 active trading days. Corresponding to this market data, we prepare two sets of simulated data for the 31 assets by sampling returns from a multivariate normal distribution with mean and covariance matrix set equal to those obtained from the S&P BSE30 data. The first set of sample returns comprises of number of samples same as the daily log-return observations in S&P BSE30 market data, namely, 193, whereas the second set comprises of 1000 samples. The two sets of simulated sample returns of different sizes were used to facilitate the study of the impact of the number of samples in simulated data on the performance of the robust portfolio optimization approaches. We make a comparative study of robust portfolio optimization approaches, in case of the historical S&P BSE 30 data, as well as the two sets of simulated data, in order to analyze whether the worst case robust portfolio optimization approaches are useful in a real market setup.

For the second scenario, we use the log-returns based on daily adjusted close price data of the 98 stocks comprising S&P BSE 100 (data source: Yahoo Finance [3]) with the period spanning from December 18, 2016 to September 30, 2018. Two sets of simulated data are constructed using multivariate normal distribution, on the similar lines as the first scenario.Similar kind of comparative study is performed for the second scenario.

The robust portfolio optimization approaches that we have taken into consideration for analyzing their performance with respect to Markowitz model without short-selling (**Mark**) are as follows:

1. Robust Model involving box uncertainty set in expected return without short-selling (**Box**).
2. Robust Model involving ellipsoidal uncertainty set in expected return without short-selling (**Ellip**).
3. Robust Model involving separable uncertainty set without short-selling (**Sep**).

For Box and Ellip model, we construct uncertainty sets in expected mean return with $100(1-\alpha)\%$ confidence level by considering $\alpha = 0.05$. Separable uncertainty set in Sep model is constructed as a $100(1-\alpha)\%$ confidence interval for both μ and Σ using Non-parametric Boostrap Algorithm with same α as in other robust models and assuming β, *i.e* the number of simulations, equal to 8000.

The performance analysis for these robust portfolio models vis-à-vis the Mark model is performed by taking into consideration the "Sharpe Ratio" of the portfolios constructed having λ representing risk-aversion in the ideal range [19] *i.e.*, $\lambda \in [2, 4]$. Since the yield for Treasury Bill in India from 2016 to 2018 has been found to oscillating around 6% [2], so, we have assumed the annualized riskfree rate to be equal to 6%. In the following sections, we present the computational results observed in case of two scenarios as discussed above.

3.1 Performance with $N = 31$ assets

We begin with the analysis for $N = 31$ assets, in the case of the simulated data with 1000 samples. In Figure 3.1 and Table 3.1, we present the efficient frontier and performance of portfolios constructed by applying the Mark model and three robust models to the simulated returns. We observe that efficient frontiers for the Ellip and Sep models lie below the one for the Mark model. This supports the argument made by Broadie [13] regarding over-estimation of efficient frontier in case of the Mark model. Overlap of efficient frontiers for the Mark and Box models indicates that utilizing box uncertainty set for robust optimization does not prove to be of much use in this case. This claim is supported quantitatively from Table 3.1 as well as since the average Sharpe ratio for portfolios constructed in the ideal range of risk-aversion is same in case of both the models. From Table 3.1, we infer that

Sep model performs at par with Mark model if we take into consideration the average Sharpe Ratio. This is evident from Figure 3.1 as well, since Mark model starts outperforming Sep model in terms of Sharpe ratio after the risk-aversion crosses 3. We also note that the Ellip model outperforms all the models including the Mark model in the entire ideal range of risk-aversion.

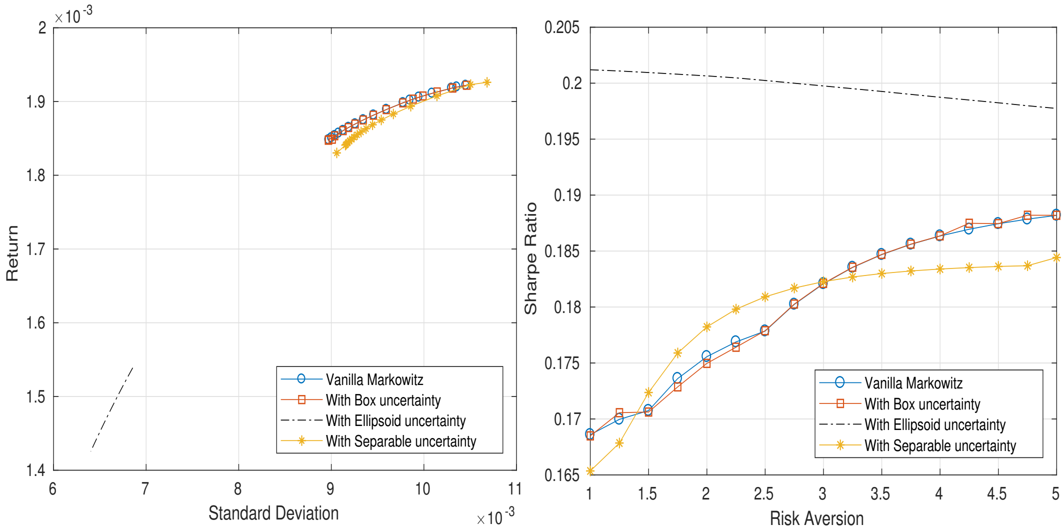

Figure 3.1: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Simulated Data with 1000 samples (31 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.176	0.175	0.201	0.178
2.5	0.178	0.178	0.2	0.181
3	0.182	0.182	0.2	0.182
3.5	0.185	0.185	0.199	0.183
4	0.186	0.186	0.199	0.183
Avg	0.181	0.181	0.2	0.182

Table 3.1: Comparison of different portfolio optimization models in case of Simulated Data with 1000 samples (31 assets)

On performing the simulation study with same number of samples as in case of market data (Figure 3.2 and Table 3.2), we observe similar results on comparing Box model with the Mark model. However, we observe slight inconsistency in performance of the Box model as evident from the plot of the Sharpe Ratio in Figure 3.2. The efficient frontiers for the Sep and Ellip model lie below that for the Mark model. We also infer that the Sep model and the Ellip model outperform the Mark model in terms of Sharpe Ratio in the ideal range of risk-aversion. However, it is difficult to compare the performance of the Sep model with that of the Ellip model in this case since the average Sharpe Ratio for both of them is almost the same.

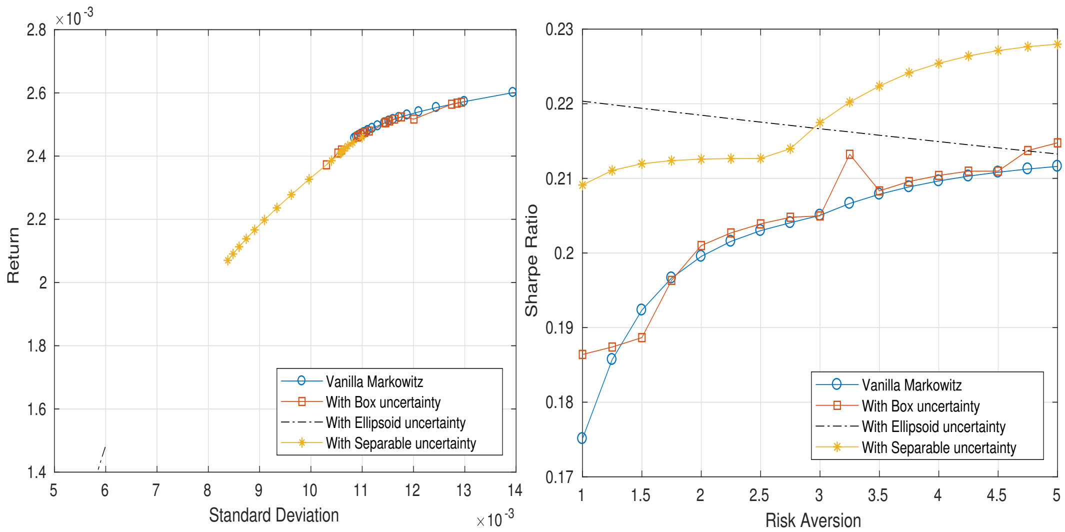

Figure 3.2: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Simulated Data with same number of samples as market data (31 assets)

In a “real” market setup involving stocks comprising S&P BSE 30, we observe that efficient frontier for the Box model almost overlaps with that for the Mark model. However, the performance of the Box model in terms of Sharpe ratio is quite inconsistent as evident from the plot in Figure 3.3. Efficient frontier for the Sep model lies below that of the Mark model and

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.2	0.198	0.218	0.213
2.5	0.203	0.204	0.218	0.213
3	0.205	0.207	0.217	0.217
3.5	0.208	0.209	0.216	0.222
4	0.21	0.21	0.215	0.225
Avg	0.205	0.206	0.217	0.218

Table 3.2: Comparison of different portfolio optimization models in case of Simulated Data with same number of samples as market data (31 assets)

the gap between the plots widens to a great extent, incase of the Ellip model. We also observe that the Sep model outperforms the Mark model in the ideal range of risk-aversion on taking the Sharpe Ratio into consideration as the performance measure. This is not true in case of the Ellip Model as evident from the Sharpe ratio plot in Figure 3.3. Even from Table 3.3, we observe that average Sharpe ratio for Ellip model is only slightly greater than that for the Mark model. Thus, unlike the simulated data, the Sep model performs superior in comparison to the Ellip model when applied to market data.

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.181	0.181	0.193	0.186
2.5	0.181	0.181	0.192	0.193
3	0.186	0.191	0.192	0.202
3.5	0.194	0.195	0.191	0.209
4	0.201	0.202	0.19	0.213
Avg	0.189	0.19	0.192	0.2

Table 3.3: Comparison of different portfolio optimization models in case of Market Data (31 assets)

A common observation that could be inferred from three cases considered in the scenario involving less number of assets ($N = 31$) is that the Sep and Ellip models perform superior or equivalent in comparison to the Mark model in the ideal range of risk-aversion.

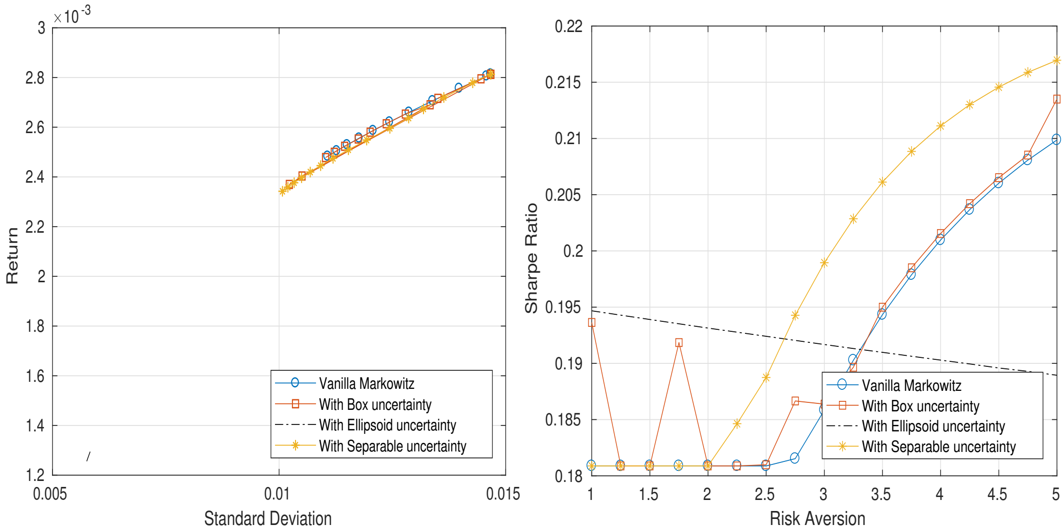

Figure 3.3: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Market Data (31 assets)

3.2 Performance with $N = 98$ assets

We now analyze the scenario involving $N = 98$ assets. On applying robust models along with the Mark model on a simulated data having 1000 samples, we observe results similar to the corresponding case for the previous scenario when we compared the Box model with the Mark model. This is evident from the coinciding plots of the efficient frontier and the Sharpe ratio for both the models in Figure 3.4. Contrary to the similar case for the scenario involving less number of assets ($N = 31$), we observe that not only does the Ellip model but also the Sep model outperforms the Mark model taking into consideration the portfolios constructed in the ideal range of risk-aversion. Additionally, from Table 3.4, we infer that the Ellip model performs superior in comparison to the Sep model in terms of greater average value of the Sharpe ratio.

Figure 3.5 and Table 3.5 present the results of simulation study with

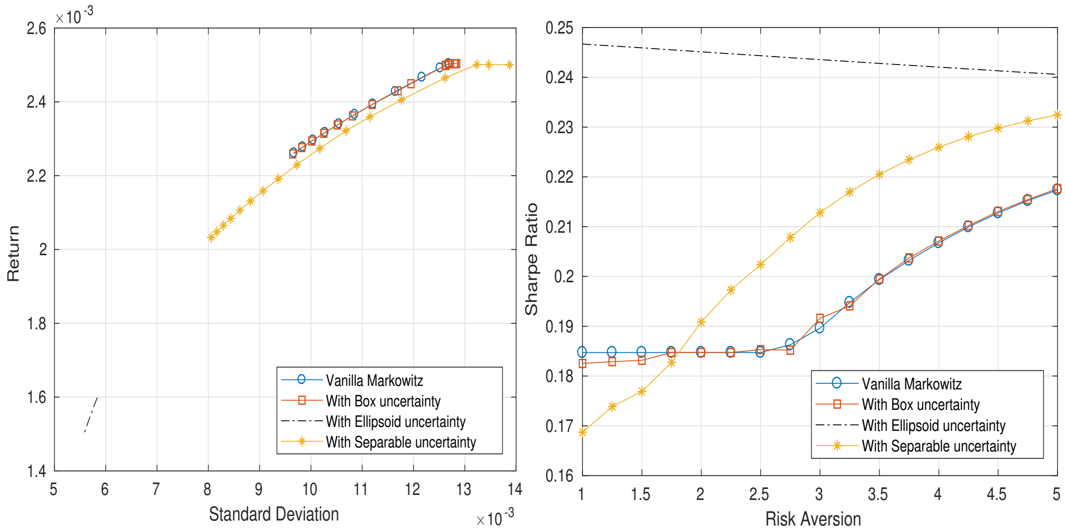

Figure 3.4: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Simulated Data with 1000 samples (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.185	0.185	0.245	0.191
2.5	0.185	0.185	0.244	0.202
3	0.19	0.192	0.244	0.213
3.5	0.199	0.199	0.243	0.221
4	0.207	0.207	0.242	0.226
Avg	0.193	0.194	0.244	0.21

Table 3.4: Comparison of different portfolio optimization models in case of Simulated Data with 1000 samples (98 assets)

same number of samples as that of log-returns of S&P BSE 100 data. Results observed on comparing the Box model with the Mark model are similar to the previous case of 1000 simulated samples. In the ideal range of risk aversion, we observe that efficient frontier for the Ellip as well as the Sep model lie below the Mark model. Additionally, both the models perform better than the Mark model in terms of the Sharpe Ratio. From the Sharpe ratio plot in Figure 3.5, it is difficult to compare the Sep model and the Ellip model since each outperforms the other in a different sub-interval of risk-aversion. The similar values of the average Sharpe ratio in Table 3.5 supports the claim of equivalent performance of these two models in this case.

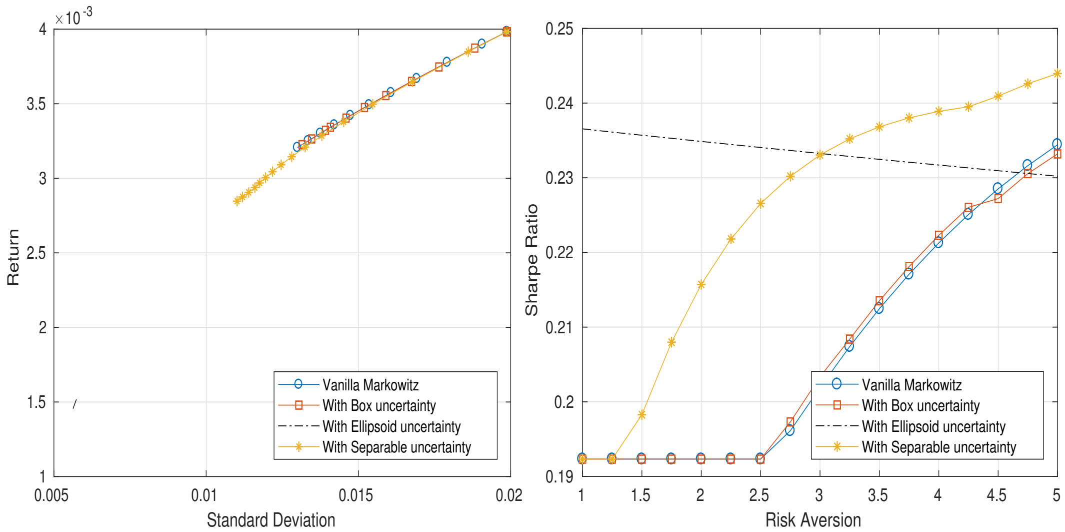

Figure 3.5: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Simulated Data with same number of samples as market data (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.192	0.192	0.235	0.216
2.5	0.192	0.192	0.234	0.227
3	0.202	0.203	0.233	0.233
3.5	0.212	0.213	0.232	0.237
4	0.221	0.222	0.232	0.239
Avg	0.204	0.205	0.233	0.23

Table 3.5: Comparison of different portfolio optimization models in case of Simulated Data with same number of samples as market data (98 assets)

The results for the case involving the market data (that contains log-returns of stocks comprising BSE 100) are presented in Figure 3.6 and Table 3.6. The efficient frontier plot leads to observations similar to the previous case. However, there is a slight inconsistency in the performance of the Box model as observed from the plot of the Sharpe Ratio in Figure 3.6. The robust portfolios constructed using the Sep and the Ellip model outperform the ones constructed using the Mark model in the ideal range of risk-aversion. Additionally, the Ellip model performs slightly better than the Sep model as evident from the Sharpe Ratio plot. Marginal difference in average Sharpe ratio between these two models supports this inference.

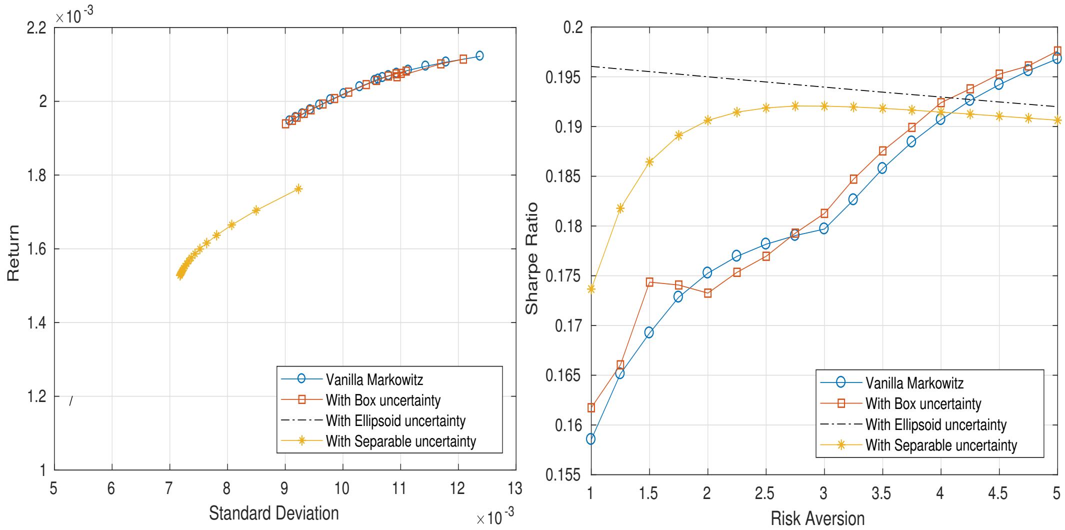

Figure 3.6: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Market Data (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.175	0.173	0.195	0.193
2.5	0.178	0.177	0.194	0.193
3	0.18	0.181	0.194	0.193
3.5	0.186	0.188	0.193	0.192
4	0.191	0.192	0.193	0.192
Avg	0.182	0.182	0.194	0.192

Table 3.6: Comparison of different portfolio optimization models in case of Market Data (98 assets)

We draw a common inference from the three cases considered in the scenario involving greater number of assets, *i.e.*, Sep and Ellip model outperform the Markowitz model in the ideal range of risk aversion.

Chapter 4

Conclusions and Comments

In this chapter, we analyze the different kinds of scenarios in the context of trends of the Sharpe Ratio. Recall that, we have considered the "adjusted closing prices" data of S&P BSE 30 and S&P BSE 100 to illustrate our analysis. Further, we have also generated simulated samples using the true mean and covariance matrix of log-returns obtained from the aforesaid actual market data of "adjusted closing prices". Since the number of instances in market data for the assets comprising the two indices, was very less, we simulated two sets of samples, one where the number of simulated samples matches the number of instances of real market data available, say $\zeta(< 1000)$ and another where the number of simulated samples is large (a constant, which in our case was taken to be 1000), irrespective of the number of stocks. The motivation behind this setup was to understand if the market data we obtained (which was limited) is able to capture the trends and results in better portfolio performance.

	#stocks = 31	#stocks = 98
#generated_simulations = 1000	0.2	0.244
#generated_simulations = ζ	0.218	0.233
Market data	0.2	0.194

Table 4.1: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

4.1 From the Standpoint of Number of Stocks

We begin with a description of the results summarized in Table 4.1, wherein for a particular row and a particular column, we presented the maximum possible Sharpe Ratio that was obtained for that particular scenario. For example, in case of the tabular entry for the case when $N = 98$ where we simulated ζ samples using true mean vector and the true covariance matrix of S&P BSE 100, we refer to Table 3.5 (which explains the simulation corresponding to S&P BSE 100 with ζ simulated samples) and take the maximum of its last row *i.e.*, maximum of average Sharpe ratios that was attained using the available robust and Mark models.

More the number of stocks, the better is the performance of the portfolios constructed using robust optimization. This claim can be supported via both qualitative and quantitative approaches. Qualitatively, the number of stocks in a portfolio represent its diversification. According to Modern Portfolio Theory (MPT), investors get the benefit of better performance from diversifying their portfolios as it reduces the risk of relying on only one security to generate returns. Value Research Online [1] provides us with the information that on an average basis, large-cap funds hold around 38 shares, mid-cap funds around 50-52 assets for balanced funds in which around 65-70% of their assets are in equity. This is because of great stability of returns in case of companies with large market capitalization, whereas this is not the case with mid-cap companies. Hence diversification requirements drives greater percentages in equities in case of mid-cap funds. From the above table, we can quantitatively justify by observing that the Sharpe ratio was more for portfolios with larger number of stocks when compared to portfolios with smaller number of stocks. However, we observe opposite behavior for the market data which can be attributed to the following two reasons:

1. The insufficient availability of market data, when it comes to larger number of stocks.

2. The error in the estimation of return and covariance matrix accumulating as the number of stocks increases, impacting the performance of the model [33].

4.2 From the Standpoint of Number of Samples Generated

In this section, we solely focus on the performance when different number of samples were generated. We tabulated Table 4.2 in the same way as we described in the preceding section. Here, we notice some interesting performance trends. One can observe that in the case of smaller number of stocks, the performance when same number of instances (ζ) were simulated is better than the scenario when large (1000) number of simulations were generated. On the contrary, exactly opposite trend can be observed when higher number of stocks are taken into consideration. We explain this type of behaviour as follows (as mentioned above): In the available real market data, the number of instances available for larger number of stocks is relatively low. So, when more number of samples were generated, we observe higher Sharpe ratios when compared to ζ number of simulations. However, the reason behind such a pattern of opposite behavior when smaller number of stocks are considered is not obvious.

	#samples = 1000	#samples = ζ
#stocks = 31	0.2	0.218
#stocks = 98	0.244	0.233

Table 4.2: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

4.3 From the Standpoint of Type of the Data

Finally, we discuss about the performance of the portfolio from the standpoint of kind of data that we have used in this work. Accordingly, the relevant results are tabulated in Table 4.3, from where the behavior is observed to be fairly consistent. For both the cases, the performance in case of the simulated data is better than in case of the real market data. This is clear from the fact that the real market data is difficult to model as it hardly follows any distribution, whereas the simulated data is generated from multivariate normal distribution with mean and covariances as the true values obtained from the data.

	Simulated data	Real Market data
#stocks = 31	0.218	0.2
#stocks = 98	0.244	0.194

Table 4.3: The maximum average Sharpe ratio compared by varying the type of the data in different kinds of scenarios.

4.4 Concluding Remarks

Robust optimization is an emerging area of portfolio optimization. Various questions have been raised on the advantages of robust methods over the Markowitz model. Through computational analysis of various robust optimization approaches followed by a discussion from different standpoints, we try to address this skepticism. We observe that robust optimization with ellipsoidal uncertainty set performs superior or equivalent as compared to the Markowitz model, in the case of simulated data, similar to the results reported by Santos [37]. In addition, we observe favorable results in the case of market data as well. Better performance of the robust formulation having separable uncertainty set in comparison to the Markowitz model is in line with the previous study on the same robust model by Tütüncü and Koenig [40]. Empirical results presented in this work advocate enhanced practical use of the robust models involving ellipsoidal uncertainty set and separable uncertainty set and accordingly, these models can be regarded as possible alternatives to the classical mean-variance analysis in a practical setup.

Chapter 5

VaR & its robust formulation

5.1 Introduction

In the above sections, the main limitations of the Mean Variance like sensitivity to errors in data and in the estimation of mean and variance of the underlying distribution. However, another criticism usually associated with the Markowitz setup is the use of standard deviation as a measure of risk. From a practitioners' point of view, the upside and downside risk are not considered the same. Most of the times, the upside risk can improve the overall performance of the portfolio whereas the downside fluctuation usually brings impactful losses. Variance can't be treated as an apt risk measure if the underlying distributions are leptokurtic. In order to address these issues, models involving other measures of risk have been developed and accordingly their corresponding robust models have been studied as well. In the upcoming chapters, we will thoroughly discuss some of the most popular risk measures like Value at Risk (VaR) and Conditional Value at Risk (CVaR) and also study their robust worst case formulations.

In this chapter, we start with the definition of VaR and formulate the optimisation problems for the computation of VaR. Furthermore, we incorporate separable uncertainty set to model the worst case formulation. Next, we analyze and compare the performance of both VaR and Worst case VaR (WVaR) with respect to S&P BSE 100 and S&P BSE 30. Finally, we conclude with insightful comments and conclusions.

5.2 Value at Risk (VaR)

Unlike Mean - Variance setup, VaR framework [29] uses the probability of losses into account. Ghaoui, Oks and Oustry [22] define value at risk as the minimum value of γ such that the probability of loss exceeding γ is less than ϵ.

$$V(\mathbf{x}) = \min \gamma \quad \text{such that } \; P\left\{\gamma \leq -r(\mathbf{x}, \boldsymbol{\mu})\right\} \leq \epsilon, \tag{5.1}$$

where $\epsilon \in (0, 1]$. When we deal with Markowitz setup, only mean and variance *i.e., first and second moments* of the asset returns are required but in VaR framework, the entire distribution is necessary for the computation part. If the underlying distribution in Gaussian with moments' pair as ($\hat{\boldsymbol{\mu}}$, Σ) then VaR can be computed via this analytical form.

$$V(x) = \kappa(\epsilon)\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} - \hat{\boldsymbol{\mu}}^\top \mathbf{x}, \tag{5.2}$$

where $\kappa(\epsilon) = -\Phi^{-1}(\epsilon)$ and $\Phi(z)\,(= P(Z \leq z))$ represents the cumulative distribution for standard normal random variable. Similarly if the distribution is known and popular then we can find the inverse cumulative function. If the distributions are unknown then we have to rely on the Chebyshev's inequality. The bound (upper) obtained by the Chebyshev's only requires the knowledge of the first two moments' pair. We call a bound to be exact if the upper bound is computationally tractable. If not, we use the bound given by Bertsimas and Popescu [8] *i.e.,* $\kappa(\epsilon) = \sqrt{\frac{1-\epsilon}{\epsilon}}$.

Finally, we formulate the generalised VaR as follows:

$$\min \kappa\sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} - \hat{\boldsymbol{\mu}}^\top \mathbf{x} \quad \text{subject to} \quad \mathbf{x} \in \mathscr{X}, \tag{5.3}$$

where κ is an appropriate factor of risk chosen according to the underlying distribution of asset returns and $\mathscr{X} = \left\{\mathbf{x} : \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0\right\}$ The function V is convex in w and the global optimum can be obtained via techniques like interior-point methods and Second order cone programming (SOCP).

Though VaR takes probability of losses into account. It has its own limitations. For the computation part, it requires the knowledge of the whole distribution. The computations also involves high dimensional numerical integration which may not be tractable at times and also not much of a study is done in using Monte Carlo simulations [29] for the design of the portfolio. Black and Litterman [12], Pearson and Ju [26] discusses the issues regarding the computational difference between the true VaR and the calculated VaR and find out that the error mainly creeps in due to the errors in the estimation of the first and second moments of the asset returns.

5.3 Worst Case VaR

The concept of worst-case VaR not only allows to approach the solution in a more tractable way but also loosen the assumptions on the information known to us beforehand. Here, we assume only partial information about the underlying distribution is known. We assume the distribution of the asset returns belong to a family of allowable probability distributions $\mathscr{P}$. For eg. Given component wise bounds of $(\hat{\boldsymbol{\mu}}, \Sigma)$,$\mathscr{P}$ could comprise of Normally distributed random variables with $\hat{\boldsymbol{\mu}}$ and Σ as the moments' pairs.

Given a probability (confidence) level ϵ, the worst-case VaR can be formulated as

$$V_{\mathscr{P}}(\mathbf{x}) = \min \gamma \quad \text{such that} \quad \sup_{P \in \mathscr{P}} P\left\{\gamma \leq -r(\mathbf{x}, \boldsymbol{\mu})\right\} \leq \epsilon, \tag{5.4}$$

and accordingly, the robust formulation can be written as

$$V_{\mathscr{P}}^{\text{opt}}(\mathbf{x}) = \min V_{\mathscr{P}}(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x} \in \mathscr{X}, \tag{5.5}$$

The above optimization problems can be computed by a semi definite programming (SDP) problem which again uses the above mentioned interior-point methods. We deal with the high dimensional problems by using bundle methods which are mainly used for large-scale (sparse) problems.

5.3.1 Polytopic uncertainty

By taking inspiration from the work of separable uncertainty sets by Tütüncü and Koenig [40], one can view the robust formulation given by Ghaoui, Oks and Oustry [22] in case of Polytopic uncertainty as a robust formulation of WVaR involving separable uncertainty. The formulation the optimization problem for worst case VaR as follows:

$$\min \kappa(\epsilon)\sqrt{\mathbf{x}^\top \overline{\Sigma} \mathbf{x}} - \underline{\hat{\boldsymbol{\mu}}}^\top \mathbf{x} \quad \text{subject to} \quad \mathbf{x} \in \mathscr{X}, \tag{5.6}$$

where $\overline{\Sigma}$ and $\underline{\hat{\boldsymbol{\mu}}}$ are higher bound for covariance matrix and lower bound for the estimated mean of asset returns. We obtain these values using Non-Parametric Bootstrap algorithm where the type of distribution is unknown. This can also be viewed as a robust formulation involving polytopic uncertainty as "Separable uncertainty" is a special case when dealing with "Polytopic uncertainty". The more complicated models involve Ellipsoidal uncertainty sets into account. The robust worst-case VaR formulation with Ellipsoidal uncertainty sets is not that trivial and in literature, it is mostly used in the setup of Factor models.

Chapter 6

CVaR & its robust formulation

6.1 Introduction

Though VaR has become popular as a measure of downside side, it has received fair share of criticism [14, 41, 28]. VaR for a diversified portfolio may exceed that for an investment in a single asset. Thus, one of the major limitations of VaR is the lack of sub-additivity in the case of general distributions. In accordance with this notion, VaR is not a coherent measure of risk as per definition laid out by Artzner et al [5]. Secondly, VaR does not provide any information on the size of losses in adverse scenarios, i.e. those beyond the threshold value. Additionally, VaR is non-convex and non-differentiable when incorporated into portfolio optimization problem. As a result, it becomes difficult to optimize VaR since global minimum may not exist.

Conditional Value-at-Risk (CVaR), introduced by Rockafellar and Uryasev [35, 36], has addressed various concerns centred around VaR. For continuous distributions, CVaR is the expected loss conditioned on the loss outcomes exceeding VaR. For $\epsilon \in (0,1)$, CVaR, at confidence level $1-\epsilon$, is defined as:

$$CVaR_\epsilon(\mathbf{x}) \triangleq \frac{1}{\epsilon}\int_{-\mathbf{r}^\top\mathbf{x}\geq VaR_\epsilon(\mathbf{x})} p(\mathbf{r})d\mathbf{r} \tag{6.1}$$

where $\mathbf{r}$ is the random return vector and $\mathbf{x}$ is the weight vector for a portfolio. In recent years, CVaR has emerged as a viable risk measure in portfolio optimization problems over its use in reducing downside risk. Since CVaR is a coherent risk measure, it follows the property of sub-additivity. Therefore, CVaR can be minimized through investment in a diversified portfolio. Unlike VaR, CVaR takes into consideration the impact of losses beyond the threshold value [14]. Also, minimization of CVaR is a convex optimization problem [28].

Similar to Markowitz optimization and VaR minimization, there is an issue of lack of robustness in the classical framework of CVaR minimization. Computing CVaR requires a perfect knowledge of the return distribution as argued by Zhu and Fukushima [41]. They have addressed this shortcoming of the classical (base-case) CVaR by proposing a new risk measure, **Worst-Case CVaR**. The worst-case CVaR (WCVaR) for a fixed weight vector $\mathbf{x}$ is defined as:

$$WCVaR_\epsilon(\mathbf{x}) \triangleq \sup_{p(\mathbf{r})\in\mathcal{P}} CVaR_\epsilon(\mathbf{x}) \tag{6.2}$$

where the density function $p(\mathbf{r})$ of returns is known to belong to a set $\mathcal{P}$ of probability distributions. The next section discusses coherent measures of risk that include both CVaR and WCVaR.

6.2 Coherent measures of risk

A risk measure ρ is a mapping of random gain X to a real value to represent the risk associated with X quantitatively. In their seminal work on

Coherent Measures of Risk, Artzner et al [5] presented and justified following consistency rules for a risk measure ρ to be coherent:

1. Monotonicity: $X \leq Y \implies \rho(X) \geq \rho(Y)$

2. Translation Invariance: For any constant $m \in \mathcal{R}$, $\rho(X+m) = \rho(X)+m$

3. Positive Homogeneity: For any positive constant $\lambda \geq 0$, $\rho(\lambda X) = \lambda\rho(X)$

4. Sub-additivity: For any X and Y, $\rho(X+Y) \leq \rho(X) + \rho(Y)$

It has been proved by Pflug [34] and Acerbi and Tasche [4] that CVaR is a coherent measure of risk. Zhu and Fukushima [41] have proved the coherence of WCVaR as a risk measure by analyzing it in terms of worst-case risk measure ρ_w:

$$\rho_w(X) \triangleq \sup_{p(\mathbf{r}) \in \mathcal{P}} \rho(X) \tag{6.3}$$

6.3 Mathematical Formulations

In this section, we discuss mathematical formulations of the problems of optimizing CVaR along with WCVaR using mixture distribution uncertainty. We skip discussion on WCVaR formulations using box uncertainty set and ellipsoidal uncertainty set since they require a set of possible return distributions to be assumed so as to obtain bounds and scaling matrix respectively. However, our problem setup doesn't involves a set of return distributions since we make use of only market data (where return distribution is not known) and simulated data (where return distribution is perfectly known).

6.3.1 Minimizing Base-Case CVaR

As proved by Rockafeller and Uryasev [35] , $CVaR_\epsilon(\mathbf{x})$, defined in equation (6.1), can be transformed into:

$$CVaR_\epsilon(\mathbf{x}) = \min_{\gamma \in \mathcal{R}^n} F_\epsilon(\mathbf{x}, \gamma) \tag{6.4}$$

where n is number of assets in the portfolio and $F_\epsilon(\mathbf{x}, \gamma)$ is defined as:

$$F_\epsilon(\mathbf{x}, \gamma) \triangleq \gamma + \frac{1}{\epsilon} \int_{\mathbf{r} \in \mathcal{R}^n} [-\mathbf{r}^\top \mathbf{x} - \gamma]^+ p(\mathbf{r}) d\mathbf{r} \tag{6.5}$$

In above equation, $[t]^+ = \max\{t, 0\}$. The problem of approximation of the integral involved in the equation (6.5) can be dealt by sampling the probability distribution of $\mathbf{r}$ as per its density $p(\mathbf{r})$. Assuming there are S samples, $\{\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_i, .., \mathbf{r}_S\}$ for the return vector $\mathbf{r}$, $F_\epsilon(\mathbf{x}, \gamma)$ can be approximated as [35]:

$$F_\epsilon(\mathbf{x}, \gamma) \approx \gamma + \frac{1}{S\,\epsilon} \sum_{i=1}^{S} [-\mathbf{r}_i^\top \mathbf{x} - \gamma]^+ \tag{6.6}$$

In accordance with above approximation, the problem of minimization of classical CVaR, assuming no short-selling constraints, can be formulated as the following Linear Programming Problem (LLP) [35, 41]:

$$\begin{aligned}
&min_{(\mathbf{x}, \mathbf{u}, \gamma, \theta)}\, \theta \ s.t. \\
&\mathbf{x}^\top \mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}, \\
&\gamma + \frac{1}{S\,\epsilon} \mathbf{1}^\top \mathbf{u} \leq \theta, \\
&u_i \geq -\mathbf{r}_i^\top \mathbf{x} - \gamma,\ u_i \geq 0,\ i = 1, 2, .., S.
\end{aligned} \tag{6.7}$$

where the auxiliary vector $\mathbf{u} \in \mathcal{R}^S$.

6.3.2 Minimizing Worst-Case CVaR using Mixture Distribution Uncertainty

This section involves formulation of the problem of optimizing WCVaR by assuming that the return distribution belongs to a set of distributions comprising of all possible mixtures of some prior likelihood distributions [41]. Mathematically, it is assumed that:

$$p(\mathbf{r}) \in \mathcal{P}_M = \triangleq \left\{ \sum_{j=1}^{l} \lambda_j p^j(\mathbf{r}) \, : \, \sum_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, .., l \right\} \tag{6.8}$$

In above equation, $p^j(\mathbf{r})$ denotes j^{th} likelihood distribution and l is the number of the likelihood distributions. In accordance with above assumption of mixture distribution uncertainty that involves $\mathcal{P}_M$ as a compact convex set, $WCVaR_\epsilon(\mathbf{x})$, defined in equation (6.2), can be rewritten as the following min-max problem [41]:

$$\begin{aligned} & WCVaR_\epsilon(\mathbf{x}) = \min_{\alpha \in \mathcal{R}} \max_{j \in \mathcal{L}} F_\epsilon^j(\mathbf{x}, \gamma) \, , \, where \\ & \mathcal{L} \triangleq \{1, 2, .., l\} \\ & F_\epsilon^j(\mathbf{x}, \gamma) \triangleq \gamma \, + \, \frac{1}{\epsilon} \int_{\mathbf{r} \in \mathcal{R}^n} [-\mathbf{r}^\top \mathbf{x} - \gamma]^+ p^j(\mathbf{r}) d\mathbf{r} \end{aligned} \tag{6.9}$$

Similar to the case involving classical CVaR in previous subsection, $F_\epsilon^j(\mathbf{x}, \gamma)$ can be approximated via discrete sampling as:

$$F_\epsilon^j(\mathbf{x}, \gamma) \approx \gamma \, + \, \frac{1}{S_j \, \epsilon} \sum_{i=1}^{S_j} [-\mathbf{r}_{i,j}^\top \mathbf{x} - \gamma]^+ \tag{6.10}$$

In above equation, $\mathbf{r}_{i,j}$ is the i^{th} sample of the return with respect to j^{th} likelihood distribution and S_j is the number of samples corresponding to

j^{th} likelihood distribution. Accordingly, assuming non-negative weights, the problem of minimization of WCVaR over a feasible set of portfolios can be formulated as the following LLP [41]:

$$\begin{aligned}
&min_{(\mathbf{x},\mathbf{u},\gamma,\theta)}\,\theta \ s.t. \\
&\mathbf{x}^\top\mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}, \\
&\gamma + \frac{1}{S_j\,\epsilon}\mathbf{1}^\top\mathbf{u}^j \leq \theta,\ j = 1, 2, .., l, \\
&u_i^j \geq -\mathbf{r}_{i,j}^\top\mathbf{x} - \gamma,\ u_i^j \geq 0,\ i = 1, 2, .., S_j,\ j = 1, 2, .., l.
\end{aligned} \tag{6.11}$$

In above equation, the auxiliary vector $\mathbf{u} = (\mathbf{u}^1; \mathbf{u}^2; ..; \mathbf{u}^l) \in \mathcal{R}^S$ where $S = \sum_{j=1}^{l} S_j$.

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