Robust Portfolio Optimization for BSE

Author 1 [*] Author 2 [†] Author 3 [‡]

Abstract

The abstract goes here
Keywords: Robust Portfolio optimization; S&P BSE 30; S&P BSE 100

1 Robust Portfolio Optimization Approaches

In absence of a known distribution for asset returns, a practical approach to estimate an uncertain parameter is to have bounds for the uncertain parameters, to be estimated using historical data [1, 2]. Ascertaining the bounds or the geometry (uncertainty sets) for the uncertain parameters for a particular optimization problem is not trivial. For the purpose of this work we will use three types of uncertainty sets, namely, box and ellipsoidal (for expected returns) and separable (for both expected returns and covariance matrix of returns [1, 2, 3]. Accordingly, we first introduce the notations to be used in this work.

1. $\mathbf{x}$: Weight vector for a portfolio.
2. $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix}$: Vector of N uncertain parameters.
3. $\hat{\mathbf{a}} = \begin{pmatrix} \hat{a}_1 & \hat{a}_2, \dots, \hat{a}_N \end{pmatrix}$: Estimate for $\mathbf{a}$.
4. $\boldsymbol{\mu}$: Vector for expected return.
5. $\hat{\boldsymbol{\mu}}$: Estimate for $\boldsymbol{\mu}$.
6. Σ: Covariance matrix.
7. Σ_μ: Covariance matrix for errors in estimation.
8. λ: Risk aversion.
9. $\mathcal{U}_{\mu,\Sigma}$: General uncertainty set with μ and Σ as uncertain parameters.
10. δ_i:
11. $\mathbf{1}$:

The classical Markowitz model formulation with no short selling constraint is given by the following problem:

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} \right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{1.1}$$

When the general uncertainty set $\mathcal{U}_{\mu,\Sigma}$ is $U_{\boldsymbol{\delta}}\left(\hat{\boldsymbol{\mu}}\right)$ the worst case classical Markowitz model formulation with no short selling constraint becomes [1, 4]:

$$\max_{\mathbf{x}} \left\{ \min_{\boldsymbol{\mu} \in U_{\boldsymbol{\delta}}(\hat{\boldsymbol{\mu}})} \boldsymbol{\mu}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} \right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0, \tag{1.2}$$

[*] Affiliation, e-mail: Author 1 Email
[†] Affiliation, e-mail: Author 2 Email
[‡] Affiliation, e-mail: Author 3 Email

1.1 Box Method

A *polytopic* uncertainty set which resembles a box, it is defined as [2]:

$$U_{\boldsymbol{\delta}}(\hat{\mathbf{a}}) = \left\{\mathbf{a} : |a_i - \hat{a}_i| \leq \delta_i, i = 1, 2, 3, \ldots, N\right\}. \tag{1.3}$$

For our purpose $\hat{\mathbf{a}}$ is $\hat{\boldsymbol{\mu}}$, which gives

$$U_{\boldsymbol{\delta}}(\hat{\boldsymbol{\mu}}) = \left\{\boldsymbol{\mu} : |\mu_i - \hat{\mu_i}| \leq \delta_i, i = 1, 2, 3, \ldots, N\right\}. \tag{1.4}$$

Accordingly, using (1.4), the problem (1.2) reduces to [1, 2]:

$$\max_{\mathbf{x}} \left\{\hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} - \boldsymbol{\delta}^\top |\mathbf{x}|\right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{1.5}$$

Here, we define δ_i for $100(1-\alpha)\%$ confidence level as $\delta_i = \sigma_i z_{\frac{\alpha}{2}} n^{-\frac{1}{2}}$ where $z_{\frac{\alpha}{2}}$ represents the inverse of standard normal distribution, σ_i is the standard deviation of returns of asset i and n is the number of observations of returns for asset i.

1.2 Ellipsoidal Method

For $\hat{\mathbf{a}}$ is $\hat{\boldsymbol{\mu}}$, the ellipsoidal uncertainty set is expressed as

$$U_{\delta}(\hat{\boldsymbol{\mu}}) = \left\{\boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \Sigma_{\boldsymbol{\mu}}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \delta^2\right\}. \tag{1.6}$$

Accordingly, using (1.6), the problem (1.2) reduces to [1, 2]:

$$\max_{\mathbf{x}} \left\{\hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma_{\boldsymbol{\mu}} \mathbf{x} - \delta \sqrt{\mathbf{x}^\top \Sigma_{\boldsymbol{\mu}} \mathbf{x}}\right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{1.7}$$

If the uncertainty set follows ellipsoid model, the underlying distribution is assumed to be tracing a χ^2 distribution with the number of assets being the degrees of freedom (df). Accordingly, for $100(1-\alpha)\%$ confidence level, δ is defined as [1, 2, 5]:

$$\delta^2 = \chi^2_N(\alpha) \tag{1.8}$$

where $\chi^2_N(\alpha)$ is the inverse of a chi square distribution with N degrees of freedom.

1.3 Separable Model

In case of separable uncertainty set, the lower bound $\underline{\Sigma}_{ij}$ and the upper bound $\overline{\Sigma}_{ij}$ can be specified for each entry Σ_{ij} of the covariance matrix resulting in the following constructed box uncertainty set for the covariance matrix [6]:

$$U_\Sigma = \{\Sigma : \underline{\Sigma} \leq \Sigma \leq \overline{\Sigma},\ \Sigma \succeq 0\}. \tag{1.9}$$

In the above equation, the condition $\Sigma \succeq 0$ implies that Σ is a symmetric positive semidefinite matrix.

Tutuncu and Koenig [6] define the uncertainty set for expected returns as

$$U_{\boldsymbol{\mu}} = \{\boldsymbol{\mu} : \underline{\boldsymbol{\mu}} \leq \boldsymbol{\mu} \leq \overline{\boldsymbol{\mu}}\}, \tag{1.10}$$

where $\underline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\mu}}$ represent lower and upper bounds on mean return vector $\boldsymbol{\mu}$ respectively. Accordingly, the problem (1.2) can be formulated as following [1, 2]:

$$\max_{\mathbf{x}} \left\{\underline{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \overline{\Sigma} \mathbf{x}\right\} \text{ such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{1.11}$$

2 Computational Results

In this section, we analyze the performance of the robust portfolio optimization approaches discussed in Section 1, using the market historical data, as well as simulated data. For the purpose of this analysis, we consider two scenarios in terms of number of stocks N, being 31 and 98, with the goal of observing the effect of increase in the number of stocks on the performance of robust portfolio optimization approaches vis-a-vis the Markowitz model. These numbers were chosen since they represent the number of stocks in S&P BSE 30 and S&P BSE 100 indices, respectively.

For the scenario of $N = 31$, we make use of the daily log-returns, based on the adjusted daily closing prices of the 31 stocks comprising the S&P BSE 30, obtained from Yahoo Finance [8]. Accordingly, we consider the period of our analysis to be from 18th December, 2017 to 30th September, 2018 (both inclusive) which had a total of 194 active trading days *i.e.,* 193 daily log-returns. Corresponding to this historical data from S&P BSE 30, we generate two sets of simulated data for all the 31 assets, by generating sample returns using a multivariate normal distribution whose mean and covariance matrix are set to be the one obtained for the historical S&P BSE 30 data. The first set of sample returns comprises of the number of samples to be the same as in the historical data, namely 193, in this case, whereas, the second set comprises of a larger number of samples, namely, 1000. The two sets of simulated sample returns of different sizes were used to facilitate the study of the impact of the number of samples in simulated data on the performance of the robust portfolio optimization approaches. In addition, we make a comparative study of robust portfolio optimization approaches vis-a-vis the Markowitz portfolio, in case of the historical S&P BSE 30 data, as well as the two sets of simulated data, in order to analyze whether the worst case robust portfolio optimization are useful in a real market setup. For the scenario of $N = 98$, we make use of the daily log-returns, based on the adjusted daily closing prices of the 98 stocks comprising the S&P BSE 100, obtained from Yahoo Finance [8], for the period 18th December, 2016 to 30th September, 2018 (both inclusive). The two sets of simulated data were generated in a manner akin to the case of $N = 31$ assets.

The robust portfolio optimization models that we have used for analyzing their performance vis-a-vis the Markowitz portfolio without short-selling (**Mark**) are as follows:

1. Robust model involving box uncertainty set in expected return without short-selling (**Box**).
2. Robust model involving ellipsoidal uncertainty set in expected return without short-selling (**Ellip**).
3. Robust model involving separable uncertainty set without short-selling (**Sep**).

For the Box and the Ellip model, we construct the uncertainty sets for the expected return at $100(1-\alpha)\%$ confidence level. In case of the Sep model, we construct the uncertainty sets for both the expected return and the covariance matrix at $100(1-\alpha)\%$ confidence level, making use of non-parametric Boostrap Algorithm. For the purpose of implementation of all the three models, the α is taken to be $\alpha = 0.05$ and in case of the Boostrap Algorithm, by assuming the number of simulations to be $\beta = 8,000$.

The performance analysis for these robust portfolio models is performed using the *Sharpe Ratio* of the constructed portfolios, with λ representing the risk-aversion in the ideal range *i.e.,* $\lambda \in [2, 4]$ [2]. Further, since the T-bill rate in India from 2016 to 2018 was observed to oscillating around 6% [9], so we have assumed the annualized riskfree rate to be equal to 6%. We now present the computational results observed in case of the two scenarios, namely $N = 31$ and $N = 98$, as discussed above.

2.1 Performance with $N = 31$ assets

We begin with the analysis for $N = 31$ assets, in case of the simulated data with 1000 samples and present the results in Figure 1 and Table 1. From Figure 1 we observe that the efficient frontiers for the Ellip and the Sep models lie below the one for the Mark model, which supports the argument made in [7] regarding over-estimation of the efficient frontier for the Mark model. Further, the observed overlap of the efficient frontiers for the Mark and the Box models suggest that the utilizing of Box uncertainty sets for robust portfolio optimization does not prove to be of much use in this case. Further, from Figure 1, we observe that the Mark model starts outperforming the Sep model in terms of the Sharpe Ratio after the risk-aversion λ crosses 3. The above observations are supported quantitatively by the results tabulated in Table 1 as well, since the average Sharpe Ratio for portfolios constructed in the ideal range of risk-aversion $\lambda \in [2, 4]$ is the same in case of both the Mark and the Box models. Also, we infer from Table 1, that the Sep model performs at par with the Mark model by taking the average Sharpe Ratio into consideration, with the Ellip model marginally outperforming the other three models.

The analysis with $N = 31$ assets in case of the number of simulated samples being the same as the number of log-returns in case of S&P BSE 30 data is presented in Figure 2 and Table 2. The efficient frontiers for the Sep and Ellip model lie below that of the Mark model. We observe results similar to 1000 simulated samples, upon comparison of the Mark model and the Box model. However, we observe a slight inconsistency in the performance of Box model as evident from the plot of the Sharpe Ratio in Figure 2. We also infer that the Ellip model and the Sep model outperform the Mark model in terms of the Sharpe Ratio in the ideal range of risk-aversion $\lambda \in [2, 4]$. However, it is slightly difficult to compare the performance of the Ellip model with that of the Sep model in this case, since the average Sharpe Ratio for both of them is almost the same.

For the historical market data involving the stocks comprising S&P BSE 30, we observe from Figure 3, that the efficient frontiers for the Mark model and the Box model almost overlap with each other. Further, the efficient frontier for the Sep model lies below that of the Mark model with further widening of the gap between the plots, in case of the Ellip model. However, the performance of the Box model, in terms of the Sharpe ratio is quite inconsistent as evident from the Figure 3. We also observe that the Sep model outperforms the Mark model in the ideal range of risk-aversion $\lambda \in [2, 4]$ upon taking the Sharpe Ratio into consideration as the performance measure. This is not true in case of the Ellip Model, as evident from the Sharpe Ratio plot in Figure 3. Even from Table 3, we observe that average Sharpe Ratio for the Ellip model is only slightly greater than that for the Mark model. Thus, unlike the simulated data, the Sep model performs superior in comparison to the Ellip model when applied to the S&P BSE 30 data.

A common observation that could be inferred from three cases considered in the scenario involving less number of assets ($N = 31$) is that the Sep and the Ellip models perform superior or equivalent in comparison to the Mark model in the ideal range of risk-aversion.

2.2 Performance with $N = 98$ assets

We now analyze the case with $N = 98$ assets. On applying robust the models along with the Mark model in case of the simulated data with 1000 samples, we observe that the results are similar to the corresponding scenario for $N = 31$ assets, when we compare Box model with Mark model. This is evident from the coinciding plots of the efficient frontier and the plots for the Sharpe Ratio for both the models in Figure 4. However, in contrast to the scenario of $N = 31$ assets, we observe that not only does the Ellip model but also the Sep model outperforms the Mark model when considering the portfolios constructed in the ideal range of risk-aversion $\lambda \in [2, 4]$. Further, from Table 4, we can infer that the Ellip model exhibits superior performance in comparison to the Sep model, in terms of greater average value of the Sharpe Ratio.

In Figure 5 and Table 5 we present the results of the study for the simulated data with the number of simulated samples being the same as that of S&P BSE 100. The comparative results observed for the Box model and the Mark model are similar to the previous case of 1000 simulated samples. In the ideal range of risk aversion $\lambda \in [2, 4]$, one observes that the efficient frontier for both the Ellip as well as the Sep model lie below Mark model, but both the models perform better than the Mark model in terms of the Sharpe Ratio. Additionally, from the plot of the Sharpe ratio plot in Figure 5, any comparative inference of the Sep model and the Ellip model is difficult, since each outperforms the other in a different sub-interval of the risk-aversion range. The similar values of the average Sharpe Ratio in Table 5 supports the claim of equivalent performance of these two models in this case.

Finally, the results for the historical market, involving stocks comprising S&P BSE 100 are presented in Figure 6 and Table 6. While the efficient frontier plot leads to observations similar to the previous case, however, there is a slight inconsistency in the performance of the Box model as can be seen from the plot of the Sharpe Ration in Figure 6. The robust portfolios constructed using the Sep and the Ellip models outperform the ones constructed using the Mark model in the ideal range of risk-aversion $\lambda \in [2, 4]$. Further, the performance of the Ellip model is marginally better than the Sep model as evident from the plot of the Sharpe Ratio plot, an inference that is supported by the marginal difference in average Sharpe Ratio of both.

We draw a common inference from the three cases considered in the scenario involving greater number of assets, *i.e.*, the Sep and the Ellip model outperform the Mark model in the ideal range of risk aversion.

3 Conclusions and Comments

In this concluding section we analyze the different kinds of scenarios in the context of trends of the Sharpe Ratio. Recall that, we have considered the “adjusted closing prices” data of S&P BSE 30 and S&P BSE 100 to illustrate our analysis. Further, we have also generated simulated samples using the true mean and covariance matrices obtained from the aforesaid actual market data of “adjusted closing prices”. Since the number of instances in market data for the assets comprising the two indices, was very less, we simulated two sets of samples, one where the number of simulated samples matches the number of instances of real market data available, say ζ and another where the number of simulated samples is large (a constant,which in our case was taken to be 1000), irrespective of the number of stocks. The motivation behind this setup was to understand if the market data we obtained (which was limited) is able to capture the trends and results in better portfolio performance.

3.1 From the Standpoint of Number of Stocks

We begin with a description of the results summarized in Table 7, wherein for a particular row and a particular column, we presented the maximum possible Sharpe Ratio that was obtained for that particular scenario. For example, in case of the tabular entry for S&P BSE 100 where we simulated ζ samples using true mean vector and the true covariance matrix, we refer to Table 5 (which explains the simulation corresponding to S&P BSE 100 with ζ simulated samples) and take the maximum of its last row *i.e.*, maximum of average Sharpe ratios that was attained using the available models.

Accordingly, from Table 7, the inference that can be drawn is that larger the number of stocks, the better is the performance of the portfolios constructed using robust optimization. This claim can be supported via both qualitative and quantitative approaches. Qualitatively, stocks in a portfolio is representative of its diversification. According to Modern Portfolio Theory (MPT), investors get the benefit of better performance from diversifying their portfolios since it reduces the risk of relying on only one (small number) security (securities) to generate returns. Based on the analysis by Value Research Online [10] one observes that on an average basis, the large-cap funds hold around 38 shares while the mid-cap funds hold around $50--52$ assets for balanced funds, in which around $65-70\%$ of the assets are held in equity. From Table 7, we can provide quantitative justification by observing that the Sharpe Ratio was more for portfolios with larger number of stocks as compared to portfolios with smaller number of stocks. On the contrary, we attribute the reason for the opposite behavior being observed when it comes to real market because of insufficient available market data, when it comes to larger number of stocks, and will be elaborated upon in the subsequent discussion. As for the second case (where the number of simulated samples is equal to available number of stocks for which the market data is available) our simulated data follows multivariate normal distribution whereas the real market data does not necessarily follow any such kind of distribution.

3.2 From the Standpoint of Number of Simulated Samples

We now focus on the performance when different number of samples were simulated and tabulate the results in Table 8 in the same way as was done in the preceding discussion. Here several interesting performance trends can be noticed. We observe that in the case of smaller number of stocks, the performance in case when the number of simulated samples is ζ is better than the case when a large (1000) simulated samples were generated. On the contrary, the exactly opposite trend can be observed when higher number of stocks are taken into consideration. This observation can be explained as follows: In case of real market data, the number of instance of real market data being available is relatively low. So, when larger number of samples were generated, we observe higher Sharpe Ratio as compared to ζ number of simulations. However, the reason behind such a pattern of behavior when smaller number of stocks are considered has not yet been explored.

3.3 From the Standpoint of the Kind of Data

Finally, we discuss about the kind of data that we have used in this work. Accordingly, the relevant results are tabulated in Table 9, from where the behavior is observed to be fairly consistent. For both the cases, the performance in case of the simulated data is better than in case of the real market data. This is evident from the fact that while real market data is difficult to obtain and does not exactly fit into any distribution, in case of the simulated data we have multivariate normal distribution with the true mean and covariances of the real market data.

References

[1] J.H. Kim, W.C. Kim and F.J. Fabozzi, Recent developments in robust rortfolios with a worst-case approach. Journal of Optimization Theory and Applications, vol. 161, no. 1, pp. 103–121, 2014.

[2] F.J. Fabozzi, P.N. Kolm, D. Pachamanova and S. M. Focardi, Robust Portfolio Optimization and Management, Wiley, 2007.

[3] Z. Lu, A new cone programming approach for robust portfolio selection. Technical report, Department of Mathematics, Simon Fraser University, Burnaby, BC, 2006.

[4] B.V. Halldorsson, R.H. Tutuncu, An interior-point method for a class of saddle-point problems. Journal of Optimization Theory and Applications, vol. 116, no. 3, pp. 559–590, 2003.

[5] B. Scherer, Can robust portfolio optimization help to build better portfolios?. Journal of Asset Management, vol. 7, no. 6, pp. 374–387, 2007.

[6] R.H. Tutuncu and M. Koenig, Robust asset allocation. Annals of Operations Research, vol. 132, no. 1-4, pp. 157–187, 2004.

[7] M. Broadie, Computing efficient frontiers using estimated parameters. Annals of Operations Research, 45(1): 21–58, 1993.

[8] Yahoo Finance, `https://in.finance.yahoo.com/`

[9] Reserve Bank of India – Weekly Statistical Supplement, `https://www.rbi.org.in/scripts/WSSViewDetail.aspx?TYPE=Section&PARAM1=4\%0A`

[10] Portfolios: Do number of stocks matter?, Value Research Online, `https://www.valueresearchonline.com/story/h2_storyView.asp?str=200396&fbclid=IwAR3YkUaNLCY3keZGXfooehh0WU90zkZ5s94h0Uazj1-rzOFSy5bw6R1E-MY`

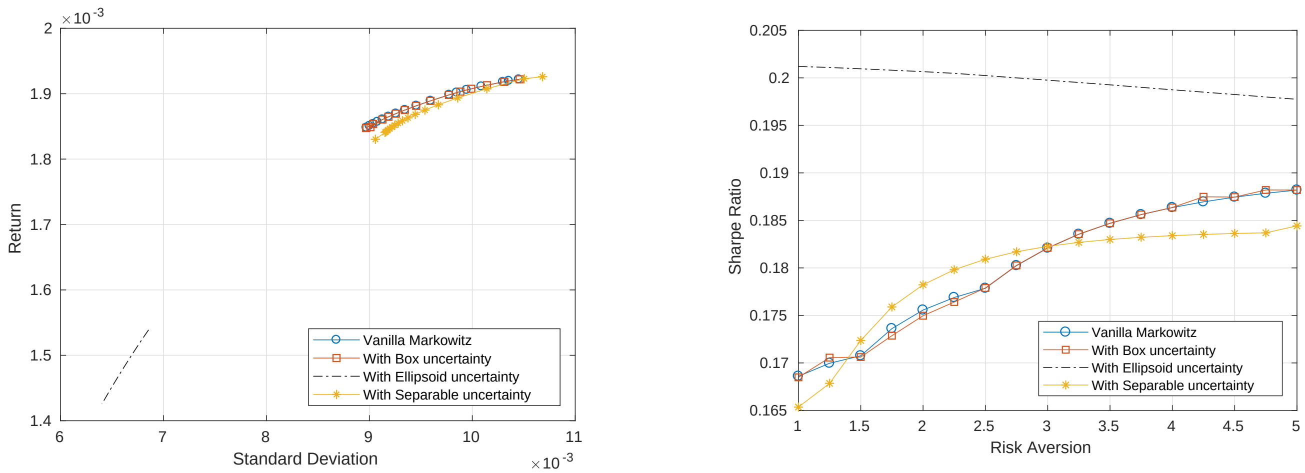

Figure 1: Efficient Frontier plot and Sharpe Ratio plot for different portfolio optimization models in case of Simulated Data with 1000 samples (31 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.176	0.175	0.201	0.178
2.5	0.178	0.178	0.2	0.181
3	0.182	0.182	0.2	0.182
3.5	0.185	0.185	0.199	0.183
4	0.186	0.186	0.199	0.183
Avg	0.181	0.181	0.2	0.182

Table 1: Comparison of different portfolio optimization models in case of Simulated Data with 1000 samples (31 assets)

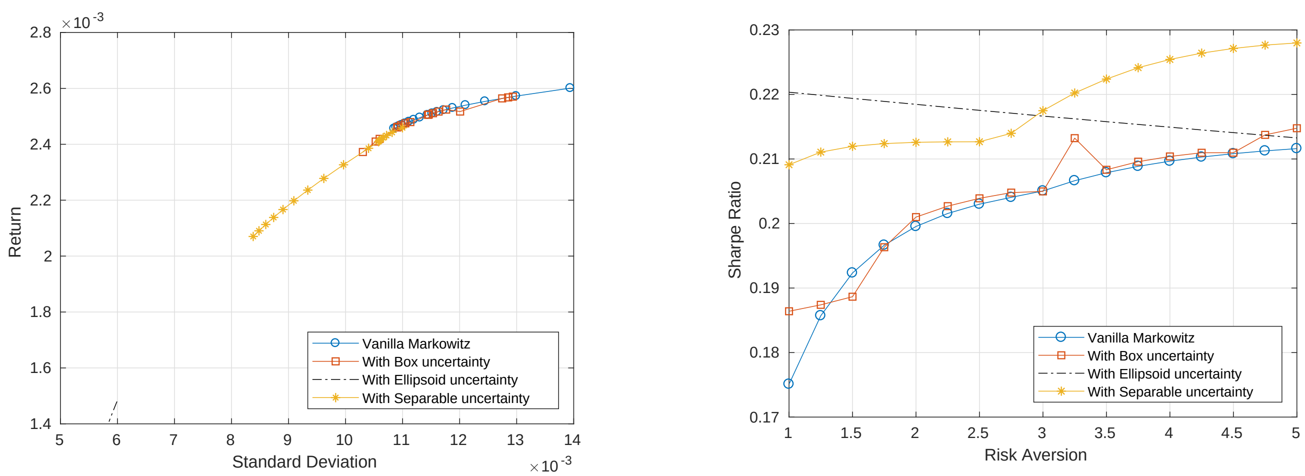

Figure 2: Efficient Frontier plot and Sharpe Ratio plot for different portfolio optimization models in case of Simulated Data with same number of samples as market data (31 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.2	0.198	0.218	0.213
2.5	0.203	0.204	0.218	0.213
3	0.205	0.207	0.217	0.217
3.5	0.208	0.209	0.216	0.222
4	0.21	0.21	0.215	0.225
Avg	0.205	0.206	0.217	0.218

Table 2: Comparison of different portfolio optimization models in case of Simulated Data with same number of samples as market data (31 assets)

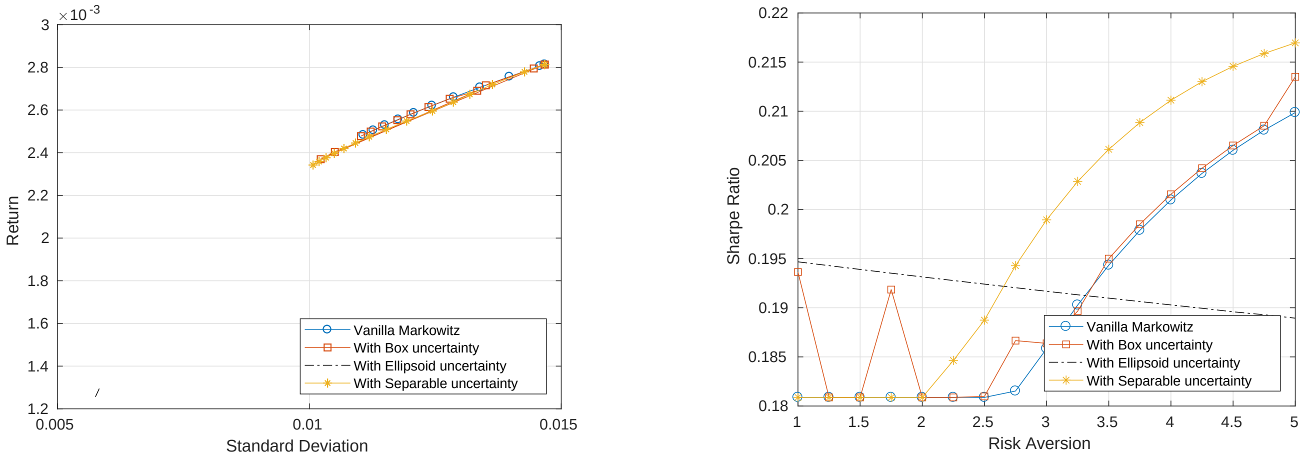

Figure 3: Efficient Frontier plot and Sharpe Ratio plot for different portfolio optimization models in case of Market Data (31 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.181	0.181	0.193	0.186
2.5	0.181	0.181	0.192	0.193
3	0.186	0.191	0.192	0.202
3.5	0.194	0.195	0.191	0.209
4	0.201	0.202	0.19	0.213
Avg	0.189	0.19	0.192	0.2

Table 3: Comparison of different portfolio optimization models in case of Market Data (31 assets)

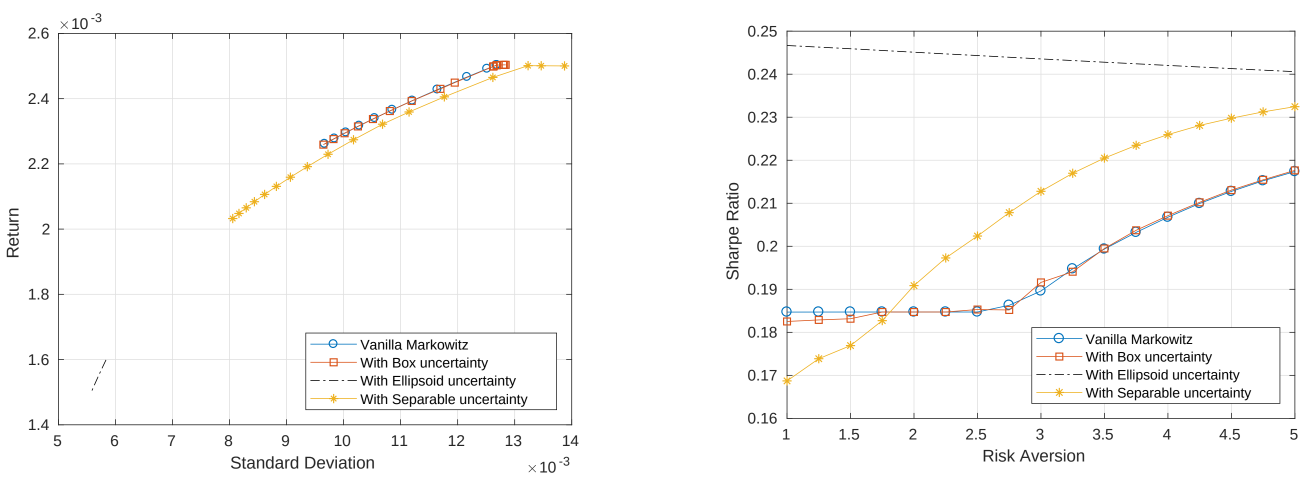

Figure 4: Efficient Frontier plot and Sharpe Ratio plot for different portfolio optimization models in case of Simulated Data with 1000 samples (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.185	0.185	0.245	0.191
2.5	0.185	0.185	0.244	0.202
3	0.19	0.192	0.244	0.213
3.5	0.199	0.199	0.243	0.221
4	0.207	0.207	0.242	0.226
Avg	0.193	0.194	0.244	0.21

Table 4: Comparison of different portfolio optimization models in case of Simulated Data with 1000 samples (98 assets)

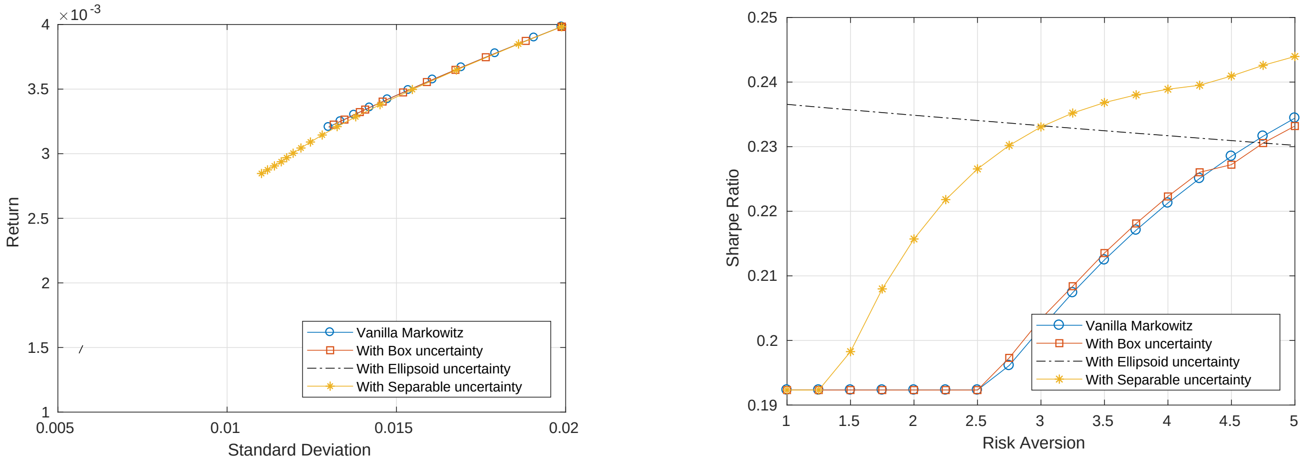

Figure 5: Efficient Frontier plot and Sharpe Ratio plot for different portfolio optimization models in case of Simulated Data with same number of samples as market data (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.192	0.192	0.235	0.216
2.5	0.192	0.192	0.234	0.227
3	0.202	0.203	0.233	0.233
3.5	0.212	0.213	0.232	0.237
4	0.221	0.222	0.232	0.239
Avg	0.204	0.205	0.233	0.23

Table 5: Comparison of different portfolio optimization models in case of Simulated Data with same number of samples as market data (98 assets)

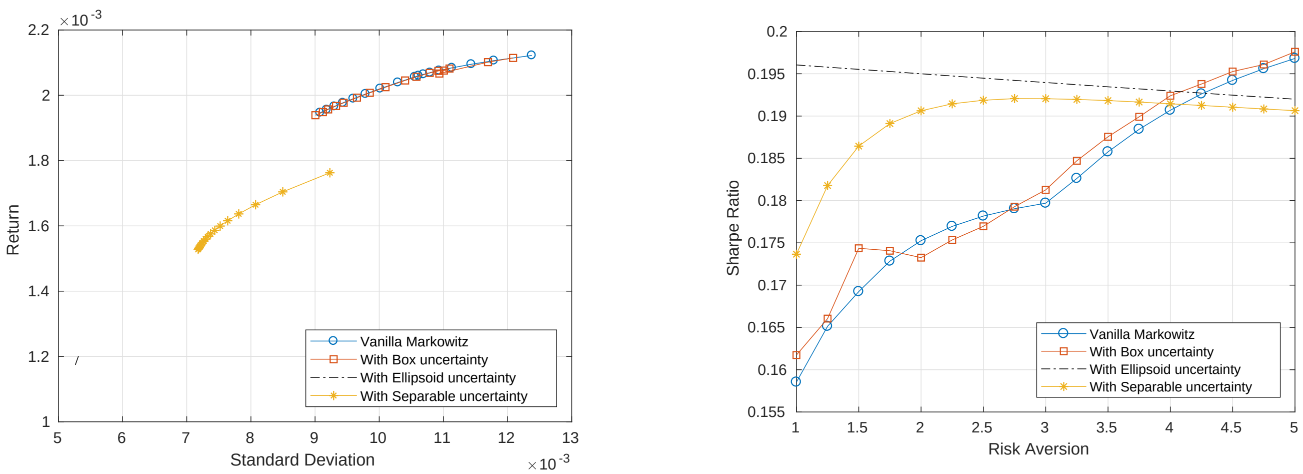

Figure 6: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Market Data (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.175	0.173	0.195	0.193
2.5	0.178	0.177	0.194	0.193
3	0.18	0.181	0.194	0.193
3.5	0.186	0.188	0.193	0.192
4	0.191	0.192	0.193	0.192
Avg	0.182	0.182	0.194	0.192

Table 6: Comparison of different portfolio optimization models in case of Market Data (98 assets)

	#stocks = 31	#stocks = 98
#generated_simulations = 1000	0.2	0.244
#generated_simulations = ζ	0.218	0.233
Market data	0.2	0.194

Table 7: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

	#samples = 1000	#samples = ζ
#stocks = 31	0.2	0.218
#stocks = 98	0.244	0.233

Table 8: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

	Simulated data	Real Market data
#stocks = 31	0.218	0.2
#stocks = 98	0.244	0.194

Table 9: The maximum average Sharpe ratio compared by varying the type of the data in different kinds of scenarios.