Can robust risk minimization offer improved portfolio performance?: An perspective from the Indian market

Mohammed Bilal Girach * Shashank Oberoi †
Siddhartha P. Chakrabarty ‡

Abstract

Keywords: Robust portfolio optimization; CVaR; VaR; S&P BSE 30; S&P BSE 100

1 Introduction

Significant progress in the area of Robust portfolio optimization has been driven primarily by the motivation to address various drawbacks of the classical mean-variance model introduced by Markowitz [17, 18]. Theoretically, Markowitz based portfolio optimization can result in assigning extreme weights such as large short positions to the securities comprising the portfolio. Such kind of counter-intuitive investments can be avoided by introducing appropriate constraints on the weights. Black and Litterman [6] argued that there is an added disadvantage since there are high chances of the optimal portfolio lying in the neighborhood of the imposed constraints, thus, leading to strong dependence of the constructed portfolio upon the constraints. Secondly, the optimal portfolio constructed using the mean-variance analysis is highly sensitive to the estimation errors of input parameters, particularly expected returns of individual securities. Based on the assumption of normal distribution of returns, the historical data is used to calculate the sample mean and sample covariance matrix for estimating the return and risk parameters. Usually, the historical data neglects various market factors and is not an accurate representation for estimates of future returns. Taking into account the above reasons, Michaud [19] labelled the Markowitz model as "estimation-error maximizers" and argued that the mean-variance analysis tends to maximize the impact of estimation errors associated with the return and the risk parameters for the securities. As a result, Markowitz portfolio optimization often overweighs (underweighs) the securities having higher (lower) expected return, lower (higher) variance of returns and negative (positive) correlation between their returns. Chopra and Ziemba [9] performed the sensitivity analysis of performance of optimal portfolios by studying the relative effect of estimation errors in means, variances and covariances of security returns, taking the investors' risk tolerance into consideration as well. They observed that at a high risk tolerance of around fifty, cash equivalent loss for estimation errors in means is about eleven times greater than that for errors in variances or covariances. Broadie [7] conducted a simulation based study to investigate the error maximization property of mean-variance analysis. He supported his argument of overestimation of expected returns of optimal portfolios obtained using the mean-variance model through his simulated results of obtaining the estimated efficient frontier lying above the actual frontier.

Major formulations proposed in the field of robust optimization have focused on optimizing the performance of the Markowitz portfolio using an uncertainty set that includes the worst possible realizations of the uncertain input parameters. Tütüncü and Koenig [23] have performed empirical analysis of robust allocation methods using the market data. In accordance with the results observed, they suggested robust

*Indian Institute of Technology Guwahati, Guwahati-781039, Assam, India, e-mail: m.girach@iitg.ac.in
†Indian Institute of Technology Guwahati, Guwahati-781039, Assam, India, e-mail: s.oberoi@iitg.ac.in
‡Indian Institute of Technology Guwahati, Guwahati-781039, Assam, India, e-mail: pratim@iitg.ac.in, Phone: +91-361-2582606, Fax: +91-361-2582649

optimization as a viable alternative for alternative investors. Significant efforts have been made toward formulating robust approaches from Markowitz based mean-variance analysis using box, ellipsoidal [10, 13] and separable uncertainty sets [16].

However, the use of standard deviation as a measure of risk in the mean-variance model and its robust alternatives has received criticism. From a practitioners' point of view, the upside and downside risk cannot be perceived in the same way, because most of the times, the upside risk can improve the overall performance of the portfolio whereas the downside fluctuation usually brings impactful losses. Variance is not a reliable or appropriate risk measure, if the underlying distribution is leptokurtic. In order to address these issues, models involving other measures of risk have been developed and accordingly their corresponding robust models have been studied as well. In the forthcoming chapters, we will elaborately discuss some of the most widely measures of risk like Value at Risk (VaR) and Conditional Value at Risk (CVaR) and also study their corresponding robust worst case formulations.

Unlike the mean-variance setup, VaR framework takes into account the probability of losses [15]. Ghaoui et al. [11] defined VaR, at confidence level $1-\epsilon$, as the minimum value of γ such that the probability of loss exceeding γ is less than ϵ. Though VaR takes probability of losses into account, it has its own limitations. For the computation part, it requires the knowledge of the whole distribution. The computation also involves high dimensional numerical integration which may not be tractable at times and also there hasn't been an extensive research on methods using Monte Carlo simulations [15] for the design of the portfolio. Black and Litterman [6], Pearson and Ju [12] discussed the issues regarding the computational difference between the true VaR and the calculated VaR and determined that the error in the computation of the VaR can be attributed to errors in the estimation of the first and second moments of the asset returns.

The concept of worst-case VaR not only allows for approaching the solution in a more tractable way, but also relaxes the assumptions on the information known to us apriori. Here, we assume that only partial information about the underlying distribution is known. We also assume that the distribution of the asset returns belong to a family of allowable probability distributions $\mathcal{P}$ [11]. For example, given component wise bounds of $(\hat{\boldsymbol{\mu}}, \Sigma)$, $\mathcal{P}$ could comprise of normally distributed random variables with $\hat{\boldsymbol{\mu}}$ and Σ as the moments' pairs.

Despite its popularity as a measure of downside risk, VaR has received a fair amount of criticism [8, 24, 14]. VaR for a diversified portfolio may exceed that for an investment in a single asset. Thus, one of the major limitations of VaR is the lack of sub-additivity in the case of general distributions. In accordance with this observation, VaR is not a coherent measure of risk as per the definition laid out by Artzner et al [4]. Secondly, VaR does not provide any information on the size of losses in adverse scenarios *i.e,* those beyond the confidence level $(1-\epsilon)$. Additionally, VaR is non-convex and non-differentiable when incorporated into portfolio optimization problem. As a result, it becomes difficult to optimize VaR since global minimum may not exist.

Conditional Value-at-Risk (CVaR), introduced by Rockafellar and Uryasev [21, 22], has addressed various concerns centred around VaR. For continuous distributions, CVaR is the expected loss conditioned on the loss outcomes exceeding VaR. In recent years, CVaR has emerged as a viable risk measure in portfolio optimization problems over its use in reducing downside risk. It has been proved by Pflug [20] and Acerbi and Tasche [3] that CVaR is a coherent measure of risk. Accordingly, it follows the property of sub-additivity. Therefore, CVaR can be minimized through investment in a diversified portfolio. Unlike VaR, CVaR takes into consideration the impact of losses beyond the confidence level $(1-\epsilon)$ [8]. Also, the minimization of CVaR is a convex optimization problem [14].

Similar to Markowitz optimization and VaR minimization, there is an issue of lack of robustness in the classical framework of CVaR minimization. Computing CVaR requires the complete knowledge of the return distribution as argued by Zhu and Fukushima [24]. They have addressed this shortcoming of the classical CVaR (henceforth, referred to as base-case CVaR) by proposing a new risk measure, namely, Worst-Case CVaR (WCVaR).

Motivated by the progress made towards incorporating robust optimization in the framework of risk minimization, this paper focuses on assessing the practical usefulness of the robust counterparts for the classical formulations of VaR and CVaR minimization. Accordingly, we perform empirical analysis of the performance of VaR and CVaR with respect to their robust formulations, Worst-Case VaR and Worst-Case CVaR using both market and simulated data. Additionally, we provide relevant insights regarding the viability of these robust models over their classical formulations from the perspective of an investment practitioner. For performing empirical study, we have used the data obtained from the Indian indices of S&P BSE 30 and S&P BSE 100.

The rest of the paper is organized as follows. In section 2, we discuss the tractable formulations of the downside risk minimization methods used in the work. Section 3 presents the empirical results observed on comparison of the performance of robust risk minimization models with their classical counterparts. Section 4 focuses on analyzing the practical viability of the robust models from the standpoint of number of stocks, sample size and type of data. In Section 5, we sum up the key inferences from this work .

2 Risk Minimization Approaches

In this section, we discuss the mathematical formulations for the problems of optimizing VaR and CVaR along with their robust counterparts, namely, WVaR and WCVaR respectively. Accordingly, we begin with a glossary of notation (to be used in the mathematical formulation) as follows:

2.1 Value at Risk (VaR)

Recall that the VaR, at confidence level $1-\epsilon$, is the minimum value of γ such that the probability of loss exceeding γ is less than ϵ. Accordingly, the VaR is mathematically defined as,

$$VaR_{\epsilon}(\mathbf{x}) = \min\left\{\gamma : P\left\{\gamma \leq -r(\mathbf{x}, \boldsymbol{\mu})\right\} \leq \epsilon\right\}, \tag{2.1}$$

where $\epsilon \in (0,1)$. When we deal with mean-variance setup, only the mean and the variance *i.e.,* the first and the second moments of the asset returns are required, but in the VaR framework, the knowledge of the entire distribution is necessary for the computation part. If the underlying distribution is Gaussian with the moments' pair as $(\hat{\boldsymbol{\mu}}, \Sigma)$, then VaR can be computed via the following analytical form:

$$VaR_{\epsilon}(\mathbf{x}) = \kappa(\epsilon)\sqrt{\mathbf{x}^{\top}\Sigma\mathbf{x}} - \hat{\boldsymbol{\mu}}^{\top}\mathbf{x}, \tag{2.2}$$

where $\kappa(\epsilon) = -\Phi^{-1}(\epsilon)$ with $\Phi(z)$ representing the cumulative distribution for standard normal random variable. In practice, the value of the $\kappa(\epsilon)$ can be determined only if the cumulative distribution function of the underlying distribution is known apriori. However, if the distribution is unknown, then we have to rely on the Chebyshev's inequality. The bound (upper) due to Chebyshev's inequality, only requires the knowledge of the first two moments' pair. We call a bound to be exact if the upper bound is attained. If not, we use the bound given by Bertsimas and Popescu [5] *i.e.,* $\kappa(\epsilon) = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$. Finally, we formulate the generalized VaR as follows:

$$\min \kappa(\epsilon)\sqrt{\mathbf{x}^{\top}\Sigma\mathbf{x}} - \hat{\boldsymbol{\mu}}^{\top}\mathbf{x}, \text{ subject to } \mathbf{x} \in \mathcal{X}, \tag{2.3}$$

where $\kappa(\epsilon)$ is an appropriate factor of risk chosen in accordance with the underlying distribution of asset returns and $\mathcal{X} = \left\{\mathbf{x} : \mathbf{x}^{\top}\mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0\right\}$. The function $VaR_{\epsilon}(\mathbf{x})$ is convex and the global optimum can be obtained using techniques like interior-point methods and second order cone programming (SOCP).

2.2 Worst-Case VaR (WVaR)

Given a probability (confidence) level ϵ, the worst-case VaR can be formulated as,

$$V_{\epsilon}^{\mathcal{P}}(\mathbf{x}) = \min\left\{\gamma : \sup_{P \in \mathcal{P}} P\left\{\gamma \leq -r(\mathbf{x}, \boldsymbol{\mu})\right\} \leq \epsilon\right\}, \tag{2.4}$$

where $\mathcal{P}$ is the family of allowable probability distributions. Accordingly, the robust formulation can be written as,

$$WVaR_\epsilon(\mathbf{x}) = \min V_\epsilon^{\mathcal{P}}(\mathbf{x}), \text{ subject to } \mathbf{x} \in \mathcal{X}, \tag{2.5}$$

The above optimization problems can be computed by a semi-definite programming (SDP) problem which again uses the above mentioned interior-point methods. We deal with the high dimensional problems by using bundle methods which are mainly used for large-scale (sparse) problems. Motivated by the work of separable uncertainty sets by Tutuncu and Koenig [23], one can view the robust formulation given by Ghaoui et al. [11] in case of Polytopic uncertainty, as a robust formulation of WVaR involving separable uncertainty (Section 2). Accordingly, the formulation of the optimization problem for worst case VaR is,

$$\min \kappa(\epsilon)\sqrt{\mathbf{x}^\top \overline{\Sigma} \mathbf{x}} - \underline{\hat{\boldsymbol{\mu}}}^\top \mathbf{x}, \text{ subject to } \mathbf{x} \in \mathcal{X}, \tag{2.6}$$

where $\overline{\Sigma}$ and $\underline{\hat{\boldsymbol{\mu}}}$ are the upper bound for covariance matrix and the lower bound for the estimated mean of asset returns, respectively. We obtain these values by using the Non-Parametric Bootstrap algorithm where the type of distribution is unknown. This can also be viewed as a robust formulation involving polytopic uncertainty because "Separable uncertainty" is a special case of "Polytopic uncertainty". In case of models involving ellipsoidal uncertainty sets, the robust WVaR formulation is not very trivial and such models mainly revolve around the assumption of factor models.

2.3 Conditional Value at Risk (CVaR)

Recall that for continuous distributions, CVaR is the expected loss conditional on the loss outcomes exceeding VaR. Therefore, for $\epsilon \in (0,1)$, CVaR, at confidence level $1-\epsilon$, is defined as:

$$CVaR_\epsilon(\mathbf{x}) \triangleq \frac{1}{\epsilon} \int\limits_{-\mathbf{r}^\top \mathbf{x} \geq VaR_\epsilon(\mathbf{x})} p(\mathbf{r}) d\mathbf{r} \tag{2.7}$$

where $\mathbf{r}$ is the random return vector having probability density function $p(\mathbf{r})$ and $\mathbf{x}$ is the weight vector for a portfolio. As proved by Rockafeller and Uryasev [21] , $CVaR_\epsilon(\mathbf{x})$, defined in equation (2.7), can be transformed into:

$$CVaR_\epsilon(\mathbf{x}) = \min_{\gamma \in \mathcal{R}^n} F_\epsilon(\mathbf{x}, \gamma) \tag{2.8}$$

where n is number of assets in the portfolio and $F_\epsilon(\mathbf{x}, \gamma)$ is defined as:

$$F_\epsilon(\mathbf{x}, \gamma) \triangleq \gamma + \frac{1}{\epsilon} \int\limits_{\mathbf{r} \in \mathcal{R}^n} \left[-\mathbf{r}^\top \mathbf{x} - \gamma\right]^+ p(\mathbf{r}) d\mathbf{r} \tag{2.9}$$

In the above equation, $[t]^+ = \max\{t, 0\}$. The problem of approximating the integral involved in equation (2.9) can be dealt by sampling the probability distribution of $\mathbf{r}$ according to its density $p(\mathbf{r})$. Assuming there are S samples, $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_S\}$ for the return vector $\mathbf{r}$, $F_\epsilon(\mathbf{x}, \gamma)$ can be approximated as [21]:

$$F_\epsilon(\mathbf{x}, \gamma) \approx \gamma + \frac{1}{S\epsilon} \sum_{i=1}^{S} \left[-\mathbf{r}_i^\top \mathbf{x} - \gamma\right]^+ \tag{2.10}$$

In accordance with the above approximation, the problem of minimization of classical CVaR, assuming no short-selling constraints, can be formulated as the following Linear Programming Problem (LPP) [21, 24]:

$$\min_{(\mathbf{x}, \mathbf{u}, \gamma, \theta)} \theta \text{ such that} \tag{2.11}$$

$$\mathbf{x}^\top \mathbf{1} = 1, \ \mathbf{x} \geq \mathbf{0},$$

$$\gamma + \frac{1}{S\epsilon}\mathbf{1}^\top \mathbf{u} \leq \theta,$$
$$u_i \geq -\mathbf{r}_i^\top \mathbf{x} - \gamma,\ u_i \geq 0,\ i = 1, 2, \ldots, S.$$

where the auxiliary vector $\mathbf{u} \in \mathcal{R}^S$ and θ is the auxiliary variable that is optimized to obtain the optimal value of base CVaR.

2.4 Worst-Case CVaR (WVaR)

For a fixed weight vector $\mathbf{x}$, at confidence level $1 - \epsilon$, WCVaR is defined as,

$$WCVaR_\epsilon(\mathbf{x}) \triangleq \sup_{p(\mathbf{r}) \in \mathcal{P}} CVaR_\epsilon(\mathbf{x}) \tag{2.12}$$

where we assume that the density function $p(\mathbf{r})$ of returns belongs to a set $\mathcal{P}$ of probability distributions. Zhu and Fukushima [24] have proved the coherence of WCVaR as a risk measure by analyzing it in terms of worst-case risk measure ρ_w, given by,

$$\rho_w(X) \triangleq \sup_{p(\mathbf{r}) \in \mathcal{P}} \rho(X) \tag{2.13}$$

We skip the discussion on WCVaR formulations using the box uncertainty set and the ellipsoidal uncertainty set, since they require a set of possible return distributions to be assumed, so as to obtain bounds and scaling matrix respectively. However, our problem setup doesn't involve a set of return distributions, since we make use of only market data (where return distribution is not known) and simulated data (where we generate a known return distribution). We describe the formulation of the optimization problem involving WCVaR by assuming that the return distribution belongs to a set of distributions comprising of all possible mixtures of some prior likelihood distributions [24]. Mathematically, it is assumed that:

$$p(\mathbf{r}) \in \mathcal{P}_M \triangleq \left\{ \sum_{j=1}^{l} \lambda_j p^j(\mathbf{r}) : \sum_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, \ldots, l \right\} \tag{2.14}$$

In the above equation, $p^j(\mathbf{r})$ denotes the j^{th} likelihood distribution and l is the number of the likelihood distributions. In accordance with above assumption of mixture distribution uncertainty (that involves $\mathcal{P}_M$ as a compact convex set), $WCVaR_\epsilon(\mathbf{x})$, defined in equation (2.12), can be rewritten as the following min-max problem [24]:

$$WCVaR_\epsilon(\mathbf{x}) = \min_{\alpha \in \mathcal{R}} \max_{j \in \mathcal{L}} F_\epsilon^j(\mathbf{x}, \gamma),\ \text{where,} \tag{2.15}$$
$$\mathcal{L} \triangleq \{1, 2, \ldots, l\}$$
$$F_\epsilon^j(\mathbf{x}, \gamma) \triangleq \gamma + \frac{1}{\epsilon} \int_{\mathbf{r} \in \mathcal{R}^n} \left[-\mathbf{r}^\top \mathbf{x} - \gamma\right]^+ p^j(\mathbf{r}) d\mathbf{r}$$

Similar to the case involving the classical CVaR in the preceding Subsection, $F_\epsilon^j(\mathbf{x}, \gamma)$ can be approximated via discrete sampling as,

$$F_\epsilon^j(\mathbf{x}, \gamma) \approx \gamma + \frac{1}{S_j \epsilon} \sum_{i=1}^{S_j} \left[-\mathbf{r}_{i,j}^\top \mathbf{x} - \gamma\right]^+ \tag{2.16}$$

In the above equation, $\mathbf{r}_{i,j}$ is the i^{th} sample of the return with respect to j^{th} likelihood distribution and S_j is the number of samples corresponding to j^{th} likelihood distribution. Accordingly, assuming non-negative weights, the problem of minimization of WCVaR over a feasible set of portfolios can be formulated as the following LLP [24],

$$\min_{(\mathbf{x}, \mathbf{u}, \gamma, \theta)} \theta,\ \text{such that} \tag{2.17}$$

$$
\begin{aligned}
&\mathbf{x}^\top \mathbf{1} = 1,\ \mathbf{x} \geq \mathbf{0},\\
&\gamma + \frac{1}{S_j \epsilon} \mathbf{1}^\top \mathbf{u}^j \leq \theta, j = 1, 2, \ldots, l,\\
&u_i^j \geq -\mathbf{r}_{i,j}^\top \mathbf{x} - \gamma,\ u_i^j \geq 0,\ i = 1, 2, \ldots, S_j,\ j = 1, 2, \ldots, l.
\end{aligned}
$$

In the above equation, the auxiliary vector $\mathbf{u} = (\mathbf{u}^1; \mathbf{u}^2; \ldots; \mathbf{u}^l) \in \mathcal{R}^S$ where $S = \sum_{j=1}^{l} S_j$ and θ is the auxiliary variable whose optimization yields the optimal value of the Worst-Case CVaR..

3 Computational Results

In this Section, we present the computational results in detail. A comparative analysis of VaR and CVaR, with their respective worst case counterparts, namely WVaR and WCVaR, is presented both in case of $N = 31$ and $N = 98$ assets.

3.1 VaR vis-a-vis WVaR

We carry out a comparative analysis on the performance of the optimal portfolios obtained when VaR (referred as "base VaR" from hereon) and WVaR are incorporated as measures of risk in the robust portfolio optimization problem. We use the Sharpe Ratio (SR) as a metric to compare the performance of the portfolios. Since the yield for Treasury Bill in India from 2016 to 2018 was observed to be around 6% [1], therefore, we assumed the annualized risk-free rate to be equal to 6%.

We perform the empirical analysis for the available historical data for S&P BSE 30 ($N = 31$) and S&P BSE 100 ($N = 98$). For the first scenario ($N = 31$), we use the daily log-returns based on daily adjusted closing price of the 31 stocks comprising S&P BSE 30 (data source: Yahoo Finance [2]). Accordingly, we have considered the period from December 18, 2017 to September 30, 2018 (both inclusive) comprising of a total of 194 active trading days. For the second scenario ($N = 98$), we use the log-returns based on daily adjusted close price data of the 98 stocks comprising S&P BSE 100 (data source: Yahoo Finance [2]) with the period spanning from December 18, 2016 to September 30, 2018. The reason behind this setup is to observe the trends/patterns in the performance of the portfolio when the number of stocks taken into consideration for constructing the optimal portfolio are changed. Furthermore, we also analyze the performance of the portfolio, when instead of real market data, simulated data with true moments' pair is fed to the robust optimization problem. We delve one step further into the above setup and vary the number of simulations to observe the changes in the performance of the optimal portfolio by increasing the number of simulations. For the above case, we simulate two data-sets with true moments' pair, one with exact number of samples as in the available real market data, say, $\zeta(\leq 1000)$ and another with a large number of samples, say 1000. We also use the same setup in order to analyze the performance of the portfolio when different type of data is taken into consideration *i.e.,* real market and simulated data. For the computational part of the robust formulation, we choose $\mathcal{P}$ to be family of distribution with the first and the second moments as the true mean and variance of the historical data. As the knowledge of the distribution is unknown, we use κ from equation (2.2), where we only consider values of ϵ lying in $(0, 0.1)$ *i.e.,* the confidence level is greater than or equal to 90%. In the following subsections, we analyze each scenario in detail with appropriate figures and tables.

3.1.1 Performance with $N = 31$ Assets

We begin by analyzing the performance of the portfolio when real market data is used to construct the optimal portfolio. From Figure 1 one can observe that the performance of the portfolio is better when the base VaR is used in the optimization problem, with the base VaR model outperforming the WVaR model over the complete range of ϵ. For a better understanding of the trends, we tabulated the values of Sharpe Ratio of the optimal portfolios obtained for different values of ϵ, in Table 1. We can observe that the average value of Sharpe Ratio is more for the base VaR model when compared to the WVaR model and can also be seen from Figure 1.

Now, we move on to the case of the simulated data. Firstly, we discuss the trends in the performance of the optimal portfolio when the number of the samples (ζ) generated are the same as the real market data. We observe from the Figure 2 that there is little difference in performance trends, when we used ζ number of simulations, as compared to when we used the real market data. Also from Table 2, we can observe that the base VaR model exhibits superior performance over WVaR model over the complete range of ϵ.

Since the model with ζ number of samples results in the same trends as in the real market data, we now consider the case where we simulated larger number of samples, say 1000, a choice made on the premise that larger the number of samples, less is the difference between true moments' pair and their computed counterparts. In this case, from Figure 3, we observe that the WVaR model exhibits better performance than the base VaR model in the ϵ interval of $(0, 0.04]$. This is suggestive that, for the more conservative investors, the WVaR model will be more beneficial than the corresponding VaR model. In Table 3, one can validate the observation made above, the WVaR model performing better than the base VaR model in the interval $(0, 0.04]$ of ϵ. However, the average values remains almost equal because, the base VaR model outperforms the WVaR model beyond the the interval of (00.04]. So, we conclude that in this case, both base VaR and WVaR models are indifferent from the perspective of a rational investor.

We now extend a similar type of analysis for a larger set of stocks for which we obtain data from S&P BSE 100 ($N = 98$).

3.1.2 Performance with $N = 98$ Assets

We now analyze the performance of the base VaR and WVaR by using real market data from S&P BSE 100. We present the empirical results in Figure 4 and tabulate the observations in Table 4. One can observe from Figure 4 that the WVaR model exhibits superior performance in comparison to the corresponding base VaR model on the entire range of ϵ. This observation can also be quantitatively justified by Table 4 where the Sharpe Ratios of the portfolios obtained from WVaR model is greater than that of base VaR model.

Similar trend is observed for $N = 98$ stocks, in the case of simulated data as well, when the number of simulations equals ζ *i.e.,* the number of instances available in the real market data of S&P BSE 100. From Figure 5 and the Table 5, we infer that the WVaR model performs better than the base VaR model when Sharpe ratio is used as performance measure for every $\epsilon \in$ (0,0.1).

Finally, we consider the case where we simulated larger number of samples in order to distinguish the fluctuations in the performance of the portfolio when number of simulations are varied. Similar kind of inferences can be drawn from Figure 6 and Table 6. The optimal portfolios obtained from the WVaR model have greater values of Sharpe ratio as compared to those obtained from base VaR model, irrespective of the value of ϵ.

3.2 CVaR vis-a-vis WCVaR

Similar to the computational analysis of robust methods in VaR minimization, we perform empirical analysis of the WCVaR model vis-a-vis the base CVaR model, making use of the historical market data ($N = 31$ and $N = 98$ representing the number of stocks in S&P BSE 30 and S&P BSE 100 indices, respectively) and simulated data.

For each scenario, we use the same three sets of data as used in the preceding Subsection *i.e,* a set of historical market data comprising daily log-returns and two sets of simulated data. The first set of simulated data comprises of the number of sample returns matching the number of return instances of the historical market data, say $\zeta(< 1000)$. On the other hand, the second set has a larger number of samples, namely, 1000.

As before, during computation for the base CVaR model, S in equation (2.12) is set equal to the number of return samples, *i.e,* either ζ or 1000, depending upon the set of data used. We perform computation of the WCVaR model, as formulated in equation (2.18), for values of $l \in \{2,3,4,5\}$ by setting $S_j = \dfrac{S}{l}$ where $j \in \{1,2,\ldots,l\}$. Note that $l = 1$ yields the same case as the classical CVaR model. We do not include larger values of l in the discussion on the empirical results since this leads to a great decrease in the sample size for the likelihood distributions.

We perform comparative analysis of the WCVaR model with respect to the base CVaR model using the Sharpe Ratio of the constructed portfolios having confidence level greater than 90%, *i.e,* $\epsilon \in (0, 0.1)$, assuming (as before) the annualized risk free rate equal to 6%. In the following subsections, we present the computational results for the two scenarios.

3.2.1 Performance with $N = 31$ Assets

We begin with the analysis for $N = 31$ assets, in the case of the historical market data having log-returns of the stocks comprising S&P BSE 30. In Table 7, we present the average Sharpe Ratio of the portfolios constructed for the base CVaR and WCVaR models (denoted by $Avg.SR_{CVaR}$ and $Avg.SR_{WCVaR}$, respectively) for different values of l. **For this case as well as other cases, we choose the value of l with maximum difference in the average Sharpe Ratio between the WCVaR and CVaR model.** Accordingly, we perform comparative study based on the plot of the Sharpe Ratio and tabulation of results for some selected values of ϵ. The motivation behind the adoption of this kind of methodology for empirical analysis is to assess the practical viability of the robust technique over the classical method in CVaR minimization. In accordance with our methodology, the empirical results are presented for $l = 2$ in Figure 7 and Table 8. From Figure 7, we observe that the base CVaR model performs better than its robust counterpart in terms of the Sharpe Ratio for the constructed portfolios with $\epsilon \in (0, 0.1)$. Table 8 supports the above observation quantitatively.

On performing the simulation study with ζ samples, we draw an inference from Table 9 that the performance of the WCVaR model vis-à-vis the CVaR model is maximal for $l = 5$. Similar to the methodology followed in the previous case, we present the relevant results for $l = 5$ in Figure 8 and Table 10. From the plot of the Sharpe Ratio in Figure 8, we observe that the WCVaR model starts outperforming the base CVaR model after ϵ crosses 0.05 *i.e.,* incorporating robust optimization in CVaR minimization is advantageous in this case only for less conservative investors. We infer that the performance of the two models is almost equivalent, which is evident from Table 10 as well.

The comparative analysis in terms of the average Sharpe Ratio with the number of simulated samples being 1000 is presented in Table 11. Accordingly, we select the value of l equal to 4 and perform empirical study of the base CVaR and WCVaR models, as presented in Figure 9 and Table 12. It is difficult to draw any comparative inference from the Sharpe Ratio plot of the two models in Figure 9, since each outperforms the other in a different sub-interval of the range of ϵ. The similar values of the Sharpe Ratio in Table 12 as well as the marginal difference in the average Sharpe Ratio for $l = 4$ in Table 11 supports the claim of almost equivalent performance of the CVaR and WCVaR models in this case.

3.2.2 Performance with $N = 98$ Assets

In this subsection, we consider the scenario involving $N = 98$ assets. Table 13 summarizes the comparative results observed using the base CVaR and WCVaR models on the historical market data (involving stocks comprising S&P BSE 100). Since the difference between the average Sharpe Ratio of the WCVaR and CVaR models is maximum for $l = 3$, so, we plot and tabulate the relevant results summarizing the empirical analysis of the two models for the same value of l (Figure 10 and Table 14). Similar to the corresponding case for the previous scenario, Figure 10 and Table 14 lead to an observation that the base CVaR model exhibits superior performance in comparison to the Worst-Case CVaR model, taking into consideration the Sharpe Ratio of the constructed portfolios as the performance measure.

Table 13 presents the comparison of the average Sharpe Ratio for the simulation study with ζ samples *i.e,* same number of samples as that of log-returns of S&P BSE 100 data. In accordance with the discussed methodology, we choose the value of l equal to 4 and present the empirical results in Figure 11 and Table 16. From Figure 11, we observe that the Worst-Case CVaR model almost outperforms the base CVaR model in terms of the Sharpe Ratio of the constructed portfolios having $\epsilon \in (0, 0.1)$. From Table 16, we can support this inference quantitatively by observing the greater Sharpe Ratio of the portfolios for the WCVaR model vis-à-vis the CVaR model.

Finally, the comparative analysis on the basis of the average Sharpe Ratio for the simulated data with 1000 samples is presented in Table 17. Accordingly, the results corresponding to the empirical study for $l = 5$ are presented in Figure 12 and Table 18. As observed from the plot of the Sharpe Ratio in Figure 12, the Worst-Case CVaR model performs superior with respect to the base CVaR model almost entirely for $\epsilon \in (0, 0.1)$. This observation is supported by the results presented in Table 18.

4 Discussion

We now present the discussion.

4.1 Robust Optimization in VaR Minimization

We discuss regarding the practical viability of incorporating robust optimization in VaR minimization from the point of view of number of stocks, sample size and types of data. For the sake of convenience, we tabulate all the results obtained in preceding Sections in Table 19, where for a particular scenario, we tabulated the average Sharpe Ratio obtained over the chosen range of ϵ.

4.1.1 From the Standpoint of Number of Stocks

From Table 19, we observe a common inference that irrespective of the type of the data, the WVaR model exhibits superior (inferior) performance than the base VaR model in case of larger number of stocks ($N = 98$) (in case of smaller number of stocks ($N = 31$)). The qualitative argument for this kind of behaviour cannot be attributed to the diversification of the portfolio because at times, VaR may not be sub-additive. We justify this behaviour on the lines of Michaud [19] and Ghaoui et al. [11]. As Michaud points out, the errors in the estimation of mean and covariances of the asset returns accumulates as the number of stocks increase. This implies that the data uncertainty in case of larger number of stocks is more in comparison to smaller number of stocks. The WVaR can handle the data uncertainty in a better way than the base VaR model [11]. Therefore, we observe an increment in the Sharpe Ratio for both the models, but the increment is more in case of WVaR model so that it outperforms the base VaR in all types of data environments. In the scenario, where the simulated data is taken into consideration, the same argument explains the behaviour, as the estimated moments' pair are used as true moments' pair, for the generation of the data. So the error in the estimation of the mean and the variance of the asset returns effects the behaviour in the same way as in the market data.

4.1.2 From the Standpoint of Number of Simulations

When we observe the results from the perspective of number of simulations, we can draw some insightful conclusions. For the case $N = 98$, the better performance of WVaR model can be attributed to the reason above. But when $N = 31$, one can observe that the performance of the optimal portfolio, when 1000 samples were simulated is better than that of the portfolio obtained when ζ number of samples were simulated. The reason for this sort of behaviour lies in the subroutine of the Non Parametric Bootstrap Algorithm, where we use sampling with replacement. Therefore more the number of samples, better are the bounds that one can obtain from the algorithm. So when we generate more number of samples (1000), the WVaR model which uses these bounds, performs better than the case where ζ number of samples are used for computing the bounds. In this setup, the increment in the number of simulation transits the robust portfolio from under-performing (ζ case) to be at par with the portfolio obtained from base VaR model.

4.1.3 From the Standpoint of Type of the Data

As explained in the previous section, when $N = 31$, the equivalent performance of the WVaR model with base VaR model in case of simulated data can be attributed to the following reason: The real market data is difficult to model and may not follow any distribution, whereas the simulated data follows the multivariate normal distribution with their true moments' pair as the tuple of estimated mean and covariance matrix of the asset returns from the real market data. Therefore, the base VaR model exhibits superior performance in the case of real market data. The reason for the out-performance of WVaR model over the base VaR model when $N = 98$ stocks have already been discussed in the preceding Sections.

4.2 Robust Optimization in CVaR Minimization

Similar to VaR minimization, we conduct the performance analysis of the WCVaR with respect to its base case counterpart from different standpoints. Table 20 presents the average Sharpe Ratio of the base CVaR and the WCVaR models for each scenario based on the chosen value of l, as per the methodology discussed in the preceding Section.

4.2.1 From the Standpoint of Number of Stocks

We begin with a discussion of the results presented in Table 20 from the standpoint of number of stocks. For the case involving market data, we observe that the CVaR models performs better than the WCVaR model in the scenario of less number of stocks ($N = 31$). Even after increasing N to 98, we draw the same inference. The reason behind this trend can be attributed to the lack of knowledge regarding the distribution of the returns in the real market data. Computation of CVaR assumes that the probability distribution is perfectly known and optimization of WCVaR is based on the assumption of the return distribution belonging to a mixture of some prior likelihood distributions. Since the market data hardly follows any distribution, so, there is a sense of ambiguity associated with optimizing CVaR and WCVaR using the discrete sampling technique (discussed in the preceding Section) for the market data.

On the other hand, we observe an expected trend for the case of simulated data with 1000 samples. For $N = 31$, the WCVaR model performs at par with the VaR model. As N increases to 98, the WCVaR model exhibits superior performance vis-à-vis the CVaR model. Since CVaR and WCVaR are coherent risk measures, so, increase in the number of stocks enhances diversification. As a result, an uptrend is observed in the performance of these two models for $N = 98$. But, WCVaR, being a robust risk measure, diversifies over worst-case scenarios as well through mixture distribution uncertainty, as defined in equation (2.14). As a result, the WCVaR model performs better than the CVaR model when larger number of stocks are taken into consideration.

However, in the case of simulated data with ζ samples, an unusual trend is observed, despite involving comparative inference similar to the previous case. On increasing N to 98, we observe a decline in the performance of the CVaR and WCVaR models. The reason behind such observation is not obvious.

4.2.2 From the Standpoint of Number of Simulations

We now compare the performance of the two models based on the number of simulated samples. From Table 20, for the case involving ζ simulations, we infer that the WCVaR model performs at par with the CVaR model irrespective of the number of stocks. Same inference can be drawn with 1000 simulated samples as well. Hence, we note equivalent performance of the two models in each simulation study.

4.2.3 From the Standpoint of Type of the Data

Due to similar reasons as in VaR minimization, an opposite trend is observed in the case of real market data (as discussed above) when the number of stocks is less ($N = 31$). Similar observation is inferred for the market data on taking into account the larger number of stocks ($N = 98$).

5 Conclusion

Akin to mean variance analysis, there is a problem of lack of robustness in the classical formulations of VaR and CVaR minimization. We discuss and assess the performance of the robust counterparts for these optimization problems that have been formulated to address this concern. Motivated by the results by Ghaoui et al. [11], we formulate the worst case robust version of the VaR model using separable uncertainty set. Regardless of the type of the data, be it from real market or from a simulated environment, we observe favourable results for the worst case VaR model with Sharpe ratio as the performance measure when the portfolio comprises higher number of stocks.

In contrast to the results reported by Zhu [24], we observe that the base case CVaR performs better than the robust counterpart (formulated by incorporating mixture distribution uncertainty) in the case of Market data irrespective of the number of stocks comprising in the optimal portfolio. This could be attributed to the following two reasons:

- Incorporation of different weight constraints in our optimization problem.
- In contrast to [24], our work uses Sharpe Ratio as a performance measure.

On the other hand, in the case of simulated data, we draw a favourable inference by noting superior or equivalent performance of the WCVaR vis-a-vis the base CVaR. In accordance with these results, we advocate for consideration of worst case models as a viable alternative to their classical counterparts especially in the case of higher number of stocks and in a simulated environment.

References

[1] Reserve bank of india - weekly statistical supplement. `https://www.rbi.org.in/scripts/WSSViewDetail.a`

[2] Yahoo finance. `https://in.finance.yahoo.com/`.

[3] C. Acerbi and D. Tasche. On the coherence of expected shortfall. *Journal of Banking & Finance*, 26(7):1487–1503, 2002.

[4] P. Artzner, F. Delbaen, J. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

[5] D. Bertsimas and I. Popescu. Optimal inequalities in probability theory: A convex optimization approach. *SIAM Journal on Optimization*, 15(3):780–804, 2005.

[6] F. Black and R. Litterman. Global portfolio optimization. *Financial Analysts Journal*, 48(5):28–43, 1992.

[7] M. Broadie. Computing efficient frontiers using estimated parameters. *Annals of Operations Research*, 45(1):21–58, Dec 1993.

[8] M. J. Capiński and E. Kopp. *Portfolio theory and risk management*, volume 5. Cambridge University Press, 2014.

[9] V. K. Chopra and W. T. Ziemba. The effect of errors in means, variances, and covariances on optimal portfolio choice. *The Journal of Portfolio Management*, 19(2):6–11, 1993.

[10] F. J. Fabozzi, P. N. Kolm, D. Pachamanova, and S. M. Focardi. *Robust Portfolio Optimization and Management*. Wiley, 2007.

[11] L. E. Ghaoui, M. Oks, and F. Oustry. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. *Operations Research*, 51(4):543–556, 2003.

[12] X. Ju and N. D. Pearson. Using value-at-risk to control risk taking: how wrong can you be? *OFOR Working Paper Series, no. 98-08*, 1998.

[13] J. H. Kim, W. C. Kim, and F. J. Fabozzi. Recent developments in robust portfolios with a worst-case approach. *Journal of Optimization Theory and Applications*, 161(1):103–121, 2014.

[14] C. Lim, H. D. Sherali, and S. Uryasev. Portfolio optimization by minimizing conditional value-at-risk via nondifferentiable optimization. *Computational Optimization and Applications*, 46(3):391–415, 2010.

[15] T. J. Linsmeier and N. D. Pearson. Risk measurement: An introduction to value at risk. Technical report, 1996.

[16] Z. Lu. A new cone programming approach for robust portfolio selection. *Optimization Methods and Software*, 26(1):89–104, 2006.

[17] H. M. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[18] H. M. Markowitz. *Portfolio Selection: Efficient Diversification of Investments*. Yale University Press, 1959.

[19] R. O. Michaud. The markowitz optimization enigma: Is 'optimized' optimal? *Financial Analysts Journal*, 45(1):31–42, 1989.

[20] G. Pflug. Some remarks on the value-at-risk and the conditional value-at-risk, probabilistic constrained optimization: Methodology and applications. *Kluwer Academic Publishers*, 38:272–281, 2000.

[21] R. T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *Journal of Risk*, 2:21–42, 2000.

[22] R. T. Rockafellar and S. Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7):1443–1471, 2002.

[23] R. H. Tütüncü and M. Koenig. Robust asset allocation. *Annals of Operations Research*, 132(1-4):157–187, 2004.

[24] S. Zhu and M. Fukushima. Worst-case conditional value-at-risk with application to robust portfolio management. *Operations Research*, 57(5):1155–1168, 2009.

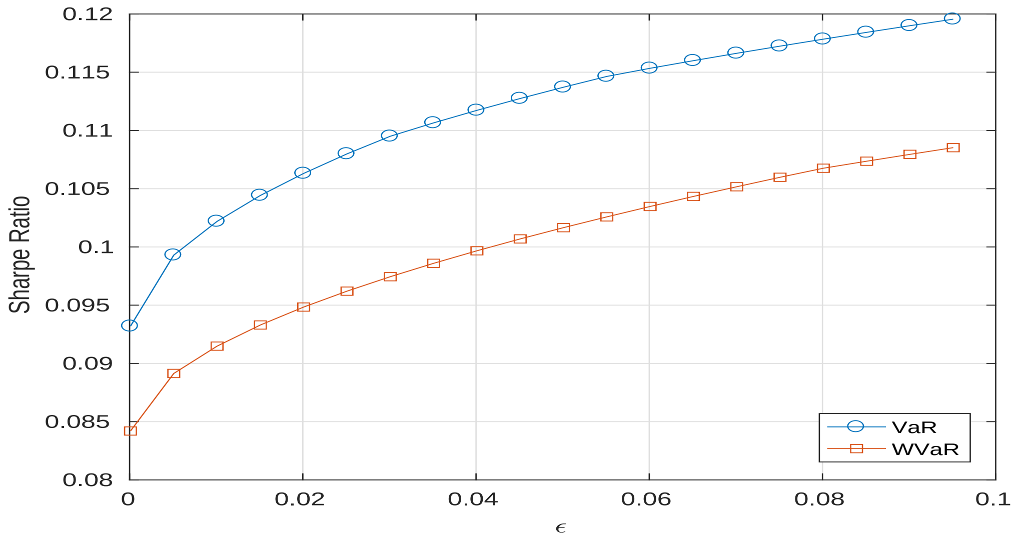

Figure 1: Sharpe ratio plot for Base VaR and WVaR models in case of Market data (31 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000646	0.00522	0.000603	0.00528	0.0932	0.0839
0.0201	0.000715	0.00522	0.00066	0.00529	0.106	0.0947
0.0401	0.000744	0.00523	0.000687	0.0053	0.112	0.0995
0.0601	0.000763	0.00523	0.000708	0.0053	0.115	0.103
0.0801	0.000777	0.00524	0.000726	0.00531	0.118	0.107
				Avg	0.111	0.0998

Table 1: Empirical Analysis of Base VaR and WVaR models in case of Market Data (31 assets)

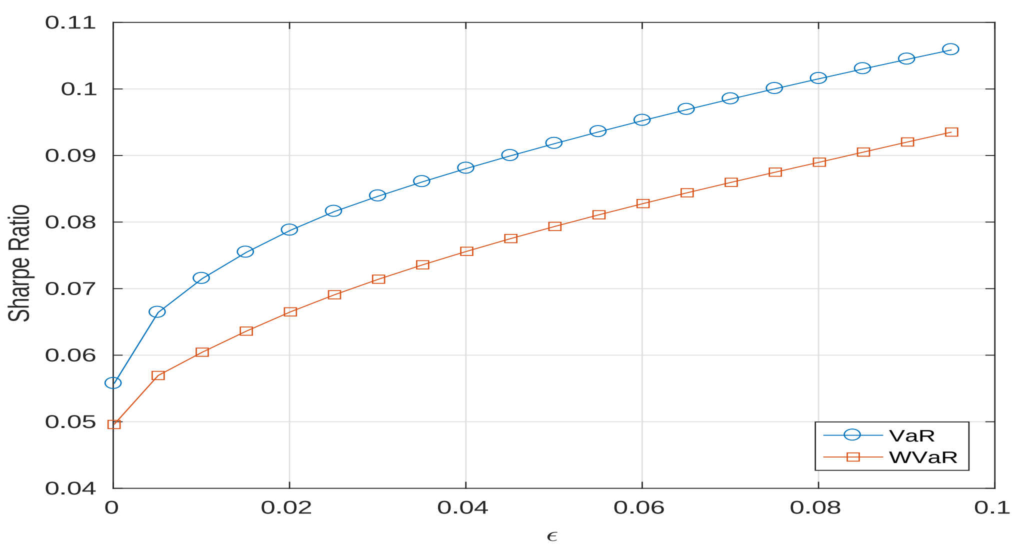

Figure 2: Sharpe ratio plot for Base VaR and WVaR models in case of simulated data with ζ number of samples (31 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000444	0.00509	0.000417	0.00519	0.0557	0.0496
0.0201	0.000562	0.0051	0.000506	0.0052	0.0788	0.0665
0.0401	0.00061	0.00511	0.000554	0.00521	0.0881	0.0756
0.0601	0.000647	0.00512	0.000592	0.00522	0.0953	0.0828
0.0801	0.000681	0.00513	0.000625	0.00523	0.102	0.089
				Avg	0.0884	0.0765

Table 2: Empirical Analysis of Base VaR and WVaR models in case of simulated data with ζ number of samples (31 assets)

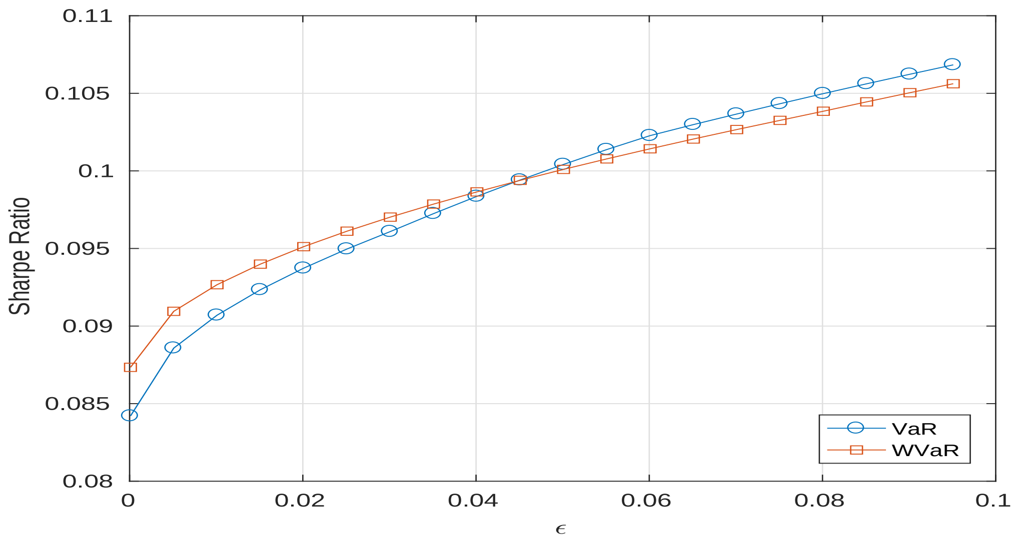

Figure 3: Sharpe ratio plot for Base VaR and WVaR models in case of simulated data with 1000 samples (31 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000587	0.00507	0.000605	0.0051	0.0842	0.0873
0.0201	0.000636	0.00508	0.000645	0.0051	0.0937	0.0951
0.0401	0.00066	0.00508	0.000663	0.00511	0.0984	0.0986
0.0601	0.00068	0.00509	0.000678	0.00511	0.102	0.101
0.0801	0.000694	0.00509	0.000691	0.00512	0.105	0.104
				Avg	0.0987	0.0989

Table 3: Empirical Analysis of Base VaR and WVaR models in case of simulated data with 1000 samples (31 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000667	0.00484	0.00071	0.00496	0.105	0.111
0.0201	0.000713	0.00485	0.000755	0.00497	0.114	0.12
0.0401	0.000735	0.00485	0.000776	0.00498	0.119	0.124
0.0601	0.000751	0.00486	0.000793	0.00499	0.122	0.127
0.0801	0.000765	0.00486	0.000807	0.005	0.124	0.13
				Avg	0.119	0.124

Table 4: Empirical Analysis of Base VaR and WVaR models in case of market data (98 assets)

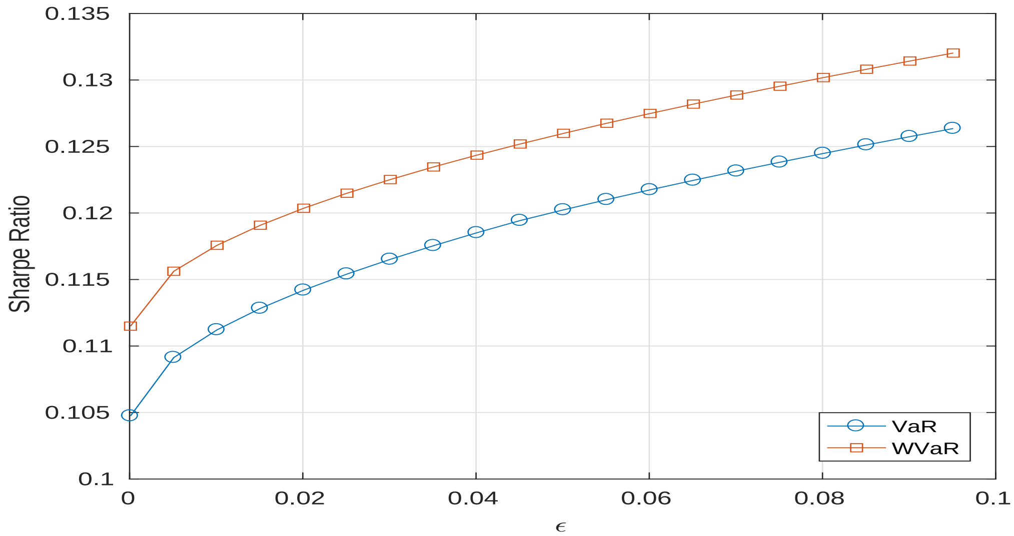

Figure 4: Sharpe ratio plot for Base VaR and WVaR models in case of market data (98 assets)

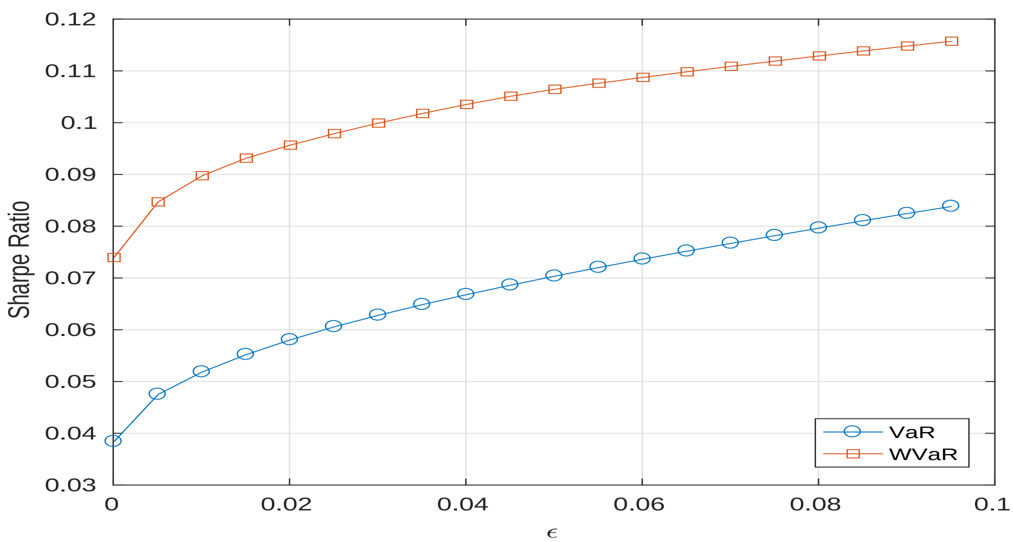

Figure 5: Sharpe ratio plot for Base VaR and WVaR models in case of simulated data with ζ number of samples (98 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000341	0.00471	0.00052	0.00487	0.0384	0.074
0.0201	0.000434	0.00472	0.000629	0.00491	0.0581	0.0957
0.0401	0.000475	0.00473	0.00067	0.00493	0.0668	0.104
0.0601	0.000508	0.00473	0.000697	0.00494	0.0736	0.109
0.0801	0.000537	0.00474	0.000719	0.00495	0.0796	0.113
				Avg	0.0674	0.103

Table 5: Empirical Analysis of Base VaR and WVaR models in case of simulated data with ζ number of samples(98 assets)

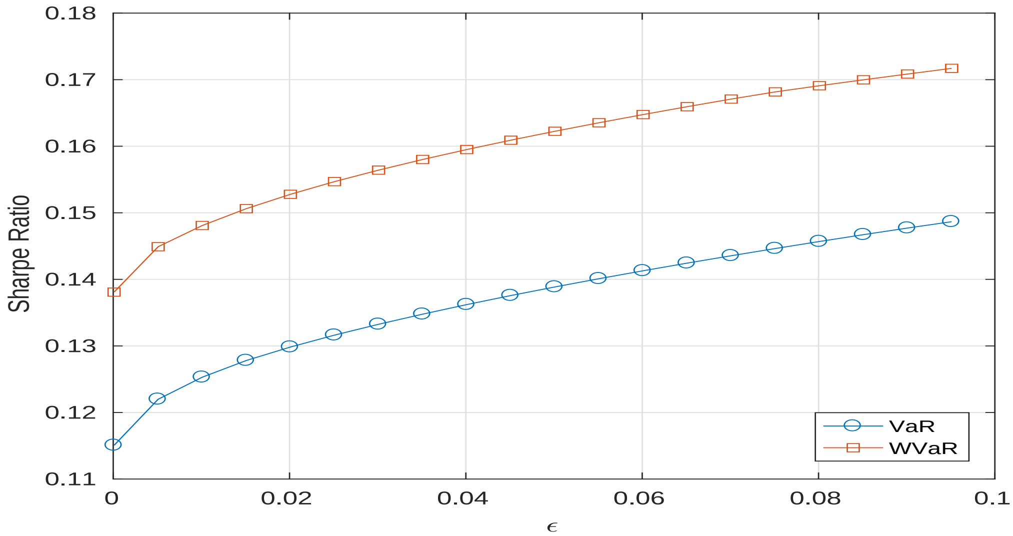

Figure 6: Sharpe ratio plot for Base VaR and WVaR models in case of simulated data with 1000 samples (98 assets)

ϵ	μ_{VaR}	σ_{VaR}	μ_{WVaR}	σ_{WVaR}	SR_{VaR}	SR_{WVaR}
0.0001	0.000702	0.00471	0.000822	0.0048	0.115	0.138
0.0201	0.000772	0.00472	0.000897	0.00483	0.13	0.153
0.0401	0.000803	0.00472	0.000931	0.00484	0.136	0.16
0.0601	0.000828	0.00473	0.000959	0.00485	0.141	0.165
0.0801	0.00085	0.00474	0.000982	0.00486	0.146	0.169
				Avg	0.137	0.16

Table 6: Empirical Analysis of Base VaR and WVaR models in case of simulated data with 1000 samples (98 assets)

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.0856	0.0611	-0.0245
3	0.0856	0.0345	-0.0511
4	0.0856	0.0439	-0.0417
5	0.0856	0.031	-0.0546

Table 7: Comparison of CVaR and WCVaR in case of Market Data (31 assets) for different values of l

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000266	0.00661	0.000266	0.00661	0.0162	0.0162
0.0201	0.000545	0.00595	0.000266	0.00661	0.0648	0.0162
0.0401	0.000706	0.00569	0.000514	0.00603	0.096	0.0587
0.0601	0.000832	0.00557	0.000645	0.00598	0.121	0.0812
0.0801	0.000877	0.00576	0.000696	0.00597	0.125	0.0897

Table 8: Empirical Analysis of CVaR and WCVaR in case of Market Data (31 assets) for $l = 2$

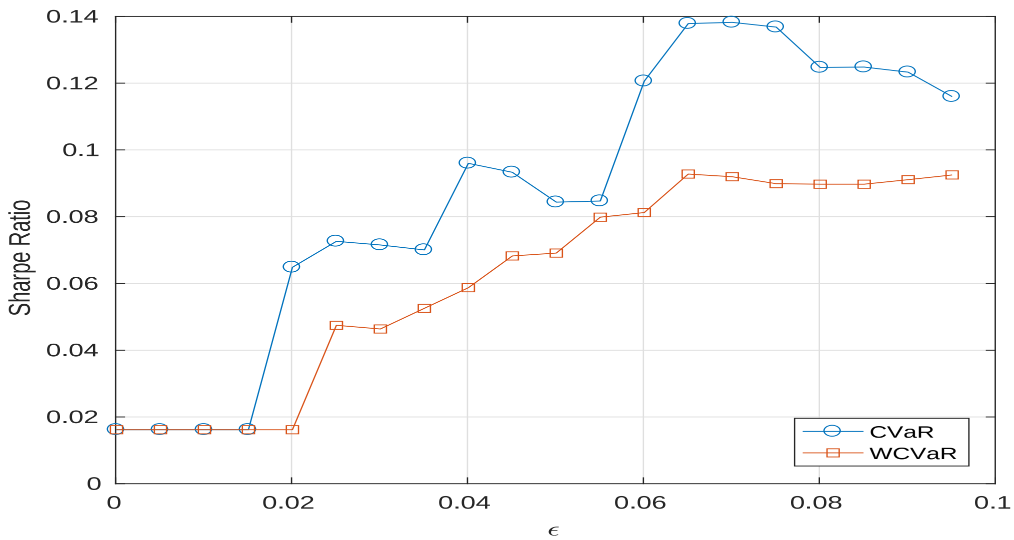

Figure 7: Sharpe ratio plot for CVaR and WCVaR in case of Market Data (31 assets) for $l = 2$

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.103	0.102	-0.000411
3	0.103	0.103	0.000195
4	0.103	0.104	0.00124
5	0.103	0.106	0.00358

Table 9: Comparison of CVaR and WCVaR in case of Simulated Data with ζ samples (31 assets) for different values of l

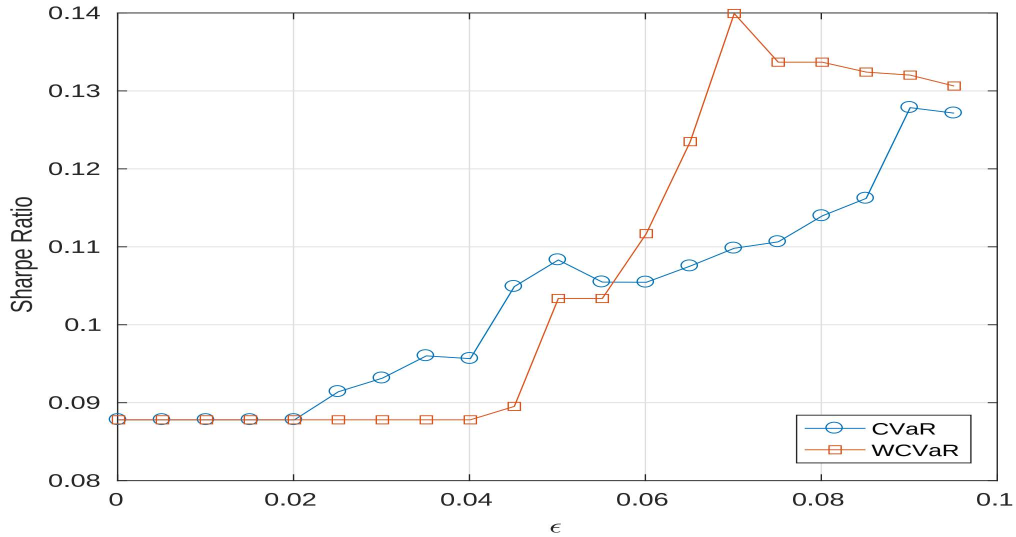

Figure 8: Sharpe ratio plot for CVaR and WCVaR in case of Simulated Data with ζ samples (31 assets) for $l = 5$

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000675	0.00587	0.000675	0.00587	0.0878	0.0878
0.0201	0.000675	0.00587	0.000675	0.00587	0.0878	0.0878
0.0401	0.000694	0.00559	0.000675	0.00587	0.0957	0.0878
0.0601	0.000731	0.00542	0.000838	0.00607	0.105	0.112
0.0801	0.000776	0.00541	0.000889	0.00545	0.114	0.134

Table 10: Empirical Analysis of CVaR and WCVaR in case of Simulated Data with ζ samples (31 assets) for $l = 5$

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.0929	0.0967	0.00374
3	0.0929	0.0945	0.00161
4	0.0929	0.0969	0.00402
5	0.0929	0.0954	0.00249

Table 11: Comparison of CVaR and WCVaR in case of Simulated Data with 1000 samples (31 assets) for different values of l

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000677	0.00563	0.000677	0.00563	0.0919	0.0919
0.0201	0.000658	0.0054	0.000661	0.0054	0.0923	0.0929
0.0401	0.000629	0.00534	0.000722	0.00531	0.0879	0.106
0.0601	0.00067	0.00523	0.000667	0.00526	0.0976	0.0965
0.0801	0.000701	0.0052	0.000692	0.00519	0.104	0.103

Table 12: Empirical Analysis of CVaR and WCVaR in case of Simulated Data with 1000 samples (31 assets) for $l = 4$

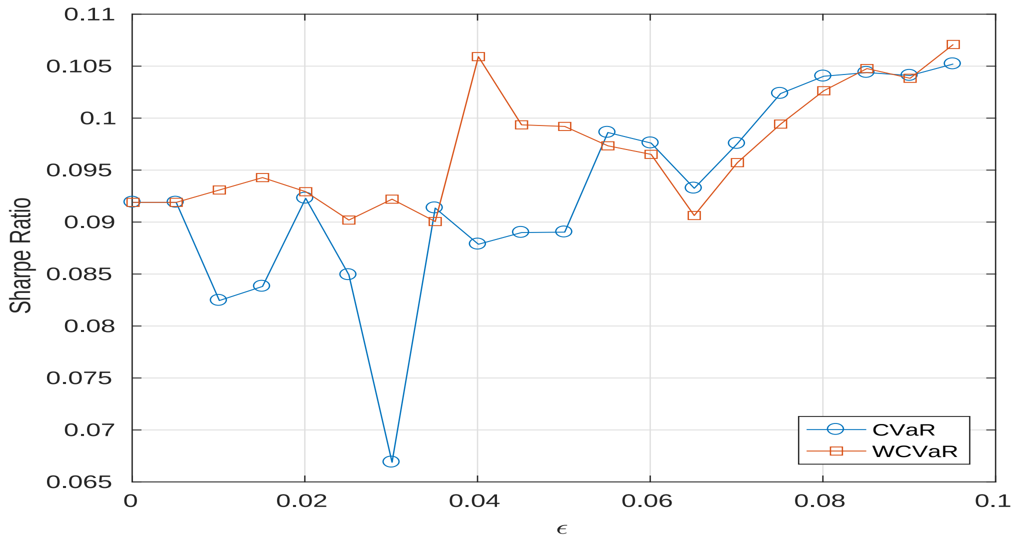

Figure 9: Sharpe ratio plot for CVaR and WCVaR in case of Simulated Data with 1000 samples (31 assets) for $l = 4$

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.121	0.102	-0.0189
3	0.121	0.105	-0.0165
4	0.121	0.0991	-0.0219
5	0.121	0.0927	-0.0283

Table 13: Comparison of CVaR and WCVaR in case of Market Data (98 assets) for different values of l

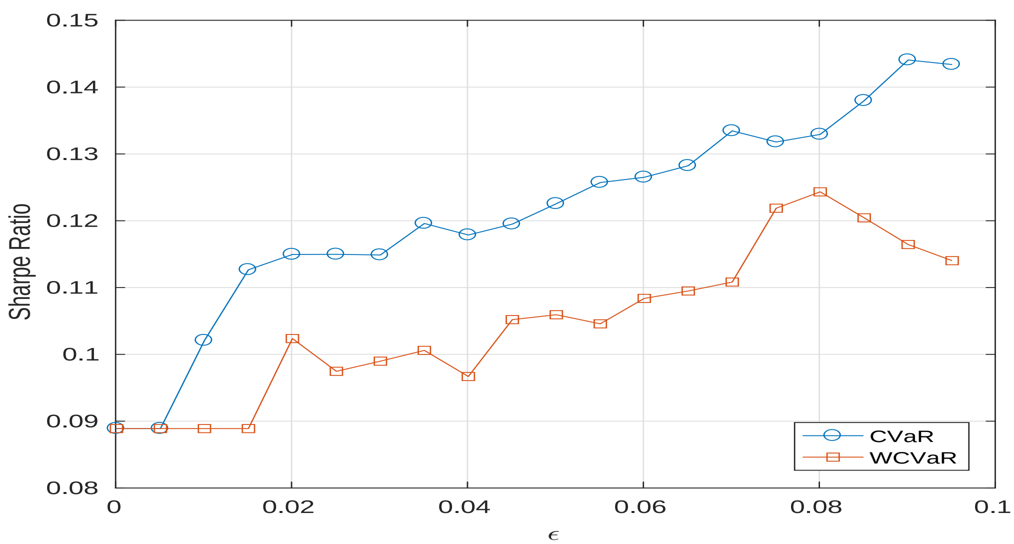

Figure 10: Sharpe ratio plot for CVaR and WCVaR in case of Market Data (98 assets) for $l = 3$

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000687	0.00593	0.000687	0.00593	0.0889	0.0889
0.0201	0.000786	0.00544	0.000755	0.00582	0.115	0.102
0.0401	0.00079	0.00535	0.000692	0.0055	0.118	0.0967
0.0601	0.000839	0.00537	0.000738	0.00533	0.127	0.108
0.0801	0.000847	0.00517	0.00082	0.00531	0.133	0.124

Table 14: Empirical Analysis of CVaR and WCVaR in case of Market Data (98 assets) for $l = 3$

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.0963	0.0978	0.00156
3	0.0963	0.0968	0.000541
4	0.0963	0.102	0.00581
5	0.0963	0.0915	-0.00479

Table 15: Comparison of CVaR and WCVaR in case of Simulated Data with ζ samples (98 assets) for different values of l

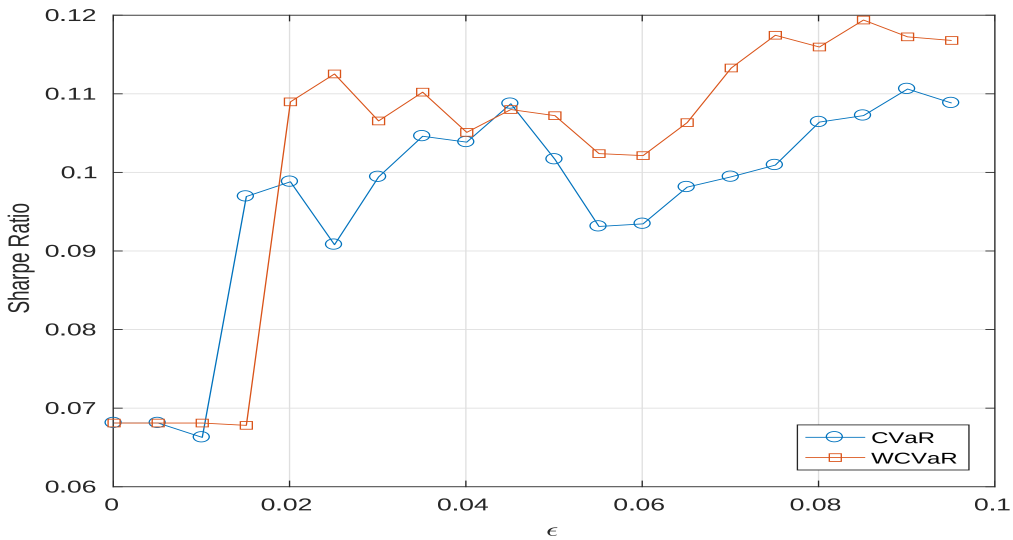

Figure 11: Sharpe ratio plot for CVaR and WCVaR in case of Simulated Data with ζ samples (98 assets) for $l = 4$

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000554	0.00579	0.000554	0.00579	0.0681	0.0681
0.0201	0.000703	0.0055	0.000755	0.00547	0.0988	0.109
0.0401	0.000727	0.00546	0.000723	0.00536	0.104	0.105
0.0601	0.000648	0.00522	0.000698	0.00527	0.0935	0.102
0.0801	0.000706	0.00514	0.000768	0.00525	0.106	0.116

Table 16: Empirical Analysis of CVaR and WCVaR in case of Simulated Data with ζ samples (98 assets) for $l = 4$

l	$Avg.\ SR_{CVaR}$	$Avg.\ SR_{WCVaR}$	$Diff.\ in\ Avg.\ SR$
2	0.156	0.156	-0.000382
3	0.156	0.16	0.0037
4	0.156	0.163	0.00667
5	0.156	0.165	0.00868

Table 17: Comparison of CVaR and WCVaR in case of Simulated Data with 1000 samples (98 assets) for different values of l

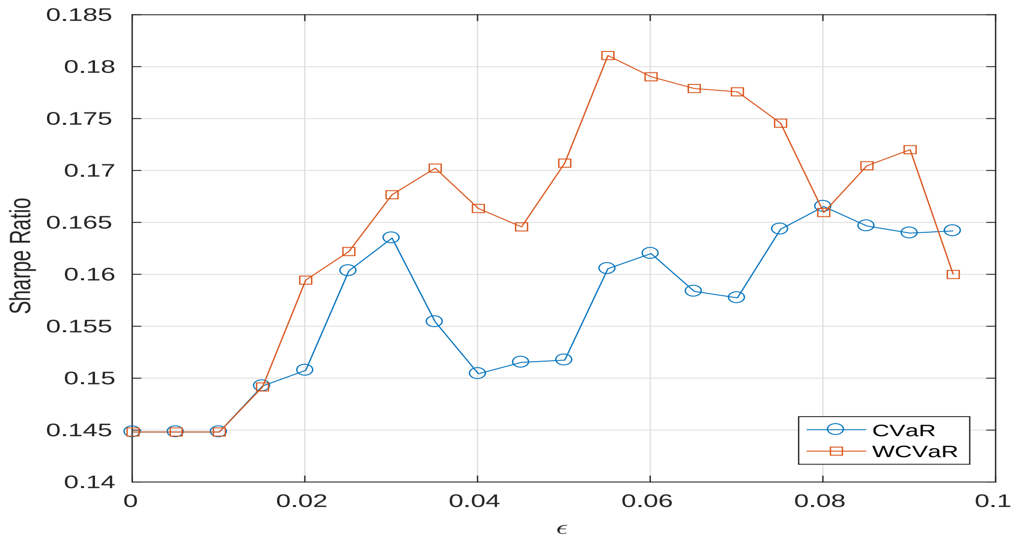

Figure 12: Sharpe ratio plot for CVaR and WCVaR in case of Simulated Data with 1000 samples (98 assets) for $l = 5$

ϵ	μ_{CVaR}	σ_{CVaR}	μ_{WCVaR}	σ_{WCVaR}	SR_{CVaR}	SR_{WCVaR}
0.0001	0.000935	0.00536	0.000935	0.00536	0.145	0.145
0.0201	0.00097	0.00537	0.00101	0.00536	0.151	0.159
0.0401	0.000927	0.0051	0.00104	0.00531	0.15	0.166
0.0601	0.000963	0.00496	0.00106	0.00504	0.162	0.179
0.0801	0.000974	0.00489	0.000981	0.00495	0.167	0.166

Table 18: Empirical Analysis of CVaR and WCVaR in case of Simulated Data with 1000 samples (98 assets) for $l = 5$

N	$N = 31$			$N = 98$		
Type of data	Market data	Sim. data ζ samples	Sim. data 1000 samples	Market data	Sim. data ζ samples	Sim. data 1000 samples
VaR	0.111	0.0884	0.0987	0.119	0.0674	0.137
WVaR	0.0998	0.0765	0.0989	0.124	0.103	0.16

Table 19: Comparison of the average Sharpe ratio for the VaR and WVaR models in various scenarios.

N	$N = 31$			$N = 98$		
Type of data	Market data	Sim. data ζ samples	Sim. data 1000 samples	Market data	Sim. data ζ samples	Sim. data 1000 samples
CVaR	0.0856	0.103	0.0929	0.121	0.0963	0.156
WCVaR	0.0611	0.106	0.0969	0.105	0.102	0.165

Table 20: Comparison of the average Sharpe ratio for the CVaR and WCVaR models in various scenarios.