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Chapter 5

VaR & its robust formulation

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~~5.1 Introduction~~

preceding chapter, we observed several limitations mean-variance framework, such as

In the ab~~ove~~ ~~sections~~, ~~the main limitations~~ of the ~~Mean Variance like~~ sensitivity to errors in data ~~and in~~ as well in the estimation of mean and variance of the underlying distribution. ~~However~~ ~~In addition~~ Furthermore, another criticism usually associated with the ~~Markowitz~~ mean-variance setup is the use of standard deviation as a measure of risk. On the other hand, from a practitioners' point of view, the upside and downside risk are not ~~considered the same~~, ~~because~~ most of the times, the upside risk can improve the overall performance of the portfolio, whereas the downside fluctuation usually brings impactful losses. Variance ~~can't be treated~~ is not very reliable as an ~~apt risk~~ appropriate measure, if the underlying distribution~~s~~ ~~are~~ is leptokurtic. In order to address these issues, models involving other measures of risk have been developed and accordingly their corresponding robust models have been studied as well. In the ~~upcoming~~ forthcoming chapters, we will ~~thoroughly~~ elaborately discuss some of the most ~~popular risk~~ widely used risk measures like Value at Risk (VaR) and Conditional Value at Risk (CVaR) and also study their robust corresponding worst case formulations.

In this chapter, we start with the definition of VaR and formulate the

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associated with

optimisation problems for ~~the computation of~~ VaR. Furthermore, we incorporate separable uncertainty set to model the worst case formulation. Next, we analyze and compare the performance of both VaR and Worst case VaR (WVaR) with respect to S&P BSE 100 and S&P BSE 30. Finally, we conclude with insightful comments and conclusions.

relevant .

5.2 Value at Risk (VaR)

the mean-variance takes into account

Unlike ~~Mean - Variance~~ setup, the VaR framework [29] uses the probability of losses ~~into account~~. Ghaoui, Oks and Oustry [22] defined VaR ~~value at risk~~ as the minimum value of γ such that the probability of loss exceeding γ is less than ϵ.

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$$V(\mathrm{x}) = \min \gamma \quad \text{such that} \quad P\{\gamma \le -r(\mathrm{x}, \mu)\} \le \epsilon, \tag{5.1}$$

\min \left\{ \gamma : P[\gamma \le ...] \right\}

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where $\epsilon \in (0, 1]$. When we deal with ~~Markowitz~~ the mean variance setup, only the mean and the variance *i.e.*, the *first and second moments* of the asset returns are required but in VaR framework, the knowledge of entire distribution is necessary for the ~~purpose of~~ computation part. If the underlying distribution ~~in~~ is Gaussian with the moments' pair as ($\hat{\mu}$, Σ), then VaR can be computed ~~via this~~ using the following analytical form:

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$$V(x) = \kappa(\epsilon)\sqrt{\mathrm{x}^\top \Sigma \mathrm{x}} - \hat{\mu}^\top \mathrm{x}, \tag{5.2}$$

$\cancel{= P(Z \le x)}$).

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where $\kappa(\epsilon) = -\Phi^{-1}(\epsilon)$ and $\Phi(z)$ ($\cancel{= P(Z \le z)}$) represents the cumulative distribution for standard normal random variable. Similarly if the distribution is known and popular then we can find the inverse cumulative function. However, if ~~If~~ the distribution ~~s are~~ is unknown, then we have to rely on the Chebyshev's inequality. The bound (upper) obtained due to ~~by the~~ Chebyshev's only requires the knowledge of the first two moments' pair. We call a bound to be exact if the upper bound is computationally tractable. If not, we use the bound given by Bertsimas and Popescu [8], i.e., $\kappa(\epsilon) = \sqrt{\frac{1-\epsilon}{\epsilon}}$. $\displaystyle{\kappa(\epsilon) = \dots}$

Finally, we formulate the generalised VaR as follows:

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$$\min \kappa\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}} - \hat{\mu}^\top\mathbf{x} \quad \text{subject to} \quad \mathbf{x} \in \mathcal{X}, \tag{5.3}$$

\mathcal{H}

where κ is an appropriate factor of risk chosen in accordance with ~~according to~~ the underlying distribution of asset returns and $\mathcal{H} = \{\mathbf{x} : \mathbf{x}^\top\mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0\}$. The function $V(x)$ is convex in w and the global optimum can be obtained using ~~via~~ techniques like interior-point methods and second order cone programming (SOCP).

Though VaR takes probability of losses into account, it has its own limitations. For the computation part, it requires the knowledge of the whole distribution. The computation also involves high dimensional numerical integration which may not be tractable at times and also not much ~~of a study~~ ~~is done in~~ using Monte Carlo simulations [29] for the design of the portfolio. Black and Litterman [12], Pearson and Ju [26] discussed the issues regarding the computational difference between the true VaR and the calculated VaR and find determined ~~out that~~ the error ~~mainly creeps in due~~ to the errors in the estimation of the first and second moments of the asset returns, in the computation of VaR can be attributed.

5.3 Worst Case VaR

The concept of worst-case VaR not only allows to approach the solution in a more tractable way, but also relaxes ~~loosens~~ the assumptions on the information known to us ~~beforehand~~. Here, we assume that only partial information about the underlying apriori distribution is known. We assume also that the distribution of the

\mathcal {P}.

\mathcal {P}

asset returns belong to a family of allowable probability distributions $\mathcal{P}$. For example of ~~e.g.~~ given component wise bounds of $(\hat{\mu}, \Sigma)$, $\mathcal{P}$ matched could comprise of normally distributed random variables with $\hat{\mu}$ and Σ as the moments' pairs.

Given a probability (confidence) level ϵ, the worst-case VaR can be formulated as   Space

$$V_{\mathcal{P}}(\mathbf{x}) = \min \gamma \quad \text{such that} \quad \sup_{P \in \mathcal{P}} P\{\gamma \leq -r(\mathbf{x}, \mu)\} \leq \epsilon, \tag{5.4}$$

and accordingly, the robust formulation can be written as   mathcal

$$V_{\mathcal{P}}^{\text{opt}}(\mathbf{x}) = \min V_{\mathcal{P}}(\mathbf{x}) \quad \text{subject to} \quad \mathbf{x} \in \mathcal{X}, \tag{5.5}$$

The above optimization problems can be computed by a semidefinite programming (SDP) problem which again uses the above mentioned interior-point methods. We deal with the high dimensional problems by using bundle methods which are mainly used for large-scale (sparse) problems.

5.3.1 Polytopic Uncertainty

Motivated by ~~By taking inspiration from~~ the work of separable uncertainty sets by Tütüncü and Koenig [40], one can view the robust formulation given by Ghaoui, Oks and Oustry [22], in case of Polytopic uncertainty as a robust formulation of WVaR involving separable uncertainty. The formulation of the optimization problem for worst case VaR as follows:

$$\min \kappa(\epsilon)\sqrt{\mathbf{x}^{\mathsf{T}}\hat{\Sigma}\mathbf{x}} - \underline{\hat{\mu}}^{\mathsf{T}}\mathbf{x} \quad \text{subject to} \quad \mathbf{x} \in \mathcal{X}, \tag{5.6}$$

where $\overline{\Sigma}$ and $\underline{\hat{\mu}}$ are higher bound for covariance matrix and lower bound for the estimated mean of asset returns. We obtain these values using Non-

Which is mainly used in case of

Parametric Bootstrap algorithm ~~where the~~ type of distribution ~~is~~ unknown.

being

This can also be viewed as a robust formulation involving polytopic uncertainty as "Separable uncertainty" is a special case when dealing with "Polytopic uncertainty". The more complicated models involve Ellipsoidal uncertainty sets into account. The robust worst-case VaR formulation with Ellipsoidal uncertainty sets is not that trivial and in literature, it is mostly used in the setup of Factor models.