1 Introduction

Investment in an individual security such as stock always has an associated risk. Such risk can be minimized through diversification that involves investing in a portfolio consisting of several securities. In accordance with this notion, Markowitz introduced the mean-variance model for optimum allocation of weights to the securities comprising a portfolio, utilizing mean and covariance matrix of returns of securities as measures for giving a quantitative sense to the return and the risk of portfolio. Significant research efforts have been made in introducing several variants of the Markowitz model and till date, improvements upon the model are being explored. Despite the theoretical simplicity of the model's approach based on risk-return trade-off, there are several drawbacks associated on incorporating it into practical use.

Theoretically, Markovitz based portfolio optimization can result in assigning extreme weights to the securities comprising a portfolio. However, investment in securities can't be made in such extreme positions like large short positions if we take into account the active trading. Such kind of scenarios can be avoided by introducing appropriate constraints on the weights in the Markowitz model. But, there is an added disadvantage as well since there are high chances of the optimal portfolio constructed lying in the neighborhood of the imposed constraints thus leading to strong dependence of the constructed portfolio upon the constraints. For example, on imposing constraint that disallows short sales, it often results in assigning zero weights to many securities that comprise the portfolio and largely positive weights to the securities having small market capitalizations.

One of the most significant demerits of the mean-variance model is the sensitivity issue associated with risk and return parameters of the individual securities of a portfolio. These parameters are computed by using expected return and standard deviation of returns as the estimates. While computing these estimates, historical data comprising returns is usually taken into account in order to calculate the sample mean and the sample variance that neglects various other market factors and isn't an accurate representation of estimates for future returns. Including above reasons in his arguments, Michaud points out that the mean-variance analysis tends to maximize the impact of estimation errors associated with the return and parameters for the securities. As a result, Markowitz portfolio optimization often leads to assigning larger weights than expected to the securities having higher expected return, lower variance of returns and negative correlation between their returns and smaller weights than expected to the securities in the opposite case. Labeling the model as 'estimation-error maximizers', he states that it often leads to financially meaningless portfolios which in some cases performs inferior in comparison to equal weighting. Broadie investigates the error maximization property of mean-variance analysis by conducting a simulation study to compare the estimated efficient frontier with the actual frontier computed using true parameter values. He observes that points on the estimated efficient frontier show superior performance as compared to the corresponding points on the actual frontier. He supports his argument of over-estimation of expected returns of optimal portfolios by the estimated frontier through his experiment results of obtaining the estimated frontier lying below the actual frontier. Additionally, he points out that non-stationarity in the data comprising returns can further increase the errors in computing the efficient frontier. Chopra and Ziemba perform the sensitivity analysis of performance of optimal portfolios by studying the impact of estimation errors in means, variances and covariances of returns on securities of portfolio individually, taking into consideration the investors' risk tolerance as well. They observe that at a high risk tolerance of around 50, cash equivalent loss for estimation errors in means is about eleven times greater than that for errors in variances or covariances. Accordingly, they point out that if the investors have superior estimates for means of security returns, they should prefer using them over the sample means calculated from historical data. Best and Grauer also came up with similar conclusions on studying the sensitivity of weights of optimal portfolios with respect to changes in estimated means of returns on individual securities. Further on imposing no short selling constraint on securities, they observe that a small change in estimated mean return of an individual security can assign zero weights to almost half the securities comprising a portfolio.

So, a common conclusion that can be derived from the discussed research works is that the optimal portfolios are extremely sensitive towards the estimated values of input parameters, especially expected values of returns on individual securities. In order to overcome this issue, there has been significant progress in recent years in the area of robust portfolio optimization. Several methods have been proposed in this area. We are particularly interested in the approaches falling in the category related to enhancing robustness by optimizing the portfolio performance in worst-case scenarios.

Significant efforts have been made towards formulating these kinds of approaches from Markowitz based mean-variance analysis. However, no concrete positive results have been observed that could establish these approaches at par with Markowitz optimization in terms of portfolio performance. Ceria and Stubbs introduce a methodology for robust portfolio optimization, taking into account the estimation errors in input parameters while formulating the optimization problem. The approach involves minimizing the worst case return for a given confidence region. They observe that the constructed robust portfolios perform superior in comparison to those constructed using mean-variance analysis in most of the cases but not in each month with certainty. Utilizing the above framework, Scherer shows that robust methods don't lead to significant change in the efficient set. Constructing an example, he shows that robust portfolio underperforms out of the sample in comparison to Markowitz portfolio, especially in the case of low risk aversion and high uncertainty aversion. He also argues that performance of robust portfolio is dependent upon the consistency between uncertainty aversion and risk aversion which is quite complicated. Santos performs similar experiments and concludes stating better performance of robust optimization in comparison to the portfolios constructed using mean-variance analysis incase of simulated data unlike the real market data.

2 Robust Optimization

All the real world optimizing problems inevitably have uncertain parameters embedded in them. In order to tackle such problems, a framework called Stochastic Programming is used, which can model such problems having uncertain parameters. These models take the probability distributions of the underlying data into consideration. To improve the stability of the solutions, robust methods such as re-sampling techniques, robust estimators and Bayesian approaches were developed. One of the approaches in Robust optimization, which is used when the parameters are known to be in certain bounds. In this section, we attempt to explain some robust models with worst-case optimization approaches for a given objective function within a predefined "uncertainty" set.

The concept of uncertainty sets was introduced by Soyster, where he uses a different definition for defining a feasible region of a convex programming problem. The convex inequalities are replaced by convex sets with a condition that the finite sum of convex sets again should be within another convex set. In another way, he defines a new linear programming problem with uncertain truth value, but it is bound to lie within a defined convex set. Later, El Ghaoui and Lebret extends these uncertainty sets to define a robust formulation while tackling the least-squares problem having uncertain parameters, but they are bounded matrices. In their work, they describe a problem of finding a worst-case residual and refer the solution as a robust least-squares solution and show that it can be computed via semi-definite or second order cone programming. El Ghaoui, Oustry, and Lebret further study how to integrate bounded uncertain parameters in semidefinite programming. They introduce robust-formulations for semidefinite programming and provide sufficient conditions to guarantee the existence of such robust solutions. Ben-Tal and Nemirovski mainly focus on the uncertainty related with *hard* constraints and which are ought to be satisfied, irrespective of the representation of the data, they suggest a methodology where they replace an actual uncertain linear programming problem by its robust counterpart. They show that the robust counterpart of a linear programming problem with the ellipsoidal uncertainty set is computationally attractive, as it reduces to a polynomial time solvable conic quadratic program. They also used interior points methods to compute efficiently, On the same lines, Goldfarb and Iyengar, focus on the robust convex quadratically constrained programs which are a subclass of the robust convex programs of Ben-Tal and Nemirovski. They mainly work on finding uncertainty sets which make this subclass of programs to be expressed as second-order cone programs.

In its earliest phase, the major directions of research were to introduce robust formulations and to build uncertainty sets for robust counterparts of the LPP as they are computationally attractive. As the basic framework of robust optimization is established, it is now applied across various domains such as learning, statistics, finance and numerous areas of engineering.