ROBUST PORTFOLIO OPTIMIZATION: A STUDY OF BSE30 AND BSE100

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Outline

Motivation

Drawbacks of Markowitz Portfolio Optimization

- Can assign extreme weights to the securities.
  1. Large negative weights (practically unviable).
  2. Zero weights to many securities and large positive weights to small-cap stocks (with no-shortselling constraints).
- Sensitivity issue with estimates of input parameters.
  1. "Estimation-error maximizers" property: Overweighs securities having higher estimated mean, lower variance and negative correlation between returns.
  2. Over-estimation of optimal portfolios: Estimated efficient frontier lies above the actual frontier.
  3. Estimation errors in means of asset returns have a greater impact on portfolio performance in comparison to that of errors in variances/covariances of asset returns.

Robust Portfolio Optimization

Class of methods proposed to enhance the robustness of Markowitz portfolios by optimizing the portfolio performance in worst-case scenarios. Most of the robust models deal with optimizing a given objective function with a predefined “uncertainty set” for obtaining computationally tractable solutions.

- Robust optimization involves defining the proper structure of uncertainty sets that will provide tractable solutions.
- Since the distribution of asset returns is not known with certainty, a general approach is to define some geometry around an estimate of the uncertain parameter.
- Empirically, the uncertain parameters are often estimated using historical returns.
- Even though there has been intensive study on the structure and geometry of uncertainty sets that are suitable for various optimization problems, we only cover a few types of uncertainty sets which are widely used in portfolio optimization.

A *polytopic* uncertainty set which resembles a "box" is defined as:

$$U_\delta(\hat{\mathbf{a}}) = \{\mathbf{a} : |a_i - \hat{a}_i| \leq \delta_i, i = 1, 2, 3, ..., N\}\,, \tag{1}$$

where $\mathbf{a} = (a_1, a_2, ..., a_N)$ is a vector of values of uncertain parameters of dimension N and $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_N)$ is generally the estimate for $\mathbf{a}$.

The robust formulation of the mean-variance model that focuses on optimizing the portfolio in the worst-case is expressed as:

$$\max_{\mathbf{x}} \left\{ \min_{\boldsymbol{\mu} \in U_{\boldsymbol{\delta}}(\hat{\boldsymbol{\mu}})} \boldsymbol{\mu}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma\, \mathbf{x} \right\} \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0, \tag{2}$$

where $\boldsymbol{\mu}$ represents the vector of expected returns, Σ represents the variance-covariance matrix, $\mathbf{x}$ stands for the vector of weights for an individual asset in the optimal portfolio, λ represents the risk-aversion of the investor and $\mathbf{1}$ represents the unity vector.

Box uncertainty in terms of expected returns can be expressed as

$$U_{\boldsymbol{\delta}}(\hat{\boldsymbol{\mu}}) = \{\boldsymbol{\mu} : |\mu_i - \hat{\mu}_i| \leq \delta_i, i = 1, 2, 3, ..., N\}, \tag{3}$$

where N represents the number of stocks and δ_i represents the value which determines the confidence interval region for individual assets. On incorporating the box uncertainty set, the formulation in (2) transforms to a maximization problem

$$\max_{\mathbf{x}} \quad \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} - \boldsymbol{\delta}^\top |\mathbf{x}| \quad \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0, \tag{4}$$

While dealing with the box uncertainty, it is assumed that the returns follow normal distribution. We define δ_i for $100(1-\alpha)\%$ confidence level as follows:

$$\delta_i = \sigma_i \, z_{\frac{\alpha}{2}} \, N^{-\frac{1}{2}}$$

Robust Models Using Uncertainty Sets

Ellipsoidal Uncertainty in Expected Returns (Without Short-Selling)

On the same lines, in this context, we define Ellipsoidal uncertainty sets in terms of expected returns as follows:

$$U_\delta(\hat{\boldsymbol{\mu}}) = \left\{ \boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^\top \Sigma_{\boldsymbol{\mu}}^{-1} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \leq \delta^2 \right\}, \tag{5}$$

where Σ_μ is the covariance matrix of the estimated errors in expected returns. Again, by this model, the optimization problem (2) transforms to

$$\max_{\mathbf{x}} \left\{ \hat{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \Sigma \mathbf{x} - \delta \sqrt{\mathbf{x}^\top \Sigma_\mu \mathbf{x}} \right\} \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{6}$$

For the same reason, if the uncertainty set follows ellipsoid model, the underlying distribution is assumed to be tracing a χ^2_N distribution. Accordingly, for $100(1-\alpha)\%$ confidence level, δ can be computed as:

$$\delta^2 = \chi^2_N(\alpha)$$

where $\chi^2_N(\alpha)$ is the inverse of a chi square distribution with N degrees of freedom.

In this type, we obtain the bounds for both the covariance matrix and the expected returns. $U_{\boldsymbol{\mu}} = \{\boldsymbol{\mu} : \underline{\boldsymbol{\mu}} \leq \boldsymbol{\mu} \leq \overline{\boldsymbol{\mu}}\}$, where $\underline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\mu}}$ represent lower and upper bounds on mean return vector $\boldsymbol{\mu}$ respectively . Lower bound $\underline{\Sigma}_{ij}$ and upper bound $\overline{\Sigma}_{ij}$ can be specified for each entry Σ_{ij} of the covariance matrix.

From this we transform the problem (2) to the following:

$$\max_{\mathbf{x}} \left\{ \underline{\boldsymbol{\mu}}^\top \mathbf{x} - \lambda \mathbf{x}^\top \overline{\Sigma} \mathbf{x} \right\} \text{such that } \mathbf{x}^\top \mathbf{1} = 1 \text{ and } \mathbf{x} \geq 0. \tag{7}$$

We obtain the bounds of both mean vectors and the covariance matrix by using non-parametric bootstrap algorithm.

- Analysis performed under two scenarios:
  1. Number of stocks $N = 31$
  2. Number of stocks $N = 98$
- For each scenario, following data used:
  1. Log-returns based on daily Adjusted Close Price of stocks comprising BSE30 for Scenario 1 and BSE100 for Scenario 2 (Yahoo Finance).
  2. Simulated Data with #samples for returns same as in market data.
  3. Simulated Data with #samples$= 1000$
- Sharpe Ratio used as performance measure for comparing robust models with Markowitz Model (Mark) with annualized risk-free rate assumed equal to 6%.
- Portfolio optimization within ideal range of risk-aversion ($\lambda \in [2, 4]$).
- Uncertainty sets constructed with 95% confidence level.

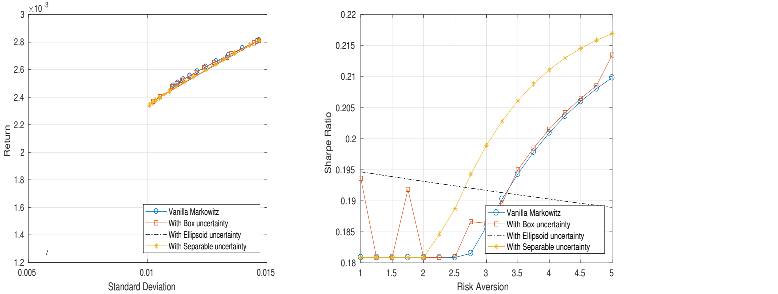

Figure: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Market Data (31 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.181	0.181	0.193	0.186
2.5	0.181	0.181	0.192	0.193
3	0.186	0.191	0.192	0.202
3.5	0.194	0.195	0.191	0.209
4	0.201	0.202	0.19	0.213
Avg	0.189	0.19	0.192	0.2

Table: Comparison of different portfolio optimization models in case of Market Data (31 assets)

Common observation inferred from three cases

Sep and Ellip model perform superior or equivalent in comparison to Markowitz model in the ideal range of risk-aversion.

Computational Results

Performance with $N = 98$ assets

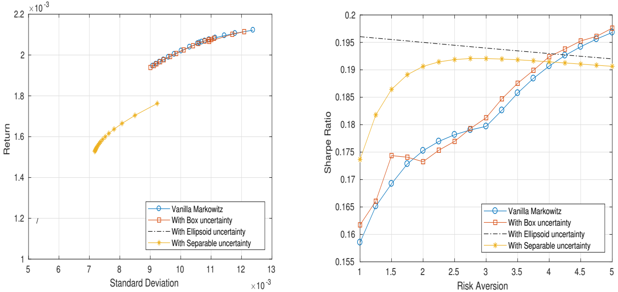

Figure: Efficient Frontier plot and Sharpe ratio plot for different portfolio optimization models in case of Market Data (98 assets)

λ	SR_{Mark}	SR_{Box}	SR_{Ellip}	SR_{Sep}
2	0.175	0.173	0.195	0.193
2.5	0.178	0.177	0.194	0.193
3	0.18	0.181	0.194	0.193
3.5	0.186	0.188	0.193	0.192
4	0.191	0.192	0.193	0.192
Avg	0.182	0.182	0.194	0.192

Table: Comparison of different portfolio optimization models in case of Market Data (98 assets)

Common observation inferred from three cases

Sep and Ellip model outperform the Markowitz model in the ideal range of risk-aversion.

Conclusions and Comments

From the Standpoint of Number of Stocks

	$\#$stocks $= 31$	$\#$stocks $= 98$
$\#$generated_simulations $= 1000$	0.2	0.244
$\#$generated_simulations $= \zeta$	0.218	0.233
Market data	0.2	0.194

Table: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

Qualitative argument for this observation

More the number of stocks, more is the diversification of the portfolio. This is the principle behind this type of observation. However, in market data, the Sharpe ratio in case of more number of stocks is less than that of the case with smaller number of stocks. We believe that this kind of behaviour is due to relatively low amount of data available for higher number of stocks.

Conclusions and Comments

From the Standpoint of Number of Samples Generated

	#samples = 1000	#samples = ζ
#stocks = 31	0.2	0.218
#stocks = 98	0.244	0.233

Table: The maximum average Sharpe ratio compared by varying the number of stocks in different kinds of scenarios.

Qualitative argument for this observation

We explain this type of behaviour as follows: In the available real market data, the number of instances available for larger number of stocks is relatively low. So, when more number of samples were generated, we observe higher Sharpe ratios when compared to ζ number of simulations. We are yet to explore the reason behind such type of behaviour when smaller number of stocks are considered.

Conclusions and Comments

From the Standpoint of Type of the Data

	Simulated data	Real Market data
#stocks = 31	0.218	0.2
#stocks = 98	0.244	0.194

Table: The maximum average Sharpe ratio compared by varying the type of the data in different kinds of scenarios.

Qualitative argument for this observation

Here the behaviour is straight forward. In both the cases, the performance in the case of simulated data is better than the real market data. This is clear from the fact that the real market data are difficult to model and hardly may follow any distribution, whereas the simulated data simply follows multivariate normal distribution with mean and covariances as the true values obtained from the data.

Take-Aways

- In contrast to the results reported by Scherer (used simulated data), we observe that robust optimization with ellipsoidal uncertainty set performs superior or equivalent as compared to Markowitz model in the case of simulated data as well as market data. This could be attributed to incorporation of no short-selling constraints in the optimization problem.
- Better performance of robust formulation having separable uncertainty set in comparison to Markowitz model is in line with the previous study on the same robust model by Tütüncü and Koenig.

Future Directions

- Extending robust optimization techniques to a framework involving multi-period investment.
- Incorporating coherent measures of risk like AVaR in the robust models

References

H. M. Markowitz. Portfolio selection. The journal of finance, 7(1):7791, 1952.

R. O. Michaud. The markowitz optimization enigma: Is optimized optimal? Financial Analysts Journal, 45(1):3142, 1989.

M. Broadie. Computing efficient frontiers using estimated parameters. Annals of Operations Research, 45(1):2158, Dec 1993.

F. Black and R. Litterman. Global portfolio optimization. Financial Analysts Journal, 48(5):2843, 1992.

R. H. Tütüncü and M. Koenig. Robust asset allocation. Annals of Operations Research, 132(1-4):157187, 2004.

B. Scherer. Can robust portfolio optimisation help to build better portfolios? Journal of Asset Management, 7(6):374387, Mar 2007.

F. J. Fabozzi, P. N. Kolm, D. Pachamanova, and S. M. Focardi. Robust Portfolio Optimization and Management. Wiley, 2007.

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