$+ \quad \times$

$- \quad \div$

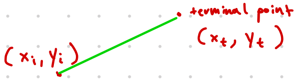
Vector Components

$v = \langle x, y \rangle$ } describe a vector

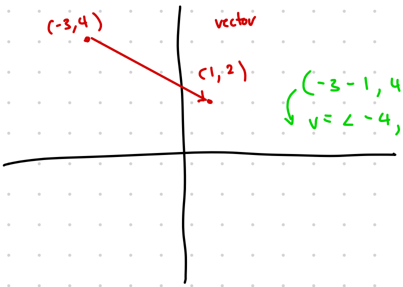
$x \rightarrow$ scalar

$y \rightarrow$ scalar

$v = (x, y)$ } point in a plane

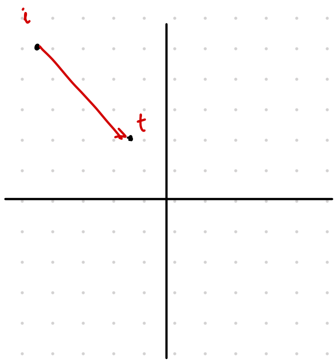


$\langle x_t - x_i, y_t - y_i \rangle$



$(-3-1, 4-2)$

$v = \langle -4, 2 \rangle$



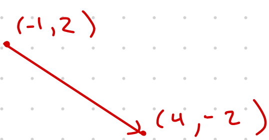
a:

$v = \langle x_t - x_i, y_t - y_i \rangle$

$v = \langle -1-(-4), 2-(-5) \rangle$

$= \langle 3, 7 \rangle$

b.



$\langle 4-(-1), -2-(2) \rangle$

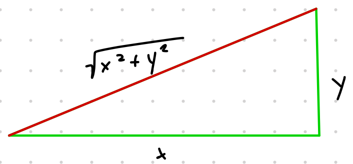
$\langle 5, -4 \rangle$

Vector Magnitude and Vector Operations

magnitude of a vector: distance between initial point and its terminal point

$\|v\| = \sqrt{x^2 + y^2}$

$\|v\| =$



Scalar multiplication: $kv = \langle kx_i, ky_i \rangle$

Vector addition: $v + m = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

$$\|kv\| = \sqrt{(kx_1)^2 + (ky_1)^2}$$
$$= \sqrt{k^2 (x_1 + y_1)^2}$$
$$= |k| \sqrt{x_1^2 + y_1^2}$$
$$= |k| \; \|v\|$$

Terminal point of w: $v + w \rightarrow (x_1 + x_2, y_1 + y_2)$

a. component form: $v = \langle 8 - 2, 13 - 5 \rangle$

$v = \langle 6, 8 \rangle$

start point

$$\|v\| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100}$$
$$= 10$$

b. $v + w = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$

$= \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle$

HW

c. $3v$

$3 \cdot v$

$3 \cdot \langle 6, 8 \rangle$

$\langle 18, 24 \rangle$

d. $v - 2w$

$= \langle 6, 8 \rangle - 2\langle -2, 4 \rangle$

$= \langle 6, 8 \rangle - \langle -4, 8 \rangle$

$= \langle 10, 0 \rangle$

a.

$a = \langle 7, 1 \rangle$

$b = (3, 2) \quad (-1, -1)$

$x_i, y_i \quad x_t, y_t$

$\langle x_t - y_t, x_i - y_i \rangle$

$b = \langle -1 - (2), 3 - (-1) \rangle$

$b = \langle -3, 4 \rangle$

$\|b\| = \sqrt{(-3)^2 + (4)^2}$

$= \sqrt{9 + 16}$

$\|b\| = \sqrt{25} = 5$

$3a = 3\langle 7, 1 \rangle$

$3a = \langle 21, 3 \rangle$

$4b = 4\langle b \rangle$

$= 4\langle -3, 4 \rangle = \langle -12, 16 \rangle$

$3a - 4b = \langle 21, 3 \rangle - \langle -12, 16 \rangle$

$= \langle 33,$

2.2.3 Applications of vectors in 3D

$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$P = (3, 12, 6)$

$Q = (-4, -3, 2)$

$= \langle -4-3, -3-12, 2-6 \rangle$

$= \langle -7, -15, -4 \rangle$

$\overrightarrow{PQ} = -7i - 15j - 4k$

$\overrightarrow{ST} = \langle -1, -9, 1 \rangle$

$= -i - 9j + k$

$u = \langle 2\cos(t), -2\sin(t), 3 \rangle$

$2u$ magnitude of a vector

$$\|\vec{v}\| = \sqrt{(x)^2 + (y)^2 + (z)^2}$$

$$k\vec{u} = k\langle a, b, c \rangle$$

$$= \langle ka, kb, kc \rangle$$

$4v - 2w$

$= 4\langle -2, 2, -3 \rangle - 2\langle 0, -1, 4 \rangle$

$= \langle -8, 8, -12 \rangle - \langle 0, -2, 8 \rangle$

$= \langle -8, 10, -20 \rangle$

$5v + 3w$

$= 5\langle -1, -1, 1 \rangle + 3\langle 2, 0, 1 \rangle$

$= \langle -5, -5, 5 \rangle + \langle 6, 0, 3 \rangle$

$= \langle 1, -5, 8 \rangle$

Find unit vector: divide by magnitude

$$\frac{\langle 1, -5, 8 \rangle}{\sqrt{1^2 + (-5)^2 + (8)^2}}$$

64

$$= \frac{\langle 1, -5, 8 \rangle}{\sqrt{1 + 25 + 64}} = \frac{\langle 1, -5, 8 \rangle}{\sqrt{90}} = \frac{\langle 1, -5, 8 \rangle}{3\sqrt{10}}$$

$u = \langle 2\cos(t), -2\sin(t), 3 \rangle$

$2u = \langle 4\cos(t), -4\sin(t), 6 \rangle$

$$\|2u\| = \sqrt{(4\cos(t))^2 + (-4\sin(t))^2 + 6^2}$$
$$= \sqrt{16\cos^2 t + 16\sin^2 t + 36}$$
$$= \sqrt{16(1) + 36}$$
$$= \sqrt{16 + 36} = \sqrt{52}$$

$4 \quad 13$

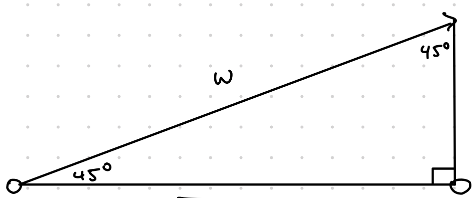
$$= 2\sqrt{13}$$

$4v - 2w$

$$= 4\langle -2, 2, -3 \rangle - 2\langle 0, -1, 4 \rangle$$
$$= \langle -8, 8, -12 \rangle - \langle 0, -2, 8 \rangle$$
$$= \langle -8, 10, -20 \rangle$$

uw

$$\sqrt{15^2 + 10^2} = \sqrt{225 + 100}$$
$$= \sqrt{325}$$
$$= 5\sqrt{13}$$



$\dfrac{5\sqrt{52}}{2}$

$\sqrt{15^2 + 10^2}$
$= 5\sqrt{13}$

$$\cos 45° = \frac{5\sqrt{13}}{\|w\|}$$

$$\|w\| = \frac{5\sqrt{13}}{\cos 45°} = \frac{5\sqrt{13}}{\frac{\sqrt{2}}{2}} = \frac{10\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\overset{5}{\cancel{10}}\sqrt{26}}{\cancel{2}} = 5\sqrt{26}$$

$$w = \|w\| \sin 45^\circ$$

$$= 5\sqrt{26}\ \sin 45^\circ$$

$$= 5\sqrt{26}\ \frac{\sqrt{2}}{2}$$

$$= \frac{5}{2}\sqrt{52} = \frac{5\sqrt{52}}{2}$$

$$w = \left< 15,\ 10,\ \frac{5\sqrt{52}}{2} \right>$$

$$\|kw\| = k\|w\| = k \cdot 5\sqrt{26}$$

$$k \cdot 5\sqrt{26} = 40$$

$$k = \frac{\cancel{40}^{\,8}}{\cancel{5}\sqrt{26}} = \frac{8\sqrt{26}}{26}$$

$$v = kw = k\left< 15,\ 10,\ \frac{5\sqrt{52}}{2} \right>$$

$$= \frac{4\sqrt{26}}{13}\left< 15,\ 10,\ \frac{5\sqrt{52}}{2} \right>$$

$$= \left< \frac{60\sqrt{26}}{13},\ \frac{40\sqrt{26}}{13},\ \frac{\cancel{20}^{\,10}\sqrt{52}\cdot\sqrt{26}}{\cancel{26}\ 13} \right>$$

$$= \left< \frac{60\sqrt{26}}{13},\ \frac{40\sqrt{26}}{13},\ \frac{\cancel{260}^{\,20}\sqrt{2}}{\cancel{13}} \right>$$

$(4j)\cdot(i - 2.5j)$

$4ji \cdot -2.5jj$

$v_1 w_1 + v_2 w_2$

$= 1\cdot(-5) + (3\cdot 4) + (6\cdot -3)$

$= -5 + 12 + -18$

$= -23 + 12 = -11$

$\langle 1, -2.5, 0 \rangle \quad \langle 0, 4, 0 \rangle$

$= 0\cdot 1 - 2.5\cdot 4 + 0\cdot 0$

$= 0 - 10 + 0$

$(a\cdot c)b$

$= ab \cdot cb$

$=$

$$= b\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet b\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} ba_1 + bc_1 \\ ba_2 + bc_2 \\ ba_3 + bc_3 \end{bmatrix}$$

$$\langle \underline{a}, \underline{b}, \underline{c} \rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$$

$-14 + 3y = 7$

$y = \dfrac{21}{3}$

$u\cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

$-\dfrac{9}{2} = -4 - 9y + -\dfrac{5}{2}$

$-\dfrac{9}{2} + 4 + \dfrac{5}{2} = -9y$

$y = \dfrac{-\frac{9}{2} + 4 + \frac{5}{2}}{-9} = -\dfrac{2}{9}$

$u \cdot (-5v) = \frac{1}{3}$

$u \cdot (-5v) = \frac{1}{3}$

$\frac{1}{3} + 2\left(\frac{1}{3}\right)$

$\frac{1}{3} + \frac{2}{3} = 1$

$a \cdot (2b + a) - b \cdot 2a$

$= a \cdot 2b + a \cdot a - b \cdot 2a$

$= 2(a \cdot b) + a \cdot a - 2(b \cdot a)$

$= a \cdot a$

$= ||a||^2$

$= 10^2$

$= 100$

$\cancel{b \cdot 3a} + b \cdot b \cancel{- b \cdot 3a} + b \cdot b$

$= ||b||^2 + ||b||^2$

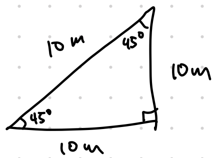
$= 25 + 25 = 50$

$-5(u \cdot v) + 10(u \cdot v)$

$v = 10$ m/sec $\quad v = kw \quad w = \langle 10, 10,$

$k\|w\| = k \quad \sqrt{12^2 + 5^2} =$

$k \cdot \quad =$



$\sqrt{10^2 + 10^2} = 10$

$c = 10 \quad \dfrac{10}{\|w\|} = \cos 45°$

$$\|w\| = \frac{10}{\cos 45°} = \frac{10}{\sqrt{2}} = \frac{\overset{5}{\cancel{10}}\sqrt{2}}{\cancel{2}}$$

$$= 5\sqrt{2} \text{ m}$$

$$\|w\| \sin 45° = 5\sqrt{2} \cdot \sqrt{2} = 10 \text{ m}$$

$$w = \langle 10, 10, 10 \rangle$$

$\|v\| = 10$ m/sec

$k\|w\| = k \cdot \sqrt{10^2 + 10^2 + 10^2}$

$k \cdot 10\sqrt{3} = 10$

$$k = \frac{\cancel{10}\,1}{\cancel{10}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$v = kw$

$$= \frac{1}{\sqrt{3}} \cdot \langle 10, 10, 10 \rangle$$

Magnitude of $u-v$

$\langle 2\cos(t), -2\sin(t), 0 \rangle$

$\sqrt{(2\cos t)^2 + (-2\sin(t))^2 + 0^2}$

$= \sqrt{4\cos^2 t + 4\sin^2 t}$

$= \sqrt{4(1)} = 2$

$\sqrt{15^2 + 10^2} = \sqrt{325}$

325 → 5, 65; 65 → 5, 13

$= 5\sqrt{13}$

$\frac{5\sqrt{13}}{\|w\|} = \cos 45^\circ$

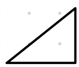
$\|w\| = \frac{5\sqrt{13}}{\cos 45^\circ}$

$= 25.5$

$\frac{5\sqrt{13}}{\sqrt{2}} \cdot 2 = \frac{10\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\overset{5}{\cancel{10}}\sqrt{26}}{\cancel{2}} = 5\sqrt{26}$

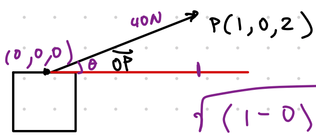
26 → 2, 13

$w = \|w\| \sin 45^\circ = 5\sqrt{26}$



$w = \langle 15, 10, 5\sqrt{26} \rangle$

F of 40N



$\sqrt{(1-0)^2 + (0-0)^2 + (2-0)^2} = \sqrt{1+0+4}$

$= \sqrt{5}$

$40\cos\theta = \sqrt{5}$

$\cos\theta = \frac{\sqrt{5}}{40}$

$\theta = \cos^{-1}\left(\frac{\sqrt{5}}{40}\right)$

$\theta = 86.8^\circ$

$= \cancel{87^\circ}$

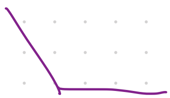
$\langle 1,-1,-1 \rangle$

$a = \langle 3,5,-1 \rangle$
$b = \langle x,-1,6 \rangle$

$a \cdot b = 0$
$3(x) + 5(-1) + 6(-1) = 0$
$3x - 5 - 6 = 0$
$3x - 11 = 0$
$x = \frac{11}{3}$

$a = \langle 3, -1, -2 \rangle \quad b = \langle -2,-3,2 \rangle + \langle 1,0,2 \rangle$
$= \langle 3, -1, -2 \rangle \quad = \langle -1, -3, 4 \rangle$

radians :



$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$

$\|u\| = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2}$
$= \sqrt{3^2 + (-1)^2 + (-2)^2}$
$= \sqrt{10+4} = \sqrt{14}$

$\|v\| = \sqrt{(-1)^2 + (-3)^2 + (4)^2}$
$= \sqrt{10 + 16}$
$= \sqrt{26}$

$\cos\theta = \frac{\cancel{-3+3}-8}{\sqrt{14} \cdot \sqrt{26}}$

$\cos\theta = \frac{-8}{\sqrt{364}}$

$\theta = 114.8°$

$\sqrt{3^2+4^2+5^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$

$\frac{3}{5\sqrt{2}} \quad \frac{4}{5\sqrt{2}} \quad \frac{1}{\sqrt{2}}$

$\langle 1,-1,-1 \rangle$

$u = 5i$

$v = -6i + 6j$

$\langle 5, 0, 0 \rangle$

$\langle -6, 6, 0 \rangle$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos\theta = \frac{5(-6) + 0(6) + 0(0)}{\sqrt{5^2+0^2}\cdot\sqrt{(-6)^2+(6)^2}}$$

$$= \frac{-\cancel{30}^{6}}{\cancel{5}\cdot\sqrt{36+36}} = \frac{-6}{\sqrt{72}}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{72}}\right)$$

$$\cos\theta = \frac{-5(2) + 2(-3) + (-6)(1)}{\sqrt{(-5)^2+2^2+(-6)^2}\cdot\sqrt{2^2+(-3)^2+(1)^2}}$$

$$\cos\theta = \frac{-10 - 6 - 6}{} = \frac{-22}{}$$

$\cancel{2.4}$ 136.8

Vector Valued Function

$r(t) = f(t)\,i + g(t)\,j$

OR

$r(t) = \langle f(t), g(t) \rangle$

For a 2D curve or plane curve

$r(t) = f(t)i + g(t)j + h(t)k$

OR

$r(t) = \langle f(t), g(t), h(t) \rangle$

1. Evaluate the vector valued function

$r(0) = 0i + (-0^2 + 5)j$

$= 0i + 5j \qquad \langle 0, 5 \rangle$

$r(2) = 2i + (-4 + 5)j$

$= 2i + 1j \qquad \langle 2, 1 \rangle$

$r(-3) = -3i + (-(-3)^2 + 5)j$

$= -3i + (-4)j = -3i - 4j \qquad \langle -3, -4 \rangle$

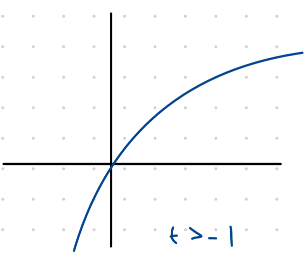
Determine domain of the vector valued functions

t

$r(t) = f(t)i + g(t)j + h(t)k$

$r(t) = \langle f(t), g(t), h(t) \rangle$

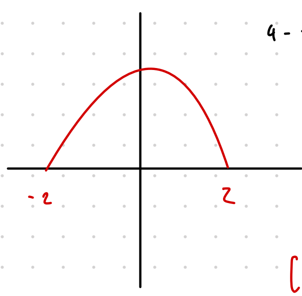
$r(t) = \langle \ln(t+1), \sqrt{4 - t^2} \rangle$



$t > -1$

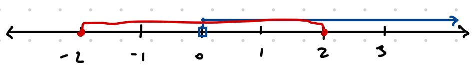
$(-1, \infty)$

$4 - t^2 \geq 0$

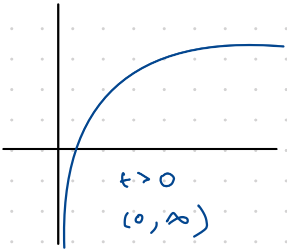


$[-2, 2]$

$-2 \leq t \leq 2$

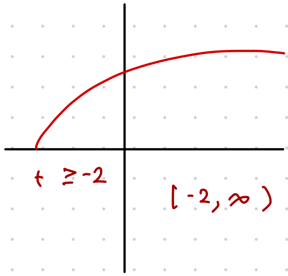


$r(t) = \langle \ln(t), \sqrt{t+2}, \cos(t) \rangle$



$t > 0$

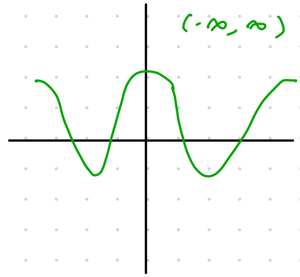
$(0, \infty)$

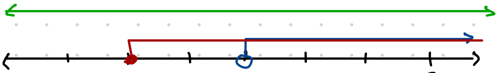


$t \geq -2$

$[-2, \infty)$

$(-\infty, \infty)$

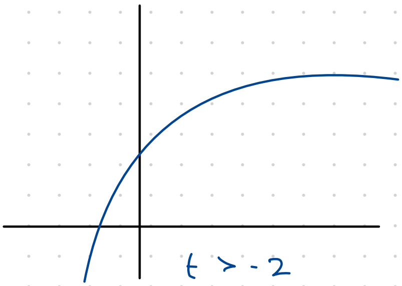




Domain: $t > 0$

$(0, \infty)$

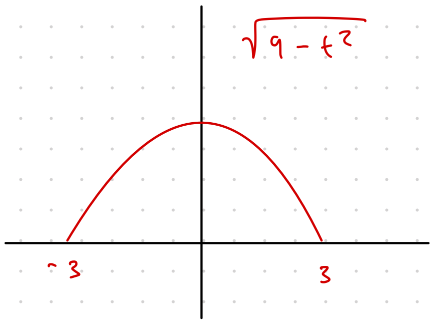
$r(t) = \langle \ln(t+2), \sqrt{9-t^2} \rangle$



$t > -2$

-2

$\sqrt{9-t^2}$

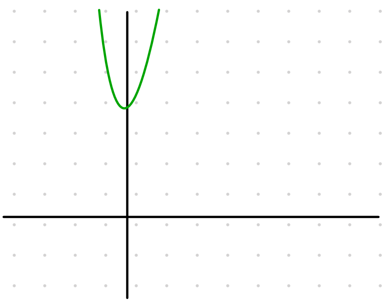


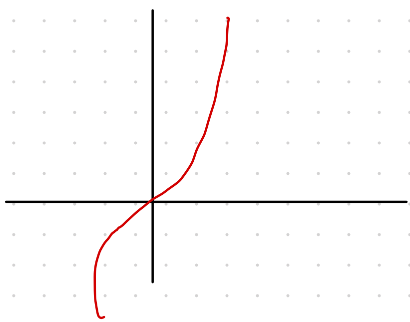
-3 3

$(-2, 3]$

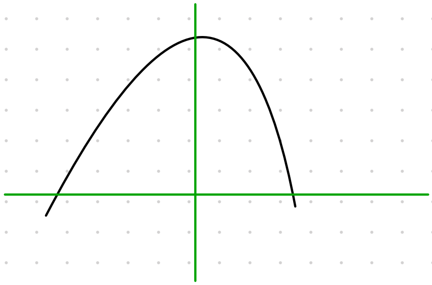
$r(t) = 3\sec(t)\,i + 2\tan(t)\,j$

$= \langle 3\sec(t), 2\tan(t) \rangle$

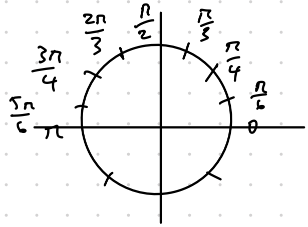




Graph a vector valued function



1. Evaluate the vector value function for each value of t



$r(t) = 2\cos(t)\,i + 2\sin(t)\,j + t\,k$

OR

$r(t) = \langle 2\cos(t), 2\sin(t), t \rangle$

$r(0) = 2\cos(0)\,i + 2\sin(0)\,j + 0\,k \quad = \langle 2, 0, 0 \rangle$

OR

$r\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) i + 2\sin\left(\frac{\pi}{4}\right) j + \left(\frac{\pi}{4}\right) k \quad = \sqrt{2}\,i + \sqrt{2}\,j + \frac{\pi}{2}\,k$

$= \langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \rangle$

$r\left(\frac{2\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}\right) i + 2\sin\left(\frac{2\pi}{3}\right) j + \left(\frac{2\pi}{3}\right) k$

$= \langle 2\left(-\frac{1}{2}\right) i + 2\left(\frac{\sqrt{3}}{2}\right) j + \left(\frac{2\pi}{3}\right) k \rangle$

$= \langle -i + \sqrt{3}\,j + \frac{2\pi}{3}\,k \rangle$

$= \langle -1, \sqrt{3}, \frac{2\pi}{3} \rangle$

$r(t) = (t^3 + 1)\,i + (t - 1)\,j, \quad 0 \le t \le 2$

$r(0) = 1\,i + -1\,j = \langle 1, -1 \rangle$

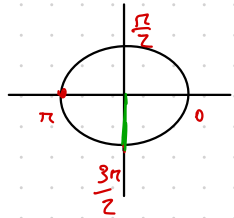
$r(1) = 2\,i + 0\,j = \langle 2, 0 \rangle$

$r(2) = 9\,i + 1\,j = \langle 9, 1 \rangle$

$r(t) = \langle 3\sin t, 3\cos t \rangle$

$r(0) = \langle 3\sin 0, 3\cos 0 \rangle$ $3j$

$= \langle 0, 3 \rangle$



$r(\frac{\pi}{2}) = \langle 3\sin\frac{\pi}{2}, 3\cos\frac{\pi}{2} \rangle$

$= \langle 3 \cdot \frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2} \rangle$

$= \langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$

$r(\pi) = \langle 3\sin\pi, 3\cos\pi \rangle$

$= \langle 0, -3 \rangle$

$r(\frac{3\pi}{2}) = \langle -\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2} \rangle$

$r(t) = \langle \sin(3t), \ln(2t) \rangle$

$r(0) =$

$r(\frac{\pi}{2}) = \langle \sin(3 \cdot \frac{\pi}{2}), \tan(2 \cdot \frac{\pi}{2}) \rangle$

$= \langle \sin(\frac{3\pi}{2}), \tan(\pi) \rangle$

$\langle -1, 0 \rangle$

Limits of vector valued functions

$$\lim_{t \to a} r(t) = L$$

$$\lim_{t \to a} \left| r(t) - L \right| = 0$$

$$\lim_{t \to a} r(t) = \left[\lim_{t \to a} f(t) \right] i + \left[\lim_{t \to a} g(t) \right] j$$

$$\lim_{t \to a} r(t) = \left[\lim_{t \to a} f(t) \right] i + \left[\lim_{t \to a} g(t) \right] j + \left[\lim_{t \to a} h(t) \right] k$$

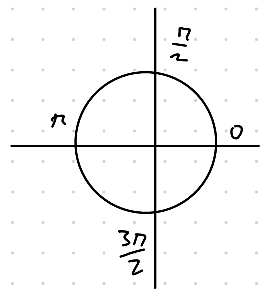
calculate $\lim_{t \to 3} r(t)$

a. $r(t) = (t^2 - 3t + 4) i + (4t + 3) j$

$$\lim_{t \to 3} r(t) = \lim_{t \to 3} \left[(t^2 - 3t + 4) i + (4t + 3) j \right]$$

$$= \left[\lim_{t \to 3} (t^2 - 3t + 4) \right] i + \left[\lim_{t \to 3} (4t + 3) \right] j$$

$$= 4i + 15j$$



b. $$\lim_{t \to 3} r(t) = \lim_{t \to 3} \left(\frac{2t - 4}{t + 1} i + \frac{t}{t^2 + 1} j + (4t - 3) k \right)$$

$$= \frac{1}{2} i + \frac{3}{10} j + 9k$$

3. $$\lim_{t \to -2} r(t) = \lim_{t \to -2} \left(\sqrt{t^2 - 3t - 1}\, i + (4t + 3) j + \sin\left(\frac{(t+1)\pi}{2} \right) k \right)$$

$$= \sqrt{4 + 6 - 1}\, i + (-8 + 3) j + \sin\left(-\frac{\pi}{2} \right) k$$

$$= \sqrt{9}\, i - 5j - 1k$$

$$= 3i - 5j - 1k$$

$$r(t) = \left\langle \sqrt{t-3},\ \frac{\sqrt{t}-2}{t-4},\ \tan\left(\frac{\pi}{t}\right) \right\rangle$$

$$\lim_{t \to 4} r(t) = \left\langle \sqrt{4-3},\ \frac{0}{} \right.$$

Determine if a Vector Valued Function is Continuous

A vector valued function $r(t) = f(t)i + g(t)j$ is in two dimensions
$r(t) = f(t)i + g(t)j + h(t)k$

1. $r(a)$ exists

2. $\lim_{t \to a} r(t)$ exists

3. $\lim_{t \to a} r(t) = r(a)$

Determine if each of the following vector-valued functions is continuous at point $t = 1$.

a) $r(t) = 3t\,i + t^2 j$

1. $r(1)$ exists

2. $\lim_{t \to 1} r(1)$ exists

3. $\lim_{t \to 1} r(t) = r(1) = 3\vec{i} + \vec{j}$

continuous

b) $r(t) = 3t\,i + t^2 j + \frac{5}{t-1} k$

1. $r(1)$ doesn't exist

not continuous

$$r(t) = \left\langle \sqrt{t-3},\ \frac{\sqrt{t}-2}{t-4},\ \tan\left(\frac{\pi}{t}\right) \right\rangle$$

$$= \left\langle \sqrt{t-3},\ \frac{\cancel{\sqrt{t}-2}}{\cancel{(\sqrt{t}-2)}(\sqrt{t}+2)},\ \tan\left(\frac{\pi}{t}\right) \right\rangle$$

$$r(2) = 2e^{-2}$$

$$= e^{-2}$$

$$= \ln(2-1) = \ln(1)$$

Is the vector function continuous at $t = 0$?

$$r(0) = e^0 i + \frac{\sin(0)}{0 - 1} j + \ln(0) k$$

$$= i + 0 j +$$

$r(0)$ doesn't exist

$r(t) = \langle \cos(t), t, \sin(t) \rangle$

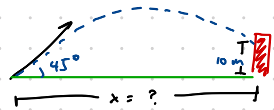
Solve Problem Involving Projectile Motion

$$s(t) = -\frac{1}{2} g t^2 j + v_0 t \cos\theta\, i + v_0 t \sin\theta\, j$$

$$= v_0 t \cos\theta\, i + v_0 t \sin\theta\, j - \frac{1}{2} g t^2 j$$

$$= v_0 t \cos\theta\, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

Steve throws a ball at a 45° angle above the horizontal at a speed of 20 m/s to his friend who is on a cliff 10 meters high

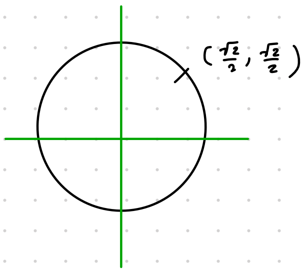


$$s(t) = v_0 t \cos\theta\, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

$$s(t) = 20 t \cos 45^\circ\, i + \left(20 t \sin 45^\circ - \frac{1}{2}(9.8) t^2\right) j$$

$$s(t) = 10\sqrt{2}\, t + \left(20 t \sin(45^\circ) - \frac{1}{2}(9.8) t^2\right) j$$

$$= 10\sqrt{2}\, t + \left(10\sqrt{2}\, t - 4.9 t^2\right) j$$



$$10\sqrt{2}\, t - 4.9 t^2 = 10$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$0 = 4.9 t^2 - 10\sqrt{2}\, t + 10$$

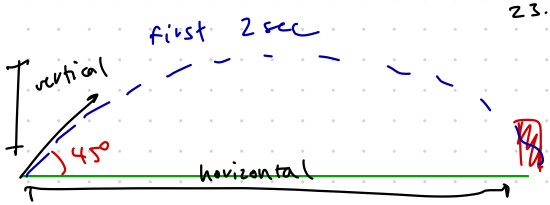
$$t = \frac{10\sqrt{2} \pm \sqrt{(10\sqrt{2})^2 - 4(4.9)}}{2(4.9)}$$

$$= 1.23,\ 1.65$$

$$s(1.65) = 10\sqrt{2}(1.65)\, i + \left(10\sqrt{2}(1.65) - 4.9(1.65)^2\right) j$$

$$\approx 23.33\, i + 9.99\, j$$

23.33 meters away.

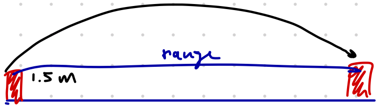


throws a ball at a 45° angle at a speed of 500 ft/s

$$s(t) = v_0 t \cos\theta\, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

$$s(t) = 500 t \cos(45^\circ)\, i + \left(500 t \sin(45^\circ) - \frac{1}{2}(9.8) t^2\right) j$$

$$s(2) = 707.11\, i + 687.51\, j$$



$$s(t) = v_0 t \cos\theta \, i + (v_0 t \sin\theta - \tfrac{1}{2} g t^2) j$$

$$s(t) = 100\,t \cos(30°) i + (100t \sin(30°) - \tfrac{1}{2}(9.8)t^2) j$$

$$100t \cos 30°$$

$$\cancel{\left[100t \sin(30°) - \tfrac{1}{2}(9.8)t^2\right]} = \cancel{1.5} \; 0$$

$$0 = \tfrac{1}{2}(9.8)t^2 - 100t \sin(30°) + 1.5$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$0 = 4.9t^2 - 50t + 1.5$$

$$t = \frac{50 \pm \sqrt{(-50)^2 - 4(4.9)(1.5)}}{2(4.9)}$$

$$t = \underline{10.17}, \; 0.030$$

$$s(10.17) = 1.69$$

$$= \cancel{1.7} \quad 1.499$$

$$1.5$$

$$s(100 t \cos(30°))$$

$$s(t) = v_0 t \cos\theta \, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

$$= 370\, t \cos 85^\circ \, i + \left(370\, t \sin 85^\circ - \frac{1}{2} 9.8\, t^2\right) j$$

$$370 t \sin 85^\circ - \frac{1}{2} 9.8 t^2 = 0$$

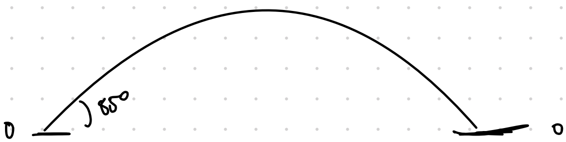
$$0 = \frac{1}{2} 9.8 t^2 - 370 t \sin 85^\circ$$

$$0 = 4.9 t^2 - 370 \sin 85^\circ t$$

$$t = \frac{-(-370 \sin 85^\circ) \pm \sqrt{(-370 \sin 85^\circ)^2 - 4(4.9)(0)}}{2(4.9)}$$

$$t = 0,\ 75.229$$

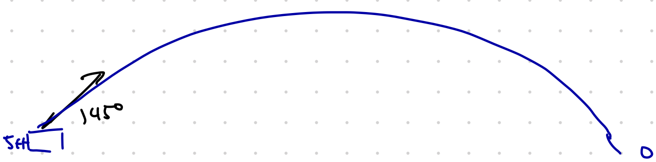
$$t = 75.22$$

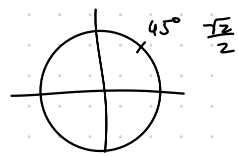


$$370\, t \cos 85^\circ \, i \qquad 370 \cos 85^\circ \, t = 0$$

$$t =$$

$$32.25$$





$$s(t) = v_0 t \cos\theta \, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

$$s(t) = 98\, t \cos 45^\circ \, i + \left(98\, t \sin 45^\circ - \frac{1}{2} 9.8\, t^2\right) j$$

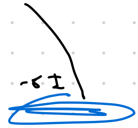
$$98 \frac{\sqrt{2}}{2} t - 4.9 t^2 = -5$$

$$4.9 t^2 + 49\sqrt{2}\, t - 5 = 0$$

$$t = \frac{-49\sqrt{2} \pm \sqrt{(49\sqrt{2})^2 - 4(4.9)(-5)}}{2(4.9)}$$

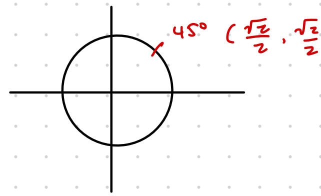
$$=$$

$s(t) = v_0 t \cos\theta \, i + (v_0 t \sin\theta - \frac{1}{2} g t^2) j$



$v_0 t \sin\theta - \frac{1}{2} g t^2 = -5$

$g = 32 \text{ ft/sec}$



$\theta = 45°$

$v_0 = 98 \text{ ft/sec}$

$98 t \sin 45° - \frac{1}{2}(32) t^2 = -5$

$\frac{1}{2}(32) t^2 - 98 t \sin 45° - 5 = 0$

$16t^2 - 98 \cdot \frac{\sqrt{2}}{2} t - 5 = 0$

$16t^2 - 49\sqrt{2}\, t - 5 = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{49\sqrt{2} \pm \sqrt{(49\sqrt{2})^2 - 4(16)(-5)}}{2(16)}$$

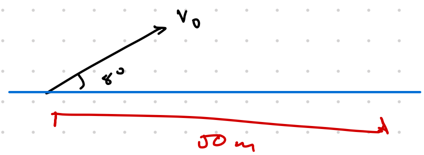
$t = 4.4020192 , 0$

$s(t) = v_0 t \cos\theta \, i + (v_0 t \sin\theta - \frac{1}{2} g t^2) \, j$

$v_0 = ?$

$g = 9.8$

$\theta = 8°$



$$v_0 t \sin(8°) - \frac{1}{2}(9.8) t^2 = 0$$

$$v_0 t \sin(8°) = 4.9 t^2$$

$$v_0 t = \frac{4.9 t^2}{\sin(8°)}$$

$$v_0 t = v_0 t$$

$$\frac{50}{\cos(8°)} = \frac{4.9 t^2}{\sin(8°)}$$

$$\frac{50 \sin(8°)}{4.9 \cos(8°)} = t^2$$

$$t = \sqrt{\frac{50 \sin(8°)}{4.9 \cos(8°)}}$$

$$t = 1.1975 \text{ sec}$$

"range" means i

$$v_0 t \cos\theta = 50$$

$$v_0 t \cos(8°) = 50$$

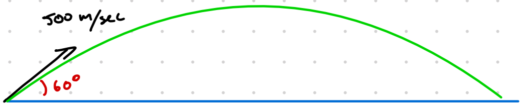
$$v_0 = \frac{50}{\cos(8°)} \cdot \frac{1}{t}$$

$$v_0 t = \frac{50}{\cos(8°)}$$

$$v_0 (1.1975) \cos(8°) = 50$$

$$v_0 = \frac{50}{1.1975 \cos(8°)}$$

$$v_0 = 42.1640$$



max range in meters?

$$s(o) = v_0 t \cos\theta\, i + \left(v_0 t \sin\theta - \frac{1}{2} g t^2\right) j$$

$$v_0 t \cos\theta = ?$$

$g = 9.8\ m/s^2$

$V_0 = 500$

$\theta = 60^\circ$

$$v_0 t \sin\theta - \frac{1}{2} g t^2 = 0$$

$$500 t \sin 60^\circ - \frac{1}{2} \cdot 9.8\, t^2 = 0$$

$$4.9 t^2 - 500 t \sin 60^\circ = 0$$

$$t(4.9t - 500 \sin 60^\circ) = 0$$

$$4.9t = 500 \sin 60^\circ$$

$$t = \frac{500 \sin 60^\circ}{4.9}$$

$$t = 88.3699 \text{ sec}$$

$$v_0 t \cos\theta = ?$$

$$500(88.3699)\cos 60^\circ = \text{max range}$$

$$= 22092.475$$

Arc Length

3.3.1 Arc Length

$r(t) = f(t)i + g(t)j + h(t)k$ on $[a, b]$

the arc length on the interval is given by

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$s = \int_a^b \|\vec{r}'(t)\|\, dt$$

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$= \int_a^b \|\vec{r}'(t)\| dt$$

Arc length defined by $\vec{r}(t) = \langle 2\cos t, 2\sin(t), 2t \rangle$

$\vec{r}'(t) = \langle -2\sin(t), 2\cos(t), 2 \rangle$

$$s = \int_0^\pi \sqrt{\underbrace{4\sin^2(t) + 4\cos^2(t)}_{4(\sin^2(t) + \cos^2(t))} + 4}\; dt \qquad = \sqrt{4+4} = \sqrt{8}$$

$\sqrt{8} = \sqrt{2 \cdot 4} = 2\sqrt{2}$

$$= \int_0^\pi 2\sqrt{2}\; dt$$

$$= 2\sqrt{2}\, t \Big|_0^\pi = 2\sqrt{2}\,\pi - 2\sqrt{2}(0)$$

$$= 2\sqrt{2}\,\pi = 8.9$$

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$

$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 4 \rangle$

$$s = \int_0^\pi \sqrt{9\sin^2 t + (9\cos^2 t) + 16}\; dt$$

$$= \int_0^\pi \sqrt{9(\sin^2 t + \cos^2 t) + 16}\; dt$$

$$= \int_0^\pi \sqrt{9+16}\; dt = \int_0^\pi \sqrt{25} = 5\pi$$

$v'(t) = \langle 3\cos t, -2\sin t, \frac{1}{5} \rangle$

$$S = \int_{-\pi}^{\frac{13\pi}{6}} \sqrt{9\cos^2 t + 4\sin^2 t + \left(\frac{1}{5}\right)^2} \, dt$$

$$= 25.410$$

$r(t) = 5\cos t \, i + 7\sin t \, j + 0k$

$r'(t) = -5\sin t \, i + 7\cos t \, j + 0k$

$$s = \int_0^{2\pi} \sqrt{25\sin^2 t + 49\cos^2 t + 0^2} \, dt$$

$$= 37.961$$

$r(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$

$r'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$

$$s = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} \, dt$$

$$= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}} \, dt$$

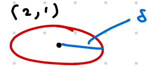
$$= 2.350$$

4.2.1 Limits and Continuity

Limit of a Function of Two Variables

Find the Limit of a Function of Two Variables using direct substitutio

Definition: $\lim_{x \to a} f(x) = L$

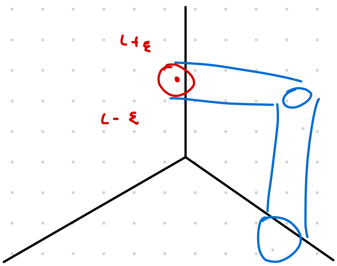


point $(a, b) \in \mathbb{R}^2$ A δ disk at (a, b)

$$\{ (x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 < \delta^2 \}$$

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$



Constant Law: $\lim_{(x,y) \to (a,b)} c = c$

Identity Law: $\lim_{(x,y) \to (a,b)} x = a$

$\lim_{(x,y) \to (a,b)} y = b$

Sum Law: $\lim_{(x,y) \to (a,b)} (f(x,y) + g(x,y)) = L + M$

Difference Law: $\lim_{(x,y) \to (a,b)} (f(x,y) - g(x,y)) = L - M$

Constant Multiple Law: $\lim_{(x,y) \to (a,b)} (c f(x,y)) = cL$

Product Law: $\lim_{(x,y) \to (a,b)} (f(x,y) g(x,y)) = LM$

Quotient Law $\lim_{(x,y) \to (a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ for $M \neq 0$

Power Law: $\lim_{(x,y) \to (a,b)} (f(x,y))^n = L^n$ for positi n

Root Law $\lim_{(x,y) \to (a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

a. $\lim_{(x,y) \to (2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$

$\lim x^2 - \lim 2xy + \lim 3y^2 - \lim 4x + \lim 3y - \lim 6y$

$= (\lim x)^2 - 2 \lim x \lim y + 3(\lim y)^2 - 4 \lim x + 3 \lim y - 6 \lim y$

$= 2^2 - 2 \cdot 2 \cdot (-1) + 3 - 8 - 3 + 6$

$= \cancel{4 + 4} \cancel{+3} \cancel{-8} \cancel{-3} + 6$

$= 6$

6. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

verify first the denominator is zero

$\lim 4x - \lim 3y$

$4 \lim x - 3 \lim y$

$4(2) - 3(-1) = 11$

$8 + 3 = 11$ non zero, soln

$\frac{2 \lim(x) + 3 \lim(y)}{11} = \frac{4-3}{11} = \frac{1}{11}$

$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2-y}{y^2+x-1}}$

$\lim y^2 + \lim x - 1$

$(\lim y)^2 + \lim x - 1$

$(-2)^2 + 5 - 1 = 8$

$(\lim x)^2 - \lim y$

$5^2 - (-2)$

$25 + 2 = 27$

$\sqrt[3]{\frac{27}{8}} = \frac{3}{2}$

Find the $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

$\lim x^2y^3 - \lim x^3y^2 + \lim 3x + \lim 2y$

$8 \cancel{-4} + 3 \cancel{+4} = 11$

$\lim_{(x,y)\to(4,4)} x \ln y = 4 \ln 4$

$\lim_{(x,y)\to(4,4)} e^{-x^2-y^2} = \lim e^{-x^2} + \lim e^{-y^2}$

$= e^{-16} + e^{-16}$

$= \frac{1}{e^{32}}$

Use Paths to Determine if the Limit of a Function of Two Variables Exists

Show that neither of the following limits exist

a. $$\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$$

Domain is all points on xy plane except (0,0)

Along $y=0$ the function is equal to zero

Along $y=x$, function is equal to $\frac{1}{2}$

Therefore, the definition of a limit at a point is never satisfied and fails to exist.

b. $$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \frac{\lim 4xy^2}{\lim x^2 + \lim(3y^4)} \qquad y = ax$$

$$= \frac{\lim 4x(ax)^2}{\lim x^2 + \lim(3(ax)^4)}$$

$$= \lim \frac{4x^3a^2}{x^2+3a^4x^4} = \frac{\cancel{x^2}(4xa^2)}{\cancel{x^2}(1+3a^4x^2)}$$

$$= 0$$

DNE

Use $x=y^3$ $\qquad x=2y^3$

$$\lim_{(x,y)\to(0,0)} \frac{6xy^2}{3xy^2+3y^5}$$

$x=y^3$

$$\lim_{(x,y)\to(0,0)} \frac{6y^3y^2}{3y^3y^2+3y^5} = \lim \frac{6y^5}{3y^5+3y^5}$$

$$= \lim \frac{\cancel{y^5}(6)}{\cancel{y^5}(3+3)} = 1$$

$x=2y^3$ $$\lim_{(x,y)\to(0,0)} \frac{6(2y^3)y^2}{3(2y^3)y^2+3y^5} = \lim_{(x,y)\to(0,0)} \frac{12y^5}{6y^5+3y^5}$$

$$= \lim \frac{12\cancel{(y^5)}}{\cancel{y^5}(6+3)} = \frac{12}{9}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2} \quad \text{along} \quad y=2x$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+(2x)^3}{x^2+(2x)^2} = \frac{2x^2+8x^3}{x^2+4x^2}$$

$$= \frac{\cancel{x^2}(2+8x)}{\cancel{x^2}(1+4)}$$

$$= \frac{2+8x}{5} = \frac{2}{5}$$

4.2.2 Continuity of a Function of Two Variables

Understand the conditions for continuity of a function of two variables at a point

Continuous if . . .

1. $f(a,b)$ exists
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Show $f(x,y) = e^{x+y}(2x-y)$ cont at $(1,3)$

1. $f(1,3) = e^{1+3}(2-3) = -e^4$ ✓

2. $\lim_{(x,y)\to(a,b)} e^{x+y}(2x-y) = \lim_{(x,y)\to(1,3)} e^{x+y} \cdot \lim_{(x,y)\to(1,3)} 2x-y$

$= e^{1+3} \quad (2-3) = e^4(-1) = -e^4$ ✓

3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ ✓

cont at $(1,3)$

Show $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$ cont. at $(1,1)$

1. $f(1,1) = \dfrac{1\cdot 1}{1+1} = \dfrac{1}{2}$

2. $\lim_{(x,y)\to(1,1)} \dfrac{x^2y}{x^4+y^2} = \dfrac{\lim_{(x,y)\to(1,1)} x^2y}{\lim_{(x,y)\to(1,1)} x^4+y^2} = \dfrac{1}{2}$

3 ✓

Show $f(x,y) = \ln(x^2y^2)$ continuous at point $(0,0)$?

1. $f(0,0) = \ln(0\cdot 0) = \ln 0$

Show $f(x,y) = \frac{x-y}{x+y}$ continuous at point $(-1,-1)$?

1. $f(-1,-1) = \frac{-1-(-1)}{-1+-1} = \frac{0}{-2} = 0$

2. $\lim\limits_{(x,y)\to(-1,-1)} \frac{x-y}{x+y} = \frac{\lim(x-y)}{\lim(x+y)} = \frac{-1-(-1)}{-2} = 0$

Is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ cont at $(-1,1)$

1. $f(-1,1) = \frac{1-1}{1+1} = \frac{0}{2} = 0$

2. $\lim\limits_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2+y^2} = \frac{\lim(x^2-y^2)}{\lim(x^2+y^2)} = \frac{1-1}{1+1} = \frac{0}{2}$

Determine the region of the coordinate plane where a function of two variables is continuous

Theorem 1: The sum of continuous Functions is Continuous

If $f(x,y)$ is continuous at (x_0, y_0) and $g(x,y)$ is continuous at (x_0, y_0)

$f(x,y) + g(x,y)$ is continuous at (x_0, y_0)

Theorem 2: The Product of Continuous Function is Continuous

$f(x,y)g(x,y)$ at (x_0, y_0)

Theorem 3: The Composition of Continuous Functions is Continuous

$$f \circ g$$

Test $\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \overset{?}{=} 1$

Test $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} \overset{?}{=} 0$

$$\frac{\lim x^2 y}{\lim x^2+y^2} = 0$$

Test $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \overset{?}{=} 0$

DNE $\overset{?}{=} 0$

$$\left.\begin{aligned} \lim \frac{x^2-0}{x^2+0} &= 1 \\ \lim \frac{0^2-y^2}{0^2+y^2} &= -1 \end{aligned}\right\} \text{Not equal}$$

Find the Limit of a Function of Three Variables

$$\{ (x, y, z) \in \mathbb{R}^3 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta \}$$

$$\lim_{(x,y,z) \to (3,4,8)} \sqrt{3xy + z^2}$$

$$\sqrt{\lim_{(x,y,z) \to (3,4,8)} (3xy + z^2)}$$

$$\sqrt{3 \lim x \cdot \lim y + (\lim z)^2}$$

$$= \sqrt{3 \cdot 3 \cdot 4 + 8^2} = \sqrt{36 + 64}$$

$$= \sqrt{100} = 10$$

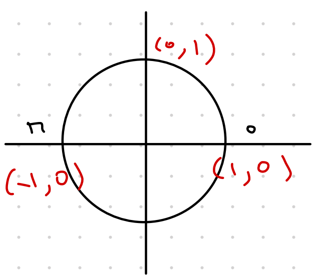
$$\lim_{(x,y,z) \to (2,4,1)} \frac{x^2 y z^2}{x + y} = \frac{(\lim x)^2 \lim y (\lim z)^2}{\lim x + \lim y}$$

$$= \frac{4 \cdot 4 \cdot 1^2}{2 + 4} = \frac{16}{6} = \frac{8}{3}$$

$$\lim_{(x,y,z) \to (\pi,3,1)} \frac{\sin(xz)}{\cos(xy)} = \frac{\lim(\sin(xz))}{\lim(\cos(xy))}$$

$$= \frac{(\sin(\pi \cdot 1))}{(\cos(\pi \cdot 3))} = \frac{\sin \pi}{\cos(3\pi)}$$

$$= \frac{0}{-1}$$



$$\lim_{(x,y,z) \to (4,-1,3)} \sqrt{13 - x^2 - 2y^2 + z^2} = 2$$

$$13 - 16 - 2 + 9$$

$$-3 - 2 + 9$$

$$-5 + 9 = \sqrt{4}$$

$$\lim_{(x,y,z)\to(0,1,2)} z\ln(x^2+y^2)$$

$$2\cdot \ln(0+1)$$

$$2\cdot \ln(1)$$

$$2\cdot 0 = 0$$

4.6.1 - 5.2.2

Exam 4 Review (Directional derivatives and the Gradient, tangent planes and linear approximation, optimization of a function of two variables, double integrals):

Directional Derivatives and the Gradient

Tangent Planes and Approximation

Optimization of a Function of Two Variables

Double Integrals

$$\iint f(x,y)\, dA$$

$$\int_0^1 \int_{y-1}^{\arccos(y)} 2 \quad dx\, dy$$

$$\int_0^1 2\arccos(y) - 2(y-1) \quad dy$$

$$\int_0^1 2\arccos(y) - 2y + 2$$

$$2\int_0^1 \cos^{-1}(y) - 2\int_0^1 y + 2\int_0^1 1$$

$$\left(2\, y\cos^{-1}(y) - 2\sqrt{1-y^2}\right)\Big|_0^1 - y^2\Big|_0^1 + 2(1-0)$$

$$2 - 1 + 2$$

$$3$$

Double Integral in Polar coordinates

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $\theta = \arctan\left(\frac{y}{x}\right)$

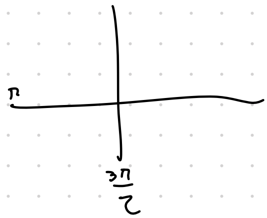
$r = x^2 + y^2$ $\quad$ $\tan\theta = \frac{y}{x}$

$$\iint f(x,y)\,dA = \iint f(r\cos\theta, r\sin\theta)\,dA$$

$$\iint f(r\cos\theta, r\sin\theta)\,dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

EX: Volume below $f(x,y) = 4 - x^2 - y^2$ above xy-plane over region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

$r = 6$ $\quad$ $r = 8$



$$\int_{\pi/4}^{\pi/3}\int_{2}^{3} \theta\, r\,dr\,d\theta$$

$$\int_{\pi/4}^{\pi/3} \frac{r^2}{2}\theta \Big|_2^3 \, d\theta$$

$$\int_{\pi/4}^{\pi/3} \left(\frac{9}{2} - \frac{4}{2}\right)\theta \, d\theta$$

$$\frac{1}{2}\cdot\frac{5}{2}\,\theta^2 \Big|_{\pi/4}^{\pi/3}$$

$$\frac{5}{4}\left(\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{4}\right)^2\right)$$

Polar Rectangular Regions of Integration

Evaluate $\iint 3x\, dA$ with $1 \le r \le 2$, $0 \le \theta \le \pi$

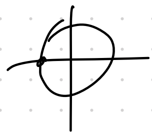
$$\int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} 3r^2 \cos\theta \, dr\, d\theta$$

$$\cos\theta \; r^3 \Big|_1^2$$

$$\int_0^\pi 7\cos\theta \; d\theta$$

$$7 \sin\theta \Big|_0^\pi = 0$$

$$\iint 3r^2 \sin\theta \; dr\; d\theta$$



$$\sin\theta \; r^3 \Big|_1^3$$

$$\int_0^\pi 26 \sin\theta \; d\theta$$

$$-26 \cos\theta \Big|_0^\pi = -26(-1) - (-26 \cdot 1)$$

$$26 + 26 = 52$$

Convert an Integral from Rect. to Polar and Evaluate

Ex 1: $\iint (1-x^2-y^2)\, dA$

$$\int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r-r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$$

$$= \int_0^{2\pi}\frac{1}{4}\,d\theta$$

$$= \frac{1}{4}\theta\Big|_0^{2\pi} = \frac{\pi}{2}$$

$$\int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)r\,dr\,d\theta$$

$$\int_1^2 r^2\,dr \int_{\pi/2}^{3\pi/2}\cos\theta + \sin\theta\,d\theta$$

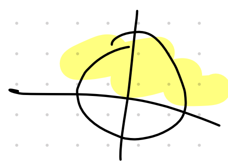
$$\frac{r^3}{3}\Big|_1^2 \qquad \sin\theta - \cos\theta\Big|_{\pi/2}^{3\pi/2}$$

$$= -\frac{14}{3}$$

$0 \le \theta \le \pi$

$\iint 2y - x\,dA \qquad 4 \le r^2 \le 9 \qquad \cancel{r\sin\theta \ge 0}$

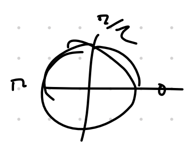
$2 \le r \le 3$



$$\int_0^{\pi}\int_2^3 (2r\sin\theta - r\cos\theta)r\,dr\,d\theta$$

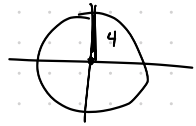
$$\left(\int_2^3 r^2\,dr\right)\left(\int_0^{\pi} 2r\sin\theta - \cos\theta\,d\theta\right)$$

$$\frac{r^3}{3}\Big|_2^3 \quad \left[-2\cos\theta - \sin\theta\right]_0^{\pi}$$



$$= \left(\frac{27-8}{3}\right)$$

$$\int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2+y^2)\, dy\, dx$$



$$\cancel{\int_0^{2\pi} \int_{-4}^{4} \sin(r^2)\, r\, dr\, d\theta}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \sin(r^2)\, r\, dr\, d\theta$$

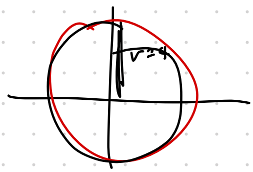
$$y = \sqrt{16-x^2}$$

$$y = \sqrt{16-x^2}$$
$$y^2 = 16 - x^2$$
$$x^2 + y^2 = 16$$

$$r^2 = 16$$
$$r = 4$$

$$\int_0^{2\pi} \int_0^4 (12 - 3(r^2))\, r\, dr\, d\theta$$

$\times 24\pi$

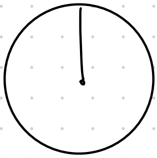


$$\int_0^{2\pi} \int_0^4 12r - 3r^3 \, dr\, d\theta$$

$$\int_0^{2\pi} \left(6r^2 - \frac{3}{4} r^4\right)\Big|_0^4 \, d\theta$$

$$2\pi\left(\left(96 - \frac{3}{4}(4)^4\right)\right)$$

$$2\pi(-96)\, d\theta$$



$3 \le r \le 4$

$$\int_3^4 \int_{\pi/3}^{2\pi/3} (x^2+y^2)(x^2+y^2)$$

$$\cancel{x^4 + 2y^2x^2 + y^4} \qquad r^2 \cdot r^2$$

$$\int_3^4 \int_{\pi/3}^{2\pi/3} \cancel{r^2}\ r\,dr\,d\theta \qquad r^4$$

$$\int_{\pi/3}^{2\pi/3} \int_3^4 \cancel{r^3}\ dr\ d\theta \qquad r^5$$

$$\int_{\pi/3}^{2\pi/3} \cancel{\frac{r^4}{4}}\Big|_3^4\ d\theta \qquad \frac{r^6}{6}$$

$$\frac{2\pi}{3}\left(\frac{4^4}{4} - \frac{3^4}{4}\right) - \frac{\pi}{3}\left(\frac{4^4}{4} - \frac{3^4}{4}\right)$$

$$\frac{\pi}{3}\left(\frac{4^4}{4} - \frac{3^4}{4}\right)$$

$$\frac{175}{12}\pi \quad \approx \quad 38.746$$

$$\int_0^{2\pi} \int_3^5 (r\cos\theta + r\sin\theta)\ r\ dr\ d\theta$$

$$\int_0^{2\pi} \int_2^5 r^2(\cos\theta + \sin\theta)\ dr\ d\theta$$

$$\int_0^{2\pi} \frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_3^5\ d\theta$$

$$\left(\frac{125}{3} - \frac{27}{3}\right)(\cos\theta + \sin\theta)$$

$$\int_0^{2\pi} \frac{98}{3}(\cos\theta + \sin\theta)\ d\theta$$

$$\frac{98}{3}(\sin\theta - \cos\theta)\Big|_0^{2\pi} = 0$$

$$\int_{\pi/4}^{\pi/3}\int_1^2 \left(e^{r^2} + r^4\right)\theta \, r\,dr\,d\theta$$

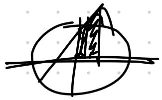
$$\int_{\pi/4}^{\pi/3}\int_1^2 \left(re^{r^2} + r^5\right)\theta \, dr\,d\theta \approx 8.747$$

$$\int_1^2 \left(re^{r^2} + r^5\right)dr \cdot \int_{\pi/4}^{\pi/3} \theta \, d\theta$$

$$\left(\frac{e^{r^2}}{2} + \frac{r^6}{6}\right)\Bigg|_1^2 \cdot \frac{\theta^2}{2}\Bigg|_{\pi/4}^{\pi/3}$$

$$\left[\left(\frac{e^4}{2} + \frac{2^6}{6}\right) - \left(\frac{e^1}{2} + \frac{1}{6}\right)\right] \cdot \left[\frac{\frac{\pi^2}{9}}{2} - \frac{\frac{\pi^2}{16}}{2}\right]$$

$$\int_2^3 \int_{y=0}^{y=x} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$



$$\frac{\cancel{r}\cos\theta}{\cancel{r}}$$

$$\int_0^{\pi/4} \int_{2\sec\theta}^{3\sec\theta} \cos\theta \quad r\, dr\, d\theta$$

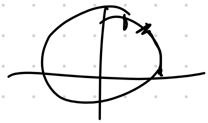
$y = r\cos\theta$ $r =$

$y = \overset{r\sin\theta}{0}$

$0 \leq \sin\theta \leq \cos\theta$

$x = 2$

$x = 3$



$r\cos\theta = 2$

$r = 2\sec\theta$

$r\cos\theta = 3$

$r = 3\sec\theta$