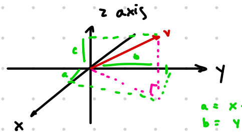
$+$ $\times$

$-$ $\div$

Vectors

Wed. Sep 22
TA: Jeremy Lilly
Filix Maisc

$\langle -1, 2 \rangle$



a = x-axis
b = y-axis
c = z-axis

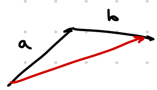
$c\,\mathbf{u}$
italic ↑bold

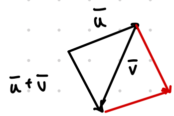
a vector u can be multiplied by a real number c called a scalar.
the product is denoted by $c\mathbf{u}$
and is the result of scaling the length of u by $|c|$
and keeping the same direction if $c > 0$ or reversing the direction if $c < 0$.

parallel: non-zero scalar multiples of each other

vector addition: the sum of vectors $\bar{u}$ and $\bar{v}$ is determined componentwise by adding corresponding components.
done by placing the tail of one vector at the head of the other and drawing the arrow from the tail of the first vector to the head of the second vector.

Notation: $\bar{u} + \bar{v}$
Order doesn't matter: $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ (parallelogram law)





Magnitude of a vector is denoted by $|v|$. In 2D one can use the Pythagorean Theorem to show that the magnitude of $v = \langle a, b \rangle$ is $|v| = \sqrt{a^2 + b^2}$.

3D
$$\vec{w} = \langle a, b, c \rangle \text{ is } |\vec{w}| = \sqrt{a^2 + b^2 + c^2}$$

Vector u is a <u>unit vector</u> if it has magnitude 1. We use $\hat{u}$ denote that a vector is a unit vector

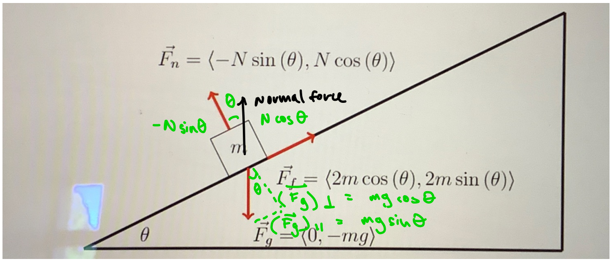
Engineering notation: We define some basic unit vectors, below:
$i = \langle 1, 0 \rangle$ or $\langle 1, 0, 0 \rangle$
$j = \langle 0, 1 \rangle$ or $\langle 0, 1, 0 \rangle$
$k = \langle 0, 0, 1 \rangle$

Then by scalar multiplication and vector addition, $\langle a, b \rangle = ai + bj$
and $\langle a, b, c \rangle = ai + bj + ck$

Working With Vectors

- Forces can be represented as vectors, direction and magnitude
- when an object is at rest, the sum of all forces acting on the object (net force) is $\vec{0}$

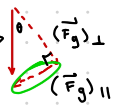
Ex: Example: Suppose a block of mass m kg is at rest on an incline whose angle of elevation is θ. Assume that the forces acting on the object are a gravitational force, a normal force (of magnitude N) and a frictional force (shown in red).



$$\vec{F_g} = (\vec{F_g})_{\parallel} + (\vec{F_g})_{\perp}$$

Note: the normal force is at an angle θ to the left of the positive vertical direction. Horizontal component is $-N\sin\theta$ and its vertical component is $N\cos\theta$

$\vec{F_g} = \langle 0, -mg\rangle$



$(\vec{F_g})_\perp = mg\cos(\theta)$

$(\vec{F_g})_{\parallel} = mg\sin(\theta)$ equal to $2m$

Find θ and N as a multiple of m

$g = 9.8\ m/s^2$

So $mg\sin\theta = 2m$

$$\sin\theta = \frac{2\cancel{m}}{\cancel{m}g}$$

$$\theta = \arcsin\left(\frac{2}{g}\right) = 0.2$$

Vector Basics !

- non zero vector $\vec{v}$ is a quantity with magnitude + direction

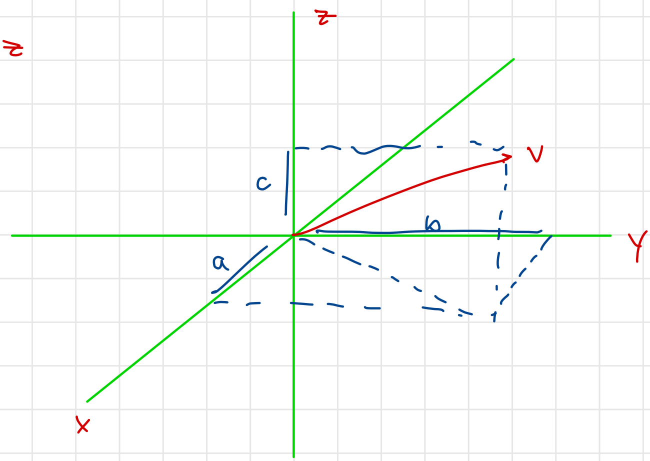
2D plane $(\mathbb{R}^2)$

zero vector $\vec{0}$ represents 0 magnitude (geometrically represented by a point)

$v = \langle a, b, c \rangle$

x - , y - , z

3D plane $(\mathbb{R}^3)$

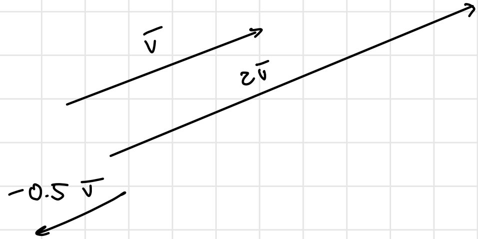


vector is a undefinable quantity

place at the origin

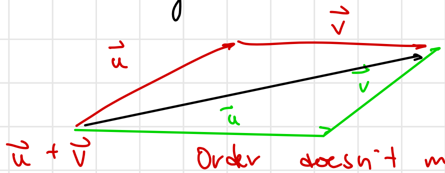
Scalar multiplication

scalar multiplication



Two vectors are parallel if they are non-zero multiples of each other

Vector addition : sum of vectors $\bar{u}$ and $\bar{v}$



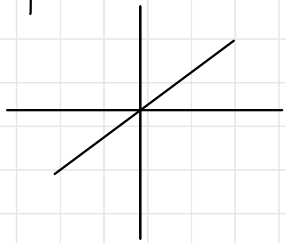
Order doesn't matter :

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (parallelogram law)

$v = \langle a, b \rangle \qquad |v| = \sqrt{a^2 + b^2}$

$\vec{w} = \langle a, b, c \rangle$

$|\vec{w}| = \sqrt{a^2 + b^2 + c^2}$



u = magnitude 1

$\hat{u}$ = denote vector is a unit vector

Recitation Activity MTH 254
Thu. Sep 23

1. a. -4
   b. 4
   c. 5
   d. 7
   e. $\sqrt{\sin^2(t)+\cos^2(t)}$ ~~not~~ $\tan(t)$
   $= 1$
   f. $\sin(t)+\cos(t)$

2. $\frac{\pi}{3}$

3. a. $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)} = \frac{1}{2}$

   b. $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2$

   c. $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x\to 4} \frac{\cancel{(2-\sqrt{x})}}{\cancel{(2-\sqrt{x})}(2+\sqrt{x})} = \lim_{x\to 4} \frac{1}{2+\sqrt{x}} = \frac{1}{4}$

   d. $\lim_{x\to 1} \frac{3x-2}{x^2+1} = \frac{1}{2}$

   e. $\lim_{x\to\infty} \arctan(x) = 1.5$

   f. $\lim_{x\to\infty} \frac{1}{x} = 0$

4. a. $\frac{d}{dx}(x^2\sin(x))$

   $= x^2 \cdot \frac{d}{dx}(\sin(x)) + \sin(x)\cdot 2x$

   $= x^2 \cdot \cos x + \sin(x)\cdot 2x$

   $= x^2\cos x + 2x\sin(x)$

sin →
cos
tan

   b. $\frac{d}{dt}(17t^5) = 85t^4 + C$

   c. $\frac{d}{dx}(y^2\sin(y)) = 0$

   d. $\frac{d}{dt}(\ln(t)) = \frac{1}{t}$

   e. $\frac{d}{dy}\left(\frac{y^2}{y^3+1}\right) = \frac{\frac{d}{dy}(y^2)(y^3+1) - \frac{d}{dy}(y^3+1)\cdot y^2}{(y^3+1)^2}$

   $= \frac{2y(y^3+1) - 3y^2\cdot y^2}{(y^3+1)^2}$

   $= \frac{2y^4+2y-3y^4}{y^6+2y^3+1}$

f. $\frac{d}{dx}\left(x^2 \cdot e^{5x}\right)$

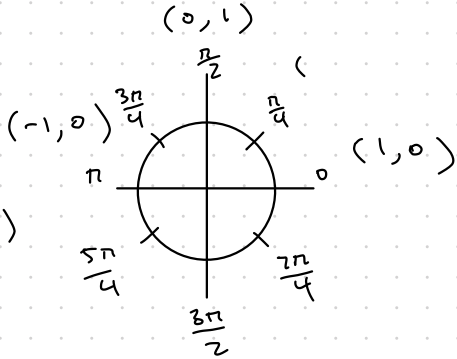
$= x^2 \cdot 5e^{5x} + 2x \cdot e^{5x}$

$= \boxed{x^2\, 5e^{5x} + 2xe^{5x}}$

5.

a. $\int_0^{\sqrt{\pi}} h\,dy = F(\sqrt{\pi}) - F(0)$

$=$



b. $\int_0^{\sqrt{\pi}} \sin(x^2)\,dy$

$= \sin(\pi) - \sin(0)$

$= -1 - 1 = -2$

c. $\int_{-1}^{1} y^2\,dy$

$= (y^3 - y^2)\Big|_{-1}^{1} = (y^3 - y^2)\Big|_{-1}^{1}$

$= (1^3 - 1^2) - ((-1)^3 - (-1)^2)$

$= 0 - (-1 - 1) = 0 - (-2)$

$= 2$

d. $\int_{x^2}^{x} at^2 + bt\,dt$

$=$

6. Use the radian settings on their calculator.

7. $\int_1^3 \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$

$= \int_1^3 \sqrt{(2t)^2 + \frac{{}^{2}\cancel{4}\sqrt{2}}{\cancel{8}} \cdot \frac{\cancel{3}}{\cancel{2}}\, t^{1/2} + 2^2}\, dt$

$= \int_1^3 \sqrt{4t^2 + 2\sqrt{2}\, t^{1/2} + 4}\, dt$

$= \sqrt{4 + 8 + 4}$

$= \sqrt{64} - \sqrt{16}$

$= 8 - 4 = 4$

8. $\int \cos(x \cdot y)\, dy$

Let $u = xy$

then $du = x \cdot dy$

so $dy = \frac{1}{x} \cdot du$

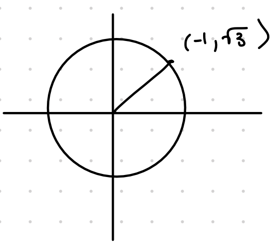
$\int \cos(u) \cdot dy = \int \cos(u) \cdot \frac{1}{x}\, du$

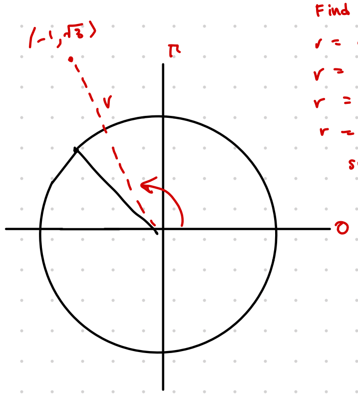
$= \frac{1}{x} \int \cos(u) \cdot du$

$= \frac{1}{x} (\sin(u)) + C$

9. Hundredth $0.0\underline{0}$

radians + hundredths





Find r:

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$r = \sqrt{1 + 3}$

$r = 2$

so $(r, \theta) = \left(2, \frac{2\pi}{3}\right)$

10.

a. Convert from radians to degrees using the ratio $\frac{180}{\pi}$

$$A = \frac{\pi}{12}$$

$$A = \frac{\cancel{\pi}}{12} \cdot \frac{180}{\cancel{\pi}}$$

$$A = \frac{180}{12} = 15^\circ$$

$$C = \frac{\pi}{6}$$

$$C = \frac{\pi}{6} \cdot \frac{180}{\pi}$$

$$= 30^\circ$$

$$B = 180^\circ - 15^\circ - 30^\circ$$

$$= 185^\circ$$

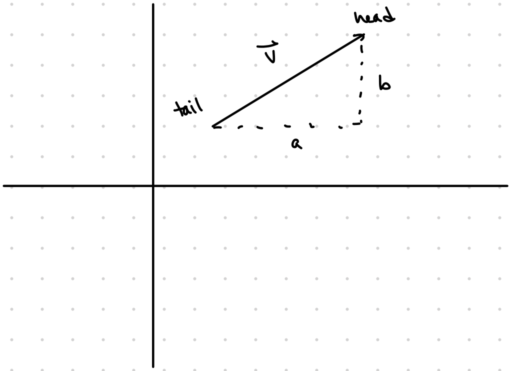
Then using Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{3}{\sin 15^\circ} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Fri Sep 24

$\mathbb{R}^2$ (xy-plane)



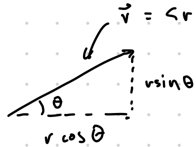
$\vec{v} = \langle a, b \rangle$

Magnitude

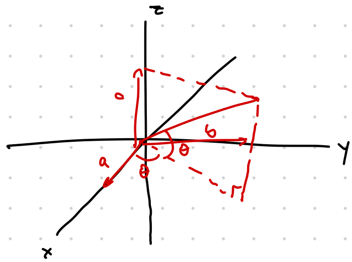
$|\vec{v}| = \sqrt{a^2 + b^2}$

or $\|\vec{v}\|$

$\vec{v} = \langle r\cos\theta, r\sin\theta \rangle$



$\mathbb{R}^3$ (xyz-space



$\vec{v} = \langle a, b, c \rangle$

$|\vec{v}| = \sqrt{a^2 + b^2 + c^2} = r$

$a = r\sin\theta\cos\theta$

$b = r\cos\theta\sin\theta$

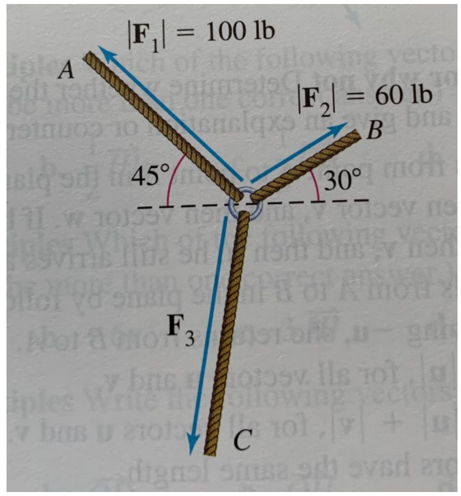
$c = r\sin\theta$

Exemplar

1a. $\vec{F}_1 = \langle -100\cos 45°, 100\sin 45° \rangle = \langle -50\sqrt{2}, 50\sqrt{2} \rangle$

$\vec{F}_2 = \langle 60\cos 30°, 60\sin 30° \rangle = \langle 30\sqrt{3}, 30 \rangle$

(1) Ropes are attached to a central ring. Three people begin pulling on the ropes in different directions, exerting forces $\vec{F_1}, \vec{F_2}, \vec{F_3}$ (see figure).



(a) Using trigonometry, let's find the components of $\vec{F_1}$ and $\vec{F_2}$.

(b) Let's find the force vector $\vec{F_3}$ that the person at point C must apply so that no one moves (the system is at equilibrium).

(c) Let's find the magnitude of the vector found in part (b). (Does the answer surprise you?)

(d) Suppose a machine is attached to the central ring that applies a force of 200 lb in the $\vec{k}$ direction (i.e. straight up). The three people must now alter the forces they apply to maintain equilibrium. Assume the people at points A and B increase the total magnitude of the force they exert by 50% each while maintaining the force they exert in the horizontal directions. Also assume that the person at point C maintains the same horizontal components of the force they apply.

(i) Let's find the new force vectors for the persons at points A and B.

$$\vec{F_4} = \langle 0, 0, 200 \rangle$$

$$\vec{F_1} = 1.5 \langle -50\sqrt{2}, 50\sqrt{2}, 0 \rangle$$

$$= \langle -75\sqrt{2}, 75\sqrt{2}, 0 \rangle$$

(ii) Let's determine how the person at point C should adjust their force to maintain equilibrium

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} = 0$$

$$\vec{F_3} = -\vec{F_1} - \vec{F_2} - \vec{F_4}$$

$$= \langle 75\sqrt{2} - 45\sqrt{3}, -75\sqrt{2} - 45, -200 \rangle$$

Inclass activity

1. let $\vec{v} = -i - 2j + 2k$

$\vec{u} = c\vec{v}$ is 12

Find scalar c

$\vec{u} = c\vec{v}$

$\vec{u} = c\vec{v}$

$\vec{u} = 12$

$12 = c \cdot 3$

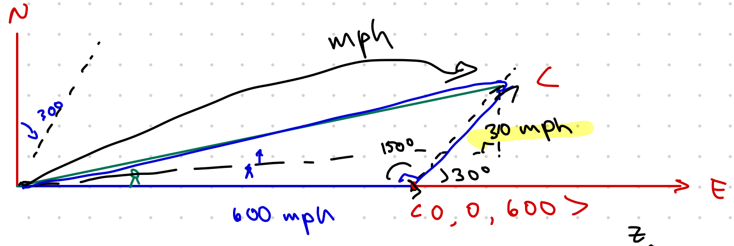
$c = -4$

$$\vec{v} = \sqrt{(-1)^2 + (-2)^2 + (2)^2}$$
$$= \sqrt{1 + 4 + 4}$$
$$= \sqrt{9}$$
$$= 3$$

$\sqrt{(-1)^2 + (-2)^2 + 2^2}$

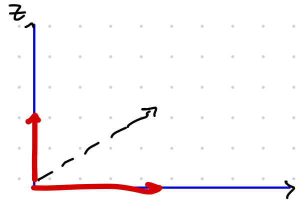
$c = \boxed{-4}$ because the force must counteract v.

3. 600 mph



$\vec{W} = \langle 0, 0, 600 \rangle$.

$\vec{D} = \langle$

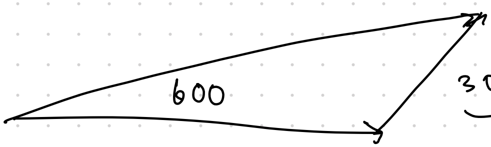


$(x, y) = (r\cos\theta, r\sin\theta)$

$\vec{w} = 30\cos$

3. (3) A plane at cruising altitude heads due east at 600 mph while experiencing a 30 mph wind blowing 30° east of north. What is the plane's groundspeed in mph, which is the speed as observed from the ground? Round to the nearest tenth.
Hint: Speed is the magnitude of velocity.

626.16



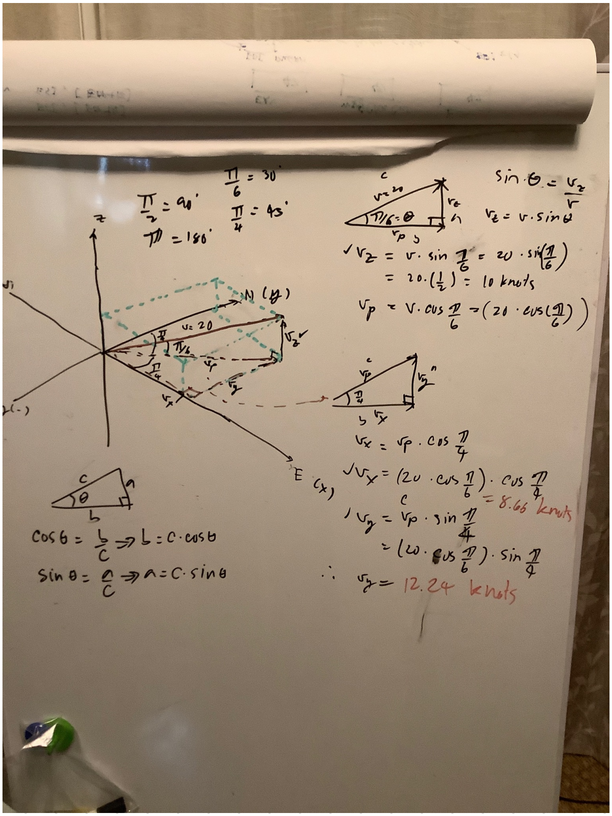
$30 \cdot \sin 30° \rightarrow 15.5$

615.5

$600 + 15.5$

4.

2.


π/2 = 90°
π/6 = 30°
π/4 = 45°
π = 180°
sin θ = vz / v
vz = v · sin θ
v = 20
π/6 = θ
vp
✓ vz = v · sin π/6 = 20 · sin(π/6)
= 20 · (1/2) = 10 knots
vp = v · cos π/6 = (20 · cos(π/6))
N (y)
E (x)
z
vx = vp · cos π/4
✓ vx = (20 · cos π/6) · cos π/4
= 8.66 knots
✓ vy = vp · sin π/4
= (20 · cos π/6) · sin π/4
∴ vy = 12.24 knots
cos θ = b/c ⇒ b = c · cos θ
sin θ = a/c ⇒ a = c · sin θ

$\frac{1}{2}u - 3v$

$\frac{1}{2}\langle 6, 14\rangle - 3\langle -1, 5\rangle$

$\langle 3, 7\rangle - \langle -3, 15\rangle$

$\langle 6, -8\rangle$

$x = \|v\| \cos\theta$

$y = \|v\| \sin\theta$

$x = 10(\cos 120°) = -5$

$y = 10(\sin 120°) = \frac{\cancel{10}^{5}\sqrt{3}}{\cancel{2}} = 5\sqrt{3}$

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||}\vec{v}$$

normalizing v

$$\vec{v} = \langle 4, -3 \rangle$$

$$\vec{v} = 4\vec{i} - 3\vec{j}$$

$$||\vec{v}|| = \sqrt{(4)^2 + (-3)^2}$$

$$||\vec{v}|| = \sqrt{16 + 9}$$

$$||\vec{v}|| = \sqrt{25} = 5$$

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||} \cdot \vec{v}$$

$$= \frac{1}{5} \cdot \langle 4, -3 \rangle$$

$$= \langle \frac{4}{5}, \frac{-3}{5} \rangle$$

$$||\vec{v}|| = \sqrt{(\tfrac{1}{4})^2 + (-\tfrac{3}{4})^2}$$

$$||\vec{v}|| = \sqrt{\frac{1}{16} + \frac{9}{16}}$$

$$||\vec{v}|| = \sqrt{\frac{10}{16}}$$

component form

$$\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||} \cdot \vec{v}$$

$$= \frac{1}{\sqrt{\frac{10}{16}}} \cdot \langle \frac{1}{4}, \frac{-3}{4} \rangle$$

$$= \langle \frac{4}{4\sqrt{10}}, \frac{-3}{\cancel{4}} \cdot \frac{\cancel{4}}{\sqrt{10}} \rangle$$

$$= \langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$$

Normalize

$$\vec{v} = \langle -2, 5 \rangle$$

$$||\vec{v}|| = \sqrt{(-2)^2 + (5)^2}$$

$$||\vec{v}|| = \sqrt{4 + 25}$$

$$||\vec{v}|| = \sqrt{29}$$

$$v = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||} \cdot \vec{v}$$

$$= \frac{1}{\sqrt{29}} \cdot \langle -2, 5 \rangle$$

$$= \langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$$

$$= \langle \frac{-2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29} \rangle$$

2.1.2 Unit Vectors

$$u = \frac{1}{\|v\|} v$$

$$\| kv \| = |k| \cdot \|v\|$$

For $u = \frac{1}{\|v\|} v$, it follows that $\|u\| = \frac{1}{\|v\|}(\|v\|) = 1$

Question

$v = \langle 1, 2 \rangle$

a. Find unit vector with the same direction as v

First: we need the magnitude

$$\|v\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

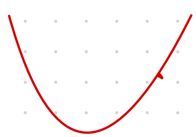
Next:

$$u = \frac{1}{\|v\|} \cdot v = \frac{1}{\sqrt{5}} \cdot \langle 1, 2 \rangle$$

$$= \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

b. $w = 7u = 7 \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \right\rangle$

Q: $v = \langle 9, 2 \rangle$



z_0

$Q(2, z_0)$

$P(1, 1)$

$z_0 = f(x_0) + f'(x_0)$

$= 1 + 4$

$= 5$ i t

unit vector, P Q

$\langle 2 - 1, 16 - 1 \rangle$

$\langle 1, 15 \rangle$

$\|v\| = \sqrt{1^2 + 15^2}$

$= \sqrt{226}$

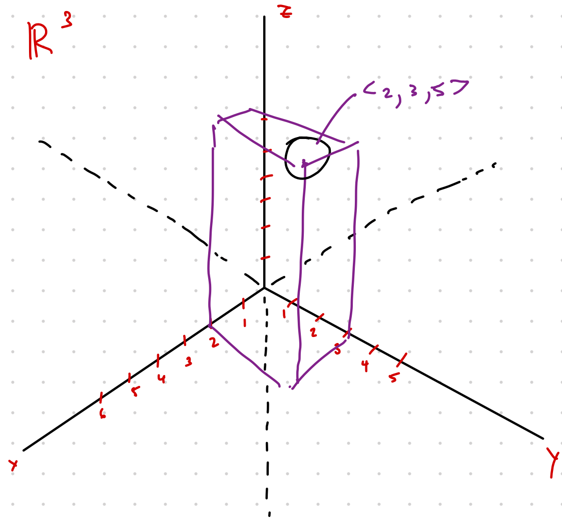
$u = \frac{1}{\|v\|} \cdot v$

$= \frac{1}{\sqrt{226}} \cdot \langle 1, 15 \rangle = \left\langle \frac{1}{\sqrt{226}}, \frac{15}{\sqrt{226}} \right\rangle$

$$\frac{30}{600} = \frac{1}{}$$

$v = \langle 9, 2 \rangle$

$\mathbb{R}^3$



$\frac{1}{2} u - 3v$

$\frac{1}{2} \langle 6, 14 \rangle - 3 \langle -1, 5 \rangle$

$\langle 3, 7 \rangle - \langle -3, 15 \rangle$

$\langle 6, -8 \rangle$

unit vector $v = \langle \frac{1}{4}, -\frac{3}{4} \rangle$

$$\|v\| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{-3}{4}\right)^2}$$

$$= \sqrt{\frac{5}{8}}$$

$$u = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

$$= \frac{1}{\sqrt{\frac{5}{8}}} \cdot \langle \frac{1}{4}, \frac{-3}{4} \rangle$$

$$= \langle \frac{1}{4\sqrt{\frac{5}{8}}}, \frac{-3}{4\sqrt{\frac{5}{8}}} \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

$\vec{v} = \langle 4, -3 \rangle$

normalize:

$$\|\vec{v}\| = \sqrt{4^2 + (-3)^2}$$

$$= \sqrt{16 + 9} = 5$$

$$\vec{u} = \frac{\vec{v}}{5} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$$

$v = \langle 1, 2 \rangle$

a. unit vector:

$$\|v\| = \sqrt{1^2 + 2^2}$$

$$= \sqrt{5}$$

$$u = \frac{1}{\|u\|} v = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$$

$$= \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

b. $w = 7u = 7 \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$$= \langle \frac{7}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

Find the component form of this vector

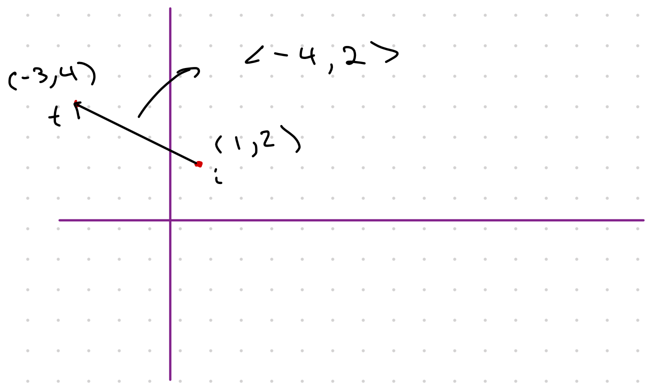
$(x_i, y_i) \qquad (x_t, y_t)$

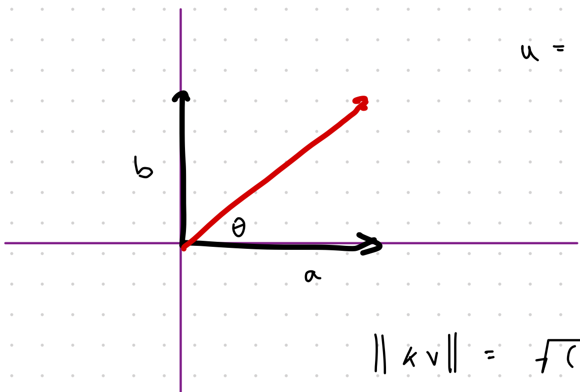
$\langle x_t - x_i , \; y_t - y_i \rangle$

$(1,2)$ initial $\qquad (-3,4)$ terminal

$\langle -3-(1), 4-2 \rangle$

$\langle -4 , 2 \rangle$





$u = \langle a, b \rangle$

component form

$$\|kv\| = \sqrt{(kx_1)^2 + (ky_1)^2}$$
$$= \sqrt{k^2 (x_1^2 + y_1^2)}$$
$$= |k| \sqrt{x_1^2 + y_1^2}$$
$$= |k| \, \|v\|$$

Find magnitude of a Vector and Perform Vector Operations in Component Form

a. v in component form and find $\|v\|$

$$v = \langle 8-2, 13-5 \rangle$$
$$= \langle 6, 8 \rangle$$

$$\|v\| = \sqrt{6^2 + 8^2}$$
$$= \sqrt{36 + 64}$$
$$= \sqrt{100} = 10$$

b. $v + w$

- add x-components and y-components separately

$$v = \langle 6, 8 \rangle$$
$$w = \langle -2, 4 \rangle$$
$$= \langle 4, 12 \rangle$$

let $a = \langle 7, 1 \rangle$ $b = \langle -4, -3 \rangle$

$$\|b\| = \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{16 + 9} = 5$$

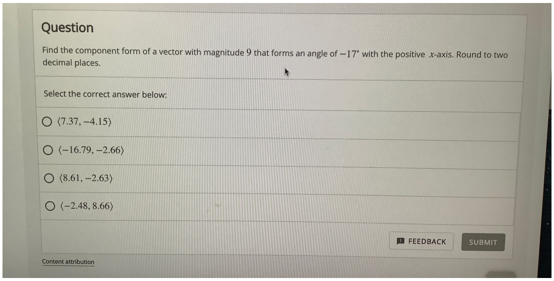
$$3a - 4b = \langle 21, 3 \rangle - \langle -16, -12 \rangle$$
$$= \langle 37, 15 \rangle$$

$\frac{1}{2}u \cdot 3v$

$\langle 3, 7 \rangle - 3\langle -1, 5 \rangle$

$\langle 3, 7 \rangle - \langle -3, 15 \rangle$

$\langle 6, -8 \rangle$



Question

Find the component form of a vector with magnitude 9 that forms an angle of −17° with the positive x-axis. Round to two decimal places.

Select the correct answer below:

- ⟨7.37, −4.15⟩
- ⟨−16.79, −2.66⟩
- ⟨8.61, −2.63⟩
- ⟨−2.48, 8.66⟩

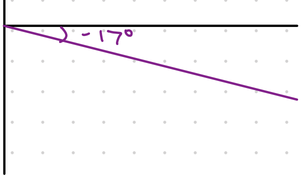
FEEDBACK SUBMIT

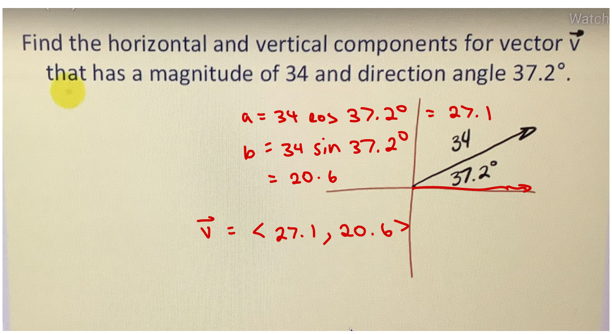
Content attribution

$a = |\vec{u}| \cos\theta \qquad b = |\vec{u}| \sin\theta$

$a = 9\cos(-17°) = 8.61$

$b = 9\sin(-17°) = -2.63$





Find the distance between two points in space

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}, \quad \frac{z_1 + z_2}{2}$$

$$\sqrt{(-3-2)^2 + (-2-(-4))^2 + (1-7)^2}$$

$$= \sqrt{(-5)^2 + (2)^2 + (-6)^2}$$

$$= \sqrt{25 + 4 + 36}$$

$$= \sqrt{65}$$

5 13

$$d = \sqrt{(1-(-1))^2 + (-4-(-4))^2 + (6-(-1))^2}$$

$$= \sqrt{2^2 + (0)^2 + (7)^2}$$

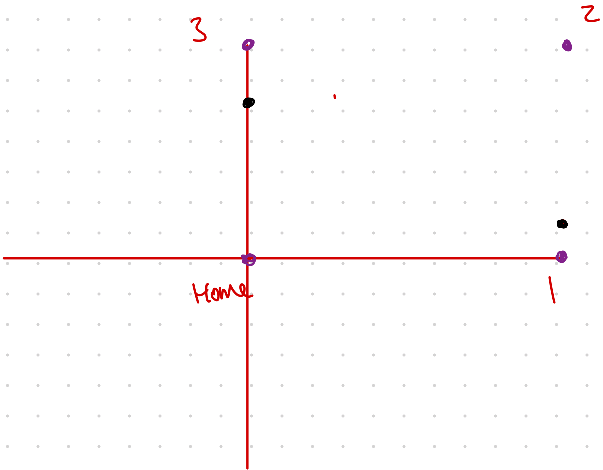
$$= \sqrt{4 + 0 + 49}$$

$$= \sqrt{53}$$

(90, 0, 5) (0, 75, 35)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{}$$



vector v with magnitude $||v|| = 10$ and the same direction as vector $u = \langle 2, -1 \rangle$

$$u = \frac{1}{||\vec{v}||} \cdot \vec{v}$$

unit

$u = \langle 2, -1 \rangle$

$$\langle 2, -1 \rangle = \frac{1}{10} \cdot \vec{v}$$

$$10 \langle 2, -1 \rangle = \vec{v}$$

$$\langle 20, -10 \rangle = \vec{v}$$

$||v|| = 7$ $\quad u = \langle 3, -5 \rangle$

$||v|| = 7$ $\quad u = \langle 3, -5 \rangle$

$$||u|| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

$$w = \frac{1}{||u||} u$$

$$= \frac{1}{\sqrt{34}} \langle 3, -5 \rangle$$

$$= \left\langle \frac{3}{\sqrt{34}}, \frac{-5}{\sqrt{34}} \right\rangle$$

$$\frac{1}{||v||} v = w$$

$$\frac{1}{7} v = \left\langle \frac{3}{\sqrt{34}}, \frac{-5}{\sqrt{34}} \right\rangle$$

$$v = \left\langle \frac{21}{\sqrt{34}}, \frac{-35}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{21\sqrt{34}}{34}, \frac{-35\sqrt{34}}{34} \right\rangle$$

$\|v\| = \sqrt{29} \quad u = \langle 3, -5 \rangle$

$\|v\| = 7 \qquad u = \langle 3, -5 \rangle$

$\|u\| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$

$w = \frac{1}{\|u\|} u$

$= \frac{1}{\sqrt{34}} \langle 3, -5 \rangle$

$= \langle \frac{3}{\sqrt{34}}, \frac{-5}{\sqrt{34}} \rangle$

$\frac{1}{\|v\|} v = w$

$\frac{1}{7} v = \langle \frac{3}{\sqrt{34}}, \frac{-5}{\sqrt{34}} \rangle$

$v = \langle \frac{21}{\sqrt{34}}, \frac{-35}{\sqrt{34}} \rangle$

$= \langle \frac{21\sqrt{34}}{34}, \frac{-35\sqrt{34}}{34} \rangle$

$\|v\| = \sqrt{29} \quad w = \langle 4, -10 \rangle$

$\frac{1}{\|v\|} v = w$

$\frac{1}{\sqrt{29}} v = \langle 4, -10 \rangle$

$v = \langle 4\sqrt{29}, -10\sqrt{29} \rangle$

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(4-1)^2 + (-1-(-5))^2 + (-1-4)^2}$$

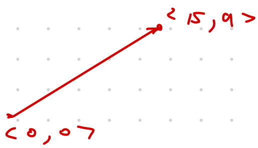
$$= \sqrt{3^2 + 4^2 + (-5)^2}$$

$$= \sqrt{9 + 16 + 25}$$

$$= \sqrt{50}$$

magnitude

$\langle 15, 9 \rangle$



$\sqrt{15^2 + 9^2} =$

$\overrightarrow{RP}$

$\langle -1-(-3), 3-7 \rangle$

$\langle 2, -4 \rangle$

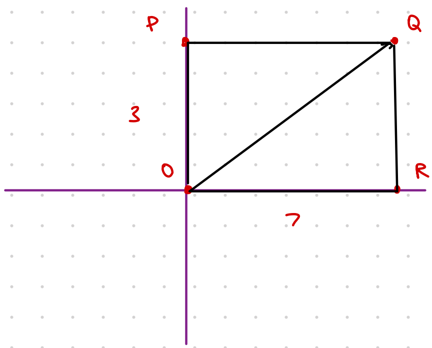
vector v

67%

u

$\|u\| = 8$

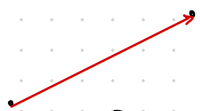
$\theta = \pi$



$\langle 3, 7\rangle - 3\langle -1, 5\rangle$

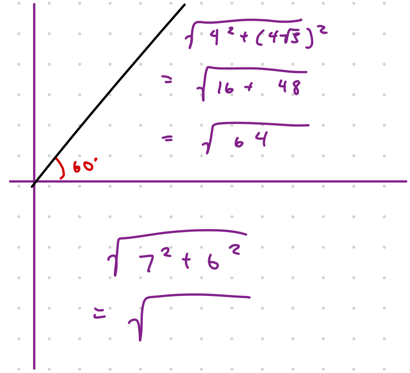
$\langle 3, 7\rangle - \langle -3, 15\rangle$

$\langle 6, -8\rangle$



$\sqrt{7^2 + 6^2}$

$= \sqrt{}$



$\sqrt{4^2 + (4\sqrt{3})^2}$

$= \sqrt{16 + 48}$

$= \sqrt{64}$

$\sqrt{7^2 + 6^2}$

$= \sqrt{}$

Tuesday Sep 27

Given a vector in space with initial point $(5,-1,1)$ and terminal point $(-1,1,4)$

a. Position vector

$\langle -1-(5),\ 1-(-1),\ 1-4 \rangle$

$\langle -6,\ 2,\ -3 \rangle$

b Vector as a linear combination of i, j, k

$\vec{v} = -6\vec{i},\ 2\vec{j},\ -3\vec{k}.$

$\overrightarrow{PQ}$

$Q(1,-4,3)$

terminal

$P(-3,5,-3)$

$\overline{PQ} = \langle 1-(-3),\ -4-5,\ 3-(-3) \rangle$

$= \langle 4,\ -9,\ 6 \rangle$

standard unit vectors

$4i,\ -9j,\ 6k$

Find terminal point $\overrightarrow{PQ} = \langle 2,0,-3 \rangle$

$P(0,2,-3)$

Q?

$\overline{PQ} = \langle 2,\ 0,\ -3 \rangle$

$= \langle x_1-0,\ y_1-2,\ z_1-(-3) \rangle$

$Q = (2,\ 2,\ -6)$

$u = \langle 2\cos(t), -2\sin(t), 3 \rangle$

magnitude: $2u$

$\|u\| = \sqrt{(2\cos(t))^2 + (-2\sin t)^2 + (3)^2}$

$= \sqrt{}$

$2u =$

3.2.1 Calculus of Vector Valued Functions

Derivative of Vector Valued Function

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

If $r'(t)$ exists, then r is diffentiable at t.
If $r'(t)$ exists for all t in an open interval (a,b) then r is differentiable over the interval (a,b)

$$r'(a) = \lim_{\Delta t \to 0^+} \frac{r(a+\Delta t) - r(a)}{\Delta t} \quad \text{and} \quad r'(b) = \lim_{\Delta t \to 0^-} \frac{r(b+\Delta t) - r(b)}{\Delta t}$$

$$r(t) = (3t+4)i + (t^2-4t+3)j$$

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[(3(t+\Delta t)+4)i + ((t+\Delta t)^2 - 4(t+\Delta t)+3)j\right] - \left[(3t+4)i + (t^2-4t+3)j\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(\cancel{3t} + 3\Delta t \cancel{+4} \cancel{-3t} \cancel{-4})i + (\cancel{t^2} + 2t\Delta t + (\Delta t)^2 \cancel{-4t} - 4\Delta t \cancel{+3} \cancel{-t^2} \cancel{+4t} \cancel{-3})j}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(3\cancel{\Delta t})i + (2t\cancel{\Delta t} + (\Delta t)^2 - 4\cancel{\Delta t})j}{\cancel{\Delta t}}$$

$$= \lim_{\Delta t \to 0} 3i + (2t + \Delta t - 4)j$$

$$= 3i + (2t-4)j$$

$$r(t) = (6t+8)i + (4t^2+2t-3)j$$

$$r'(t) = 6i + (8t+2)j$$

$$r(t) = (t\ln(t))i + (5e^t)j + (\cos(t) - \sin(t))k$$

$$r'(t) = (1+\ln t)i + 5e^t j + (-\sin(t) - \cos(t))k$$

$$= (1+\ln t)i + 5e^t j + (-\sin(t) - \cos(t))k$$

Use linear properties of derivatives of vector-valued functions

theorem: properties of the derivative of vector-valued functions

$$\frac{d}{dt} c r(t) = c r'(t)$$

$$\frac{d}{dt}[r(t) \pm u(t)] = r'(t) \pm u'(t)$$

$$\frac{d}{dt}[f(t)u(t)] = (f'(t)u(t))i + (f(t)u'(t))j$$

$$r(t) = 3\cos(t)i - 4t^3 j$$

$$\frac{d}{dt}[cr(t)] = cr'(t)$$

$$\frac{d}{dt}[r(t) \pm u(t)] = r'(t) \pm u'(t)$$

$$\frac{d}{dt}[3\cos(t)i - 4t^3 j] = \frac{d}{dt}[3\cos(t)i] - \frac{d}{dt}[4t^3 j]$$

$$= 3\frac{d}{dt}[\cos(t)i] - 4\frac{d}{dt}[t^3 j]$$

$$= -3\sin(t)i - 12t^2 j$$

$$r(t) = 5e^{2t}i + 3tj$$

$$r'(t) = 5 \cdot 2e^{2t}i + 3j$$

$$= 10e^{2t}i + 3j$$

$$\frac{d}{dt}\left[e^{-4t}(\cos(t))i + e^{-4t}(\sin(t))j\right]$$

$$= -4e^{-4t}\sin(t) - e^{-4t}\sin(t) - 4e^{-4t}\cos(t) + e^{-4t}\cos(t)$$

$$\cancel{e^{-4t}(-\sin(t)) - e^{-4t}(\cos(t))}$$

$-e^{-kt}(\sin(t) + k\cos(t))i + e^{-kt}(k\sin(t) - \cos(t))j$

$A = 3ti - 2t^2 j + 5t^3 k$ $\quad$ $B = -ti + 3t^2 j - t^3 k$

$A' = 3i - 4t j + 15t^2 k$ $\quad$ $B' = -i + 6t j - 3t^2 k$

$A'' = 0i - 4j + 30t k$ $\quad$ $B'' = 0i + 6j - 6t k$

$A'' - B''$ at $t = 2$

$\langle 0, -4, 60 \rangle - \langle 0, 6, -12 \rangle$

$\langle 0, -10, 72 \rangle$

$0i - 10j + 72k.$

Use dot product, cross product, and chain rule properties of derivatives of vector valued functions.

4. $\frac{d}{dt}[r(t)\cdot u(t)] = r(t)\cdot u'(t) + r'(t)\cdot u(t)$

5. $\frac{d}{dt}[r(t)\times u(t)] = r(t)\times u'(t) + r'(t)\times u(t)$

6. $\frac{d}{dt}[r(f(t))] = r'(f(t))\,f'(t)$

$\vec{r}(t) = 3\tan(7t)\,i + \sin^2(t)\,j - 4\ln(t)\,k$

$\vec{r}'(t) = 3\sec^2(7t)\cdot 7i + 2\sin(t)\cos(t)\,j - \frac{4}{t}\,k$

$\vec{r}'(t) = \langle 21\sec^2(7t),\ 2\sin(t)\cos(t),\ -\frac{4}{t}\rangle$

$\frac{d}{dt}[r(t)\times u(t)]$

$\vec{r} = \langle t^2, 2t^3, -t\rangle$ $\qquad$ $\vec{r}' = \langle 2t, 6t^2, -1\rangle$

$\vec{u} = \langle t, t^4, 4\rangle$ $\qquad$ $\vec{u}' = \langle 1, 4t^3, 0\rangle$

$$\vec{r}\times\vec{u}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & 2t^3 & -t \\ 1 & 4t^3 & 0 \end{vmatrix} = \begin{vmatrix} 2t^3 & -t \\ 4t^3 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} t^2 & -t \\ 1 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} t^2 & 2t^3 \\ 1 & 4t^3 \end{vmatrix}\vec{k}$$

$\vec{r}\times\vec{u}' = \langle 4t^4, -t, 4t^5 - 2t^3\rangle$

$$\vec{r}'\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 6t^2 & -1 \\ t & t^4 & 4 \end{vmatrix} = \begin{vmatrix} 6t^2 & -1 \\ t^4 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 2t & -1 \\ t & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 2t & 6t^2 \\ t & t^4 \end{vmatrix}k$$

$$\vec{r}'\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 6t^2 & -1 \\ t & t^4 & 4 \end{vmatrix} = \langle 24t^2 + t^4,\ -9t,\ 2t^5 - 6t^3\rangle$$

$\vec{r}\times\vec{u}' + \vec{r}'\times\vec{u} = \langle 24t^2 + 5t^4,\ -10t,\ 6t^5 - 8t^3\rangle$

$r(t) = 3\tan(\underbrace{7t}_{u})\, i + \sin^2(t)\, j - 4\ln t\, k$

$a' = 7$

$\underbrace{(\sin t)^2}_{u}$

$\vec{r}'(t) = 3\sec^2(7t)\cdot 7\vec{i} + 2\sin(t)\cdot\cos(t)\vec{j} - 4\frac{1}{t}\vec{k}$

$\vec{r}'(t) = \langle 21\sec^2(7t),\ 2\sin(t)\cdot\cos(t),\ -\frac{4}{t} \rangle$

$r(t) = 5\sec(4t)\, i - e^{t^2} j$

$r'(t) = 20\sec(4t)\tan(4t)\, i - 2t e^{t^2} j$

$r(t) = \ln(t)\, j - \overbrace{e^{-9t}}^{v}\, \overbrace{\sin(-5t)}^{u}\, k$

$u = \sin(-5t)$

$u' =$

$v = e^{-9t}$

$r'(t) = \frac{1}{t} j + \left(5e^{-9x}\cos(5t) - 9e^{-9x}\sin(5t)\right) k$

$r(t) = 3t\,i - 4t^2\,j$

$s(t) = e^t r(t) = e^t(3t\,i - 4t^2\,j)$

$= 3te^t\,i - 4t^2e^t\,j$

$s'(t) = (3t+3)e^t\,i - 4t(t+2)e^t\,j$

$\begin{array}{r} {}^{4}180 \\ 5 \\ \hline 900 \end{array}$

-15
-180

$\begin{array}{r} 180 \\ 10 \\ \hline \end{array}$

$\begin{array}{r} 1800 \\ 900 \\ \hline 2700 \end{array}$

$r(t) = -3t^5\,i + 5t\,j + 2t^2\,k$

$r'(t) = -15t^4\,i + 5\,j + 4t\,k$

$r''(t) = -60t^3\,i + 0\,j + 4\,k$ $\quad r'''(t) = -180t^2\,i + 0\,j + 0\,k$

$\frac{d}{dt}[r(t)\cdot u(t)] = r(t)\cdot u'(t) + r'(t)\cdot u(t)$

$\frac{d}{dt}[r'(t)\cdot r''(t)] = r''(t)\cdot r''(t) + r'(t)\cdot r'''(t)$

$= ((-60t^3)(-60t^3) + 0 + 16) + (2700t^6)$

$= 3600t^6 + 16 + 2700t^6$

$6300t^6$

To prove property iv. let $\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j}$ and $\mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$. Then

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\
&= f_1'(t)f_2(t) + f_1(t)f_2'(t) + g_1'(t)g_2(t) + g_1(t)g_2'(t) \\
&= f_1'(t)f_2(t) + g_1'(t)g_2(t) + f_1(t)f_2'(t) + g_1(t)g_2'(t) \\
&= (f_1'\,\mathbf{i} + g_1'\,\mathbf{j})\cdot(f_2\,\mathbf{i} + g_2\,\mathbf{j}) + (f_1\,\mathbf{i} + g_1\,\mathbf{j})\cdot(f_2'\,\mathbf{i} + g_2'\,\mathbf{j}) \\
&= \mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t).
\end{aligned}$$

$u(t) = \langle t^2, 2t-6, 4t^5-12 \rangle$

$u'(t) = \langle 2t, 2, 20t^4 \rangle$ $\qquad u''(t) = \langle 2, 0, 80t^3 \rangle$

$u'(t) \times u(t) =$

$$\frac{d}{dt}\left[r(t) \times u(t) \right] = r(t) \times u'(t) + r'(t) \times u(t)$$

$$\frac{d}{dt}\left[u'(t) \times u(t) \right] = u'(t) \times u'(t) + u''(t) \times u(t)$$

$$= \qquad\qquad + $$

$$u'(t) \times u'(t) = \begin{array}{ccc} i & j & k \\ 2t & 2 & 20t^4 \\ 2t & 2 & 20t^4 \end{array}$$

$$= (40t^4 - 40t^4)i - (40t^5 - 40t^5)j$$

$$+ (4t - 4t)k = \langle 0, 0, 0 \rangle$$

$$u''(t) \times u(t) = \begin{array}{ccc} i & j & k \\ 2 & 0 & 80t^3 \\ t^2 & 2t-6 & 4t^5-12 \end{array}$$

$$= (0 - (80t^3)(2t-6))i -$$

$$(2(4t^5 - 12) - 80t^5)j +$$

$$(4t - 12)k$$

$$= (-(160t^4 - 480t^3))i -$$

$$(8t^5 - 24 - 80t^5)j +$$

$$(4t - 12)k$$

$$= -(-72t^5 - 24)$$

$$\frac{d}{dt}(r(t) \times u(t)) = r(t) \times u'(t) + r'(t) \times u(t)$$

$$=$$

$$r(t) = t\,i + 2\sin t\,j + 2\cos t\,k$$

$$r'(t) = 1\,i + 2\cos t\,j + 2\sin t\,k$$

$$u(t) = \frac{1}{t}\,i + 2\sin t\,j + 2\cos t\,k$$

$$u'(t) = -\frac{1}{t^2}\,i + 2\cos t\,j + 2\sin t\,k$$

$$r \times u' = \begin{array}{ccc} i & j & k \\ t & 2\sin t & 2\cos t \\ -\frac{1}{t^2} & 2\cos t & 2\sin t \end{array} = \left[4\sin^2(t) - 4\cos^2(t)\right]i - \left[2t\sin t + \frac{2}{t^2}\cos t\right]j + \left[2t\cos t + \frac{2\sin t}{t^2}\right]k$$

$$r' \times u = \begin{array}{ccc} i & j & k \\ 1 & 2\cos t & 2\sin t \\ \frac{1}{t} & 2\sin t & 2\cos t \end{array} = \left[4\cos^2 t - 4\sin^2 t\right]i - \left[2\cos t - \frac{2\sin t}{t}\right]j + \left[2\sin t - \frac{2\cos(t)}{t}\right]k$$

$$-2t\sin t - \frac{2}{t^2}\cos t - 2\cos t + \frac{2\sin t}{t}$$

$$2\sin t\left(\frac{1}{t} - t\right)$$

$$-2\cos t\left(\frac{1}{t^2} + 1\right)$$

$$2t\cos t + \frac{2\sin t}{t^2} + \left(2\sin t - \frac{2\cos(t)}{t}\right)$$

$$2t\cos t + \frac{2\sin t}{t^2} + 2\sin t - \frac{2\cos(t)}{t}$$

$$2\sin t\left(\frac{1}{t^2} + 1\right) + 2\cos t\left(t - \frac{1}{t}\right)$$

$$r(t) = \frac{1}{\sqrt{t^2-1}}\, i - \arcsin(t)\, j + e^{t^2}\, k$$

$$r'(t) = \frac{-t}{(x^2-1)^{3/2}} - \frac{1}{\sqrt{1-t^2}}\, j + 2t\, e^{t^2}\, k$$

$$= (t^2-1)^{-1/2}$$

$$= -\frac{1}{2}(t^2-1)^{-\frac{3}{2}}$$

$$r(t) = 3i + 4\sin(3t)\, j + t\cos(t)\, k$$

$$r'(t) = 0i + 12\cos(3t)\, j + (\cos(t) - t\sin(t))\, k$$

3.2.2 Tangent Vectors and Unit Tangent Vectors

$r(t) = \langle f(t), g(t) \rangle$

$r(t) = f(t)i + g(t)j + h(t)k$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$r'(t) \neq 0$

$r(t) = \langle t^3, 2t^2 \rangle$ $\quad\quad \langle 3t^2, 4t \rangle$

$\vec{r}(t) = \langle 1^3, 2(1)^2 \rangle$

$= \langle 1, 2 \rangle$

The point of tangency is $(1, 2)$

when $t = 1$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 3t^2, 4t \rangle}{\sqrt{(3t^2)^2 + (4t)^2}}$$

$$= \frac{\langle 3, 4 \rangle}{\sqrt{9 + 16}}$$

$$= \frac{\langle 3, 4 \rangle}{\sqrt{25}}$$

$$= \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

When $t = 1$ what is the unit tangent vector $r(t) = \langle 2t^3, t^2 \rangle$?

$r(1) = \langle 2, 1 \rangle$

$$\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 6t^2, 2t \rangle}{\sqrt{(6t^2)^2 + (2t)^2}}$$

$$= \frac{\langle 6t^2, 2t \rangle}{\sqrt{36t^4 + 4t^2}} = \frac{\langle 6, 2 \rangle}{\sqrt{36 + 4}}$$

$$= \frac{\langle 6, 2 \rangle}{\sqrt{40}} = \left\langle \frac{6}{\sqrt{40}}, \frac{2}{\sqrt{40}} \right\rangle$$

$$= \left\langle \frac{\cancel{6}\,3}{\cancel{2}\sqrt{10}}, \frac{\cancel{2}}{\cancel{2}\sqrt{10}} \right\rangle$$

$r(t) = 3e^{t} i + 2e^{-3t} j + 4e^{2t} k \quad \text{at} \quad t = \ln 2$

$r'(t) = 3e^{t} i + -6e^{-3t} j + 8e^{2t} k$

$r'(\ln 2) = 3e^{\ln 2} i + -6e^{-3\ln 2} j + 8e^{2\ln 2} k$

$= 6i - 0.75 j + 32 k$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{6i - 0.75j + 32k}{\sqrt{6^2 + (0.75)^2 + 32^2}} = \frac{6i - 0.75j + 32k}{32.6}$$

$$= \frac{6}{32.6} i - \frac{0.75}{32.6} j + \frac{32}{32.6} k$$

$$= 0.184i - 0.023j + 0.983k$$

$r(t) = e^{-\sin^2(t)} k, \quad 0 < t < \frac{\pi}{2}$

Find the unit tangent vector of $r(t) = e^{-\sin^2(t)} k, \quad 0 < t < \frac{\pi}{2}$

$r'(t) = -2e^{-\sin^2(x)} \cos(x) \sin(x) k$

$r'(0) = -2e^{-\sin^2(0)} \cos(0) \sin(0) k = 0k$

$r'(\frac{\pi}{2}) = -2e^{-\sin^2(\frac{\pi}{2})} \cos(\frac{\pi}{2}) \sin(\frac{\pi}{2})$

$r'(\frac{\pi}{4}) = -2e^{-0.5} \cos(\frac{\pi}{4}) \sin(\frac{\pi}{4})$

$r'(\frac{\pi}{4}) = -0.6065$

$\|r'(\frac{\pi}{4})\| = 0.60653$

$$\frac{r'(\frac{\pi}{4})}{\|r'(\frac{\pi}{4})\|} \cancel{= -1}$$

not neg. one

$r(t) = e^{-t}\sin(t)\, i + e^{-t}\cos(t)\, j + k$

$r'(t) = \left[-e^{-t}(\sin(t) - \cos(t))\right] i + \left[-e^{-t}(\sin(t) + \cos(t))\right] j + 0k$

$\|r'(t)\| = \sqrt{\left[-e^{-t}(\sin(t) - \cos(t))\right]^2 + \left[-e^{-t}(\sin(t) + \cos(t))\right]^2}$

3.2.3 The indefinite integral

$$\int \left(\frac{5}{t^2} i - 4\sqrt{t}\, j \right) dt$$

$$= \left(\int 5t^{-2}\, dt\right)\vec{i} - \left(\int 4\sqrt{t}\, dt\right)\vec{j}$$

$$= \left(\frac{5t^{-1}}{-1} + C_1\right)\vec{i} - \left(4 \cdot \frac{2}{3} t^{3/2} + C_2\right)\vec{j}$$

$$= \left(\frac{-5}{t} + C_1\right)\vec{i} - \left(\frac{8}{3} t^{3/2} + C_2\right)\vec{j}$$

$$\vec{C} = C_1\vec{i} - C_2\vec{j}$$

$$= \frac{-5}{t}\vec{i} - \frac{8}{3} t^{3/2} + C$$

$$\left(\int 6t\right) i + \left(\int \frac{1}{t}\right) j$$

$$3t^2\, i + \ln|t|\, j + C$$

Evaluate the integral

$$\int \langle \sqrt{t}, \frac{1}{t+1}, e^{-t} \rangle \, dt$$

$$\int \langle t^{1/2}, (t+1)^{-1}, e^{-t} \rangle$$

$$\langle \frac{2}{3} t^{3/2}, \ln(t+1), -e^{-t} \rangle + C$$

$$\langle \frac{2}{3} t\sqrt{t}, \ln(t+1), -e^{-t} \rangle + C$$

$$r'(t) = \left(\frac{2}{t}\right) i + \left(8t^3\right) j$$

$$r(1) = 2i - 2j$$

$$r'(t) = \frac{2}{t} i + 8t^3 j$$

$$r'(t) = \left(\frac{2}{t}\right) i + \left(8t^3\right) j$$

$$r'(t) = \langle e^t, e^{-t}, 1 \rangle$$

$$r(-1)$$

2.2.2 Graphs of Equations in 3D

Equation of a plane parallel to a coordinate plane

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$

2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$

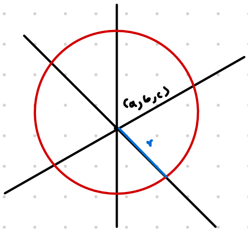
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by $x = a$

Q

1. write an equation passing through $(3, 11, 7)$ that is parallel to the yz-plane
   only y and z coordinates may vary, x coordinate will be constant
   $x = 3$
2. Find equation of the plane through $(6, -2, 9)$ $(0, -2, 4)$ $(1, -2, -3)$
   $y = -2$

$z = 8$

Find the equation of a sphere Z



center (a, b, c) radius r

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$(x-1)^2 + (y-6)^2 + (z+c)^2 =$$

$$\sqrt{(-3-1)^2 + (4-6)^2 + (z-(-2))^2}$$

$$\sqrt{16 + 4 + 16}$$

$$\sqrt{36} = 6 \qquad 6 = r$$

$$r^2 = 36$$

Find the equation of the largest sphere that is centered at $(5, 4, 9)$ and has interior in the first octant.

"interior in the first octant"

bound by xy plane where z is 0

distance from the center of the plane to the xy plane is 9

bound by xz plane when y is 0

distance from the center of the plane to the xz plane is 4

bound by yz plane when x is 0

distance from the center of the plane to the yz plane is 5

If the radius is higher than 4, then the sphere will cross into a different octant.

$$4^2 = 16$$

$\overline{PQ}$ when $P(-16, -3, 9)$ and $Q(-2, 3, 5)$

midpoint formula

$$\frac{-16 + (-2)}{2}, \quad \frac{(-3) + (3)}{2}, \quad \frac{(9) + 5}{2}$$

$$\frac{-18}{2}, \quad \frac{0}{2}, \quad \frac{14}{2}$$

center: $-9, \quad 0, \quad 7$

radius: distance from center to point

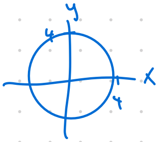
$$\sqrt{7^2 + 3^2 + (-2)^2} = \sqrt{62}$$

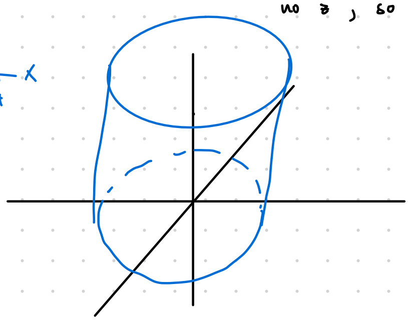
$$(x - (-9))^2 + (y - 0)^2 + (z - 7)^2 = 62$$

Graph Other Equations in 3D

$x^2 + y^2 = 16$ 　　 x-y plane

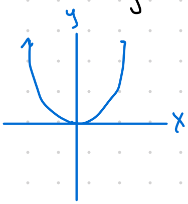
no z, so parallel to z

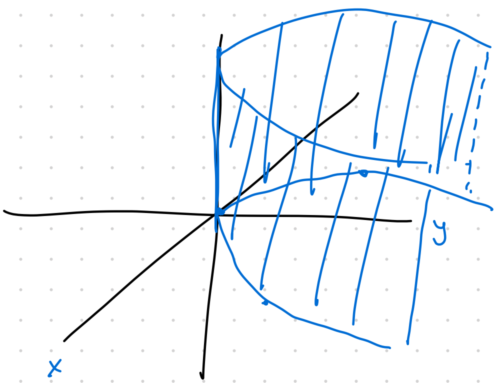




xy - plane

$y = x^2$





$y^2 + (z-2)^2 = 9$

4.1.1 Functions of Several Variables

function of two variables $z = (x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z

The set D is called the <u>domain</u> of the function

The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the following figure

Find domain and range of

a. $f(x,y) = 3x + 5y + 2$

no values of x and y that cause $f(x,y)$ to be undefined

domain : $\mathbb{R}^2$

range: $\mathbb{R}^2$

$3x + 5y + 2 = z$

$y = 0$ so $3x + 2 = z$

$3x = z - 2$

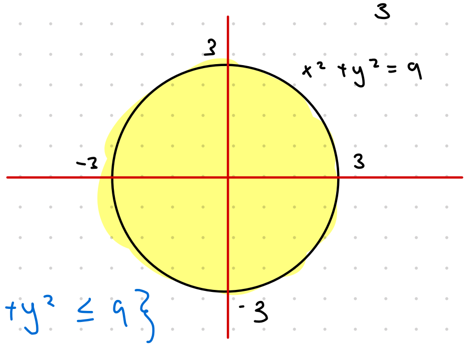
all Real numbers $x = \frac{z-2}{3}$

b. $g(x,y) = \sqrt{9 - x^2 - y^2}$

$9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$

$x^2 + y^2 \leq 9$



domain: $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$

range (x_0, y_0)

$g(x_0, y_0) = \sqrt{9 - x_0^2 - y_0^2}$

$= \sqrt{9 - (x_0^2 + y_0^2)}$

$x^2 + y^2 = 9$

$= \sqrt{9 - 9} = 0$

$[0, 3]$

if $x^2 + y^2 = 0$

$g(x_0, y_0) = \sqrt{9 - (x_0^2 + y_0^2)} = \sqrt{9 - 0} = 3$

Domain + range of $f(x,y) = \sqrt{36 - 9x^2 - 9y^2}$

Domain

$$36 - 9x^2 - 9y^2 \geq 0$$

$$36 \geq 9x^2 + 9y^2$$

$$\frac{36}{9} \geq x^2 + y^2$$

$$4 \geq x^2 + y^2$$

$$x^2 + y^2 \leq 4$$

Range

$$f(x,y) = \sqrt{36 - 9(x^2 + y^2)} \quad \text{if } x^2 + y^2 = 4$$

$$= \sqrt{36 - 9(4)} = 0$$

$$f(x,y) = \sqrt{36 - 9(x^2 + y^2)} \quad \text{if } x^2 + y^2 = 0$$

$$= \sqrt{36} = 6 \qquad [0, 6]$$

$4x^2 + y^2$

$f(x,y) = 4x^2 + y^2$ Domain: all real numbers

Range : $y = 0$

$4x^2 = z$

$[0, \infty)$

At the origin $V(0,0) = 0$

for any positive c, $V(0, \sqrt{c}$

$g(x, y) = -\sqrt{x^2 + y^2 - 4}$

Domain

$$x^2 + y^2 - 4 \geq 0$$

$$x^2 + y^2 \geq 4$$

Range $g(x,y) = -\sqrt{x^2+y^2-4}$ if $x^2+y^2=4$

$= -\sqrt{4-4}$

$= -0 = 0$

$g(x,y) =$ if $x^2+y^2>4$

like 5

$f(x,y) = \frac{y+2}{x^2}$

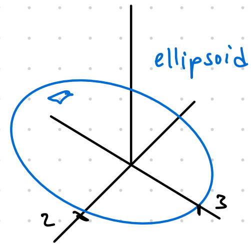
Domain $x \neq 0$

Range: all real numbers

Identify the graph of a two variable function

Sketch graph of the function $f(x,y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$

$$f(x,y) = \sqrt{1 - \left(\frac{x^2}{4} + \frac{y^2}{9}\right)}$$



domain : $\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \right\}$

range : $\{ z \in \mathbb{R}^2 \mid 0 \leq z \leq 1 \}$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

vertices at $(0, \pm 3)$ co-vertices at $(\pm 2, 0)$

$$f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{16}} = \sqrt{1 - \left(\frac{x^2}{9} + \frac{y^2}{16}\right)}$$

domain: $\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{9} + \frac{y^2}{16} \leq 1 \right\}$

range: $\{ z \in \mathbb{R}^2 \mid 0 \leq z \leq 1 \}$

vertices at $(0, \pm 4)$ co vertices at $(\pm 3, 0)$

$$z = x^2 + y^2$$

elliptic paraboloid

4.1.3 Graphs and Level Curves of Functions of Three Variables

Find the domain of a function of 3 variables

$$f(x,y,z) = x^2 - 2xy + y^2 + 3yz - z^2 + 4x - 2y + 3x - 6$$

and

$$g(x,y,t) = (x^2 - 4xy + y^2)\sin t - (3x + 5y)\cos t$$

Domain for functions of Three Variables

a. $f(x,y,z) = \dfrac{3x - 4y + 2z}{\sqrt{9 - x^2 - y^2 - z^2}}$

denominator can't be zero

$$\sqrt{9 - x^2 - y^2 - z^2} > 0$$

$$9 - x^2 - y^2 - z^2 > 0$$

$$9 > x^2 + y^2 + z^2$$

ball with radius 3

b. $g(x,y,t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$

$$2t - 4 \geq 0$$

$$t \geq 2$$

$$x^2 - y^2 \neq 0$$

$$x^2 \neq y^2$$

$$y \neq \pm x$$

$f(x,y,z) = \ln(x+y+z)$

$$x + y + z > 0$$

$f(x,y,z) = \ln\left(\sqrt{1 - x^2 - y^2 - z^2}\right)$

$$\sqrt{1 - x^2 - y^2 - z^2} > 0$$

$$1 - x^2 - y^2 - z^2 > 0$$

$$1 > x^2 + y^2 + z^2$$

$f(x,y,z) = \sqrt[3]{x^2 + y^2 + z^2 - 25}$

Level surfaces of a function of three variables

given a function $f(x,y,z)$ and a number c in the range of f, a level surface of a function of three variables is defined to be the set of points satisfying the equation $f(x,y,z) = c$

$f(x,y,z) = 4x^2 + 9y^2 - z^2$ corresponding $c = 1$

$4x^2 + 9y^2 - z^2 = 1$ hyperboloid

$w(x,y,z) = x^2 + y^2$ corresponding $c = 4$

$x^2 + y^2 = 4$ cylinder

$w(x,y,z) = 9x^2 + 4y^2 + 36z^2 = 0$

$(0, 0, 0)$ point

$w(x,y,z) = 36z^2 - 9x^2 - 4y^2 = -9$

$9x^2 + 4y^2 = 36z^2 + 9$

4.1.2 Level Curves of Functions of Two Variables

Find the level curves of a function of two variables

level curve of a function of two variables for the value c is defined to be the set of points satisfying $f(x,y) = c$

$$g(x,y) = \sqrt{9 - x^2 - y^2}$$ range is $[0,3]$

$$x^2 - 6x + y^2 + 2y = 15$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 15 + 9 + 1$$

$$(x-3)^2 + (y+1)^2 = 25$$

$$g(x,y) = \frac{x}{x+y}$$ corresponding to $c = 2$

$$\frac{x}{x+y} = 2$$

$$x = 2x + 2y \qquad x + y \neq 0$$

$$0 = x + 2y$$

$$x^2 = 0$$

$$x = 0$$ vertical line

$$f(x,y) = \frac{y+2}{x^2}$$

4.3.1

Partial Derivatives of a Function of Two Variables

Find the partial derivative of a function of two variables

$$f(x,y) = -2x^6 + 6xy^2 + 5y^3$$

$$f_x(x,y) = -12x^5 + 6y^2$$

$$f_y(x,y) = 12xy + 15y^2$$

$3x^2 + y$

$f_y(x,y)$ for $f(x,y) = e^{xy}\cos(x)\sin(y)$

$$= \cos(x)\frac{\partial}{\partial y}\left(e^{xy}\sin y\right)$$

$$= ye^{xy}\cos(x)\cdot\cos(y) + e^{xy}\cos(x)\cos(y)$$

$f_x(2,-2)$ for $f(x,y) = \arctan\left(\frac{y}{x}\right)$

$\tan^{-1}\left(\frac{y}{x}\right)$ $\qquad \frac{1}{1+x^2}$ $\qquad yx^{-1}$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot -yx^{-2} = \frac{-y}{y^2+x^2}$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2} = \frac{-(-2)}{4+4}$$

$$= \frac{-y}{x^2+y^2} \qquad = \frac{2}{8} = \frac{1}{4}$$

$\frac{\partial z}{\partial y}$ for $z = \sin(3x)\cos(3y)$

$= \sin(3x) \cdot 3(-\sin(3y))$

$= -3\sin(3x)\sin(3y)$

Estimate the partial derivative of a Function at a Point from a Graph or Contour Map

$f_y(3,30) = 360$

$f(3,30) = 360$

$f(3,21) = 270$

$$\frac{\Delta f}{\Delta y} = \frac{360-270}{30-21} = \frac{90}{9} = 10$$

$f_x(2,5) =$

$f(2,5) = 90$

$f(4,5) = 135$

$$\frac{\Delta f}{\Delta x} = \frac{135-90}{4-2} = \frac{45}{2} = 22.5$$

$f_x(2,7) =$

$f(2,7) = 80$

$f(4,7) = \cancel{90}\ \cancel{100}\ 95$

$$\frac{\Delta f}{\Delta x} = \frac{\cancel{40}\,15}{4-2} = \cancel{8}$$

$c = -1$

$c = -2$

$$\frac{\partial f}{\partial y}$$

$f(x,y) = x^2 - y^2$

$c = 0 \quad (0,0)$

$c = -1 \quad (0,1)$

$c = -2 \quad (0,\sqrt{2})$

$f_y(0,1) =$

X - 1

$f(0,1) = -1$

$f(0,\sqrt{2}) = -2$

$$\frac{\Delta f}{\Delta y} = \frac{-2-(-1)}{\sqrt{2}-1} = \frac{-1}{\sqrt{2}-1} =$$

$f_y(x, y)$ $\qquad f(x,y) = e^{xy}\cos(x)\sin(y)$

$$f(0,1) = 0.84147$$

$$f(0,4) = -0.7568$$

$$\frac{\Delta f}{\Delta y} = \frac{(-0.7568)-(0.84147)}{4-1} = \frac{-1.59827}{3}$$

$$= -0.53275$$

$$g(x,y) = \sqrt{9 - x^2 - y^2} \qquad g_x(\sqrt{5}, 0)$$

$c = 2$ $\qquad \frac{\partial g}{\partial x}$

$c = 0$

$x = \sqrt{5}$
$y = 0$
$c = 2$

$$g_x = \frac{\Delta g}{\Delta x} = \frac{0-2}{\sqrt{9}\cdot\sqrt{5}}$$

$x = \sqrt{9}$
$y = 0$
$c = 0$

$$= \frac{-2}{0.764}$$

$$= -2.618$$

$f_y = (1, -1)$

$\frac{\partial f}{\partial y}$ represents the slope of the tangent line passing through

$(x, y, f(x, y))$

It can be observed from the graph that the slope of the tangent line is positive at point $(1, -1)$

that means

$f(x,y) = x^2 - y^2$ $\qquad \dfrac{\partial f}{\partial x}$ $\qquad$ at point $(2,0)$

$c = 4$

$c = 5$

$x = 2$

$y = 0$ $\qquad c = 4$

$$f_x = \frac{\Delta f}{\Delta x} = \frac{5 - 4}{\sqrt{5} - 2}$$

$x = \sqrt{5}$ $\qquad c = 5$

$y = 0$

$$= \frac{1}{\sqrt{5} - 2} = 4.236$$

4.3.2 Partial Derivatives of a Func. 3+ Variables

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}$$

$$\frac{\partial f}{\partial z} = \lim_{m \to 0} \frac{f(x, y, z+m) - f(x, y, z)}{m}$$

Ex 1: Calculating Partials for a Function of 3 Variables

$$f(x,y,z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z$$

$$f(x+h,y,z) = (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z$$

$$f_x = 2x - 3x - 4z - 12$$

$$f_y = -3x + 4y + 5z^2 + 4$$

$$f_z = -4x + 10yz - 3$$

EX 2

a. $$f(x,y,z) = \frac{x^2y - 4xz + y^2}{x - 3yz}$$

$$f_x(x,y,z) = \frac{(2xy - 4z)(x - 3yz) - (x^2y - 4xz + y^2)(1)}{(x - 3yz)^2}$$

$$f_y(x,y,z) = \frac{(x^2 + 2y)(x - 3yz) - (x^2y - 4xz + y^2)(-3z)}{(x - 3yz)^2}$$

$$= \frac{x^3 + 2yx \cancel{- 3x^2yz} - 6y^2z \cancel{+ 3x^2yz} - 12xz^2 - 3y^2z}{(x - 3yz)^2}$$

$$f_z = \frac{(-4x)(x-3yz) - (x^2y-4xz+y^2)(-3y)}{(x-3yz)^2}$$

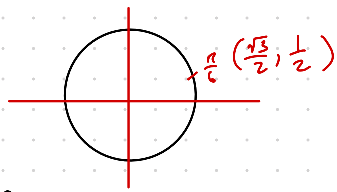
$$= \frac{-4x^2+12xyz+3x^2y^2+12xyz-3y^3}{(x-3yz)^2}$$

$$\sin(x^2y-z) + \cos(x^2-yz)$$

$$\frac{\partial f}{\partial x} = \cos(x^2y-z)\cdot(2xy) - \sin(x^2-yz)(2x)$$

$$f(x,y,z) = \sin(x^3+4y^2-z) - x$$

$$\frac{\partial f}{\partial x} = \cos(x^3+4y^2-z)(3x^2) - 1$$



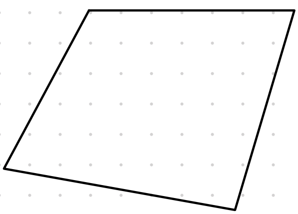
$$F(x,y,\theta) = x^2 + y^2 - 2xy\cos\theta$$

$$\frac{\partial F}{\partial \theta} = -2xy(-\sin\theta) = 2xy\sin\theta$$

$$= 2(2\cdot 3)\sin\left(\frac{\pi}{6}\right)$$

$$= 12\sin\left(\frac{\pi}{6}\right)$$

$$= 12\cdot\frac{1}{2} = 6$$



$$A(a,b,x) = ba\sin x$$

$$= a\sin x$$

$$\frac{\partial w}{\partial y} = e^{-2x}\left(\cos(z^2y)\cdot z^2\right)$$

Find the higher order partial derivatives of a function of two variables

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \frac{\partial f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{y}\left(\frac{\partial f}{\partial y}\right)$$

$f(x,y) = 5x^6 + 4xy^3 + 6y^2$

$f_x(x,y) = 30x^5 + 4y^3$ $\qquad f_y = 12xy^2 + 12y$

$f_{xx}(x,y) = 150x^4$

$f_{yy}(x,y) = 24xy + 12$

$f_{xy}(x,y) = 12y^2$ $\qquad f_{xy} = f_{yx}$

$f_{yx}(x,y) = 12y^2$

How a partial in one variable is changing in the direction of the other

how the rate of change of $f(x,y)$ in x-direction changes as we move in the y-direction

$f(x,y) = -2x^3y^2 + 7x^2 - 9y^4$

$f_x(x,y) = -6x^2y^2 + 14x$

$f_{xx}(x,y) = -12xy^2 + 14$

$f(x,y) = \ln(x-y)$

$f_y(x,y) = \frac{1}{x-y} \cdot -1 = \frac{-1}{x-y} = -(x-y)^{-1}$

$f_{yy}(x,y) = +(x-y)^{-2} \cdot -1$

$= \frac{-1}{(x-y)^2}$

$f(x,y) = e^x \tan y$

$f_y = e^x \cdot \sec^2(y)$

$f_{yy} = e^x \cdot 2\sec(y) \cdot \sec(y)\tan(y)$

$f(x,y) = \ln(x-y)$

$f_y(x,y) = \frac{1}{x-y} \cdot -1 = \frac{-1}{(x-y)} = -(x-y)^{-1}$

$f_{yx}(x,y) = (x-y)^{-2} \cdot 1 = \frac{1}{(x-y)^2}$

4.5.1 The Chain Rule

Use the chain rule for one independent variable

$\frac{dz}{dt}$ given $z = xe^{2y}$ $x = t^3$ $y = -3+4t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^{2y} \cdot 3t^2 + xe^{2y} \cdot 2 \cdot 4$$

$$= 3t^2 e^{2y} + 8xe^{2y}$$

$$= 3t^2 e^{2(-3+4t)} + 8t^3 e^{2(-3+4t)}$$

$$\frac{dz}{dt} = 3t^2 e^{-6+8t} + 8t^3 e^{-6+8t}$$

$z = f(x,y) = x^2 + xy + y^2$ $x = \sin(t)$ $y = \cos t$

$$\frac{dz}{dt} = (2x+y)(\cos t) + (x+2y)(-\sin t)$$

$$\frac{dz}{dt} = 2x\cos t + y\cos t - x\sin t - 2y\sin t$$

$$= \cancel{2\sin(t)\cos(t)} + \cos^2 t - \sin^2 t \cancel{- 2\cos t\sin t}$$

$\frac{dP}{dt}$ $P = \frac{kT}{V}$ $k = 1$ $\frac{dV}{dt} = 2 \text{ cm}^3/\text{min}$ $\frac{dT}{dt} = \frac{1}{2} \text{ k/min}$

$= kT(V^{-1})$ $V = 20 \text{ cm}^3$ $T = 300 \text{ K}$

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt}$$

$$= \frac{-kT}{V^2}\frac{dV}{dt} + \frac{k}{V} \cdot \frac{dT}{dt}$$

$$= -\frac{300\text{ k}}{\cancel{400}_{200}} \cdot \frac{2}{\text{min}} + \frac{1}{20 \text{ cm}^3} \cdot \frac{1}{2}\frac{\text{k}}{\text{min}}$$

$$= -\frac{3}{2} + \frac{1}{40} = -\frac{59}{40} = -1.475$$

$z = e^{1-xy} \qquad x = t^{1/3} \qquad y = t^3$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= e^{1-xy}\cdot(-y)\cdot\frac{1}{3}t^{-2/3} + e^{1-xy}\cdot(-x)\cdot 3t^2$$

$$= -ye^{1-xy}\cdot\frac{1}{3}t^{-2/3} - x\ e^{1-xy}\cdot 3t^2$$

$$= -t^3 e^{1-(t^{1/3})(t^3)}\cdot\frac{1}{3}t^{-2/3} - t^{1/3}\ e^{1-t^{10/3}}\cdot 3t^2$$

$$= -t^3 e^{1-t^{10/3}}\frac{1}{3}t^{-2/3} - t^{1/3}\ e^{1-t^{10/3}}\cdot 3t^2$$

$$= -\frac{1}{3}t^{7/3}\ e^{1-t^{10/3}} - 3t^{7/3}e^{1-t^{10/3}}$$

$$= e^{1-t^{10/3}}\ t^{7/3}\left(-\frac{1}{3} - 3\right)$$

$$= \frac{-10}{3}e^{1-t^{10/3}}\ t^{7/3}$$

Calculate $\frac{dz}{dt}$

$z = f(x,y) = x^2 - 3xy + 2y^2 \qquad \frac{\partial z}{\partial x} = 2x - 3y \qquad \frac{\partial z}{\partial y} = -3x + 4y$

$x = x(t) = 3\sin 2t \qquad x'(t) = 6\cos 2t$

$y = y(t) = 4\cos 2t \qquad y'(t) = -8\sin 2t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x-3y)\,6\cos 2t + (-3x+4y)(-8\sin 2t)$$

$$= (6\sin 2t - 12\cos 2t)\,6\cos 2t + (-9\sin 2t + 16\cos 2t)(-8\sin 2t)$$

$$= 36\sin 2t\cos 2t - 72\cos^2 2t + 72\sin^2 2t - 128\cos 2t\sin 2t$$

$$= 36\sin 2t\cos 2t - (72)\cancel{(\cos^2 2t + \sin^2 2t)} - 128\cos 2t\sin 2t$$

Two independent variables

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$z = f(x,y) = \frac{2x-y}{x+3y}$ $\quad x(u,v) = e^{2u}\cos(3v)$ $\quad y(u,v) = e^{2u}\sin(3v)$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{\partial v}$$

$$= \frac{7y}{(x+3y)^2}\left(e^{2u} \cdot -\sin(3v)\cdot 3\right) + \frac{-7x}{(x+3y)^2}\left(e^{2u}\cos(3v)\cdot 3\right)$$

$$= \frac{7y}{(x+3y)^2}\left(e^{2u} \cdot -\sin(3v)\cdot 3\right) - \frac{7x}{(x+3y)^2}\left(3e^{2u}\cos(3v)\right)$$

Substitute x and y

$$= \frac{7(e^{2u}\cos(3v))}{(e^{2u}\cos(3v) + 3(e^{2u}\sin(3v)))^2}\cdot(-3e^{2u}\sin(3v)) + \frac{-7(e^{2u}\cos 3v)}{(x+3y)^2}\cdot(3e^{2u}\cos(3v))$$

$$= 21e^{2u}$$

$\frac{\partial z}{\partial u}$ $\quad z = f(x,y) = \frac{2x-y}{x+3y}$ $\quad \frac{\partial z}{\partial x} = \frac{7y}{(x+3y)^2}$ $\quad \frac{\partial z}{\partial y} = \frac{-7x}{(x+3y)^2}$

$x(u,v) = e^{2u}\cos 3v$ $\quad e^{2u}\cos 3v \cdot 2$

$y(u,v) = e^{2u}\sin 3v$ $\quad e^{2u}\sin 3v \cdot 2$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= \frac{7y}{(x+3y)^2}\,2e^{2u}\cos 3v + \left(\frac{-7x}{(x+3y)^2}\right)2e^{2u}\sin 3v$$

$$= \frac{7(e^{2u}\sin 3v)}{(e^{2u}\cos 3v + 3e^{2u}\sin 3v)^2}\,2e^{2u}\cos 3v - \frac{7(e^{2u}\cos 3v)}{(e^{2u}\cos 3v + 3\sin 3v)^2}\,2e^{2u}\sin 3v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= e^{x/y}(x+y)\cos v + \frac{x e^{x/y}(y-x)}{y}\sin v$$

$$= e^{u\cos v/u\sin v}(u\cos v + u\sin v)\cos v + \frac{(u\cos v\, e^{u\cos v/u\sin v})(u\sin v - u\cos v)\sin v}{u\sin v}$$

forgot to replan

$\sqrt{3}\, e^{\sqrt{3}}$

$$\frac{dz}{du} = \frac{\partial z}{dx}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du} \qquad x = (uv)^{-1/2} \qquad y = v^{-1}$$

$$= e^{x^2 y}\cdot 2xy\cdot\left(2\frac{v}{\sqrt{uv}}\right) + \cancel{e^{x^2y} x^2 \cdot 0}$$

$$= e^{u}\cdot \cancel{2}\left(\cancel{\sqrt{uv}}\cdot\cancel{\frac{1}{v}}\right)\cdot \cancel{\frac{v}{2\sqrt{uv}}}$$

$$f(u,v) = u^2 - v^2 \qquad u = xy + z \qquad v = 2x + y^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial u}\frac{du}{dy} + \frac{\partial f}{\partial u}\frac{du}{dz}$$

$$= 2u\cdot y + 2u\cdot x + 2u$$

Use the generalized chain rule

$\frac{\partial w}{\partial s}$ in terms of r, s, t

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} + \frac{\partial w}{\partial z}\frac{dz}{ds}$$

$$= 4 \cdot e^{rs^2} \cdot 2rs + 2y \cdot \frac{1}{\left(\frac{r+s}{\cancel{t}}\right)} \cdot \frac{1}{\cancel{t}} + 3z^2 \cdot rt^2$$

$$= 8rse^{rs^2} + \frac{2y}{r+s} + 3z^2rt^2$$

$$= 8rs \cdot e^{rs^2} + 2\left(\ln\left(\frac{r+s}{t}\right)\right) \cdot \frac{1}{r+s} + 3(rst^2)^2 rt^2$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (2z)(2t) + (z^2)2t + (2x + 2yz)2t$$

$$= 2t^2(2t) + t^4(2t) + (2(t^2+1) + 2(t^2-1)(t^2))2t$$

$f(u,v) = u^2 - v^2 \qquad u = xy + z \qquad v = 2x + y^2$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{du}{dy} + \frac{\partial f}{\partial v}\frac{dv}{dy}$$

$$= 2u \quad x + (-2v)\,2y$$

$$= 2(xy+z)(x) + -2(2x+y^2)(2y)$$

$$= 2x^2y + 2xz + -8xy + 4y^3$$

$f(x,y,z)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} + \frac{\partial f}{\partial z}\frac{dz}{du}$$

$$= 2(y+z)v + 2x \cdot 0 + 2x \cdot 0$$

$$= 2(v+w+w^2)v$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}$$

$$= 1 \cdot y + (-2v)(1)$$

$$= y - 2(z+x)$$

4.6.1 Find directional derivative of a function of two variables

The directional derivative of f in the direction of u is given by

$$D_u f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta,\ b + h\sin\theta) - f(a,b)}{h}$$

Example 1: Finding a directional derivative from the direction

$\theta = \arccos\left(\frac{3}{5}\right)$

Theorem 1: Let $z = f(x,y)$ be a function of two variables x and y, assume that f_x and f_y exist

Directional derivative of f in direction of $u = \cos\theta\, i + \sin\theta\, j$ is given by

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Find the directional derivative $D_u f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $u = \cos\theta\, i + \sin\theta\, j$. What is $D_u f(-1,2)$

First, calculate the partial derivatives of f:

$$f_x = 2x - y$$

$$f_y = -x + 6y$$

$$\theta = \arccos\frac{3}{5} = \cos^{-1}\left(\frac{3}{5}\right)$$

$$\cos\theta = \frac{3}{5}$$

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

$$= (2x - y)\frac{3}{5} + (-x + 6y)\frac{4}{5}$$

$$D_u f(-1,2) = (-2 - 2)\frac{3}{5} + (+1 + 12)\frac{4}{5}$$

$$-\frac{12}{5} + \frac{52}{5} = \frac{40}{5} = 8$$

If the vector that is given is not a unit vector, only necessary to divide

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

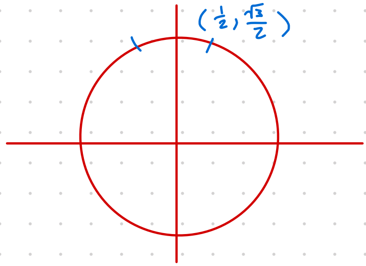
$$=$$

directional derivative

$D_u f(x,y)$

$f_x = 6xy - 4y^3 - 4$

$f_y = 3x^2 - 4x \cdot 3y^2 + 6y$



$D_u f(x,y) = (6xy - 4y^3 - 4)\cos\frac{\pi}{3} + (3x^2 - 12xy^2 + 6y)\sin\frac{\pi}{3}$

$= \frac{\qquad}{2} + \frac{\sqrt{3}}{2}$

gradient of $f(x,y,z) = 7\cos y^4 - 5\sin z^3 + 3\sin x^5$

$15\sin^4(x)\cos(x)\,i - 28\sin y\cos^3 y\,j - 15\sin^2 z\cos z$

$f(x,y,z) = x\sin(y+z) + y\sin(x+z) + z\sin(x+y)$

$f_x = z\cos(x+y) + y\cos(x+z) + \sin(y+z)$

$f_y =$

Directional derivatin

$\nabla f(x,y,z) \cdot u$ unit vector

$= f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma$

$D_u f(3,4,5)$ $\quad u = 5i + 4j - 3k$

$\|u\| = \sqrt{5^2 + 4^2 + (-3)^2} = \sqrt{50}$

$\hat{u} = \frac{5}{\sqrt{50}} i + \frac{4}{\sqrt{50}} j + \frac{-3}{\sqrt{50}}$

$f_x = 2z - 4x$

$f_y = 3z - 4y$

$f_z = 2x + 3y + 4$

$\cos\alpha = \frac{5}{\sqrt{50}}$

$\cos\beta = \frac{4}{\sqrt{50}}$

$\cos\gamma = \frac{-3}{\sqrt{50}}$

$D_u f(3,4,5) = (2z - 4x)\frac{5}{\sqrt{50}} + (3z - 4y)\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}(2x + 3y + 4)$

$= (10 - 12)\frac{5}{\sqrt{50}} + (15 - 16)\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}(6 + 12 + 4)$

$= \frac{-10}{\sqrt{50}} - \frac{4}{\sqrt{50}} - \frac{66}{\sqrt{50}}$

$= \frac{-14 - 66}{\sqrt{50}} = -\frac{80}{\sqrt{50}}$

Gradient of a function of two variables

4.4.1 Tangent Planes

Find equ. of tangent to surface given by $z = 2x^2 - y^2 + 5y$ at $(-2, 2, 14)$
$x_0 \quad y_0 \quad z_0$

$F(x, y, z) = 0$ Point to tangency: (x_0, y_0, z_0)

Equation:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$$

$F(x, y, z) = 2x^2 - y^2 + 5y - z = 0$

$F_x = 4x \rightarrow F_x(-2, 2, 14) = -8$

$F_y = -2y + 5 \longrightarrow F_y(-2, 2, 14) = 1$

$F_z = -1 \longrightarrow F_z(-2, 2, 14) = -1$

$$-8(x + 2) + (y - 2) + -(z - 14) = 0$$

$$-8x - 16 + y - 2 - z + 14 = 0$$

$$-8x + y - z - 4 = 0$$

$$-8x + y - 4 = z$$

$0 = x^3 + y - 2x - z$ at $(1, 0, -1)$
$x_0 \quad y_0 \quad z_0$

$F_x = 3x^2 - 2 \longrightarrow 1$

$F_y = 1$

$F_z = -1$

$$1(x - 1) + 1(y - 0) + -1(z - (-1)) = 0$$

$$x - 1 + y - z - 1 = 0$$

$$x + y - 2 = z$$

$$x^3 + y^3 = 3xyz$$

at $P(1, 2, \frac{3}{2})$

$$F(x, y, z) = x^3 + y^3 - 3xyz = 0$$

$F_x = 3x^2 - 3yz \qquad F_x(1, 2, \frac{3}{2}) = 3 - 9 = -6$

$F_y = 3y^2 - 3xz \qquad F_y(1, 2, \frac{3}{2}) = 12 - \frac{9}{2} = \frac{24}{2} - \frac{9}{2} = \frac{15}{2}$

$F_z = -3xy \qquad F_z(1, 2, \frac{3}{2}) = -3(1)(2) = -6$

$$-6(x - 1) + \frac{15}{2}(y - 2) - 6(z - \frac{3}{2}) = 0$$

$$-12(x - 1) + 15(y - 2) - 12(z - \frac{3}{2}) = 0$$

$$-4(x - 1) + 5(y - 2) - 4(z - \frac{3}{2}) = 0$$

$$-4x \cancel{+4} + 5y \cancel{-10} - 4z \cancel{+\frac{12}{2}} = 0$$

$z = axy$ at point $P(1, \frac{1}{a}, 1)$

$F(x,y,z) = axy - z$

$F_x = ay \rightarrow 1$

$F_y = ax \rightarrow a$

$F_z = -1 \rightarrow -1$

$$(x-1) + a(y - \tfrac{1}{a}) - (z-1) = 0$$

$$x - 1 + ay \cancel{-1} - z \cancel{+1} = 0$$

$$x - 1 + ay - z = 0$$

$$x + ay - z = 1$$

$$-x - ay + z = -1$$

$$z - x - ay = -1$$

$h(x,y) = \ln(\sqrt{x^2+y^2}) = z$

$\ln(\sqrt{x^2+y^2}) - z = 0$ at $P(3,4)$

$\ln(\sqrt{9+16})$

$\ln(\sqrt{25}) = \ln(5)$

$F_x = \frac{x}{x^2+y^2} \rightarrow \frac{3}{9+16} = \frac{3}{25}$

$F_y = \frac{y}{x^2+y^2} \rightarrow \frac{4}{9+16} = \frac{4}{25}$

$F_z = -1 \rightarrow -1$

$$\frac{3}{25}(x-3) + \frac{4}{25}(y-4) - 1(z - \ln 5) = 0$$

$$3(x-3) + 4(y-4) - 25(z - \ln 5) = 0$$

$$3x - 9 + 4y - 16 - 25z + 25\ln 5 = 0$$

$$25z - 3x - 4y = 25\ln 5 - 25$$

4.4.2 Linear Approximation of Functions of Several Variables

$$y \approx f(a) + f'(a)(x-a)$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$f(x,y) = \sqrt{41 - 4x^2 - y^2}$ $\quad f(2.1, 2.9)$ $\quad$ using point $(2,3)$ for (x_0, y_0)

what is $f(2.1, 2.9)$ to 4 decimal places?

Calculate $f(x_0, y_0)$ $f_x(x_0,y_0)$ $f_y(x_0,y_0)$ using $x_0 = 2$ and $y_0 = 3$

$$f(x_0,y_0) = f(2,3) = \sqrt{41 - 4(2)^2 - (3)^2} = \sqrt{41-16-9} = \sqrt{16} = 4$$

$$f_x(x,y) = \frac{-4x}{\sqrt{41-4x^2-y^2}} \quad \text{so} \quad \frac{-4(2)}{\sqrt{41-4(2)^2-(3)^2}} = -2$$

$$f_y(x,y) = \frac{-y}{\sqrt{41-4x^2-y^2}} \quad \text{so} \quad \frac{-3}{\sqrt{41-4(2)^2-3^2}} = -\frac{3}{4}$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$4 - 2(x-2) - \frac{3}{4}(y-3)$$

$$L(2.1, 2.9) = \frac{41}{4} - 2(2.1) - \frac{3}{4}(2.9) = 3.875$$

$$f(x_0,y_0) = f(4,1) = e^{5-2(4)+3(1)} = e^{5-8+3} = e^0 = 1$$

$$f_x(x_0,y_0) = -2e^{-2x+3y+5} \longrightarrow -2e^{-8+3+5} = -2$$

$$f_y(x_0,y_0) = 3e^{5-2x+3y} \longrightarrow 3e^{5-8+3} = 3$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L = 1 - 2(x-4) + 3(y-1)$$

$$L(4.1, 0.9) = 1 - 2(4.1-4) + 3(0.9-1)$$

$$= 1 - 2x + 8 + 3y - 3$$

$$= -2x + 3y + 6$$

$f(x, y, z) = xyz$ $\quad f(3.1, 4.9, 6.9)$ $\quad P(3, 5, 7)$

$f(x_0, y_0, z_0) = xyz \rightarrow 105$

$F_x = yz \rightarrow 35$

$F_y = xz \rightarrow 21$

$F_z = xy \rightarrow 15$

$L(x,y) = 105 + 35(x-3) + 21(y-5) + 15(z-7)$

$L(x,y) =$

$f(x,y) = \dfrac{11 - xy}{y + x} \rightarrow \dfrac{9}{3} = 3$ $\quad P(1, 2)$

$f_x = -\dfrac{y^2 + 11}{(x+y)^2} \rightarrow \dfrac{-15}{9} = -\dfrac{5}{3}$

$f_y = -\dfrac{x^2 + 11}{(x+y)^2} \rightarrow -\dfrac{12}{9} = -\dfrac{4}{3}$

$L(x,y) = 3 - \dfrac{5}{3}(x - 1) - \dfrac{4}{3}(y - 2)$

$= 3 - \dfrac{5}{3}(1.1 - 1) - \dfrac{4}{3}(2.1 - 2)$

$= 2.7$

$f(x,y) = -\dfrac{8x}{7} - \dfrac{3y}{7} + \dfrac{19}{7}$ $\quad$ approx. $f(1.1, -0.9)$ using $P(1, -1)$

$f_x = -\dfrac{8}{7}$ $\quad f(x,y) = -\dfrac{8}{7} + \dfrac{3}{7} + \dfrac{19}{7}$

$f_y = -\dfrac{3}{7}$ $\quad = -\dfrac{8}{7} + \dfrac{22}{7} = \dfrac{14}{7}$

$L(x,y) = 2 - \dfrac{8}{7}(x - 1) - \dfrac{3}{7}(y + 1)$ $\quad = 2$

4.7.1 Maxima / Minima Problems

Critical Points

$f_x(x,y) = 0 \qquad f_y(x,y) = 0$

$f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ do not exist

$$d = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

If
- $d > 0$, $f_{xx}(a,b) < 0$, f has a relative maximum at (a,b)
- $d > 0$, $f_{xx}(a,b) > 0$, f has a relative minimum at (a,b)
- $d < 0$, then $(a, b, f(a,b))$ is a saddle point)

Inconclusive if $d = 0$

$f(x,y) = x^4 + y^4 - 2xy$ find any relative extrema.

$f_x = 4x^3 - 2y$

$f_{xx} = 12x^2$

$f_y = 4y^3 - 2x$

$f_{yy} = 12y^2$

f_{xy}

$f_x = 4x^3 - 2y = 0$

$f_y = 4y^3 - 2x = 0$

$= 4(2x^3)^3 - 2x = 0$

$= 4 \cdot 8x^9 - 2x = 0$

$32x^9 - 2x = 0$

$2x(16x^8 - 1) = 0$

$x = 0 \qquad 16x^8 - 1 = 0$

$x^8 = \frac{1}{16}$

$x = \frac{1}{2^4}$

$(x^8)^{1/8} = \pm\left(\frac{1}{2^4}\right)^{1/8}$

$x = \pm\frac{1}{\sqrt{2}}$

$2y = 4x^3$

$y = 2x^3$

$2 \cdot \left(\frac{1}{\sqrt{2}}\right)^3$

$= \cancel{2}\left(\frac{1}{\cancel{2}\sqrt{2}}\right)$

$2 \cdot \left(-\frac{1}{\sqrt{2}}\right)^3$

Critical Points

$(0, 0)$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Find the critical point of the function $f(x,y)$

$$x^3 + 2xy - 2x - 4y$$

$f_x = 3x^2 + 2y - 2 = 0$

$f_{xx} = 6x$

Critical Points

$(2, -5)$

$f_y = 2x - 4 = 0$

$2x = 4$

$x = 2$

$f_{yy} = 0$

$3(2)^2 + 2y - 2 = 0$

$12 + 2y - 2 = 0$

$10 + 2y = 0$

$2y = -10$

$y = -5$

$$\sqrt{x^3 + y^3 + 1} = (x^3 + y^3 + 1)^{1/2}$$

$$f_x = \frac{1}{2}(x^3 + y^3 + 1)^{-1/2} \cdot 3x^2$$

$$f_y = \frac{3y^2}{2}(x^3 + y^3 + 1)^{-1/2}$$

$$0 = 2\sqrt{x^3 + y^3 + 1}$$

$$0 = x^3 + y^3 + 1$$

$$f(x,y) = \sqrt{x^2 - 3x + 4y^2 + y} = (x^2 - 3x + 4y^2 + y)^{1/2}$$

$$f_x = \frac{1}{2}(x^2 - 3x + 4y^2 + y)^{-1/2} \cdot (2x - 3) = 0$$

$$x = \frac{3}{2}$$

$$f_y = \frac{1}{2}(x^2 - 3x + 4y^2 + y)^{-1/2} \cdot (8y + 1) = 0$$

$$y = -\frac{1}{8}$$

$$\left(\frac{3}{2}, -\frac{1}{8}\right)$$

Second derivative test

Fermat's theorem:

let $z = f(x, y)$ be a function of two variables that is defined and continuous on (x_0, y_0)

Suppose f_x and f_y both exist at (x_0, y_0). If f has local extrem at (x_0, y_0), then (x_0, y_0) is a critical point of f

$(x_0, y_0, f(x_0, y_0))$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

f has no local extremum at (x_0, y_0)

1. Determine critical points (x_0, y_0) where $f_x = f_y = 0$
   Discard any where at least one of the partial derivatives do not exist
2. Calculate the discriminant $D = f_{xx} f_{yy} - f_{xy}^2$
3. Apply 2^{nd} Derivative Test

$f_x = 3x^2 + 2y - 6$

$f_y = 2x - 8y$

$f_{xx} = 6x$

$f_{yy} = -8$

$f_{xy} = 2$

$\left(\frac{4}{3}, \frac{1}{3}, -\frac{140}{27}\right)$ saddle

$\left(-\frac{3}{2}, -\frac{3}{8}, \frac{99}{16}\right)$ local max

$\cancel{6}^{2}\left(\frac{4}{\cancel{3}}\right)(-8) - 4$

$-64 - 4 = -68$

saddle

$\cancel{6}^{3}\left(-\frac{3}{\cancel{2}}\right)(-8) - 4$

$72 - 4 = 68$ local max

$f = x^2 + 4xy + y^2$

$f_x = 2x + 4y$ $\quad f_y = 4x + 2y$

$f_{xx} = 2$ $\quad f_{yy} = 2$

$f_{xy} = 4$

$D = f_{xx}f_{yy} - f_{xy}^2$

$D = 2 \cdot 2 - 4^2$

$= 4 - 16 = -12$

$f = x^3 + y^3 - 300x - 75y - 3$

$f_x = 3x^2 - 300$

$f_{xx} = 6x$

$f_y = 3y^2 - 75$

$f_{yy} = 6y$

$f_{xy} = 0$

$D = f_{xx} f_{yy} - f_{xy}^2$

$= 6(10)\,6(5) - 0$

$= 60 \cdot 30$

$= 1800$

$+$ positive 60

$f = -x^3 + 4xy - 2y^2 + 1$

$f_x = -3x^2 + 4y$ $\quad f_y = 4x - 4y$

$f_{xx} = -6x$ $\quad f_{yy} = -4$

$f_{xy} = 4$

$D = \left(-\cancel{6}^{2}\left(\frac{4}{\cancel{3}}\right)\right) \cdot (-4) - 4^2$

$= 32 - 16 = 16$ $\quad$ positive D

$-$neg, 8 $\quad$ neg f_{xx}

$$f(x,y) = e^{-(x^2+y^2+2x)}$$

$$f_x = e^{-(x^2+y^2+2x)} \cdot (2x+2) = 0$$

$$x = -1$$

$$f_y = e^{-(x^2+y^2+2x)} \cdot -(2y) = 0$$

$$y = 0$$

Find the absolute extrema of a function of two variables on a closed region

EVT : continuous function $f(x,y)$ on a closed and bounded set D
attains an absolute maximum value

Theorem 2: Finding Extreme Values of a Function

i. The values at critical points

ii. The values of f on the boundary of D

Find absolute max + min:

1. Determine the critical points of f in D
2. Calculate f at each of these critical points
3. Determine the max and min values of f on the boundary of its domain
4. The maximum and minimum of f will occur at one of its values

$f = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$

$f_x = 8x - 2y - 8 = 0$

$f_y = -2x + 12y + 2 = 0$

$$\begin{aligned} -8x + 48y + 8 &= 0 \\ 8x - 2y - 8 &= 0 \\ \hline 46y &= 0 \\ y &= 0 \\ x &= 1 \end{aligned}$$

critical point:

$(1, 0)$

$4 - 8 + 3$

$-4 + 3 = -1$

Test

$(0, -1)$

$(2, -1)$ 11

$(2, 3)$ 51

$(0, 3)$

$6(9) + 2(3) + 3$

$54 + 6 + 3$

63

$f(x,y) = e^x - ye^x$

Test	$(-1, 2)$	$-\frac{1}{e}$	$(-1, 2, -\frac{1}{e})$
	$(1, 2)$	$-e$	
	$(-1, 3)$	$-\frac{2}{e}$	
	$(1, 3)$	$-2e$	

$f(x,y) = y\sqrt{xy}$

$(-4, -2)$	$-2\sqrt{8}$	≈ -5.65
$(-4, -1)$	$-1\sqrt{4}$	≈ -2
$(-1, -2)$	$-2\sqrt{2}$	≈ -2.83
$(-1, -1)$	$-1\sqrt{1}$	≈ -1

$f(x,y) = xy - x - 3y$ Abs. minimum

$(0,0)$	0
$(0,4)$	-12
$(5,0)$	-5

Double Integrals over Rectangular Regions

find $z = f(x,y) = 3x^2 - y$ over $R = [0,2] \times [0,2]$

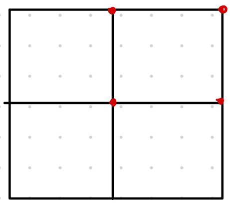
a. set up a double integral

$$V = \iint_R (3x^2 - y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[3(x^*_{ij})^2 - y^*_{ij} \right] \Delta A$$

b. Approximate the signed volume of the solid S

- divide R into four squares with $m = n = 2$
- choose the sample point as the upper right corner
  $(1,1)$ $(2,1)$ $(1,2)$ $(2,2)$

$$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$$



$f(1,1)(1) + f(2,1)(1)$ ·

$f(1,2)(1) + f(2,2)(1)$

$(3\cdot 1)(1) + (12-1)(1) +$

$(3-2)(1) + (12-2)(1)$

$2 + 11 + 1 + 10 = 24$

c. $V = \sum \sum f \, \Delta A$

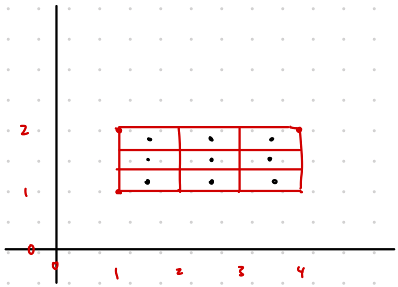
$$= f\left(\tfrac{1}{2}, \tfrac{1}{2}\right)(1) + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right)(1) + f\left(\tfrac{1}{2}, \tfrac{3}{2}\right)(1) + f\left(\tfrac{3}{2}, \tfrac{3}{2}\right)(1)$$

$$= \frac{1}{4} + \frac{21}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{40}{4} = 10$$

a. $\iint_R 2x^2 + 6y \, dA$

$R = [1,4] \times [1,2]$

$\Delta A = \Delta x \Delta y = 1 \times \frac{1}{3} = \frac{1}{3}$



sample points

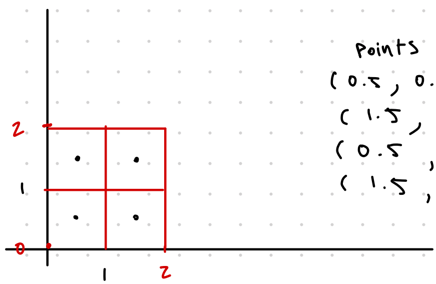
$(1.5, 1\frac{1}{6})$	11.5
$(1.5, 1\frac{1}{2})$	13.5
$(1.5, 1\frac{5}{6})$	15.5
$(2.5, 1\frac{1}{6})$	19.5
$(2.5, 1\frac{1}{2})$	21.5
$(2.5, 1\frac{5}{6})$	23.5
$(3.5, 1\frac{1}{6})$	31.5
$(3.5, 1\frac{1}{2})$	33.5
$(3.5, 1\frac{5}{6})$	35.5

Area:

$f \Delta A + f \Delta A + \ldots$

68.5

b. $\iint x^2 y^2 \, dA$ $R = [0,2] \times [0,2]$ $\Delta A = 1$

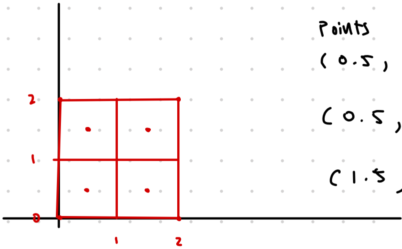


Points

$(0.5, 0.5)$	0.0625
$(1.5, 0.5)$	0.5625
$(0.5, 1.5)$	0.5625
$(1.5, 1.5)$	5.0625

$= 6.25$

$z = f(x,y) = 2x^2 - 2y$ $\qquad \Delta A = 1$



Points

$(0.5, 0.5) \rightarrow -0.5$

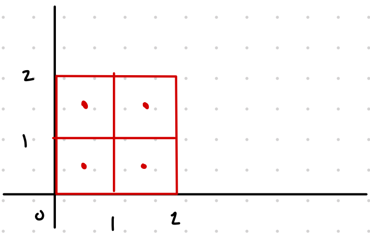
$(0.5, 1.5) \rightarrow -2.5$

$(1.5, 0.5) \rightarrow 3.5$

$(1.5, 1.5) \rightarrow 1.5$

$= 2$

$z = xy^2$ $\qquad \Delta A = 1$



$(0.5, 0.5) \longrightarrow 0.125$

$(0.5, 1.5) \longrightarrow 1.125$

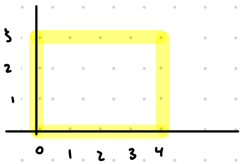
$(1.5, 0.5) \longrightarrow 0.375$

$(1.5, 1.5) \longrightarrow 3.375$

$= 5$

5.1.2. Iterated Integrals and Properties of Double Integrals

$\iint_R 2x \, dA$ over $R = \{(x,y) \mid 0 \le x \le 4, \ 0 \le y \le 3\}$



$$\int_0^3 \int_0^4 2x \, dx \, dy$$

$$\int_0^4 \int_0^3 2x \, dy \, dx$$

$$\int_0^4 2x \, y \Big|_{y=0}^{y=3} dx$$

$$\int_0^4 6x - 0 \, dx$$

$$\frac{6x^2}{2}\Big|_0^4 = 3x^2\Big|_0^4$$

$$= 3(16) - 0$$

$$= 48$$

$$\int_0^3 \int_0^2 x^7 \, dx \, dy = \int_0^2 \int_0^3 x^7 \, dy \, dx$$

$$\int_0^2 x^7 y \Big|_0^3 dx$$

$$\int_0^2 3x^7 \, dx$$

$$\frac{3}{8} x^8 \Big|_0^2$$

$$\frac{3}{8}(2)^8 = 96$$

$$\int_0^1 \int_{-1}^0 y e^{x+1} \, dx \, dy = \int_{-1}^0 \int_0^1 y e^{x+1} \, dy \, dx$$

$$= \int_{-1}^0 e^{x+1} \frac{y^2}{2} \Big|_{y=0}^{y=1} dx$$

$$= \int_{-1}^0 e^{x+1} \left(\frac{1}{2}\right) dx$$

$$= \frac{1}{2}\left[e^{x+1} \right]_{-1}^0 = \frac{1}{2}\left[e^1 - e^0 \right]$$

$\frac{1}{2}(e-1)$

$\frac{e}{2} - \frac{1}{2}$

Fubini Theorem

$$\iint_R f(x,y)\, dA$$

$$= \int_1^2 \int_{-1}^3 x^3 + y^3 + 1 \quad dx\, dy$$

$$= \int_{-1}^3 \int_1^2 x^3 + y^3 + 1 \quad dy\, dx$$

$$= \int_{-1}^3 \frac{1}{4}y^4 + y x^3 + y \Big|_{y=1}^{y=2} \quad dx$$

$$= \int_{-1}^3 \left(\frac{16}{4} + 2x^3 + 2\right) - \left(\frac{1}{4} + x^3 + 1\right) \quad dx$$

$$= \int_{-1}^3 \left(\frac{15}{4} + x^3 + 1\right) dx$$

$$= \left(\frac{19}{4}x + \frac{x^4}{4}\right)\Bigg|_{-1}^3$$

$$= \left(\frac{19}{4}(3) + \frac{3^4}{4}\right) - \left(\frac{-19}{4} + \frac{1}{4}\right) = 39$$

$$\int_0^1 \int_\pi^{2\pi} \sin(x) + 5 \quad dx \; dy$$

$$= \int_0^1 -\cos(x) + 5x \Big|_\pi^{2\pi} \; dy$$

$$= \int_0^1 \left(-\cos(2\pi) + 10\pi\right) - \left(-\cos(\pi) + 5\pi\right) dy$$

$$= \int_0^1 (-1 + 10\pi) - (1 + 5\pi) \, dy$$

$$= \int_0^1 -2 + 10\pi - 5\pi \; dy = \int_0^1 -2 + 5\pi \; dy$$

$$= -2 + 5\pi \left[\; y \; \right]_0^1$$

$$= -2 + 5\pi = 13.708$$

Example: Find the volume under a surface

Find the volume of the solid bounded above by the graph of

$f(x,y) = xy\sin(x^2 y)$ and below by the xy-plane on the

rectangular region $R = [0,1] \times [0,\pi]$

$$V = \iint_R z\,dA = \int_0^\pi \int_0^1 xy\sin(x^2 y)\,dx\,dy$$

$$= \int_0^\pi \left. \frac{-\cos(x^2 y)}{2} \right|_0^1 dy$$

$$= \int_0^\pi \frac{1-\cos(y)}{2}\,dy = \frac{1}{2}\int_0^\pi 1-\cos(y)\,dy$$

$$= \frac{1}{2}\Big[y - \sin(y) \Big]_0^\pi$$

$$V = \int_0^2 \int_0^1 2x + 4y^3\,dx\,dy$$

$$= 18$$

$$V = \int_0^2 \int_0^2 x + 4y^2\,dx\,dy$$

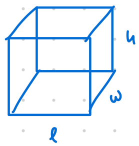
$$= \frac{76}{3}$$

$$V = \int_0^\pi \int_0^3 x^2 y\,dy\,dx$$

$$= \frac{3\pi^3}{2}$$

Find the average value of a function over a rectangular region

$$f_{ave}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$



Find average value $f(x,y) = e^{x+y}$ over $R = [0,2] \times [0,2]$

$$f_{ave}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$

$$= \frac{1}{4}\int_0^2\int_0^2 e^{x+y}\,dx\,dy$$

$$= \frac{1}{4}\int_0^2 e^{x+y}\Big|_0^2\,dy$$

$$= \frac{1}{4}\int_0^2 e^{2+y} - e^{y}\,dy$$

$$= \frac{1}{4}\left[e^{2+y} - e^{y}\right]_0^2$$

$$= \frac{1}{4}\left[\left(e^{4} - e^{2}\right) - \left(e^{2} - e^{0}\right)\right]$$

$$= 10.205$$

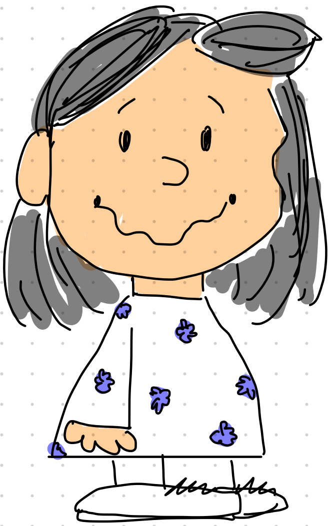
$$f_{ave}(x,y) = \frac{1}{A}\iint_R f(x,y)\,dA$$

$$= \frac{1}{4}\int_0^2\int_0^2 2x^2y\,dx\,dy$$

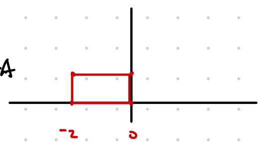
$$= \frac{1}{4}\int_0^2\left[\frac{\cancel{2}x^2y^2}{\cancel{2}}\right]_0^2\,dx$$

$$= \frac{1}{4}\int_0^2 4x^2\,dx$$

$$= \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$



$$f_{ave} = \frac{1}{A} \iint f(x,y)\, dA$$



$$= \frac{1}{2} \int_0^1 \int_{-2}^0 x e^y \, dx \, dy$$

$$f_{ave} = \frac{1}{3} \int_0^1 \int_0^3 -y^3 + xy^2 + 2x \, dy \, dx$$

$$f_{ave} = \frac{2}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\pi} \pi(\sin(x) + \sin(y))\, dx\, dy$$

$$= \frac{2}{\pi} \int_0^{\pi/2} -\cos(x) + x\sin(y) \Big|_0^{\pi} dy$$

$$= \frac{2}{\pi} \int_0^{\pi/2} (2 + \pi \sin y)\, dy$$

Double Integrals Over Nonrectangular Regions

Ex: volume of $f(x,y) = y^2$ in the region bounded by $x \geq 0$, $y = x-1$, $y = \frac{1}{2}x$

$$\iint_R f(x,y)\,dA$$

$$\int_0^2 \int_{x-1}^{1/2 x} y^2 \, dy \, dx$$

$$\int_0^2 \frac{y^3}{3} \Big|_{x-1}^{\frac{1}{2}x} dx$$

$$= \frac{1}{3}\int_0^2 \left(\left(\frac{1}{2}x\right)^3 - (x-1)^3\right) dx$$

$$= \frac{1}{24}\frac{x^4}{4} - \frac{1}{3}\frac{(x-1)^4}{4}\Big|_0^2$$

$$= \frac{1}{\cancel{24}_6}\cdot\frac{\cancel{16}^4}{\cancel{4}} - \frac{1}{3}\cdot\frac{1}{4} - \left(-\frac{1}{3}\cdot\frac{1}{4}\right)$$

$$= \frac{1}{6} - \frac{1}{12} + \frac{1}{12} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

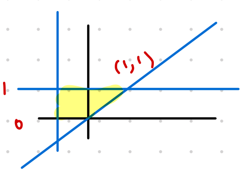
$$x - 1 = \frac{1}{2}x$$
$$2x - 2 = x$$
$$x = 2$$
$$y = 1$$

$$\int_2^3 \int_{3-y}^{\sqrt{y-1}} xy^3 + 2xy \; dx\, dy$$

$$= \int_2^3 \frac{x^2}{2}y^3 + 2\frac{x^2}{2}y \Big|_{3-y}^{\sqrt{y-1}} dy$$

$$= \int_2^3 \left(\frac{y-1}{2}y^3 + (y-1)y\right) - \left(\frac{(3-y)^2}{2}y^3 + (3-y)^2 y\right)$$

$$= \frac{847}{60}$$



$y = 0$
$y = 1$
$y = x$
$x = -1$

$x = 1$
$y = 1$

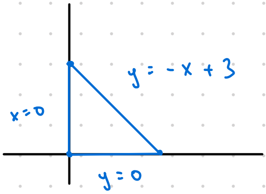
$$-\frac{1}{8} - \frac{1}{4} - \frac{1}{3} =$$

$$\int_0^1\int_{-1}^{y} (xy - y^2)\,dx\,dy = \int_0^1 \frac{x^2}{2}y - xy^2 \Big|_{-1}^{y} dy$$

$$= \int_0^1 \left(\frac{y^3}{2} - y^3\right) \cdot \left(\frac{1}{2}y - -1y^2\right) dy$$

$$\int_0^1 \left(-\frac{1}{2}y^3 - \frac{1}{2}y - y^2\right) dy$$

$$= \frac{-1}{4\cdot 2}y^4 - \frac{1}{2\cdot 2}y^2 - \frac{1}{3}y^3\Big|_0^1$$



$x = 3 - y$

$$\int_0^3 \int_0^{3-y} \sin(y) \quad dx \quad dy$$

$$\int_0^3 \sin(y)\, x \Big|_0^{3-y} \quad dy$$

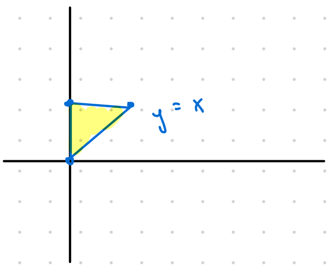
$$\int_0^3 \sin(y)(3-y)\ dy$$

$$\int_0^3 3\sin(y) - y\sin(y) \ dy$$

$$-3\cos(y) - \left(-y\cos(y) + \sin(y)\right) \Big|_0^3$$

$$\left(\cancel{-3\cos(3)} + \cancel{3\cos(3)} - \sin(3)\right) - \left(-3\cos(0) - (\sin 0)\right)$$

$$3 - \sin(3)$$



$y = 2 \qquad x = 0$

$y = x$

$$\int_0^2 \int_0^y -x + 1 \quad dx \quad dy$$

$$-\frac{x^2}{2} + x \Big|_0^y$$

$$\int_0^2 -\frac{y^2}{2} + y \quad dy$$

$$-\frac{y^3}{6} + \frac{y^2}{2} \Big|_0^2$$

$$-\frac{8}{6} + 2$$

$$\frac{12 - 8}{6} = \frac{4}{6} = \frac{2}{3}$$

Double Integrals by Decomposing Regions or Changing the Order of Integrands

Theorem: $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

$D_1 \quad 0 \le y \le 1, \quad 1 \le x \le e^y$

$D_2 \quad 1 \le y \le e, \quad 1 \le x \le 2$

$D_3 \quad e \le y \le e^2, \quad \ln y \le x \le 2$

$$\iint_D (3x-y)\,dA = \iint_{D_1} (3x-y)\,dA + \iint_{D_2} (3x-y)\,dA + \iint_{D_3} (3x-y)\,dA$$

$$= \int_{y=0}^{y=1}\int_{x=1}^{x=e^y} (3x-y)\,dx\,dy + \int_1^e\int_1^2 (3x-y)\,dx\,dy + \int_e^{e^2}\int_{\ln y}^{2} (3x-y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \tfrac{1}{2}(e^y-1)(-2y+3e^y+3) + \int_{y=1}^{y=e} \tfrac{9}{2} - y + \int_{y=e}^{y=e^2} -2y - \tfrac{3}{2}\log^2(y) + y\log(y) + 6$$

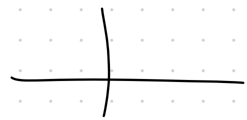
$$= \tfrac{1}{4}(3e^2-11) + \tfrac{1}{2}(-8+9e-e^2) - \tfrac{1}{4}e(18-15e+e^3)$$

$$\approx 9.16$$

$\iint_D y\,dA \qquad y=1 \qquad y=x \qquad y=\ln(x)$

$y=0 \qquad x=e^y$

$x=y$



$$\int_0^1\int_{e^y}^{y} y \quad dx\,dy$$

$$\int_0^1 yx\Big|_{e^y}^{y} \, dy$$

$$\int_0^1 y^2 - ye^y \, dy$$

$$\frac{y^3}{3} - (ye^y - e^y)\Big|_0^1$$

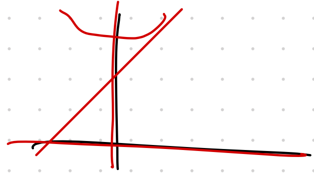
$$\left(\frac{1}{3} - e + e\right) - (0 - 0 - 1)$$

$$\frac{1}{3} + 1 = \frac{4}{3}$$

$\iint (x+2y)\, dA$

$y=0$
$x=0$
$y=x+2$
$y=(x-4)^2$

$x=y-2$

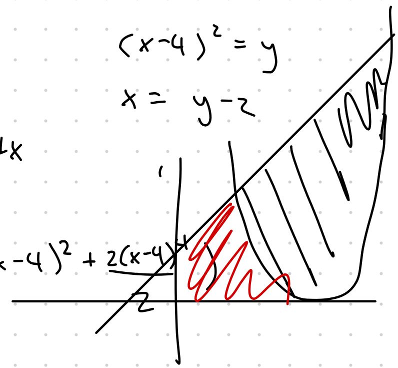


$x+2=(x-4)^2$
$x+2 = x^2-8x+16$
$= x^2-9x+14$
$(x-2)(x-7)$
$x=2$
$x=7$

$$\int_2^7 \int_{(x-4)^2}^{x+2} x+2y \; dy\, dx$$

$(x-4)^2 = y$

$x = y-2$



$$\int_2^7 xy + \frac{2y^2}{2} \Big|_{(x-4)^2}^{x+2} dx$$

$$\int_2^7 x(x+2) + \frac{2(x+2)^2}{2} - \left(x(x-4)^2 + \frac{2(x-4)^4}{2}\right)$$

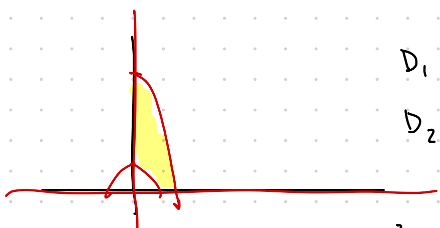
$$= \cancel{\frac{3125}{12}}$$

$D_1 = 0 \le x \le 2 \quad 0 \le y \le x+2$

$D_2 = 2 \le x \le 4 \quad 0 \le y \le (x-4)^2$

$$\iint x+2y\, dA = \iint_{D_1} x+2y\, dA + \iint_{D_2} x+2y\, dA$$

$$\int_0^2 \int_0^{x+2} (x+2y)\, dy\, dx + \int_2^4 \int_0^{(x-4)^2} (x+2y)\, dy\, dx$$



$D_1 = 0 \le x \le 1, \quad 0 \le y \le 1-x^2$

$D_2 = 1 \le x \le 2, \quad 0 \le y \le 4-x^2$

$$\iint x\, dA = \cancel{\int_0^1 \int_0^{1-x^2} x\, dy\, dx + \int_1^2 \int_0^{4-x^2} x\, dy\, dx}$$

$$\cancel{\frac{1}{4} + \frac{9}{4} = \frac{10}{4}}$$

$$\frac{9}{4} - \frac{1}{4} = \frac{8}{4}$$

$$= \int_0^1 \int_{1-x^2}^{4-x^2} x\, dy\, dx + \int_1^2 \int_0^{4-x^2} x\, dy\, dx$$

$$\int_{x=\sqrt{y}}^{x=\frac{4-y}{3}} \int_{y=0}^{y=1} 2x^2y\,dy\,dx$$

$$= \int_0^1 \int_0^{x^2} 2x^2y\,dy\,dx + \int_1^{\frac{4}{3}} \int_0^{4-3x} 2x^2y\,dy\,dx$$

$$= \int_0^1 x^2y^2\Big|_0^{x^2} dx + \int_1^{4/3} x^2y^2\Big|_0^{4-3x} dx$$

$$= \int_0^1 x^6\,dx + \int_1^{4/3} x^2(4-3x)^2\,dx$$

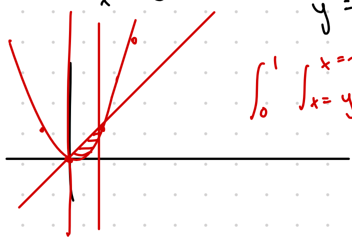
$$= \frac{x^7}{7}\Big|_0^1 + \frac{9}{5}x^5 - 6x^4 + \frac{16}{3}x^3\Big|_1^{4/3}$$

$$= \frac{776}{2835}$$

$x = 1$
$x = 0$

$y = x$
$y = x^2$

$uv - \int v\,du$



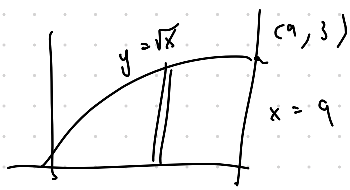
$$\int_0^1 \int_{x=y}^{x=\sqrt{y}}$$

Changing order of integration

$$\int_0^3 \int_{y^2}^9 \sqrt{x}\cos(x)\,dx\,dy$$

$9 \geq x \geq y^2 \qquad \sqrt{x} \geq y$

$3 \geq y \geq 0$



$$\int_0^9 \int_0^{\sqrt{x}} x^{1/2}\cos x\,dy\,dx$$

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} e^{y^2}\,dy\,dx$$

$1 \geq y \geq x \qquad y \geq x$

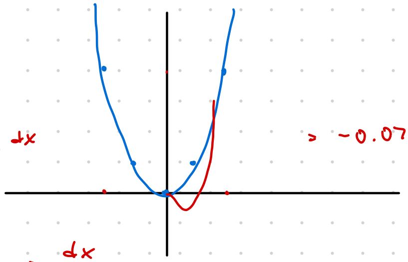
$1 \geq x \geq 0$

$$\int_0^1 \int_0^y e^{y^2}\,dx\,dy \approx 0.859$$

$$\int_0^1 x e^{y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 y e^{y^2}\,dy = \int_0^1 \frac{1}{2}e^u\,du$$

$$\iint x^2 - 2y$$

$$\int_{x=0}^{x=1.414} \int_{y=x^4-x^2}^{y=x^2} (x^2 - 2y) \, dy \, dx$$



$$> -0.072$$

$$\int_0^{\sqrt{2}} x^2 y - y^2 \Big|_{x^4-x^2}^{x^2} \, dx$$

$$\int_0^{\sqrt{2}} x^8 - 3x^6 + 2x^4 \, dx$$

$$\frac{x^9}{9} - \frac{3x^7}{7} + \frac{2x^5}{5} \Big|_0^{\sqrt{2}}$$

$$\int_{y=0}^{y=1} \int_{x=\sqrt[3]{y}}^{x=\sqrt[4]{y}} f(x,y) \, dx \, dy$$

$$\sqrt[4]{y} \geq x \geq \sqrt[3]{y} \qquad x^3 \geq y \qquad x^3 \geq x^4$$

between 0 and 1

$$1 > y > 0$$

$$0 \leq x \leq 1 \quad , \quad x^4 \leq y \leq x^3$$

$$\int_0^1 \int_0^{e^x} x \, dy \, dx + \int_1^2 \int_{x^2-1}^{e^x} x \, dy \, dx$$

$$\int_0^1 \int_0^x y \quad dy \, dx + \int_1^e \int_{\ln x}^1 y \quad dx$$

$$\int_0^1 \int_{-2x+3}^3 \left(\frac{6}{5}x^2 - 8y\right) dy \, dx + \int_{\frac{1}{2}x+\frac{1}{2}}^3 \int_1^5 \left(\frac{6}{5}x^2 - 8y\right) dx \, dy$$

$$\frac{-271}{15} + -\frac{664}{15} = \frac{-935}{15}$$

$$= \frac{-187}{3}$$

$$\int_0^1 \int_{-x^2+1}^{-\frac{1}{2}x^2+2} (x+y) \, dy \, dx + \int_1^2 \int_0^{-\frac{1}{2}x^2+2} (x+y) \, dy \, dx$$

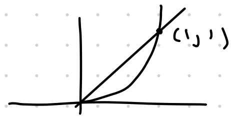
$$\frac{41}{20} + \frac{47}{30} = \frac{217}{60}$$

5.2.3 Applications of Double Integrals Over General Regions

- use $\iint$ to calculate the area of a general plane region

Volume of $f(x,y) = xy$ in region $0 \le x \le 1$, $x^2 \le y \le x$

$y \le x \qquad y \ge x^2$



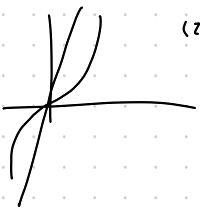
$$\int_0^1 \int_{x^2}^{x} xy \, dy \, dx$$

$$\int_0^1 \frac{xy^2}{2} \Big|_{x^2}^{x} dx = \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2} \right) dx$$

$$= \frac{1}{2} \left(\frac{x^4}{4} - \frac{x^6}{6} \right) \Big|_0^1$$

$$= \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{1}{2} \left(\frac{6}{24} - \frac{4}{24} \right)$$

$$= \frac{1}{\cancel{2}} \left(\frac{\cancel{2}}{24} \right) = \frac{1}{24}$$



$$\int_0^2 (4x - x^3) \, dx$$

$$\frac{4x^2}{2} - \frac{x^4}{4} \Big|_0^2$$

$$\frac{16}{2} - \frac{16}{4} = 8 - 4 = 4$$

or $$\int_0^2 \int_{x^3}^{4x} 1 \, dy \, dx = 4$$

$$\int_{-4}^{0} \int_{x^2+4x-7}^{-x^2-4x-7} 1 \, dy \, dx = \frac{64}{3}$$

$$\int_{-4}^{0} -x^2 - 4x - 7 - (x^2 + 4x - 7) \, dx$$

$$\int_{-4}^{0} -2x^2 - 8x \, dx$$

$$-\frac{2x^3}{3} - 4x^2 \Big|_{-4}^{0}$$

$$-\left(-\frac{2(4)^3}{3} - 4(16) \right) = \frac{64}{3}$$

- Use double integrals to calculate volume of a region between two surfaces over a general plane region

Find volume of solid bounded by $f(x,y) = 4-3x-y$ over the region enclosed by $y=0$
$y = \cos x$ where
x is in $[0,\pi]$

$$\int_0^{\pi} \int_0^{\cos x} (4-3x-y)\, dy\, dx$$

$$\int_0^{\pi} 4y - 3xy - \frac{y^2}{2} \Big|_0^{\cos x} dx$$

$$\int_0^{\pi} 4\cos x - 3x\cos x - \frac{\cos^2 x}{2}$$

$$4\sin x - 3(x\sin x + \cos x) - \frac{1}{4}(x + \sin x\cos x) \Big|_0^{\pi}$$

$$= 3 - \frac{1}{4}\pi - (-3)$$

$$\approx 5.21 \text{ unit}$$

$$z = 2x^2 y$$

$y = x^2$ $\quad y = -x^2+2$ $\quad x = 0$ $\quad z = 0$

$$0 \le x \le 1$$

$$x^2 \le y \le -x^2+2$$

$$\int_{x^2}^{-x^2+2} \int_0^1 2x^2 y \, dx\, dy = \frac{8}{15}$$

$$\int_0^1 x^2 y^2 \Big|_{x^2}^{-x^2+2} dx$$

$$\int_0^1 (-4x^4 + 4x^2)\, dx$$

$$-\frac{4}{5}x^5 + \frac{4}{3}x^3 \Big|_0^1 = -\frac{4}{5} + \frac{4}{3} = \frac{-12}{15} + \frac{20}{15}$$

$$= \frac{8}{15}$$

$$z = 2 - 2x - 2y \qquad z = 1 - x - y$$

$$x = 0 \qquad y = 0$$

S_1 $\quad 2x + 2y + z = 2$ when $z = 0$

$$2x + 2y = 2$$

$$y = 1 - x$$

$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\, dx = \frac{1}{6}$$

S_2

$$V = \int_0^1 \int_0^{1-x} 2(1 - x - y)\, dy\, dx$$

$$= \frac{1}{3}$$

$$S_2 - S_1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

$$V_3 = \int_0^1 \int_0^{1-x} 1 - x - y \; dy\, dx = \frac{1}{6}$$

$z_1 - z_2$

$= 2(1-x-y) - (1-x-y)$

$= 1 - x - y$

$2x + 2y + z = 2$ $\qquad z = 2 - 2x - 2y$

$$\int_0^1 \int_0^{1-x} 2(1 - x - y) \, dy \, dx$$

$2x + 2y + 0 = 2$

$2(x + y) = 2$

$y = 1 - x$

$$2 \int_0^1 (1 - x)^2 \, dx$$

$$= \frac{1}{3}$$

$y^2 = x \qquad x^2 = y$

$x + y + z = 3 \qquad z = 3 - x - y$

$$\int_0^1 \int_{x^2}^{\sqrt{x}} 3 - x - y \, dy \, dx$$

$$= \frac{7}{10}$$

$$\int_0^2 \int_1^{e^y} 1 \, dx \, dy \rightarrow e^2 - 3$$

$\ln(2) =$

$e^2 - 3 + 1 +$
$2 - \log 4$

$$\int_0^2 \int_{\ln(2)}^1 1 \, dx \, dy \rightarrow 2 - \log 4$$

$e^2 - \log 4$

$$\int_0^{\ln 2} \int_0^{e^x} 1 \, dy \, dx \rightarrow 1$$

$y = x^2 - 4$

$x^2 - 4 = x^2 - 9$

$y = x^2 - 9$

$$\int_{-3}^{3} \int_{x^2-9}^{x^2-4} 1 \, dy \, dx$$

$$\int_{-3}^{3} \cancel{x^2} - 4 \cancel{- (x^2} - 9) \, dx$$

$$5x \Big|_{-3}^{3}$$

$-18 \cdot 2 = -36$

$-3 \cdot 2 = -6$

30

$15 + 15 = 30$

-30

$$\int_0^1 \int_0^{e^y} 1 \, dx \, dy + \int_1^2 \int_{\ln y}^{e^y} 1 \, dx \, dy$$

$$= e^2 - 2\ln 2$$

$$\int_0^9 \int_0^{\sqrt{y}} 1 \, dx \, dy$$

$$\int_0^9 \sqrt{y} \, dy$$

$$\frac{2}{3} y^{3/2} \Big|_0^9$$

$$\int_0^4 \int_0^{\sqrt{x}} 1 \, dy \, dx = \frac{16}{3}$$

$$\int_0^4 \sqrt{x} \, dx$$

$$\frac{2}{3} x^{3/2} \Big|_0^4 = \frac{16}{3}$$

$$\iint_D f = \int_0^1 \int_{1-x^2}^{4-x^2} 1 \, dy \, dx + \int_1^2 \int_0^{4-x^2} 1 \, dy \, dx$$

$$= 3 + \frac{5}{3} = \frac{14}{3}$$

5.3.2 Double Integrals Over a General Polar Region

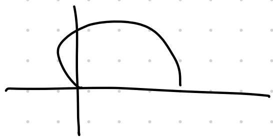
Evaluate a double integral over a general polar region

$$R = \{ (r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta) \}$$

$$\iint f(r, \theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

$$\iint r^2 \sin\theta \; r\, dr\, d\theta$$

$r = 1 + \cos\theta$

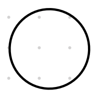


$0 \le \theta \le \pi$

$$\int_0^{\pi} \int_0^{1+\cos\theta} r^2 \sin\theta \, r\, dr\, d\theta$$

$$\int_0^{\pi} \left. \frac{r^4}{4} \right|_0^{1+\cos\theta} \sin\theta \; d\theta$$

$$\frac{1}{4} \int_0^{\pi} \left[(1+\cos\theta)^4 \sin\theta \right]$$



$$-\frac{1}{4} \left. \frac{(1+\cos\theta)^5}{5} \right|_0^{\pi}$$

$$-\frac{1}{4} \left[\frac{(1-1)^5}{5} - \frac{2^5}{5} \right]$$

$$-\frac{1}{\cancel{4}} \left(-\frac{\cancel{32}\,8}{5} \right) = \frac{8}{5}$$

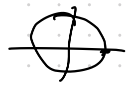
$$\int_0^{\pi}\int_0^{2\sqrt{\cos 2\theta}} r^3 \sin^2 2\theta \, dr\, d\theta$$

$$\int_0^{\pi} \frac{r^4}{4} \sin^2 2\theta \Big|_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$\int_0^{\pi} 4\cos^2 2\theta \ \sin^2 2\theta \ d\theta \qquad u = 2\theta$$

$$\int_0^{2\pi} 2\cos^2 u \ \sin^2 u \ d\theta \qquad du = 2 d\theta$$

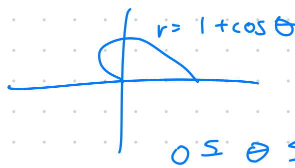
$$2 \cdot \frac{1}{8}\left(u - \frac{1}{4}\sin(4u)\right)\Big|_0^{2\pi}$$



$$\frac{1}{4} \cdot \left(\left[2\pi - \frac{1}{4}\sin(8\pi)\right] - \left[0 - \frac{1}{4}\sin(0)\right]\right)$$

$$\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$\int_0^{\pi}\int_0^{1+\cos\theta} r, \ dr\, d\theta = \frac{3\pi}{4}$$

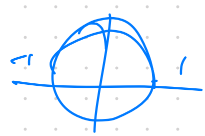


$0 \leq \theta \leq \pi$

$$\int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta$$

$$\int_0^{\pi} \left[\frac{(1+\cos\theta)^2}{2}\right] d\theta$$

$$\frac{1}{2}\int_0^{\pi} (1+\cos\theta)^2 \ d\theta$$

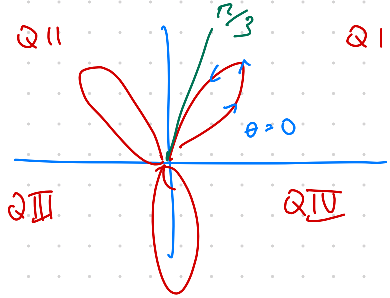


$$\frac{1}{2}\int_0^{\pi}\left[\cos^2\theta + 2\cos\theta + 1\right]$$

$$\frac{1}{2}\cdot\frac{1}{4}\left[6\theta + 8\sin\theta + \sin 2\theta\right]\Big|_0^{\pi}$$

$$\frac{1}{8}\left[6\pi\right] = \frac{3\pi}{4}$$

$$\int_0^{\pi/2}\int_0^{3\sin(3\theta)} r \quad dr \quad d\theta$$



$$\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{3\sin(3\theta)} d\theta$$

$$\int_0^{\pi/2} \frac{9\sin^2(3\theta)}{2} \, d\theta$$

$$\int \sin^2(3\theta)$$

$$= \frac{\theta}{2} - \frac{1}{12}\sin(6\theta)$$

take the derivative

$$= \frac{1}{2} - \frac{6}{12}\cos(6\theta)$$

$$\frac{9\pi}{8}$$

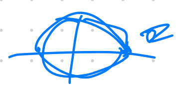
$$\int_0^{\pi/6}\int_0^{3\cos(3\theta)} r \, dr \, d\theta = \frac{3\pi}{8}$$

$$\int_0^{\pi/6} \frac{r^2}{2}\Big|_0^{3\cos(3\theta)} d\theta$$

$$\int_0^{\pi/6} \frac{9\cos^2(3\theta)}{2} \, d\theta$$

$$\frac{9}{2}\int_0^{\pi/6} \cos^2(3\theta) \, d\theta$$

$$\frac{3}{8}\Big[6\theta + \sin(6\theta) \Big]_0^{\pi/6}$$



$$\int_0^{\pi/6}\int_0^{3\cos 3\theta} 3r\cos(3\theta)\,dr\,d\theta$$

$$\int_0^{\pi/6} \frac{3r^2}{2}\cos(3\theta)\Big|_{r=0}^{r=3\cos 3\theta} d\theta$$

$$\frac{3}{2}\int_0^{\pi/6} \left(9\cos^2(3\theta) - 0\right)(\cos 3\theta)\,d\theta$$

$$\frac{27}{2}\int_0^{\pi/6} \cos^3(3\theta)\,d\theta$$

$$= 3$$

5.4.1 Triple Integrals

$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$\int_a^b \int_{h_1(x)}^{h(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

$$\int_0^2\int_0^4\int_0^3 xyz\;dz\,dy\,dx$$

$$\frac{xyz^2}{2}\Big|_0^3$$

$$\int_0^4 \frac{9}{2}xy\;dy$$

$$\frac{9}{4}xy^2\Big|_0^4$$

$$\int_0^2 36x\;dx$$

$$\frac{36}{2}x^2\Big|_0^2$$

$$72$$

$$\int_1^3\int_0^2\int_1^2 xy+yz+xz\;dx\,dy\,dz$$

$$\frac{x^2y}{2}+yzx+\frac{x^2z}{2}\Big|_1^2$$

$$2y+2yz+2z-\frac{y}{2}-yz-\frac{1}{2}z$$

$$\int_0^2 \frac{3y}{2}+yz+\frac{3}{2}z\;dy$$

$$\frac{3}{4}y^2+\frac{y^2z}{2}+\frac{3}{2}yz\Big|_0^2$$

$$\int_1^3 3+2z+3z\;dz$$

$$\int_1^3 3+5z\;dz$$

$$3z+\frac{5z^2}{2}\Big|_1^3$$

$$9+\frac{45}{2}-3-\frac{5}{2}$$

$$6+\frac{40}{2}=\frac{12}{2}+\frac{40}{2}=\frac{52}{2}=26$$

$$\int_0^2\int_0^1\int_0^2 3xz+2y^2z^2+xyz\;dx\,dy\,dz$$

$$\frac{3x^2z}{2}+2y^2z^2x+\frac{x^2yz}{2}\Big|_0^2$$

$$\int_0^1 6z+4y^2z^2+2yz\;dy$$

$$6zy+\frac{4y^3z^2}{3}+\frac{2y^2z}{2}\Big|_0^1$$

$$\int_0^2 6z+\frac{4}{3}z^2+z\;dz$$

$$\frac{6z^2}{2}+\frac{4}{3}\frac{z^3}{2}+\frac{z^2}{2}\Big|_0^2$$

$$12+\frac{32}{9}+2=\frac{158}{9}$$

$$\int_{-1}^{2}\int_{0}^{1}\int_{0}^{\pi} z\sin x + y^2 \; dx\, dy\, dz$$

Evaluate a triple integral over a general bounded region

$f(x,y,z) = y$

$z = 9 - x^2 - y^2$

$x^2 + y^2 + z = 9$

$y = 0 \qquad y \geq 0$

$z = 0 \qquad z \geq 0$

$$\int_{-3}^{3}\int_{0}^{9-x^2}\int_{0}^{9-x^2-y^2} y \; dz\, dy\, dx \quad X$$

$$\int_{-3}^{3}\int_{0}^{3}\int_{0}^{9} y \; dz\, dy\, dx \quad X$$

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} y \; dz\, dy\, dx$$

$$\int_{-2}^{1}\int_{0}^{2}\int_{0}^{y^2} x\,y\,z \;dx\;dy\;dz \qquad x = y^2$$

$$\frac{x^2}{2}\,yz\,\Big|_0^{y^2} \qquad \int_{-2}^{1}\int_{0}^{2}\int_{y^2}^{4}$$

$$\frac{y^4}{2}\,yz$$

$$\int_0^2 \frac{y^5}{2}z\;dy$$

$$\frac{y^6}{6\cdot 2}z\,\Big|_0^2 = \frac{32}{6}z$$

$$\int_{-2}^{1}\frac{32}{6}z\;dz$$

$$\frac{32}{6}\,\frac{z^2}{2}\,\Big|_{-2}^{1}$$

$$\frac{32}{6}\cdot\frac{1}{2} - \left(\frac{32}{6}\cdot\frac{4}{2}\right) = -8$$

$$\frac{x-1}{2} = y$$

$$x = 0$$

$$x = 15$$

$$\int_0^{15}\int_{\frac{x-1}{2}}^{7}\int_0^{1} (x+2)^4 - 3 \quad dz\,dy\,dx$$

$$\int_0^1 \int_{x^2}^{x} \int_0^3 3 \quad dz\, dy\, dx$$

$$3z \Big|_0^3$$

$$9y \Big|_{x^2}^{x}$$

$$\int_0^1 9x - 9x^2 \quad dx$$

$$\frac{9x^2}{2} - \frac{9x^3}{3} \Big|_0^1$$

$$\frac{9}{2} - \frac{9}{3} = 4.5 - 3$$
$$= 1.5$$

$$\int_0^2 \int_0 \int_0^{x-yy^2} z^2 y \quad dx\, dz\, dy$$

$$\int_0^2 \int_0^1 \int_0^{1-x} x + y + z \quad dy\, dx\, dz$$

$$xy + \frac{y^2}{2} + zy \Big|_0^{1-x}$$

$$\int_0^1 x(1-x) + \frac{(1-x)^2}{2} + z(1-x) \quad dx$$

$$x - x^2 + \frac{(1-x)^2}{2} + z - xz$$

$$\frac{x^2}{2} - \frac{x^3}{3} - \frac{(1-x)^3}{6} + zx - \frac{x^2 z}{2} \Big|_0^1$$

$$\frac{3}{6} - \frac{2}{6} \quad \left(\frac{1}{2} - \frac{1}{3} + z - \frac{1}{2}z\right) - \left(-\frac{1}{6}\right)$$

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{6} + \frac{1}{2}z$$

$$\int_0^2 \frac{1}{3} + \frac{1}{2}z \quad dz$$

$$\frac{1}{3}z + \frac{1}{2}\frac{z^2}{2} \Big|_0^2$$

$$\frac{2}{3} + \frac{4}{4} = \frac{5}{3}$$

$$\int_0^1 \int_1^e \int_0^{\ln x} y \ln x + z \quad dy\ dx\ dz$$

$$\frac{y^2}{2} \ln x + z y \Big|_0^{\ln x}$$

$$\frac{(\ln x)^2 \ln x}{2} - z(\ln x)$$

$$\int_0^1 \frac{(\ln x)^3}{2} - (\ln x) z \quad dz$$

$$\frac{(\ln x)^3}{2} z - \frac{(\ln x) z^2}{2} \Big|_0^1$$

$$\frac{(\ln x)^3}{2} - \frac{\ln(x)}{2} = \frac{(\ln x)^3 - (\ln x)}{2}$$

$$\int_1^e \frac{(\ln x)^3 - \ln x}{2} \quad dx$$

$$\int_1^2 \int_{-1}^1 \int_{y^2-1}^{1-y^2} x^2 \quad dx\ dy\ dz$$

$$\frac{x^3}{2} \Big|_{y^2-1}^{1-y^2}$$

$$\int_{-1}^1 -\frac{2y^6}{3} + 2y^4 - 2y^2 + \frac{2}{3} \quad dy$$

$$\frac{-2y^7}{21} + \frac{2y^5}{5} - \frac{2y^3}{3} + \frac{2}{3} y \Big|_{-1}^1$$

$$\left(\frac{-2}{21} + \frac{2}{5} - \frac{2}{3} + \frac{2}{3}\right) - \left(\frac{2}{21} + \frac{-2}{5} + \frac{2}{3} - \frac{2}{3}\right)$$

$$\int_1^2 \frac{64}{105} \quad dz$$

$$x^2 = 1 - y^2$$
$$x = \pm\sqrt{1 - y^2}$$

$$y = -1$$
$$y = 1$$

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{1}^{2} x + z \, dz \, dx \, dy = \frac{3\pi}{2}$$

$$x^2 + y^2 \leq 4$$
$$x = \pm\sqrt{4 - y^2}$$

$$y = -2$$
$$y = 2$$

$$\int_{1}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{0}^{4x^2+4y^2} y \, dz \, dx \, dy = \frac{50\sqrt{3}}{5}$$

Cyl Sph. Coord

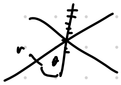
Cylindrical Coordinates r, θ, z

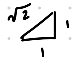
$x = r\cos\theta$ $\quad r^2 = x^2 + y^2$

$y = r\sin\theta$ $\quad \tan\theta = \frac{y}{x}$

$z = z$

$z = z$





Point $(1, 1, 3)$ — $x\ y\ z$

$r^2 = 1^2 + 1^2$

$r = \sqrt{2}$ $\quad z = 3$

$\theta = \frac{\pi}{4}$

$\tan\theta = \frac{1}{1}$ $\quad (\sqrt{2}, \frac{\pi}{4}, 3)$

$\theta = 45^\circ = \frac{\pi}{4}$

$r\ \ \theta\ \ z$

Point $(4, \pi/3, -2)$ **to rectangular**

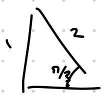
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$4\cos(\frac{\pi}{3}),\ 4\sin\frac{\pi}{3},\ -2$

$= 2 \quad = 2\sqrt{3}$



$(2, 2\sqrt{3}, -2$

$4\cos\frac{\pi}{2}$ $\quad 4\sin\frac{\pi}{2}$

$0,\ 4,\ -3$

$(0, 1, 5)$ to cyl.

$r^2 = 0^2 + 1^2 = 1$

$r = 1$

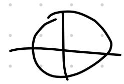
$\tan\theta = \frac{1}{0}$

$z = 5$

$(-2\sqrt{2}, 2\sqrt{2}, 4)$ $\quad r = 4$

$r^2 = (-2\sqrt{2})^2 + (2\sqrt{2})^2$

$r^2 = 4\ 2 + 4 \cdot 2 = 16$



$\tan\theta = -1 \quad \theta = \frac{3\pi}{4}$

$z = 4$

Spherical coords (ρ, θ, φ)

$x = \rho \sin\varphi \cos\theta$
$y = \rho \sin\varphi \sin\theta$ sph → rect
$z = \rho \cos\varphi$

$\rho^2 = x^2 + y^2 + z^2$
$\tan\theta = \frac{y}{x}$ rect → sph
$\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$r = \rho\sin\varphi$
$\theta = \theta$ sph → cyl
$z = \rho\cos\varphi$

$\rho = \sqrt{r^2+z^2}$ cyl → sph
$\theta = \theta$
$\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

$(8, \frac{\pi}{3}, \frac{\pi}{6})$ → rect and cyl

$x = \rho\sin\varphi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 2$

$y = \rho\sin\varphi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{\pi}{3} = 8\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ $(2, 2\sqrt{3}, 4\sqrt{3})$

$z = \rho\cos\varphi = 8\cos\frac{\pi}{6} = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$

$r = \rho\sin\varphi = 8\left(\sin\frac{\pi}{6}\right) = 4$

$\theta = \theta = \frac{\pi}{3}$ $(4, \frac{\pi}{3}, 4\sqrt{3})$

$z = \rho\cos\varphi = 4\sqrt{3}$

$(-1, 1, \sqrt{6})$ → sph and cyl.

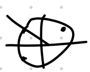
$r = \rho\sin\varphi = \sqrt{2}$
$= 2\sqrt{2}\sin\frac{\pi}{6}$

sph. $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$

$\rho^2 = x^2 + y^2 + z^2$
$\rho^2 = (-1)^2 + (1)^2 + (\sqrt{6})^2 = 1 + 1 + 6$
$= 8$
$\rho = \sqrt{8} = 2\sqrt{2}$

cyl. $(\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$

$\tan\theta = \frac{y}{x} = \frac{1}{-1}$



$\cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$

$\tan\theta = -1$
$\theta = \frac{3\pi}{4}$

$\varphi = \frac{\pi}{6}$

$z = \rho\cos\varphi$ $\sqrt{6} = 2\sqrt{2}\cos\varphi$

$\left(2, -\frac{5\pi}{6}, \frac{\pi}{6}\right)$ sph → ~~cyl.~~ rect

ρ, θ, φ

$r = \rho \sin\varphi = 2 \sin\frac{\pi}{6} = 2\left(\frac{1}{2}\right) = 1$

$\theta = \theta = -\frac{5\pi}{6}$

$z = \rho\cos\varphi = 2\cos\frac{\pi}{6} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$x = \rho \sin\varphi \cos\theta = 2\sin\left(\frac{\pi}{6}\right)\cos\left(-\frac{5\pi}{6}\right) = 2\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\sqrt{3}}{2}$

$y = \rho \sin\varphi \sin\theta = 2\sin\left(\frac{\pi}{6}\right)\sin\left(-\frac{5\pi}{6}\right) = 2\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{2}$

$z = \rho\cos\varphi = 2\cos\frac{\pi}{6} = \sqrt{3}$

$\rho^2 = x^2 + y^2 + z^2$

$\rho^2 = 2^2 + 0^2 + 1^2$

$\rho = \sqrt{4+1} = \sqrt{5}$

$\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right) \rightarrow$ rect

5.5.1 Triple Integrals in Cylindrical

Integration in Cylindrical Coordinates

- Evaluate 3 Integral over Cyl. Box

	Cir. Cyl	Cir. Cone	Sphere	Paraboloid
Rect	$x^2+y^2=c^2$	$z^2=c^2(x^2+y^2)$	$x^2+y^2+z^2=c^2$	$z=c(x^2+y^2)$
Cylin	$r=c$	$z=cr$	$r^2+z^2=c^2$	$z=cr^2$

EX

$$B=\{(r,\theta,z) \mid 0\le r\le 2,\ 0\le\theta\le\frac{\pi}{2},\ 0\le z\le 4\}$$

$$\int_0^{\pi/2}\int_0^2\int_0^4 z\,r\sin\theta\, r\,dz\,dr\,d\theta$$

$$\int_0^4 z\,dz \int_0^{\pi/2}\sin\theta\,d\theta\int_0^2 r^2\,dr$$

$$\frac{z^2}{2}\Big|_0^4 \quad -\cos\theta\Big|_0^{\pi/2} \quad \frac{r^3}{3}\Big|_0^2$$

$$\frac{16}{\cancel{2}}\cdot\left(-\cos\frac{\pi}{2}+\cos 0\right)\cdot\frac{\cancel{8}\,4}{3}$$

$$\frac{64}{3}(1)=\frac{64}{3}$$

Practice

$$\int_0^{\pi}\int_0^1\int_0^4 r z\sin\theta\, r\,dz\,dr\,d\theta$$

$$\int_0^{\pi}\sin\theta\,d\theta\int_0^1 r^2\,dr\int_0^4 z$$

$$-\cos\theta\Big|_0^{\pi} \quad \frac{r^3}{3}\Big|_0^1 \quad \frac{z^2}{2}\Big|_0^4$$

$$(-\cos\pi+\cos 0)\left(\frac{1}{3}\cdot\frac{16}{2}\right)=\frac{16}{6}(2)=\frac{16}{3}$$

$$\int_{-1}^{2}\int_{0}^{\pi}\int_{1}^{2} r^3 \sin\theta \, r \, dr \, d\theta \, dz$$

$$\int_{-1}^{2} \int_{1}^{2} r^4 \, dr \quad \int_{0}^{\pi} \sin\theta \, d\theta \quad dz$$

$$\int_{-1}^{2} \frac{r^5}{5}\Big|_{1}^{2} \quad -\cos\theta \Big|_{0}^{\pi} \quad dz$$

$$\int_{-1}^{2} \left(\frac{32}{5} - \frac{1}{5}\right)\left(-\cos\pi + \cos 0\right) dz$$

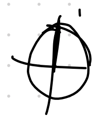
$$\int_{-1}^{2} \frac{31}{5} \cdot 2 \, dz$$

$$\frac{62}{5} z \Big|_{-1}^{2}$$

$$\frac{62}{5} \cdot 2 + \frac{62}{5}$$

$$\frac{124}{5} + \frac{62}{5} = \frac{186}{5}$$

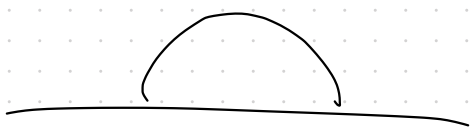
$$\int_{0}^{4}\int_{0}^{\pi/2}\int_{0}^{2} r^2 z \sin^2\theta \, dr \, d\theta \, dz$$



$$\frac{r^3}{3}\Big|_{0}^{2} \quad \frac{z^2}{2}\Big|_{0}^{4} \quad \frac{\theta - \frac{\sin(2\theta)}{2}}{2}\Big|_{0}^{\pi/2}$$

$$\cancel{4} \cdot \frac{\cancel{8}}{3} \cdot \frac{16}{\cancel{2}} \cdot \frac{\pi}{\cancel{4}}$$

$$\frac{16\pi}{3}$$



Convert a 3 integral from rect. to cyl. coord + evaluate

$x = r\cos\theta \qquad r^2 = x^2 + y^2$

$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$z = z \qquad z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{z = x^2+y^2}^{\sqrt{x^2+y^2}} xyz \; dz\,dx\,dy$$

$-1 \le y \le 1 \qquad 0 \le x \le \sqrt{1-y^2} \qquad x^2+y^2 \le z \le \sqrt{x^2+y^2}$

right half of circle w/ $r = 1$

$r^2 \le z \le r$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

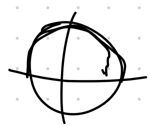
$0 \le r \le 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} \int_{r^2}^{r} r\,(r\cos\theta)\,(r\sin\theta)\, z \; dz\,dr\,d\theta$$

$$= 0$$

$$\int_{x=0}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=0}^{z=2} 2z(x^2+y^2) \; dz\,dy\,dx$$

$$\int_{\pi/2}^{\pi/2} \int_{0}^{3} \int_{0}^{2} 2z\,r^2 \; r\,dz\,dr\,d\theta =$$



$r^2 - x^2 = y^2$

$y = \sqrt{r^2 - x^2}$

$y = \sqrt{9 - x^2}$

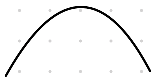
$$\frac{r^4}{\cancel{4}\;2} \cdot \cancel{2} z \,\Big|_0^3$$

$$\pi \int_0^2 \frac{81}{2} z \; dz$$

$$\pi \; \frac{81}{2} \, \frac{z^2}{2} \Big|_0^2 = 81\pi$$

$f = r\cos\theta$

$0 \le z \le a + \sqrt{a^2 - r^2}$



$0 \le r \le a$ from $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$0 \le z \le a + \sqrt{a^2-r^2} \rightarrow 0 \le \theta \le 2\pi$

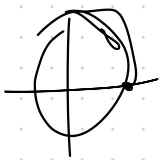
$0 \le x^2 + y^2 \le 9$ $\quad x \ge 0 \quad y \ge 0 \quad 0 \le z \le x+3$

$0 \le r^2 \le 9$

$0 \le r \le 3$ $\quad\quad 0 \le z \le r\cos\theta + 3$

$0 \le \theta \le \frac{\pi}{2}$

r, θ, z



$$\int_0^{\pi/2} \int_0^3 \int_0^{r\cos\theta+3} \frac{1}{r\cos\theta+3} \, r \, dz \, dr \, d\theta$$

$$\frac{1}{r\cos\theta+3} \, r \, z \Big|_0^{r\cos+3}$$

$$\int_0^{\pi/2} \int_0^3 r \, dr \, d\theta$$

$$\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^3 d\theta$$

$$\frac{9}{2} \cdot \frac{\pi}{2} = \frac{9\pi}{4}$$

$$\int_0^9 \int_0^{\pi/2} \int_0^3 r^3 \sqrt{z} \cos^2\theta \, dr \, d\theta \, dz$$

$$\int_0^9 z^{1/2} dz \int_0^3 r^3 \, dr \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$\frac{2}{3} z^{3/2} \Big|_0^9 \cdot \frac{r^4}{4} \Big|_0^3 \cdot \frac{\pi}{4}$$

$$\frac{2 \cdot \cancel{27}^{9}}{\cancel{3}} \quad \frac{81}{4} \cdot \frac{\pi}{4}$$

$$\cancel{18}^{9} \cdot \frac{81}{\cancel{4}_{2}} \cdot \frac{\pi}{4} = \frac{729}{8} \pi$$

$$\int_0^2 \int_0^{2\pi} \int_0^{1-r^2} z \quad r \, dz \, d\theta \, dr$$

(r, θ, z)

$0 \le r^2 \le 4$ $\quad 0 \le z \le 1 - r^2$ $\quad 0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$2\pi \left[\int_0^2 \frac{z^2}{2} \Big|_0^{1-r^2} r \, dr \right.$$

$$\frac{(1-r^2)^2}{2} r = \frac{(1 - 2r^2 + r^4)}{2} r$$

$$= \frac{r - 2r^3 + r^5}{2}$$

$$\pi \left(\frac{r^2}{2} - \frac{2}{4} r^4 + \frac{r^6}{6} \right) \Big|_0^2$$

$$\pi \left(\frac{\cancel{4}}{\cancel{2}}^{2} - \frac{1}{\cancel{2}} \cdot \cancel{16}^{8} + \frac{64}{6} \right)$$

$$\left(2 - 8 + \frac{64}{6} \right) \pi$$

$$\frac{14}{3} \pi$$

$$\int_0^1 \int_{1-r^2}^1 \int_0^{2\pi} z \quad r \quad d\theta \, dz \, dr$$

$$0 \le y^2 + z^2 \le 1$$

$$0 \le r_c^2 \le 1$$

$$0 \le r_c \le 1$$

$$1 - r^2 \le z_c \le 1$$

$$2\pi \left(\int_0^1 \int_{1-r^2}^1 z r \, dz \, dr \right)$$

$$2\pi \int_0^1 \frac{z^2}{2} \Big|_{1-r^2}^1 r \, dr$$

$$\cancel{2}\pi \int_0^1 \left(\frac{1}{\cancel{2}} - \frac{((1-r^2))^2}{\cancel{2}} \right) r \, dr$$

$$\pi \int_0^1 (1 - (1 - 2r^2 + r^4)) r \, dr$$

$$\pi \int_0^1 (2r^2 - r^4) r \, dr$$

$$\pi \int_0^1 2r^3 - r^5 \, dr$$

$$\pi \left(\frac{2r^4}{4} - \frac{r^6}{6} \right) \Big|_0^1$$

$$\pi \left(\frac{1}{2} - \frac{1}{6} \right)$$

$$\pi \left(\frac{3}{6} - \frac{1}{6} \right) = \frac{1}{3} \pi$$

$$\int_0^{\pi} \int_0^2 \int_0^{3 - r\cos\theta} r^2 \, r \quad dz \, dr \, d\theta$$

$$r^3 z \Big|_0^{3 - r\cos\theta}$$

$$\int_0^{\pi} \int_0^2 r^3 (3 - r\cos\theta) \, dr \, d\theta$$

$$3r^3 - r^4 \cos\theta$$

$$\frac{3r^4}{4} - \frac{r^5}{5} \cos\theta \Big|_0^2$$

$$\int_0^{\pi}\int_0^{2}\int_0^{3-r\cos\theta} r^2 \; r \quad dz\,dr\,d\theta$$

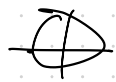
$$r^3 z \Big|_0^{3-r\cos\theta}$$

$$\int_0^{\pi}\int_0^{2} r^3(3-r\cos\theta)\; dr\, d\theta$$

$$3r^3 - r^4\cos\theta$$

$$\frac{3r^4}{4} - \frac{r^5}{5}\cos\theta \Big|_0^{2}$$

$$\int_0^{\pi} 12 - \frac{32}{5}\cos\theta \quad d\theta$$



$$12\theta - \frac{32}{5}\sin\theta \Big|_0^{\pi}$$

$$12\pi - \frac{32}{5}(0)$$

$$12\pi$$

Triple Integrals in Spherical Coord.

$$\iiint f(\rho, \theta, \varphi)\,\rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

Ex.

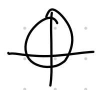
$$\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/2} \int_{\rho=0}^{\rho=1} \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_{\rho=0}^{\rho=1} \rho^2 \, d\rho \quad \int_0^{\pi/2} \sin\varphi \, d\varphi$$

$$= 2\pi \cdot \left.\frac{\rho^3}{3}\right|_0^1 \cdot \left. -\cos\varphi \right|_0^{\pi/2}$$

$$= \frac{2\pi}{3}$$

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=1} 3\sin\theta\, \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$



$$3 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^1 \rho^2 \, d\rho \int_0^{\pi/2} \sin\varphi \, d\varphi$$

$$3 \left.(-\cos\theta)\right|_0^{\pi/2} \quad \left.\frac{\rho^3}{3}\right|_0^1 \quad \left.(-\cos\theta)\right|_0^{\pi/2}$$

$$\cancel{3}\left(-\cos\frac{\pi}{2} + \cos 0\right) \frac{1}{\cancel{3}} \cdot \left(-\cos\frac{\pi}{2} + \cos 0\right)$$

Convert 3I from rect. to sph.

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$\rho^2 = x^2 + y^2 + z^2$

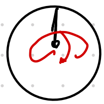
$\tan\theta = \frac{y}{x}$

$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}$

Integrate

$f(x,y,z) = x^2+y^2+z^2$ over V inside $x^2+y^2+z^2=4$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^2 \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



$$2\pi \int_0^{\pi}\int_0^{2} \rho^4 \sin\varphi \, d\rho \, d\varphi$$

$$2\pi \quad \frac{\rho^5}{5}\Big|_0^2 \quad (-\cos\varphi)\Big|_0^{\pi}$$

$$2\pi \cdot \frac{32}{5} \cdot 2 = \frac{64\pi}{5} \cdot 2 =$$

Convert

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \frac{\sqrt{x^2+y^2+z^2}}{1+(x^2+y^2+z^2)^2} \, dz \, dy \, dx$$

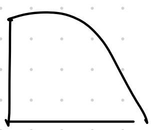
$\leftarrow$ half sphere

$z^2 = 16 - x^2 - y^2$

$\rho^2 = 16$

$$\int_0^{\cancel{2\pi}\ \pi/2} \int_0^{\pi/2} \int_0^{4} \frac{\rho}{1+\rho^4} \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

half sphere



$$\iiint f(\rho,\theta,\varphi) = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

let E bounded below

$$x^2+y^2+(z-1)^2 = 1$$

convert bounding sphere to sp. coord

$$\rightarrow \quad x^2+y^2+z^2 = 2z$$

$$\rho^2 = 2\rho\cos\varphi$$

$$\rho = 2\cos\varphi$$

bounded above

$$z = \sqrt{x^2+y^2}$$

$$\downarrow$$

$$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\varphi + \rho^2\sin^2\varphi\sin\varphi}$$

$$\rho\cos\varphi = \rho\sin\varphi$$

$$\cos\varphi = \sin\varphi$$

$$\varphi = \frac{\pi}{4}$$

Find the limits and solve

$$V(E) = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\frac{\pi}{4}}^{\cos^{-1}\frac{\rho}{2}} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

$$2\pi \int_0^{\sqrt{2}} \rho^2\left(\frac{\sqrt{2}}{2} - \frac{\rho}{2}\right) d\rho$$

$$2\pi \quad \frac{1}{6} = \frac{1}{3}\pi$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$\frac{\pi}{96}$$

$\rho = \cos(\varphi)$ mets $\varphi = \frac{\pi}{3}$ $\varphi = \frac{\pi}{2}$ at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ $\left(\frac{\pi}{2}, 0\right)$

$$0 \le \rho \le \frac{1}{2})$$

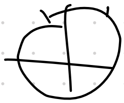
$$\frac{\pi}{3} \le \varphi \le \arccos(\rho)$$

$$\int_0^{2\pi/3} \int_0^{\pi/2} \int_0^1 \rho^2 \cos\varphi \sin\theta \, d\rho \, d\varphi \, d\theta$$

$$\int_0^{2\pi/3} \int_0^{\pi/2} \frac{1}{3} \cos\varphi \sin\theta \, d\varphi \, d\theta$$

$$\int_0^{2\pi/3} \frac{1}{3} \sin\theta \, d\theta$$

$$\frac{1}{2}$$



$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 (\sin\varphi)^2 \, d\rho \, d\varphi \, d\theta$$

$$\frac{\pi}{2} \cdot \frac{2^3}{3} \cdot \left[\frac{\varphi}{2} - \frac{1}{4}\sin(2\varphi) \right]_0^{\pi/2}$$

$$\frac{\pi}{\cancel{2}} \cdot \frac{\cancel{8}}{3} \cdot \frac{\pi}{\cancel{4}} = \frac{\pi^2}{3}$$

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_{\sin\varphi - 1}^{0} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{-5\pi^2}{4}$$

$$\int_0^{2\pi} \int_0^2 \int_{-\pi/2}^{\sin^{-1}(\rho - 1)} \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta = -\frac{5\pi^2}{4}$$

$$\int_0^{2\pi} \int_0^{\pi/8} \int_0^{\cos 2\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 0.101384$$

0.0563657

0.0450183

Correct answer:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\frac{\sqrt{2}}{2}}\int_{\varphi=0}^{\varphi=\frac{\pi}{8}}\rho^2\sin(\varphi)d\varphi d\rho d\theta+\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\frac{\sqrt{2}}{2}}^{\rho=1}\int_{\varphi=0}^{\varphi=\frac{1}{2}\arccos(\rho)}\rho^2\sin(\varphi)d\varphi d\rho d\theta$$

In the $\varphi\rho$ planar coordinate system, the curve $\rho=\cos(2\varphi)$ meets lines $\varphi=0$ and $\varphi=\frac{\pi}{8}$ at points $(0,1)$ and $\left(\frac{\pi}{8},\frac{\sqrt{2}}{2}\right)$, respectively, and the region of integration is below and to the right of the curve. To change the order, we need two pieces

$$\begin{array}{ccccc} 0 & \le & \varphi & \le & \frac{\pi}{8} \\ 0 & \le & \rho & \le & \frac{\sqrt{2}}{2} \end{array} \quad \text{and} \quad \begin{array}{ccccc} 0 & \le & \varphi & \le & \frac{1}{2}\arccos(\rho) \\ \frac{\sqrt{2}}{2} & \le & \rho & \le & 1 \end{array}$$

to integrate over the same region. Therefore, the triple integral becomes

$$\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\frac{\sqrt{2}}{2}}\int_{\varphi=0}^{\varphi=\frac{\pi}{8}}\rho^2\sin(\varphi)d\varphi d\rho d\theta+\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\frac{\sqrt{2}}{2}}^{\rho=1}\int_{\varphi=0}^{\varphi=\frac{1}{2}\arccos(\rho)}\rho^2\sin(\varphi)d\varphi d\rho d\theta.$$

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{\sqrt{4-x^2-y^2}} x y^2 z^2 \, dz\, dx\, dy$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho\sin\varphi\cos\theta\,(\rho\sin\varphi\sin\theta)^2(\rho\cos\varphi)^2\rho^2\sin\varphi$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^7\sin^4\varphi\cos^2\varphi\sin^2\theta\cos\theta \, d\rho\, d\varphi\, d\theta$$

$x = \rho\sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$
$z = \rho\cos\varphi$

$\rho^2 = x^2+y^2+z^2$
$\tan\theta = \frac{y}{x}$
$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}$