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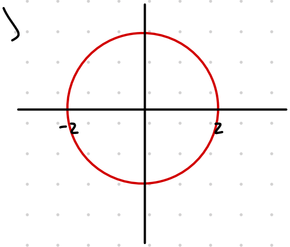
1. a)



$$A(x) = \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - x^2 - y^2 \, dy$$

$$A(x) = 4 - x^2 - \frac{y^3}{3} \Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$x^2 + y^2 = 4$

$y = \pm\sqrt{4 - x^2}$

$$= \cancel{4 - x^2} - \frac{(4-x^2)(\sqrt{4-x^2})}{3} - \left(\cancel{4-x^2} + \frac{(4-x^2)(\sqrt{4-x^2})}{3}\right)$$

$$= -\frac{2(4-x^2)(\sqrt{4-x^2})}{3}$$

b. $$E = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - x^2 - y^2 \, dy \, dx$$

c. $$E = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - (x^2 + y^2) \, dy \, dx$$

$$E = \int_{0}^{2\pi} \int_{0}^{2} (4 - r^2) \, r \, dr \, d\theta$$

$$= 2\pi \int_0^2 4r - r^3 \, dr$$

$$= 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2$$

$$= 2\pi \left[8 - \frac{16}{4} \right] = 2\pi [4]$$

$$= 8\pi$$

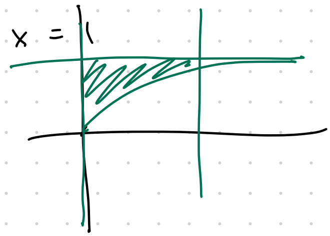
2. $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$

$y = \sqrt{x} \to x = y^2$

$y = 1 \to x = 1$

$x = 0 \to$

$x = 1$



$$\int_0^1 \int_0^{y^2} e^{y^3}\, dx\, dy$$

$$= \int_0^1 e^{y^3} x \Big|_0^{y^2} dy$$

$$= \int_0^1 e^{y^3} y^2\, dy$$

$u = y^3$

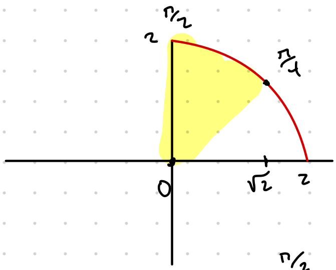
$du = 3y\, dy$

Changing order of integrands

$$= \frac{1}{3}\int_0^1 e^u\, du$$

$$= \frac{1}{3}\left(e^1 - 1\right)$$

3. $\displaystyle\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \frac{1}{1+\sqrt{x^2+y^2}} \, dy \, dx$



$$\frac{r}{1+r} = 1 - \frac{1}{1+r}$$

$$4 - x^2 = y^2$$
$$y^2 + x^2 = 4$$

$$\int_{\pi/4}^{\pi/2} \int_0^2 1 - \frac{1}{1+r} \, dr \, d\theta$$

$u = 1+r$
$du = dr$

$$= \frac{\pi}{4} \left(r - \ln(1+r) \right) \Big|_0^2$$

$$= \frac{\pi}{4} \left(2 - \ln(3) \right)$$

4. $\iint_{D_a} e^{-x^2-y^2}\, dA$

$$\int_0^{2\pi}\int_0^a e^{-r^2}\, r\, dr\, d\theta$$

$$2\pi \int_0^a r e^{-r^2}\, dr$$

$u = -r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$$2\pi \int_0^{-a^2} -\frac{1}{2} e^u\, du$$

$$-\pi \left[e^u \right]_0^{-a^2} = -\pi \left[e^{-a^2} - 1 \right]$$

$$= \pi - \pi e^{-a^2}$$

$$= \pi$$

Exam 4 Review

5. $f(x,y) = \arctan\left(\frac{y}{x}\right)$ $\qquad P = (2,2)$

$\nabla f(x,y) = \langle F_x, F_y \rangle$ $\qquad \frac{d}{du}\arctan(u) = \frac{1}{1+u^2}$

$\|\nabla f\|$ = rate of increase $\qquad u = \frac{y}{x}$

$$F_x = \frac{\frac{d}{dx}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{-\left(y/x^2\right)}{1+\left(\frac{y}{x}\right)^2}$$

$\frac{du}{dx} = -y$

$u = \frac{y}{x}$

$$= -\frac{y}{x}\left(\frac{1}{1+\frac{y^2}{x^2}}\right)$$

$\frac{du}{dy} = \frac{1}{x}$

$$= \frac{-y}{x^2+y^2} = \frac{-2}{8} = -\frac{1}{4}$$

$$F_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} = \frac{2}{8} = -\frac{1}{4}$$

$$\|\nabla F\| = \sqrt{\left(\frac{x}{x^2+y^2}\right)^2 + \left(\frac{-y}{x^2+y^2}\right)^2}$$

$$= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = 0.354$$

$$= \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{8}}$$

6. $f(x,y) = x^2 + x - 3xy + y^3$

$f_x = 2x + 1 - 3y$

$f_{xx} = 2$ $\quad f_{xy} = -3$

$f_y = -3x + 3y^2$

$f_{yy} = 6y$

To find CP: both partials = 0

$2x + 1 - 3y = 0$ $\quad 2x + 1 = 3y$

$x = \frac{3y - 1}{2}$

$-3x + 3y^2 = 0$

$(1, 1) \quad \left(\frac{1}{4}, \frac{1}{2}\right)$

$f(1,1) = 2 - 3 + 1$

$= 0$ inconclusive

$d = f_{xx} f_{yy} - (f_{xy})^2 = 2\left(\frac{6}{2}\right) - 9$

$= -3$ negative

saddle point

$\left(\frac{1}{4}, \frac{1}{2}\right)$

7.
optional

$yz = 2\lambda$

$\lambda = \frac{32}{9}$

$x = \frac{4}{3}$

$y = \frac{8}{3}$

Legrange Multiplier

$V(x,y,z) = xyz$

$\nabla V = \langle yz, xz, xy \rangle$

$\nabla f = \langle 2, 1, 1 \rangle$

$\nabla V = \lambda \nabla f$

$yz = 2\lambda$

$xz = \lambda$

$xy = \lambda$

$2x + y + z - 8 = 0$

constraint: $2x + y + z - 8 = 0$

f ↗

$z = 2 \cdot \frac{\cancel{32}^{4}}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{\cancel{8}}$

$V\left(\frac{4}{3}, \frac{8}{3}, \frac{8}{3}\right) = \frac{256}{27}$

8. 1) Find the critical pts

2) Solve constrained region

$f(x,y) = \ln(2 + y^2 + x^2 - 2x)$

$g(x) = \ln(2 + (16 - 4x^2) + x^2 - 2x)$

Region:

$4x^2 + y^2 = 16$

$y^2 = 16 - 4x^2$

Critical points:

$f_x = \dfrac{2x - 2}{2 + y^2 + x^2 - 2x}$ $\qquad$ $f_y = \dfrac{2y}{2 + y^2 + x^2 - 2x}$

$f_x = 0 \rightarrow x = 1$ $\qquad$ $f_y = 0 \rightarrow y = 0$

CP: $(1, 0)$

$f(1,0) = \ln(2 + 0 + 1 - 2) = \ln(1) = 0$

Constrained region:

$g(x) = \ln(2 + 16 - 3x^2 - 2x)$

$= \ln(-3x^2 - 2x + 18)$

$g'(x) = \dfrac{-6x - 2}{-3x^2 - 2x + 18}$; $g'(x) = 0 = -6x - 2$

$x = \dfrac{-2}{6} = \dfrac{-1}{3}$

$g\left(-\frac{1}{3}\right) = \ln\left(-3\left(\frac{1}{3}\right)^2 + \frac{2}{3} + 18\right)$

$= \ln\left(-\frac{1}{3} + \frac{2}{3} + 18\right)$

$= \ln\left(\frac{55}{3}\right)$

Extreme values of $f(x,y)$: 0, $\ln\left(\frac{55}{3}\right)$

9. double integral

bounded $z = e^y$, $z = 1$ over $[0,1] \times [0, \ln(2)]$

$$\int_0^1 \int_0^{\ln(2)} e^z - 1 \; dy\, dx$$

$$\int_0^{\ln(2)} \int_0^1 e^z - 1 \; dx\, dy$$