Lecture 1

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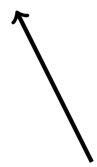
Vectors!

A <u>non-zero vector</u> $\vec{v}$ is a quantity with a magnitude and a direction; they are often represented geometrically by a directed line segment (aka arrow) where the length represents the vector's magnitude and the orientation represents the vector's direction.



The <u>zero vector</u> $\vec{0}$ represents 0 magnitude (geometrically represented by a point).

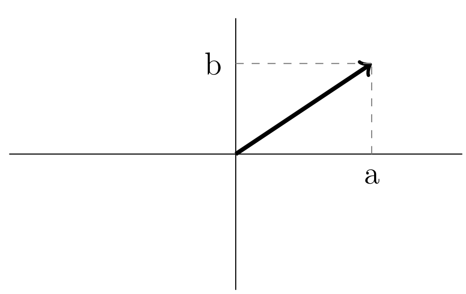
In this course we will primarily focus on vectors in the 2D plane ($\mathbb{R}^2$) or in 3D space ($\mathbb{R}^3$). Often we draw representations in 2D for convenience.

2D: One can algebraically represent a vector in the xy-plane by $\vec{u} = \langle a, b\rangle$ where a and b are the x- and y-components, representing the horizontal and vertical displacement respectively (from the "tail" of the arrow to the "head" of the arrow). Displacement can be negative! Using the usual orientation, the vector above could be $\langle -1, 2\rangle$.

3D: We do it similarly with $\mathbf{v} = \langle a, b, c\rangle$ representing the vector in xyz-space whose x-, y-, and z- components are a, b, and c respectively. Each one is a displacement along one of three perpendicular axes. Orientation convention: If the positive x-axis represents East and the positive y-axis represents North, then the positive z-axis represents up ("right-hand rule").

The zero vector of course would have all components equal to 0.

Using these conventions, the point (a, b) in the xy-plane is the position of the head of $\vec{x} = \langle a, b\rangle$ when its tail is placed at the origin $(0, 0)$.



A similar idea works for xyz-space, where the point (a, b, c) is the position of the head of $\mathbf{y} = \langle a, b, c\rangle$ when its tail is placed at the origin $(0, 0, 0)$ which is where the x, y, z axes meet.

Lecture 1 Supplemental
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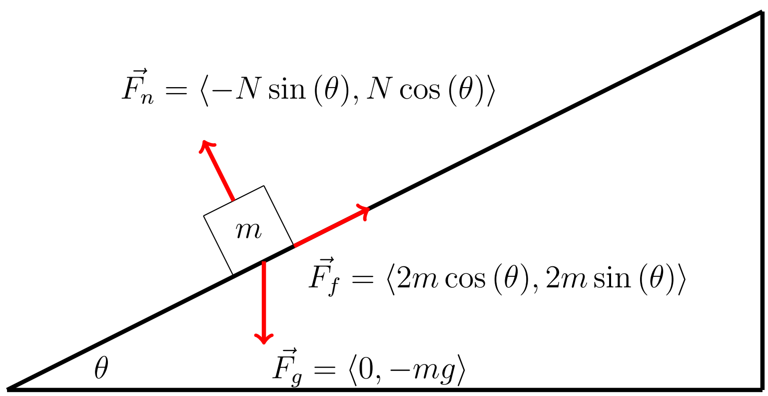
Working with Vectors!

There are many applications of vectors (as we will see). Here we look at one example.

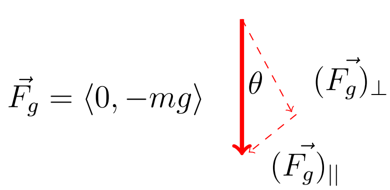
Forces can be represented as vectors, as they have a direction and a magnitude. In particular, when an object is at rest (or moving at a constant velocity) the sum of all the forces acting on the object (called the net force) is $\vec{0}$.

Example: Suppose a block of mass m kg is at rest on an incline whose angle of elevation is θ. Assume that the forces acting on the object are a gravitational force, a normal force (of magnitude N) and a frictional force (shown in red). Suppose the frictional force has magnitude $2m$. Let's find θ and N as a multiple of m (using $g = 9.8$ m/s^2 as the acceleration due to gravity).



Note: The normal force (of magintude N) is in a direction that is at an angle of θ to the left of the positive vertical direction. So its horizontal component is $-N\sin(\theta)$ and its vertical component is $N\cos(\theta)$.

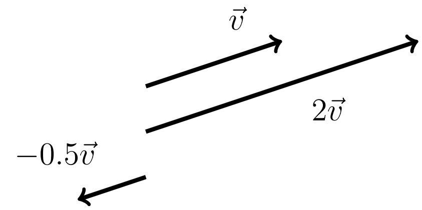
While we can set the sum of these vectors to $\vec{0}$ it turns out to be easier to decompose $\vec{F_g}$ into the sum of two perpendicular vectors, $\vec{F_g} = (\vec{F_g})_{||} + (\vec{F_g})_{\perp}$:



The magnitude of $(\vec{F_g})_{\perp}$ is $mg\cos(\theta)$ and is equal to the magnitude of $\vec{F_n}$ (so that they cancel). So $N = mg\cos(\theta)$. Similarly, the magnitude of $(\vec{F_g})_{||}$ is $mg\sin\theta$ and is equal to $2m$ (the magnitude of $\vec{F_f}$, so that they cancel).

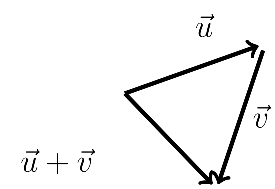
So $mg\sin(\theta) = 2m$ implying $\theta = \arcsin(2/9.8) \approx 0.2$ radians (or about 12 degrees) and $N \approx 9.8m\cos(0.2) \approx 9.6m$.

Scalar Multiplication: A vector $\mathbf{u}$ can be multiplied by a real number c called a *scalar*. The product is denoted by $c\mathbf{u}$, and is the result scaling the length of $\mathbf{u}$ by $|c|$ and keeping the same direction if $c > 0$ or reversing the direction if $c < 0$. In terms of components, each component of the vector gets multiplied by the scalar c.



Two vectors are parallel if they are non-zero scalar multiples of each other.

Vector Addition: The sum of vectors $\vec{u}$ and $\vec{v}$ is determined componentwise by adding corresponding components. It can be drawn by placing the tail of one vector at the head of the other and drawing the arrow from the tail of the first vector to the head of the second vector. Notation: $\vec{u} + \vec{v}$. Order doesn't matter, so $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (parallelogram law).



The magnitude of a vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$. In 2D one can use the Pythagorean Theorem to show that the magnitude of $\mathbf{v} = \langle a, b\rangle$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$. In $3D$ one can still use the Pythagorean Theorem (applying it more than once) to determine that the magnitude of $\vec{w} = \langle a, b, c\rangle$ is $|\vec{w}| = \sqrt{a^2 + b^2 + c^2}$.

A vector $\mathbf{u}$ is a unit vector if it has magnitude 1. We use $\hat{\mathbf{u}}$ denote that a vector is a unit vector.

Engineering notation: We define some basic unit vectors, shown below.

$\mathbf{i} = \langle 1, 0\rangle$ or $\langle 1, 0, 0\rangle$,

$\mathbf{j} = \langle 0, 1\rangle$ or $\langle 0, 1, 0\rangle$,

$\mathbf{k} = \langle 0, 0, 1\rangle$.

Then by scalar multiplication and vector addition, $\langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$ and $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Mon - Week 1

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Dot and Cross Products

2D: The dot product of two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is the scalar

$$u_1 v_1 + u_2 v_2.$$

$u = \langle u_1, u_2 \rangle$ $\quad v = \langle v_1, v_2 \rangle$ $\quad u_1 v_1 + u_2 v_2$

3D: The dot product of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$u_1 v_1 + u_2 v_2 + u_3 v_3.$$

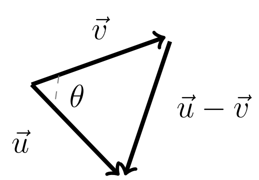
$u = \langle u_1, u_2, u_3 \rangle$ $\quad v = \langle v_1, v_2, v_3 \rangle$ $\quad u_1 v_1 + u_2 v_2, u_3 v_3$

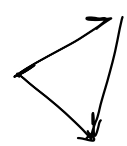
The dot product of two vectors $\vec{u}, \vec{v}$ is denoted by $\vec{u} \cdot \vec{v}$. The identities below (in either 2D or 3D) are left for the reader to verify, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors and c is a scalar.

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} \\ (c\vec{u}) \cdot \vec{v} &= c(\vec{u} \cdot \vec{v}) \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ \vec{u} \cdot \vec{u} &= |\vec{u}|^2 \end{aligned}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} \\ c\vec{u} \cdot \vec{v} &= c(\vec{u} \cdot \vec{v}) \\ \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ \vec{u} \cdot \vec{u} &= |\vec{u}|^2 \end{aligned}$$

Let θ be the angle between $\vec{u}$ and $\vec{v}$. Consider the picture below:





By the Law of Cosines, $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos\theta$.

$|u - v|^2 = |u|^2 + |v|^2 - 2|u||v| \cos\theta$

By properties of the dot product, with some details left to the reader,

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

$(u - v) \cdot (u - v)$

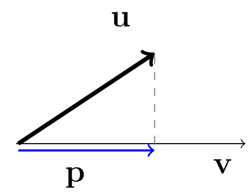
$u \cdot u - 2u \cdot v + v \cdot v$

$|u|^2 + |v|^2 - 2u \cdot v$

Hence $\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.}$

$$\boxed{\vec{u} \perp \vec{v} \text{ if and only if } \vec{u} \cdot \vec{v} = 0.}$$

The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v} \neq \mathbf{0}$, denoted $\mathbf{p} = \text{proj}_{\mathbf{v}}(\mathbf{u})$, is defined as the scalar multiple of $\mathbf{v}$ that is orthogonal to $\mathbf{u} - \mathbf{p}$ as seen here:



$\mathbf{p} = c\mathbf{v}$ for some scalar c and satisfies that $(\mathbf{u} - \mathbf{p}) \perp \mathbf{v}$. Therefore we can solve for c by using $(\mathbf{u} - c\mathbf{v}) \cdot \mathbf{v} = 0$ which implies that $c = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$.

So we get the formula

$$\boxed{\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}.}$$

What is the difference between a cross product and dot product?

The cross product of two 3D vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ (in that order) is denoted $\vec{u} \times \vec{v}$ and is defined as follows:

$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$

$$\vec{u} \times \vec{v} = \langle\, u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \,\rangle.$$

In my opinion, the best way to remember how to calculate a cross product is to use the following template, where each component of the cross product is the 2×2 matrix determinant $\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\right)$ of the matrix that is the result from deleting the corresponding components above, with an alternating sign:

$$\begin{array}{l} \langle\, u_1,\ u_2,\ u_3 \,\rangle \\ \times \langle\, v_1,\ v_2,\ v_3 \,\rangle \\ \hline = \langle\, u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \,\rangle. \end{array}$$

$$\begin{array}{l} \langle u_1, u_2, u_3 \rangle \\ \times \langle v_1, v_2, v_3 \rangle \\ \hline \end{array}$$

$= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle$

Let θ be the angle between $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$. With some effort, one can show that

How do you show that $|u \times v| = |u||v|\sin\theta$?

$\boxed{|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta}$ by showing that

$|u \times v| = |u||v|\sin\theta$

$|\mathbf{u} \times \mathbf{v}|^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ and using $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.

$|u \times v|^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$

As a consequence, one can show that $|\vec{u} \times \vec{v}|$ is the area of the parallelogram whose non-parallel sides are $\vec{u}$ and $\vec{v}$.

$= |u|^2|v|^2 - (u \cdot v)^2$

$u \cdot v = |u||v|\cos\theta$

$|\vec{u} \times \vec{v}|$

The identities below (in 3D) are left for the reader to verify, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors and c is a scalar.

$$\begin{aligned}
\vec{u} \times \vec{v} &= -(\vec{v} \times \vec{u}) \\
(c\vec{u}) \times \vec{v} &= c(\vec{u} \times \vec{v}) \\
\vec{u} \times (\vec{v} + \vec{w}) &= (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})
\end{aligned}$$

Geometrically, one can show that the cross product is perpendicular to both inputs and its direction is given by the "right-hand" rule.

a. Using the given angle, find a unit vector painting down the plane and parallel to it

$$\hat{v} = \langle -\cos\left(\frac{\pi}{6}\right), -\sin\left(\frac{\pi}{6}\right) \rangle = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

b. $$(\vec{F_g})_{\parallel} = \text{proj}_{\hat{v}}(\vec{F_g}) = \frac{\vec{F_g} \cdot \hat{v}}{\hat{v} \cdot \hat{v}} \cdot \hat{v} = \left[\frac{0 \cdot \frac{-\sqrt{3}}{2} + (-20)\cdot\left(\frac{1}{2}\right)}{1}\right]$$

$$\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

$$= \langle -5\sqrt{3}, -5 \rangle$$

c. $$(\vec{F_g})_{\parallel} + (\vec{F_g})_{\perp} = \vec{F_g} \rightarrow$$

d. Find

$$\vec{F_N} = -(\vec{F_g})_{\perp} = \langle -5\sqrt{3}, 15 \rangle$$

$$\vec{F_f} = -(\vec{F_g})_{\parallel} = \langle 5\sqrt{3}, 5 \rangle$$

e. $$|F_{g_{\parallel}}| = 10\sqrt{3}$$

$$\hat{v} = \langle -\cos\theta, -\sin\theta \rangle$$

$$(\vec{F_g})_{\parallel} = \frac{F_g \cdot \hat{v}}{\hat{v} \cdot \hat{v}} \quad (\text{proj}_{\hat{v}}(F_g))$$

$$= \frac{0 \cdot (-\cos\theta) + (-20)(-\sin\theta)}{1} \langle -\cos\theta, -\sin\theta \rangle$$

$$= |(\vec{F_g})_{\parallel}| = 20 \sin\theta \cdot 1$$

In Class Activity

1. $(5i + 3j + 7k) \cdot (i - 3j)$

$= \langle 5, 3, 7 \rangle \cdot \langle 1, -3, 0 \rangle$

$= 5 \cdot 1 + 3 \cdot (-3) + 7 \cdot 0$

$= 5 + -9 + 0 = -4.0$

2. Find the value t for which $\langle 9-t, -17+t, -6+3t \rangle$ is orthogonal to $\langle -1, 1, 3 \rangle$ Round to nearest hundredth

$0 = \langle -1, 1, 3 \rangle \cdot \langle 9-t, -17+t, -6+3t \rangle$

$0 = (-9+t) + (-17+t) + (-18+9t)$

$0 = -9 + t - 17 + t - 18 + 9t$

$0 = 11t - 26 - 18$

$0 = 11t - 44$

$t = \frac{44}{11} = 4.00$

3. Find the magnitude of the orthogonal projection $i + j + k$ (V) onto $2i - j + k$ (u) to the nearest hundredth

$\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \vec{v} =$

$V = \langle 1, 1, 1 \rangle$

$u = \langle 2, -1, 1 \rangle$

Magnitude:

$\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$

$= 0.82$

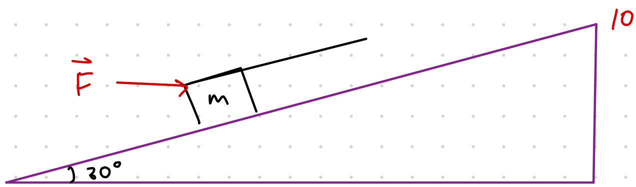
$\text{proj}_u V = \frac{u \cdot V}{\|u\|^2} u$

$= \frac{\langle 2, -1, 1 \rangle \cdot \langle 1, 1, 1 \rangle}{\langle 2, -1, 1 \rangle \cdot \langle 2, -1, 1 \rangle} \cdot \langle 2, -1, 1 \rangle$

$= \frac{2 \cdot 1 + -1 \cdot 1 + 1 \cdot 1}{2^2 + (-1)^2 + (1)^2} \cdot \langle 2, -1, 1 \rangle$

$= \frac{2 - 1 + 1}{6} \langle 2, -1, 1 \rangle = \langle \frac{2}{3}, \frac{-1}{3}, \frac{1}{3} \rangle$

4.



magnitude of force

magnitude of

$W = \vec{F}\vec{d}$

$30° - 15°$

$F = 500 \cdot 10 \cos(15°)$ $\quad N \cdot m \cos(15°)$

$= 4829.6$ J $\quad N \cdot m = J$

$30 \cdot 10 \cos(60°)$

Cross Products

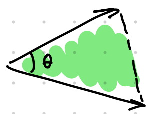
* Only works in xyz-space $\mathbb{R}^3$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta = A$$



$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$

$$\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

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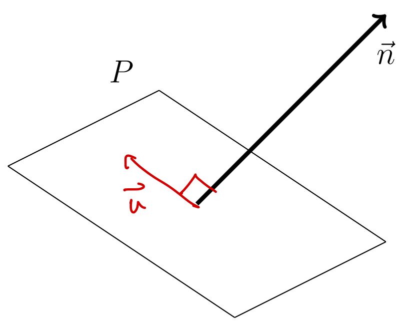
Planes, Lines and Curves (Vector-Valued Functions)

Planes: A plane in $\mathbb{R}^3$ is a copy of $\mathbb{R}^2$ embedded in space. It is uniquely determined by a point in it and an orthogonal direction to it.

Suppose (x_0, y_0, z_0) is a point in the plane P. Suppose $\mathbf{n} =< a, b, c >$ is a non-zero vector that is orthogonal to the plane, called a normal vector. Note: The angle between two planes is the angle θ between normal vectors to the planes chosen so that $0 \leq \theta \leq \pi/2$.



perpendicular

$u \perp n$

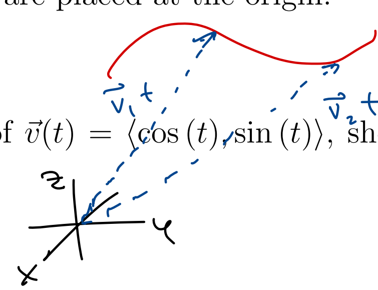
$\vec{u} \cdot \vec{n} = 0$

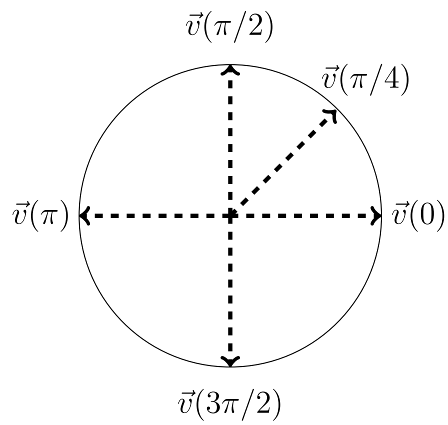
$\langle x-x_0, y-y_0, z-z_0 \rangle = 0$

Let (x, y, z) represent an arbitrary point in the plane. Then the vector $\mathbf{u} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is perpendicular to $\mathbf{n}$. So the plane is determined by $\mathbf{n} \cdot \mathbf{u} = 0$, equivalently $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$, or $ax+by+cz = d$ where $d = ax_0+by_0+cz_0$.

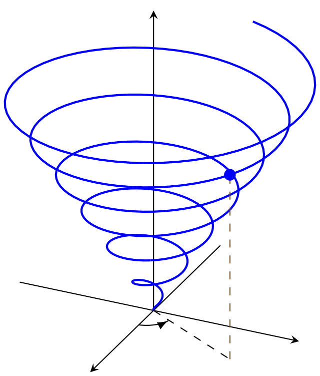
A function of one variable that outputs a vector is a vector-valued function. Such a function has a form $\mathbf{u}(\mathbf{t}) = \langle x(t), y(t) \rangle$ (2D) or $\mathbf{v}(\mathbf{t}) = \langle x(t), y(t), z(t) \rangle$ (3D) where each component is a function. The graph of such a function is the curve traced out by the positions of the heads of the vectors $\mathbf{v}(\mathbf{t})$ when their tails are placed at the origin.

For example, below could be the graph of $\vec{v}(t) = \langle \cos(t), \sin(t) \rangle$, shown with some of the vector outputs.





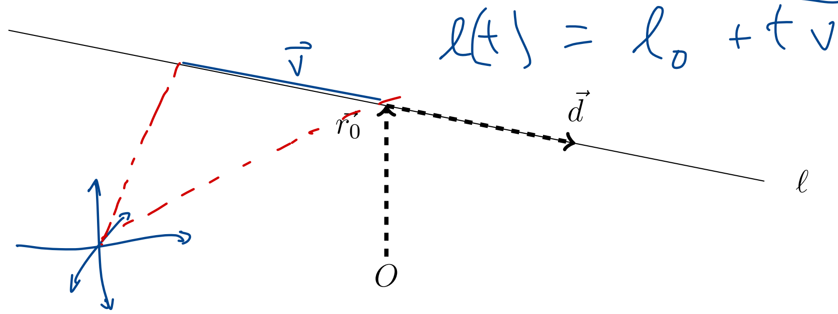
Here could be the graph of $\vec{r}(t) = \langle t\sin(t), t\cos(t), t\rangle$ for $t \geq 0$.



Lines in a plane or in space: $\vec{l}(t) = \langle x(t), y(t), z(t)\rangle$

Let ℓ be a line in $\mathbb{R}^2$ or $\mathbb{R}^3$. The line ℓ is uniquely determined by two distinct points on it. Let A and B be two points on ℓ. Let $\vec{r_0} = \vec{OA}$ where O is the origin and $\vec{d} = \vec{AB}$, which is parallel to the line ℓ. Then the line ℓ is the graph of $\vec{r}(t) = \vec{r_0} + t\vec{d}$.

$\vec{l}(t) = \vec{l_0} + t\vec{v}$



In terms of coordinates, the graph of $\vec{r}(t) = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0)\rangle$ is the line through (x_0, y_0, z_0) and (x_1, y_1, z_1).

A vector-valued function $\mathbf{v}(t)$ approaches a vector limit $\mathbf{L}$ as t approaches a, if $|\mathbf{v}(t) - \mathbf{L}|$ is arbitrarily close to 0 for t sufficiently close to a, but not equal to a. If $\mathbf{v}(a)$ equals the limit of $\mathbf{v}(t)$ as $t \to a$, then $\mathbf{v}(t)$ is continuous at $t = a$.

The limit of a vector-valued function exists if and only if the corresponding limit of each component function exists.

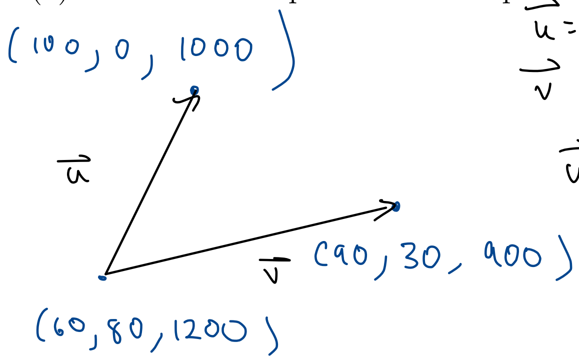
$\lim_{t \to a} \vec{v}(t) = \vec{L}$ if

$f(t) = |\vec{v}(t) - \vec{L}| \to 0$ as $t \to a$

$\vec{v}(t)$ is cont.

Suppose a certain slope on a mountain can be modelled as a plane going through the points $(100, 0, 1000)$, $(90, 30, 900)$ and $(60, 80, 1200)$ where the positive x direction is east, the positive y direction is north, and the positive z direction is up (all in feet).

(a) Let's find an equation for this plane.



$$\vec{u} = \langle 40, -80, -200 \rangle$$

$$\vec{v} = \langle 30, -50, -300 \rangle$$

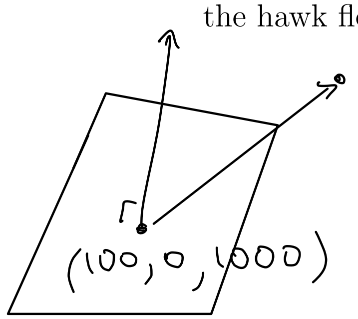
$$\vec{u} \times \vec{v} = \langle 1400, +6000, 400 \rangle$$

set $\vec{n} = \langle 35, 15, 1 \rangle$

$$35x + 15y + 1z = 35(100) + 0 + 1000$$

$$= 1000 + 3500 = 4500$$

(b) A hawk is at $(90, 50, 1600)$ when it begins to fly straight towards a mouse that is at the point on this slope that is nearest to the hawk. Let's find the distance the hawk flew from $(90, 50, 1600)$ to the mouse.



$(90, 50, 1600)$

$$\vec{h} = \langle -10, 50, 600 \rangle$$

$$\text{proj}_{\vec{n}}(\vec{h}) = \frac{\vec{h}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{\vec{h}\cdot\vec{n}}{\|n\|^2}\vec{n} = \frac{1000}{|\vec{n}|}\cdot\underbrace{\frac{1}{|\vec{n}|}\vec{n}}_{\text{unit vector}}$$

$$D = |\vec{p}| = \frac{1000}{|\vec{n}|} = \frac{1000}{\sqrt{35^2 + 15^2 + 1^2}} = 26.25$$

(c) At what point on the slope is the mouse?

$$\vec{m} = \underbrace{\langle 90, 50, 1600 \rangle}_{\text{point of hawk}} - \frac{1000}{|n|^2}\langle 35, 15, 1 \rangle$$

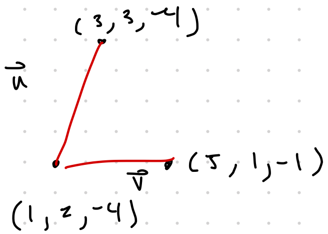
$$= \langle 90, 50, 1600 \rangle - \frac{1000}{1451}\langle 35, 15, 1 \rangle$$

1.

CLASS DISCUSSION EXERCISES (submit answers in corresponding Canvas quiz question):

(1) Find z-coordinate of the point of intersection of the line that is the graph of $\vec{\ell}(t) = \langle 2+7t, 1-t, 3t \rangle$ and the plane that contains the points $(1,2,-4)$, $(3,3,-4)$ and $(5,1,-1)$. Round the nearest tenth.

(2) Find the distance from the point $(1,2,3)$ to the line that is the graph of $\vec{v}(t) = \langle 1-t, 2t, 3+2t \rangle$. Round to the nearest hundredth.



$\vec{u} = \langle 2, 1, 0 \rangle$

$\vec{v} = \langle 4, -1, 3 \rangle$

$\vec{u} \times \vec{v} = \langle 3, -6, -6 \rangle$

$\vec{n} = \langle 1, -2, -2 \rangle$

$x - 2y - 2z = d$

$3 - 2(3) - 2(-4) = -3 + 8 = 5$

$x - 2y - 2z = 5$

$\vec{\ell}(t) = \langle 2+7t, 1-t, 3t \rangle$

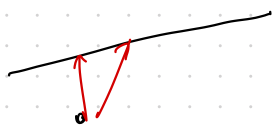
$2 + 7t - 2(1-t) - 2(3t) = 5$

$\cancel{2} \; 7t \; \cancel{-2} + 2t - 6t = 5$

$3t = 5$

$t = \frac{5}{3}$

$z = 5.0$



$\boxed{5.0}$ ✓

2. distance between point and line

$(1,2,3)$ $\qquad$ $\vec{v}(t) = (1-t, 2t, 3+2t)$

M

$(1, 0, 3)$

P point on the line

$v(1) \quad v(0)$

$\vec{w} = \vec{v}(1) - \vec{v}(0)$

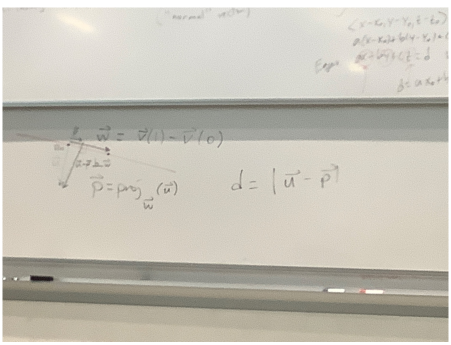
$v(0) = (1,0,3)$

find the vector

$(1,2,3)$

$\text{proj}_{\vec{w}}\, \vec{u}$

$d = |\vec{u} - \vec{p}|$



2.

$$v(1) = (0, 2, 5)$$

$$v(0) = (1, 0, 3)$$

$$\vec{w} = \langle -1, 2, 2 \rangle$$

$$\vec{u} = \langle 0, 2, 0 \rangle$$

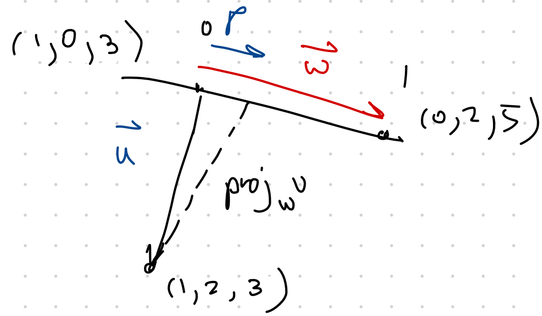
$$\text{proj}_{\vec{w}} \vec{u} = \frac{u \cdot w}{\|w\|^2} \vec{w}$$

$$\vec{p} = \frac{0 + 4 + 0}{(-1)^2 + 2^2 + 2^2} \langle -1, 2, 2 \rangle$$

$$= \frac{4}{1+4+4} \langle -1, 2, 2 \rangle$$

$$= \frac{4}{9} \langle -1, 2, 2 \rangle = \langle -\frac{4}{9}, \frac{8}{9}, \frac{8}{9} \rangle$$

$\vec{v}(t) = (1-t, 2t, 3+2t)$



$$d = |\vec{u} - \vec{p}| = \left| \langle 0, 2, 0 \rangle - \langle -\frac{4}{9}, \frac{8}{9}, \frac{8}{9} \rangle \right|$$

$$= \left| \langle \frac{4}{9}, \frac{10}{9}, -\frac{8}{9} \rangle \right|$$

$$\approx \sqrt{\left(\frac{4}{9}\right)^2 + \left(\frac{10}{9}\right)^2 + \left(-\frac{8}{9}\right)^2} = 1.49$$

3. 0.65 ✓

4. i) d
ii) f
iii) c

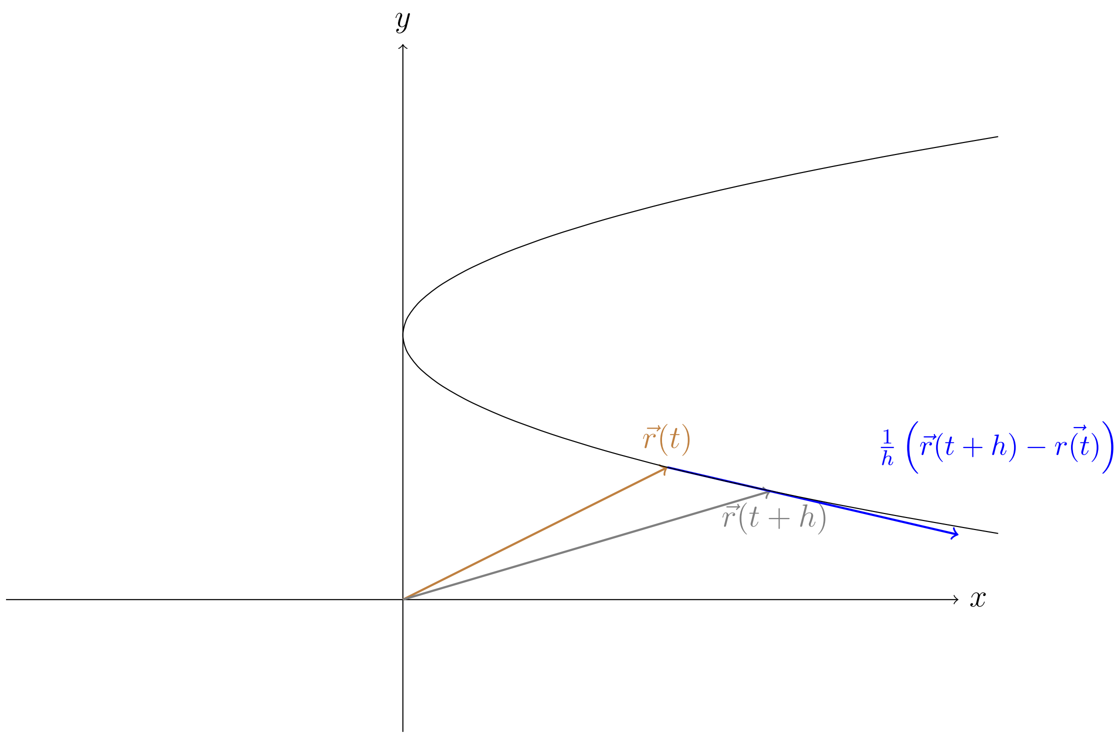
...

Calculus of Curves (and Motion)

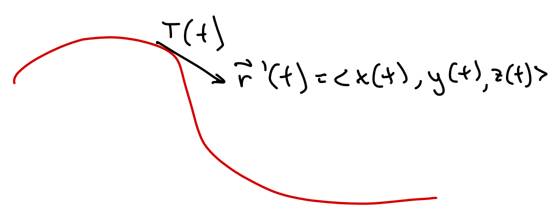
Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ (or $\langle x(t), y(t), z(t) \rangle$) be a vector-valued function. The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \lim_{h \to 0} \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ (limit of the difference quotients). Wherever this exists $\mathbf{r}(t)$ is said to be differentiable.

Since the limit is componentwise, $\mathbf{r}(t)$ is differentiable at $t = a$ iff each component function is differentiable at $t = a$. Where $\mathbf{r}(t)$ is differentiable, $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ (or $\langle x'(t), y'(t), z'(t) \rangle$)

Where $\mathbf{r}(t)$ is differentiable, a non-zero derivative at t results in a vector that is tangent to the curve at the corresponding point along the graph of $\mathbf{r}(t)$ (see picture of the difference quotient) in a direction consistent with the orientation of the parametrization $\mathbf{r}(t)$.



Unit Tangent Vector function: When $\mathbf{r}'(t)$ exists and is nonzero, we can define the unit tangent vector function, a function that outputs a unit vector tangent to the curve (at each point along the curve):



$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$

$R'(t) = \vec{r}(t)$

positive or negative, one unit

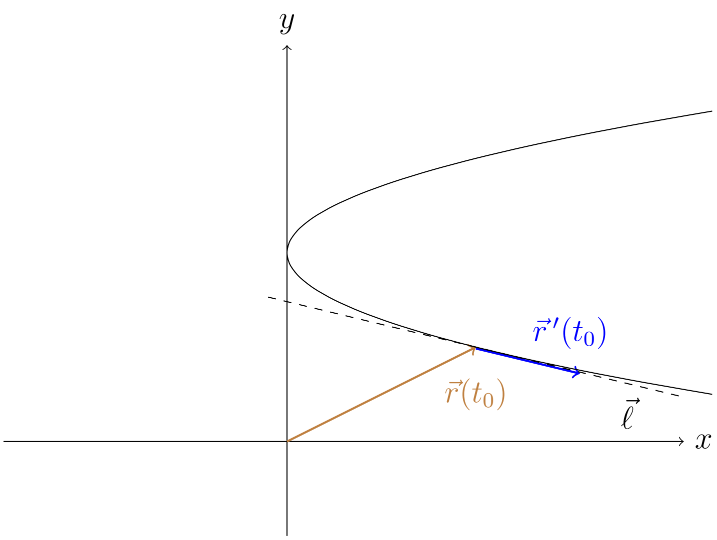
Let $\mathbf{u}(t), \mathbf{v}(t)$ be vector-valued functions, and let $f(t)$ be a scalar-valued function of t. Whenever the cross product is involved assume these are in 3D. Wherever all derivatives exist:

- $\dfrac{d}{dt}(\mathbf{u}(t)+\mathbf{v}(t)) = \mathbf{u}'(t)+\mathbf{v}'(t)$.
- $\dfrac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$.
- $\dfrac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$.
- $\dfrac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$.

You can verify each of these by working with the component functions.

Let C be a curve traced out by a vector-valued function $\mathbf{r}(t)$ that is differentiable at t_0 with $\mathbf{r}'(t_0) \neq \vec{0}$. Then the *tangent line* to C at the point $\mathbf{r}(t_0)$ is given by

$$\vec{\ell}(t) = \mathbf{r}(t_0) + t\,\mathbf{r}'(t_0).$$



Let $\mathbf{v}(t)$ be a vector-valued function. If $\mathbf{V}(t)$ is a vector-valued function such that $\mathbf{V}'(t) = \mathbf{v}(t)$ then, we call $\mathbf{V}$ an antiderivative of $\mathbf{v}$. Since derivatives are componentwise, to compute an antiderivative, just compute an antiderivative for each of the component functions.

The indefinite integral is the set of all antiderivatives denoted in the following way:

$$\int \mathbf{v}(t)dt = \mathbf{V}(t) + \mathbf{C}, \text{ where } \mathbf{V} \text{ is an antiderivative of } \mathbf{v} \text{ and } \mathbf{C} \text{ is a constant.}$$

The definite integral is the calculated in the following way:

$$\int_a^b \mathbf{v}(t)dt = \mathbf{V}(b) - \mathbf{V}(a), \text{ where } \mathbf{V} \text{ is an antiderivative of } \mathbf{v}.$$

Let $\mathbf{r}(t)$ represent the position of an object against time t. The graph of $\mathbf{r}(t)$ represents the path or trajectory of the object.

Then $\mathbf{v}(t) = \mathbf{r}'(t)$ represents velocity of the object and $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ represents acceleration of the object. The speed of the object is measured by $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$.

From the velocity $\mathbf{v}(t)$, the definite integral, $\displaystyle\int_a^b \mathbf{v}(t)\;dt = \mathbf{r}(b) - \mathbf{r}(a)$, calculates the net displacement of the object over the time interval $[a, b]$.

Assume $\mathbf{r}(t)$ is a trajectory of an object with constant positive speed: $|\mathbf{r}'(t)| = c > 0$.

Then $0 = \dfrac{d}{dt}\left(c^2\right) = \dfrac{d}{dt}\left(|\mathbf{r}'(t)|^2\right) = \dfrac{d}{dt}\left(\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right) = \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2\mathbf{r}''(t) \cdot \mathbf{r}'(t)$,

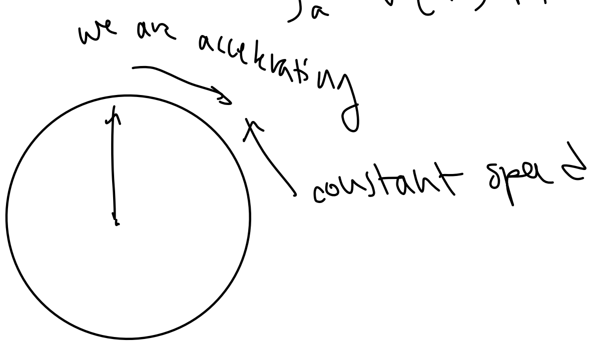
That implies $0 = \mathbf{r}''(t) \cdot \mathbf{r}'(t)$, and thus, $\mathbf{r}''(t) \perp \mathbf{r}'(t)$.

For motion with constant speed we have that the velocity and acceleration vectors are orthogonal.

$$\int \vec{a}(t)\,dt = \vec{v}(t) + C_1$$

$$\int \vec{v}(t)\,dt = \vec{p}(t) + C_2$$

$$\int_a^b \vec{v}(t)\,dt = \vec{r}(b) - \vec{r}(a)$$

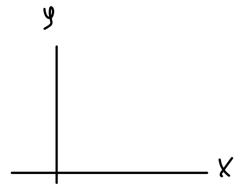


$$\int_0^1 v(t)\,dt = r(1) - r(0)$$

$$\int_0^1$$

A soccer ball is kicked so that it initially (at time $t = 0$) it has a velocity of 12 meters per second at an angle of 45° above the horizon over ground that slopes downward at 30° below the horizon.

(a) Let's find the initial velocity vector.



$v_0 = \langle 12 \cos 45, 12 \sin 45 \rangle$

$= \langle 6\sqrt{2}, 6\sqrt{2} \rangle$

(b) Let's find a vector-valued function for the velocity of the soccer ball, using $g = 10$ m/s^2 for the acceleration due to gravity.

$\vec{a} = \langle 0, -10 \rangle$

$\int \vec{a}\, dt = \langle c_1, -10t + c_2 \rangle = \tilde{v}$

$c_1 = 6\sqrt{2}$ $\quad \tilde{v} = \langle 6\sqrt{2}, -10t + 6\sqrt{2} \rangle$

$c_2 = 6\sqrt{2}$

(c) Let's find a vector-valued function for the position of the soccer ball, assuming that it begins at the origin of our coordinate system.

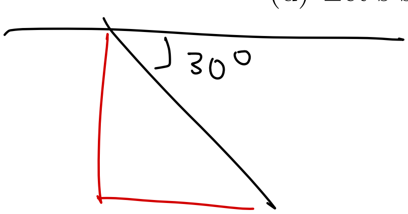
assume zero bc origins

$(0, 0)$

$\vec{r} = \int \tilde{v}\, dt = \langle 6\sqrt{2}t + c_1, -5t^2 + 6\sqrt{2}t + c_2 \rangle$

$= \langle 6\sqrt{2}t, -5t^2 + 6\sqrt{2}t \rangle$

(d) Let's set up a vector valued function for the sloping ground.



$\langle 2\cos 30°, -2\sin 30° \rangle$ vector in the direction of the ground

$\langle \sqrt{3}, -1 \rangle$

$\tilde{g} = \langle \sqrt{3}s, -s \rangle$

(e) Let's find the points at which vector-valued functions for the position of the soccer ball and the sloping ground intersect.

set x and y equal to each other

$$6\sqrt{2}\,t = \sqrt{3}\,s \rightarrow s = \frac{6\sqrt{2}}{\sqrt{3}} = \frac{\cancel{6}\sqrt{6}}{\cancel{3}} = 2\sqrt{6}t$$

$$-s = -5t^2 + 6\sqrt{2}\,t$$

$$-2\sqrt{6}\,t = -5t^2 + 6\sqrt{2}\,t$$

$$0 = -5t^2 + 6\sqrt{2}\,t + 2\sqrt{6}\,t$$

$$0 = t\,(-5t + 6\sqrt{2} + 2\sqrt{6})$$

$t = 0$ $\quad = 0$

$=$

$$t = \frac{-6\sqrt{2} - 2\sqrt{6}}{-5}$$

$$\vec{r}(t) \approx (22,\ 7,\ -13,\ 0)$$

(f) How long does it take for the ball to hit the ground?

$t \approx 2.7$ sec

1. C ✓

an object moving in a circle at constant speed is indeed accelerating. therefore constant speed meaning zero acceleration is indeed false

2. T(1) to nearst hundredth

T(t) is the unit tangent

$u(t) = \langle t, \ln(1+t^2) \rangle$ $\quad u'(t) = \langle 1, \frac{1}{1+t^2} \cdot 2t \rangle$

$T(t) = \frac{r'(t)}{\|r'(t)\|}$ $\quad \|u'(t)\| = \sqrt{1^2 + \left(\frac{2t}{1+t^2}\right)^2}$

$T(1) = \frac{u'(t)}{\|u'(t)\|}$ $\quad = \sqrt{1^2 + \left(\frac{2\cdot}{1+1}\right)^2}$

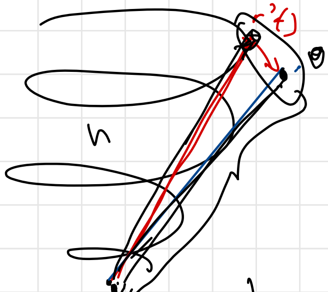
$= \sqrt{1^2 + 1^2} = \sqrt{2}$

$= \frac{\langle 1, \frac{1}{2} \cdot 2 \rangle}{\sqrt{2}} = \frac{\langle 1, 1 \rangle}{\sqrt{2}}$

$= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ ✓

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

3.



$\vec{r}(t) = \langle \cos(t^2), \sin(t^2), t^2 \rangle$

$\vec{r}\,'(t) = \langle -\sin(t^2) \cdot 2t, \cos(t^2) \cdot 2t, 2t \rangle$

$\vec{r}\,'\left(\sqrt{\frac{3\pi}{4}}\right) = \langle -\sin\left(\frac{3\pi}{4}\right) \cdot 2\left(\sqrt{\frac{3\pi}{4}}\right), \cos\left(\frac{3\pi}{4}\right) \cdot 2\left(\sqrt{\frac{3\pi}{4}}\right), 2\left(\sqrt{\frac{3\pi}{4}}\right)$

$\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3\pi}{4} \rangle$

5.63

$$\int_a^b \hat{v}(t)\,dt = \vec{r}(b) - \vec{r}(a) = \vec{r}(b) - \vec{r}(0)$$

different way

$$T\left(\sqrt{\frac{3\pi}{4}}\right) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\left\langle -\frac{\sqrt{6\pi}}{2},\ -\frac{\sqrt{6\pi}}{2},\ \sqrt{3\pi} \right\rangle}{\sqrt{\left(-\frac{\sqrt{2}}{2}\cdot\sqrt{3\pi}\right)^2 + \left(-\frac{\sqrt{2}}{2}\cdot\sqrt{3\pi}\right)^2 + \left(\sqrt{3\pi}\right)^2}}$$

$$= \frac{\left\langle -\frac{\sqrt{6\pi}}{2},\ -\frac{\sqrt{6\pi}}{2},\ \sqrt{3\pi} \right\rangle}{\sqrt{6\pi}}$$

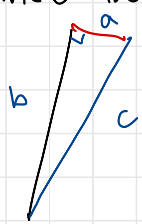
$$T\left(\sqrt{\frac{3\pi}{4}}\right) = \left\langle -\frac{\cancel{\sqrt{6\pi}}}{2}\cdot\frac{1}{\cancel{\sqrt{6\pi}}},\ -\frac{\cancel{\sqrt{6\pi}}}{2}\cdot\frac{1}{\cancel{\sqrt{6\pi}}},\ \sqrt{3\pi}\cdot\frac{1}{\sqrt{6\pi}} \right\rangle$$

$$T\left(\sqrt{\frac{3\pi}{4}}\right) = \left\langle -\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{\sqrt{2}} \right\rangle$$

$$a = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$b = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{3\pi}{4}\right)^2}$$

distance from the unit vector to the origin



$$c = \sqrt{7.55} = 2.748$$

$a = 1$ $\qquad c^2 = 1^2 + 1 + \frac{9\pi^2}{16}$

$b = \sqrt{1 + \frac{9\pi^2}{16}}$ $\qquad c^2 = \frac{9\pi^2}{16} + 2 = 7.55$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{8}}{4}$$

Vector Addition

$$T(1) + h = \left\langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle +$$

$$\left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3\pi}{4} \right\rangle$$

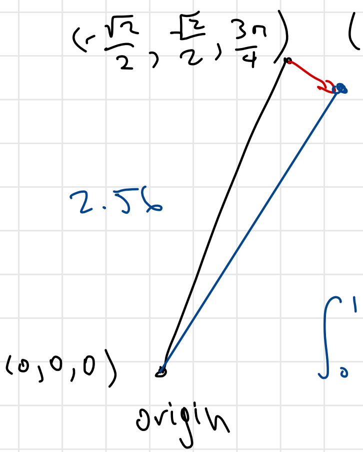
$$= \left\langle \frac{-1-\sqrt{2}}{2}, \frac{-1+\sqrt{2}}{2}, \frac{3\pi+\sqrt{8}}{4} \right\rangle$$

distance :

$$d = \sqrt{\left(\frac{-1-\sqrt{2}}{2}\right)^2 + \left(\frac{-1+\sqrt{2}}{2}\right)^2 + \left(\frac{3\pi+\sqrt{8}}{4}\right)^2}$$

$$\approx 3.299$$

$$\approx 3.30$$



$$\int_0^1 v(t) = s(1) - s(0)$$

$$\int_0^1 \langle -\sin(t^2)\,2t, \cos(t^2)\cdot 2t, 2t \rangle =$$

$$\int_0^1 -\sin(t^2)\,2t$$

$$-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}, \frac{3\pi}{4} =$$

3. $\vec{r}\,'\left(\sqrt{\frac{3\pi}{4}}\right) = \left(-\sin(t^2)\cdot 2t,\ \cos(t^2)\cdot 2t,\ 2t\right)$

$$r'\left(\sqrt{\frac{3\pi}{4}}\right) = \left(-\sin\left(\frac{3\pi}{4}\right)\left(2\cdot\sqrt{\frac{3\pi}{4}}\right),\ \cos\left(\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{\frac{3\pi}{4}}\right),\ 2\sqrt{\frac{3\pi}{4}}\right)$$

$$= \left(-\sqrt{3\pi}\sin\left(\frac{3\pi}{4}\right),\ -\sqrt{3\pi}\cos\left(\frac{3\pi}{4}\right),\ \sqrt{\frac{3\pi}{2}}\right)$$

position @ 1 second later

$$\ell(t) = r(t_0) + t\left(r'(t_0)\right)$$

$$\ell(t) = \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ \frac{3\pi}{4}\right) + 1\cdot\left(-\sqrt{3\pi}\sin\left(\frac{3\pi}{4}\right),\ \ldots\right)$$

$$\left(\underbrace{\left[-\frac{\sqrt{2}}{2} - \sqrt{3\pi}\sin\left(\frac{3\pi}{4}\right)\right]}_{X_1},\ \underbrace{\left[\frac{\sqrt{2}}{2} - \sqrt{3\pi}\cos\left(\frac{3\pi}{4}\right)\right]}_{Y_1},\ \frac{\sqrt{3\pi}}{2} + \frac{3\pi}{4}\right)$$

distance:

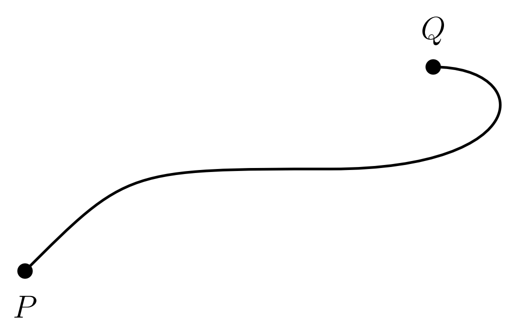
$$d = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$= \cancel{4.465}$ $\approx \cancel{4.47}$ $= 5.63$

...

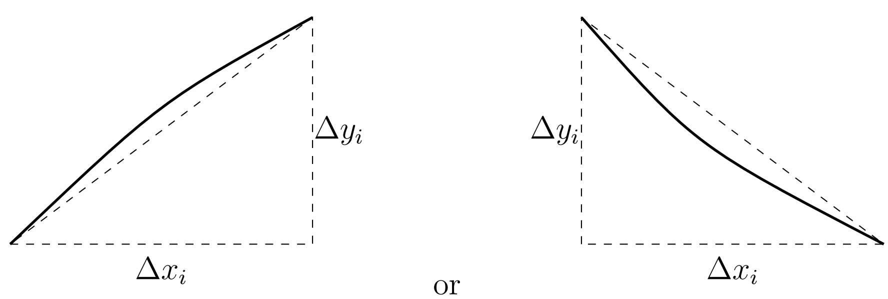
Arc Length & Curvature

Consider the curve that is the graph of a differentiable vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ on the domain $[a, b]$ such that $|\mathbf{r}'(t)| \neq 0$ for all $a \leq t \leq b$. Let P, Q be the corresponding points on the curve to $\mathbf{r}(a)$,$\mathbf{r}(b)$ respectively.



The length of this curve from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ called the arc length of the curve from P to Q.

We can partition the curve into n little pieces such that each of the n pieces looks like:



Then by using the hypotenuses to estimate the arc length of each of the n little pieces, we get a Riemann sum approximation to the arc length of the whole curve:

$$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\, \Delta t.$$

Then by taking the limit as $n \to \infty$ (such that the length of all the pieces tend towards 0) we converge on the exact arc-length with the integral

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt \text{ (similar result in 3D).}$$

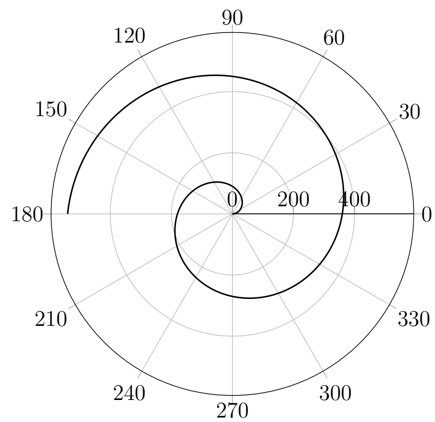
In the context of $\mathbf{r}(t)$ representing an object's position as it moves from P towards Q without stopping, the integral of the speed equals the distance travelled by the object along the trajectory, which, of course, is also the arc-length of the curve from P to Q.

integral of speed = distance

$L = \int_a^b |r'(t)|\, dt$

Recall: One can use polar coordinates (r, θ) to describe points in the xy-plane, where r is the (signed) distance to the origin and θ is an angle measured counter-clockwise from the positive x-axis. In particular, $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Suppose $r = f(\theta)$, which describes a polar curve (shown below is the graph of $r = \theta$).



Then $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. So the length of the curve from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\,d\theta.$$

One can simplify this down to

for a polar curve

$$L = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\,d\theta.$$

Let $\mathbf{r}(t)$ be a differentiable vector-valued function on $[a, b]$ such that $|\mathbf{r}'(t)| \neq 0$ for $a \leq t \leq b$.

Recall: The unit tangent vector function for $\mathbf{r}(t)$ is $\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$. $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}(t)$ only changes direction, never magnitude.

Define the <u>curvature function</u> $\kappa(t)$ as the function that outputs the magnitude of the rate of change in $\mathbf{T}(t)$ with respect to arc-length s. Symbolically, $\kappa(t) = \left|\frac{d\vec{T}}{ds}\right|$. We would like to avoid having to use this formula!

Since the arc-length from a to t along the graph of $\mathbf{r}(t)$ is given by $s(t) = \int_a^t |\mathbf{r}'(u)|\,du$ we see that $\frac{ds}{dt} = |\mathbf{r}'(t)|$. Then by the Chain Rule, $\left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right| = \kappa(t)|\mathbf{r}'(t)|$.

So $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} \qquad \frac{\text{curve}}{\text{speed}}$$

The circle of radius R centered at (x_0, y_0) is the graph of $\mathbf{r}(t) = \langle x_0 + R\cos(t), y_0 + R\sin(t)\rangle$. We have that $\mathbf{r}'(t) = R\langle -\sin(t), \cos(t)\rangle$, with $R = |\mathbf{r}'(t)|$, and hence $\mathbf{T}(t) = \langle -\sin(t), \cos(t)\rangle$ is the unit tangent vector function. Then $\mathbf{T}'(t) = \langle -\cos(t), -\sin(t)\rangle$ and $|\mathbf{T}'(t)| = 1$. Hence the curvature is $\kappa(t) = 1/R$.

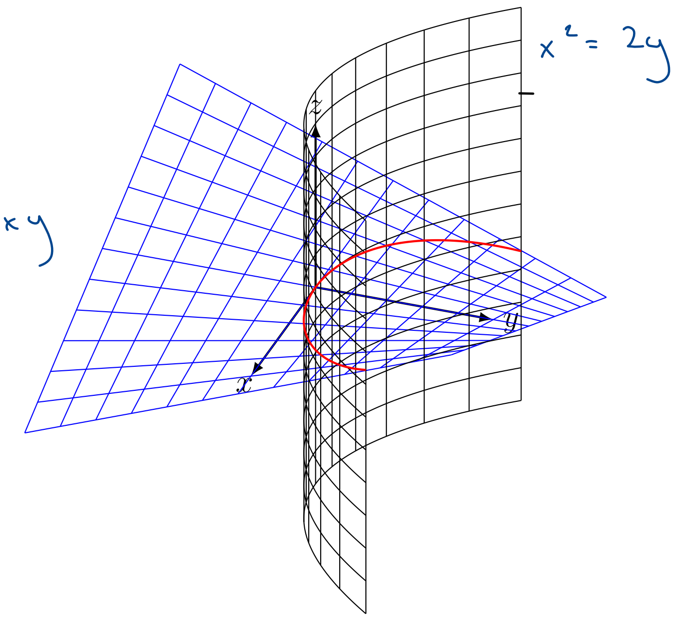
One can also derive the following in $\mathbb{R}^3$: $\boxed{\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.}$

Let $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ (which is called the principal unit normal vector function). Since $\mathbf{T}$ is constant in magnitude, $\mathbf{N}$ is perpendicular to $\mathbf{T}$ (and hence orthogonal to the graph of $\mathbf{r}(t)$).

Let $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{v}' = \mathbf{r}''$.

$\mathbf{v} = |\mathbf{v}|\mathbf{T}$ so $\mathbf{a} = \frac{d}{dt}(|\mathbf{v}|)\,\mathbf{T} + |\mathbf{v}|\mathbf{T}'(t)$ so $\mathbf{a} = \frac{d}{dt}(|\mathbf{v}|)\,\mathbf{T} + |\mathbf{v}||\mathbf{T}'(t)|\mathbf{N} = \frac{d}{dt}(|\mathbf{v}|)\,\mathbf{T} + \kappa(t)|\mathbf{v}|^2\mathbf{N}$. So the *tangential component of the acceleration* is $\frac{d}{dt}(|\mathbf{v}|)\,\mathbf{T}$ with magnitude given by the absolute value of $\frac{d}{dt}(|\mathbf{v}|)$ and the *orthogonal component of acceleration* is $\kappa(t)|\mathbf{v}|^2\mathbf{N}$ with magnitude given by $\kappa(t)|\mathbf{v}|^2$.

Consider the curve of intersection between the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$.



$x = t$

(a) Let's parameterize this curve!

describe as vector value function

$x = t \qquad y = \frac{t^2}{2} \qquad z = \frac{t^3}{6}$

$\vec{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6} \rangle$

(b) Now let's find the arc length along this curve from the origin to $(\underbrace{6, 18, 36}_{t=6})$.

$t = 0$

$\vec{r}\,'(t) = \langle 1, t, \frac{t^2}{2} \rangle$

$|\vec{r}\,'(t)| = \sqrt{1 + t^2 + \frac{t^4}{4}} = \sqrt{\left(\frac{t^2}{2} + 1\right)^2} = \frac{t^2}{2} + 1$

$L = \int_0^6 \frac{t^2}{2} + 1 \, dt = t + \frac{t^3}{6} \Big|_0^6 = 6 + 36 = 42$

(c) Now let's find the curvature function along this curve.

$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ $\qquad r'' = \langle 0, 1, t \rangle$

$r' \times r'' = \begin{matrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \end{matrix}$

$\langle \frac{t^2}{2}, -t, 1 \rangle$

$K =$

$\frac{|\vec{r}\,' \times \vec{r}\,''|}{|r'|^3} = \frac{1 + \frac{t^2}{2}}{\left(1 + \frac{t^2}{2}\right)^3} = \frac{1}{\left(1 + \frac{t^2}{2}\right)^2}$

(d) Where is the curvature at a maximum? What is the maximum curvature?

max: when $t = 0$ at the origin!

$K_{max} = \frac{1}{(1+0)^2} = 1$

1. graph of

$\vec{r}(t) = \langle t, 2t+1, 2t^{3/2} \rangle$ from $(0,1,0)$ to $(4,9,16)$

$t=0$ $\quad$ $t=4$

$\vec{r}\,'(t) = \langle 1, 2, 3t^{1/2} \rangle$

$$|\vec{r}\,'(t)| = \sqrt{1^2 + 2^2 + (9t)}$$

$$= \sqrt{1+4+9t} = \sqrt{9t+5}$$

$$L = \int_0^4 \sqrt{9t+5}\,dt = \frac{2}{3(9)}(9t+5)^{3/2}\Big|_0^4$$

$$= \frac{2}{27}(36+5)^{3/2} - \frac{2}{27}(5)^{3/2}$$

$$= 18.62$$

2. arc length of the curve with $(0,1)$ $(\ln 2, 0.75)$

$t=0$ $\quad$ $t=\ln 2$

$\vec{r}(t) = \langle t, \frac{e^t + e^{-t}}{2} \rangle$

$\vec{r}\,'(t) = \langle 1, \frac{1}{2}(e^t - e^{-t}) \rangle$

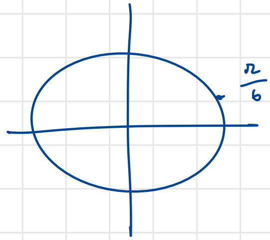
$$|\vec{r}\,'(t)| = \sqrt{1^2 + \left(\frac{1}{2}(e^t - e^{-t})\right)^2}$$

$$= \sqrt{1 + \left(\frac{e^t + e^{-t}}{2}\right)^2} = \frac{e^t + e^{-t}}{2}$$

$$L = \int_0^{\ln 2} \frac{e^t + e^{-t}}{2}\,dt = \frac{e^{\ln 2} + e^{-\ln 2}}{2} - \frac{e^0 + e^{-0}}{2}$$

$$L = \frac{2 + -2}{2} - \frac{e^{0} + e^{-0}}{2}$$

$= \cancel{0.25}$? 1.75



3. $$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$x^2 + 4y^2 = 4 \qquad \frac{x^2}{4} + y^2 = 1 \quad \text{at } \left(\sqrt{3}, \frac{1}{2}\right)$$

parameterize the curve

$x = 2\cos(t)$ $\qquad y = \sin(t)$ $\qquad z = 0$

$2\cos(t) = \sqrt{3}$ $\qquad \sin(t) = \frac{1}{2}$

$\cos(t) = \frac{\sqrt{3}}{2}$ $\qquad t = \frac{\pi}{6}$

$t = \frac{\pi}{6}$

$$\frac{(2\cos(t))^2}{4} + (\sin(t))^2 = 1$$

$$\cos^2(t) + \sin^2(t) = 1$$

$\vec{r}(t) = \langle 2\cos(t), \sin(t), 0 \rangle$

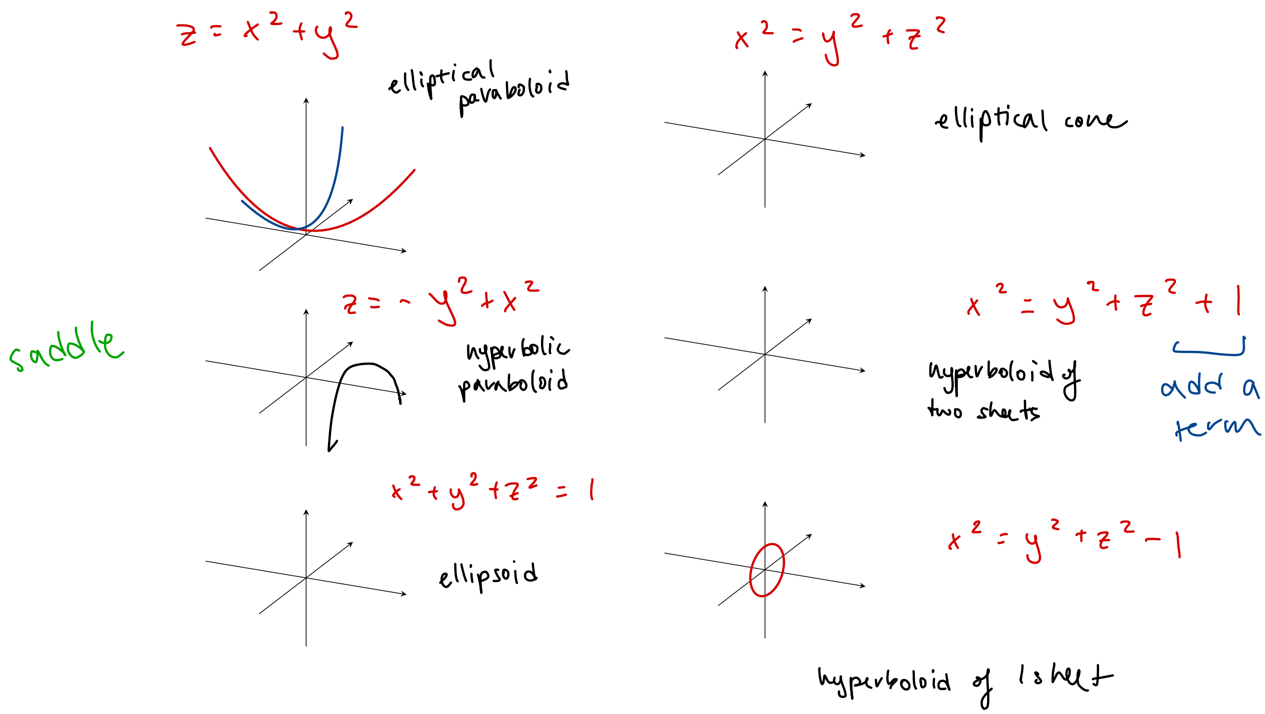
$\vec{r}'(t) = \langle \quad ,$

1.16

...

Functions of Several Variables

The following are examples of common quadratic surfaces in $\mathbb{R}^3$:

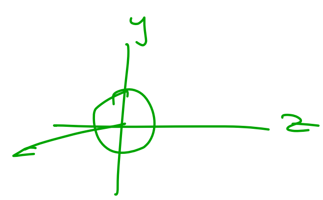


elliptical paraboloid, elliptical cone,
hyperbolic paraboloid, hyperboloid of 2 sheets,
ellipsoid, hyperboloid of 1 sheet

when zero

$x^2 + 1 = y^2 + z^2$

$1 = y^2 + z^2$



ellipsoid hyperboloid of 1 sheet elliptical cone

Let's match each to one of these: $x^2 + y^2 + z^2 = 1$, $x^2 - y^2 - z^2 = -1$, $x^2 - y^2 - z^2 = 0$, $x^2 - y^2 - z^2 = 1$, $x^2 - y^2 - z = 0$, and $x^2 + y^2 - z = 0$.

hyperboloid of two sheets hyperbolic paraboloid elliptical paraboloid

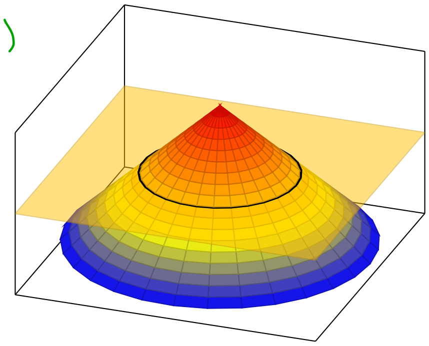
Note: A hyperbolic paraboloid is also called a "saddle".

A <u>trace</u> of a surface in $\mathbb{R}^3$ is the cross-section of the surface cut by a plane (intersection of the surface and plane). When the plane corresponds to $x = k$, $y = k$ or $z = k$ the resulting trace is called an x-trace, y-trace, or z-trace.

Here is the z-trace at $z = k$ along the surface $z = C - \sqrt{x^2 + y^2}$ where $k < C$ are constants.

Domain: set of (x, y) inputs

Range: set of $f(x, y)$ outputs



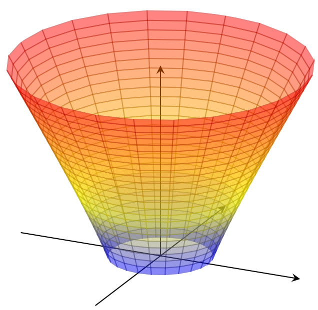
$z = C - \sqrt{x^2+y^2}$

$z = k$

$k < C$ so you get less than the apex

A function f of two variables x and y is often expressed with z set to the output, $z = f(x, y)$ and the graph of that surface is the graph of $f(x, y)$. The <u>domain</u> is a subset D of the xy-plane at which f is defined. The <u>range</u> is $R = \{f(x, y) | (x, y) \in D\}$, a subset of the real numbers. Since each (x, y) in the domain $f(x, y)$ has a unique output, every vertical line through the graph of $f(x, y)$ intersects it at most once.

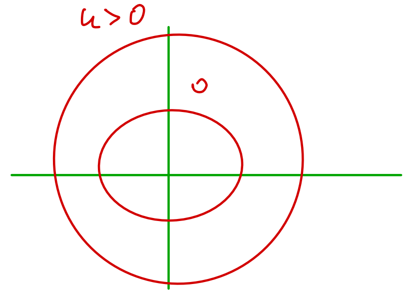
Here is the graph the function $f(x, y) = \sqrt{-1 + x^2 + y^2}$. Note the domain is $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \geq 1\}$ and the range is $[0, \infty)$.



Level Set of $f(x, y)$ is the set of points where $f(x, y) = k \leftarrow$ CONSTANT

$$z = \sqrt{\underbrace{-1 + x^2 + y^2}_{x^2+y^2=1}} = \overset{k}{0}$$

$k > 0$

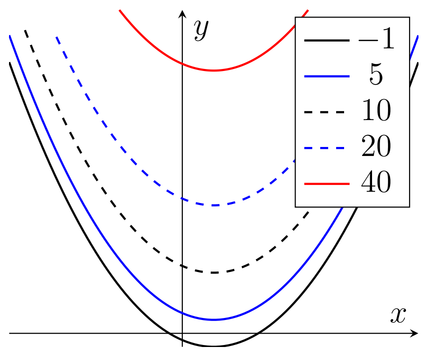


0

$x^2 + y^2 = k^2 + 1$

Now consider a surface defined by $z = f(x, y)$. Imagine the intersection of the surface with various planes parallel to the xy-plane (where $z = k$). These z-traces in xyz-space are called contours, and projected into the xy-plane, they are called level sets (aka level curves since they are usually curves). Several level sets together is called a contour map.

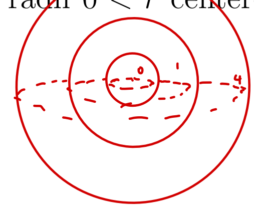
Here is a contour map for $f(x, y) = 2x + y - x^2$:



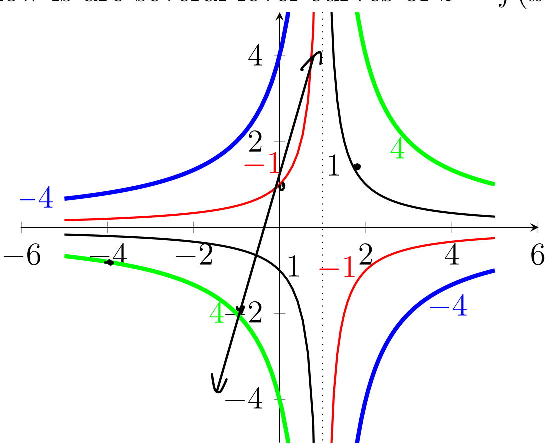
Similar ideas exist for functions $f(x, y, z)$ of three variables. Their graphs are in 4-dimensions so we don't draw them, but their *level sets* (aka *level surfaces*) where $f(x, y, z) = k$ for various constants k are in 3D.

$f(x,y,z) = x^2 + y^2 = k$

For example, the level sets of $f(x, y, z) = e^{-x^2-y^2-z^2}$ are spheres of radii $0 < r$ centered at (0.0,0)



(1) Below is are several level curves of $z = f(x, y)$.



(a) Using these level curves let's estimate $f(2, 1)$.

$f(2,1) = 1$

(b) Using these level curves let's estimate $f(-4, -1)$.

$f(-4,-1) = 4$

(c) Using these level curves let's estimate the slope of a path on the surface $z = f(x, y)$ over the domain point $(0, 1)$ towards the domain point $(-1, -2)$.

$\approx \frac{2}{1.3}$

(2) Let's plot and label level curves of $f(x, y) = 8 - \sqrt{4x^2 + y^2}$ corresponding to $k = 0, 4, 6$.

k

$\sqrt{4x^2+y^2} = 8-k$

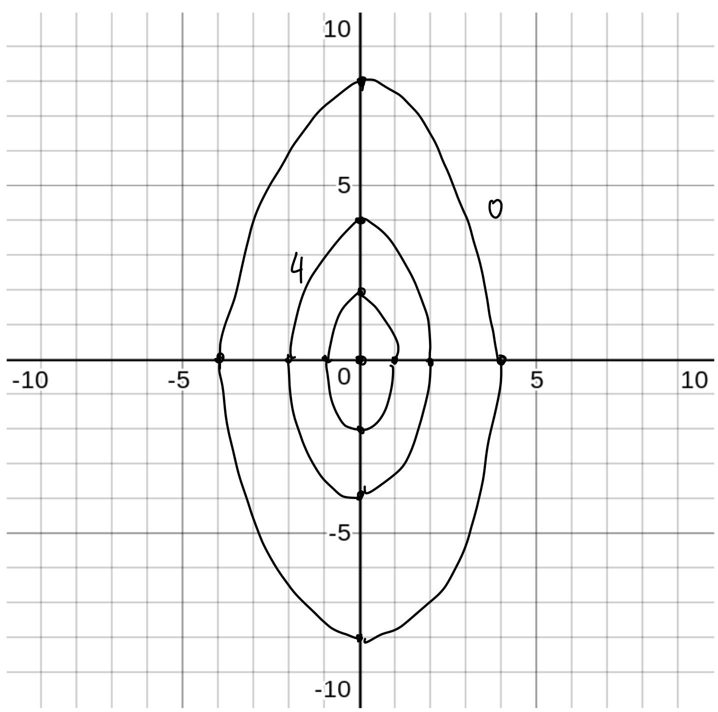
$4x^2+y^2 = (8-k)^2$

$64 \ (k=0)$

$16 \ (k=4)$

$4 \ (k=6)$

$0 \ (k=8)$



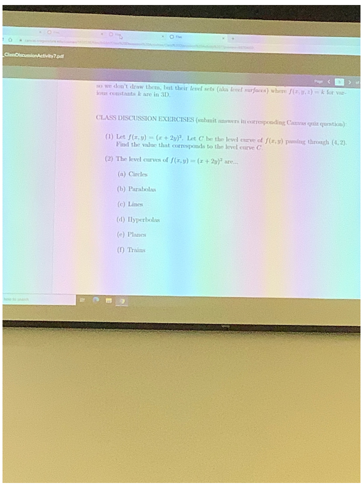
Class Discussion 7.

1. $f(x,y) = (x+2y)^2$

$f(4,2) = (4+2(2))^2$

$= 8^2 = 64$

2. level curves of $f(x,y) = (x+2y)^2$

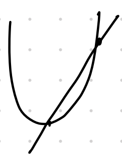


$\boxed{C}$

square root of

$\sqrt{(x+2y)^2} = x+2y$

which is a line



3. 1.

4. $\dfrac{3}{\text{length of vector} = 2} = 1.5$

$\underline{\text{change in } z}$

5. b. hyperbolic paraboloid.

Functions of several variables + level sets

Scalar " two $f(x,y)$

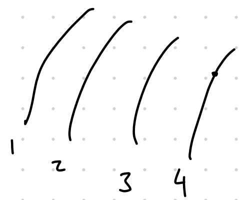
Domain: set of inputs (x,y) in R^2 (xy-plane)
Range: set of scalar outputs

scalar " three " $f(x,y,z)$

Domain: set of inputs (x,y,z) in R^3 (xyz-space)
Range: set of scalar outputs

Level set:

contour map

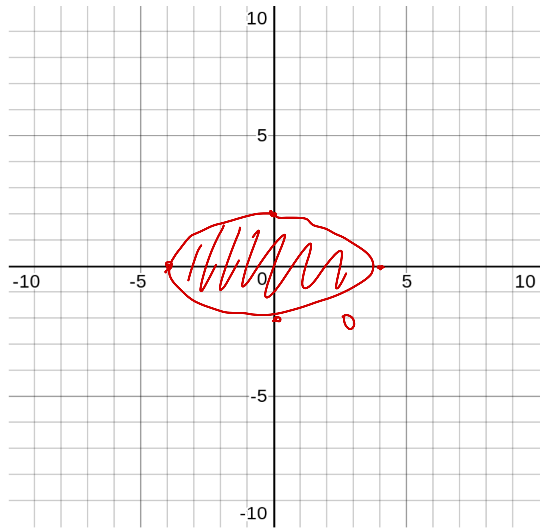


range

(1) Consider the function $f(x,y) = \sqrt{16 - x^2 - 4y^2}$. ≥ 0

domain non-negative

(a) Let's draw the domain:



domain

(b) Let's determine the range of this function. What are the maximum and minimum values of f?

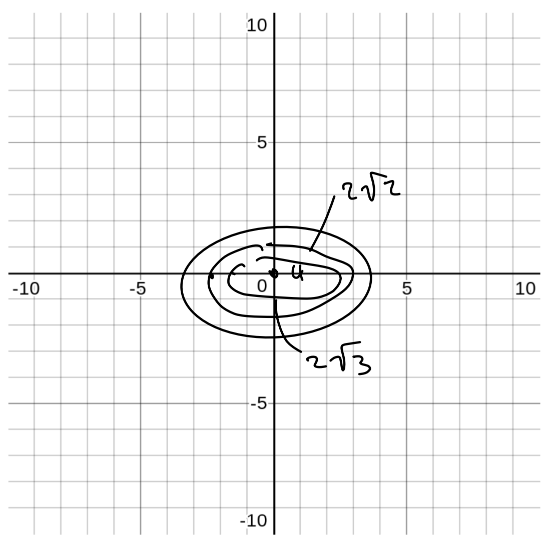
$[0, 4]$

min and max

(c) Let's draw and label the level sets (mostly curves) for $f(x,y) = k$ with $k = 0, 2\sqrt{2}, 2\sqrt{3}, 4$.

$k = \sqrt{16 - x^2 - 4y^2}$

$k^2 = 16 - x^2 - 4y^2$

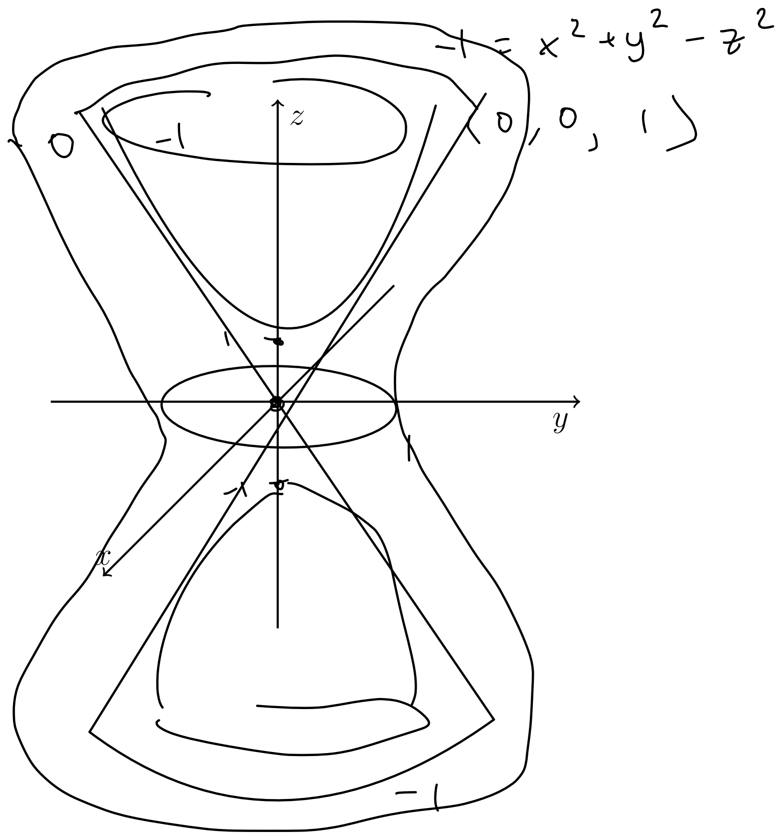


$x^2 + 4y^2 = 16 - k^2$

$8 \quad (2\sqrt{2} = k)$

$4 \quad (2\sqrt{3} = k)$

(2) Consider the function $g(x, y, z) = x^2 + y^2 - z^2$. Let's draw and label the level sets (surfaces) for $g(x, y, z) = k$ with $k = -1, 0, 1$.



1. Level curve

$\sqrt{25-16} = \sqrt{9} = \boxed{3}$

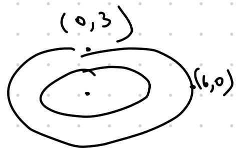
2. $f(x,y) = \sqrt{18 - \frac{1}{2}x^2 - 2y^2}$

$0 = \sqrt{18 - \frac{1}{2}x^2 - 2y^2}$

$0 = 18 - \frac{1}{2}x^2 - 2y^2$

$\frac{1}{2}x^2 + 2y^2 = 18$

$\frac{36}{2} + 2 \cdot 9 = 18$



$x = 6$
$y = 0$

$y = 3$
$x = 0$

$\boxed{6}$

$\frac{8}{10} = \frac{2a}{5}$

$2 = a$

3. $x = 2$ $y =$

$f(x,y) = \frac{ax^2y}{x^4 + by^2}$

$0.8 = \frac{a(1)^2(2)}{1^4 + 1(2)^2} = \frac{2a}{1+4}$

$= \frac{2a}{5}$

$\frac{a(1)^2(2)}{1^4 + b(2)^2} = \frac{a(2)^2(2)}{(2)^4 + b(2)^2}$

$\frac{2a}{1 + 4b} = \frac{8a}{16 + 4b} = \frac{2a}{4+b}$

$\frac{32a + 8ab}{2a} = \frac{8a + 32ab}{2a}$

$\frac{2a}{1+4b} = \frac{2a}{4+b}$

$b = 1$

$8a + 2ab = 2a + 8ab$

$6a = 6ab$

$a = ab$

4. $f(x, y, z) = \sqrt{x^2 + 2x + y^2 - 4y + z^2 + 10}$

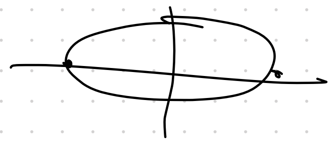
$$= \sqrt{x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 + 10 - 1 - 4}$$

$$= \sqrt{(x+1)^2 + (y-2)^2 + z^2 + 5}$$

$$= x + 1 + y - 2 + z + \sqrt{5}$$

$$= x + 1 + y - 2 + z + \boxed{\sqrt{5}}$$

Determine the minimum value of $f(x, y, z)$ to the nearest hundredth.



$$\boxed{\sqrt{5}} \approx 2.24$$

...

Limits of Functions of Several Variables & Continuity

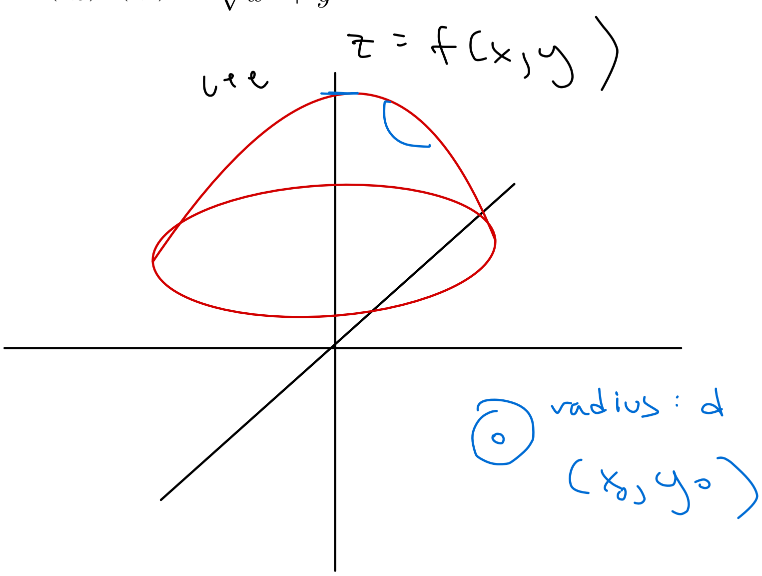
A function $f(x, y)$ has a <u>limit</u> of L as (x, y) approaches (x_0, y_0) if $|f(x, y) - L|$ can be made arbitrarily small for all points in a sufficiently small punctured disk centered at (x_0, y_0). Notation: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$.

A function $f(x, y, z)$ has a <u>limit</u> of L as (x, y, z) approaches (x_0, y_0, z_0) if $|f(x, y, z) - L|$ can be made arbitrarily small for all points in a sufficiently small punctured ball centered at (x_0, y_0, z_0). Notation: $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L$. To evaluate limits, one can use various techniques including algebra, polar coordinates and concepts from single-variable limits.

Example: Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$. One can speculate based on the graph (shown below) that this limit exists (as a finite number) even though the function $f(x, y) = \dfrac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}$ is undefined at the origin.

Using the polar substitution, $r = \sqrt{x^2 + y^2}$ and that $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0$ we can convert this limit to a single-variable limit and just recall that $\sin(\alpha)/\alpha \to 1$ as $\alpha \to 0$:

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}} = \lim_{r\to 0} \frac{\sin(r)}{r} = 1.$$



can never show limit exists using paths

2-Path Tests: (2D) If there are two distinct paths approaching (x_0, y_0) in the domain of $f(x, y)$ such that the limits of $f(x, y)$ as (x, y) approaches (x_0, y_0) along these paths are distinct then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

non-existent

(3D) If there are two distinct paths approaching (x_0, y_0, z_0) in the domain of $f(x, y, z)$ such that the limits of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) along these paths are distinct then $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$ does not exist.

infinitely many ways

if you find the same path, then you know nothing

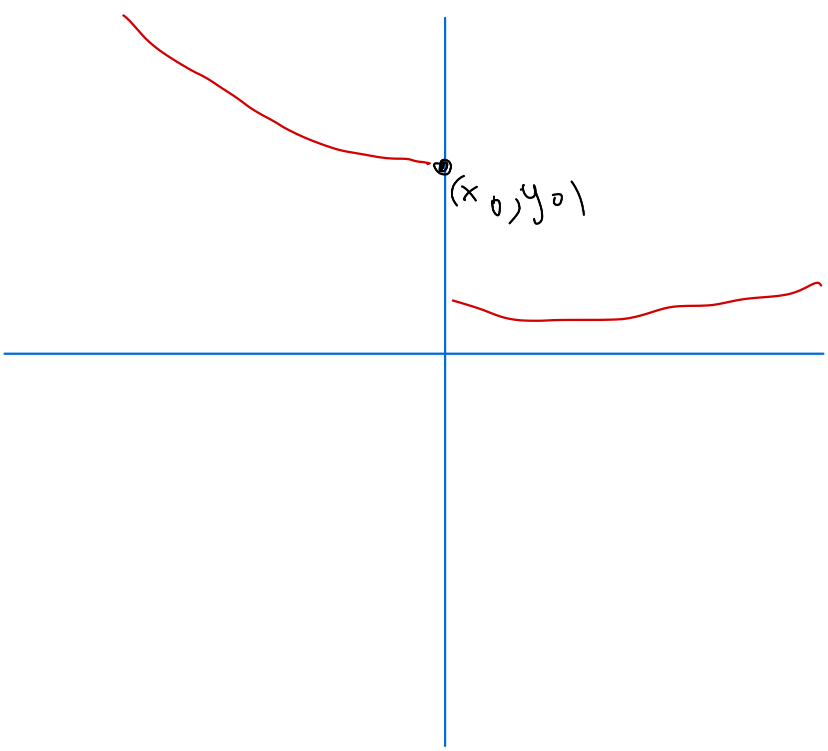
Example: Justify that the limit $\lim_{(x,y,z)\to(0,0,1)} \frac{2xy(z-1)}{x^3 + (z-1)^3}$ does not exist (dne).

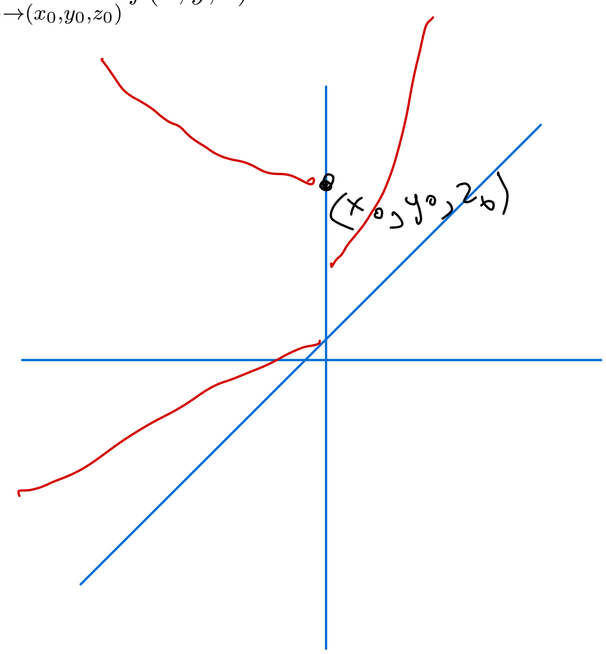
Consider approaching $(0, 0, 1)$ along the line $\vec{r}(t) = \langle t, t, 1\rangle$ as $t \to 0$. Along this path $\lim_{(t,t,1)\to(0,0,1)} \frac{2tt(1-1)}{t^3 + (1-1)^3} = \lim_{t\to 0} \frac{0}{t^3} = 0.$

Consider approaching $(0, 0, 1)$ along the line $\vec{v}(t) = \langle t, t, t + 1\rangle$ as $t \to 0$. Along this path $\lim_{(t,t,t+1)\to(0,0,1)} \frac{2tt(t+1-1)}{t^3 + (t+1-1)^3} = \lim_{t\to 0} \frac{2t^3}{2t^3} = 1.$

Therefore, $\lim_{(x,y,z)\to(0,0,1)} \frac{2xy(z-1)}{x^3 + (z-1)^3}$ does not exist.

A function $f(x, y)$ is <u>continuous</u> at (x_0, y_0) if $f(x_0, y_0) = \lim_{(x,y)\to(0,0)} f(x, y)$. A function $f(x, y, z)$ is <u>continuous</u> at (x_0, y_0, z_0) if $f(x_0, y_0, z_0) = \lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z)$.





$f(P) = \lim_{Q \to P}$ if $f(P)$

(1) Consider the function $f(x,y) = \dfrac{x^4 + 6x^2y + 2y^2}{x^4 + 2y^2}$.

$f(x_0, y_0) = \lim_{(x,y)\to(0,0)} f(x_0, y_0)$

(a) Where is $f(x,y)$ continuous?

everywhere except origin $(0,0)$

(b) Consider the line $x = 0$. Let's determine the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along $x = 0$, that is, $\lim_{(0,y)\to(0,0)} f(0,y)$.

$$\lim_{(0,y)\to(0,0)} f(x,y) = \frac{2y^2}{2y^2} = 1$$

(c) Consider the line $y = mx$ where m is a scalar. Let's determine the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along $y = mx$, that is, $\lim_{(x,mx)\to(0,0)} f(x,mx)$.

$$\lim_{(0,mx)\to(0,0)} \frac{x^4 + 6x^2mx + 2(mx)^2}{x^4 + 2(mx)^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}[x^2 + 6mx + 2m^2]}{\cancel{x^2}[x^2 + 2m^2]} = 1$$

(d) Can a conclusion about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$ be drawn from the above?

None

(e) Consider the parabola $y = x^2$ where m. Let's determine the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along $y = x^2$, that is, $\lim_{(x,x^2)\to(0,0)} f(x,x^2)$.

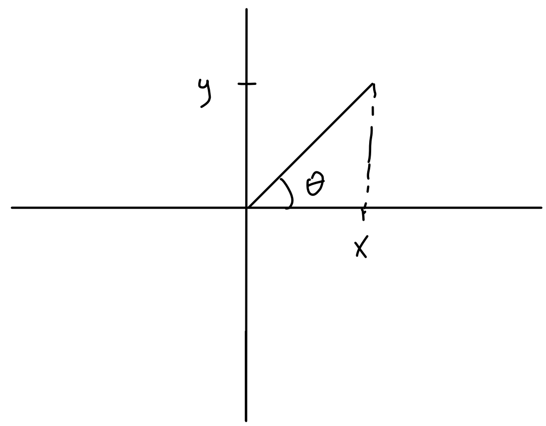
$$\lim_{(x,x^2)\to(0,x)} \frac{x^4 + 6x^2(x^2) + 2(x^2)^2}{x^4 + 2(x^2)^2} = \frac{9x^4}{3x^4} = 3 \neq 1$$

(f) Can a conclusion about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$ be drawn from the above? The limit doesn't exist,

because we found two distinct paths

(2) Let's show that $\lim_{(x,y)\to(0,0)} \dfrac{6x^2y^2}{x^2+y^2}$ exists by using polar coordinates and the Squeeze Theorem!

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{ Polar coord.}$$



Consider $\lim_{r\to 0} \dfrac{6(r\cos\theta)^2(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$

$= \lim_{r\to 0} 6r^2\cos^2\theta\sin^2\theta$

$0 \le \cos^2\theta\sin^2\theta \le 1 \to$

$0 \le 6r^2\cos^2\theta\sin^2\theta \le 6r^2$

Since $\lim_{r\to 0} 0 \to 0$ + $\lim_{r\to 0} 6r^2 = 0$

it follows that $\lim_{r\to 0} 6r^2\cos^2\theta\sin^2\theta \to 0$

by squeeze theorem.

$$f(x,y)=\frac{2x^3y}{x^6+y^2}$$

1. $\lim_{(0,y)\to(0,0)} f(0,y) = \frac{2y}{y^2} = \frac{0}{y^2} = 0$

2. $y=x$ $\quad f(x,y)$ as $\frac{2x^3x}{x^6+x^2} = \frac{2x^4}{x^6+x^2} = \frac{2(\cancel{x^2})x^2}{\cancel{x^2}(x^4+1)}$

$$\lim_{(x,x)\to(0,0)} f(x,y) = \frac{2x^2}{x^4+1}$$

No conclusin

Doesn't mean anything

$$= \frac{0}{1}$$

$$= 0$$

3. $\lim_{(x,x^3)\to(0,0)} f(x,x^3) = 1$

$$\frac{2x^3(x^3)}{x^6+(x^3)^2} = \frac{2x^6}{x^6+x^6} = \frac{2x^6}{2x^6} = 1$$

$\boxed{\text{DNE}}$

4. Evaluate $\lim_{(x,y)\to(8,8)} \frac{x^{1/3}-y^{1/3}}{x^{2/3}-y^{2/3}}$

$$A^2-B^2 = (A+B)(A-B)$$

$$\lim_{(x,y)\to(8,8)} \frac{\cancel{(x^{1/3}-y^{1/3})}}{\cancel{(x^{1/3}-y^{1/3})}(x^{1/3}+y^{1/3})}$$

$$= \frac{1}{x^{1/3}+y^{1/3}}$$

$$= \frac{1}{8^{1/3}+8^{1/3}} = \frac{1}{2+2} = \frac{1}{4}$$

1. 0

2. $$\frac{2x^4}{x^6+x^2} = \frac{2x^2}{x^4+1} \qquad 0$$

No conclusion

3. $$\frac{2x^6}{x^6+x^6} = \frac{\cancel{x^6}(2)}{\cancel{x^6}(1+1)} = 1$$

the limit DNE

4. $$\frac{\cancel{x^{1/3}-y^{1/3}}}{(x^{1/3}+y^{1/3})\cancel{(x^{1/3}-y^{1/3})}}$$

$$\frac{1}{x^{1/3}+y^{1/3}} = \frac{1}{2+2} = \frac{1}{4}$$

...

Mon.

Partial Derivatives & Chain Rules

The partial derivative of $f(x,y)$ with respect to x is given by $\dfrac{\partial f}{\partial x} = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.
This x-partial derivative is also given the subscript notation f_x.

The partial derivative of $f(x,y)$ with respect to y is given by $\dfrac{\partial f}{\partial y} = \lim_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$.
This y-partial derivative is also given the subscript notation f_y.

Analogous definitions exists for the x-, y- and z-partial derivatives of $f(x,y,z)$.

$$\frac{\partial}{\partial y}\left(xy^2\right) = \lim_{h\to 0}\frac{x(y+h)^2 - xy^2}{h} = \lim_{h\to 0}\frac{x(y^2+2hy+h^2)-xy^2}{h} = \lim_{h\to 0}\frac{2hxy+xh^2}{h} =$$
$$= \lim_{h\to 0} 2xy + xh = 2xy.$$

So to calculate a partial derivative of a function f with respect to a certain variable, regard the other variable(s) as constant(s) and take the derivative. All the rules of differential calculus apply with the other variable(s) treated as constants.

Example: $\dfrac{\partial}{\partial y}\left(\cos(2y)e^{x^2/y}\right) = -2\sin(2y)e^{x^2/y} - \dfrac{x^2}{y^2}\cos(2y)e^{x^2/y}$.

The $x-$ and $y-$ partial derivative of $f(x,y)$ evaluated at a (x_0,y_0) is the slope at $(x_0,y_0,f(x_0,y_0))$ along the y- and x-traces on $z=f(x,y)$ at $y=y_0$ and $x=x_0$ respectively:

In practice take the derivative while holding all other variables constant

$$\frac{\partial}{\partial z}\left(xze^{y^2+z^2}\right) = xe^{y^2+z^2} + xze^{y^2+z^2}\cdot 2z$$

$$= [1+2x^2]x\, e^{y^2+z^2}$$

Higher-order partial derivatives are partial derivatives of partial derivatives. The order corresponds to how many derivatives have been taken.

For instance, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$ are examples of second order partial derivatives, while $\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial g}{\partial z}\right)\right) = \dfrac{\partial^3 g}{\partial x^2 \partial z} = g_{zxx}$ is an example of a third-order partial derivative.

Higher-order partial derivatives involving more than one variable are called mixed partial derivatives.

Theorem (Equality of Mixed Partials): Let $f(x, y)$ be a function and assume the mixed partials f_{xy} and f_{yx} are continuous. Then $f_{xy} = f_{yx}$.

$f_{xy} = f_{yx}$

The function $z = f(x, y)$ is differentiable at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and that $\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where ϵ_1 and ϵ_2 both approach 0 as Δx and Δy approach 0.

Theorem: If a function $f(x, y)$ has partial derivatives defined on an open region containing (x_0, y_0) where both f_x and f_y continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Theorem: (Differentiability Implies Continuity) If a function $f(x, y)$ is differentiable at (x_0, y_0) then it is continuous at (x_0, y_0).

Consider the function $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Then $f_x(0, 0) = \displaystyle\lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = 0$ and $f_y(0, 0) = \displaystyle\lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = 0$.

However, this function is not differentiable at $(0, 0)$ as it isn't even continuous at $(0, 0)$. Just consider the limit as $(x, y) \to (0, 0)$.

Chain Rules:

Suppose z is a function of x and y, which are each functions of t. Then we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Suppose w is a function of x, y and z, which are each functions of t. Then we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Suppose z is a function of x and y, which are each functions of s and t. Then we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Suppose w is a function of x, y and z, which are each functions of s and t. Then we have

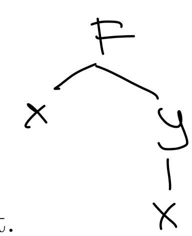
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \text{ and } \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

To determine the appropriate Chain Rule one can draw a tree diagram, where each variable that directly depends on other variables is a node with edges emanating downward to those variables it directly depends on until we reach the independent variables at the bottom. For example, the diagram for the Chain Rules directly above is....

For each downward path from w to s there is a term in $\frac{\partial w}{\partial s}$ and that term is the product of the partial derivatives encoded along each edge.

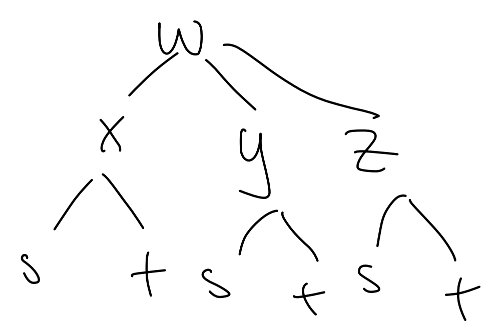
Implicit differentiation: Suppose $y = f(x)$ is implicitly defined by $F(x, y) = C$, where C is a constant.

By differentiating both sides with respect to x, $F_x + F_y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

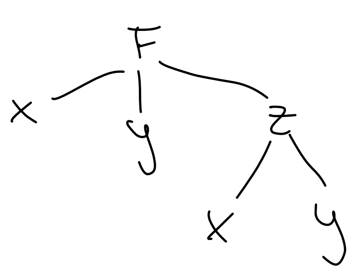


Suppose $z = f(x, y)$ is implicitly defined by $F(x, y, z) = k$ where k is a constant.

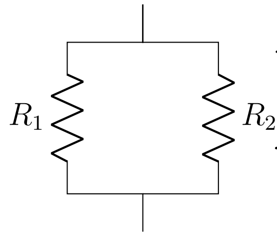
Then one can show that $f_x = -\frac{F_x}{F_z}$ and $f_y = -\frac{F_y}{F_z}$.



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$



Two resistors in an electrical circuit with resistances R_1 and R_2 wired in parallel (see figure below) give an effective resistance of R where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



$$\frac{1}{R} = \frac{1}{R_1}\frac{R_2}{R_2} + \frac{1}{R_2}\frac{R_1}{R_1}$$

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

(a) Let's solve for R as a function of R_1 and R_2.

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

(b) Let's find $\frac{\partial R}{\partial R_1}$.

$$\frac{\partial R}{\partial R_1} = \frac{R_2(R_2 + R_1) - R_1 R_2 (1)}{(R_2 + R_1)^2} = \frac{R_2^2}{(R_1 + R_2)^2}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)^2$$

(c) Let's evaluate $\frac{\partial R}{\partial R_1}$ when $R_1 = 200$ ohms and $R_2 = 100$ ohms.

$$\left.\frac{\partial R}{\partial R_1}\right|_{(200,100)} = \left(\frac{100}{200 + 100}\right)^2 = \frac{1}{9} \ \frac{\text{ohms}}{\text{ohms}}$$

(d) How does an increase in R_1 with R_2 held constant change the effective resistance in general?

It increases it.

$$\frac{\partial R}{\partial R_1} = \left(\frac{R_2}{R_1 + R_2}\right)^2 > 0$$

Note: $\frac{\partial R}{\partial R_2} = \left(\frac{R_1}{R_1 + R_2}\right)^2$

Class Discussion

1. $(1,2,4)$ $\quad z = 2x^2 + \frac{1}{2}xy^2 \quad y=2$

<u>y is constant</u>

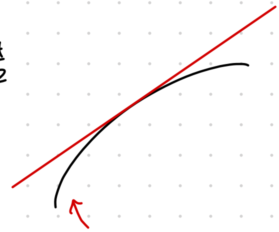
$$\frac{\partial}{\partial x} = 4x + \frac{1}{2}y^2$$

$y=2$

$$\frac{\partial}{\partial x} = 4x + \frac{4}{2}$$

$$\frac{\partial}{\partial x} = 4x + 2$$

$$= 6$$



2. $f(x,y) = 4x\sqrt{1+\frac{y}{x}}$ $\quad$ Evaluate $f_{xy}(1,3)$

$$f_{xy}(1,3) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial f}{\partial x\,\partial y} = f_{xy}$$

$$f(x,y) = 4x\left(1+\frac{y}{x}\right)^{1/2}$$

$$\frac{\partial}{\partial x} = 4\sqrt{\frac{y}{x}+1} - \frac{2y}{\sqrt{\frac{y}{x}+1}\,x}$$

$$\boxed{\frac{\partial}{\partial y} =} \quad 4x\sqrt{1+\frac{y}{x}} = 4x\left(1+\frac{y}{x}\right)^{1/2}$$

$$4\cancel{x} \cdot \frac{1}{2}\left(1+\frac{y}{x}\right)^{-1/2} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{2}{\sqrt{\frac{y}{x}+1}}$$

yx^{-1}

$-yx^{-2}$

$$\frac{\partial}{\partial x}\left(\frac{2}{\sqrt{\frac{y}{x}+1}}\right) \longrightarrow 2\cdot\left(\frac{y}{x}+1\right)^{-1/2}$$

$$= \cancel{-}1 \cdot \left(\frac{y}{x}+1\right)^{-3/2} \cdot \cancel{-}yx^{-2}$$

$$= \frac{y}{\left(\frac{y}{x}+1\right)^{3/2}x^2}$$

$$\frac{y}{\left(\frac{y}{x}+1\right)^{3/2} x^2} = \frac{3}{\left(\frac{3}{1}+1\right)^{3/2}(1)} = \frac{3}{4^{3/2}(1)}$$

$$= \frac{3}{8} = 0.375$$

$$\sqrt{4 \cdot 4 \cdot 4}$$

3. -0.0001

CLASS DISCUSSION EXERCISES (submit answers in corresponding Canvas quiz question):

(1) Calculate the slope at the point $(1, 2, 4)$ along the y-trace of the surface $z = 2x^2 + \frac{1}{2}xy^2$ where $y = 2$ in the positive x-direction. Round to the nearest thousandth.

(2) Let $f(x, y) = 4x\sqrt{1 + \frac{y}{x}}$. Evaluate $f_{xy}(1, 3)$. Round to the nearest thousandth.

Hint: Due to the Equality of Mixed Partials, there are two ways to calculate f_{xy}. One way makes the calculation easier.

(3) According to the ideal gas law a 1-mole sample of gas with volume V (in cubic meters), at pressure P (in pascals) and temperature T (in Kelvin) satisfies $PV = kT$ where $k \approx 8.314$ (Joules per Kelvin). At what rate does the volume of the 1-mole sample of gas change with respect to pressure (in cubic meters per Pa) when the pressure is 5×10^3 Pa (about 1/20 of an atm) if it is held at a constant temperature of 300 K (about 80 degrees Fahrenheit)? Round to the nearest ten-thousandth.

$PV = kT$ $\qquad$ $k \approx 8.314$

$$\frac{\partial}{\partial}$$

1. along the y trace $\rightarrow$ y is constant

$$\frac{\partial}{\partial x} = 4x + \frac{1}{2}y^2$$

at $(1, 2, 4)$

$$\frac{\partial}{\partial x} = 6$$

2. $f = 4x\left(1+\frac{y}{x}\right)^{1/2}$

$$f_y = \cancel{2}\cancel{4x} \cdot \frac{1}{\cancel{2}}\left(1+\frac{y}{x}\right)^{-1/2} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{2}{\sqrt{1+\frac{y}{x}}} = 2\left(1+\frac{y}{x}\right)^{-1/2}$$

$\frac{1}{x}$

$$f_{yx} = -1\left(1+\frac{y}{x}\right)^{-3/2} \cdot \frac{-y}{x^2}$$

$$= \frac{y}{\left(1+\frac{y}{x}\right)^{3/2} x^2}$$

at $(1, 3)$ $\sqrt{4 \cdot 4 \cdot 4}$

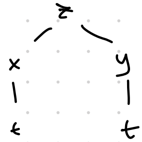
$$\frac{3}{4^{3/2}} = \boxed{\frac{3}{8}}$$

3. $V = \frac{kT}{P}$

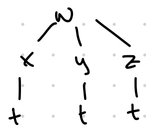
$$V_P = \frac{-kT}{P^2} = \frac{-8.314 \cdot 300}{(5 \cdot 10^3)^2} \approx -0.0001$$

Chain Rules

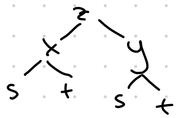
$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$



$$z_s = z_x x_s + z_y y_s$$

$$z_t = z_x x_t + z_y y_t$$

Implicit diff.

$$y' = -\frac{F_x}{F_y}$$

1) If $f(x,y)$ is impl. defined by $F(x,y) = C$ where $F(x,y)$ is differentiable

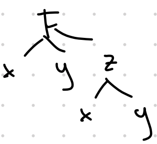
$$y' = -\frac{F_x}{F_y}$$

$$F_x + F_y y' = 0$$

$$y' = \frac{-F_x}{F_y}$$

2) Suppose $z = f(x,y)$ is implicitly defined by $F(x,y,z) = C$
assuming F is diff then

$$z_x = \frac{-F_x}{F_z} \quad z_y = \frac{-F_y}{F_z}$$



Wed.

Math 254 **Exemplar Examples 11**

(1) Suppose that a certain terrain can be modeled by the surface $z = 0.5x^2 - x - 0.25y^2 + 700$ where z is the elevation (in ft) and x and y (also in ft) are eastward and northward displacements respectively from a starting position. Suppose a dog runs along this terrain directly above the curve C in the xy-plane that is parameterized by $\vec{r}(t) = \langle \underbrace{t\cos(t/2)}_{x}, \underbrace{t\sin(t/2)}_{y} \rangle$ where $0 \le t \le 4\pi$ is in seconds.

(a) Let's write down an expression for the rate of change in the dog's elevation with respect to time.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (x-1)\left[\cos\left(\frac{t}{2}\right) - \frac{t}{2}\sin\left(\frac{t}{2}\right)\right] + (-0.5y)\left[\sin\left(\frac{t}{2}\right) + \frac{t}{2}\cos\left(\frac{t}{2}\right)\right]$$

$$= \left(t\cos\left(\frac{t}{2}\right) - 1\right)\left[\quad\right] - \frac{t}{2}\sin\left(\frac{t}{2}\right)\left[\quad\right]$$

(b) Let's find the rate of change in the dog's elevation with respect to time at $t = \pi$. Is the dog going uphill or downhill at this time?

$$\left.\frac{dz}{dt}\right|_{t=\pi} = [-1]\left[-\frac{\pi}{2}\right] - \frac{\pi}{2}[1]$$

$$= 0 \ \text{ft/s} \longrightarrow \text{neither downhill or uphill}$$

(c) Let's find the rate of change in the dog's elevation with respect to time at $t = 2\pi$. Is the dog going uphill or downhill at this time?

$$\left.\frac{dz}{dt}\right|_{t=2\pi} = (-2\pi - 1)(-1) + 0$$

$$= 2\pi + 1 \ \text{ft/sec}$$

uphill

(2) Let $z = x^2 + 4xy + 4y^2$ where $x = g(t)$ and $y = h(t)$ are differentiable functions with values of $g(t)$, $h(t)$, $g'(t)$ and $h'(t)$ given in the table below. Let's determine $\frac{dz}{dt}$ evaluated at $t = 2$.

t	$g(t)$	$h(t)$	$g'(t)$	$h'(t)$
1	2	5	1	1
2	4	-1	7	13

$$\left.\frac{dz}{dt}\right|_2 = \left.\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right|_{t=2} = \left.(2x+4y)\frac{dx}{dt} + (4x+8y)\frac{dy}{dt}\right|_{t=2}$$

$$= 4 \cdot 7 + 8 \cdot 13 = 132$$

(3) Suppose $z = f(x, y)$ is above the xy-plane and is defined implicitly by $3z^2 - xy - xyz = 5$ near the domain point $(-1, 1)$. Let's determine z_x at $(-1, 1)$.

$$z_x = -\frac{F_x}{F_z} = -\frac{-y - yz}{6z - xy}$$

Evaluate at $x = -1$, $y = 1$, and $z = ??$

$$3z^2 + 1 + z = 5$$

$$3z^2 + z - 4 = 0$$

$$(3z + 4)(z - 1) = 0$$

$$z = \cancel{-\frac{4}{3}} \text{ or } 1$$

$$\left. z_x \right|_1 = -\frac{-1-1}{6+1} = \frac{2}{7}$$

1. $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$

e

2. $\left.\frac{dz}{dt}\right|_{t=1} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$= (4xy+1)(2) + (2x^2-3y^2)(-1)$

$= 25 \cdot 2 + (18-12)(-1)$

$= 50 + (-6) = 44$

$\frac{\partial z}{\partial x} = 4xy + 1$

$\frac{\partial z}{\partial y} = 2x^2 - 3y^2$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = -1$

3. $\frac{\partial z}{\partial t} = z_x x_t + z_y y_t$

$x = 3 - 1 = 2$
$y = 3$

when $s = 1$, $t = -1$: $(5)(-2t) + (-1)\left(\frac{1}{s}\right)$

$(5)(+2) + (-1)(1) = 10 - 1$

$= 9$

$z_x =$

$f_x(1,-1) = -2 \qquad f_x(x,y) \qquad x = 2, \; y = 3$

$x_t = -2t$

$y_t = \frac{1}{s}$

4. $z_y = \dfrac{-F_y}{F_z}$

$$3x^2z + 3y^2z + z^3 = 7$$

$$F_y = 6yz$$

$$F_z = 3x^2 + 3y^2 + 3z^2$$

$$= \boxed{\dfrac{-6yz}{3x^2 + 3y^2 + 3z^2}}$$

$$3x^2z + 3y^2z = 7 - z^3$$

$x, y \ (1, 1)$

$$3x^2z + 3y^2z - 7 + z^3 = 0$$

$$z^3 + 3x^2z + 3y^2z - 7 = 0$$

$$z^3 + 3z + 3z - 7 = 0$$

$$z^3 + 3z + 3z = 7$$

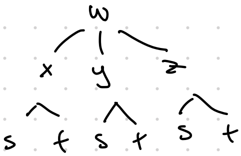
$$\begin{array}{r|rrrr} 1 & 1 & 0 & 6 & -7 \\ + & & 1 & 1 & 7 \\ \hline & 1 & 1 & 7 & 0 \end{array}$$

$$(z^2 + z + 7)(z - 1) = 0$$

$$z = 1$$

$$z_y = \frac{-6}{3+3+3} = -\frac{2}{3}$$

$$-\frac{6}{9} = -\frac{2}{3} = \boxed{-0.67}$$

1. 

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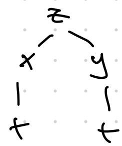
2. $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= (4xy + 1)(2) + (2x^2 - 3y^2)(-1)$$
$$= (25)(2) + (18 - 12)(-1)$$
$$= 50 - 6 = 44$$

3. $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Directional Derivatives & the Gradient

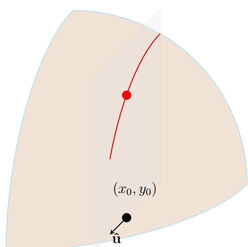
Suppose $z = f(x, y)$ (or $w = g(x, y, z)$) and you want to know how z (or w) changes at the point (x_0, y_0) (or (x_0, y_0, z_0)) with respect to a direction (within the domain) given by a unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ (or $\hat{\mathbf{v}} = \langle a, b, c\rangle$).

The directional derivative of f (or g) in the direction $\hat{\mathbf{u}}$ (or $\hat{\mathbf{v}}$) is given by

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

$$\left(\text{or } D_{\hat{\mathbf{v}}} g(x_0, y_0, z_0) = \lim_{h \to 0} \frac{g(x_0 + ha, y_0 + hb, z_0 + hc) - g(x_0, y_0, z_0)}{h}.\right)$$

In the case of $z = f(x, y)$, the value of $D_{\hat{\mathbf{u}}} f(x_0, y_0)$ is the slope of the curve that is the intersection of the surface $z = f(x, y)$ with the vertical plane $bx - ay = bx_0 - ay_0$ at the point $(x_0, y_0, f(x_0, y_0))$, while moving along the curve in the direction determined by $\hat{\mathbf{u}}$.



As usual we avoid using the limit definition, and instead derive a nice formula:

Consider the function $p(s) = f(x_0 + as, y_0 + bs)$ (or $q(s) = g(x_0 + as, y_0 + bs, z_0 + cs)$). Then:

$$p'(0) = \lim_{h\to 0} \frac{p(0+h) - p(0)}{h} = D_{\hat{\mathbf{u}}} f(x_0, y_0) \text{ [or } \left(q'(0) = \lim_{h\to 0} \frac{q(0+h) - q(0)}{h} = D_{\hat{\mathbf{v}}} g(x_0, y_0, z_0) \right) \text{]}.$$

And by the chain rule:

$$p'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b \text{ (or } q'(0) = g_x(x_0, y_0, z_0)\, a + g_y(x_0, y_0, z_0)\, b + g_z(x_0, y_0, z_0)\, c)$$

All that motivates the following important definition:

The gradient of $f(x, y)$ (or $g(x, y, z)$) is given by $\vec{\nabla} f = \langle f_x, f_y \rangle$ (or $\vec{\nabla} g = \langle g_x, g_y, g_z \rangle$). Note: Sometimes we just use ∇ instead of $\vec{\nabla}$.

So, using that notation, $\boxed{D_{\hat{\mathbf{u}}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}}$ (or $\boxed{D_{\hat{\mathbf{v}}} g(x_0, y_0, z_0) = \nabla g(x_0, y_0, z_0) \cdot \hat{v}}$).

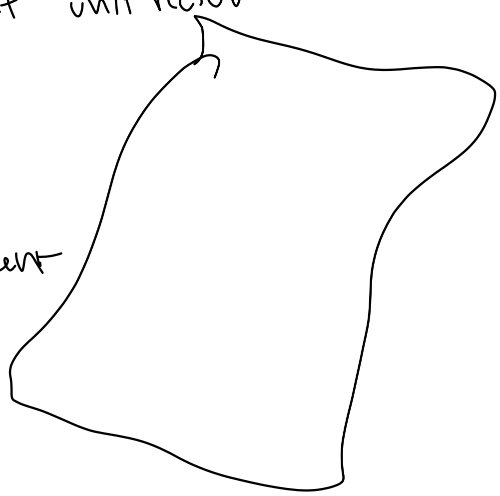
$D_{\hat{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$ ← for direction vector

$D_{\hat{v}} g(x_0, y_0, z_0) = \nabla g(x_0, y_0, z_0) \cdot \hat{v}$

gradient "dot product" unit vector

steepest incline

negative max descent



orthogonal to level curves

gradient $f(x,y)$ is $\vec{\nabla} f = \langle f_x, f_y \rangle$

gradient $f(x,y,z)$ is $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

(1) Suppose a very hot object (at the origin) is radiating heat and that the temperature T at a nearby point is inversely proportional to the distance from the very hot object. A thermometer is placed at $(2, 1, 2)$ and the temperature there is measured to be 150° C. Assume x, y, z are in units of meters.

(a) Let's determine a function for the temperature $T(x, y, z)$ at (x, y, z).

$$T = \frac{1}{\sqrt{x^2+y^2+z^2}} \qquad 150 = \frac{k}{\sqrt{2^2+1^2+2^2}} \qquad k = 450$$

$$T = \frac{450}{\sqrt{x^2+y^2+z^2}}$$

(b) Let's find the gradient of $T(x, y, z)$.

$$\vec{\nabla} T = \langle T_x, T_y, T_z \rangle = \frac{-225}{(x^2+y^2+z^2)^{3/2}} \langle 2x, 2y, 2z \rangle$$

$$= \frac{-450}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

(c) Let's find the rate of change in the temperature at $(2, 1, 2)$ in the direction of the point $(5, 5, 2)$.

$$\vec{u} = \langle 3, 4, 0 \rangle \qquad |\vec{u}| = 5$$

$$\hat{u} = \frac{1}{5} \langle 3, 4, 0 \rangle$$

$$\vec{\nabla} T(2,1,2) = \frac{-450}{27} \langle 2, 1, 2 \rangle$$

dot

$$= -\frac{50}{3} \langle 2, 1, 2 \rangle$$

$$D_{\hat{u}} T(2,1,2) = -\frac{10}{3}(6 + 4 + 0)$$

$$= -\frac{100}{3}$$

$$\approx -33\ ^\circ C/M$$

(d) Let's find the rate of change in the temperature at $\underset{\text{head}}{\underline{(2, 1, 2)}}$ in the direction of the point $\underset{\text{tail}}{\underline{(1, -1, 0)}}$.

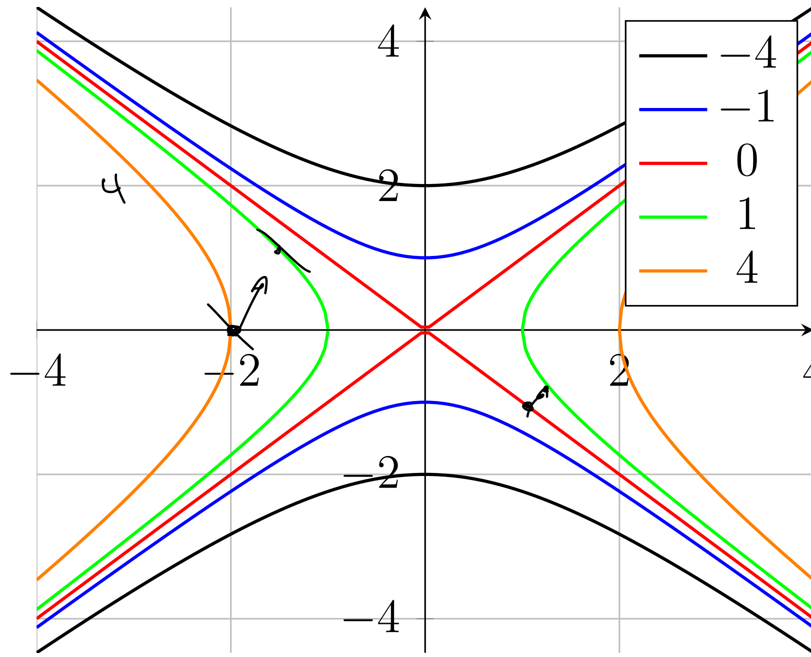
$$\vec{v} = \langle -1, -2, -2 \rangle \qquad |\vec{v}| = 3$$

$$\hat{v} = \frac{1}{3} \langle -1, -2, -2 \rangle$$

$$D_{\hat{v}} T(2,1,2) = -\frac{50}{9}(-2 - 2 - 4)$$

$$= -\frac{50}{9}(-8) = \frac{400}{9} \approx 44\ ^\circ C/M$$

(2) Below are several level curves of $f(x, y)$.



(a) Let's estimate the directional derivative of $f(x, y)$ at $(-2, 0)$ in the direction of the vector $\vec{u} = \langle 1, 2\rangle$.

$\frac{-3}{0.8} = -3.75$

(b) Let's estimate the directional derivative of $f(x, y)$ at $(1, -1)$ in the direction of the point $(2, 0)$.

$\frac{1}{0.2} = 5$

1. $f(x,y) = \sqrt{4x^2+y^2-9} = (4x^2+y^2-9)^{1/2}$

$\vec{\nabla} f = \langle f_x , f_y \rangle$

$f_x = \frac{1}{2}(4x^2+y^2-9)^{-1/2} \cdot 8x$

$f_y = \frac{1}{2}(4x^2+y^2-9)^{-1/2} \cdot 2y$

$\frac{1}{\sqrt{4x^2+y^2-9}} \langle 4x, y \rangle$

at $(2,3)$

$\frac{1}{\sqrt{16 \cancel{+9-9}}} \langle 8, 3 \rangle$

$\frac{1}{4} \langle 8, 3 \rangle$

$\vec{u} = (-3, -4)$ $\quad |\vec{u}| = \sqrt{3^2+4^2}$

$= \sqrt{9+16} = 5$

$\hat{u} = \langle \frac{-3}{5}, \frac{-4}{5} \rangle$

dot product

$\frac{1}{5} \langle -3, -4 \rangle \cdot \frac{1}{4} \langle 8, 3 \rangle$

$\frac{1}{20}(-24 + -12) = \frac{-36}{20} = -\frac{9}{5}$

$= -1.8$

2. $z = \frac{y}{\sqrt{x^2+y^2}}$ $z = y(x^2+y^2)^{-1/2}$

$f_x = y \cdot -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2x$

$f_y = (x^2+y^2)^{-1/2} + y(-\frac{1}{\cancel{2}}(x^2+y^2)^{-3/2} \cdot \cancel{2}y)$

at (3,4)

$f_x = -xy\,(x^2+y^2)^{-3/2} = -xy \cdot \frac{1}{(x^2+y^2)^{3/2}} = \frac{-12}{(25)^{3/2}}$

at (3,4)

$f_y = (x^2+y^2)^{-1/2} - \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{5} - \frac{16}{25^{3/2}}$

$\frac{x^2}{(x^2+y^2)^{3/2}}$

$\frac{9}{(9+16)^{3/2}}$

$\vec{u} = \langle 12, 5 \rangle$ $|\vec{u}| = \sqrt{144+25}$
$= \sqrt{169}$
$= 13$

$\hat{u} = \frac{1}{13}\langle 12, 5\rangle$

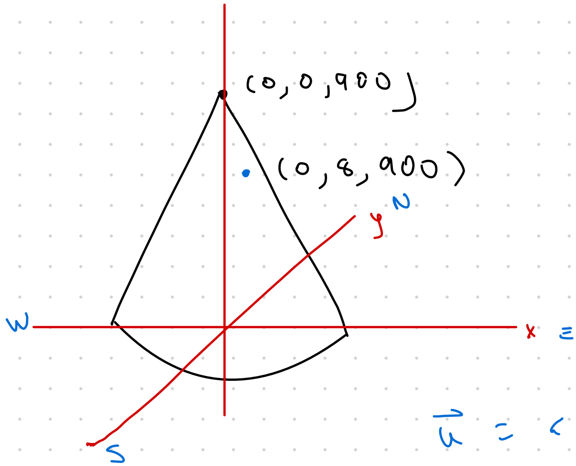
$\frac{1}{13}\langle 12, 5\rangle \cdot \langle -\frac{12}{125}, \frac{1}{5} - \frac{16}{125}\rangle$

$\frac{1}{13}\langle 12, 5\rangle \cdot \langle -\frac{12}{125}, \frac{25}{125} - \frac{16}{125}\rangle$

$\frac{1}{13}\langle 12, 5\rangle \cdot \frac{1}{125}\langle -12, 9\rangle$

$\frac{1}{1625}(-144+45) = \frac{-99}{1625}$
$= -0.06$

$h(x,y) = 900 - \sqrt{5x^2+4y^2}$



point

$(0, 8, 884)$

$\vec{u} = \langle -1, -1 \rangle$

$|\vec{u}| = \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

$h(x,y) = 900 - \sqrt{5x^2+4y^2} = 900 - (5x^2+4y^2)^{1/2}$

$h_x = -\frac{1}{2}(5x^2+4y^2)^{-1/2} \cdot 10x$

$h_y = -\frac{1}{2}(5x^2+4y^2)^{-1/2} \cdot 8y$

$\frac{-1}{2\sqrt{5x^2+4y^2}} \langle 10x, 8y \rangle$ at $(0, 8, \cancel{884})$

$\frac{-1}{2\sqrt{4 \cdot 64}} \langle 0, 64 \rangle$

$\frac{-1}{2 \cdot 16} \langle 0, 64 \rangle = \frac{-1}{32} \langle 0, 64 \rangle$.

$$= \frac{-1}{32} \langle 0, 64 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, -1 \rangle$$

$$= -\frac{1}{32 \cdot \sqrt{2}} (-64) = \frac{2}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$$

$$= 1.41$$

Q4. 0.2 optional

Example: Let $f(x,y) = x^3y^2 + \tan^{-1}(xy)$. Find the directional derivative of $f(x,y)$ at the point $(1,3)$ in the direction $\langle 3,-4\rangle$.

Notice the given vector is not a unit vector, so we scale it into a unit vector in the same direction: $\hat{\mathbf{u}} = \frac{1}{5}\langle 3,-4\rangle$.

The gradient of f is $\vec{\nabla} f = \langle \overbrace{3x^2y^2 + y/(1+x^2y^2)}^{f_x}, \overbrace{2x^3y + x/(1+x^2y^2)}^{f_y}\rangle$, which when evaluated at $(1,3)$ is $\underline{\vec{\nabla} f(1,3)} = \langle 273/10, 61/10\rangle$. So the directional derivative is

$$\underline{D_{\hat{\mathbf{u}}} f(1,3)} = \vec{\nabla} f(1,3) \cdot \hat{\mathbf{u}} = \left\langle \frac{273}{10}, \frac{61}{10} \right\rangle \cdot \frac{1}{5}\langle 3,-4\rangle = \frac{1}{50}\left(273(3) + 61(-4)\right) = \frac{575}{50} = \frac{23}{2}.$$

Let f be scalar function (of two or three variables) with continuous partial derivatives on its domain, and let P be a point in the domain of f. For what domain direction $\hat{\mathbf{u}}$ is $D_{\hat{\mathbf{u}}} f(P)$ at a maximum? Minimum? What are the maximum and minimum values?

By the dot product identity, $D_{\hat{\mathbf{u}}} f(x_0,y_0) = |\nabla f(P)||\hat{\mathbf{u}}|\cos\theta = |\nabla f(P)|\cos\theta$, where θ is the angle between $\nabla f(P)$ and $\hat{\mathbf{u}}$.

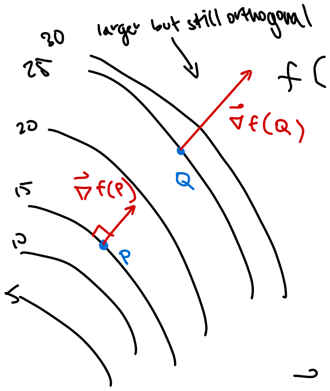
So it is at a maximum when $\theta = 0$ and at a minimum when $\theta = \pi$. The maximum and minimum values are $\pm|\nabla f(P)|$ respectively. Those occur when $\hat{\mathbf{u}}$ is in the direction of $\pm\nabla f(P)$ respectively. Hence the gradient points in the direction of maximum increase (and the negative of the gradient points in the direction of maximum decrease) and its length is that max. rate of increase (or decrease).

Now consider a function $f(x,y)$, with continuous partial derivatives on its domain such that $\vec{r}(t) = \langle x(t), y(t)\rangle$ is a differentiable parametrization of the level curve $f(x,y) = C$ (where C is an arbitrary constant in the range of f).

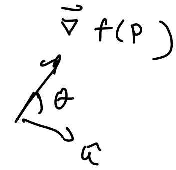
Then $f(\vec{r}(t)) = C$, which by differentiating both sides and using a Chain Rule gives $f_x(\vec{r}(t))x'(t) + f_y(\vec{r}(t))y'(t) = 0$. That can be rewritten as $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$, which means that $\nabla f(\vec{r}(t))$ and $\vec{r}\,'(t)$ are orthogonal! Since $\vec{r}\,'(t)$ is tangent to the graph of $\vec{r}(t)$, the gradient evaluated at a point in the domain is perpendicular to the level curve through that point.

$f(x,y) \rightarrow \vec{\nabla} f = \langle f_x, f_y \rangle$

$f(x,y,z) \rightarrow \vec{\nabla} f = \langle f_x, f_y, f_z \rangle$



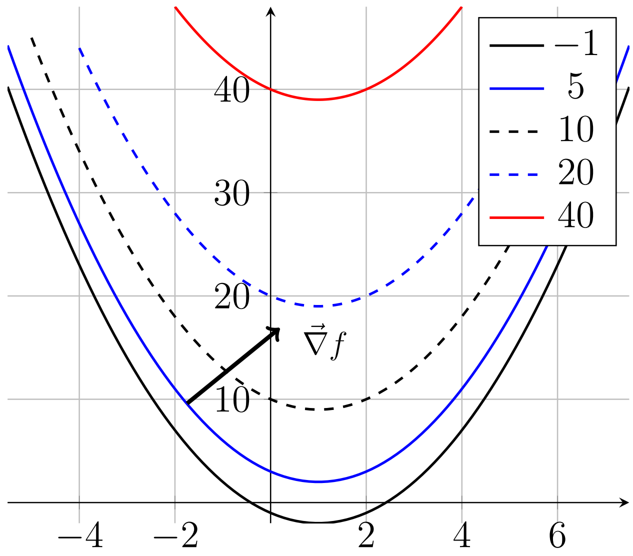
$$D_{\hat{u}} f(P) = \vec{\nabla} f(P) \cdot \hat{u}$$
$$= |\vec{\nabla} f(P)| \cos\theta$$



$\vec{\nabla} f(P)$: points in the direction of max increase in f at point P
$-$ decrease

$|\vec{\nabla} f(P)|$ is that max rate of incr. / decr

Here is a gradient drawn amongst level curves of $f(x, y) = 2x + y - x^2$:



Note: A similar result holds for the gradient of a scalar function $g(x, y, z)$ (with cont. first order partials) and level surfaces of $g(x, y, z)$.

Suppose the elevation x meters east and y meters north of the entrance to a sno-park is given by $h(x,y) = 6522 - 2\sqrt{0.5(x-600)^2 + (y-400)^2}$ (in meters). Suppose that at a certain instant, a skier on this terrain is 200 meters east and 300 meters north of the entrance and is following the path of maximum descent.

(a) Find the gradient of h

$$\vec{\nabla} h = -\frac{1}{2}\cdot 2\left(0.5(x-600)^2 + (y-400)^2\right)^{-1/2} \langle x-600,\ 2(y-400)\rangle$$

$$= \frac{1}{\sqrt{0.5(x-600)^2 + (y-400)^2}} \langle x-600,\ 2(y-400)\rangle$$

(b) Evaluate the gradient at the point at the point in the domain corresponding to where the skier is on the terrain.

x = east

y = north

$$\vec{\nabla} h(200,300) = -\frac{1}{\sqrt{0.5(-400)^2 + (-100)^2}} \langle -400,\ 2(-100)\rangle$$

$$= \cancel{-}\frac{1}{300}\langle \cancel{-}400, \cancel{-}200\rangle$$

$$= \left\langle \frac{4}{3}, \frac{2}{3}\right\rangle = \frac{2}{3}\langle 2,1\rangle$$

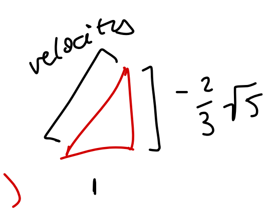
(c) Determine the slope of the path followed by the skier at the instant described.

$$-|\vec{\nabla} h(200, 00)| = -\frac{2}{3}\sqrt{5}$$

(d) If the skier is skiing at a speed of $\sqrt{5}$ meters per second, then determine the skier's velocity vector at the instant described.

unit vector in same direction

$$-\vec{\nabla} h(200,300): \quad -\frac{1}{\sqrt{5}}\langle 2, 1\rangle \text{ (run)}$$



velocity:

$$\vec{v} \parallel \left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5}\right\rangle$$

$$\vec{v} = k\left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5}\right\rangle \quad (k>0)$$

$$|\vec{v}| = \sqrt{5} = k\sqrt{1 + \left(-\frac{2}{3}\sqrt{5}\right)^2}$$

$$= k\sqrt{\frac{29}{9}} = k\frac{\sqrt{29}}{3}$$

$$k = \frac{3\sqrt{5}}{\sqrt{29}} \qquad \vec{v} = \frac{3\sqrt{5}}{29}\left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5}\right\rangle$$

1. b.

2. $\vec{\nabla} f$

$f_x = 1 + 2xy + 3x^2y^2z$ $\quad 1 + -2 + 6 = 5$

$f_y = x^2 + x^3 2yz$ $\quad 1 + (-1)(2)(2) = -3$

$f_z = x^3y^2$ $\quad$ at $(-1,1,2)$ $\quad (-1)(1) = -1$

$\vec{\nabla} f(-1,1,2) = \langle 5, -3, -1 \rangle$

$|\vec{\nabla} f(-1,1,2)| = \sqrt{35} \approx 5.92$

$25+9+1$

3. unit vector

$\vec{\nabla} f$ $\quad -(x - yx^2)\, y^{-1}$

$f_x = 2x - \frac{1}{y}$ $\quad 4 - 1 = 3$

at $(2,1)$

$f_y = \frac{x}{y^2}$ $\quad \frac{2}{1} = 2$

$\vec{\nabla} f(2,1) = \langle 3, 2 \rangle \rightarrow \langle -3, -2 \rangle$

$|\vec{\nabla} f(2,1)| = \sqrt{13}$ $\quad 9+4$ $\quad$ ↳ decreasing

$\langle \frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \rangle$ or $-\frac{1}{\sqrt{13}} \langle 3, 2 \rangle$

$\boxed{-\frac{2}{\sqrt{13}}} = -0.55$

4.

unit vector in same direction

$$-\vec{\nabla} h(200, 200): \quad -\frac{1}{\sqrt{5}} \langle 2, 1 \rangle$$

velocity:

$$\vec{v} \parallel \left\langle \frac{-2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5} \right\rangle$$

$$\vec{v} = k \left\langle -\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5} \right\rangle \quad (k > 0)$$

$$|\vec{v}| = \sqrt{5} = k\sqrt{1 + \left(-\frac{2}{3}\sqrt{5}\right)^2}$$

$$= k\sqrt{\frac{29}{9}} = k\frac{\sqrt{29}}{3}$$

$$k = \frac{3\sqrt{5}}{\sqrt{29}} \qquad \vec{v} = \frac{3\sqrt{5}}{29}\left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -\frac{2}{3}\sqrt{5} \right\rangle$$

$$f(x,y) = k\left(\frac{x}{(x+y)^2}\right) \qquad (8, -6)$$

$$f_x = (x+y)^{-2} + (x)(-2(x+y)^{-3}) = \frac{y-x}{(x+y)^2} = \frac{-14}{4}$$

$$f_y = 0 + x(-2(x+y)^{-3})(1) = -\frac{2x}{(x+y)^3} = \frac{-16}{8}$$

$$f_x = 2^{-2} + 8(-2(2)^{-3})$$

$$f_y = 0 + 8(-2(2)^{-3})$$

$$= \cancel{8}\left(-\frac{2}{\cancel{8}}\right)$$

$$\left\langle \frac{1}{4} + -2, -2 \right\rangle$$

$$\vec{a} = \left\langle -\frac{7}{4}, -2 \right\rangle$$

$$\sqrt{113} = k \left\langle -\frac{7}{4}, -2 \right\rangle$$

$$\sqrt{\frac{49}{16} + 4}$$

$$k = \frac{\sqrt{113}}{\frac{\sqrt{113}}{4}}$$

$$k = 4$$

...

Tangent Planes, Linear Approximation & Optimization of Functions of Two Variables

Consider the surface in 3-dimensions defined implicitly by $F(x, y, z) = C$ (a constant), which is a level set of F. Note that an explicit function $z = f(x, y)$ can always be defined implicitly with $F(x, y, z) = 0$ by having $F(x, y, z) = z - f(x, y)$.

Recall that the gradient of F evaluated at a point on a level set of F is orthogonal to the level set at the point. So it can be used as a normal vector to give the tangent plane to the surface at the point: $\boxed{\vec{\nabla} F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.}$

In the case the tangent plane to $z = f(x, y)$ at (x_0, y_0, z_0) (so $z_0 = f(x_0, y_0)$ at this point) the above reduces to $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

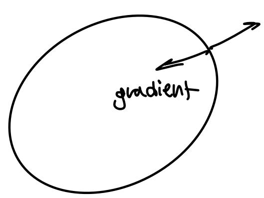
An important associated definition here is what's called the linear approximation $L(x, y)$, to $z = f(x, y)$ at (x_0, y_0):

$\boxed{f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$ near (x_0, y_0).

can add z_0 dimension

Yet another equivalent form of this is in the language of differentials, where we define the differential of $z = f(x, y)$ as $\boxed{dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$ where dx and dy represent small changes in x and in y and dz approximates the corresponding small change in z.

Any surface in xyz-space defined by $F(x,y,z) = C$ where F is diff. satisfies that $\vec{\nabla} F(x_0, y_0, z_0)$ is $\perp$ to the surface at (x_0, y_0, z_0)



Note: For $z = f(x,y)$
set $F(x,y,z) = f(x,y) - z$
So $z_0 = f(x_0, y_0)$

Tangent plane

$$\vec{\nabla} F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear

Optimization

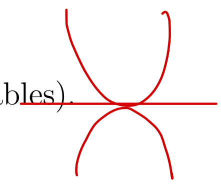
Let's find an equation for the tangent plane to the graph of $f(x, y) = 100 - x^2 - y^4$ at the point on the surface above where $x = 3$ and $y = 1$.

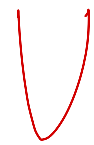
For $F(x, y, z) = z + x^2 + y^4 = 100$, $\vec{\nabla} F = \langle 2x, 4y^3, 1 >$ and so $\vec{n} = \vec{\nabla} F(3, 1, 90) = \langle 6, 4, 1\rangle$. So one can determine that the plane is $6x + 4y + z = 112$.

$f(x, y)$ has a local maximum/minimum at (x_0, y_0) if $f(x_0, y_0)$ is the largest/smallest value of f in an open disk about (x_0, y_0).

(x_0, y_0) is a critical point of $f(x, y)$ if $\nabla f(x_0, y_0) = \vec{0}$ (horizontal tangent plane) or one of the partial derivatives does not exist.

(x_0, y_0) is a saddle point of $f(x, y)$ if every disk about (x_0, y_0) contains values of f that are smaller and larger than $f(x_0, y_0)$.





(All these ideas extend naturally to functions of 3 variables).

Second Derivative Test: (proof omitted)

Suppose the second order partial derivatives of $f(x, y)$ are continuous throughout an open disk about a critical point (x_0, y_0). Let $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$. Then

or f_{yy}

(1) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).

(2) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).

(3) If $D(x_0, y_0) < 0$, then f has a saddle at (x_0, y_0).

(4) If $D(x_0, y_0) = 0$, then the test is inconclusive.

Let's classify all the critical points of $f(x, y) = x^2y^2 - xy^2 + 2x^2y - 2xy + 5$.

$\vec{\nabla} f = \langle 2xy^2 - y^2 + 4xy - 2y, 2x^2y - 2xy + 2x^2 - 2x\rangle = \vec{0}$.

So $y(2x - 1)(y + 2) = 0$ and $2x(x - 1)(y + 1) = 0$.

$y(2x-1)(y+2) = 0$ implies $y = 0$, $x = 0.5$ or $y = -2$. If $y = 0$, $x(x-1)(1) = 0$ yields $x = 0$ or $x = 1$. If $y = -2$, $x(x - 1)(-1) = 0$ yields the same. If $x = 0.5$, $0.5(-0.5)(y + 1) = 0$ yields $y = -1$.

$D = f_{xx}f_{yy} - (f_{xy})^2 = (2y^2 + 4y)(2x^2 - 2x) - (4xy - 2y + 4x - 2)^2.$

(x_0, y_0)	$D(x_0, y_0)$	$f_{xx}(x_0, y_0)$	Conclusion
$(0, 0)$	-4	-	Saddle
$(1, 0)$	-4	-	Saddle
$(0. - 2)$	-4	-	Saddle
$(1, -2)$	-4	-	Saddle
$(0.5, -1)$	1	-2	Local maximum

$f(x, y)$ has an <u>absolute maximum/minimum</u> at (x_0, y_0) if $f(x_0, y_0)$ is the largest/smallest value in the range.

Extreme Value Theorem: Suppose f is defined on a closed and bounded set. Find the values of f at all critical points in the interior. Find the maximum and minimum value of f on the boundary. The largest/smallest values correspond to the absolute maximum/minimum.

(Proof is skipped.)

Example: Find global extremes of $f(x, y) = e^{4x+y^2-x^2}$ on the region $x^2 + y^2 \leq 9$.

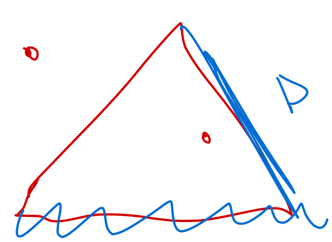
$\vec{\nabla} f = \langle 2(2-x)e^{4x+y^2-x^2}, 2ye^{4x+y^2-x^2} \rangle = \vec{0}$ implies $x = 2$ and $y = 0$.

The point $(2, 0)$ lies within the interior (not on the boundary). So we keep it.

Set $g(x) = f(x, \pm\sqrt{9 - x^2}) = e^{4x+9-2x^2}$. So $g'(x) = 2(1 - x)e^{4x+9-2x^2} = 0$ at the point $x = 1$ (with $y = \pm 2\sqrt{2}$).

With the endpoints at $x = \pm 3$ and the interior point found above we get the following potential extreme points: $(-3, 0)$, $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$, $(2, 0)$, $(3, 0)$.

(x, y)	$f(x, y)$	Conclusion
$(-3, 0)$	e^{-21}	Global minimum
$(1, 2\sqrt{2})$	e^{11}	Global maximum
$(1, -2\sqrt{2})$	e^{11}	Global maximum
$(2, 0)$	e^{4}	Not a global extreme
$(3, 0)$	e^{3}	Not a global extreme



To find the abs./global extremes of $f(x,y)$ on a bounded domain including its boundary, find critical pts & compare the values at these pts w/ critical points/end points on the boundary

(1) Consider the function $f(x, y) = -\dfrac{50x}{x^2 + y^2}$.

(a) Let's determine a function $F(x, y, z)$ for which the graph of $f(x, y)$ is (all or part of) a level surface of $F(x, y, z)$.

$$z = -\frac{50x}{x^2+y^2} \rightarrow zx^2 + y^2 z = -50x$$

$$F(x,y,z) = x^2 z + y^2 z + 50x = 0$$

(b) Let's find an equation for the tangent plane to the graph of $f(x, y)$ at the point where $x = 3$ and $y = 4$.

$$(3,4,-6) \qquad z_0 = f(3,4) = -\frac{150}{9+16} = -\frac{150}{25} = -6$$

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \langle 2xz^{+50}, 2yz, x^2+y^2 \rangle$$

$$\vec{\nabla} F(3,4,-6) = \langle 14, -48, 25 \rangle = \vec{n}$$

$$14x - 48y + 25z = -300$$

(c) Let's find the linear approximation to $f(x, y)$ based at the point where $x = 3$ and $y = 4$. Solve for z

$$z = \frac{-14x + 48y - 300}{25}$$

$$L(x,y) = -\frac{14}{25}x + \frac{48}{25}y - \frac{300}{25}$$

$$= -\frac{14}{25}x + \frac{48}{25}y - 12$$

(d) Let's use this linear approximation to approximate $f(2.95, 4.1)$.

$$L(2.95, 4.1) = -\frac{14}{25}(2.95) + \frac{48}{25}(4.1) - 12$$

$$= -5.78$$

(2) Let's find an equation for the tangent plane to the hyperboloid of two sheets given by $x^2 + z^2 = 4y^2 - 16$ at the point $(2, 3, 4)$.

$F = x^2 - 4y^2 + z^2 = -16$

$\vec{\nabla} F = \langle 2x, -8y, 2z \rangle$

$\vec{\nabla} F(2, 3, 4) = \langle 4, -24, 8 \rangle$

set $\vec{n} = \langle 1, -6, 2 \rangle$

$x - 6y + 2z = -8$ plug in the point

$\nabla F = \langle 2x, -8y, 2z \rangle$

$\nabla F(2,3,4)$

$= \langle 4, -24, 8 \rangle$

$\vec{n} = \langle 1, -6, 2 \rangle$

$x - 6y + 2z =$

1. $f(x,y) = 4$

$f(3,4) = 4$

$$z = \sqrt{x^2+y^2-9}$$

$$z^2 = x^2+y^2-9$$

$$F(x,y,z) = x^2+y^2-z^2 \underset{d}{\underline{-9}}$$

$$\vec{\nabla} F = \langle 2x, 2y, -2z \rangle \qquad (3,-4,4)$$

$$\vec{\nabla} F = \langle 6, -8, -8 \rangle$$

$$\langle 3, -4, -4 \rangle$$

where $x=0$
$y=0$

$$3x - 4y - 4z = 9$$

$$z = \frac{-9}{4} = -2.25$$

$$f(x,y) = \sqrt{x^2y^4+9}$$

2.

Point $(1,2,5)$

$$z = \sqrt{x^2y^4+9}$$

$$z^2 = x^2y^4+9$$

32

$$F(x,y,z) = x^2y^4 - z^2 + 9$$

$$\vec{\nabla} F = \langle 2xy^4, 4y^3x^2, -2z \rangle$$

$$= \langle 32, 32, -10 \rangle$$

$$\vec{n} = \langle 16, 16, -5 \rangle$$

$$16x + 16y - 5z = 23$$

$$z = \frac{23-16x-16y}{-5}$$

$f(1.1, 2.1)$

$$z = 5.64$$

3. $f(x,y) = \dfrac{10 + xy}{x - y}$

$(2, 1, 12)$

$$z = \frac{10 + xy}{x - y}$$

$$zx - zy = 10 + xy$$

$$F(x,y,z) = xy - zx + zy - 10$$

$$\vec{\nabla} F = \langle y - z, x + z, -x + y \rangle$$

$$= \langle -11, 14, -1 \rangle$$

$$-11x + 14y - 1z = -20$$

$$-11x + 14y + 20 = z$$

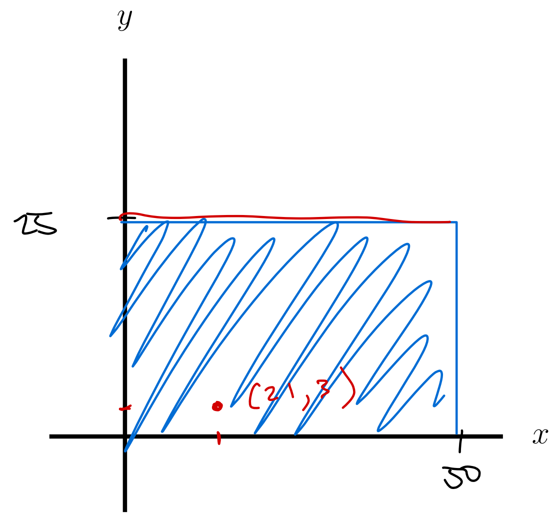
$$z = 11.6$$

1.

A tennis ball manufacturer has developed a profit model that depends on the number x of thousands of tennis balls produced each month and the amount y of hours of radio advertisements each month. Their model predicts that the monthly profit will be given by $P(x, y) = 48x + 96y - x^2 - 2xy - 9y^2$ (in thousands of dollars) where the maximum number of balls that can be produced is $50,000$ and their monthly budget allows for up to 25 hours of advertisements.

0 to 25

(a) Draw the domain!



(b) Let's find the critical point(s) of $P(x, y)$ that is/are within the interior of the domain.

$$\vec{\nabla} P = \langle 48 - 2x - 2y,\ 96 - 2x - 18y \rangle = \langle 0, 0 \rangle$$

two lines w/ distinct slopes

$$\begin{aligned} 48 - 2x - 2y &= 0 \\ -96 + 2x + 18y &= 0 \\ \hline -48 + 16y &= 0 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} 48 - 6 - 2x &= 0 \\ 42 &= 2x \\ x &= 21 \end{aligned}$$

$(21, 3)$

(c) Let's classify these critical points according to the Second Derivative Test. (This is technically not necessary for this exercise, but we do it anyway.)

$$\begin{aligned} D &= (-2)(-18) - (-2)^2 \\ &= f_{xx} - f_{yy} - (f_{xy})^2 = 32 > 0 \end{aligned}$$

$$f_{xx} = -2 < 0$$

Local max

(d) Clearly the profit will not be a maximum when $x = 0$ or when $y = 0$. So let's assume $x > 0$ and $y > 0$. Let's plug in the top boundary to the domain into P and determine any critical points along that boundary.

$y = 25$

$$g(x) = P(x, 25) = -2x - x^2 - 3225$$

$$g'(x) = -2 - 2x = 0 \qquad \text{None}$$

$$\cancel{x = -1}$$

(e) Let's plug in the right boundary to the domain into P and determine any critical points along that boundary.

$x = 50$

$$h(y) = P(50, y) = -4 - 9y^2 - 100$$

$$h'(y) = -4 - 18y = 0 \qquad \text{None}$$

(f) Together with the boundary point at $(50, 25)$, let's compare the value of $P(x, y)$ at critical points inside and along the top/right boundaries to determine where $P(x, y)$ is maximized and what that maximum monthly profit is.

$$P(50, 25) < 0$$

$$P(21, 3) = 648 > 0$$

Max

1. 4

2. Find the number of critical points

$$f(x,y) = 2xy^2 - x^2y - 3xy$$

∇

$$= 0$$

$$\langle 2y^2 - 2xy - 3y,\ 4xy - x^2 - 3x \rangle$$

$$y(2y - 2x - 3) = 0$$

$$2\ (2y^2 - 2xy - 3y) = 0$$

$$4xy - x^2 - 3x = 0$$

$$4y^2 - 4xy - 6y = 0 \qquad x(4y - x - 3) = 0$$

$$4y^2 - x^2 - 6y - 3x = 0$$

$$y(4y - 6) = x^2 + 3x$$

$$y(4y-6) = x(x+3)$$

$y = 0$ $\qquad$ $x = 0$

$y = \frac{6}{4} = \frac{3}{2}$ $\qquad$ $x = -3$

$(0,0)$ $(-3,0)$ $(-1, \frac{1}{2})$

$(0, \frac{3}{2})$

$\boxed{4}$

3. $f_x = 6(x^2-1)(y+1)$ $f_{xx} =$

$f_y = 2x(x^2-3)$

$f_{xx} = (6x^2-6)(y+1)$

$6x^2y - 6y + 6x - 6$

$\cancel{12xy + 6}$

$f_{yy} = 0$ $f_{xy} = 6x^2 - 6$

$(6x^2-6)(6x^2-6)$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$0 - (6x^2-6)^2$

$0 > -(36x^4 - 72x^2 + 36)$

$0 > -36x^4 + 72x^2 - 36$

$> -36(x^4 - 2x^2 + 1)$

$> -36(x^2-1)(x^2-1)$

$x = \pm 1$

4. 14.97

3. $f_x = 6(x^2-1)(y+1) = 0$

$y = -1$

$x = +1, -1$ $\quad 6(2)(y+1) = 0$

$f_y = 2x(x^2-3) = 0$

$x = 0$

$x = \sqrt{3}, -\sqrt{3}$

$(0,-1)\ (\sqrt{3},-1)\ (-\sqrt{3},-1)$

$\boxed{3}$

4. EVT

...

Lagrange Multipliers

Let $f(x,y)$ be a differentiable function in a region of $\mathbb{R}^2$ that contains the "smooth" curve C given by $g(x,y)=k$ where k is a constant and $\vec{\nabla} g(x,y) \neq \vec{0}$ for all points on C.

Suppose we want to find the local extremes values of f restricted to inputs on the curve C, called a <u>constraint</u>.

Let C be parameterized by $\mathbf{r}(\mathbf{t}) = \langle x(t), y(t)\rangle$ where $x(t)$ and $y(t)$ are differentiable. Then $h(t) = f(x(t),y(t))$ is differentiable.

We want the extreme values of h. By a Chain Rule $h'(t) = f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t)$. h has critical points where $0 = f_x\,x'(t) + f_y\,y'(t)$. This could be rewritten as $0 = \vec{\nabla} f(x(t),y(t)) \cdot \mathbf{r}'(\mathbf{t})$.

So the extreme values of f restricted to C occur where the gradient of f at a point P on C is orthogonal to the curve C at P ($\mathbf{r}'(\mathbf{t})$ is tangent to C at P). The gradient of g evaluated at a point P on C is also orthogonal to C at P. That means that the gradient of f must be a scalar multiple of the gradient of g.

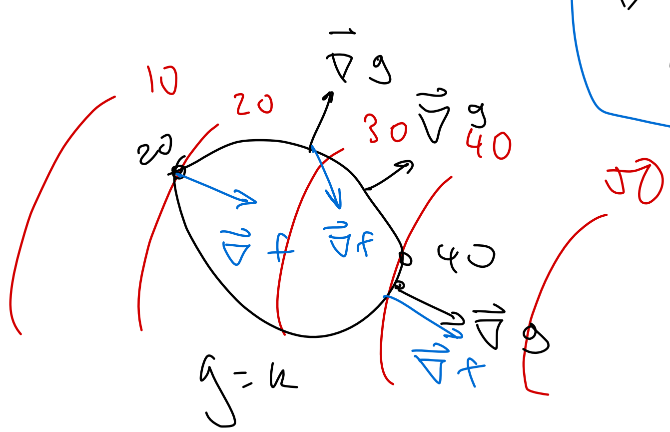
So from this we get the "Method of Lagrange multipliers" for constrained optimization. To find the minima and/or maxima of $f(x,y)$ restricted to a "smooth" curve C given by $g(x,y)=k$ where $\vec{\nabla} g(x,y) \neq 0$ for all points on C, solve the system given by the equations:

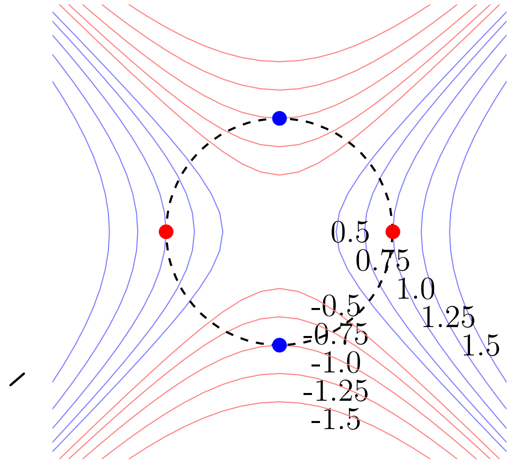
$$\boxed{\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \text{ and } g(x,y)=k.}$$

This generalizes to a function $f(x,y,z)$ restricted to a "nice" surface S given by $g(x,y,z)=k$ where $\nabla g(x,y,z) \neq \vec{0}$ on S. To find the minima and/or maxima of $f(x,y,z)$ restricted to the surface S, solve the system given by the equations:

$$\boxed{\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z) \text{ and } g(x,y,z)=k.}$$

$$\boxed{\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g \\ g &= k \end{aligned}}$$





Example (see picture): Suppose we want to know the extreme values of $f(x,y) = x^2 - y^2$ restricted to only the points (x,y) on the unit circle $x^2 + y^2 = 1$.

We set $g(x,y) = x^2 + y^2$ and require $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$ and $g(x,y) = 1$ which is equivalent to $\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$ and $x^2 + y^2 = 1$.

Then $2x = 2\lambda x \to x = 0$ or $\lambda = 1$. Suppose $x = 0$. Then $y = \pm 1 \to \lambda = -1$. If $x \neq 0$ then $\lambda = 1 \to -2y = 2\lambda y$ becomes $-2y = 2y \to y = 0$, in which case $x = \pm 1$.

When $x = 0$, $y = \pm 1$, we get $f = -1$, which is a minima. When $y = 0$, $x = \pm 1$, we get $f = 1$, which is maxima. This confirms what we see in the picture above.

Consider finding the distance from a point (x_1, y_1, z_1) to a plane $ax + by + cz = d$. Recall that earlier in this class we had a method for finding this distance using orthogonal projection. With Lagrange multipliers we now have another way to do it:

Let $f(x,y,z) = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$ (square distance from (x,y,z) to (x_1, y_1, z_1)). Let $g(x,y,z) = 2ax + 2by + 2cz - 2d$ (for convenience). Clearly there is no maximum value of f on $g = 0$ since (x,y,z) on the plane $g = 0$ can get arbitrarily far away from any fixed point. We find the minimum distance by solving the following system:

$2(x - x_1) = 2a\lambda$, $2(y - y_1) = 2b\lambda$, $2(z - z_1) = 2c\lambda$, & $ax + by + cz - d = 0$.

Then $x = a\lambda + x_1$, $y = b\lambda + y_1$ and $z = a\lambda + z_1$. Then $a(a\lambda + x_1) + b(b\lambda + y_1) + c(c\lambda + z_1) = d$.

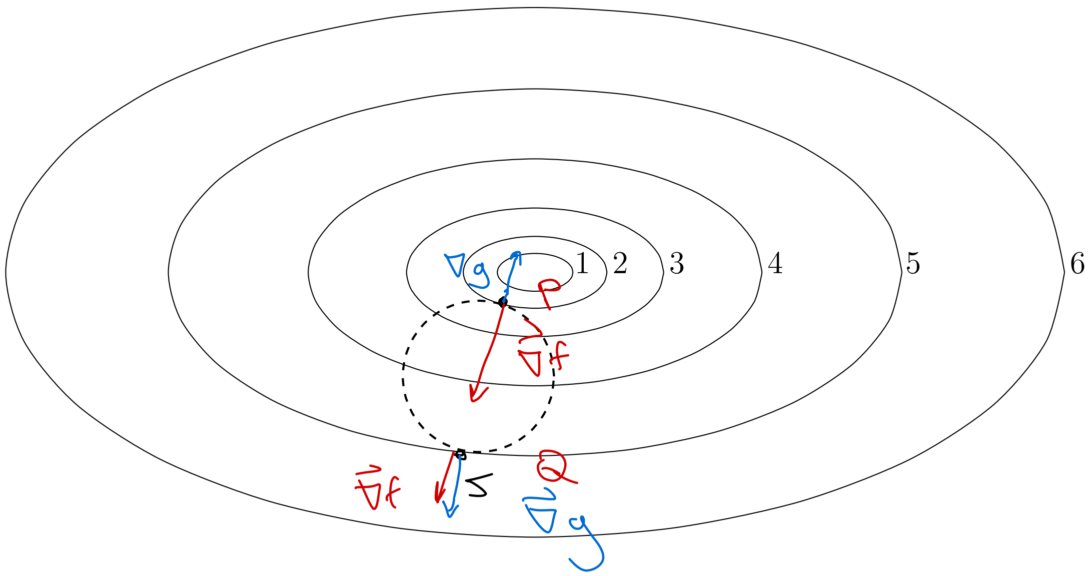
So $(a^2 + b^2 + c^2)\lambda + ax_1 + by_1 + cz_1 = d$

$\lambda = -\dfrac{ax_1 + by_1 + cz_1 - d}{a^2 + b^2 + c^2}$.

Then $x - x_1 = \lambda a$, $y - y_1 = \lambda b$ and $z - z_1 = \lambda c$ for λ above finds (x,y,z) on the plane closest to (x_1, y_1, z_1).

So the distance is $\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = \sqrt{\lambda^2 (a^2 + b^2 + c^2)} = \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

(1) Suppose $f(x, y)$ has following (solid) level curves, the graph of $g(x, y) = 1$ is the dashed curve and that the gradient of $g(x, y)$ evaluated at any point along the curve $g(x, y) = 1$ points towards the outside of this circle.



(a) Let's mark the points where $f(x, y)$ has extreme value(s) and draw possible vectors for gradients of $f(x, y)$ and $g(x, y)$ at these points.

P + Q

(b) What are the extreme values?

min 2 at P

max 5 at Q

(2) Let's find the point on the paraboloid $2z = 2x^2 + 2y^2 + 1$ closest to $(2, 4, 1)$.

optimize distance

(a) Let's determine the square of the distance from (x, y, z) to $(2, 4, 1)$ and call this function $f(x, y, z)$.

$$f = (x-2)^2 + (y-4)^2 + (z-1)^2$$

(b) The surface is the constraint, let's re-express it as $g(x, y, z) = k$ for some function g and constant k.

$$g = -2x^2 - 2y^2 + 2z = 1$$

(c) Let's apply Lagrange Mulipliers to find the closest point on this paraboloid to $(2, 4, 1)$. (Note: the surface is not bounded, there is no farthest point.)

$$\vec{\nabla} f = \langle 2(x-2), 2(y-4), 2(z-1) \rangle$$

$$\vec{\nabla} g = \langle -4x, -4y, 2 \rangle$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$g = 1$$

$$2x - 4 = -4x(\lambda) \quad (x \neq 0)$$

$$2y - 8 = -4y(\lambda) \quad (y \neq 0)$$

$$\lambda = -\frac{1}{2} + \frac{1}{x} = \frac{-x+2}{2x}$$

$$\lambda = -\frac{1}{2} + \frac{2}{y} = \frac{-y+4}{2y}$$

$$2z - 2 = 2\lambda$$

$$\lambda = z - 1$$

$$-2x^2 - 2y^2 + 2z = 1$$

$$y = 2x$$

$$z = \frac{1}{2} + \frac{1}{x}$$

$$-2x^2 - 2(2x)^2 + 2\left(\frac{1}{2} + \frac{1}{x}\right) = 1$$

$$-10x^2 \cancel{+1} + \frac{2}{x} = \cancel{1}$$

$$-10x^2 + \frac{2}{x} = 0$$

$$10x^2 = \frac{2}{x}$$

$$x^3 = \frac{1}{5}$$

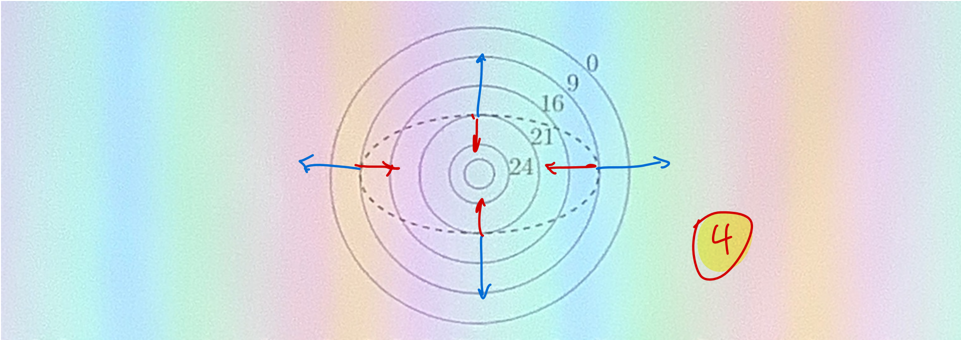
$$x = \frac{1}{\sqrt[3]{5}}$$

$$y = \frac{2}{\sqrt[3]{5}}$$

$$z = \frac{1}{2} + \sqrt[3]{5}$$

1. 21 ?

2.



3. $f(x,y) = x - 4y$ $\quad \vec{\nabla} f \ \langle 1, -4 \rangle$

$g = 2x^2 + y^2 = 66$ $\quad \vec{\nabla} g \ \langle 4x, 2y \rangle$

$$\langle 1, -4 \rangle = \langle 4x, 2y \rangle \lambda$$

$$1 = 4x\lambda$$

$$-4 = 2y\lambda$$

$$-4 = -16x\lambda$$

$$2y\cancel{\lambda} = -16x\cancel{\lambda}$$

$$y = -8x$$

$$2x^2 + (-8x)^2 = 66$$

$$2x^2 + 64x^2 = 66$$

$$x = 1$$

$$y = -8$$

$f(1,-8) = 1 - 4(-8)$

$= 1 + 32$

$= 33$

4. $\vec{\nabla} f = \langle 2x, 4y+8 \rangle$

$\vec{\nabla} g = \langle y, x \rangle$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\langle 2x, 4y+8 \rangle = \langle y, x \rangle \lambda$$

$$2x = y\lambda$$

$$4y+8 = x\lambda$$

$$\frac{2x}{y} = \frac{4y+8}{x}$$

$$2x^2 = 4y^2 + 8y$$

$$0 = 4y^2 + 8y - 2x^2$$

$yx = 8$

$$0 = \cancel{2}(2y^2 + 4y - x^2)$$

$$x^2 \cancel{+4} = (2y^2 + 4y \cancel{+4})$$

$16 + 8 + 16 + 2$

$$x = \sqrt{2y^2 + 4y}$$

$\boxed{42}$

$$\sqrt{2y^2 + 4y} \cdot y = 8$$

$$2y^2 + 4y = \left(\frac{8}{y}\right)^2$$

$\boxed{42}$ goal

$$2y^2 + 4y = \frac{64}{y^2}$$

$$2y^4 + 4y^3 = 64$$

$$2y^3(y+2) = 64$$

$y = 2$

$x = 4$

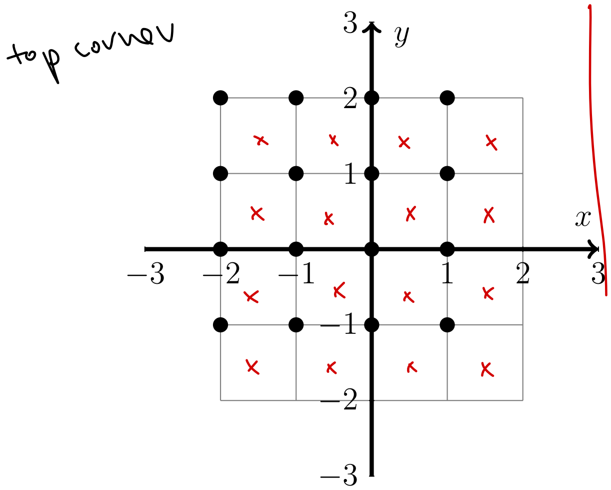
...

Double Integrals

Consider $f(x, y) = x^2 + y^2$. The graph is a paraboloid with a vertex at the origin. Suppose we'd like to know how much volume is bounded between the paraboloid and the xy-plane over the square $S = [-2, 2] \times [-2, 2]$.

Let's estimate it.

Here is one particular estimate: We can split S into 16 sub-rectangles, each of area $\Delta A = 1$, using gridlines at $x = -1, 0, 1$ and at $y = -1, 0, 1$, and then evaluate f at upper-left corners ("sample points") to get an estimate of the value of f over each sub-rectangle.



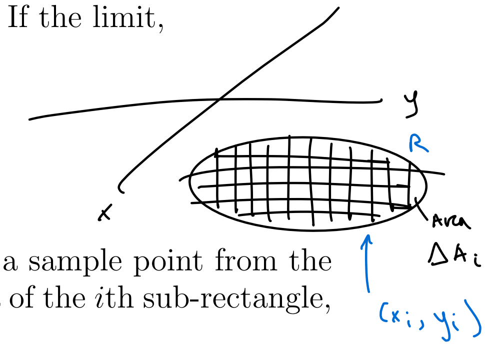
$$\left(\begin{array}{ccc} 4.5 & + & 4.5 \\ 2.5 & + & 2.5 \\ 2.5 & + & 2.5 \\ 0.5 & + & 0.5 \end{array}\right) \times 2 = 40$$

So an estimate for the volume is given by evaluating f at each sample point and multiplying that value of the area of the sub-rectangle:

$V \approx (8 + 5 + 4 + 5 + 5 + 2 + 1 + 2 + 4 + 1 + 0 + 1 + 5 + 2 + 1 + 2) \cdot 1 = 48.$

Such a sum is called a Riemann sum.

Let $f(x, y)$ be defined over a rectangular region R in the xy-plane. If the limit,



$$\lim_{n\to\infty, \Delta A_i \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i,$$

exists for all partitions of R into n sub-rectangles, where (x_i, y_i) is a sample point from the ith sub-rectangle, and ΔA_i (such that $\Delta A_i \to 0$) represents the area of the ith sub-rectangle, then the limit is the double integral of f over R.

Notation: $\displaystyle\int\int_R f(x, y)\, dA.$

So on the previous page, 48 is an estimate of $\displaystyle\int\int_{[-2,2]\times[-2,2]} x^2 + y^2\, dA.$

Since f may sometimes be negative on R, the double integral is the net volume of f over R, which is the volume between $z = f(x, y)$ and $z = 0$ when f is non-negative MINUS the volume between $z = f(x, y)$ and $z = 0$ when f is negative.

Of course the Riemann sum estimates for double-integrals get better with more rectangles! In the following figures, we visualize Riemann sum estimates for $\displaystyle\int\int_{[-2,2]\times[-2,2]} x^2 + y^2\, dA$ made with various numbers of equal-sized sub-rectangles, and rectangle centers as sample points.

Fubini's Theorem (over rect. regions): Suppose $f(x, y)$ is continuous over a rectangular region $R = [x_1, x_2]\times[y_1, y_2]$. Then $\displaystyle\int\int_R f(x, y)\, dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dydx = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\, dxdy.$

(The proof is skipped.)

$R = [a, b] \times [c, d]$

Sometimes we refer to this process as "iterating the integral" and refer to integrals inside of integrals as "iterated integrals."

For example,

$$\int\int_{[-2,2]\times[-2,2]} x^2 + y^2\, dA = \int_{-2}^{2}\int_{-2}^{2} x^2 + y^2\, dydx = \int_{-2}^{2}\left(x^2 y + \frac{1}{3}y^3 \Big|_{y=-2}^{y=2} \right) dx =$$

$$= \int_{-2}^{2}\left(4x^2 + \frac{16}{3}\right) dx = \frac{4}{3}x^3 + \frac{16}{3}x \Big|_{-2}^{2} = \frac{128}{3} \approx 42.6667.$$

$$\int_c^d \int_a^b f(x,y)\, \underbrace{dx\, dy}_{dA} = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Let's evaluate $\displaystyle\iint_{[0,2]\times[1,2]} \frac{x}{(1+xy)^2}\, dA.$

First we choose an order for the iteration. Since it appears to be easier to integrate with respect to y first, we choose the $dydx$ order:

$$\int_0^1 \int_1^2 \frac{x}{(1+xy)^2}\, dydx = \int_0^1 -\frac{1}{1+xy}\bigg|_{y=1}^{y=2} dx = \int_0^1 \frac{1}{1+x} - \frac{1}{1+2x}\, dx = \ln(2) - \frac{1}{2}\ln(3) = \ln\left(\frac{2}{\sqrt{3}}\right).$$

The definition of a double integral for a rectangular region can be extended to general regions that are closed and bounded, and a generalized version of Fubini's Theorem can be applied. There are two basic types of regions:

(1) R is bounded by $a \le x \le b$ and $g(x) \le y \le h(x)$. Then if $f(x,y)$ is continuous over R,
$$\iint_R f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dydx.$$

(2) R is bounded by $c \le y \le d$ and $\ell(y) \le x \le m(y)$. Then if $f(x,y)$ is continuous over R,
$$\iint_R f(x,y)\, dA = \int_c^d \int_{\ell(y)}^{m(y)} f(x,y)\, dxdy.$$

Notes:

-A region might need to be split into more than one of these types of regions above! If a region R is partitioned into subregions R_1 and R_2 then
$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.$$

-If the integrand is 1, the double integral recovers the area of the region (since it's the volume under a surface that is height 1 over the region).

-The average value of $f(x,y)$ over a region R is $\displaystyle\frac{1}{A}\iint_R f(x,y)\, dA$ where A is the area of R.

Let's evaluate $\displaystyle\int_0^4 \int_{x/2}^2 e^{-y^2}\, dydx.$

This one is already presented as an iterated integral, but since we cannot
find an elementary antiderivative of e^{-y^2} with respect to y
we switch the order of integration!

The region described by $0 \le x \le 4$ and $\frac{x}{2} \le y \le 2$
can also be described by $0 \le y \le 2$ and $0 \le x \le 2y$.

So $\displaystyle\int_0^4 \int_{x/2}^2 e^{-y^2}\,dydx = \int_0^2 \int_0^{2y} e^{-y^2}\,dxdy = \int_0^2 2ye^{-y^2}\,dy = -e^{-y^2}\Big|_0^2 = 1 - \frac{1}{e^4}.$

Math 254 **Exemplar Examples 17**

(1) Suppose a research team is trying to estimate the number of trees in a particular 120,000 square-meter patch of a forest. Starting from a particular point, samples of the tree densities (trees per square meter) are taken on a grid with sample points at every NE corner in a grid using gridlines at every 100 meters east and 100 meters north, over a region that goes 400 meters east and 300 meters north from the starting point. The data is put into a table:

northward $\longrightarrow$ east (x)

	100	200	300	400
300	0.5	1.2	0.8	1
200	0.7	0.5	1.4	0.9
100	1	0.6	0.4	0.2

(y)

(a) Suppose we had a function $f(x, y)$ which gave the density of trees, in trees per square meter, at x meters east and y meters north of the starting point for the survey. What double integral would yield the number of trees in this 120,000 square-meter patch of a forest?

$$\int_0^{400} \int_0^{300} f(x,y)\, dy\, dx$$

(b) Let's use a Riemann sum to estimate the total number of trees in the 120,000 square-meter patch of forest.

$\Delta A = 100^2$ for each 100×100

$$(1 + 0.6 + 0.4 + 0.2 + 0.7 + 0.5 + 1.4 + 0.9 + 0.5 + 1.2 + 0.8 + 1) \cdot 10^4$$

$= 92{,}000$

(2) Let's evaluate $\int_0^1 \int_1^{\sqrt{3}} \frac{y}{(x^2+y^2)^2}\, dx\, dy$.

$$\int_1^{\sqrt{3}} \int_0^1 \frac{y}{(x^2+y^2)^2}\, dy\, dx$$

$$= \int_1^{\sqrt{3}} -\frac{1}{2(x^2+y^2)} \Bigg|_{y=0}^{y=1} dx$$

$$= \int_1^{\sqrt{3}} -\frac{1}{2} \frac{1}{x^2+1} + \frac{1}{2} \cdot \frac{1}{x^2}\, dx$$

$$= -\frac{1}{2} \tan^{-1}(x) - \frac{1}{2} \cdot \frac{1}{x} \Bigg|_1^{\sqrt{3}}$$

$$= -\frac{1}{2} \frac{\pi}{3} - \frac{1}{2} \cdot \frac{1}{\sqrt{3}} - \left(-\frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \right)$$

$$= 0.08$$

$u = x^2 + y^2$

$du = 2y\, dy$

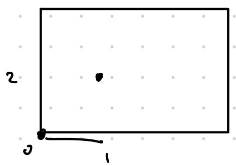
$\frac{1}{2} du = y\, dy$

$\int \frac{\frac{1}{2} du}{u^2} =$

$-\frac{1}{2} u^{-1} + C$

1. ~~48~~ 40

2. Find a



$R = [0, 1] \times [0\ 2]$

$$\int_0^2 \int_0^1 \frac{ay}{(1+xy)^3}\, dx\, dy = 1$$

$u = 1 + xy$

$du = y$

$\frac{1}{u^3} \cdot du$

u^{-3}

$\frac{-1}{u^2}(y)$

$$\int_0^2 -\frac{a}{(1+xy)^2} \cdot \frac{1}{2} \Big|_{x=0}^{x=1} dy = 1$$

$$-\frac{1}{2} \frac{a}{(1+xy)^2} \Big|_{x=0}^{x=1}$$

$$\left[-\frac{1}{2} \frac{a}{(1+y)^2} - \frac{1}{2} a \right]$$

$$\int_0^2 \left(-\frac{1}{2} \frac{a}{(1+y)^2} - \frac{1}{2} a \right) dy = 1$$

$$\int_0^2 \left[-\frac{1}{2} a \cdot (1+y)^{-2} - \frac{1}{2} a \right] dy = 1$$

$$\frac{1}{2} a (1+y)^{-1} - \frac{1}{2} a y \Big|_0^2 = 1$$

$$\left[\frac{1}{2} \cdot \frac{1}{3} a - \frac{1}{2} a (2) \right] - \left[\frac{1}{2} a - \right] = 1$$

$$\frac{1}{6} a - a - \frac{1}{2} a = 1$$

$$\frac{1}{6} - \frac{12}{12} - \frac{6}{12} = 1$$

$a = 1.5$

3.

$f \quad [0,1] \times [0,1]$

$f(x) = x^2 + y^2$

$$\int_0^1 \int_0^1 x^2 + y^2 \, dx \, dy$$

$$\int_0^1 \frac{1}{3}x^3 + y^2 x \Big|_{x=0}^{x=1} dy$$

$$\int_0^1 \frac{1}{3} + y^2 \, dy$$

$$\frac{1}{3}y + \frac{1}{3}y^3 \Big|_{y=0}^{y=1}$$

$$\frac{1}{3} + \frac{1}{3}$$

$$\boxed{\frac{2}{3}}$$

4.

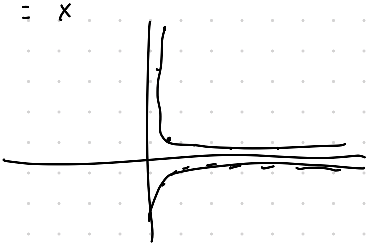
i iii vii

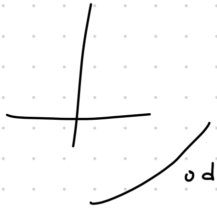
v. vi

4

$\frac{1}{y} = x$

$y = \frac{1}{x}$





odd y

i. $\iint_D xy \, dA$ ✓ ✓

ii $\iint_D x^2 \, dA$ ✗ ✗

iii $\iint_D x^4 y^3 \, dA$ ✓ ✓

iv. $\iint_D e^{xy} \, dA$ ✗ ✗ $-e^{xy} \neq e^{x \cdot -y}$

v. $\iint_D e^x \sin\left(\frac{\pi}{8} y\right) dA$ ✓ ✓

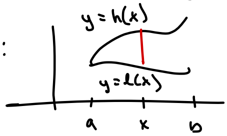
vi. $\iint_D \frac{x^2 y}{1+x^2 y^2} \, dA$ ✓ ✓

vii $\iint_D \frac{x(y+1)^2}{1+x^2 y^2} \, dA$ ✓ ✗

$$\frac{x(y^2+2y+1)}{1+x^2y^2} \qquad -\frac{x^2 y}{1+x^2 y} = \frac{x^2(-y)}{1+x^2 y}$$

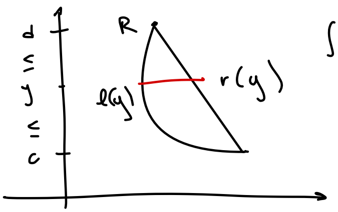
Double Integrals over General Regions

I:

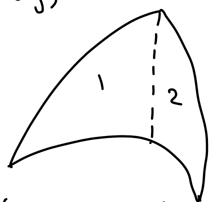


$$\iint_R f(x,y)\,dA = \int_a^b \int_{l(x)}^{h(x)} f(x,y)\,dy\,dx$$

II:



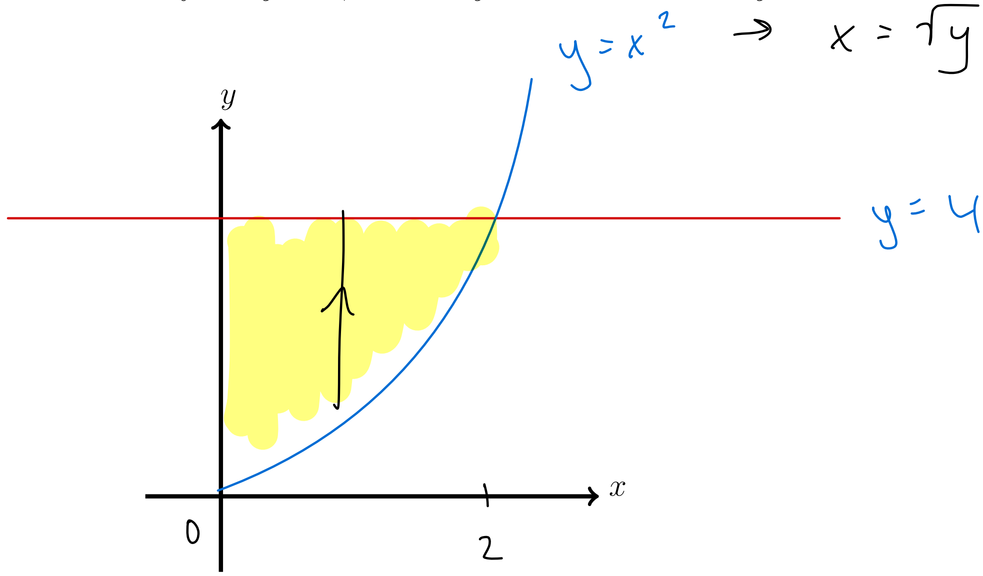
$$\iint_R f(x,y)\,dA = \int_c^d \int_{l(y)}^{r(y)} f(x,y)\,dx\,dy$$



$$\iint_R f\,dA = \iint_{R1} f\,dA + \iint_{R_2} f\,DA$$

(1) Let's consider $\iint_R f(x,y)\,dA$ where f is a continuous function and R is the region in the first quadrant bounded by the y-axis, the line $y = 4$ and the curve $y = x^2$.

(a) Sketch R.



(b) Write down an iterated double integral that is equal to $\iint_R f(x,y)\,dA$ in the $dA = dy\,dx$ order by describing the region above in the form $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$.

$$0 \leq x \leq 2$$

$$x^2 \leq y \leq 4$$

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx$$

(c) Write down an iterated double integral that is equal to $\iint_R f(x,y)\,dA$ in the $dA = dx\,dy$ order by describing the region above in the form $a \leq y \leq b$ and $g(y) \leq x \leq h(y)$.

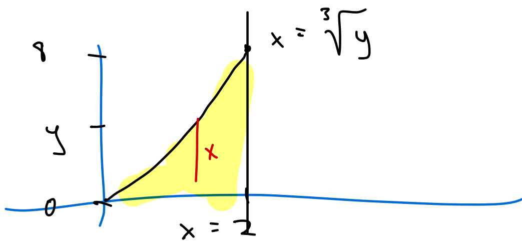
$$0 \leq y \leq 4$$

$$0 \leq x \leq \sqrt{y}$$

$$\iint_R f(x,y)\,dA = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$

(2) Let's evaluate $\int_0^8 \int_{y^{1/3}}^2 \sqrt{1+x^4}\,dx\,dy$ by swapping the order of integration.

$$\int_{y^{1/3}}^{2} \int_0^8 \sqrt{1+x^4}\;dy\,dx$$



$0 \le x \le 2$

$0 \le y \le x^3$

$$\int_0^2 \int_0^{x^3} \sqrt{1+x^4}\;dy\,dx$$

$$\int_0^2 y\sqrt{1+x^4}\,\Big|_{y=0}^{y=x^3}\;dx$$

$$\int_0^2 x^3\sqrt{1+x^4}\;dx$$

$u = 1+x^4$

$du = 4x^3$

$\frac{du}{4} = x^3$

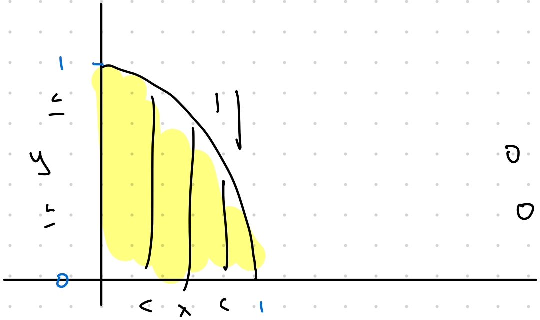
$\int \sqrt{u}\cdot\frac{du}{4}$

$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C$

$$= \frac{1}{6}\left(1+x^4\right)^{3/2}\Big|_0^2$$

$$= \frac{1}{6}\left(17\sqrt{17}-1\right)$$

1.



$$0 \le x \le 1$$
$$0 \le y \le \sqrt{1-x^2}$$

$$\iint_R y\sqrt{x^2+y^2} \qquad y = \sqrt{1-x^2}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$$

$$u = x^2+y^2$$
$$du = 2y$$
$$\frac{du}{2} = y$$

$$\int_0^{\sqrt{1-x^2}} \frac{1}{2}\sqrt{u}\,du$$

$$\frac{1}{2}u^{1/2}$$

$$\frac{2}{3}\frac{1}{2}u^{3/2} + C$$

$$\frac{1}{3}u^{3/2}\Big|_0^{\sqrt{1-x^2}}$$

☆

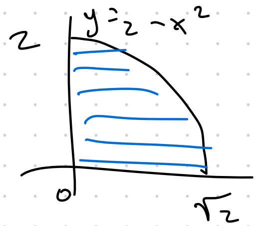
$$\frac{1}{3}\left(x^2 + (1-x^2)\right)^{3/2} - \frac{1}{3}\left(x^2 + 0^2\right)^{3/2}$$

$$\frac{1}{3}\int_0^1 (1 - x^3)\,dx$$

$$\frac{1}{3}\left[x - \frac{x^4}{4}\right]_0^1$$

$$\frac{1}{3}\left[1 - \frac{1}{4}\right] = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

2. $\displaystyle\int_0^{\sqrt{2}} \int_0^{2-x^2} \frac{x e^{2y}}{2-y} \, dy \, dx$



$$\int_0^{2-x^2} \int_0^{\sqrt{2}} \frac{x e^{2y}}{2-y} \, dx \, dy$$

$$\int_0^{2} \int_0^{\sqrt{2}} \frac{x e^{2y}}{2-y} \, dx \, dy$$

$x^2 = 2 - y$

$x = \sqrt{2-y}$

$$\int_0^{2} \int_0^{\sqrt{2-y}} \frac{x e^{2y}}{2-y} \, dx \, dy$$

$$\frac{1}{2} \frac{x^2 \cdot e^{2y}}{(2-y)} \Bigg|_0^{\sqrt{2-y}}$$

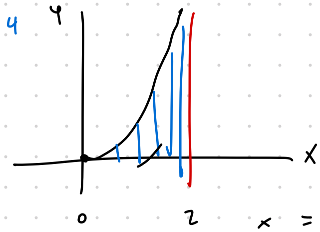
$$\int_0^{2} \frac{\cancel{2-y}}{2} \cdot \frac{e^{2y}}{\cancel{(2-y)}} \, dy$$

$$\int_0^{2} \frac{e^{2y}}{2} \, dy$$

$$\frac{1}{4} e^{2y} \Bigg|_0^{2} = \frac{e^4}{4} - \frac{1}{4}$$

$$= 13.4$$

3. $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{y}{x^5+1}\, dx\, dy$



$x = 0 \quad x = 2$

$\sqrt{y} = x$

$y = x^2$

$$\int_0^2 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx$$

$$= \int_0^2 \frac{1}{2} \frac{y^2}{x^5+1} \Bigg|_0^{x^2} dx$$

$$= \int \frac{1}{2} \frac{(x^2)^2}{x^5+1} = \int_0^2 \frac{1}{2} \frac{x^4}{x^5+1}\, dx$$

$$= \frac{1}{2} \int_0^2 \frac{x^4}{x^5+1}\, dx$$

$u = x^5 + 1$

$du = 5x^4\, dx$

$dx = \dfrac{du}{5x^4}$

$$= \frac{1}{2} \int_1^{33} \frac{1}{u} \frac{1}{5}\, du$$

$$= \frac{1}{10} \left[\ln(33) - \overset{0}{\cancel{\ln(1)}} \right]$$

$$= 0.35$$

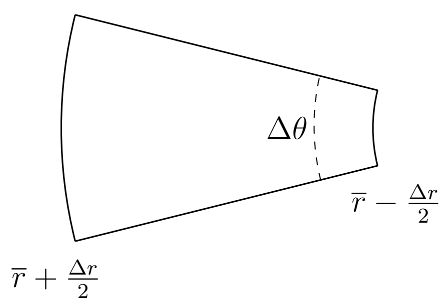
...

Double Integrals in Polar Coordinates

Consider the iterated integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$. As an exercise left for the viewer, attempt to evaluate this iterated integral as written. It turns out that it is much more easily done in polar coordinates. But first we need some theory!

A polar "sub-rectangle" can be given by the bounds, $\overline{r} - \frac{\Delta r}{2} \le r \le \overline{r} + \frac{\Delta r}{2}$ and $\alpha \le \theta \le \alpha + \Delta\theta$. To compute the area of these "sub-rectangles" we use the formula for the area of a sector, which is equal to the fraction $\frac{\Delta\theta}{2\pi}$ of the area of the circle πr^2: $A = \frac{\Delta\theta}{2\pi}(\pi r^2) = \frac{r^2 \Delta\theta}{2}$.

Let's determine that area ΔA bounded by $\overline{r} - \frac{\Delta r}{2} \le r \le \overline{r} + \frac{\Delta r}{2}$ with a central angle of $\Delta\theta$:

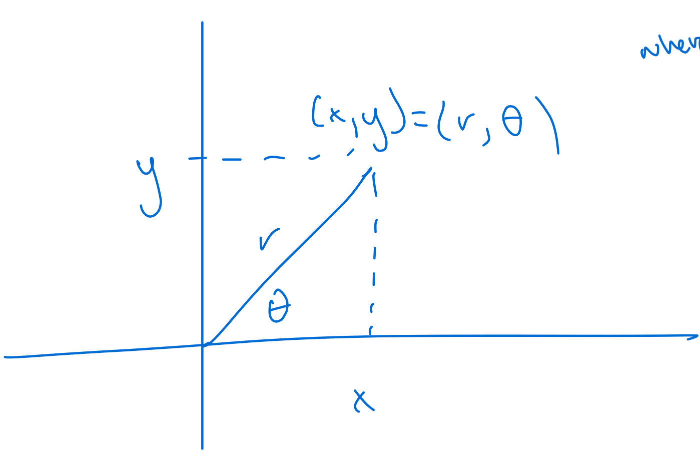


$$\Delta A = \frac{(\overline{r} + \frac{\Delta r}{2})^2 \Delta\theta}{2} - \frac{(\overline{r} - \frac{\Delta r}{2})^2 \Delta\theta}{2} = \overline{r}\Delta r \Delta\theta.$$

As $\Delta r \to 0$ and $\Delta\theta \to 0$, $\overline{r} \to r$. Thus in polar coordinates, $\boxed{dA = r\, dr\, d\theta}$.

Of course $f(x, y)$ needs to be converted to $f(r, \theta)$ by using $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

So $\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = \frac{\pi}{2}(1/2 - 1/4) = \frac{\pi}{8}$.



where $x^2 + y^2 = r^2$ $\left(r = \sqrt{x^2+y^2}\right)$

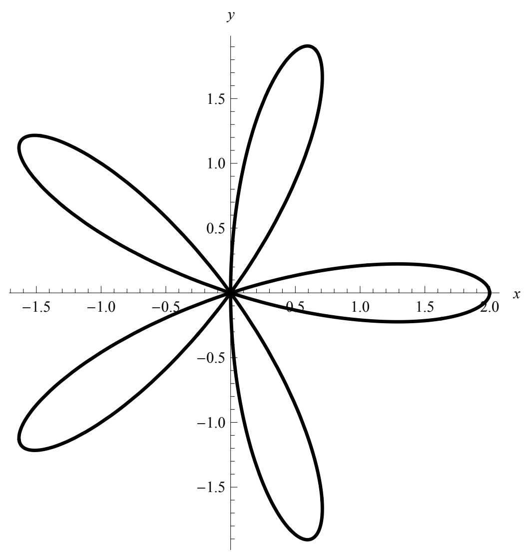
and $x \tan\theta = y$

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, \underbrace{(r\, dr\, d\theta)}_{dA}$$

If a polar region R is given by $g(\theta) \le r \le h(\theta)$ and $\alpha \le \theta \le \beta$ then

$$\int\int_R f(r,\theta)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

Let's find the area of the region bounded by one "leaf" of the rose $r = 2\cos(5\theta)$:



Notice that $\cos(5\theta)$ is an even function. We find the first time after $\theta = 0$ that the function equals 0. That occurs at $\theta = \frac{\pi}{10}$. Hence the area is (using symmetry)

$$2\int_0^{\frac{\pi}{10}} \int_0^{2\cos(5\theta)} r\, dr\, d\theta = 2\int_0^{\frac{\pi}{10}} \frac{1}{2}(2\cos(5\theta))^2 d\theta = 4\int_0^{\frac{\pi}{10}} \frac{1}{2}(1+\cos(10\theta))d\theta = \frac{\pi}{5}.$$

Let's evaluate the integral of $\displaystyle\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dA$ where R is the region given by $1 \le x \le \sqrt{4-y^2}$.

The region is to the right of $x = 1$, which in polar coordinates is $r = \sec(\theta)$, and to the left of $x^2 + y^2 = 4$, which in polar coordinates is $r = 2$. These curves intersect when $\sec(\theta) = 2$, which occurs at $\theta = \pm\frac{\pi}{3}$.

So the integral is $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\sec(\theta)}^{2} \frac{1}{r} r\, dr\, d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2 - \sec(\theta)\, d\theta = 2\left(\frac{2\pi}{3} - \ln(2+\sqrt{3})\right)$, using even-function symmetry.

The graph of $f(x) = e^{-x^2}$ has the classic "bell-shape" to it. Suppose we would like to know the area under this bell-shaped curve on $(-\infty, \infty)$. That is, $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \lim_{t\to\infty} \int_{-t}^{t} e^{-x^2}\, dx = \lim_{t\to\infty} \sqrt{\int_{-t}^{t} e^{-x^2}\, dx \int_{-t}^{t} e^{-y^2}\, dy} =$$

$$= \lim_{t\to\infty} \sqrt{\int_{-t}^{t}\int_{-t}^{t} e^{-x^2-y^2}\, dx dy} = \lim_{t\to\infty} \sqrt{\int_{0}^{2\pi}\int_{0}^{t} e^{-r^2} r dr d\theta} =$$

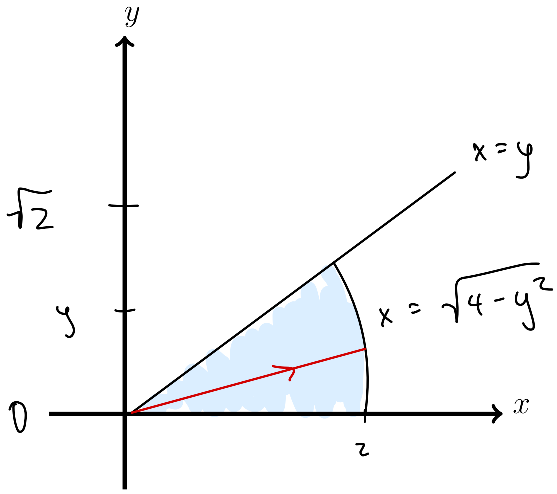
$$= \lim_{t\to\infty} \sqrt{\int_{0}^{2\pi} \left(-\frac{1}{2}\left(e^{-t^2}-1\right)\right) d\theta} = \lim_{t\to\infty} \sqrt{-\pi\left(e^{-t^2}-1\right)} = \sqrt{\pi}.$$

(1) Let's consider $\iint_R e^{\sqrt{x^2+y^2}}\, dA$ where $R = \{(x,y) | 0 \le y \le \sqrt{2}, y \le x \le \sqrt{4-y^2}\}$.

(a) Sketch R.

$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$



(b) Let's write down an iterated double integral that is equal to $\iint_R e^{\sqrt{x^2+y^2}}\, dA$ in rectangular coordinates in the $dA = dx\, dy$ order. Can we integrate this? Could we if we swapped the order?

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$$

??

(c) Now let's write down an iterated double integral that is equal to $\iint_R e^{\sqrt{x^2+y^2}}\, dA$ in polar coordinates in the $dA = r\, dr\, d\theta$ order.

$r\, dr\, d\theta$

$0 \le \theta \le \frac{\pi}{4}$

$0 \le r \le 2$

$$\int_0^{\pi/4} \underbrace{\int_0^2 e^r\, r\, dr}_{\text{no } \theta}\, d\theta$$

(d) This latter one we can integrate! Let's evaluate it.

int by parts

$u = r \qquad du = dr$

$dv = e^r\, dr \qquad v = e^r$

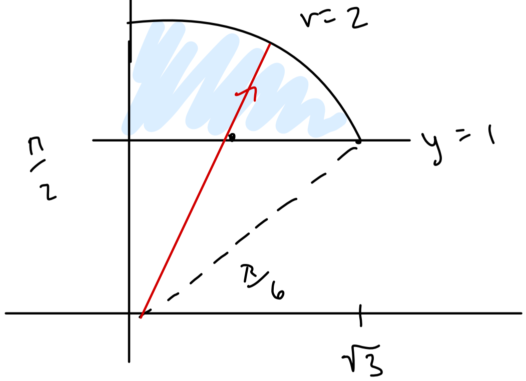
$\int u\, dv = uv - \int v\, du$

$$\frac{\pi}{4} \int_0^2 e^r r\, dr$$

$$= \frac{\pi}{4}\left(re^r - \int e^r\, dr \right)\Big|_0^2$$

$$= \frac{\pi}{4}\left(2e^2 - e^2 + 1 \right)$$

(2) Let's evaluate $\int_0^{\sqrt{3}}\int_1^{\sqrt{4-x^2}} \sqrt{x^2+y^2-1}\,dy\,dx$ by converting it to polar coordinates.



$y=1 \leftarrow r\sin\theta = 1$

$r = \csc\theta$

$\frac{\cos^2}{\sin^2} + \frac{\sin^2}{\sin^2} = \frac{1}{\sin^2}$

$\cot^2 + 1 = \csc^2$

$$\int_{\pi/6}^{\pi/2}\int_{\csc\theta}^{2} \sqrt{r^2-1}\,(r\,dr\,d\theta)$$

$$\int_{\pi/6}^{\pi/2} \frac{1}{3}(r^2-1)^{3/2}\Big|_{\csc\theta}^{2}\,d\theta$$

$$= \frac{1}{3}\int_{\pi/6}^{\pi/2} 3^{3/2} - (\csc^2\theta - 1)^{3/2}\,d\theta$$

$\hookrightarrow \cot^3\theta$

$$= \frac{1}{\cancel{3}}\left[\cancel{3}\sqrt{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) - \int_{\pi/6}^{\pi/2}\frac{\cos^3\theta}{\sin^3\theta}\,d\theta\right]$$

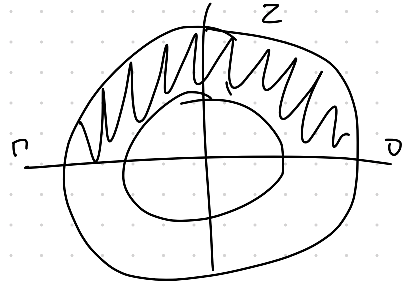
$$= \sqrt{3}\cdot\frac{\pi}{3} - \frac{1}{3}\int_{\pi/6}^{\pi/2}\frac{\cos\theta\cdot(1-\sin^2\theta)}{\sin^3\theta}\,d\theta$$

$$= \frac{\pi\sqrt{3}}{3} - \frac{1}{3}\int_{1/2}^{1}\frac{1-u^2}{u^3}\,du$$

1. $\iint_R \frac{1}{x^2+y^2}$ $\quad 1 \le r^2 \le 4$ $\quad x^2+y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$$\int_0^{\pi}\int_1^2 \frac{1}{r^2}\, r\, dr\, d\theta$$



$$\int_0^{\pi}\underbrace{\int_1^2 \frac{1}{r}\, dr}\, d\theta$$

$$\pi\left(\ln(r)\Big|_1^2 \right)$$

$0 < \theta < \pi$

$1 < r < 2$

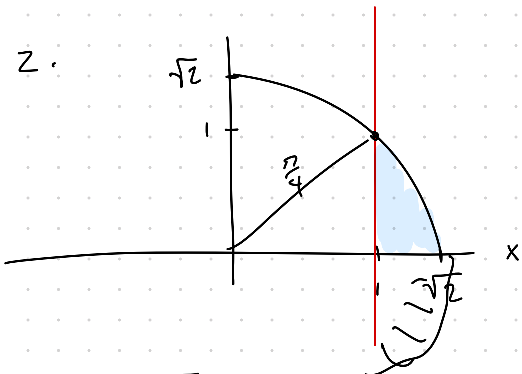
$$\pi\left(\ln(2) - \ln(1)\right)$$

$$\pi \ln 2 = 2.18$$

$x^2+y^2 = 2$

$r = \sqrt{2}$

2.



$y = r\sin\theta$

$0 \le \theta \le \frac{\pi}{4}$

$\frac{1}{\cos\theta} \le r \le \sqrt{2}$

$$2 \cdot \int_0^{\pi/4} \int_{\frac{1}{\cos\theta}}^{\sqrt{2}} 1 \quad r\, dr\, d\theta$$

$x = 1$

$x = r\cos\theta = 1$

$r = \frac{1}{\cos\theta}$

$$2 \cdot \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_{\frac{1}{\cos\theta}}^{\sqrt{2}} d\theta$$

$$2 \int_0^{\pi/4} \frac{2}{2} - \frac{\sec^2\theta}{2}\, d\theta = 2 \cdot \int_0^{\pi/4} 1 - \frac{\sec^2\theta}{2}\, d\theta$$

$$2\int_0^{\pi/4} 1-\frac{\sec^2\theta}{2}\,d\theta$$

$$\int_0^{\pi/4} 2-\sec^2\theta\,d\theta$$

$$\left[2\theta-\tan\theta\right]_0^{\pi/4}$$

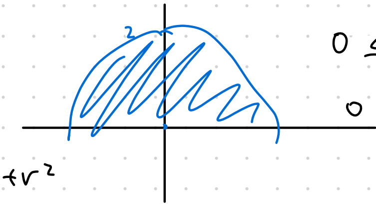
$$\left(\frac{\pi}{2}-\tan\frac{\pi}{4}\right)-\left(0-\tan 0\right)$$

$$=0.57$$

3. $\iint_R \frac{1}{1+x^2+y^2}\,dA$

$y=0 \qquad y=\sqrt{4-x^2}$

$y^2+x^2=4$



$0 \le r \le 2$

$0 \le \theta \le \pi$

$$\int_0^{\pi}\int_0^{2} \frac{1}{1+r^2}\, r\,dr\,d\theta$$

$$\int_0^{2} \frac{r}{1+r^2}\,dr$$

$u = 1+r^2$

$du = 2r\,dr$

$r\,dr = \frac{1}{2}\,du$

$$\int_1^{5} \frac{1}{u}\,\frac{1}{2}\,du$$

#4 $\boxed{0.06}$

$$\int_0^{\pi} \frac{1}{2}\left[\ln u\right]_1^5\,d\theta$$

$$\pi\left(\frac{1}{2}\left(\ln 5 - \ln 1\right)\right) = \frac{\pi}{2}\left(\ln 5\right)$$

$$= 2.53$$

...

Triple Integrals

Suppose a large solid box occupies the region $[0,2]\times[0,2]\times[0,1]$ in xyz-space, and that the density of the object is variable, modeled by $\rho(x,y,z)=\frac{10x+10y}{3-2z}$ in kg per cubic meter at (x,y,z) (assume x,y,z are in meters). Suppose we would like to estimate the mass of the object!

We can divide $[0,2]\times[0,2]\times[0,1]$ into 4 sub-cubes of volume $\Delta V=1$ cubic meter by slicing the object with the planes $x=1$ and $y=1$. Then we can use, as "sample points," sub-cube centers at $(0.5,0.5,0.5)$, $(1.5,0.5,0.5)$, $(0.5,1.5,0.5)$ and $(1.5,1.5,0.5)$ to estimate the mass:

$$m\approx\left(\frac{5+5}{3-2(0.5)}+\frac{15+5}{3-2(0.5)}+\frac{5+15}{3-2(0.5)}+\frac{15+15}{3-2(0.5)}\right)\cdot 1=40.$$

This is an example of a Riemann sum of a function of three variables.

Let $f(x,y,z)$ be defined over a solid region E in xyz-space. If the limit,

$$\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i,y_i,z_i)\Delta V_i,$$

exists for all partitions of E into n sub-solids,where (x_i,y_i,z_i) is a sample point from the ith sub-solid, and ΔV_i (such that $\Delta V_i\to 0$) is the volume of the ith sub-solid, then the limit is the triple integral of f over E.

Notation:

$$\int\int\int_E f(x,y,z)dV.$$

Fubini's theorem: We can iterate the integral when f is continuous over E.

Let's see how good our mass estimate was: The mass of the large box is

$$
\begin{aligned}
m = 10\int_0^1\int_0^2\int_0^2 \frac{x+y}{3-2z}\,dx\,dy\,dz &= 10\int_0^1\int_0^2 \left.\frac{0.5x^2+xy}{3-2z}\right|_{x=0}^{x=2} dy\,dz \\
&= 10\int_0^1\int_0^2 \frac{2+2y}{3-2z}\,dy\,dz \\
&= 10\int_0^1 \left.\frac{2y+y^2}{3-2z}\right|_{y=0}^{y=2} dz \\
&= 10\int_0^1 \frac{8}{3-2z}\,dz \\
&= 10\left(-4\ln(3-2z)|_0^1\right) \\
&= 40\ln(3) \approx 44\text{kg}
\end{aligned}
$$

There are six basic types of solids (sometimes a solid has to be broken into basic pieces):

(1) E: $a \le x \le b$, $g(x) \le y \le h(x)$, $u(x,y) \le z \le v(x,y)$.

$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(x)}^{h(x)}\int_{u(x,y)}^{v(x,y)} f(x,y,z)dzdydx.$$

(2) E: $a \le x \le b$, $g(x) \le z \le h(x)$, $u(x,z) \le y \le v(x,z)$.

$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(x)}^{h(x)}\int_{u(x,z)}^{v(x,z)} f(x,y,z)dydzdx.$$

(3) E: $a \le y \le b$, $g(y) \le x \le h(y)$, $u(x,y) \le z \le v(x,y)$.

$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(y)}^{h(y)}\int_{u(x,y)}^{v(x,y)} f(x,y,z)dzdxdy.$$

(4) E: $a \le y \le b$, $g(y) \le z \le h(y)$, $u(y,z) \le x \le v(y,z)$.

$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(y)}^{h(y)}\int_{u(y,z)}^{v(y,z)} f(x,y,z)dxdzdy.$$

(5) E: $a \le z \le b$, $g(z) \le x \le h(z)$, $u(x,z) \le y \le v(x,z)$.

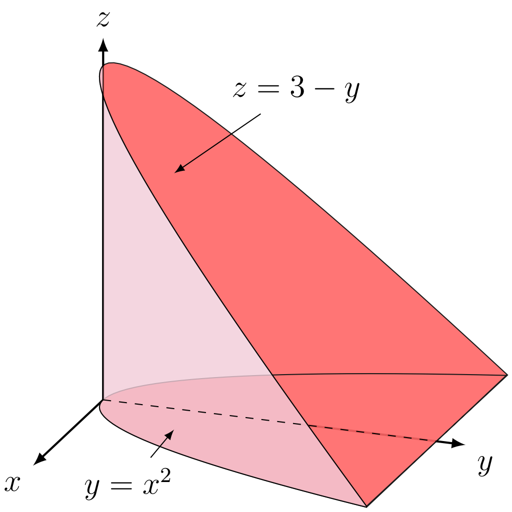
$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(z)}^{h(z)}\int_{u(x,z)}^{v(x,z)} f(x,y,z)dydxdz.$$

(6) E: $a \le z \le b$, $g(z) \le y \le h(z)$, $u(y,z) \le x \le v(y,z)$.

$$\int\int\int_E f(x,y,z)dV = \int_a^b\int_{g(z)}^{h(z)}\int_{u(y,z)}^{v(y,z)} f(x,y,z)dxdydz.$$

If the integrand is 1 then the triple integral finds the volume of E. Also, the average value of $f(x, y, z)$ over E is $\dfrac{1}{V}\iiint_E f(x, y, z)\, dV$ where V is the volume of E.

Let's find the average value of $f(x, y, z) = x^2$ over the solid E inside the parabolic cylinder $y = x^2$ between the planes $z = 3 - y$ and $z = 0$.



First let's calculate the volume (observing the symmetry across the yz-plane).

$$\begin{aligned}
V = \iiint_E 1\, dV &= 2\int_0^{\sqrt{3}}\int_{x^2}^{3}\int_0^{3-y} 1\, dz\, dy\, dx \\
&= 2\int_0^{\sqrt{3}}\int_{x^2}^{3} 3 - y\, dy\, dx \\
&= 2\int_0^{\sqrt{3}} 3y - \frac{1}{2}y^2\Big|_{y=x^2}^{y=3}\, dx \\
&= 2\int_0^{\sqrt{3}} \frac{9}{2} - 3x^2 + \frac{1}{2}x^4\, dx \\
&= 2\left(\frac{9}{2}(\sqrt{3}) - (\sqrt{3})^3 + \frac{1}{10}(\sqrt{3})^5\right) = 2\sqrt{3}\left(\frac{9}{2} - 3 + \frac{9}{10}\right) = \frac{24\sqrt{3}}{5}.
\end{aligned}$$

Now let's calculate the triple integral of $f(x, y, z) = x^2$ over the solid region E (observing the symmetry across the yz-plane and that f is an even function with respect to x).

$$
\begin{aligned}
V = \iiint_E x^2\,dV &= 2\int_0^{\sqrt{3}}\int_{x^2}^{3}\int_0^{3-y} x^2\,dz\,dy\,dx \\
&= 2\int_0^{\sqrt{3}}\int_{x^2}^{3} x^2(3-y)\,dy\,dx \\
&= 2\int_0^{\sqrt{3}} x^2\left(3y - \frac{1}{2}y^2\Big|_{y=x^2}^{y=3}\right)dx \\
&= 2\int_0^{\sqrt{3}} \frac{9}{2}x^2 - 3x^4 + \frac{1}{2}x^6\,dx \\
&= 2\left(\frac{3}{2}(\sqrt{3})^3 - \frac{3}{5}(\sqrt{3})^7 + \frac{1}{14}(\sqrt{3})^7\right) = 2\sqrt{3}\left(\frac{9}{2} - \frac{27}{5} + \frac{27}{14}\right) = \frac{72\sqrt{3}}{35}.
\end{aligned}
$$

So the average value of $f(x, y, z) = x^2$ over the solid region E is $\dfrac{72\sqrt{3}/35}{24\sqrt{3}/5} = \dfrac{3}{7}$.

What if we decided to evaluate the integral above in the $dy\,dx\,dz$ order instead? Well, the bounds would be $0 \le z \le 3$, $-\sqrt{3-z} \le x \le \sqrt{3-z}$ and $x^2 \le y \le 3-z$.

$$
\begin{aligned}
V = \iiint_E x^2\,dV &= 2\int_0^{3}\int_0^{\sqrt{3-z}}\int_{x^2}^{3-z} x^2\,dy\,dx\,dz \\
&= 2\int_0^{3}\int_0^{\sqrt{3-z}} x^2(3-z) - x^4\,dx\,dz \\
&= 2\int_0^{3} \frac{1}{3}(\sqrt{3-z})^3(3-z) - \frac{1}{5}(\sqrt{3-z})^5\,dz \\
&= 2\int_0^{3} \frac{1}{3}(3-z)^{5/2} - \frac{1}{5}(3-z)^{5/2}\,dz \\
&= 2\int_0^{3} \frac{2}{15}(3-z)^{5/2}\,dz \\
&= 2\left(-\frac{4}{105}(3-z)^{7/2}\Big|_0^3\right) = 2\,\frac{4(27\sqrt{3})}{105} = \frac{72\sqrt{3}}{35}.
\end{aligned}
$$

Monday Nov 15

Let E be a solid region in xyz space, which will be a rect. box

Divide into subboxes $E_1, E_2, \ldots E_n$ of volumes $\Delta V_1, \Delta V_2, \ldots \Delta V_n$
choose from each a sample point (x_i, y_i, z_i)

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i \qquad \text{Reimann sum}$$

$$\iiint_E f(x,y,z)\,dV = \lim_{\substack{n\to\infty \\ \Delta V_i \to 0}} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i$$

Fubini's

$$\int_a^b \int_c^d \int_e^m f(x,y,z)\,\underbrace{dz\,dy\,dx}_{\text{5 other ways}}$$

Wednesday Nov 17

Triple Integrals over general regions :
6 basic types

$E = \{(x,y,z) \mid a \le x \le b,\ g(x) \le y \le h(x),\ u(x,y) \le z \le v(x,y)\}$

Then $\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\,dy\,dx$

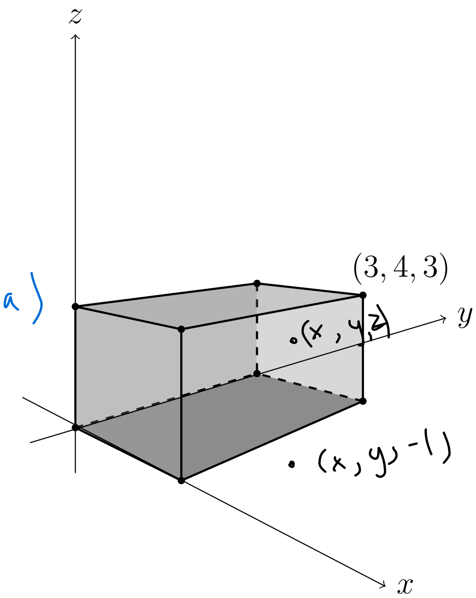
$E = \{(x,y,z) \mid a \le y \le b,\ g(y) \le z \le h(y),\ u(y,z) \le x \le v(y,z)\}$

Then $\iiint_E f(x,y,z)\,dV = \int \int \int \quad dx\,dz\,dy$

If an object occupies a solid region E and the density of the object at (x, y, z) in E is given by a continuous function $\rho(x, y, z)$ then one can find the mass of object using $m = \iiint_E \rho \, dV$.

(a) Suppose at each point (x, y, z) in the block, shown below, the density of the block in grams per cubic centimeters at (x, y, z) is given by $10 - xz$. Assume x, y, z are in units of centimeters. Let's find the mass of this block, in grams.

$\int_a^b$ ✱ $d\ell = (b-a)$



$d = z + 1$

$$\int_0^3 \int_0^4 \int_0^3 10 - zx \; dz \, dy \, dx$$

$$\int_0^3 \int_0^4 10z - \frac{z^2}{2} x \Big|_0^3 \; dy \, dx$$

$$\int_0^3 4\left(30 - \frac{9}{2} x\right) \; dx$$

$$\int_0^3 120 - 18x \; dx = 120x - \frac{18}{2} x^2 \Big|_0^3$$

$= 360 - 81$

$= 279$

(b) Suppose instead that the density of the block above was inversely proportional to the distance (x, y, z) is from the plane $z = -1$ and that the density at the top of the block is 3 grams per cubic centimeters. Let's find the mass of the block.

$$\rho = \frac{k}{z+1} \leftarrow 3 = \frac{k}{3+1} \text{ so } k = 12$$

$$m = \int_0^3 \int_0^4 \int_0^3 \frac{12}{z+1} \; dz \, dy \, dx = 12 \int_0^3 \frac{12}{z+1} \; dz$$

$= 144 \ln(4)$

≈ 200 g

1. $\int_1^2 \int_0^2 \int_0^3 \frac{2xy^2}{z^2}\, dx\, dy\, dz$

$$\int_1^2 \int_0^2 \frac{x^2 y^2}{z^2} \Big|_0^3 \, dy\, dz$$

$$\int_0^2 \frac{9y^2}{z^2}\, dy$$

$$\frac{3y^3}{z^2} \Big|_0^2$$

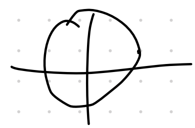
$$\int_1^2 \frac{24}{z^2}\, dz \qquad z^{-2} \qquad -z^{-1}$$

$$24 \left[-\frac{1}{z} \right] \Big|_1^2$$

$$24\left(-\frac{1}{2} + 1 \right)$$

$$12$$

2. $\int_0^2 \int_{1/6}^{1/3} \int_0^{\ln 2} \pi x e^{x+z} \sin(\pi y)\, dz\, dy\, dx$



$$= \pi x e^{x+z} \sin(\pi y) \Big|_0^{\ln 2} \qquad u = x \quad du = 1$$

$$v = e^x$$

$$\pi x e^{x+\ln 2} \sin(\pi y) - \pi x e^x \sin(\pi y) \qquad dv = e^x$$

$$= \int_{1/6}^{1/3} \pi x \sin(\pi y)(e^{x+\ln 2} - e^x)\, dy$$

$$-\cancel{\frac{1}{\pi}}\, \cancel{\pi} x \cos(\pi y)(e^{x+\ln 2} - e^x) \longrightarrow e^x \cdot 2 - e^x$$

$$-x \cos(\pi y)(e^x) \Big|_{1/6}^{1/3}$$

$$= -x\cos\left(\frac{\pi}{3}\right) e^x + x \cos\left(\frac{\pi}{6}\right) e^x$$

$$\int_0^2 x e^x \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) dx$$

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \left[x e^x - \int e^x\, du \right] \Big|_0^3 = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(xe^x - e^x\right)\Big|_0^2$$

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(2e^2 - e^2 - (-1)\right) = 3.07$$

3. $$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1} x^2 y^3 e^{xy^2 z}\, dz\, dy\, dx$$

$$x^{\cancel{2}} y^{\cancel{3}} e^{xy^2 z} \cdot \frac{1}{\cancel{xy^2}}$$

$$xy e^{xy^2 z} \Big|_0^1$$

uv

$$\int_{-1}^{1} xye^{xy^2} - xy \quad dy \qquad 2yx \qquad \frac{1}{2}y\cancel{x}$$

$$\frac{1}{2}e^{xy^2} - \frac{xy^2}{2} \Big|_{-1}^{1}$$

$$\left(\frac{1}{2}e^x - \frac{x}{2}\right) - \left(\frac{1}{2}e^x - \frac{x}{2}\right)$$

$$0$$

4.

xyz

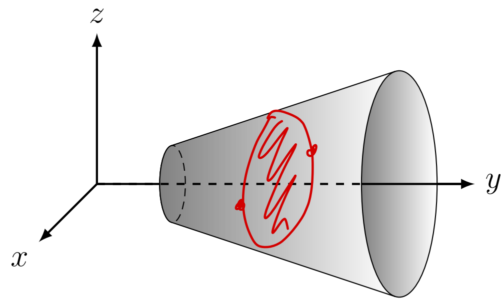
$$\int_0^1 \int_0^1 \int_0^1 x^2 + y^2 + z^2 = 1$$

Math 254 **Exemplar Example 21**

The average value of a continuous function $f(x, y, z)$ over a solid region E is $\frac{1}{V}\iiint_E f\, dV$ where V is the volume of E.

know:
Draw the picture

(a) Let E be the solid region where $y \geq 1$ and $\sqrt{x^2+z^2} \leq y \leq 3$, depicted below. Let's use a triple integral to find the volume of E.



$x^2 + z^2 \leq y^2$

↓ polar coordinates

$$\int_1^3 \int_{-y}^{y} \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} 1 \; dz\, dx\, dy$$

use polar: $x^2 + z^2 = r^2$

$x \tan\theta = z$

$r = y$

$$\int_1^3 \int_0^{2\pi} \int_0^{y} 1 \; r\, dr\, d\theta \; dy$$

$$\int_1^3 \cancel{2}\pi \; \frac{y^2}{\cancel{2}} \; dy$$

$$\pi \; \left.\frac{y^3}{3}\right|_1^3 = \pi\left(\frac{27}{3} - \frac{1}{3}\right)$$

$$= \pi\left(\frac{26}{3}\right)$$

(b) Now let's find the average distance away from the y-axis within E.

distance from y-ax

$$AV = \frac{1}{\left(26\pi/3\right)} \int_1^3 \int_0^{2\pi} \int_0^y r \, (r\,dr\,d\theta)\,dy$$

$$= \frac{3}{\cancel{2}6\cancel{\pi}_{\,13}}\,\cancel{2}\cancel{\pi} \int_1^3 \int_0^y r^2 \, dr \, dy$$

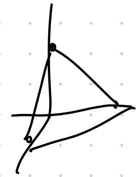
$$= \frac{\cancel{3}}{13} \int_1^3 \frac{y^3}{\cancel{3}}$$

$$\frac{1}{13}\left(\frac{1}{4}y^4\right)\Big|_1^3$$

$$= \frac{1}{13}\left(\frac{80}{4}\right) = \frac{20}{13}$$

1.

$$z = 1 - x - y$$



$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy \quad dz\, dy\, dx$$

$$\int_0^{1-x} xy(1-x-y) \quad dy$$

$$\int xy - x^2y - xy^2 \quad dy$$

$$\frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3} \Big|_0^{1-x}$$

$$\frac{x(1-x)^2}{2} - \frac{x^2(1-x)^2}{2} - \frac{x(1-x)^3}{3}$$

$$\int_0^1 \frac{x(1-x)^2}{2} - \frac{x^2(1-x)^2}{2} - \frac{x(1-x)^3}{3} \, dx$$

integration by parts

$$\int_0^1 (1-x)^2 \left[\frac{x}{2} - \frac{x^2}{2} - \frac{x(1-x)}{3} \right] dx$$

$$\int_0^1 (1-x)^2 \left[\frac{x(1-x)}{2} - \frac{x(1-x)}{3} \right]$$

$$\int_0^1 (1-x)^2 \left(\frac{1}{2} - \frac{1}{3}\right) \left[x(1-x) \right]$$

$$\int_0^1 x(1-x)^3 \quad \frac{1}{6} \quad dx$$

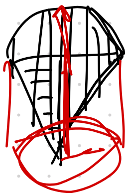
$$\frac{1}{6} \Big[$$

$$= 0.008$$

$$= 0.01$$

2. polar coordinates

$3x^2+3y^2 \le z^2 \le 4-x^2-y^2$



$3(x^2+y^2) \le z^2$

$3(r^2) \le z^2$

$z = \sqrt{3}r$

$4-(x^2+y^2)$

$z^2 \le 4-r^2$

$z \le \sqrt{4-r^2}$

Find r

$3x^2+3y^2 = 4-x^2-y^2$

$3(x^2+y^2) = 4-x^2-y^2$

$r^2 = \frac{2}{3}$

$r = \sqrt{\frac{2}{3}}$

$\frac{2z}{1+r^2}$

$4x^2+4y^2$

$4r^2 = 4$

$r^2 = 1$

$r = \pm 1$

$$\int_0^{2\pi}\int_0^1\int_{\sqrt{3}r}^{\sqrt{4-r^2}} \frac{2z}{1+r^2}\, r\, dz\, dr\, d\theta$$

$$r \cdot \frac{2}{1+r^2}\left(\frac{z^2}{2}\right)\Bigg|_{\sqrt{3}r}^{\sqrt{4-r^2}}$$

$$r \cdot \frac{2}{1+r^2}\left(\frac{4-r^2}{2} - \frac{3r^2}{2}\right)$$

$$r \cdot \frac{\cancel{2}}{1+r^2}\left(\frac{4-4r^2}{\cancel{2}}\right)$$

$$2\pi\int_0^1 \frac{4-4r^2}{1+r^2}\, r\, dr$$

4.85

#3. 0

#4 7.996

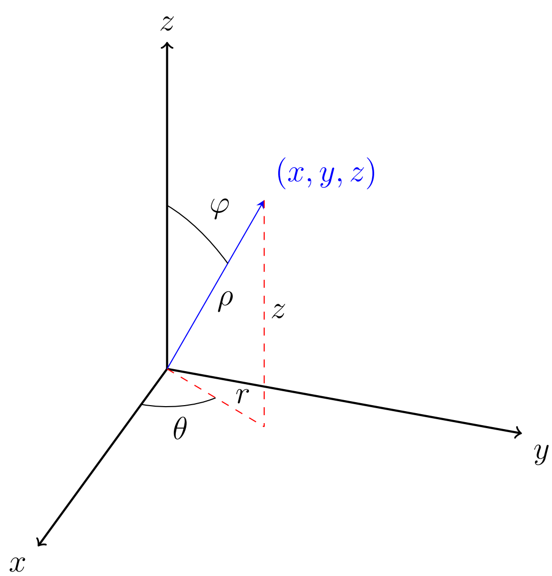
...

Triple Integrals in Spherical & Cylindrical Coordinates

Cylindrical coordinates: (r, θ, z) where (r, θ) are polar coordinates for the xy-plane.

Note: Whenever convenient, one can permute the role of z with either x, y.

Spherical coordinates: (ρ, θ, φ) where ρ is the signed distance to the origin along the ray where θ is the polar angle of the point $(x, y, 0)$ made in the xy-plane and φ is the angle between the point (x, y, z) regarded as a vector and the positive z-axis.



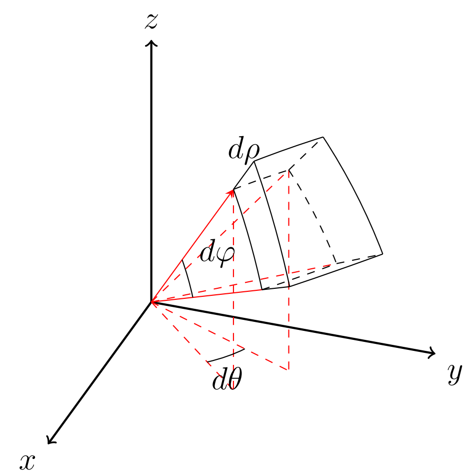
Here are some key conversions:

$r^2 + z^2$

$\rho^2 = x^2+y^2+z^2$, $r = \rho \sin(\varphi)$, $x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$, $y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$, and $z = \rho\cos(\varphi)$.

Cylindrical coordinates are just polar coordinates plus a rectangular coordinate, so differential volume in cylindrical coordinates is $\boxed{dV = r\,dz\,dr\,d\theta}$.

For spherical coordinates, the volume ΔV of a spherical box (see picture below) can be approximated by $\Delta V \approx \Delta\rho(\rho\Delta\varphi)(\rho\sin(\varphi)\Delta\theta)$ and the error in the approximation goes towards 0 as $\Delta\rho \to 0$, $\Delta\theta \to 0$ and $\Delta\varphi \to 0$. So the differential volume of a spherical box satisfies $dV = d\rho\,(\rho d\varphi)(\rho\sin(\varphi)\,d\theta)$. Equivalently $\boxed{dV = \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta}$.



Let's evaluate $\displaystyle\iiint_E z\,dV$ where E is the part of the ball $\{(x,y,z)|x^2+y^2+z^2 \le 4\}$ above the plane $z = 1$.

First let's do it in cylindrical coordinates. When $z = 1$, $x^2 + y^2 \le 3$. Thus region can be described in cylindrical coordinates as $1 \le z \le \sqrt{4-r^2}$, $0 \le r \le \sqrt{3}$, $0 \le \theta \le 2\pi$.

$$\begin{aligned}
\iiint_E z\,dV &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} z\,r\,dz\,dr\,d\theta \\
&= 2\pi\int_0^{\sqrt{3}} \frac{1}{2}z^2 r\Big|_{z=1}^{z=\sqrt{4-r^2}}\,dr \qquad 2\pi \int \\
&= \pi\int_0^{\sqrt{3}} (3-r^2)r\,dr \\
&= \pi\int_0^{\sqrt{3}} 3r - r^3\,dr \\
&= \pi\left(\frac{3}{2}r^2 - \frac{1}{4}r^4\Big|_0^{\sqrt{3}}\right) \\
&= \pi\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9\pi}{4}.
\end{aligned}$$

Now let's do it in spherical coordinates. The region in the ball $\{(x,y,z)|x^2+y^2+z^2 \leq 4\}$ above the plane $z=1$ is equivalent to $\rho \leq 2$ and $\rho\cos(\varphi) \geq 1$. So $\sec(\varphi) \leq \rho \leq 2$ and the bounding surfaces intersect where $\sec(\varphi) = 2$, or equivalently $\cos(\varphi) = \frac{1}{2}$, implying $\varphi = \frac{\pi}{3}$. So $0 \leq \varphi \leq \frac{\pi}{3}$. Finally $0 \leq \theta \leq 2\pi$.

$$\begin{aligned}
\iiint_E z\,dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec(\varphi)}^2 \rho\cos(\varphi)\,\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta \\
&= 2\pi\int_0^{\frac{\pi}{3}}\int_{\sec(\varphi)}^2 \rho^3\cos(\varphi)\sin(\varphi)\,d\rho\,d\varphi \\
&= 2\pi\int_0^{\frac{\pi}{3}} \frac{1}{4}\rho^4\cos(\varphi)\sin(\varphi)\Big|_{\sec(\varphi)}^2\,d\varphi \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{3}}(16-\sec^4(\varphi))\cos(\varphi)\sin(\varphi)\,d\varphi \\
&= \frac{\pi}{2}\int_0^{\frac{\pi}{3}}\left(16\cos(\varphi)\sin(\varphi) - \frac{\sin(\varphi)}{\cos^3(\varphi)}\right)d\varphi \\
&= \frac{\pi}{2}\left(-8\cos^2(\varphi) - \frac{1}{2\cos^2(\varphi)}\Big|_0^{\frac{\pi}{3}}\right) \\
&= \frac{\pi}{2}\left(-8(1/4-1) - \frac{1}{2}(4-1)\right) = \frac{9\pi}{4}.
\end{aligned}$$

Let's evaluate $\iiint_E \frac{1}{1+x^2+y^2+z^2}dV$ where E is the solid bounded by $z=\sqrt{3x^2+3y^2}$ and $z=\sqrt{25-x^2-y^2}$ where $y \geq 0$. We will do it in spherical coordinates. The cone $z=\sqrt{3}\sqrt{x^2+y^2}$ is $\rho\cos(\varphi) = \sqrt{3}\rho\sin(\varphi)$, equivalently $\varphi = \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{6}$ and $z=\sqrt{25-x^2-y^2}$ is the top half of the sphere $\rho=5$. Also as $y\geq 0$, $0\leq\theta\leq\pi$. Hence

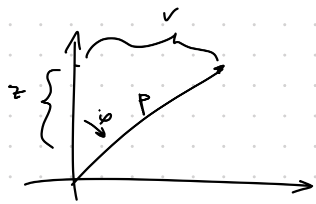
$$\begin{aligned}
\iiint_E \frac{1}{1+x^2+y^2+z^2}\,dV &= \int_0^{\pi}\int_0^{\frac{\pi}{6}}\int_0^5 \frac{1}{1+\rho^2}\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta \\
&= \pi\int_0^{\frac{\pi}{6}}\int_0^5\left(1-\frac{1}{1+\rho^2}\right)\sin(\varphi)\,d\rho\,d\varphi \\
&= \pi\int_0^{\frac{\pi}{6}}\left(\rho-\tan^{-1}(\rho)\right)\sin(\varphi)\Big|_{\rho=0}^{\rho=5}\,d\varphi \\
&= \pi\int_0^{\frac{\pi}{6}}\left(5-\tan^{-1}(5)\right)\sin(\varphi)\,d\varphi \\
&= \pi\left(5-\tan^{-1}(5)\right)\left(-\cos(\varphi)|_0^{\frac{\pi}{6}}\right) \\
&= \pi\left(5-\tan^{-1}(5)\right)\left(1-\frac{\sqrt{3}}{2}\right).
\end{aligned}$$

Cylindrical Coordinates (r, θ, z) (r, θ) are polar coord. for (x,y) for (x,y,z)

$$dV = dA_{polar}\, dz = r\, dz\, dr\, d\theta$$

Spherical coordinates (ρ, θ, φ) where $\rho = \sqrt{x^2+y^2+z^2}$

$x \tan\theta = y$ and $z = \cos\varphi$. $\uparrow = \rho\cos\varphi$



$r = \rho\sin\varphi$

$$x = r\cos\theta = \rho\sin\varphi\cos\theta$$

$$y = r\sin\theta = \rho\sin\varphi\sin\theta$$

$$z = \rho\cos\varphi$$

$$dV = (r\, d\theta)(\rho\, d\varphi)\, d\rho$$

$$= \rho\sin\varphi\, d\theta \cdot \rho\, d\varphi\, d\rho$$

$$= \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

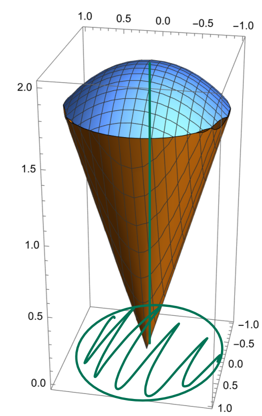
$$\rho \sin\phi \; \rho\cos\phi \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\rho^2 \rho \sin^2\phi \; \rho\cos\phi \, d\rho \, d\phi \, d\theta$$

$$\int \int_0 \rho^4 \sin^2\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

Math 254 **Exemplar Example 22**

Consider the integral $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} \frac{z}{1+\sqrt{x^2+y^2+z^2}}\,dz\,dx\,dy.$



$$\sqrt{4-x^2-y^2} = \sqrt{3x^2+3y^2}$$

$$4-(x^2+y^2) = 3x^2+3y^2$$

$$4-r^2 = 3(r^2)$$

$$4 = 4r^2$$

$$1 = r^2$$

(a) Let's convert this integral into one using cylindrical coordinates.

$0 \le \theta \le 2\pi \qquad 0 \le r \le 1 \qquad r\sqrt{3} \le z \le \sqrt{4-r^2}$

$$\int_0^{2\pi}\int_0^{1}\int_{r\sqrt{3}}^{\sqrt{4-r^2}} \frac{z}{1+\sqrt{r^2+z^2}}\,r\,dz\,dr\,d\theta$$

$$\frac{1}{\sqrt{3}} = \tan\phi$$

$$4 = \rho^2$$

$z^2 = 4-x^2-y^2 \qquad 4 = x^2+y^2+z^2 \qquad z = r\sqrt{3}$

$$\cancel{\rho}\cos\phi = \cancel{\rho}\sin\phi\sqrt{3}$$

(b) Let's convert this integral into one using spherical coordinates.

rho phi

$\rho = 2 \qquad \phi = \frac{\pi}{6} \qquad 0 \le \theta \le 2\pi$

$0 \le \rho \le 2 \qquad 0 \le \phi \le \frac{\pi}{6}$

$$\int_0^{2}\int_0^{\pi/6}\int_0^{2\pi} \frac{\rho\cos\theta}{1+\rho}\,\rho^2\sin\phi\,d\theta\,d\rho\,d\phi$$

$\Phi = \varphi$

(c) Now let's pick whichever looks nicer and evaluate!

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \frac{\rho^3 \cos\varphi \sin\varphi}{1+\rho}\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/6}\int_0^{2} \left[\rho^2 - \rho + 1 - \frac{1}{1+\rho}\right] \sin\varphi \cos\varphi\, d\rho\, d\varphi$$

$$= 2\pi \int_0^{\pi/6} \frac{\rho^3}{3} - \frac{\rho^2}{2} + \rho - \ln\left|1+\rho\right|\Big|_0^2 \sin\varphi \cos\varphi\, d\varphi$$

$$= 1.23$$

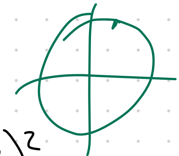
1. $x^2+y^2+z^2=4$ $\qquad z=1$

$\rho^2=4$ $\qquad \rho\cos\phi=1$

$\cos\phi=\frac{1}{2}$

$\phi=\cos^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{3}$



$r^2=(\rho\sin\phi)^2$

x^2+y^2

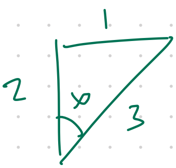
$\rho\ r\ \phi$

1.047

$\boxed{1.05}$

2. Evaluate $\iiint z\sqrt{x^2+y^2}\,dV$

$x^2+y^2\le 1$ $\qquad x^2+y^2+z^2\le 4$ $\qquad z\ge 0$



$\phi=\frac{\pi}{6}$ $\qquad \rho^2\le 4$

$0\le\phi\le\frac{\pi}{6}$ $\qquad 0\le\rho\le 2$

phi $\qquad$ rho

$0\le\theta\le 2\pi$

$$\int_0^{2\pi}\int_0^{2}\int_0^{\pi/6}\rho\cos\phi\,\rho\sin\phi \qquad \rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

$$2\pi\int_0^{\pi/6}\rho^4\,\underline{\cos\phi\sin\phi\sin\phi}\,d\phi\,d\rho.$$

$\frac{32}{5}\cdot\frac{1}{\cancel{24}_{12}}\cdot 2\pi$ $\qquad \rho^4\,\frac{\sin^3\phi}{3}\Big|_0^{\pi/6}\,d\rho$ $\qquad =1.676$

$\underline{1.68}$

$\frac{1}{24}\left[\frac{\rho^5}{5}\right]_0^2\,2\pi$ $\qquad \rho^4\,\frac{\left(\frac{1}{2}\right)^3}{3}=\frac{1}{8}\cdot\frac{1}{3}$

$\rho^4\,\frac{1}{24}$

$\boxed{3.56}$

$z \to \rho \cos\phi \qquad z\ \sqrt{x^2+y^2}$

$$\int_0^{2\pi}\int_0^{2}\int_0^{\pi/6} \rho\cos\phi \cdot \rho\sin\phi\, \rho^2\sin\phi\, d\phi\, d\rho\, d\theta$$

$$2\pi \int_0^{2}\int_0^{\pi/6} \rho^4 \cos\phi \sin^2\phi \; d\phi\, d\rho$$

$$2\pi \left[\frac{\rho^5}{5}\bigg|_0^2 d\rho \cdot \int_0^{\pi/6} \cos\phi \sin^2\phi \; d\phi \right]$$

$$\frac{32}{5} \cdot \frac{\sin^3\phi}{3}\bigg|_0^{\pi/6}$$

$$2\pi \cdot \frac{32}{5}\left[\frac{\sin^3\left(\frac{\pi}{6}\right)}{3} - \frac{\sin^3(0)}{3} \right]$$

3. 0.36

4. 8.38

Cylindrical Coordinates (r, θ, z)

$$dV = r\,dz\,dr\,d\theta$$

Spherical (ρ, ϕ, θ)

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

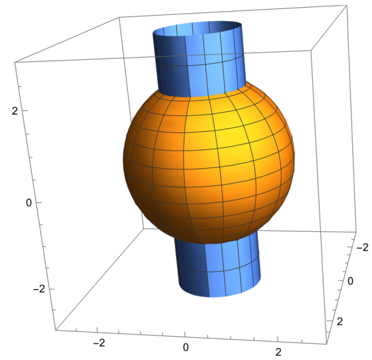
$$x = \rho \sin\phi \cos\theta$$

$$\rho\cos\phi = z \qquad \rho\sin\phi = r$$

$$y = \rho \sin\phi \sin\theta$$

Math 254 **Exemplar Example 23**

Consider $\displaystyle\iiint_E \frac{4}{\sqrt{x^2+y^2}}\, dV$ where $E = \{(x,y,z) | x^2+y^2 \geq 1, x^2+y^2+z^2 \leq 4\}$.



(a) Let's set up this integral using cylindrical coordinates. r, θ, z

$r^2 \geq 1$, $r \geq 1$ $r^2 + z^2 \leq 4$

$r \leq \sqrt{4 - z^2}$

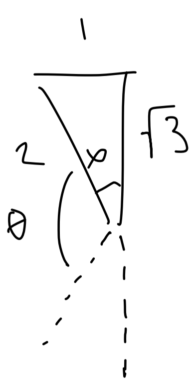
$x^2 + y^2 = 1$ $1 + z^2 = 4$

$z^2 = 3$

$z = \pm\sqrt{3}$ $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} \frac{4}{\cancel{r}}\, \cancel{r}\, dr\, dz\, d\theta$$

(b) Let's set up this integral using spherical coordinates. (ρ, ϕ, θ)



$\rho^2 \leq 4 \rightarrow \rho \leq 2$ $r \geq 1 \rightarrow \rho \sin\phi \geq 1$

$\rho \geq \csc\phi$

$\frac{\pi}{6} \leq \phi \leq \frac{5\pi}{6}$, $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc\phi}^{2} \frac{4}{\cancel{\rho}\,\cancel{\sin\phi}}\, \rho^{\cancel{2}}\, \cancel{\sin\phi}\, d\rho\, d\phi\, d\theta$$

Spherical

(c) Now let's pick whichever looks nicer and evaluate!

$$\int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc\phi}^{2} 4\rho \, d\rho \, d\phi \, d\theta$$

$$2\pi \int_{\pi/6}^{5\pi/6} 2\rho^2 \Big|_{\csc\phi}^{2} \, d\phi$$

$$2\pi \int_{\pi/6}^{5\pi/6} 8 - 2\csc^2\phi \, d\phi$$

$$2\pi \left(8\phi + 2\cot\phi \Big|_{\pi/6}^{5\pi/6} \right)$$

$$2\pi \left(8 \cdot \frac{2\pi}{3} - 2\sqrt{3} - 2\sqrt{3} \right]$$

$$= 2\pi \left[\frac{16\pi}{3} - 4\sqrt{3} \right]$$

$$\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$

$$= \csc^2\theta$$

1. cylindical

a $$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r z \, dz \, dr \, d\theta$$

2. spherical

d $$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec(\phi)}^{2} \rho^3 \sin(\phi)\cos(\phi) \, d\rho \, d\phi \, d\theta$$

3. Evaluate

$$2\pi \int_0^{\sqrt{3}} \frac{r z^2}{2} \Big|_1^{\sqrt{4-r^2}} dr$$

$$2\pi \int_0^{\sqrt{3}} \left(\frac{r(4-r^2)}{2} - \frac{r}{2} \right)$$

$$\frac{4r - r^3}{2} - \frac{r}{2}$$

$$2\pi \int_0^{\sqrt{3}} \frac{3r - r^3}{2} \, dr$$

$$\pi \int_0^{\sqrt{3}} 3r - r^3 \, dr$$

$$\pi \left(\frac{3r^2}{2} - \frac{r^4}{4} \right) \Big|_0^{\sqrt{3}}$$

$$\pi \left(\frac{9}{2} - \frac{9}{4} \right)$$

$$\pi \left(\frac{18}{4} - \frac{9}{4} \right) = \frac{9}{4}\pi = \underline{7.07}$$

$\underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}$

10^6

1

$\underline{0}$ 1 2 3 4 5 6 7 8 9

...

Center of Mass & Change of Variables

Suppose objects of masses m_1, m_2,...,m_n be arranged along a line at coordinates x_1, x_2,...,x_n respectively and the balance point or center of mass is at $\overline{x}$. Each mass contributes a "torque" of $m_i g(x_i - \overline{x})$ (where g is the magnitude of acceleration due to gravity) and to balance,

$$\sum_{i=1}^{n} m_i g(x_i - \overline{x}) = 0 \to \overline{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

Now consider a continuous (linear) density function $\rho(x)$ for an object occupying the interval $[a, b]$. One can take n subintervals of the "rod," each of width Δx, and use sample points $x_i = a + i\Delta x$, to get a limit for the balance condition for finding the center of mass $\overline{x}$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i)\Delta x g(x_i - \overline{x}) = 0 \to$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i)\Delta x(x_i - \overline{x}) = 0 \to$$

$$\overline{x} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \rho(x_i)x_i\Delta x}{\sum_{i=1}^{n} \rho(x_i)\Delta x}.$$

Thus the balance condition becomes,

$$\overline{x} = \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx}.$$

Note that the denominator is the mass! The numerator is called the moment.

Now let a "plate" $R = \{(x,y)|a \leq x \leq b, g(x) \leq y \leq f(x)\}$ have a continuous (planar) density function $\rho(x,y)$.

To compute the center of mass, $(\overline{x}, \overline{y})$, let's consider how to compute $\overline{x}$ (it is analogous for $\overline{y}$):

Suppose we partition $[a,b]$ into n subintervals of equal width Δx and let $x_i = a + i\Delta x$. We consider the torque about the y-axis of the vertical part of the plate that lies along the line where x is a constant as though it was a point mass at $(x,0)$. Then $\lim_{n\to\infty}\sum_{i=1}^{n} x_i\left(\int_{g(x_i)}^{f(x_i)} \rho(x_i,y)dy\right)\Delta x$ is the moment about the y-axis. So

$$M_y = \int\int_R x\rho(x,y)dA.$$

Similarly, the moment about the x-axis is

$$M_x = \int\int_R y\rho(x,y)dA.$$

Note: These formulas extend to all regions, not just ones that can be described with $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$.

Thus:

$$\overline{x} = \frac{M_y}{m} \text{ and } \overline{y} = \frac{M_x}{m} \text{ where the mass is } m.$$

A plate occupies the region R in the first quadrant where $x^2 + y^2 \leq 4$ (x, y are in cm) with the density function $\rho(x,y) = 5 - xy$ grams per cubic cm. Let's find the mass and the center of mass.

Due to the reflective symmetry of region over the line $y = x$, and since x, y play symmetric roles in $\rho(x,y)$, we know that $\overline{x} = \overline{y}$. The mass is

$$\begin{aligned} m &= \int_0^{\frac{\pi}{2}}\int_0^2 (5 - r^2\sin(\theta)\cos(\theta))\, r\, dr\, d\theta \\ &= \frac{\pi}{2}\left(\frac{5}{2}r^2\Big|_0^2\right) - \left(\frac{1}{2}\sin^2(\theta)\Big|_0^{\frac{\pi}{2}}\right)\left(\frac{1}{4}r^4\Big|_0^2\right) \\ &= 5\pi - 2. \end{aligned}$$

The moment about the y-axis is

$$\begin{aligned}
M_y &= \int_0^{\frac{\pi}{2}} \int_0^2 r\cos(\theta)\left(5 - r^2\sin(\theta)\cos(\theta)\right) r\, dr\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 5r^2\cos(\theta) - r^4\sin(\theta)\cos^2(\theta)\, dr\, d\theta \\
&= \left(\sin(\theta)\Big|_0^{\frac{\pi}{2}}\right)\left(\frac{5}{3}r^3\Big|_0^2\right) - \left(-\frac{1}{3}\cos^3(\theta)\Big|_0^{\frac{\pi}{2}}\right)\left(\frac{1}{5}r^5\Big|_0^2\right) \\
&= \left(\frac{40}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{32}{5}\right) \\
&= \frac{168}{15} = \frac{56}{5}.
\end{aligned}$$

So the center of mass is $\left(\frac{56}{5(5\pi-2)}, \frac{56}{5(5\pi-2)}\right) \approx (0.8, 0.8)$.

Suppose a solid occupies a region E in xyz-sapce with a continuous density function $\rho(x, y, z)$ on E. Then the moments are about planes. The center of mass $(\overline{x}, \overline{y}, \overline{z})$ satisfies:

$$\overline{x} = \frac{M_{yz}}{m},$$

$$\overline{y} = \frac{M_{xz}}{m},$$

$$\overline{z} = \frac{M_{xy}}{m},$$

where $m = \int\int\int_E \rho(x, y, z)dV$ is the mass and

$$M_{yz} = \int\int\int_E x\rho(x, y, z)dV,$$

$$M_{xz} = \int\int\int_E y\rho(x, y, z)dV,$$

$$M_{xy} = \int\int\int_E z\rho(x, y, z)dV.$$

A change of variables in a double integral is a transformation that relates two sets of variables, (u, v) and (x, y), where the region R in the xy-plane corresponds to a region S in the uv-plane.

Suppose $x = g(u, v)$ and $y = h(u, v)$ defines transformation where g, h have continuous first order partial derivatives over S and are one-to-one on the interior of S. The Jacobian determinant is computed in the following way:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Change of Variables Theorem (2D): Suppose $f(x, y)$ is continuous on a region R in the xy-plane that transforms (with the assumptions above) into a region S in the uv-plane. Then

$$\int\int_R f(x,y)dA = \int\int_S f(g(u,v), h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA.$$

(We skip the proof.)

Let's evaluate $\int\int_R y^2 dA$ where R is the region in the first quadrant bounded by $xy = 1$, $xy = 4$, $\frac{y}{x} = 1$ and $\frac{y}{x} = 3$.

We change the variables with $u = xy$ and $v = \frac{y}{x}$ because in the uv-plane the transformed region is $[1, 4] \times [1, 3]$.

We have to solve for x, y in terms of u, v. Since $u/v = x^2$ and $uv = y^2$ and x, y, u, v are all positive, $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$. So

$$\frac{\partial(x,y)}{\partial(u,v)} = det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = det\begin{pmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}.$$

So $\int\int_R y^2 dA = \int_1^3 \int_1^4 uv\frac{1}{2v}\,du\,dv = \int_1^4 u\,du = \frac{15}{2}$.

Suppose $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = \ell(u, v, w)$ is a transformation of a solid B in uvw-space into a solid E in xyz-space that has continuous 1st order partials and is 1-1 on the interior. The Jacobian determinant is computed in the following way:

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

Change of Varaibles Theorem (3D) Suppose $f(x, y, z)$ is continuous on a solid region E in xyz-space that transforms (with the assumptions above) into a solid region B in uvw-space. Then

$$\int\int\int_E f(x, y, z)dV = \int\int\int_B f(g(u, v, w), h(u, v, w), \ell(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV.$$

Mass and Center of Mass

Let $\rho(x,y)$ be the planar density (mass/area) at (x,y) of a flat object that occupies a region R in the xy-plane.

The mass is $\quad m = \iint_R \rho(x,y)\, dA \quad \leftarrow$ mass

$$M_y = \iint_R x\rho(x,y)\, dA \qquad \text{moment about the y-axis}$$

$$M_x = \iint_R y\rho(x,y)\, dA \qquad \text{moment about the x-axis}$$

center of mass $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

3D

$$m = \iiint_E \underset{\text{mass/vol}}{\rho(x,y,z)}\, dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)\, dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)\, dV$$

$$M_{yz} = \iiint_E x\rho(x,y,z)\, dV$$

$(\bar{x}, \bar{y}, \bar{z})$

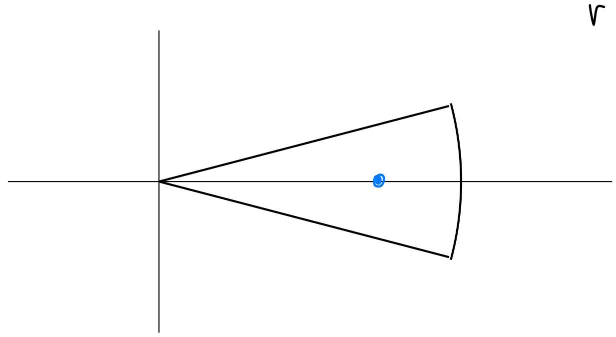
$$\bar{x} = \frac{M_{yz}}{m}$$

$$\bar{y} = \frac{M_{xz}}{m}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

A pizza slice of radius 10 cm and central angle $\frac{\pi}{6}$ radians occupies a region where the line of symmetry through the middle of the slice, from point to crust, occupies the interval [0,10] of the x-axis where x, y are in units of centimeters, and the planar density (in grams per sq. cm) is given by $\rho(x, y) = 4 + 0.1\sqrt{x^2 + y^2}$ at position (x, y) of the slice.

r



(a) Let's find the mass of the slice.

$$m = 2\int_0^{\pi/12}\int_0^{10} (4 + 0.1\, r)\, r \, dr \, d\theta$$

$$= \frac{\pi}{6}\int_0^{10} 4r + \frac{1}{10}r^2 \, dr$$

$$= \frac{\pi}{6}\left[2r^2 + \frac{1}{30}r^3\right]_0^{10}$$

$$= \frac{\pi}{6}\left[200 + \frac{100}{3}\right] = 122 \text{ g}$$

(b) Which coordinate of the center of mass is 0 because the moment about the opposite axis is 0, by symmetry?

$$\bar{y} = 0 \quad \text{since} \quad M_x = \iint_R y\rho(x,y)\, dA = 0$$

(c) Let's find the non-zero moment.

$$M_y = 2\int_0^{\pi/12}\int_0^{10} \underbrace{r\cos\theta}_{x} \underbrace{\left(4 + \frac{1}{10}r\right)}_{r} r\, dr\, d\theta$$

$$= 2\iint \left(4r^2 + \frac{1}{10}r^3\right)\cos\theta \, dr\, d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{4}{3}r^3 + \frac{1}{40}r^4\right)\Big|_0^{10} \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{4000}{3} + 250\right)\cos\theta\, d\theta = \frac{9500}{3}\sin\theta\Big|_0^{\pi/12}$$

$$= \frac{9500}{3}\sin\frac{\pi}{12} = 820$$

(d) What are the coordinates of the center of mass?

$$\bar{x} = \frac{M_y}{m} = \frac{820}{122} = 6.7$$

$$\bar{y} = 0$$

$$(6.7, 0)$$

1. $$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2}\left(12-4\rho\cos\phi\right)\underline{\rho^2\sin\phi\;d\rho\;d\phi\;d\theta}$$

spherical coor.

ρ

ϕ

θ

$$dV = \rho^2\sin^2\phi\; d\phi\, d\rho\, d\theta$$

$$\rho^2 = 4$$
$$\rho = \pm 2$$

$$2\pi\left[\int_0^{\pi/2}\int_0^{2} 12\rho^2\sin\phi - 4\rho^3\cos\phi\sin\phi\; d\rho\, d\phi\right.$$

$$\frac{12}{3}\rho^3\sin\phi - \rho^4\cos\phi\sin\phi\Big|_0^2$$

$$4\cdot 8\sin\phi - 16\cos\phi\sin\phi$$

$$2\pi\int_0^{\pi/2} 32\sin\phi - 16\cos\phi\sin\phi\; d\phi$$

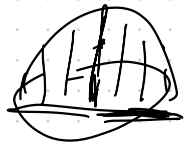
$$2\pi\cdot 16\int_0^{\pi/2} 2\sin\phi - \cos\phi\sin\underline{\phi}\; d\phi$$

$$32\pi\cdot\frac{3}{2} = 16\cdot 3\pi$$

$$m = 48\pi \approx 150.8\text{ g}$$

2. $\bar{x}$, $\bar{y}$, $\bar{z}$

2 , x and y



3. Find the z coord.

$$\bar{z} = \frac{M_{xy}}{m}$$

$$M_{xy} = \iiint z\,\rho(x,y,z)\,dV$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} z\,(12-4\rho\cos\phi)\,\rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

$$2\pi\int_0^{\pi/2}\int_0^{2} \rho\cos\phi\,(12-4\rho\cos\phi)\,\rho^2\sin\phi\;d\rho\,d\phi\,d\theta$$

$$2\pi\int_0^{\pi/2}\int_0^{2} (12\rho^3\sin\phi\cos\phi - 4\rho^4\cos^2\phi\sin\phi)\,d\rho\,d\phi$$

$$8\pi\int_0^{\pi/2}\int_0^{2} 3\rho^3\sin\phi\cos\phi - \rho^4\cos^2\phi\sin\phi \quad d\rho\,d\phi$$

$$8\pi\cdot\int_0^{\pi/2}\int_0^{2} \rho^3\sin\phi\cos\phi\,(3-\rho\cos\phi)\,\underline{d\rho}\,d\phi$$

$$8\pi\int_0^{\pi/2}\int_0^{2} \sin\phi\cos\phi\,(3\rho^3-\rho^4\cos\phi)\;d\rho\,d\phi$$

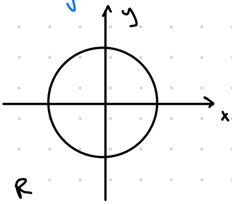
$$\sin\phi\cos\phi\left(\frac{3}{4}\rho^4 - \frac{\rho^5}{5}\cos\phi\right)\Big|_0^2$$

$$8\pi\int_0^{\pi/2} \sin\phi\cos\phi\left(12-\frac{32}{5}\cos\phi\right)d\phi$$

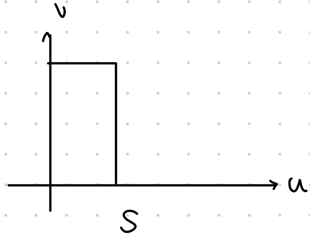
$= 97.180$

$$\frac{97.180}{150.8} = 0.64$$

Change of Variables



$$x = g(u,v)$$
$$y = h(u,v)$$



Jacobian $\quad J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

Theorem $\quad \iint_R f(x,y)\, \underbrace{dA}_{dx\,dy \text{ or } dy\,dx} = \iint_S f(g(u,v), h(u,v))\, |J(u,v)|\, \underbrace{dA}_{du\,dv \text{ or } dv\,du}$

$$x = v\cos(u)$$
$$y = v\sin(u$$

$$J(u,v) = \begin{pmatrix} -v\sin(u) & \cos u \\ v\cos(u) & \sin u \end{pmatrix}$$

$$= -v\sin^2(u) - v\cos^2(u)$$

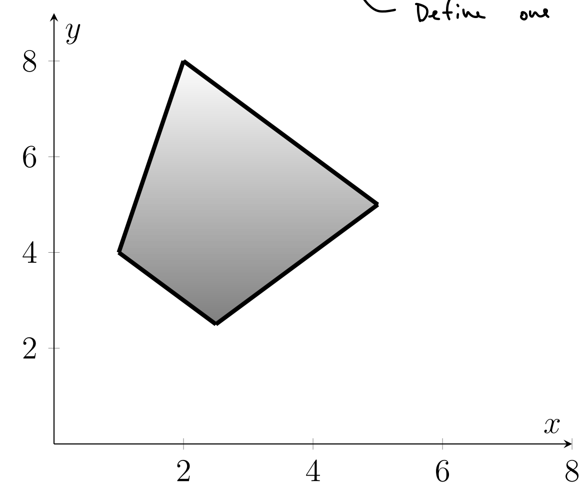
$$= -v(1) = -v$$

Let's evaluate $\iint_R \frac{1}{(x+y)^2}\,dA$ for $R = \{(x,y)|5-x \le y \le 10-x \text{ and } x \le y \le 4x\}$.

$5 \le x+y \le 10 \qquad 1 \le \frac{y}{x} \le 4$

Define one to be u and v

Here is the graph of the region:



(a) What might be the best coordinate system in which to evaluate this?

$u = \frac{y}{x}$

$v = x+y$

(b) Let's find the Jacobian of our new coordinates.

$y = v - x = ux$

$v = (u+1)x$

$x = \frac{v}{u+1}$

$y = v - x = v - \frac{v}{u+1} = v\left(\frac{u}{u+1}\right)$

Jacobian

$$J(u,v) = \det\begin{pmatrix} -v(u+1)^{-2} & (u+1)^{-1} \\ v(u+1)^{-1} - vu(u+1)^{-2} & u(u+1)^{-1} \end{pmatrix}$$

$$= \cancel{-vu(u+1)^{-3}} - v(u+1)^{-2} + \cancel{vu(u+1)^{-3}}$$

$$= -v(u+1)^{-2} = \frac{-v}{(u+1)^2}$$

(c) Let's evaluate the integral $\iint \frac{1}{(x+y)^2}\, dA$

$$\int_1^4 \int_5^{10} \frac{1}{v^{\cancel{2}}} \left| -\frac{\cancel{v}}{(u+1)^2} \right| dv\, du$$

$$= \int_1^4 \ln|v| \Big|_5^{10} \frac{1}{(u+1)^2}\, du$$

$$= \ln 2 \int_1^4 \frac{1}{(u+1)^2}\, du$$

$$= (\ln 2) \left. -\frac{1}{u+1} \right|_1^4$$

$$= \ln 2 \left(-\frac{1}{5} + \frac{1}{2} \right) = \frac{3 \ln 2}{10}$$

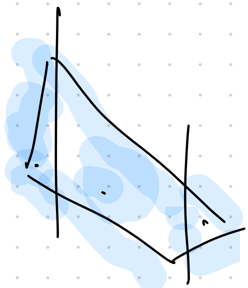
25

1. $x = u + 2v \qquad y = 3u - v$

$$J(u,v) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$

$$-1 - 6 = -7$$

2. 3



3. $y = 9x \qquad 4y = x \qquad y = 5 - x \qquad y = 10 - x$

$$\frac{1}{4} \le \underbrace{\frac{y}{x}}_{u} \le 9 \qquad 5 \le \underbrace{y + x}_{v} \le 10$$

x^2

$$\int_{1/4}^{9}\int_{5}^{10} \left(\frac{v}{u+1}\right)^2 \left| -\frac{v}{(u+1)^2} \right| \, dv\, du$$

$$\int_{1/4}^{9}\int_{5}^{10} \frac{v^3}{(u+1)^4} \, dv \quad du$$

$$\frac{1}{(u+1)^4} \left. \frac{v^4}{4} \right|_5^{10}$$

$$\int_{1/4}^{9} \frac{1}{(u+1)^4} \left(\frac{10^4}{4} - \frac{5^4}{4} \right) du$$

$$\frac{9375}{4}\int_{\frac{1}{4}}^{9}\frac{1}{(u+1)^4}\,du = 399.22 \quad X$$

$$\boxed{399.2} \quad ?$$

$$\int_{1/4}^{9}(u+1)^{-4}\,du$$

$$\left(\frac{9375}{4}\right)\cdot -\frac{(u+1)^{-3}}{3}\Bigg|_{1/4}^{9}$$

$$\left(\frac{9375}{4}\right)\cdot\left[-\frac{1}{3000} - \frac{-64}{375}\right] =$$

4. optional

0.4

Final: 7:30 AM on Wed. Dec 8^{th}

Cordley 1109

match min boundary

1. plug in, optimize on boundary

   check critical pts, endpts

2. Legrange Multiplies

up to

spherical / cylindrical coord

18 Questions — most MC (14-15), some open (3)

1 hr 50 min

post last class

no topics from this week.

no center of mass or center of variables question

learn from midterm mistakes

70 mistakes

1. (x, y)

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\vec{r}(t) = \langle t^2, t+1 \rangle$

$\vec{r}\,'(t) = \langle 2t, 1 \rangle$

$\langle 2t, 1 \rangle \langle 1, -4 \rangle = 0$

$2t - 4 = 0$

$t = 2$

$(4, 3)$

2. A B C

$\overrightarrow{AB} = \langle 1, 2, -4 \rangle$

$\overrightarrow{AC} = \langle 3, 0, -2 \rangle$

$\overrightarrow{AB} \times \overrightarrow{AC} = \langle -4, -10, -6 \rangle$

$= -2 \langle 2, 5, 3 \rangle$

mag : $\sqrt{38}$

$2^2 + 5^2 + 3^2$

$4 + 25 + 9$

3. $\vec{r}(t) = \langle \cos(t), \sin t, \frac{1}{\pi^2} t^2 \rangle$

1 0 0

-1 0 1

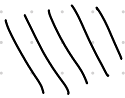
$\int_0^{\pi} \sqrt{1 + \frac{4}{\pi^4} t^2}$

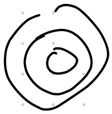
$(1, 0, 0)$ to $(-1, 0, 1)$

$\vec{r}\,' = \langle -\sin(t), \cos(t), \frac{2t}{\pi^2} \rangle$

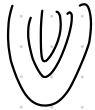
$|\vec{r}| = \sqrt{1 + \frac{4}{\pi^4} t^2}$

4. $z = x + y$ level curves

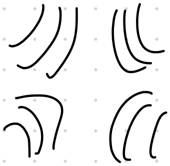




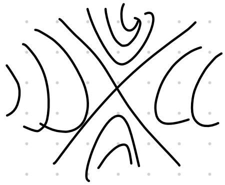
$x^2 + y^2$



$x^2 - y$

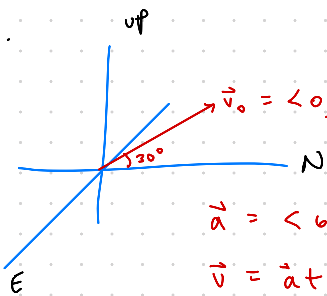


xy



$x^2 - y^2$

5.



$\vec{v}_0 = \langle 0, 128\cos 30°, 128\sin 30° \rangle$

$\qquad\quad 64\sqrt{3} \qquad\quad 64$

$\vec{a} = \langle 6, 0, -32 \rangle$

$\vec{v} = \vec{a}t + \vec{v}_0$

$\vec{v} = \langle 6t, 64\sqrt{3}, -32t + 64 \rangle$

anti deriv

$\vec{d} = \langle 3t^2, 64\sqrt{3}t, -16t^2 + 64t \rangle$

6. $E(x,y) = 3000 - 0.025x^2 - 0.1y^2$

direction of strongest ascent. at 40 m east 20 m south

$\vec{\nabla} E \big|_{(40,-20)} = \langle -0.05x, -0.2y \rangle \big|_{(40,-20)}$

$= \langle -2, 4 \rangle$

$\approx \langle -1, 2 \rangle$ scalar multiple

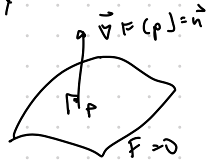
7. Find eq. for tangent plane to $z = \dfrac{xy}{x+y}$ at $(2,-4,4)$

if no z, plug it in

$$xz + yz = xy$$

$$xz + yz - xy = 0$$

$$F$$



$$\vec{\nabla} F = \langle z-y,\ z-x,\ x+y \rangle$$

$(2,-4,4)$

$$\vec{\nabla} F = \langle 8,\ 2,\ -2 \rangle$$

use $\langle 4, 1, -1 \rangle$

$$4x + y - z = ? = 0$$

$$2 \quad -4 \quad 4$$

$$8 - 4 - 4 = 0$$

8. $\iint xe^{xy}$ $\qquad 0 \le x \le 1 \qquad 0 \le y \le 1$

$$\int_0^1 \int_0^1 x e^{xy} \, dy \, dx$$

$$e^{xy} \Big|_{y=0}^{y=1}$$

$$\int_0^1 e^x - 1 \, dx$$

$$(e^x - x) \Big|_0^1$$

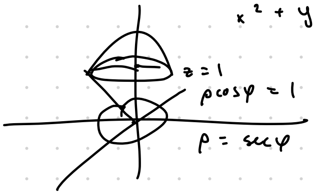
$$e - 1 - 1$$

$$\underline{e - 2}$$

9.

16. bounded above $z = \sqrt{2 - x^2 - y^2}$

bounded below $z = 1$



$\rho \cos\varphi = 1$

$\rho = \sec\varphi$

$x^2 + y^2 + z^2 = 2$

$\rho^2 = 2$

$\rho = \sqrt{2}$

$\sqrt{2} = \sec\varphi$

$\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\varphi = \frac{\pi}{4}$

$\rho \quad \theta \quad \varphi$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{4}$

$\sec\varphi \le \rho \le \sqrt{2}$ plane to hemisphere

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\varphi}^{\sqrt{2}} 1 \; \rho^2 \sin\varphi \; d\rho \, d\varphi \, d\theta$$

because volume

cylindrical

$$\int_0^{2\pi} \int_0^{1} \int_1^{\sqrt{2 - r^2}} 1 \quad r \quad dz \, dr \, d\theta$$

$2 - r^2 = 1$

$r^2 = 1$

$r = 1$

14. $f(x,y)$ find critical pts

$f_x = 6x(y-2x) = 0$ $\qquad f_x = 6xy - 12x^2$

$f_y = 3x^2 - 3y^2 + 27 = 0$

$6x(y-2x)$

$x = 0 \qquad y - 2x = 0$

first order partials $\qquad y = 2x$

$x = 0$:

plug in f_y

$-3y^2 + 27 = 0$

$y^2 = 9$

$y = \pm 3$

$(0, 3)\ (0, -3)$

$y = 2x$

$3x^2 - 3(4x^2) + 27 = 0$

$-9x^2 + 27 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

$(\sqrt{3}, 2\sqrt{3})$

$(\sqrt{3}, -2\sqrt{3})$

$D = f_{xx}f_{yy} - (f_{xy})^2 = (6y - 24x)(-6y) - (6x)^2$

	D	$6yy = -6y$	
(0, 3)	Neg	NA	saddle
	Pos		