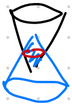
Andrea Tongsak MTH 254

1. Evaluate $\iiint_E z\,dV$ where E is

a) 

$z = \sqrt{x^2+y^2}$ $\qquad z = 2 - \sqrt{x^2+y^2}$

$z = r$ $\qquad z = 2 - r$ $\qquad 2 - z$

z

$r \quad \theta \quad z$

$$\int_0^1 \int_0^{2\pi} \int_r^{2-r} z \quad r \quad dz \, d\theta \, dr$$

$$2\pi \int_0^1 \left. \frac{z^2}{2} \right|_r^{2-r} r \, dr$$

$$\frac{(2-r)^2}{2} - \frac{r^2}{2} = \frac{4 - 4r \cancel{+r^2} \cancel{-r^2}}{2} = \frac{\overset{2}{\cancel{4}}(1-r)}{\cancel{2}}$$

$$2 \cdot 2\pi \int_0^1 (1-r)r \, dr$$

$$4\pi \cdot \int_0^1 r - r^2 \, dr$$

$$4\pi \left. \left(\frac{r^2}{2} - \frac{r^3}{3} \right) \right|_0^1$$

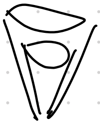
$$4\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$2 - r = r$

$2 = 2r$

$r = 1$

b) between $z = \sqrt{x^2+y^2}$ and $z = \sqrt{3x^2+3y^2}$



$z = r$ $\qquad$ $z = \sqrt{3} \cdot \sqrt{x^2+y^2} = \sqrt{3}\, r$

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{z/\sqrt{3}}^{z} z \quad r \; dr\, dz \; d\theta$$

$z = r$

$\frac{z}{\sqrt{3}} = r$

$$\left. \frac{z r^2}{2} \right|_{z/\sqrt{3}}^{z}$$

$$\frac{z^3}{2} - \frac{z \cdot \frac{z^2}{3}}{2} = \left(z^3 - \frac{z^3}{3} \right) \frac{1}{2}$$

$$\frac{1}{2} \cdot 2\pi \int_0^{\sqrt{3}} \frac{2z^3}{3} \; dz$$

$$\pi \cdot \left. \frac{\cancel{2} z^4}{2 \cdot \cancel{4} \cdot 3} \right|_0^{\sqrt{3}}$$

$$\pi \cdot \frac{9}{6} = \frac{3}{2}\pi$$

2. $\iiint \frac{1}{\sqrt{x^2+y^2}} \, dV$

a. cylindrical (r, θ, z)

$r^2 = 4 - z^2$

$r = \sqrt{4-z^2}$

$\frac{1}{r} \, r \, dr \, d\theta \, dz$

$z = \sqrt{3}$ $\quad r = 0$ $\quad \theta = 0$

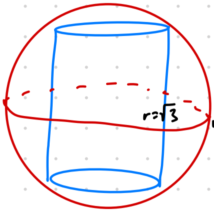
$z^2 = 4 - r^2$ $\quad r = \sqrt{4-r^2}$ $\quad \theta = 2\pi$

$z = 2$

$$\int_0^{2\pi} \int_{\sqrt{3}}^{2} \int_0^{\sqrt{4-z^2}} \frac{1}{\cancel{r}} \cancel{r} \, dr \, dz \, d\theta$$

$$2\pi \int_{\sqrt{3}}^{2} \sqrt{4-z^2} \, dz \approx 1.138$$

b.



region

$x^2 + y^2 + z^2 = 4$ outside $x^2 + y^2 = 3$

$r^2 = 4 - z^2$ $\qquad$ $r^2 = 3$

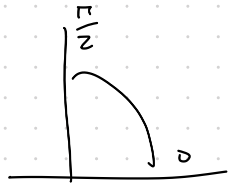
$r = \sqrt{4-z^2}$ $\qquad$ $r = \sqrt{3}$

$$\int_0^{2\pi} \int_{-2}^{2} \int_{\sqrt{4-z^2}}^{\sqrt{3}} \frac{1}{r} \, r \, dr \, dz \, d\theta$$

$$2\pi \int_{-2}^{2} \sqrt{3} - \sqrt{4-z^2} \, dz$$

$$\approx 4.053$$

3. $\iiint x^2+y^2\,dV$ (ρ, θ, φ)



$1 \le x^2+y^2+z^2 \le 9$ $z \ge 0$

$1 \le \rho^2 \le 9$ $\rho\cos\varphi \ge 0$

$1 \le \rho \le 3$ $\cos\varphi \ge 0$

$\varphi \le \frac{\pi}{2}$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^3 (\rho\sin\varphi)^2 \quad \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$2\pi \int_0^{\frac{\pi}{2}}\int_1^3 \rho^4 (\sin^3\varphi)\, d\rho\, d\varphi\, d\theta$$

$$2\pi \int_0^{\pi/2} \frac{\rho^5}{5}\Big|_1^3 \sin^3\varphi$$

$$2\pi \int_0^{\pi/2}\left(\frac{243}{5}-\frac{1}{5}\right)\sin^3\varphi\, d\varphi$$

$$2\pi\cdot\frac{242}{5}\int_0^{\pi/2}\sin^3\varphi\, d\varphi$$

$$\frac{484}{5}\pi \int_0^{\pi/2}\sin^2\varphi\,\sin\varphi\, d\varphi$$

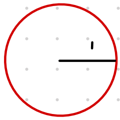
$$\int_0^{\pi/2}(1-\cos^2\varphi)\sin\varphi\, d\varphi$$ $u = \cos\varphi$

$du = -\sin\varphi$

$$\int_0^{\pi/2}(1-u^2)(-1)\,du$$

$$\frac{2}{3}\cdot\frac{484\pi}{5} = \frac{968\pi}{15}$$

4.



$$\rho^2 = 1 \qquad \varphi$$
$$\rho = 1$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (2-\rho) \quad \rho^2 \sin\varphi \, d\rho \; d\varphi \; d\theta$$

$$2\pi \int_0^{\pi}\int_0^{1} 2\rho^2 \sin\varphi - \rho^3 \sin\varphi \quad d\rho \; d\varphi$$

$$\left(\frac{2\rho^3}{3}\sin\varphi - \frac{\rho^4}{4}\sin\varphi\right)\Big|_0^1$$

$$2\pi \int_0^{\pi} \frac{2}{3}\sin\varphi - \frac{1}{4}\sin\varphi \quad d\varphi$$

$$2\pi \int_0^{\pi} \left(\frac{2}{3} - \frac{1}{4}\right)\sin\varphi \quad d\varphi$$

$$\frac{5}{12}\cdot 2\pi \int_0^{\pi} \sin\varphi \, d\varphi$$

$$\frac{5}{6}\pi \qquad (-\cos\varphi)\Big|_0^{\pi} = \frac{5}{6}\pi\left(-\cos\pi + \cos 0\right)$$

$$= 2\cdot\frac{5}{6}\pi = \frac{5}{3}\pi$$

5. $$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dy\, dz\, dx$$

is also...

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\, dz\, dy\, dx$$

$$\int_0^1 \int_0^{1-y^2} \int_0^{1-y} f(x,y,z)\, dx\, dz\, dy$$

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\, dy\, dx\, dz$$

6.

a. $f_x(P)$

0 on A

$-$ on B

$+$ on C

b. $f_y(P)$

$-$ on A

$-$ on B

$-$ on C

c. $f_{xx}(P)$

0 on A

0 on B

$-$ on C

d. $f_{yy}(P)$

$+$ on A

0 on B

$+$ on C

e. $f_{xy}(P)$

0 on A

0 on B

$+$ on C

7. $\vec{r}(t) = \langle 2 + \frac{t}{4}, \sqrt{1+t} \rangle$ $\qquad T_x(4,3) = -5 \qquad T_y(4,3) = 5$

a) $\vec{r}(8) = \langle 4, 3 \rangle$

$t = 8$

$T(x,y) = T(\vec{r}(t))$

$\frac{d}{dt} T(\vec{r}(t)) = T_x x' + T_y y'$ $\qquad \frac{1}{2}(1+t)^{-1/2}$

$= -5 \cdot \frac{1}{4} + 5 \cdot \frac{1}{2}(1+8)^{-1/2}$

$= \frac{-5}{4} + \frac{5}{6} = \frac{-30}{24} + \frac{20}{24}$

$\frac{-10}{24} = \frac{-5}{12}$

b) what rate the temp is changing

$r'(t) = \langle \frac{1}{4}, \frac{1}{2\sqrt{1+t}} \rangle$

$r'(8) = \langle \frac{1}{4}, \frac{1}{6} \rangle$

$\vec{u} = \frac{r'(t)}{\|r'(t)\|}$ $\qquad \frac{r'(8)}{\|r'(8)\|} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{6}\right)^2} \langle \frac{1}{4}, \frac{1}{6} \rangle$

$= \sqrt{\frac{1}{16} + \frac{1}{36}} \langle \frac{1}{4}, \frac{1}{6} \rangle$

$D_u T(4,3) = \nabla T \cdot \vec{u}$

$= \sqrt{\frac{1}{16} + \frac{1}{36}} \left(-5\left(\frac{1}{4}\right) + 5\left(\frac{1}{6}\right) \right)$

8. intersect or skew

$\vec{u}(t) = \langle 1-t,\ 3+2t,\ 9-5t \rangle$
$\vec{v}(s) = \langle 14-3s,\ 2+s,\ 4-s \rangle$

$1-t = 14-3s$
$3+2t = 2+s$
$9-5t = 4-s$

$12-3t = 6$
$6 = 3t$
$t = 2$
$s = 5$

$-1 = 14-3s$
$3s = 15$
$s = 5$

$(2, 5)$ so intersect.

9. $f(x,y) = x^2 + y^2 - 2y$ on $R = \{(x,y) \mid x^2 + 4y^2 = 16\}$

$f_x = 2x = 0 \quad x = 0$

$f_y = 2y - 2 = 0 \quad y = 1$

critical pt: $(0,1)$ $\quad f(x,y) = -1$

Use boundaries (Legrange multiplier)

$g(x) = x^2 + 4y^2 - 16$

$\nabla F = \langle 2x, 2y-2 \rangle \ ; \ \nabla g = \langle 2x, 8y \rangle$

$\nabla F = \lambda \nabla g$ $\quad 2x = \lambda 2x \quad \lambda = 1$

$\nabla g = 0$ $\quad 2y - 2 = \lambda 8y \quad 6y = -2 \quad y = -\frac{1}{3}$

$x^2 + 4y^2 - 16 = 0$

$x^2 + 4\left(\frac{1}{9}\right) - 16 = 0$

$x^2 + \frac{4}{9} - 16 = 0$

$x = \pm\sqrt{\frac{140}{9}}$

$\left.\begin{array}{l} \left(\sqrt{\frac{140}{9}}, -\frac{1}{3}\right) \\ \left(-\frac{\sqrt{140}}{3}, \frac{-1}{3}\right) \end{array}\right\}$ abs. max

$(0,1) \rightarrow$ abs min.