CS 381
Programming Language Fundamentals

Lecture Notes

Metalanguage: **Haskell**
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Contents

1 Introduction

1.1 The Role of Programming Languages in Computer Science

One driving force behind computer science is the goal to automate computational tasks by having them performed by machines. In this context a *programming language* is an interface between humans and machines that enables the description of tasks as algorithms, presented in a form that can be understood and performed by machines.

Since a high-level programming language can generally not be directly understood by machines, any program in such a language needs to be translated into a simpler form before it can be executed. A *compiler* is a program that translates a program in a language L into a different, often simpler, language M that can then be executed by a machine. An *interpreter* combines the translation and execution part.

A programmer who uses L typically doesn't care about how programs are translated to M; and they really shouldn't, because the purpose of L is to abstract from the details of M and empower the programmer to think about algorithms on a higher level.[1] Nevertheless, to be effective, a programmer needs to understand what the constructs of L do, alone and in combination. The importance of understanding the interaction of language constructs is often underestimated. For example, having a good understanding of the assignment operation and expressions only in isolation limits your effectiveness as a programmer. In ad-

[1] Yet, writing a compiler or interpreter for a language L can be an extremely educational endeavor. Moreover, it can be viewed as the ultimate test of understanding L, since one can't ignore any details of L and must be precise about what each construct exactly means.

dition, you need to understand the phenomenon of side effects, that is, the fact that the same expression can result in different values if it is evaluated at different places/times and if it contains variables. Knowledge of side effects and their potential harm can lead to more careful programming practices and helps avoid difficult to debug program faults.

1.2 Why Study Programming Language Fundamentals?

Programming courses teach how to effectively *use* a particular programming language. By taking such a course combined with lots of practice and experience, one can become an excellent programmer in a specific programming language.

Now the story could end right here if it weren't for the fact that computing technology is a fast-evolving field. New application areas constantly lead to requirements for new features to be supported by programming languages. Estimates of the number of programming languages range between 700 and 9,000, which is evidence for the fluidity of the field. Of course, no one can master all these languages, but that shouldn't really be the goal at all. To be a successful programmer, a viable strategy is to master the most popular language, or the one needed for a particular job, and then retrain whenever needed. This utilitarian view of programming languages is certainly adequate for programmers and software engineers, and it seems to be the strategy employed by coding boot camps.

However, a computer scientist is more than just a programmer or a software engineer. No one would expect a physicist or mechanical engineer to be just a good car mechanic who understand the particular workings of a specific set of car models, or a civil engineer able to build a specific kind of houses. Computer science is still a young discipline and incorporates many different applied subjects that over time will likely spin off into separate disciplines. This is what happened in scientific disciplines with a longer tradition. For example, physics has led to mechanical, civil, and electrical engineering, and biology has promoted medicine, agriculture, and ecology. Similarly, we will likely see specialized disciplines such as software engineering separate from computer science.

A computer scientist must understand the underlying principles of computing and how they are reflected in the languages used to describe computation. The study of programming language *fundamentals* abstracts from one programming language and looks more generally at the concepts that can be found in many or most programming languages. Such a more general understanding of programming languages makes you a better and more flexible programmer, since it allows you to learn new languages more quickly. It also supports you in assessing the scope of existing language features, puts you in a position to better judge the need to switch to a different language for a new task, and helps you select

Abstraction Level		Example
5	Meta	Grammar, Rule System
4	Feature	Syntax, Semantics (scope, types, parameter passing schemas)
3	Model/Paradigm	Lambda Calculus, Turing Machine, Predicate Calculus
2	Language	Haskell, List, C, Java, Python, Prolog
1	Program	`fac n = if n==1 then 1 else n*fac (n-1)`
0	Computation	`fac 3` $\Longrightarrow$ `3*fac 2` $\Longrightarrow$ `3*2*fac 1` $\Longrightarrow$ `3*2*1` $\Longrightarrow$ `6`

Table 1.1: The Programming Language Abstraction Hierarchy. The level of abstraction of computer science concepts that are especially relevant to the study of programming languages.

the best language for a particular task.

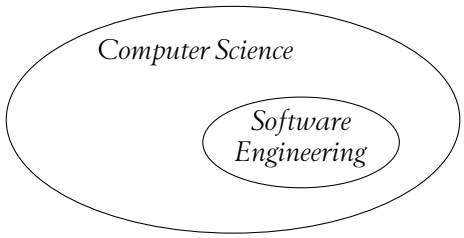
Beyond its practical use, there is another, maybe even more important, reason for studying programming language fundamentals. Computer science is essentially devoted to studying different forms of *abstraction*: An algorithm is an abstraction of computation; Turing machines, pushdown automata, and finite state machines are abstractions of physical computing machines; types are abstractions of values; and the so-called "big O" notation is an abstraction of runtime. A programming language is an abstraction itself, an abstraction of all the programs that can be written in it. The study of programming language fundamentals abstracts further from individual programming languages. Specifically, individual features of programming languages are abstractions of specific algorithmic ideas (loops, parameterization, scope, etc.). Moreover, in the description of programming language features we often use formalization tools, including grammars or rule systems, that live at an even higher level of abstraction.

Table 1.1 shows the range of different abstraction levels. As you can see, programming languages employ many abstractions.[2] In addition, Table 1.2 shows the involvement of different groups of people with the abstraction levels. In particular, we can observe that computer scientists are generally working at one abstraction level above programmers, reminding us of the fact that programming and software engineering are proper subsets of computer science.

[2]Mainly for lack of time we unfortunately cannot cover fundamental models of programming on level 3, but you can learn about Turing machines in CS 321 and lambda calculus in CS 581.

	Objects at Abstraction Level	Who Creates Them?	Who Uses Them?
4-5	Meta & Feature	CS Researchers	***Computer Scientists***
3	Model/Paradigm	CS Researchers	***Computer Scientists***
2	Language	***Computer Scientists***	***Programmers***
1	Program	***Programmers***	Everyone
0	Computation	Everyone	Everyone

Table 1.2: Abstraction users and creators. Most users of abstractions are also abstraction creators. In particular, any user of an abstraction at level $n > 0$ creates an abstraction on level $n - 1$.



The study of programming language principles provides an excellent exposure to identifying, understanding, and applying abstractions. These are crucial activities in the work of any computer scientist. If you are successful as a computer scientist, it is very likely that your are very good at dealing with abstractions.

1.3 What are the Fundamentals of Programming Languages?

How do we tell a calculator to compute the result of $(4 + 1)^2$? Assuming a sophisticated natural language interface, we might try the command C*ompute the square of 4 plus 1*. But how can the calculator decide whether to compute $(4 + 1)^2$ or $4^2 + 1$? Instead of using English, we have to recourse to a notation that is *unambiguous* and that can distinguish between the two expressions. Not relying on a semi-visual notation with exponents, etc., we could write the first expression as `sqr(4+1)` and the second one as `sqr(4)+1`. The difference matters, since the two expressions yield different results.

In general, the definition of a programming language has to specify the *syntax* of programs because only well-formed programs can be executed in a way that results in predictable behavior. For example, an expressions such as `3+sqr` should be rejected, since it is not clear what the intended meaning is.

As another example, consider the task of determining which expression denotes the greater value, 2^3 or 3^2? While calculators provide a rich set of arithmetic operations, they often lack comparison operators such as `>` or `<` and the notion of boolean values. Therefore, we can't directly express this task in terms of the operations offered by the calculator and instead have to simulate the computation. This is not difficult in this case: We can simply compute the difference $2^3 - 3^2$, and if the result is positive, the first expression is the larger one; if the result is negative, it's the second one.

While the encoding of boolean values as integers might seem like a clever idea, it can lead to unexpected program behaviors. For example, if we add a `>` comparison operator to the calculator language and have it return integers, what should be the result of the expressions `(3>2)>1` and `(3>4)+5`? A programming language definition has to decide whether such expressions are valid, and if so, what they evaluate to. It is a non-trivial task to define a large set of operations on a single type in a consistent way. The purpose of introducing *types* into programming languages is to rule out nonsensical programs and catch programming errors early on. A particular design decision for a programming language is *when* to check whether the operations in a program are used in a consistent way with respect to their types. This can be done before the program is run (*static typing*) or while it is being executed (*dynamic typing*)?

To know what can be expressed in a particular programming language and to decide how to express a specific task, one needs to know what values and computations the language provides and what the *semantics* of all the language constructs are. Machines offer only a limited set of operations that can't directly support every imaginable computing application. Therefore, programming languages offer advanced programming constructs to facilitate higher-level descriptions to make the programming of machines easier and more reliable.

Even in the case of a simple calculator, one quickly encounters situations in which one wants to use names in expressing computations. Consider, for example, the task of computing $(371 \cdot 47)^2$ with a calculator that doesn't have a predefined `sqr` operation. Of course, one can simulate `sqr` by entering `(371*47)*(371*47)`, but that's inconvenient and prone to errors. It would be nicer if we could assign a *name* to the product and then use the name for computing the square, as in `let x=371*47 in x*x`.

Factoring repeated program parts using names gets more and more important the bigger programs become. Names are used for simple values, for functions and parameters, for modules, etc., and with the definition and use of names, the question arises whether the same name can be used in different parts of a program, and if so, which of its definitions will be visible where. When a function uses a non-local name, that is, a name not defined as a parameter or locally inside the function, an important question is what non-local name

to use: the one that is visible where the function is defined (*static scoping*) or the one that is visible where the function is used (*dynamic scoping*). These two strategies yield, in general, different results.

Directly related to the concept of names is the question of when and how expressions are bound to names. For example, in `let x=371*47 in 0*x*x` the expression `371*47` could be evaluated and its resulting *value* bound to `x` *before* the expression `0*x*x` is evaluated. Alternatively, a language definition could bind the *expression* `371*47` unevaluated to `x` and evaluate it only when `x` is needed. In this example, the evaluator could determine that `x` is actually not needed to compute the value of `0*x*x`, and so the computation of `371*47` could be saved.

In summary, to understand a programming language, you have to understand its syntax and semantics (which includes a potential type system, the scoping rules, and available parameter passing schemas). How can we accomplish this task?

1.4 How to Study the Fundamentals of Programming Languages

To understand any technical subject, it is not enough to look at examples. Instead one needs to comprehend the fundamental principles that underlie the subject matter. It is no different for programming languages: To understand what a programming language is, it doesn't suffice to look at a handful of examples. In other words, programming in a few languages won't lead to a principled understanding of programming languages.

Since a programming language is defined through its syntax and semantics, we have to understand how to define the syntax and semantics of a programming language in general. The formalisms for syntax and semantics definitions happen to be themselves languages, and thus any language definition necessarily requires the use of one or more so-called *metalanguages*, that is, languages that can describe other languages. The language that is described by a metalanguage is then called the *object language*.

One important metalanguage that is used widely in computer science to describe the *syntax* of languages is the formalism of *context-free grammars*. For the definition of the semantics of languages we have a choice of several different approaches. We will employ *denotational semantics* for this purpose, which identifies a set of semantic values and then maps programs to these semantic values. Why denotational semantics? Because it reflects our intuition about (one aspect of) computation quite well: A computation transforms a representation of a problem into a one of a solution. If we are not interested in the details of *how* this is done but only in the input/output relationship, we regard the computations described by a program as a function. Denotational semantics describes exactly this func-

tion.

As mentioned before, computer science is about the automation of tasks by writing programs that can be executed by machines. Now wouldn't it be great if we could automatically execute language definitions? This would allow us to experiment directly with different variants of syntax and semantics and thus explore the space of programming languages by observing the effects of language definitions.

This is indeed possible by choosing a programming language as a metalanguage for performing language definitions, and we will do this here as well. Specifically, we will employ the functional programming language Haskell for this purpose. While one could in principle use any Turing-complete language, functional languages are particularly well suited for this task, especially for expressing denotational semantics. Since we need Haskell as a metalanguage, this will have to be the first topic to be explored after this introduction.

The approach of understanding programming language concepts by being able to define them follows Richard Feynman's maxim:[3]

> *What I cannot create, I do not understand.*

Or in the words of Albert Camus:

> *To create is to live twice.*

CS 381 is about the *fundamentals* of programming languages; it is effectively a *theory* of programming languages class. CS 381 is much like CS 321 or CS 325, except that mathematical definitions are often expressed in terms of the programming language Haskell.

1.5 About Programming Paradigms

Programming languages can differ notably by the *programming paradigm* they subscribe to. A programming paradigm defines an idiosyncratic view of what computation is and how to describe computation through programs. In other words, a programming paradigm describes a *model of computation*. Understanding different programming paradigms is an important part of understanding the landscape of programming languages and making sense of the variety of programming abstractions that exists because the view of *what* computation is determines *how* to describe it (by programs in a specific programming language).

In general, a computation is a transformation of a problem representation into a representation of a solution for that problem. A model of computation, and thus a programming paradigm, has to define the structure of these representations as well as the details

[3] https://archives.caltech.edu/pictures/1.10-29.jpg

of how transformations between representations can happen. The three major basic programming paradigms take the following points of views.

Imperative Programming. The main idea of imperative programming is to represent a problem as a state using a collection of variables and to compute a solution by manipulating these variables individually through assignment operations, under the control of conditionals and loops whose behavior is determined by expressions that are based on the values of the variables.

- *Representations* have the form of named values that are subject to repeated modification. The collection of named values is called a *state*, and parts of the state can be manipulated by individual *assignments* that access the state through the names, which are also called *program variables*.
- *Computation* is a step-by-step transformation of a state that results from a sequence of assignment statements.
- *Programs* consist of *statements* that are organized by *control structures* for selecting between different (groups of) statements and for repeating them.
- *Execution* of a program requires to define initial values for the program variables (often read through `input` operations).
- *Results* of program executions are the values of the final state.
- *Formalization* of this computation model is through the *Turing machine*.

Functional Programming. The main idea of functional programming is to represent a problem as an expression that is systematically simplified into a value that represents the solution.

- *Representations* of data have the form of atomic values and trees of values. The representation of a problem is given by an expression built from functions applied to data and other functions.
- *Computation* is a step-by-step transformation of an expression into a result value. The transformation of representation essentially happens through the systematic decomposition of trees into components and the construction of new trees, directed by function definitions.
- *Programs* consist of *function definitions*. Control structures are realized by recursion and higher-order functions that take functions as arguments and return functions as results.
- *Execution* of a program means to apply a function to arguments.
- *Results* of program executions are single values.

- *Formalization* of this computation model is through the *lambda calculus*.

Logic Programming. The main idea behind logic programming is to represent knowledge of a problem area by a collection of facts and rules, which are built from terms. A problem is formulated as a query (also called *goal*), which is a term consisting of values and variables that describe a pattern to which a solution is found by matching the pattern against the relations using the rules.[4]

- *Representations* have the form of terms built from atomic values as well as relations of terms and values. The representation of a problem is given by a term (the "goal") that may contain values and variables, and the representation of a solution is a binding for the variables in the goal (as well as an indication of whether or not the goal could be satisfied).
- *Computation* is the construction of a tree of rules instances (called a *derivation*) by repeated generation of bindings through the selection of rules. This process also creates bindings for variables. The systematic exploration of all rules and the process of backtracking allows alternative rules to be tried when the search for a solution gets stuck.
- *Programs* consist of *rule definitions* that define the relationships between inputs and outputs. Control structures are realized by the structure of rule definitions (multiple premises of a rule require the execution of all of them, whereas alternative rules provide choices).
- *Execution* of a program means to formulate a goal, which consists of a term with values and variables.
- *Results* of program executions are bindings of values to variables contained in a goal.
- *Formalization* of this computation model is through *first-order logic* (also known as *predicate logic* or *predicate calculus*).

Computations can be captured in so-called *traces*, which are structures that arrange snapshots of the states a computation went through. It is instructive to compare the different kinds of traces that are produced by the different programming paradigms. The structure of a trace reflects the essence of the paradigm and can help to get a better understanding of the differences between programming paradigms. Simplified example traces for the computation of a factorial are shown in Figure 1.1. Again, the trace for logic programming makes sense only after having studied Unit 8.

Understanding different programming paradigms not only helps with evaluating and

[4] This description may make sense only after having studied Unit 8; so you may want to re-read this explanation later.

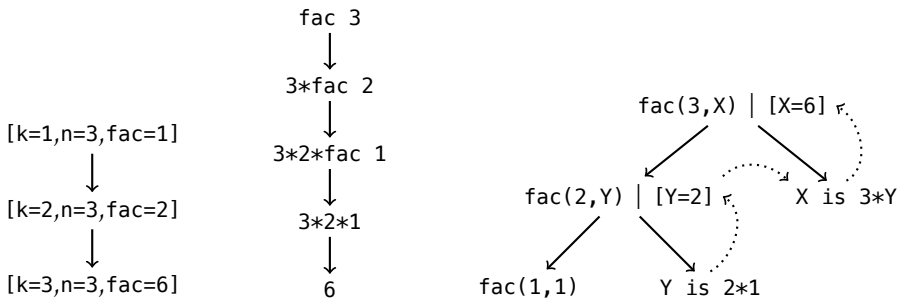


Figure 1.1 Computation traces in different programming paradigms. On the left: A sequence of states characterizes computation in imperative languages. In the middle: A sequence of expression simplifications characterizes computation in functional languages. On the right: A tree of goals and bindings captures (part of) the computation as it happens in logic languages.

selecting a programming language that is best suited for a particular project, it also provides a broader, more diverse understanding of the notion of computation itself and is therefore an important part of the computer science education.

While some languages try to support multiple paradigms, many programming languages fall into one of the three categories in the sense that most programs are written in a style that fits the underlying computational model of the paradigm. One can take a more fine-grained view of programming paradigms and identify further paradigms (such as object-oriented or constraint-based programming), but most of these classifications are built on the three mentioned paradigms.

Most students have extensive experience with imperative programming, but haven't written a functional or logic programs. We will therefore also explore the basics of functional programming in Unit 2 and logic programming in Unit 8. Those chapters also provide more details regarding the differences between paradigms (see, for example, Tables 2.1 on page 12 and 8.1 page 150).

And in case you wonder about the ordering of units (functional programming Unit 2 and logic programming in Unit 8): We start with Haskell because we need it as a metalanguage in the discussion of syntax (Unit 3) and semantics (Unit 4). On the other hand, the discussion of Prolog requires the understanding of some technical ideas that are discussed in earlier units, such as rules and rule systems (Unit 5) and bindings (Unit 6).

2 Haskell

In imperative languages, such as C or Java, computation is expressed as transformations of state, that is, the underlying programming model views a program as a series of statements that perform successive changes to parts of a state that can be accessed through variable names. For example, if a program declares an integer variable `x`, then the assignment `x = 3` has the effect that referencing the variable `x` afterwards will yield the value `3`, at least until `x` is changed by another assignment.

Haskell does not have an underlying global state that can be manipulated. Specifically, Haskell does not have an assignment operation. It doesn't have *while* or *for* loops either. Haskell is a *functional programming language*, in which every computation is expressed as a function. Thus writing a program in Haskell is defining a function, and running a program is applying a function to one or more arguments. The difference between Haskell and imperative languages with regard how they realize algorithmic concepts is summarized in Table 2.1.

2.1 Getting Started

Since you can't learn how to program in a new language by just reading about it, it's best to work through the following examples by trying them out on a computer.

To this end, the first step is to install a Haskell interpreter. There are several ways to do this. Probably the easiest approach is to go to www.haskell.org/platform/ and follow the instructions to download and install the implementation for the operation system on your

Concept	Haskell	C, Java, etc.
Algorithm	Function	Procedure/Method/Program
Instruction	Expression	Statement
Input	Function Argument	`read` Statement
Output	Function Result	`print` Statement
Iteration	Recursion	Loop
Computation step	Expression simplification	State update
Computation	Sequence of steps	

Table 2.1: How algorithmic and computation concepts are realized in Haskell compared to imperative programming languages.

laptop or desktop computer.

It is important to understand that the code presented here occurs in two principally different places, namely (1) in *Haskell files* and (2) in the *interpreter interface*.

First, definitions of values and functions are usually placed in a file (that has the file extension `.hs`). An example is shown in Figure 2.1, which defines the name `alice` to be a number, representing the number of points Alice received on a test. In addition, the file contains the definition of the function `grade` for computing the grade for a particular point score. The value definition for `alice` should be obvious, and I will explain later in Section 2.3 how function definitions work; the details are not important at this point. The file also contains definitions of the two types (the type `Points` as a synonym for `Int` and a data type containing two grade values). Again, the details will be explained later.

Second, starting the Haskell interpreter presents a command-line interface for loading Haskell files and evaluating expressions. The following description uses the Glasgow Haskell Compiler, which is provided as part of the Haskell platform. After starting the interpreter,[1] we are presented with a short message, followed by an input prompt.

```
GHCi, version 9.0.1: http://www.haskell.org/ghc/  :? for help
Loaded GHCi configuration from /Users/erwig/.ghci
ghci>
```

At the prompt, we can enter expressions that the interpreter then evaluates, like so:

```
ghci> 4+5*6
34
```

[1] On Linux and Unix systems, including Mac OS, this is done by entering the command `ghci` in a terminal window.

```
type Points = Int

data Grade = Pass | Fail deriving Show

alice :: Points
alice = 65

grade :: Points -> Grade
grade p = if p>50 then Pass else Fail
```

Figure 2.1 The file `Grading.hs` contains Haskell definitions for two types, a value, and a function.

We can also load Haskell files containing type, value, and function definitions, which are then available for inspection and evaluation.[2]

```
ghci> :l Grading.hs
[1 of 1] Compiling Main             ( Grading.hs, interpreted )
Ok, modules loaded: Main.
ghci>
```

For example, we can use the function `grade` to compute the grade for Alice's point score.[3]

```
> grade alice
Pass
```

You will notice that the syntax for applying functions is different from what you are used to in math. In Haskell (and other functional languages), applying a function `f` to a value `v` is written by simple juxtaposition, that is, `f v`. In particular, parentheses around `v` are omitted unless `f` is applied to an expression.[4]

When we change the definitions in a file and want to use them in the interpreter, we can use the command `:r` to reload the definitions. Later I will explain some additional features of the interpreter.

The important thing to remember is this: When definitions of types, values, and functions are shown, such as for `Point`, `Grade`, `alice`, and `grade`, these appear in some file that

[2] Definitions are always part of a module. When no module name is given, as in the example file, the default module name `Main` is used.

[3] Here and in the following I will use the simpler prompt `>`. The prompt is one of several parts of the interpreter that can be customized in the `.ghci` file that is loaded whenever `ghci` is started.

[4] Thus we write `grade alice` to compute the grade for Alice but `grade (alice-15)` to compute the grade when we want to deduct `15` points for a late submission. Note that the expression `grade alice-15` is type incorrect, since it is parsed as `(grade alice)-15` and tries to subtract a number from a `Grade` value.

INCREMENTAL PROGRAMMING

In many cases a functional program f can be effectively developed by following a divide-and-conquer strategy that tries to *simplify* and/or *decompose* f, whichever is applicable.

Simplify: For example, if f is too complex to be defined directly, try to identify special cases or make simplifying assumptions and implement (several versions of) f for the simplified case(s). This may include reducing the number of parameters for f and using constants instead of parameters in the definition as well as only dealing with a subset of the possible input cases. It may also include defining f for arguments that have a less general or structurally simpler type than required. Then test the function on several examples. (These test cases should be stored in the Haskell file for easier reuse.) After that, try to gradually generalize the definition.

Decompose: If the problem to be implemented consists of several subtasks, try to express f as a combination of functions $f_1, f_2, \ldots$ that solve these subtasks, and implement and test the functions $f_1, f_2, \ldots$ independently. Then test f.

Note that both these strategies can often be employed in parallel.

can be loaded into the Haskell interpreter. On the other hand, expressions to be evaluated must be entered into the Haskell interpreter, which is indicated by the `>` prompt that precedes expressions and by the result printed on the next line. For how to effectively develop functional programs, see the box INCREMENTAL PROGRAMMING.

Oh, and then there will be errors. Inevitably, you will enter an erroneous expression into the interpreter or place an incorrect definition in a file. Since machines are so picky in what input they accept, talking to a machine can be a frustrating experience. Errors can be simple misspellings, which are easy to spot and correct, as in the following example.

```
> grades

<interactive>:2:1: error:
    • Variable not in scope: grades
    • Perhaps you meant ‘grade’ (line 9)
```

But some error messages can be more difficult to understand.

```
> grade

<interactive>:3:1: error:
    • No instance for (Show (Points -> Grade))
        arising from a use of ‘print’
        (maybe you haven't applied a function to enough arguments?)
    • In a stmt of an interactive GHCi command: print it
```

This message essentially says that function values cannot be printed, which means functions must be applied to their arguments to produce printable results (which is mentioned in the eror message). In this example, `grade` is a function that produces a value of type `Grade` only when it is applied to a `Point` value. You should not hesitate to consult outside resources for help when you cannot make sense of an error message. In some cases, a simple Google query will help. In other situations, try the mailing list for beginners (beginners@haskell.org) or an IRC channel (wiki.haskell.org/IRC_channel).

2.2 Expressions, Values, and Their Types

We have already seen integer values and expressions and how expressions are evaluated to values in the interpreter. Other basic types include floating point numbers (type `Float`), characters (type `Char`), and Booleans (type `Bool`). We can instruct the interpreter to show the type of evaluated expressions in addition to their resulting value as follows.

```
> :set +t
```

Now when we evaluate expressions, the type of the result is shown as well.

```
> grade alice
Pass
it :: Grade
```

```
> 7 == 9
False
it :: Bool
```

```
> succ 'a'
'b'
it :: Char
```

```
> not False
True
it :: Bool
```

The name `it` is automatically bound to the last value in the interpreter, which comes in handy when we want to compute values through a series of steps.

```
> not it
False
it :: Bool
```

Working with number types can be a bit tricky sometimes, since the number symbols and arithmetic operations are overloaded. For example, `4` can stand for an integer or a float, and `+` denotes both integer addition and float addition. Thus if we again evaluate the earlier expression `4+5*6`, Haskell responds with the type expression `Num a => a`, which says that the result is of some type `a` as long as it is a number type.[5]

[5] Formally, `a` is a type variable that represents any possible type, and `Num a` is a so-called *type class constraint*, which means the type `a` must support the operations defined by the type class `Num`. A Haskell type class is similar to a Java interface.

```
> 4+5*6
34
it :: Num a => a
```

Haskell tries to infer the most general type of any expression, which provides some flexibility in programming. For example, the above result can be used further as part of an integer operation (such as `even it`) or float operation (as in `it + 1.1`).

The details of Haskell's type classes are not important here, but one should be aware that they exist, since error messages unfortunately often mention them. In the following, unless explicitly mentioned otherwise, I will assume an `Int` type for numbers. In addition to `Num`, we already have seen a glimpse of the type class `Show` (in the file `Grading.hs`), which contains types whose values can be printed. We will later also encounter the type class `Eq`, which contains type whose values can be compared with `==`.

Finally, note that there is an important difference between conditional *expressions* (in Haskell) and conditional *statements* as found in imperative languages: In Haskell the arguments of the `then` and `else` branches are expressions to be evaluated depending on the condition.[6]

```
> if 1+2==3 then 4 else 5
4
```

Exercise 2.1

(a) Write an expression that computes the larger value of `5` and `2*3` (without using the function `max`).

(b) Write an expression that tests whether `2*3` is equal to `3+3`.

(c) Is the expression `(1==2)==(3==4)` well defined? If so, what is the result?

(d) How about `(1<2)<(3<4)`? Is this expression well defined? If so, what is the result?

2.2.1 Names

An expression may contain names that are to be replaced by values during evaluation. A name, also called a *variable*, must start with a lowercase character and may contain characters, numbers, or the two symbols `_` or `'`.

Of course, only if the names are defined (and in scope) can such an expression be successfully evaluated. As shown earlier, names can be defined in Haskell files, but they can

[6] From now on I will generally omit the type of results in the interpreter and show it only in cases when it helps illustrate an important point.

also be defined "on the fly" with so-called `let` expressions that have the form `let x=e in b`. Such an expression binds `e` to the variable `x`, which may be referenced in `b`, the so-called *body* of the `let` expressions. Values of any type can be bound to names, that is, bindings also work for functions.

```
> alice+1
66
```

```
> let f=grade in f alice
Pass
```

```
> let late=alice-15 in grade late
Fail
```

Note that in nested definitions, inner definitions hide (temporarily) outer definitions.

```
> let x=1 in x+(let x=5 in 2*x)+x
12
```

How is this result actually obtained? A `let` expression `let x=e in b` is evaluated by substituting `e` for all occurrences of `x` in `b`. In the last example, this means to substitute `1` for `x` in `x+(let x=5 in 2*x)+x`, which first results in the expression `1+(let x=5 in 2*x)+1`. Note how the nested `let` expression shields the use of `x` in its body from the substitution. This `let` expression can be evaluated in the same way by substituting `5` for `x` in `2*x`, which yields `2*5`. Thus the overall expression simplifies to `1+2*5+1`, which then evaluates to `12`.

Haskell uses so-called *lazy evaluation*, which means that expressions are *not* evaluated before they are bound to a name. Instead, names refer to unevaluated expressions, and an expression is evaluated only when needed, at which point the expression is replaced with its result. In many cases the difference compared to Call-By-Value[7] cannot be noticed, but it can be observed when expressions are bound to a name that is never used, as in the following example.

```
> let x=1/0 in 3
3
```

Since the variable `x` is never used, the expression `1/0` is not evaluated. In contrast, under Call-By-Value where expressions are evaluated before being bound to names, the division would cause a runtime error, and the whole expression would fail to evaluate.

The `let` construct also allows the definitions of multiple variables, either in parallel using tuples or in sequence over multiple lines. This latter form is primarily for the use in program files.

```
> let (x,y) = (1,2) in (y+1,x-1)
(3,0)
```

```
let x=1
    y=x+1
in (x,y)
```

```
let f x y = x*y
    g z = f 2 z
in g 3
```

[7] Call-By-Value is the evaluation model in which expressions are evaluated before bound to variables. Call-By-Value is used in most mainstream programming languages. We will discuss different evaluation models later in Unit 7.

Finally, note that names can also be defined within the interpreter and then subsequently used repeatedly, like so.

```
> let x=3
> x
3
> x+1
4
```

Since such definitions are lost when the interpreter session ends, it is generally advisable to keep definitions as part of a Haskell file that is loaded into the current interpreter session.

Exercise 2.2

What is the result of evaluating the following expressions?

(a) `let x=1 in let x=2 in x+x`

(b) `let x=1 in (let x=2 in x)+x`

(c) `let x=1 in (let y=x in (let x=y+1 in x))+x`

(d) `let x=let y=1 in y+1 in 2*x`

2.2.2 Tracing Evaluations

The behavior of functions and expressions can be explained by tracing the evaluation of expressions step by step. This is particularly useful when expressions involve names. Consider again the expression `let x=1 in x+(let x=5 in 2*x)+x`. The challenge in this example is to resolve the different bindings for `x` correctly.

```
   let x=1 in x+(let x=5 in 2*x)+x
=> let x=1 in x+2*5+x
=> 1+2*5+1
=> 12
```

In many cases, it is possible to obtain different traces by simplifying subexpressions in a different order. Since Haskell expressions don't have any side effects (and since Haskell is lazily evaluated), the order of evaluation does not change the final result. For example, we could have just as well evaluated the outermost subexpression first.

```
   let x=1 in x+(let x=5 in 2*x)+x
=> 1+(let x=5 in 2*x)+1
=> 1+2*5+1
=> 12
```

Tracing expression evaluation is particularly useful in understanding the behavior of recursively defined functions and functions defined on data types such as lists. As an example, consider the following definition of the function `rev` for reversing the elements of a list.

```
rev :: [Int] -> [Int]
rev []     = []
rev (x:xs) = rev xs ++ [x]
```

In addition to the type signature that says that `rev` takes a list of integers and produces a list of integers, the definition consists of two equations. The first equation defines that reversing an empty list, which is written as `[]`, yields an empty list. The second equation says that reversing an non-empty list, that is, a list that starts with some element `x` and is followed by a rest list `xs`, can be achieved by reversing `xs` and appending `x` at the end.[8]

We will discuss recursion in Section 2.4 and lists in Section 2.5. At this point, we want to understand how `rev` works, and we can do that by tracing example computations. Here are the first two steps in the trace for the evaluation of the expression `rev [1,2,3]`.

```
   rev [1,2,3]
=> rev [2,3] ++ [1]
```

Note that the expression in the second line is grouped as `(rev [2,3]) ++ [1]`, since function application binds stronger than infix operations.

Each step in a trace is obtained by substituting a function application by its result. For functions defined by multiple equations this means to select the equation that matches the argument. In our example, since `rev` is applied to a non-empty list, the second equation must be used.

To apply the equation we must first match the argument list `[1,2,3]` against the pattern `x:xs`. A pattern is an expression that may contain variables. In this example, the pattern `x:xs` denotes a list with first element `x` and rest list `xs`. Matching a pattern against a value means to create bindings for the variables in the pattern, using the values from the argument. In this case, since the argument list `[1,2,3]` is just a syntactically nicer representation of `1:[2,3]`, `x` will be bound to `1`, and `xs` will be bound to `[2,3]`.

Once the pattern on the left side of the equation has been successfully matched against the argument, the equation can be applied, which means to replace the function application in the trace with the right-hand side of the equation, using the just-created bindings to substitute the variables. In our example this means to replace `rev [1,2,3]` by `rev xs ++ [x]` while substituting `x` by `1` and `xs` by `[2,3]`. This step can be made precise by using the trace notation: We first perform a substitution step, followed by a simplification of the `let`

[8] Here `[x]` creates a singleton list consisting of the element `x`, and `++` is a function that appends two lists.

expression.

```
   rev [1,2,3]
=> let x=1 in let xs=[2,3] in rev xs ++ [x]
=> rev [2,3] ++ [1]
```

The intermediate step that makes the variable bindings explicit is usually omitted, but there is nothing wrong with including it if it is helpful in explaining how the bindings are handled.

The next step in the trace is obtained in exactly the same way, except that pattern matching now yields different variable bindings, that is, we replace the application `rev [2,3]` as follows.

```
   rev [2,3]
=> let x=2 in let xs=[3] in rev xs ++ [x]
=> rev [3] ++ [2]
```

The resulting expression is placed in the context of `rev [2,3]` of the trace, which leads to the following step in the trace.

```
   rev [2,3] ++ [1]
=> rev [3] ++ [2] ++ [1]
```

At this point we[9] have a choice to either continue with evaluating `rev [3]` or evaluating the rightmost application of `++` (which is allowed because the function `++` is associative). This is often a matter of taste. In the interest of keeping traces small, let's evaluate the subexpression `[2] ++ [1]` next, which leads to the following step in the trace.

```
   rev [3] ++ [2] ++ [1]
=> rev [3] ++ [2,1]
```

We have treated `++` here as an atomic operation that concatenates two lists, which makes sense, since this effect that is easy to show in a trace. However, `++` is actually itself a recursively defined function, and we could thus inspect the details of the computation by constructing a trace (see Exercise 2.3).

The next step in the trace is to evaluate `rev [3]`. Since `[3]` is syntactic sugar for the list constructed by `3:[]`, this leads to the following steps.

```
   rev [3]
=> let x=3 in let xs=[] in rev xs ++ [x]
=> rev [] ++ [3]
```

[9] While the Haskell interpreter will always evaluate in a specific order, humans who want to make sense of evaluations through traces can very well exploit algebraic laws about operations to explore different ordering.

We can integrate this simplification into the trace and after that again evaluate another application of ++.

```
   rev [3] ++ [2,1]
=> rev [] ++ [3] ++ [2,1]
=> rev [] ++ [3,2,1]
```

At this point, rev is applied to the empty list, which means we must use the first equation that simply yields []. By evaluating another append operation we get our final result.

```
   rev [] ++ [3,2,1]
=> [] ++ [3,2,1]
=> [3,2,1]
```

As a summary, here is the complete trace we just discussed, plus an alternative that results when we evaluate recursive calls always before ++.

```
   rev [1,2,3]
=> rev [2,3] ++ [1]
=> rev [3] ++ [2] ++ [1]
=> rev [3] ++ [2,1]
=> rev [] ++ [3] ++ [2,1]
=> rev [] ++ [3,2,1]
=> [] ++ [3,2,1]
=> [3,2,1]
```

```
   rev [1,2,3]
=> rev [2,3] ++ [1]
=> rev [3] ++ [2] ++ [1]
=> rev [] ++ [3] ++ [2] ++ [1]
=> [] ++ [3] ++ [2] ++ [1]
=> [3] ++ [2] ++ [1]
=> [3,2] ++ [1]
=> [3,2,1]
```

The tracing examples highlight the simplicity of functional programming: Any computation happens by repeatedly replacing function applications by the function's definition in which all parameters are substituted by the arguments of the application (plus evaluation of basic operations, such as arithmetic). This observation is reflected by the fact that lambda calculus, the computational model that underlies functional programming, consists of only a single rule for computation (called β-reduction).

Exercise 2.3

Consider the following definition of the function ++ for appending two lists.

```
(++) :: [Int] -> [Int] -> [Int]
[]     ++ ys = ys
(x:xs) ++ ys = x:(xs ++ ys)
```

Create a trace for the evaluation of the expression `[1,2] ++ [5] ++ [8]`.

2.2.3 Tuples

A tuple is a data structure for representing aggregate data. A tuple can have two[10] or more components, which can be of potentially different types. Tuples can also be nested.

```
> (2,3+1)
(2,4)
```

```
> (2==3,5,'a')
(False,5,'a')
```

```
> (2==3,(5,'a'))
(False,(5,'a'))
```

The type of a tuple is a tuple type, which has the types of its components as argument types. The two type constraints in the following example may seem strange, but all they are saying is that both components are numbers, which could be of different number types (one could be an integer while the other one could be a float).

```
> (2,4)
(2,4)
it :: (Num a, Num b) => (a, b)
```

This is the reason for why the following example works: At the time of definition, both components of `p` have an unspecified number type, which allows the later use of the first component as an integer and the second one as a float.

```
> let (x,y) = (2,3) in (even x,sqrt y)
(True,1.7320508075688772)
```

Note also that the nesting structure of tuples matters and is part of the type of a tuple, which means that, for example, the values `(1,(2,3))`, `((1,2),3)`, and `(1,2,3)` all have different types and can't even be compared.

The components of a tuple are accessed by position, which is typically done using pattern matching (as shown in the previous example and to be discussed in more detail in Section 2.6). For pairs the two predefined functions `fst` and `snd` can access the first and second components, respectively.

```
> fst (2+1,5)
3
```

```
> snd (2+1,5) + 2
7
```

```
> fst (snd (1,(2,3)))
2
```

2.3 Functions

Functional programming is centered around the definition and use of functions. A function takes an argument (of some type `T`) and transforms it into a result (of some type `U`), and

[10] A one-tuple is indistinguishable from a single value; the null-tuple value `()` is also called *unit*; it's the only value of type `()` (also called *unit*).

the function has then the function type `T -> U`. The transformation performed by a function is defined by an expression. Functions also typically have a name that allows them to be used many times, but this isn't strictly necessary as we will see.

2.3.1 Function Application

The syntax for applying a function `f` to an argument expression `e` is `f e`, that is, it uses plain prefix notation.[11] Note that `e` needs to be in parentheses only if it is not a name or value but another expression built through function application. This means that parentheses in an application such as `f(3)` are redundant and are omitted in Haskell.

```
> even 4
True
```

```
> even(4)
True
```

```
> even (3*2)
True
```

```
> succ 3*2
8
```

As can be seen in the last example, function application binds strongest, that is, the expression is parsed as `(succ 3)*2`; to compute the successor of `3*2`, the argument must be put into parentheses. This syntactic rule is also the reason why the following example generates a type error.

```
> even 3*2

<interactive>:1:7: error:
    • No instance for (Num Bool) arising from a use of ‘*’
    • In the expression: even 3 * 2
      In an equation for ‘it’: it = even 3 * 2
```

In this expression `even 2` is of type `Bool`, which is not a numeric type (that is, `Bool` is not an instance of the type class `Num`) and thus causes the type error, since `+` requires two numbers as arguments.

How do we apply functions that take more than one argument? We have seen that some binary operations are written using infix notation in between their arguments, but that doesn't work for functions with three or more arguments. One possibility is to group all arguments into a tuple and apply the function to the tuple. We have seen that already with the functions `fst` and `snd`. In functional languages, however, the general approach is to simply list the arguments following the function, again, adding parentheses only when necessary. For example, to compute the smaller of two numbers we write the following.[12]

```
> min 7 (5*2)
7
```

[11] Binary operations such as `+` are written using infix notation.

[12] Quick test: What's the result of `min 7 5*2`?

2.3.2 Currying and Partial Function Application

What is the type of the function `min`? It can't be `(Int,Int) -> Int`, because in that case, we would have to apply the function to a pair, as in `min (7,5*2)`. It turns out that the type of the `min` function[13] is actually as follows.

```
min :: Int -> Int -> Int
```

You can read and understand this type simply as saying that `min` takes two integer arguments (separated by space) and returns an integer. But still the notation may look peculiar. In particular, one may wonder what the purpose of the two function arrows is. The function arrow associates to the right, which means the above type is parsed as follows.

```
min :: Int -> (Int -> Int)
```

Now this type says that `min` is a function that takes a value of type `Int` and returns a (unary) function of type `Int -> Int`. The `min` function therefore takes an integer n and returns a function that computes the minimum of any value compared to n. Specifically, this means we can apply `min` to only one integer, bind the resulting function to a name, and then use that name to apply the function repeatedly.

```
> let f = min 3 in (f 1,f 2,f 3,f 4,f 5)
(1,2,3,3,3)
```

Applying a function to only some of its arguments is called *partial function application*. It is a convenient feature that supports creating specialized instances of functions "on the fly." These kinds of function definitions that support partial function application are called *curried function definitions*.

The fact that curried function definitions return functions as results can be made clear by considering an alternative way of defining functions. In Haskell, we can write down a function value using so-called *lambda abstraction*. For example, the square function can be written as follows.

```
\x->x*x
```

We can directly apply the function, put it into a data structure, or give it a name and then use it through the name.

[13] As already mentioned, many functions in Haskell are overloaded. And `min` is no different; its most general type is actually `min :: Ord a => a -> a -> a`, which means that it can compute the smaller of two values for any type that has a `<` operation defined, which are many more types than numbers. For example, `min True False` = `False` and `min 'b' 'y'` = `'b'`.

```
> (\x->x*x) 4
16
```

```
> head [\x->x*x,succ] 5
25
```

```
> let sqr = \x->x*x in sqr 6
36
```

In general, any equation `f x = e` can be also written as `f = \x->e`. For a function such as `min` this means we can either define it by listing all parameters on the left of the = sign, like so.

```
min x y = if x<y then x else y
```

Or we can move arguments over to the right and turn the expression into a lambda abstraction, that is:

```
min x = \y->if x<y then x else y
```

Both definition are equivalent, but the latter emphasizes the fact that a function of n arguments can be viewed as a function of 1 argument that returns a function of $n-1$ arguments.

Partial function application works also for binary operations that are written in infix notation by placing the operation in parentheses and adding only its left or right argument. Here are some examples.[14]

```
suc   = (+1)
dbl   = (2*)
sqr   = (^2)
pow2  = (2^)
recip = (1/)
isDig = (<10)
```

```
> (suc 4,suc 5,suc 6)
(5,6,7)
> (dbl 4,dbl 5,dbl 6)
(8,10,12)
> (sqr 4,sqr 5,sqr 6)
(16,25,36)
```

```
> (pow2 4,pow2 5,pow2 6)
(16,32,64)
> (recip 4,recip 5,recip 6)
(0.25,0.2,0.16666666666666666)
> (isDig 4,isDig 5,isDig 11)
(True,True,False)
```

Note that it generally matters whether you supply the left or right argument to a binary operation (cf. examples `sqr` and `pow2`). The type reductions for these partial applications work like in the general case. For example, the type of `+` (on integers) is as follows.[15]

```
(+) :: Int -> Int -> Int
```

For the partially applied `(+1)` we have:

```
(+1) :: Int -> Int
```

Partial function application is possible in languages that have so-called *higher-order functions*. Specifically, it must be possible for functions to return functions as a result. See

[14] You can place the definitions of the left column into a file and then use the expressions in the middle and right column as shown.

[15] Note that putting an infix operation into parentheses turns it into prefix notation function, that is, we can write `3+5` also as `(+) 3 5`.

Section 2.7 for more details.

Exercise 2.4

(a) Assume that `p = (1,2)` and `q = (3,4)`. Write an expression that returns the smallest of the four values. (Hint: Use the functions `min`, `fst`, and `snd`.)

(b) A clever alternative solution to (a) is to use the expression `fst (min p q)`. Try to explain why this solution works. Why is it possible to apply `min` to pairs? Does this expression work for other pairs? How do you need to change the solution for the definitions `p = (4,3)` and `q = (2,1)`?

2.3.3 Function Definitions and Pattern Guards

A function definition consists of a name, a type, and a defining expression. Consider the function `min3` that computes the smallest of three numbers. Since the function takes three arguments and returns one result, its type signature should be as follows.

```
min3 :: Int -> Int -> Int -> Int
```

Strictly speaking, a definition doesn't always need a type signature, since the type can in most cases be inferred from the definition itself. However, starting any (non-trivial) function definition by giving a type signature is a good idea because types describe functions and programs on a high level. A type acts as documentation and gives the programmer the opportunity to express their intention about the function. The implementation of the function can then be checked against this intention by the type checker, and potential errors in the implementation can often be identified by the compiler.

Before reading any further, think about how you would define a function `min3` that computes the smallest of three numbers. An implementation that uses a nested `if-then-else` with multiple comparisons should aim at describing three conditions that have to be true for each argument to be the result. Here is a possible implementation (note that `&&` is Haskell's logical *and* operation).

```
min3 x y z = if x<=y && x<=z then x else
             if y<=x && y<=z then y else z
```

This definition is not so easy to read. One could use line breaks and indentation, but that makes the definition needlessly longer. Haskell provides so-called *pattern guards* that can be attached to (groups of) arguments to create conditional equations. Employing pattern guards, the above definition looks as follows.

```
min3 x y z | x<=y && x<=z = x
           | y<=x && y<=z = y
           | otherwise    = z
```

This syntax is less cluttered and clearly delineates the different cases. This style imitates mathematical function definitions. Note that pattern guards do not extend the expressiveness of the language; they are simply a syntactic convenience that makes function definitions easier to read and understand.

While pattern guards often help improve function definitions, the last definition for `min3` is still far from optimal. When implementing a function, you should always think about possibly reusing functions that have already been defined. For this example, we can profitably put the function `min` to work to obtain the following succinct implementation of `min3`.

```
min3 :: Int -> Int -> Int -> Int
min3 x y z = min x (min y z)
```

Such a definition is preferable in most situations, since it is shorter and (in this case) requires less effort to understand. Being more succinct offers fewer opportunities for making mistakes (it's very easy to swap variable names in the conditions, which might lead to an error that might be difficult to detect).

Exercise 2.5

Define a function `sign` that takes an integer and returns `-1` if the argument is negative, `1` if the argument is positive, and `0` otherwise. First, define the function using `if-then-else`. Then give a definition that uses pattern guards.

2.4 Iteration and Recursion

Consider the following excerpt from a C program for computing the factorial of 6. The while loop iterates over the values of `n` (downwards to 1) and accumulates the result in the variable `f`.

```
int f = 1;
int n = 6;
while (n>1) {
  f = f*n;
  n = n-1;
};
printf("%d\n",f);
```

The while loop effectively works like a conditional plus a jump,[16] that is, if the loop condition is true, the statements in the body of the loop are executed, followed by jump to the beginning of the loop. If the loop condition is false, the computation continues with the first statement after the loop. To concretely illustrate this fact, we can rewrite the program snippet into the following equivalent C code.[17]

```
int f = 1;
int n = 6;
fac:
if (n>1) {
  f = f*n;
  n = n-1;
  goto fac; }
else
  printf("%d\n",f);
```

Any while loop works by manipulating some part of the program state. In this example the state changes affect the two integer variables `f` and `n`. Specifically, `f` is updated to be `f*n`, and `n` is updated to be `n-1`.

If we had a mechanism to make this state explicit in the name for the loop (represented by the label `fac`) through, say, a list of parameters, then we could pass the updates directly to the parameters in the jump, and we could also initialize the loop in the same way. If we do that and also drop the `goto` keyword, we obtain the following program fragment.

```
fac(1,6);

fac(int f,int n) {
if (n>1)
   fac(f*n,n-1);
else
   printf("%d\n",f);
}
```

You may already have noticed that this is actually valid C code for the definition and use of a recursive function `fac`.

The point of this example is that a recursive function definition can act as a description of a while loop. In fact, it could be arguably considered a better notation for loops than the while syntax because:

[16] That's actually what while loops are compiled into on most architectures.

[17] Labels and `goto` statements are indeed part of the C programming language.

- it makes the type of state and its initialization explicit,
- it treats updates as transactions (that is, performs them independently of one another) and thus simplifies code understanding by avoiding state dependencies, and
- it makes the loop reusable by giving it a name

In Haskell we can define the same function as follows. The major conceptual difference is that the Haskell function returns a value, whereas the C function performs an output action.

```
fac :: (Int,Int) -> Int
fac (f,n) = if n>1 then
               fac (f*n,n-1)
            else
               f
```

The use of a pair for the two arguments and the multi-line conditional illustrate the similarity to the C notation. In particular, the loop has to be called in the same way by explicitly passing both arguments for `f` and `n` to `fac`. A Haskell programmer would probably define the function in the following way.

```
fac :: Int -> Int -> Int
fac f n | n>1       = fac (f*n) (n-1)
        | otherwise = f
```

Similarly, this definition requires the loop to be called as `fac 1 6`. However, we can put the curried function definition to good use here and partially apply `fac` to the initial value of the accumulator `f` and define the following function.

```
fact :: Int -> Int
fact = fac 1
```

We can now compute the factorial of 6 through the call `fact 6`.

We can observe that the accumulated factorial value does not have to be carried around explicitly in the state, since the intermediate values can be computed by the recursive function itself. This is often the case for problems that are by their very nature recursive, that is, the definition of the function tracks the inductive definition of the input. In the case of the factorial, the definition consists of two cases: one for the base case 1, and one for the general case n that is based on the factorial for the simpler value $n - 1$. Accordingly, the standard recursive implementation of the factorial function needs only one parameter and can be directly implemented as follows.

```
fac :: Int -> Int
fac n | n>1       = n * fac (n-1)
      | otherwise = 1
```

Finally, this definition can be simplified yet a little bit more by using pattern matching. We will look into pattern matching in more detail in Section 2.5 when we talk about lists, but the basic idea can also be illustrated with numbers.

A function definition can actually contain multiple equations that are tried one-by-one, from top to bottom. For example, we could define `min3` by 3 separate equations as follows.

```
min3 x y z | x<=y && x<=z = x
min3 x y z | y<=x && y<=z = y
min3 x y z                = z
```

When `min3` is applied, the arguments are bound to the parameters of the first equation, and if the attached condition evaluates to `True`, the result `x` is returned. Otherwise, the next equation is tried, which means the arguments are again bound to the parameters, and if the attached condition evaluates to `True`, the result `y` is returned. If the condition isn't true either, the last equation is used. No condition is needed here, since if the first two cases don't apply, `z` must be the minimal value.

In addition to attaching conditions to the parameters of an equation, we can also use specific values instead of variables to restrict the applicability of an equation. For example, we can rewrite the factorial function even more succinctly as follows.[18]

```
fac :: Int -> Int
fac 1 = 1
fac n = n * fac (n-1)
```

When `fac` is applied to a number, the first equation is tried, and the result `1` of the right-hand side of the equation is returned only if if the argument is `1`. Otherwise, the second equation is tried and executed.

For some general strategies for how to approach recursive function definitions, see the box RECURSIVE FUNCTION DEFINITIONS on page 32.

2.5 Lists and Pattern Matching

Lists are the most important data structure in Haskell and any functional language. Lists are a specific example of a *data type*. Each data type has one or more so-called *constructors*

[18] Note that the definitions are not equivalent: The last definition of `fac` does not terminate for arguments that are smaller than 1.

Exercise 2.6

Remember to approach the definition of a recursive functions by first identifying the base case(s) and then the inductive/recursive case.

(a) Define a function `ld :: Int -> Int` that computes base-2 logarithms of integers as (rounded down) integers. For example, `ld 512` = `ld 513` = `9` while `ld 511` = `8`.

(b) Define a function `isEven :: Int -> Bool` that yields `True` for even numbers and `False` for odd numbers. You should use pattern matching for the definition. (Hint: You need three equations: one for the base case `0`, one for the base case `1`, and one case for all other numbers `n`, which requires a recursive definition.)

(c) Define a function `fib :: Int -> Int` for computing Fibonacci numbers.

(d) Define three versions of the function `isPos :: Int -> Bool` that returns `True` for numbers that are greater than zero and `False` otherwise. For the first version use a conditional, for the second version use pattern guards, and for the third version, use partial function application with the function `(>) :: Int -> Int -> Bool`.

(also sometimes called *data constructors*), each of which carries zero or more arguments of potentially different types to represent the information for that particular case of the data type. Lists have the following two constructors.

`[]` to represent an empty list, and
`:` to add an element at the front of a list (the constructor `:` is called *cons*)

Specifically, the syntax for adding `e` at the beginning of list `l` is `e:l`. Thus, to construct a list of the number 3 followed by the number 5, we can write the following.

```
> 3:5:[]
[3,5]
```

```
> [3,5]
[3,5]
```

```
> 3:[5]
[3,5]
```

```
[3 .. 5]
[3,4,5]
```

On the left we can observe that Haskell prints lists by simply listing the elements in square brackets, and since lists are so important, this syntax is also offered as a more convenient way to write down list constants, as shown in the second example. We can also construct lists from elements within a specified range, as shown on the right.

Lists are a *polymorhic* data type, that is, lists can store elements of arbitrary types, so we can have lists of booleans, lists of characters, lists of lists of numbers, and even lists of functions.

```
> [1==2,True]
[False,True]
```

```
> ['H','e','l','l','o']
"Hello"
```

```
[[1..2],[],[2..4]]
[[1,2],[],[2,3,4]]
```

RECURSIVE FUNCTION DEFINITIONS

Any recursive definition always consist of at least a *base case* and an *inductive case*.

An effective approach for the definition of a recursive function is therefore to first identify the base case(s) by asking, *For which argument is the function definition trivial*? and, *What is the result in this case*? In some instances, there might be multiple different base cases with possibly different results. In any case, the arguments and results for a base case can be directly encoded in an equation.

The next step is to identify the inductive step by asking, *How can the definition for the general case be reduced to (or be expressed in terms of) a solution for a simpler argument*? This step actually requires figuring out two parts. First, we ask, *How is the argument simplified toward the base case*? For numbers, this often means to express the result of `f n` in terms of `f (n-1)` (as in the case of `fac`), and for lists this often means to express the result of `f (x:xs)` in terms of `f xs` (as in the case of `sum`). Second, we ask *How can the result be obtained from the result for a simpler argument*? Here, you should be guided by the result type of the function. For example, for both `fac` and `sum` the result type is `Int`, which means the result of the recursive call must be part of an `Int` operation (which is `*` in the case of `fac` and `+` in the case of `sum`).

When the function has multiple parameters you also have to decide what parameter the recursion should be defined on. In some cases the recursion is on just one argument (think of defining a function `plus` recursively using only `succ`), while in other situations the recursion works in parallel on two arguments (think of defining a function `merge` for merging two sorted lists into one sorted list.)

The example in the middle shows that lists of characters are actually the same as strings. This means that we can use list operations to manipulate strings.

Since we can build lists of different types, the type of the two list constructors is described using *type variables*.

```
[]  :: [a]
(:) :: a -> [a] -> [a]
```

These two type signatures say that the empty list `[]` can be of any type and that `:` can add an individual element of any type at the beginning of a list of elements of the same type. This constraint is expressed by using the same type variable `a` for the type of the single element and the type of the elements in the list. This constraint also means that while lists are polymorphic and thus can carry data of any type, they are also *homogeneous*, that is, all the elements in a list must have the same type, that is, lists such as `[2,True]` or `[2,[3,4]]` cannot be built. Why is the latter example incorrect? It only contains numbers, and we said that lists can contain other lists. To understand the issue here we should look at how the types of lists are described. The types of the previous examples are as follows.

```
[3,5]             :: [Int]
[False,True]      :: [Bool]
"Hello"           :: [Char]
[[1,2],[],[2,3,4]] :: [[Int]]
```

We observe that a list that contains elements of type `T` has the type `[T]`. Now what could possibly be the type of `[2,[3,4]]`? The first element suggests that it's `[Int]`, but the second element demands the type `[[Int]]`. But these are different types, and thus the list is not homogeneous.

Why is it important that all elements in a list have the same type? This is because Haskell is a statically typed language (see Section 5.5), which means that no program will be run that could lead to a runtime type error. If we allowed the construction of lists with elements of different types, the type checker could not ensure, without running a program, that a program does not lead to a type error.

To append two lists, you can use the predefined function `++`, which we have seen before. Again, the use of the same type variable for both argument lists and the result list ensures that we can only append lists of the same type.

```
(++) :: [a] -> [a] -> [a]
```

Note the slight but important difference between the types of `++` and `:`. You have to remember that `:` requires single elements as first argument whereas `++` takes a list.

Building lists is nice but gets boring quickly. What we really want to do is to compute with lists. So what can we do with lists? Since lists are constructed inductively, the most basic operation on lists is to inspect their structure, that is, we can check whether a list is empty, and we can access the first element and tail of any non-empty list.

```
> null []      | > null [2,3]   | head [5..9]  | tail [5..9]
True           | False          | 5            | [6,7,8,9]
```

With these basic access functions we can actually implement any function on lists. Consider, for example, a function for computing the sum of a list of numbers. For an empty list, the sum is 0, and for a non-empty list, the sum is obtained by adding the first element to the sum of the list tail.[19]

```
sum :: [Int] -> Int
sum xs = if null xs then 0 else head xs + sum (tail xs)
```

[19] In Haskell it is common practice to name variables for lists by a trailing `s`.

If you try this definition by putting into a file and loading it into the interpreter, you will likely encounter some error because the function `sum` is already predefined. More precisely it is defined in a module `Prelude` that contains many useful function and is loaded whenever the interpreter is started. To redefine such a function you can add the following line at the top of your file.

```
import Prelude hiding (sum)
```

This will load all definitions from the prelude, except the one for `sum`, and the definition causes no problem. This method works whenever a name conflict with a predefined function occurs.

As before, we can get rid of the `if-then-else` construct and make the definition a bit more readable by using pattern guards.

```
sum :: [Int] -> Int
sum xs | null xs   = 0
       | otherwise = head xs + sum (tail xs)
```

Pattern matching and pattern guards are convenient features to replace verbose and clumsy `if-then-else` structures by more easily readable equations.[20] Luckily, we can use pattern matching also on lists and any other data types. Specifically, instead of distinguishing between the two cases of lists (empty and non-empty) with the predicate `null`, we can use the constructors in arguments to restrict an equation to apply only for that particular constructor. For example, we can rewrite the definition of `sum` with pattern matching as follows.

```
sum :: [Int] -> Int
sum []     = 0
sum (x:xs) = x + sum xs
```

The first equation should be obvious: it says that the sum of the empty list is 0. An interesting aspect about the second equation is that the pattern for the non-empty list is built with the `:` constructor by applying it to two variables, `x` and `xs`. The pattern `x:xs` has two important effects.

- First, it matches only non-empty lists and guards the equation so that it is only applied to non-empty lists.
- Second, when matched against non-empty lists, the head of that list will be bound to `x`, and the tail will be bound to `xs`. These bindings can then be used on the right side of the equation in the result expression.

[20] Equations are also much more usable when tracing function executions, see Section 2.2.2.

It is this double-duty of pattern matching (discriminating between different cases of a data type and binding variables to the arguments of the constructor) that makes it such a convenient tool for function definitions.

Why do we need parentheses around the pattern in the second equation? Because without them Haskell would try to parse the equation as `(sum x):xs = ...`, which fails, since the compiler considers the variable `x` to be the function parameter and expects an `=` sign next.

In general, a pattern is an expression built from constructors, values, and variables, which means we can construct nested patterns, as in the following function that extracts the second element from a list.

```
f (x:y:xs) = y
```

Variables in a pattern that aren't used in the defining expression can be replaced by the wildcard symbol _, that is, we can rewrite the previous example as follows.

```
f (_:y:_) = y
```

The use of wildcards is generally good programming practice, since it already indicates in the pattern that the corresponding parts of the data structure will be discarded. And it also helps prevent certain programming errors, since you can't accidentally use the wrong part of a data structure.

The basic list functions shown earlier are actually all implemented using pattern matching.

```
null :: [a] -> Bool
null []    = True
null (_:_) = False
```

```
head :: [a] -> a
head (x:xs) = x
```

```
tail :: [a] -> [a]
tail (x:xs) = xs
```

We can observe that `head` and `tail` are only defined for non-empty lists. What happens if we apply `head` to the empty list?

```
> head []
*** Exception: Prelude.head: empty list
```

We get a we get a runtime error, which should not be surprising, since `head` is undefined for empty lists.

Note that a pattern must not contain function names, that is we cannot write function equations such as this one.

```
f (xs++ys) = ...                    -- INVALID PATTERN
```

Note also that a pattern must not contain repeated variables. For example, the following attempt to identify duplicate elements at the beginning of a list does *not* work.

```
rmDup (x:x:xs) = rmDum (x:xs)          -- INVALID PATTERN
rmDup (x:xs)   = x:rmDup xs
```

However, it is easy to express the intended constraint through a pattern guard.

```
rmDup (x:y:xs) | x==y      = rmDum (y:xs)
               | otherwise = x:rmDup (y:xs)
```

The function `sum` illustrates how to consume lists of arbitrary length through recursion. We should also examine at least one function that constructs lists of varying length through recursion. As an example, consider the definition of a function `range` that constructs a list of integers in the range between two given numbers n and m. The key, as always, is to identify the base and inductive case for the recursion. When the function to be defined builds a list, one can identify the base case by asking under which condition the result list is empty. This happens when $n > m$ because no numbers exist between n and m in this case. Otherwise, n will be part of the constructed list. Specifically, n precedes the elements of the list in the range $n + 1$ and m, which is obtained through a recursive call.

```
range :: Int -> Int -> [Int]
range n m | n>m       = []
          | otherwise = n:range (n+1) m
```

The expression `range n m` constructs the same list as the expression `[n .. m]`.

2.6 Data Types

In Haskell data structures are built with data types. As already mentioned in Section 2.5, a data type provides *constructors* that may carry arguments of specific types. In addition to lists, we have already encountered another data type, namely boolean values. The type `Bool` is actually not built into Haskell, but is defined in the prelude by the following data type definition.

```
data Bool = True | False
```

This definition says that `Bool` consists of just two values, represented by the two constructors `True` and `False`, which carry no arguments. Data types whose constructors don't have arguments are also called *enumeration types*, and another example of an enumeration type can be found in the file `Grading.hs`, which defines the data type `Grade` (see Figure 2.1).

Exercise 2.7

(a) Define the function `nth :: Int -> [a] -> a` that selects an element from a list by its position. As is common practice, the index for the first element is defined to be 0. The function definition needs two arguments: the index and the list to be selected from. Since the definition is inductive on the number argument, you can apply the same strategy as for `fac` and write two equations for the base case 0 and the general, inductive case n.

(b) Reimplement the function `nth` using pattern matching for the list argument.

(c) Give a definition of `head` that is based on `nth`.

(d) Define the function `length :: [a] -> Int`.

(e) Give a definition of `null` that is based on `length`.

(f) What pattern matches any list of exactly one element? What pattern matches any list of exactly two elements?

(g) Define the function `member :: Int -> [Int] -> Bool` that checks whether a particular number is contained in a list. Give a recursive definition using pattern guards.

(h) Define the function `delete :: Int -> [Int] -> [Int]` that deletes all occurrences of the first argument from the list given as a second argument.

(i) Define the function `insert :: Int -> [Int] -> [Int]` that inserts an integer at the correct position into a sorted list of integers.

Data constructors are similar to object constructors in C++ or Java. However, one important difference is Haskell constructors are immutable, that is, once you have called a constructor with an argument, that value is built and stays in memory until it is not referenced anymore and can be garbage-collected. This means that to replace an element in a data structure we have to actually rebuild (part of) the data structure with constructors that contain the replacement values while the old values still exist.

To see what happens, consider the following function that takes a value `x` and a list and replaces the the first element in the list by `x`.

```
replFst :: a -> [a] -> [a]
replFst x (_:xs) = (x:xs)
```

The pattern `_:xs` for the list argument matches any non-empty list and binds the tail of that list to `xs`, that is, when `replFst` is applied to `9` and `[1,2,3]`, `x` will be bound to `9`, and `xs` will be bound to `[2,3]`.[21] The result of the application will be the list `[9,2,3]` which is not

[21] The first list element `1` will be matched against the wildcard without generating a binding.

surprising. However, we have constructed a new list, and the old list still exists, as can be seen from the following interaction in `ghci`.

```
> let ys = [1,2,3]
> replFst 99 ys
[99,2,3]
> ys
[1,2,3]
```

As you can see, `replFst` has produced a new list, and the old list `ys` was not changed at all. In general, to change a value in a data structure we have to copy the the path from the root of the data structure to the element that is to be changed. The other parts of the data structure will be reused and shared among the new and old version.

The functional behavior of data structures is a good thing, since it prevents side effects. A potential downside is that it is more time- and space-consuming. The question of efficiency is an interesting topic in the context of functional data structures and compilers, but it is not relevant in the context of this class where we use Haskell primarily as a mathematical language to describe the syntax and semantics of programming languages.

Exercise 2.8

To change a value somewhere inside of a list, we cannot overwrite a list element. Instead we have to construct a new list with the new element in it and copy all elements from the front part of the list.

(a) Define the function `replK :: Int -> a -> [a] -> [a]` that replaces the k^{th} element of a list. The first `Int` argument indicates the position of the element to be replaced. If the position exceeds the length of the list, the function should do nothing. (*Hint*: You need three equations, one for the case of an empty list, one for a non-empty list and position 0, and one for a non-empty list and position n.)

(b) Give a simple definition of `replFst` in terms of `replK`.

(c) Consider the following definitions.

```
xs = [1 .. 10]
ys = replK 3 99 xs
```

Given that the definition for `xs` creates 10 nodes in main memory, How many new nodes are created by the call `replK 3 xs`? How many elements of the list `xs` are shared with the list `ys`?

Enumeration types are a special case of data type when none of its constructors has any arguments. In the more general case, constructors do take arguments, as is the case with

lists. As an example we consider a data type with two constructors for binary trees.

```
data Tree = Node Int Tree Tree
          | Leaf
          deriving Show
```

This definition introduces a constructor `Node` that takes an integer value n and two trees l and r and builds a new tree with root n and left and right subtrees l and r, respectively. The constructor `Leaf` represents an empty tree. Remember that `deriving Show` makes it possible to print values of type `Tree`. Here are the definitions of two trees, `lft`, which has 3 nodes and is used as the left subtree of the root node of tree `t`.

```
lft :: Tree
lft = Node 3 (Node 1 Leaf Leaf) (Node 5 Leaf Leaf)

t :: Tree
t = Node 6 lft (Node 9 Leaf (Node 8 Leaf Leaf))
```

Functions on the `Tree` data type are defined in a similar way as functions on lists, namely by equations that use patterns for the different constructors. As an example, consider the following definition of the function `inorder` that produces a list of nodes obtained by an in-order traversal of a tree.

```
inorder :: Tree -> [Int]
inorder Leaf         = []
inorder (Node x l r) = inorder l ++ [x] ++ inorder r
```

Since the `Node` constructor takes three arguments, the corresponding pattern has three variables to match the value in the node and the left and right subtree.

The definition of a data type directs or even dictates, to a large degree, the structure of function definitions for that type: For each constructor the function will have a separate equation with a corresponding pattern that defines how to deal with that case of the type. This is not always the case: Sometimes functions are partially defined for only some of the constructors (cf. the definition of `head` and `tail`), and sometimes a pattern for a constructor can be further refined to express additional constraints. Consider, for example, the following definition of the function `swapHead`.

```
swapHead (x:y:ys) = y:x:xs
```

First, we can observe that this definition works only for lists that contain at least two elements. Second, we can see that the pattern `x:y:ys`, which is the same as `x:(y:ys)`, is a special case of the typical list pattern `x:xs` in which `xs` has been further refined by the pattern `y:ys`,

which expresses the constraint that the rest list must not be empty and ensures that the list contains, in addition to the first element `x`, a second element `y` for swapping.

We will use data types specifically to represent the abstract syntax of languages. We will then also learn more about further aspects of data types and pattern matching.

Exercise 2.9

(a) Define the function `preorder :: Tree -> [Int]` that produces a list of nodes obtained by a pre-order traversal of the tree.

(b) Define a function `find :: Int -> Tree -> Bool` that determines whether a number is contained in a tree. Don't assume that the tree is a binary search tree!

(c) Define a function `findBST :: Int -> Tree -> Bool` that determines whether a number is contained in a tree, assuming that the tree is a binary search tree.

2.7 Higher-Order Functions

In functional programming, functions are values much like numbers or lists. Of course, we can't multiply functions like numbers, but we can't do that with lists either. Values of a particular type always support a specific set of operations. For numbers it is arithmetic operations such as multiplication, for lists (and all other data types) it is constructing values with constructors and deconstructing values using pattern matching. The two operations allowed for functions are (lambda) *abstraction* and *application*.

Since functions are values, we should be able to treat them like other values, that is, store them in lists, pass them as arguments to functions, and return them as results from functions. Functional languages, such as Haskell, that allow functions to be handled this way are said to treat functions as *first-class citizens*, and a function that takes functions as arguments and/or produces a function as a result is called a *higher-order function*.

The function `map` is such a higher-order function: It takes a function, applies it to all elements of a list, and yields the list of all the results. The definition is as follows.

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x:map f xs
```

The function `map` represents a control structure of traversing a list in a loop. Here are some example applications of `map`.

```
> map (^2) [1..5]
[1,4,9,16,25]
```

```
> map even [1..3]
[False,True,False]
```

```
> map succ "ohm"
"pin"
```

The first two examples should be obvious; the third example exploits the fact that strings are lists of characters. The second example illustrates the general type of `map`, that is, the type of the resulting list can be different from the argument list. Of course, `map` can be used to apply functions on more complex data types, such as lists.

```
> map length ["Be","a","bat"]
[2,1,3]
```

```
> map sum [[1..3],[4,5],[1..10]]
[6,9,55]
```

And we can also apply a function to all element in a nested list of lists, which then produces a list of result lists. The key to understanding the example is to take a close look at the types. Since the type of `succ` is `Int -> Int`, the type of `map succ` is `[Int] -> [Int]`, that is, `map succ` is a function that maps integer lists into integer lists. We can use this function then as an argument to `map` to apply it to a list of integer lists, like so.

```
> map (map succ) [[1..3],[4..7]]
[[2,3,4],[5,6,7,8]]
```

The importance and wide applicability of `map` lies in the fact that it implements iteration over a number of values.

Yet, the iteration offered by `map` is somewhat limited, since the individual list elements are all processed independently of one another. For aggregating the elements of a list, we can employ another higher-order function, `foldr`, which combines the elements of a list with a binary function. The function is defined as follows.[22]

```
foldr :: (a -> a -> a) -> a -> [a] -> a
foldr f u []     = u
foldr f u (x:xs) = f x (foldr f u xs)
```

The second argument, `u`, is used as an initial value for the aggregation. The effect of `foldr` can be best understood by looking at some examples. Let's start with expressing the `sum` function as an application of `foldr`.[23]

[22] The type of the function is actually more general, namely `foldr :: (a -> b -> b) -> b -> [a] -> b`, but working with the simpler type makes it initially easier to understand the function. The more general type indicates that the elements of a list can be aggregated into a value of a different type.

[23] Remember that putting parentheses around an infix operation turns it into a prefix function. In the example we need to use the parentheses because otherwise, by writing `foldr + 0`, the Haskell parser would assume that we want to apply `+` to `foldr` and `0`, which is not the case.

```
> let mySum = foldr (+) 0
> mySum [1..10]
55
```

We can also use `foldr` to obtain the following succinct implementation of the factorial function.

```
fac :: Int -> Int
fac n = foldr (*) 1 [2 .. n]
```

In case you wondered: The `r` in the name `foldr` indicates that the elements in the list are consumed from the right, and yes, there is also a function `foldl` that consumes elements from the left.

Finally, the third higher-order function that can be employed in many situation is function composition, which is written in Haskell using an infix period operation. The definition of function composition is straightforward; it says that applying the composition of `f` and `g` to a value `x` means to first apply `g` to `x` and then apply `f` to the result.

```
(.) :: (b -> c) -> (a -> b) -> a -> c
(f . g) x = f (g x)
```

Here are a few example applications for function composition. If you need a function for adding `2` to a number, you can compose the successor function with itself. If you have defined the function `even`, you can obtain the `odd` predicate by composing it with `not`. If you want to extract the second element of a list, you can do this by first taking the tail of the list and then the head of the result, or if you want to drop the first two elements from a list, you can achieve this by applying `tail` twice.

```
add2  = succ . succ          |          add2 x   = succ (succ x)
odd   = not . even           |          odd x    = not (even x)
snd   = head . tail          |          snd xs   = head (tail xs)
drop2 = tail . tail          |          drop2 xs = tail (tail xs)
```

As the equivalent definitions on the right illustrate, one can always rewrite the definitions using variables (and parentheses).

Higher-order functions are the control structures of functional programming. Notably, none of the functions `map`, `foldr`, or `(.)` are built-in primitives of Haskell; they are all defined as functions. The nice thing is that if you need a new specialized control structure, you can define it yourself. By being able to define your own control structures, you can capture specific patterns of computation, make them explicit, and then use them. We

will see examples later in Section 4.2.

Exercise 2.10

(a) Consider the following function definition.

```
th = tail . head
```

Is this function well defined? If so, what is its type, and what does it do?

(b) What does the function `map f . map g` do? How could it be rewritten?

(c) Define the function `last :: [a] -> a` as a composition of a list function and the function `reverse`.

(d) Consider the following two lists `xs` and `ys`. Note that `ys` is a nested list of lists.

```
xs = [1,2,3]
ys = [xs,[7]]
```

Which of the following expressions are type correct? What values do they evaluate to?

```
map sum xs
map sum ys
last ys
map last ys
last (last ys)
```

3 Syntax

The syntax of a language serves two major purposes. First, it defines the set of all valid sentences in that language, that is, the syntax of a programming language defines what is a valid program in that language. Second, the syntax defines the *structure* of sentences and programs. Understanding the structure of a sentence is important for understanding its meaning. This becomes clear when we look at examples of ambiguous sentences such as the following.

They opened the box with a knife.

The sentence might say that they used a knife for opening the box or that they opened a box that contained a knife. Similarly, it is important to understanding the syntactic elements of programs. Consider, for example, the following attempt to define a Haskell function for negating boolean values.

```
not :: Bool -> Bool
not true  = False
not false = True
```

If you don't immediately see what the problem is, load the definition into `ghci` and apply it to different input, and you will notice that the function `not`, as defined, is a function that always returns `False`.

```
> not True
False

> not False
False
```

The problem with the definition is that the argument used in the first equation, `true`, starts with a lowercase letter and is thus a variable that matches everything, causing the first equation to always be selected. A correct definition should instead use the constructor `True`, which starts with capital letter and leads to the intended behavior of the function. We can observe that a minor change in syntax—which is not even a syntax error—can lead to semantically incorrect programs.

In the context of this class, the second aspect, the representation of the structure of programs, is of particular importance. We will therefore first briefly review in Section 3.1 context-free grammars as a metalangauge for describing textual, concrete syntax and in Section 3.2 the notion of parse trees to represent the structure of languages. We will then focus in Section 3.3 on the concept of *abstract syntax* and how to represent abstract syntax within Haskell.

3.1 Context-Free Grammars

How can we describe the set of sentences that belong to a language? Trying to simply enumerate them is not viable for languages that contain an infinite number of sentences, which is the case for all non-trivial programming languages. What we need is a finite description for an infinite set. Grammars offer such a description formalism.

A *grammar* is a formal system that consists of a set of *productions* (or *rules*) that can be used to derive sentences from a start symbol. For example, here are two productions which can generate sequences of zeros followed by twice as many ones.

zoo ::= `0 1 1` (P1)
zoo ::= `0` *zoo* `1 1` (P2)

Each production is of the form $L ::= R$, and can be used to repeatedly replace occurrences of L (the left-hand side, or LHS) in a sequence of symbols by R (the right-hand side, or RHS). In the case of context-free grammars, L is always given by a single symbol, called a *nonterminal symbol*, written in *italics font*.[1] R is a sequence of nonterminal and terminal

[1] If we allow the left-hand sides to be sequences of symbols, the productions can depend on the context of the symbol being replaced, which yields a more powerful grammar formalism, one that is more difficult for parsers to work with.

symbols. Terminals symbols are written in `typewriter font`.

To see this grammar in action, we can apply the second production to replace the start symbol *zoo* by `0` *zoo* `11`. Then we can apply the first production to replace the new occurrence of *zoo* by `011`. Such a repeated application of productions that ends with a sequence of only terminal symbols is called a *derivation*. Here is the derivation of the sentence `001111` from the nonterminal *zoo*. For each step we show which production was used.

zoo
⇒ `0` *zoo* `11` (P2)
⇒ `001111` (P1)

The final sequence of1 terminal symbols is called a *sentence*, and the intermediate sequences of the derivation that still contain nonterminal symbols are called *sentential forms*. Sentential forms can be thought of as incomplete sentences that have the potential of becoming a sentence once all their nonterminals in have been replaced.

Formally, a grammar is defined as a four-tuple (N, Σ, P, S) where

(1) N is a set of *nonterminal symbols*,
(2) Σ is a set of *terminal symbols* with $N \cap \Sigma = \varnothing$,
(3) $P = N \times (N \cup \Sigma)^*$ is a set of *productions*, and
(4) $S \in N$ is the *start symbol*.

With this definition, a sentence is an element of Σ^*, and a sentential form is an element of $(N \cup \Sigma)^*$. Given two sentential forms $\alpha, \beta \in (N \cup \Sigma)^*$, β can be *derived in one step* from α if one of its nonterminals can be replaced using a production so that the resulting sentential form is β, that is, if

(1) $\alpha = \alpha_1 A \alpha_2$ (with $\alpha_1, \alpha_2 \in (N \cup \Sigma)^*$),
(2) $(A, \gamma) \in P$, and
(3) $\beta = \alpha_1 \gamma \alpha_2$.

This one-step derivation relationship between two sentential forms is written as $\alpha \Rightarrow \beta$. The generalization to multi-step derivations is given by the reflexive, transitive closure of the one-step derivation relationship, and we can say that a sentential form β can be *derived* from α (in several steps), written as $\alpha \Rightarrow^* \beta$, if either $\beta = \alpha$ or $\alpha \Rightarrow \alpha'$ and $\alpha' \Rightarrow^* \beta$.

Finally, the *language* $L(G)$ defined by the grammar $G = (N, \Sigma, P, S)$ is the set of all sentences that can be derived from its start symbol, that is,

$$L(G) = \{\sigma \in \Sigma^* \mid S \Rightarrow^* \sigma\}$$

Our previous example grammar is formally defined by the following four-tuple.

$$G_{zoo} = (\{zoo\}, \{\texttt{0}, \texttt{1}\}, \{(zoo, \texttt{0 1 1}), (zoo, \texttt{0}\, zoo\, \texttt{1 1})\}, zoo)$$

We have previously shown the derivation for the fact $S \Rightarrow^* \texttt{0 0 1 1 1 1}$, and the language defined by G_{zoo} is $L(G_{zoo}) = \{\, \texttt{0}^k\texttt{1}^{2k} \mid k > 0\}$.

The grammar G_{zoo} is quite simple: It has only one terminal symbol, and its two productions don't leave any choice in the derivation of sentences. In general, this is of course not the case. Consider the following productions for describing the language of binary digit sequences (BDS).

$dig ::= \texttt{0}$	(P1)
$dig ::= \texttt{1}$	(P2)
$bin ::= dig$	(P3)
$bin ::= dig\, bin$	(P4)

We observe that we have two productions for each of the nonterminals *dig* and *bin*. One consequence in this particular case is that it permits different derivations for the same sentence. Consider, for example, the following derivations of the sentence `1 0 1` from the start symbol *bin*.

	bin			*bin*			*bin*	
$\Rightarrow$	$dig\, bin$	(P4)	$\Rightarrow$	$dig\, bin$	(P4)	$\Rightarrow$	$dig\, bin$	(P4)
$\Rightarrow$	$dig\, dig\, bin$	(P4)	$\Rightarrow$	$dig\, dig\, bin$	(P4)	$\Rightarrow$	$\texttt{1}\, bin$	(P2)
$\Rightarrow$	$dig\, dig\, dig$	(P3)	$\Rightarrow$	$dig\, dig\, dig$	(P3)	$\Rightarrow$	$\texttt{1}\, dig\, bin$	(P4)
$\Rightarrow$	$\texttt{1}\, dig\, dig$	(P2)	$\Rightarrow$	$dig\, dig\, \texttt{1}$	(P2)	$\Rightarrow$	$\texttt{1 0}\, bin$	(P1)
$\Rightarrow$	$\texttt{1 0}\, dig$	(P1)	$\Rightarrow$	$dig\, \texttt{0 1}$	(P1)	$\Rightarrow$	$\texttt{1 0}\, dig$	(P3)
$\Rightarrow$	`1 0 1`	(P2)	$\Rightarrow$	`1 0 1`	(P2)	$\Rightarrow$	`1 0 1`	(P2)

The first and second derivation both replace nonterminals by terminals in their last three steps. Notably, the second derivation in the middle column applies the same rules in the same order as the first one, but it does that to different nonterminals in the sentential form. The third derivation replaces nonterminals by terminals as soon as possible.

While grammars sometimes offer a choice in which order to apply productions, the order actually doesn't matter when the goal is to derive a sentence. Ultimately, we are interested in the structure of sentences, which is captured in *parse trees* (explained in Section 3.2) and (abstract) *syntax trees* (explained in Section 3.3). These trees provide a representation of sentences that abstracts from the order in which productions have been applied.

To write down productions more economically, it is common practice to combine productions with the same LHS nonterminal by listing the different RHSs separated by a pipe or bar symbol (|). For the BDS language this looks as follows.

$$\begin{aligned} \mathit{dig} &::= \texttt{0} \mid \texttt{1} \\ \mathit{bin} &::= \mathit{dig} \mid \mathit{dig}\,\mathit{bin} \end{aligned}$$

To identify specific productions, we can use the LHS nonterminal and the position of the RHS in the list. For example, (P1) is (dig_1), and (P4) is (bin_2).

Exercise 3.1

(a) The BDS language allows sequences that start with zeros as sentences. Change the grammar so that it describes proper binary numbers, in which all sequences of more than one digit start with a `1`.

(b) Write a grammar for boolean expressions that contain the constants `True` and `False` and the unary operation `not`.

(c) Using the grammar from part (b), create a derivation of the sentence `not not True`.

If you look closely at the formal definition of a grammar, you will notice that the RHS of a production can be empty. Instead of simply leaving the RHS empty, it is common practice to use the symbol ϵ instead to make this fact explicit. Productions with an empty RHS are therefore also sometimes called *ϵ-productions*. The use of ϵ-productions can be very convenient and help simplify grammars. For example, we can rewrite the productions for the BDS language using just one nonterminal and three rules as follows.

$$\mathit{bin} ::= \texttt{0}\,\mathit{bin} \mid \texttt{1}\,\mathit{bin} \mid \epsilon$$

Since ϵ-productions essentially allow the removal of nonterminals from sentential forms during a derivation, derivations often become shorter as well. For example, here is the derivation of the sentence `1 0 1` from the start symbol *bin* using the new grammar.

$$\begin{aligned} & \mathit{bin} \\ \Rightarrow\ & \texttt{1}\,\mathit{bin} && (\mathit{bin}_2) \\ \Rightarrow\ & \texttt{1 0}\,\mathit{bin} && (\mathit{bin}_1) \\ \Rightarrow\ & \texttt{1 0 1}\,\mathit{bin} && (\mathit{bin}_2) \\ \Rightarrow\ & \texttt{1 0 1} && (\mathit{bin}_3) \end{aligned}$$

The concept of derivation explains how a sentence can be produced with a grammar. However, the internal structure of the sentence that is revealed through this process is lost.

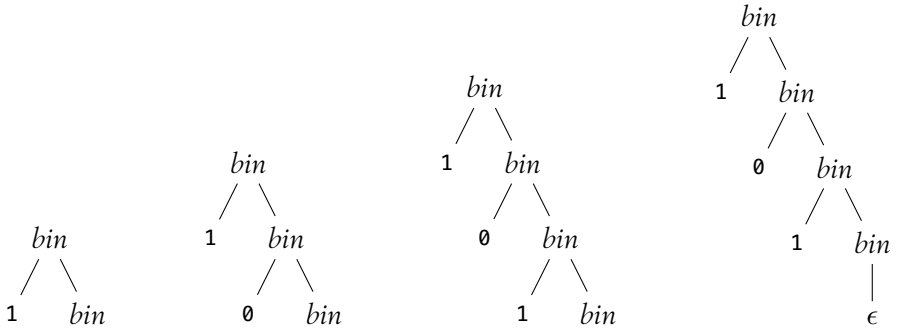


Figure 3.1 Creating a parse tree from the derivation $bin \Rightarrow \texttt{1}\,bin \Rightarrow \texttt{1 0}\,bin \Rightarrow \texttt{1 0 1}\,bin \Rightarrow \texttt{1 0 1}$. The leaves of each tree show the sentential form of the current step in the derivation. The leaves of the final tree contains only terminal symbols and thus show the derived sentence.

3.2 Parse Trees

A parse tree represents the structure of a sentence. We can think of a parse tree as a trace of a derivation. The internal nodes of a parse tree contain nonterminal symbols (with the start symbol in the root), and the leaves contain terminal symbols. A *parser* is a program, often part of a compiler, that transforms text into parse trees for further processing.

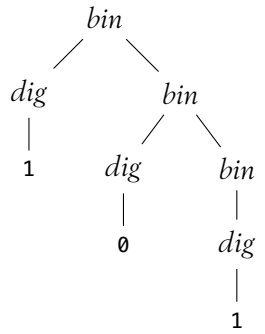
A parse tree for a sentence can be obtained from a derivation for that sentence by following derivation the steps. Specifically, for each step in the derivation that applies a production $(A, \alpha_1 \ldots \alpha_n) \in P$ (where we assume $\alpha_1 \ldots \alpha_n \in N \cup \Sigma$) to replace a nonterminal $A \in N$ in the current sentential form, we add the symbols $\alpha_1 \ldots \alpha_n$ as children to the occurrence of A in the tree.

Let's see how we can produce a parse tree for the sentence `1 0 1` using the grammar for the BDS language with ϵ-productions. We start with *bin* as the root node and first apply production (bin_2), which means we have to add the symbols of the RHS of that production, `1` and *bin*, as children to the root. We repeat the same step twice, applying productions (bin_1) and (bin_2) to the nonterminal *bin* in the leaf of the tree. In a final step, we apply the ϵ-production. The process is illustrated in Figure 3.1.

The structure of a parse tree depends on the grammar for a language and not just the language. Let's take again a look at the grammar for the BDS language without the ϵ-production.

$$
\begin{aligned}
dig &::= \texttt{0} \mid \texttt{1} \\
bin &::= dig \mid dig\, bin
\end{aligned}
$$

Here is the parse tree for the sentence `1 0 1`; its structure is quite different from the parse tree in Figure 3.1.



A crucial property of parse trees is the following: While it does contain information about *which* productions have been applied, it does *not* tell us in which *order* they were applied. Specifically, if you recall the different derivations for the sentence `1 0 1` shown earlier, you will notice that they all produce the same parse tree.

The structure of either of the parse trees for `1 0 1` is admittedly not very interesting (as is the BDS language). However, for more complex languages, parse trees contain important information about the structure of sentences that is exploited in the definition of the language's semantics.

As an example, let's consider the following grammar for a simple form of arithmetic expressions.

$$\begin{aligned} \mathit{num} &::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \ldots \\ \mathit{expr} &::= \mathit{num} \mid \mathit{expr} + \mathit{expr} \mid \mathit{expr} * \mathit{expr} \end{aligned}$$

Now consider the expression `1+2*3`. It turns out that by switching the order in which we apply the productions ($expr_2$) and ($expr_3$) we get two derivations that produce different parse trees. We only have to look at the first two derivation steps to see what's going on. Since the sentential forms contain two occurrences of *expr*, I have underlined in the derivation the nonterminal that is going to be replaced in the next step.

$$\begin{array}{lll} & \mathit{expr} & \\ \Rightarrow & \mathit{expr} + \underline{\mathit{expr}} & (\mathit{expr}_2) \\ \Rightarrow & \mathit{expr} + \mathit{expr} * \mathit{expr} & (\mathit{expr}_3) \\ & \ldots & \end{array} \qquad \begin{array}{lll} & \mathit{expr} & \\ \Rightarrow & \underline{\mathit{expr}} * \mathit{expr} & (\mathit{expr}_3) \\ \Rightarrow & \mathit{expr} + \mathit{expr} * \mathit{expr} & (\mathit{expr}_2) \\ & \ldots & \end{array}$$

As we can see in the third step, we have obtained the same sentential form through two

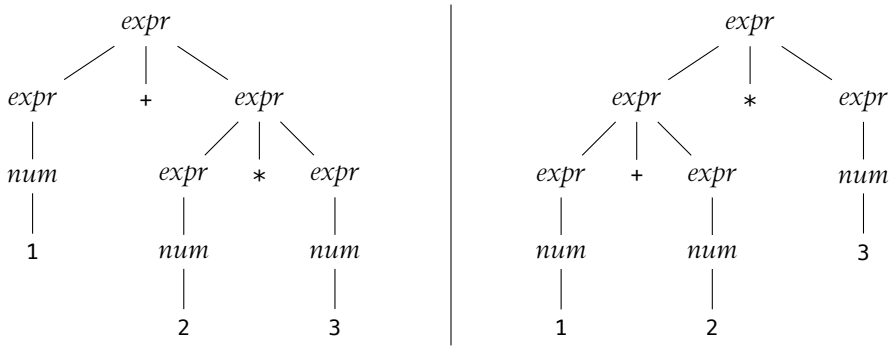


Figure 3.2 Two different parse trees for the expression `1+2*3`. The tree on the left is obtained when production ($expr_2$) is applied first, followed by the application of production ($expr_3$) to the rightmost *expr* nonterminal. The tree on the right is obtained when the order is reversed and production ($expr_3$) is applied before production ($expr_2$), applied to the leftmost *expr* nonterminal.

different derivation sequences, and while the sentential forms are identical, the construction of the parse tree reflects the different order of applying the two *expr* productions in its internal structure, see Figure 3.2.

A grammar that allows the construction of two different parse trees for the same sentence is called *ambiguous*. Ambiguity is an undesirable property for grammars, since it makes predictions about the semantics of sentences difficult or impossible. For example, the left parse tree in Figure 3.2 groups the multiplication of `2` and `3` together, and an evaluation of the expression that follows this structure would correspondingly result in `7`. In contrast, the right parse tree groups the addition of `1` and `2` together, which means that the evaluation of the expression would result in `9`.

Ambiguity in grammars can be resolved by adding extra rules about operator precedences or by refactoring the grammar. The details are not important here.

What you should remember is that programs that are often supplied in a linear form of

text must be transformed into a representation that reveals the structure of the program.

Exercise 3.2

(a) Consider the following grammar.

bool ::= `True` | `False` | `not` *bool*

Draw the syntax tree for the sentence `not not True`.

(b) Extend the grammar from part (a) so that it is possible to derive the sentence `not (True or False)` and draw a parse tree for it.

(c) Consider the following grammar.

bins ::= `0` *bins* `1` *bins* | `1` *bins* `0` *bins* | ϵ

Show that this grammar is ambiguous by drawing two different parse trees for the sentence `0 1 0 1`.

3.3 Abstract Syntax

We have seen how grammars can describe languages, but the linear form of sentences is limited and often leaves questions about the intended meaning unanswered. Remember that the goal of formalizing the syntax of a language is to facilitate the definition of its semantics. Consider again the expression `1+2*3`. Does it denote `(1+2)*3` or `1+(2*3)`? Parse trees make the structure of an expression explicit, but since they contain information that is not relevant for the definition of semantics (and other analyses one might want to perform on a program), the structure of a program is usually transformed from parse trees in a more succinct form, into a so-called (*abstract*) *syntax tree*, or AST.

An example is illustrated in Figure 3.3, which shows the left parse tree from Figure 3.2 and its corresponding abstract syntax tree.

While a parse tree keeps vestiges of the derivation, the AST retains only the information necessary to understand the structure of the sentence. To distinguish the two forms of syntax, the set-of-sentences syntax defined by grammars is called *concrete syntax*, while a set-of-ASTs is called *abstract syntax*. The two forms of syntax are compared in Table 3.1.

The visual representation of trees is quite useful in illustrating concepts and explaining specific examples, but is not well suited for use in formalizing operations on trees. Therefore, to effectively work with ASTs, we need a notation that allows us to do two things:

(1) Denote trees in a linear, text-based form

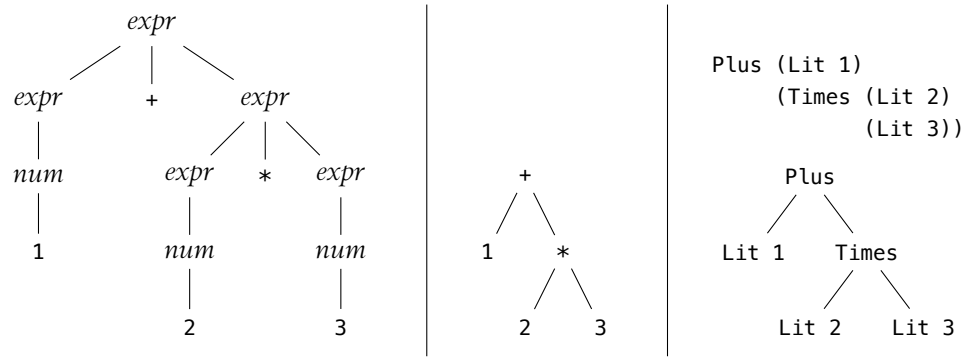


Figure 3.3 A parse tree for the expression `1+2*3` (on the left) and its corresponding abstract syntax tree (in the middle). The AST is more succinct: It eschews all nonterminal symbols and keeps instead operations in internal nodes. On the right: How the abstract syntax tree can be written as a value of a Haskell data type. Haskell data constructors are in fact operations for constructing trees.

(2) Denote tree patterns in definitions

It turns out that Haskell data types satisfy exactly these two requirements. To represent the abstract syntax of a grammar in Haskell we define, as a rule of thumb, a separate data type for each nonterminal with a constructor for each of its productions. An exception to this rule are nonterminals that represent basic lexical units that are best represented by Haskell built-in types, such as *num*, which we always represent by `Int`, or *name*, which we always represent by `String`. There are several other simplifications and special cases that can lead to more concise definitions, and we will discuss them all in due course.

As an example, consider again the very simple grammar for arithmetic expressions.

num ::= `0` | `1` | `2` | `3` | `...`
expr ::= *num* | *expr* `+` *expr* | *expr* `*` *expr*

An abstract syntax representation is given by the following Haskell data type.

	Concrete syntax	**Abstract syntax**
Descriptive formalism	context-free grammars	Haskell data types
Structure representation	parse trees	(abstract) syntax trees
Denoted language	set of sentences	set of ASTs
Ambiguity	can be a problem	ruled out by formalism

Table 3.1: Comparison of concrete and abstract syntax.

```
data Expr = Lit Int | Plus Expr Expr | Times Expr Expr
```

The *expr* nonterminal is represented by the data type `Expr`, and each *expr* production is represented by one constructor. In general, the argument types for each constructor are given by the types that represent the nonterminals that occur in the RHS of the corresponding production. For example, the RHS of the second *expr* production, *expr* `+` *expr*, contains two occurrences of *expr*. Therefore, the corresponding constructor `Plus` has two `Expr` arguments. The same applies to the third *expr* production and the corresponding `Times` constructor.

Since the first *expr* production contains the "basic" nonterminal *num*, we use the corresponding built-in type `Int` as an argument type for the constructor `Lit`. Note that we have to invent a name for the constructor, since numbers are not accompanied by a terminal symbol. But why do we need a constructor at all and not just use `Int` as one case of the data type definition? Maybe we could do something like this.

```
data Expr = Int | Plus Expr Expr | Times Expr Expr   -- INCORRECT!
```

This definition would define `Int` as one constructor of `Expr` without any arguments, but that is not what we want. To view integers as expressions we need a constructor, such as `Lit`, that injects them as one variant into the data type.

With the Haskell data type definition for `Expr`, the AST shown in Figure 3.3 can be represented as follows.[2]

```
> Plus (Lit 1) (Times (Lit 2) (Lit 3))
Plus (Lit 1) (Times (Lit 2) (Lit 3))
```

At this point, constructing trees doesn't seem very useful, but the AST representation puts us into a position to implement syntactic transformations in a succinct way.

As another example, take a look at the following grammar for a toy imperative language. Before reading further, please try to define the abstract syntax for this grammar via two data types `Cond` and `Stmt` yourself. Note that `&` is meant to represent a logical *and* operation.

cond ::= `true` | *cond* `&` *cond* | `not` *cond* | `(`*cond*`)`
stmt ::= `while` *cond* `{` *stmt* `}` | `skip`

It is important to remember the syntactic rules about constructors and type names in Haskell (they must start with a capital letter!) when defining the types. Here is one possible representation.

[2] If you want to replay this example in `ghci`, don't forget to add a `deriving Show` clause to the `data` definition for `Expr`.

Exercise 3.3

(a) Implement a simple optimizer for the `Expr` data type that replaces (sub)expression such as `0 + 5` by `5`. You will probably need to employ patterns such as the following that match `0` constants as parts of `Plus` expressions.

```
optimize :: Expr -> Expr
...
optimize (Plus (Lit 0) e) = ...
```

If this exercise is too easy, add simplifications that can also simplify the following expressions: `1 * 7` → `7` and `0 * 5` → `0`. Note that by adding the last simplification, the order in which the different replacements happen matters. To see this, consider the expression `2+(0*3)`, which should be simplified to `2`.

(b) Implement a pretty printer for `Expr`, that is, a function `pp` of type `Expr -> String` that converts expression ASTs into their textual representation. For example, `pp` should produce the following.

```
> pp (Plus (Lit 1) (Times (Lit 2) (Lit 3)))
"1+2*3"
```

The basic structure of the function definition is straightforward. However, you have to be careful to create parentheses around `Plus` expressions that occur as arguments of `Times` without creating any unnecessary parentheses. For example, we expect `pp` to produce the following.

```
> pp (Plus (Lit 0) (Plus (Times (Plus (Lit 1) (Lit 2)) (Lit 3)) (Lit 4)))
"0+(1+2)*3+4"
```

```
data Cond = T | And Cond Cond | Not Cond
data Stmt = While Cond Stmt | Skip
```

Three things deserve to be noted here. First, I have used `T` and not `True` as the name of the constructor for representing the nonterminal *true*. The reason is that `True` is already a defined constructor in Haskell and cannot be defined again.[3] The names of the constructors don't carry an intrinsic meaning, and the choice actually doesn't matter. We can represent *cond* just as well through the following definition.

```
data Conditio = Verum | Et Conditio Conditio | Non Conditio
```

Or even this one:

[3] If you really want to use the constructor `True` in your definition, you can actually hide the existing definition by including at the beginning of your file the declaration `import Prelude hiding (True)`. The point is to be aware of existing constructor and type definitions.

```
data A = B | C A A | D A
```

What matters is that the constructors can accurately represent the information from the grammar productions. Of course, it's always a good idea to choose names wisely to support the writing and understanding of definitions that use these names.

Second, `Cond` doesn't have a constructor to represent the last *cond* production. The reason is, simply said, that it is not needed, since it doesn't introduce any new operation on conditions or any new way of building conditions. Parentheses are used in linear representation to express the grouping of different parts. We do not need parentheses in trees, because the grouping happens through the hierarchical tree structure and the fact that children are associated with their parent. This resolves any ambiguity that one could have in a linear notation. The need for parentheses in the concrete syntax can be seen in the following sentence.

```
not true & true
```

What exactly are the arguments to the operations `not` and `&`? Since the grammar is ambiguous (the expression can be parsed either as `(not true) & true` or `not (true & true)`), no matter what disambiguating rule we pick, one or the other expressions *cannot* be expressed without parentheses. In contrast, we can readily represent both expression in the abstract syntax.

```
And (Not T) T
Not (And T T)
```

This works because we can use the parentheses of the metalanguage (Haskell) to express the grouping of subexpressions in the represented object language (*cond*).

Third, the curly braces in the first *stmt* production do not occur in the abstract syntax, again, because they are not needed: The production is identified in the abstract syntax by the constructor `While`, and both nonterminal arguments of the production are represented by the corresponding argument types `Cond` and `Stmt`. The same applies to other "filler" terminal symbols such as `do`, `begin`, or `end`, which are often included in concrete syntax to avoid ambiguity and aid the parser.

As in the case of `Expr`, we can use the abstract syntax for representing programs in Haskell. Consider the following program in concrete syntax.

```
while not (true & true) {
    while true {
        skip
    }
}
```

ABSTRACT SYNTAX

The abstract syntax of a language is a set of syntax trees. Using Haskell data types for representing abstract syntax means that an abstract syntax tree is given by a value of a Haskell data type, which facilitates program analyses and transformations to be realized as Haskell functions.

The language defined by a context-free grammar is the set of sentences that can be derived from the start symbol of that grammar. Correspondingly, a Haskell data type defines the language of ASTs as the set of of values that can be built using the data type's constructors.

The abstract syntax for this program is given by the following value of type `Stmt`.

```
While (Not (And T T)) (While T Skip)
```

Abstract and concrete syntax are two sides of the same coin. They both describe languages and their structure, but they have a different purpose.

Concrete syntax is meant to be read and understood by programmers and is thus generally more readable than abstract syntax. But this also makes concrete syntax more verbose, especially through the use of key words and white space. Moreover, despite the availability of structure editors and visual program representations, programmers today still edit programs with text editors, which means the concrete program representation has to be a linear sequence of symbols. This therefore often requires filler symbols in the grammar for expressing grouping.

Abstract syntax, on the other hand, is generally more parsimonious than concrete syntax. Its major purpose is to present the structure of programs to compilers and mathematical formalisms. Abstract syntax is not meant to be read and edited by programmers.[4] The abstract syntax representation lays the foundation for defining the semantics of language constructs, and it facilitates different kinds of analyses of programs, including type checking and optimization. By being more concise, abstract syntax representations consume less space and reveal essential language structures more directly than concrete syntax.

3.4 Abstract Syntax Idioms

So far the standard approach described in Section 3.3 says to define a separate data type for each nonterminal and to introduce a constructor for each production. (We have already

[4] Although computer science researchers are using it extensively.

Exercise 3.4

(a) Define the abstract syntax for the following grammar as a Haskell data type, and represent the abstract syntax tree for the sentence `flip flip on` as a value of that type.

switch ::= `on` | `off` | `flip` *switch*

(b) Assume we add the following production to the grammar for *expr*.

expr ::= ... | `(`*expr*`)`

Do we need to change the abstract syntax, that is, do we have to add a constructor to the data type definition for `Expr`)? If so, how? If not, why not?

(c) Assume we add the following *stmt* production for representing sequential composition of statements.

stmt ::= ... | *stmt*`;` *stmt*

Extend the abstract syntax by adding a constructor to the data type definition for `Stmt`.

(d) The extension of *stmt* in part (c) seems to have added effectively a binary operation `;` on two statements. Does this extension make the grammar ambiguous? If so, do we need to introduce a construct for grouping statements? If you think the grammar is ambiguous but we don't need syntax for grouping statements, explain why.

seen an exception to the last rule and can simply ignore productions that introduce nonterminals for expressing grouping.) In addition, there are a few grammar patterns that occur quite frequently and that can be represented more succinctly in abstract syntax.

3.4.1 Factoring

Reusing existing constructors and embedding them into a new data type helps with managing names and also makes relationships between different types explicit. Consider, for example, the following definition of the `Cond` data type.

```
data Cond = T | F | And Cond Cond | Not Cond
```

We have already discussed the need to use `T` instead of `True`, since the latter is predefined. But instead of introducing new constructors for something that already exists, we could also reuse the type `Bool` and embed it into the definition of `Cond` as follows.

```
data Cond = Const Bool | And Cond Cond | Not Cond
```

Note that we need to introduce the constructor `Const` to inject boolean values into the type `Cond`. (This is basically the same situation that we've seen before when we used `Lit` to inject `Int` values into the data type `Expr`.) The use of `Bool` instead of `T` and `F` amounts to factoring the two constructors into a separate data type. The advantage of this representation is that we can reuse `Bool`, the disadvantage is that we need to use an additional constructor for boolean constants when constructing `Cond` expressions. For example, instead of `And (Not T) T`, we now hat to write the more verbose:

```
And (Not (Const True)) (Const True)
```

This not a huge problem in practice, since (a) we construct such expressions rarely (often only once for creating test cases) and (b) we can always introduce so-called *smart constructors*, that is, we can introduce abbreviations such as `tt = Const True` and then use them in expression such as `And (Not tt) tt`.

Factoring is not limited to constructs that are pre-defined in the metalanguage, and it is not limited to constants either. For example, suppose the `Cond` data type contains several other binary operations, such as `Or`, `Xor`, and `Nand`. Instead of introducing each operation separately as a constructor of `Cond` we can combine all binary operations into one generalized constructor that takes the operation as an additional parameter and define the operations in a separate enumeration type `Op`.

```
data Op = And | Or | Xor | Nand
data Cond = Const Bool | Bin Op Cond Cond | Not Cond
```

This generalization captures the common structure of all binary operation in a single constructor `Bin`.

This last definition looks more economical from the perspective of the type definition. Having fewer constructors can be an advantage for later function definitions, since we have fewer cases to consider. However, expressions get bigger and more nested, and the added levels of indirections through constructors might be less attractive in other parts of the code. There's no general rule of which approach is better. The point is that it is possible, and that the metalanguage provides the language designer with options to chose from.

Finally, we can also factor the argument types of operations. For example, we could introduce a type definition for pairs of conditions and use that type as the argument type for binary operations. Combining this idea with the previous one would lead to the following definitions.

```
type CondPair = (Cond,Cond)
data Op = And | Or | Xor | Nand
data Cond = Const Bool | Bin Op CondPair | Not Cond
```

Note that `CondPair` is not a new type in the sense that it doesn't introduce any new values. It's simply a name for an already existing type and it is therefore defined using a type definition. The factoring of argument types is less common, and in this specific example it doesn't seem to enhance the abstract syntax definition, if only because `CondPair` is just used once.

3.4.2 Replacing Grammar Recursion by Lists

Most languages contain repetitions of specific syntactic categories, and programming languages are no exception. For example, a function can have zero or more parameters, a procedure may have an arbitrary number of local variable declarations, or a statement in an imperative language can be itself a sequence of one or more statements.

The repetition of syntactic structures is often represented in grammars through recursion. In many cases, we can simplify the abstract syntax representation by using Haskell built-in lists to represent repetition more directly. Suppose we want to define the syntax for function definitions so that a function definition consists of a name, a sequence of zero or more parameters, and an expression defining its return value. Here is a possible definition.

$$\begin{aligned} \mathit{fundef} &::= \mathit{name}\ \mathit{params} = \mathit{expr} \\ \mathit{params} &::= \mathit{name}\ \mathit{params} \mid \epsilon \end{aligned}$$

What is important to note are the two *params* productions for defining a list of parameter names. The only purpose of the nonterminal *params* is to be able to generate an arbitrary number of *name* terminals as parameter names for the function definition. This recursive pattern for defining a list is so common that many grammars have adopted a special notation for it: The repetition of a nonterminal can be expressed by simply attaching a $*$ to it. With this notational extension, we can simplify the grammar as follows.

$$\mathit{fundef} ::= \mathit{name}\ \mathit{name}^* = \mathit{expr}$$

The star notation is really just an abbreviation for the extended definition. It can be used to refactor grammar definitions by identifying (groups of) productions that define repeating syntactic structures and simplify them using the star. The star notation allows us to express repetition directly without having to invent an additional nonterminal.

If we translated the original grammar into abstract syntax using our standard method, we would obtain something like the following.[5]

[5] Since this example has only one production for function definitions, we could also use the following type definition `type FunDef = (Name,Params,Expr)`.

```
type Name = String
data FunDef = FunDef Name Params Expr
data Params = Param Name Params | Empty
```

This definition is fine, but it can be quite inconvenient to work with. Consider the following function definition.

```
f m x y = m*x + y
```

The initial part of the AST for it looks as follows.[6]

```
FunDef "f" (Param "m" (Param "x" (Param "y" Empty))) (Plus (Times ...) ...)
```

The need for the nested application of the `Param` constructor is quite annoying. It is difficult to look at and work with, especially taking care of the correct matching of parentheses. Fortunately, we can simplify the definition considerably, using a mechanism that mimics the star notation of the concrete syntax. Recognizing that `Params` is simply defining a list of `Name`s, we can use Haskell's list data type to express this type more directly.

```
data FunDef = Fun Name [Name] Expr
```

Like the use of the star notation obviates the need for the *params* nonterminal, the use of the built-in lists saves us the definition of the `Params` data type, and, importantly, the definition of ASTs can make use of the built-in list notation.

```
Fun "f" ["m","x","y"] (Plus (Times ...) ...)
```

In addition to the star notation, grammars also sometimes attach a $^+$ to indicate the repetition of nonterminals that must occur at least once, that is, while A^* stands for *zero* or more repetitions of A, A^+ stands for *one* or more repetitions of A. Thus, if the language required a function definition to have at least one parameter, we would change the original grammar by replacing the ϵ-production for *params* by *name*, or we could the plus notation.

$$\mathit{fundef} ::= \mathit{name}\ \mathit{name}^+ = \mathit{expr}$$

How should we address the change in the abstract syntax? Since lists can be empty, the use of `[Name]` as an argument type of the `Fun` constructor does not exactly reflect the grammar anymore. We could enforce the presence of at least one name by using a pair of a single name and a potentially empty list, like so.

[6] To follow this example in `ghci`, you need to add a constructor such as `Var` to the `Expr` data type for representing variables in expressions such as `Times (Var "m") (Var "x")`.

```
data FunDef = Fun Name (Name,[Name]) Expr
```

While this is technically an accurate representation of the abstract syntax, the more permissive but simpler `[Name]` is usually an acceptable or even better representation, in particular, in contexts where we can assume that a parser ensures that the lists are not empty.

The star or plus notation can also be applied to groups of syntactic elements. Suppose that the concrete syntax for function definitions requires parentheses around the parameters, which also must be separated by commas, that is, the definition of `f` would have to have the following form.

```
f (m,x,y) = m*x + y
```

The corresponding change in the grammar productions is as follows.

fundef ::= *name* `(` *params* `)` `=` *expr*
params ::= *name* `,` *params* | *name*

Note that this definition requires the parameter list to be non-empty. To replace the use of *params* by the star notation we have to be a bit careful, since a list of n variables is accompanied by only $n - 1$ commas. In particular, we *cannot* use the plus notation in this situation.

fundef ::= *name* `(` (*name* `,`)* *name* `)` `=` *expr*

Applying the star to the group of two symbols (*name* and `,`) means to repeat this group zero or more times.

However, the representation of the abstract syntax doesn't change, since the comma is a filler symbol that doesn't add any information and is thus to be ignored.

To see how the use of lists for expressing repetition can be combined with tuples to express grouping, we examine the syntax of `let` expressions. A typical concrete syntax introduces the construct with the keyword `let`, followed by a list of name/expression pairs, which are separated by `=`, followed by the keyword `in` and the body of the `let` expression.

expr ::= `...` | `let` (*name* `=` *expr*)* `in` *expr*

With the star notation, we can conveniently express the repetition of the "*name* `=` *expr*" group within the `let` production. On the abstract syntax level, we can express the grouping of the types corresponding type `Name` and `Expr` as a pair and the repetition, again, using a list type.

```
data Expr = ... | Let [(Name,Expr)] Expr
```

This definition allows a `let` expression to have an empty list of definitions, which is curiously indeed the case in Haskell.

```
> let in 3
3
```

3.4.3 Grouping Associative Operations using Lists

Consider again the definition of `Cond` from page 55. The `&` operator is meant to denote the logical *and* operation, which is an associative operation. Therefore, the grouping of nested applications of `&` doesn't matter, that is, `a & (b & c)` denotes the same value as `(a & b) & c`. Thus, instead of defining `And` with two arguments, we could also define is with a list of arguments.

```
data Cond = ... | And [Cond]
```

With this definition we can avoid nested applications of `And` and represent both previous expressions as follows.

```
And [a,b,c]
```

As another example consider the extension of *stmt* by a production for sequential composition (cf. Exercise 3.4**(c)**).

stmt ::= ... | *stmt*; *stmt*

The list-based representation looks as follows.

```
data Stmt = ... | Seq [Stmt]
```

Suppose now we have to implement a syntax analysis that checks whether a property `p` holds for all statements. Having `Seq` defined as a binary operation requires a recursive definition, something like the following.

```
check :: (Stmt -> Bool) -> Stmt -> Bool
...
check p (Seq b1 b2) = check p b1 && check p b2
```

This doesn't seem too bad. But what we are doing here is essentially implementing a control structure for iterating over statements using recursion, whereas with the list-based representation we can reuse control structures for lists that we already have. For example, we have a function `all` that checks whether a predicate holds for all elements of a list.

```
all :: (a -> Bool) -> [a] -> Bool
```

Using `all` we can implement the case for `check` as follows.

```
check :: (Stmt -> Bool) -> Stmt -> Bool
...
check p (Seq bs) = all check bs
```

For this example, the gain of reusing list control structures doesn't seem very compelling. But consider the tasks of removing statements from a sequence of statements until a statement is reached that satisfies a specific property. (We will see a concrete example that will need this kind of operation in Unit 4, Exercise 4.1**(a)**). Concretely, we are seeking a function `removeUntil` of type `(Stmt -> Bool) -> Stmt -> Stmt`. An implementation that works for binary trees built using `Seq` is not at all obvious (try it!). In contrast, we can relatively easily compose a solution using list functions by working on the reverse of a list.

```
keepUntil :: (Stmt -> Bool) -> [Stmt] -> Bool
keepUntil p []     = []
keepUntil p (s:ss) | p s       = []
                   | otherwise = s:keepUntil p ss

removeUntil :: (Stmt -> Bool) -> Stmt -> Stmt
removeUntil p (Seq ss) = (Seq . reverse . keepUntil p . reverse) ss
```

The same strategy is not so easy to realize for the alternative representation.

To conclude this unit, the box TRANSLATING GRAMMARS INTO DATA TYPES gives a summary of how to generate an abstract syntax definition from a concrete syntax. While the general method for obtaining abstract syntax is not complex, careful analysis can in many cases improve the representation significantly with great benefit to all operations that have to use it.

TRANSLATING GRAMMARS INTO DATA TYPES

The following table shows the correspondence between different elements of the grammar formalism for concrete syntax and data types for abstract syntax. Nonterminals, which can generally be expanded to many different sequences of terminals, correspond to types, which contain many different values. Sequences of terminal symbols correspond to abstract syntax trees, and terminal symbols that represent operations (and are not just helpers for the parser to indicate grouping) correspond to tree constructors.

Grammar		**Data Type**	
Basic nonterminal	*num*, *name*, ...	Predefined type	`Int`, `String`, ...
Non-basic nonterminal	*expr*, *stmt*, ...	Data type	`Expr`, `Stmt`, ...
Terminal representing an operation	`+`, `while`, ...	Constructor	`Plus`, `While`, ...
Grouping/filler terminals	`begin`, `(`, `)`, `do`, ...	—	—

The translation of a context-free grammar into an abstract syntax representation given by data types follows four basic rules.

(1) Represent each *basic nonterminal* by a *built-in type*.
(2) For each *other nonterminal*, define a separate *data type*.
(3) For each *production*, define a *constructor*.
(4) The *nonterminals* in a production determine the *argument types* of its constructor.

Exercise 3.5

(a) Consider the following grammar that defines the syntax of a simple assembly language.

reg ::= `A` | `B` | `C`
op ::= `MOV` *num* `TO` *reg* | `MOV` *reg* `TO` *reg* | `INC` *reg* `BY` *num* | `INC` *reg* `BY` *reg*
prog ::= *op* | *op* `;` *prog*

Define the abstract syntax for this grammar through Haskell data types.

(b) Does your solution to (a) contains a data type `Op` with four constructors, corresponding to the four *op* productions? Such a definition can be factored further by capturing the choice between a *num* and *reg* argument in a separate nonterminal (in the grammar) or data type (in the abstract syntax).

Define the refactored grammar and abstract syntax.

(c) Some grammar formalisms allow the definition of optional syntax elements by enclosing them in square brackets. Consider, for example, the definition of numbers that may be preceded by a `+` or `-` sign.

sign ::= `+` | `-`
snum ::= *num* | *sign num*

Using the square-bracket notation, the two *snum* productions can be replaced by a single one.

snum ::= [*sign*] *num*

In the abstract syntax, optional elements can be represented by using the `Maybe` data type, which is defined as follows.

```
data Maybe a = Just a | Nothing
```

Define the abstract syntax for *snum* using `Maybe` to express the optionality of a sign. Note that since we are left with only one production we can use a simple type instead of a data type definition.

(d) Consider the following language for describing movements in the 2D plane.

point ::= `(`*num*`,`*num*`)`
move ::= [`from` *point*] (`via` *point*)* `to` *point*

Define the abstract syntax for *move*.

4 Denotational Semantics

Syntax is about the *form* and *structure* of languages. Semantics is about the *meaning* of languages, which is what ultimately matters. Speaking a language effectively does not merely mean being able to form syntactically correct sentences. To communicate ideas through a language, sentences must also represent coherent ideas or concepts. While it is of course obvious that you have to understand the semantics of a language before you can effectively use it, the question remains why semantics, and especially formal semantics, is such an important topic for programming languages?

Why Semantics?

Language semantics provide a number of important benefits.

First and foremost, the semantics of a programming language defines *precisely* the *meaning of individual programming constructs* and helps us understand what the constructs of the language do. In particular, a formal definition cannot leave any corner case uncovered and resolves any uncertainty about how a specific construct behaves in any situation.

Second, semantics allows us to *judge the correctness of a program* and compare expected with observed behavior. Based on an understanding of individual language constructs, we can understand what a whole program is doing and thus make sense of unexpected program behavior and correct mistakes in the program.

Third, a semantics definition for a language enables us to *prove properties about the language*. One such important property is the so-called *type soundness*, that is, the fact that

the absence of type errors in a program guarantees the absence of a large class of runtime errors. We will look at this aspect in more detail in Unit 5. The fact that a type checker can automatically prove the absence of a large class of errors and thus effectively delivers a partial correctness proof for a program is an extremely important language feature. But type soundness can only be established for a language that has a semantics definition.

Fourth, semantics definitions helps with *comparing languages*. For example, the way a language realizes parameter passing (call-by-value or lazy evaluation) has implication on its expressiveness and efficiency and is thus an important consideration in picking a language for a particular project. The parameter passing style of a language is part of its semantics.

Finally, semantics definitions can *guide the design of languages*, which is particularly helpful for domain-specific languages. Ultimately, semantics can serve as a *specification for language implementations*.

What is Semantics?

Having established that semantics play an important role in many aspects of programming, the next questions are:

- *What actually is the semantics of a program*?
- *What is the semantics of a programming language*?

It turns out that the answers depend on the language. For example, the meaning of an arithmetic expression is a number, that is, the meaning of an individual program from the language of arithmetic expression is a number. In other words, the semantics of an arithmetic expression is a value of a particular type ($\mathbb{Z}$ or `Int`). In contrast, the meaning of a boolean expression (see page 55) is either true or false, that is, the semantics of a boolean expression is a value of type $\mathbb{B}$ or `Bool`.

These two languages are quite simple. What about a fully-fledged general-purpose programming language such as C or Java? What is the meaning of programs in these languages? Since a program in an imperative language works by manipulating state through assignment operations, its semantics can be described as the final state it produces when starting with a particular initial state. This effect can be described as a function that maps an input state to an output state. In other words, given a type $\mathbb{S}$ (or `State`) capturing all possible states, the semantics of an imperative language is a function of type $\mathbb{S} \to \mathbb{S}$ or `State -> State`.

The sets $\mathbb{Z}$, $\mathbb{B}$, and $\mathbb{S} \to \mathbb{S}$ (as well as the corresponding Haskell types `Int`, `Bool`, and `State -> State`) are called *semantic domains*; they capture the range of all possible program meanings for a specific programming language. Thus the semantics of an individual pro-

gram is given by a value of its language's semantic domain. More generally, the semantics of a programming language is then given by a function, called *semantic function*, that maps each program to a corresponding semantic value.

This approach of identifying and formally defining the semantic domain of a language and then defining a mapping from sentences or ASTs to elements of the semantic domain is called *denotational semantics*. We will look into how that works in more detail in this unit. There are also other styles of defining semantics, notably *operational semantics* in which the semantics of programs is described by transformation rules that simplify a program into some resulting value, and *axiomatic semantics* in which the effect of programs (or parts of programs) are described by pairs of pre- and post-conditions.

It may seem from the order of presentation that a language definition starts with the definition of the syntax and after that defines the semantic domain. However, that is not really the case. The semantic domain for a language must actually be known (at least informally) before the syntax of a language can be defined. For example, if you decide to define a new imperative programming language, then you already have chosen $\mathbb{S} \to \mathbb{S}$ as a domain, and your decision on the syntax for an assignment operation presumes this domain because without state transformation as the underlying semantic domain for the language, an assignment operation makes no sense and can't even be defined. Here is another example to make this point clear. Consider the language of chemical formulae, which are used to describe the proportions of atoms in chemical substances. Using H_2O to denote water makes sense to us know, but would not have made any sense before 1661 when Robert Boyle proposed a model of chemical substances of atomic particles.

About this Unit

In Section 4.1 I will illustrate the three basic steps to defining denotational semantics with a simple example, a language of arithmetic expressions. In Section 4.2 I will describe a few general constructions for semantic domains that capture some prototypical structures that occur frequently in programming languages. These examples demonstrate that significant part of the effort in defining semantics is devoted to the design of the semantic domain.

4.1 Defining Semantics in Three Steps

Since we represent abstract syntax as Haskell data types, the semantics function will be given by as a Haskell function and, consequently, the semantic domain will be given by a Haskell type.

In the formal, mathematical version of denotational semantics, domains are defined as

DENOTATIONAL SEMANTICS DEFINITION

The denotational semantics of a language consists of three components.

(1) *Syntax* (a set S)
(2) *Semantic domain* (a CPO D)
(3) *Semantic function* (a function $[\![\cdot]\!] : S \to D$)

Performing a denotational semantics definition in Haskell means to define the representation of the three components and involves the following three steps.

(1) Define a type `S` of ASTs.
(2) Define a type `D` of semantic values.
(3) Define the function `sem :: S -> D` that maps ASTs to semantic values.

Note: While it seems natural to define the semantic domain in step 2, it must be known, even if only implicitly, before the definition of the syntax in step 1.

so-called *complete partial orders*, or CPOs, which are sets plus a partial order relation $\sqsubseteq$ and a least element $\bot$. The need for using CPOs arises from recursion (or iteration). To give a proper definition for the meaning of arbitrary programs we need to account for nontermination and assign different values from the domain to terminating and nonterminating functions. For lack of time we will not pursue this any further and instead focus mostly on how to define denotational semantics directly in Haskell. The process is summarized in the box DENOTATIONAL SEMANTICS DEFINITION.

Let's consider as an example the language of arithmetics expression from Unit 3. We have a type for the abstract syntax already.

```
data Expr = Lit Int | Plus Expr Expr | Times Expr Expr
```

The next step is to determine the semantic domain, which is the type of all values that can be denoted by an expression of type `Expr`. In this example, the answer is fairly obvious: We consider arithmetic expressions to denote the integers they evaluate to. The semantic domain is thus given simply by the type `Int`.

Finally, we map ASTs (represented by values of type `Expr`) to semantics values (that is, integers) by defining a function `sem`. (The function name is, of course, arbitrary.)

```
sem :: Expr -> Int
sem (Lit i)       = i
sem (Plus e1 e2)  = sem e1 + sem e2
sem (Times e1 e2) = sem e1 * sem e2
```

The function definition is mostly self-explanatory: First, a literal integer (injected into the type `Expr` with the constructor `Lit`) denotes itself. Second, a plus expression, represented by an AST with a `Plus` constructor as its root, denotes the integer that is obtained by adding (using the integer operation `+`) the integers that are denoted by the expressions represented by the two argument ASTs of `Plus`. These integers can be obtained by recursively applying `sem` to the argument ASTs. Third, the semantics of a times expression is obtained in the same ways as a plus expression, except that the integers resulting from the argument ASTs are multiplied.

The mathematical version of this definition looks very similar. Instead of an AST representation as a data type, the syntax is typically defined by a context-free grammar. In this example, we use the grammar given on page 54. In the context of semantics definitions, the grammar notation is typically extended to introduce so-called *metavariables* that can stand for sentences that can be derived from a particular nonterminal. For our example, we write $n \in num$ to express that n stands for (or "ranges over") terminal symbols representing numbers, and we write $e_1, e_2 \in expr$, to express that e_1 and e_2 range over expression sentences.[1] Those metavariables are then used in defining the semantics, but they can also be used in place of nonterminals in productions. The expression grammar using this extended notation looks as follows.

$$\begin{aligned} n \in num &::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \ldots \\ e_1, e_2 \in expr &::= n \mid e_1\texttt{+}e_2 \mid e_1\texttt{*}e_2 \end{aligned}$$

The semantic domain D would be in this case the CPO $\mathbb{Z}_\bot = (\mathbb{Z} \cup \{\bot\}, \sqsubseteq)$, which is the set of integers $\mathbb{Z}$, extended by a so-called *bottom element* $\bot$ to represent an "undefined" value plus the partial order $\sqsubseteq = \{(\bot, x) \mid x \in \mathbb{Z}\}$.

The semantic function is defined for each *expr* production, much like `sem` is defined by an equation for each `Expr` constructor. In the first equations, we employ the notation $\underline{n}$ that maps a *num* nonterminal to the integer it represents. Even though this is a trivial mapping ($\underline{\texttt{0}} = 0$, $\underline{\texttt{1}} = 1$, $\underline{\texttt{2}} = 2$, etc.), it is nevertheless important because the syntax of a numbers is different from the number it stands for.[2] With this mapping we can finally give the equations for the denotational semantics.

[1] We essentially use a nonterminal A as a shorthand for the set $L(A)$.

[2] Formally, we are using a function $\underline{\cdot} : num \to \mathbb{Z}$. If we were to use, for example, roman numerals as syntax for numbers, the need for the mapping would be more obvious. For example, we then would have $\underline{\texttt{III}} = 3$, $\underline{\texttt{IV}} = 4$, or $\underline{\texttt{CCCLXXXI}} = 381$.

$$
\begin{aligned}
[\![\cdot]\!] &: \mathit{expr} \to \mathbb{Z}_\perp \\
[\![n]\!] &= \underline{n} \\
[\![e_1\texttt{+}e_2]\!] &= [\![e_1]\!] + [\![e_2]\!] \\
[\![e_1\texttt{*}e_2]\!] &= [\![e_1]\!] \cdot [\![e_2]\!]
\end{aligned}
$$

The hollow square bracket notation helps us with parsing the notation: Everything within the brackets belongs to the syntax, everything outside belongs to the semantic domain (which also means that the results produced by the function $[\![\cdot]\!]$ are semantic values). Note some of the subtleties of the notation: For example, on the LHS of the second equation, the `+` symbol inside the brackets is part of the syntax; it's the nonterminal that stands for the plus operation. In contrast, the $+$ symbol on the RHS of the equation denotes the plus operation on integers (the semantic domain); it is applied to $[\![e_1]\!]$ and $[\![e_2]\!]$, the integers denoted by the syntactic expressions e_1 and e_2, respectively.

In the Haskell definition, the distinction between syntax and semantic domain is clearly expressed through the different types: Syntactic objects, that is, ASTs, are values of type `Expr` whereas semantic values are of type `Int`. Specifically, the syntax for addition is expressed by the constructor `Plus` whereas the addition operation on semantic values is the Haskell function `+`, defined on `Int` values, and the syntax for numbers is delineated from the semantic values by the constructor `Lit`.

You may have noticed that for the current limited set of expression, which only contain a plus and times operation, the bottom element is not needed, and the simpler set $\mathbb{Z}$ would suffice as semantic domain. However, consider what happens if we add an operation for division.

$$e_1, e_2 \in \mathit{expr} ::= \ldots \mid e_1\texttt{/}e_2$$

Now the situation changes because the semantics of an expression such as $5/0$ cannot be a value from the set $\mathbb{Z}$. To capture this case we can amend the semantic function as follows.

$$[\![e_1\texttt{/}e_2]\!] = \begin{cases} [\![e_1]\!] \div [\![e_2]\!] & \text{if } [\![e_2]\!] \neq 0 \\ \perp & \text{otherwise} \end{cases}$$

In a corresponding extension of the Haskell definition, we could add simply continue to work with `Int` and use integer division in the definition of `sem`. We extend the syntax:

```
data Expr = ... | Div Expr Expr
```

and the semantic function:

```
sem (Div e1 e2) = div (sem e1) (sem e2)
```

	Math	Haskell
Syntax	context-free grammar	data type
Semantic domain	CPO	type
Semantic function	function	function
undefined	$\bot$	`Nothing` (or other constructor)

Table 4.1: Comparing Math and Haskell as a metalanguage for describing denotational semantics.

However, if we try to determine the semantics of an expression such as $5/0$, we encounter a runtime error.

```
> sem (Div (Lit 5) (Lit 0))
*** Exception: divide by zero
```

While one could argue that producing a runtime error is equivalent to an undefined semantics, handing over part of the semantic definition over to the runtime system does not seem to be a very principled approach. Since Haskell types are not CPOs, we therefore need to deal with this situation in a different way, which we will do in Section 4.2.1.

The need for the partial order arises when the semantic domain includes partial functions, which are needed to represent the semantics of recursive function definitions in a language. In the following, we will not go into the details of the mathematical foundations of the formalization and accept the fact that recursive function definitions can be given a mathematical meaning. Table 4.1 compares the concepts that are used in math with those used in Haskell for expressing denotational semantics.

Exercise 4.1

The goal of this exercise is to define the language of boolean expressions. The syntax of is given by the following grammar.

$b, b_1, b_2 \in \mathit{bexpr} ::= \texttt{T} \mid \texttt{F} \mid b_1 \vee b_2 \mid b_1 \wedge b_2 \mid \neg b$

(a) Give the abstract syntax for *bexpr* by defining a data type `BExpr`.

(b) Identify the type `D` for the semantic domain.

(c) Define the semantic function `sem :: BExpr -> D` in Haskell.

(d) Define the semantic function $[\![\cdot]\!] : \mathit{bexpr} \to \mathbb{B}$ in math notation.

Much, if not most, of the work in defining denotational semantics is devoted to the

identification and definition of the semantic domains. This observation is consistent with computer science more generally, namely, that finding the right representation for a problem is often the key to a solution. For the Haskell-based approach to denotational semantics this means that a key step in defining denotational semantics is the definition of types for the semantic domains. We will look at this part next.

Exercise 4.2

The following grammar describes a language for controlling the moves of a robot in the 2D plane. In the productions for *move*, we use the metavariable $n \in num$.

$m_1, m_2 \in move ::=$ `go up` n | `go right` n | m_1`;`m_2

(a) Define the abstract syntax for *move* as a data type `Move`. (Remember to either hide constructors that are already predefined or choose different names; see the footnote on page 56.)

(b) The semantics of a *move* program m is the point in the plane the robot ends up after performing all the steps in m, starting at the home position $(0, 0)$. Identify the type `D` for the semantic domain.

(c) Define the semantic function `sem :: Move -> D`.

Hint 1: First, think about what the final position of the robot is when the program contains only a single move is, say, `go up 3`. This will tell you the definition for `sem` for this case. Then do the same for the operation `go right`. Finally, think about what effect the sequencing of two moves has on their respective positions. This will tell you how to define `sem` for this case.

Hint 2: Consider defining an auxiliary function for combining values of `D` that you could use here.

4.2 Systematic Construction of Semantic Domains

The languages considered in the previous section had very simple semantic domains such as `Int` or `Bool` (or `(Int,Int)` in the case of Exercise 4.2). For defining the semantics of more realistic languages we need to express more advanced domains, describing a richer set of semantic values. It turns out that semantic domains can often be built systematically, employing only a few basic principles. When defining semantic domains within Haskell, these principles essentially correspond to specific *type constructors*, which are operators on types that build new types from existing ones.

Table 4.2 provides a brief overview of language features and how to incorporate then (that is, using which type constructions) into semantic domains of languages. Note that

Exercise 4.3

Assume that we add a move `home` to the robot-move language from Exercise 4.2 that has the effect of bringing the robot back to the home position.

move ::= `...` | `go home`

(a) Extend the language definition to account for this new operation. The definition is not as easy as it might seem at first. Think about what the correct values of all of the following programs should be.

```
go up 4; go home
go home; go up 3
go up 4; go home; go up 3
go up 4; go home; go up 3; go home; go right 2
```

Hint 1: What effect does the `go home` move have on all the preceding moves?

Hint 2: Employ a helper function `simplify :: Move -> Move` to simplify a `Move` program without changing its semantics.

Hint 3: The simplification should try to get rid of `go home` moves.

(b) If you have represented `;` as a binary operation, try the list-based representation described in Section 3.4.3.

designing semantic domains is still not a trivial exercise, because, in particular, the question of how to combine multiple features requires careful thought about who to compose the corresponding type constructors. Consider, for example, adding errors to a simple product domain `(D1,D2)`. This task already presents a choice of seven possible combinations.

```
      (Maybe D1,D2)  |       (D1,Maybe D2)  |      (Maybe D1,Maybe D2)
Maybe (Maybe D1,D2)  | Maybe (D1,Maybe D2)  | Maybe (Maybe D1,Maybe D2)
                     |                      |                Maybe (D1,D2)
```

Which of these types provides the correct definition of the semantic domain depends on where errors can occur, how they are propagated, and whether they should be distinguished or identified.

4.2.1 Error Domains

When a program execution goes wrong and can't continue, this situation is often resolved by aborting a computation while reporting a runtime error. From the denotational seman-

Domain Construction	Type Representation
Adding Errors	Use `Maybe` (or `Either`)
Product of domains	Use tuple types
Union of domains	Define data type (or use `Either`)
Adding state	Use function types

Table 4.2: Domain constructions and corresponding type representation.

tics point of view, this means two things.

First, a computation that can't continue amounts to a case in the semantic function definition that can't be assigned a proper semantic value from the semantic domain. Second, to capture this information we need to add a special value to the semantic domain that can be clearly distinguished from all other values.

In the mathematical formulation of denotational semantics, the undefined value $\perp$ plays the role of this special value. It is already part of any domain because in the mathematical approach domains are CPOs that contain $\perp$ as their least value. Thus when the semantic function ordinarily returns values of some type D, we can have it return $\perp$ from the domain $D_\perp$ in undefined cases when an ordinary D value can't be given. We have seen this earlier with handling the division by zero in the semantics for *expr*.

Since Haskell types don't have an undefined value to fall back on, we have to add such a value explicitly when needed. We can do this conveniently using the `Maybe` data type, which is defined as follows.

```
data Maybe a = Nothing | Just a
```

The `Just` constructor represents "regular" values, and the `Nothing` constructor represents an "undefined" value that is to be used in error situations. Like Haskell lists, `Maybe` is parameterized and thus is a type constructor, that is, `Maybe D` represents the type of values of type `D`, extended by the value `Nothing`.

The `Maybe` type constructor provides a means to define the denotational semantics for the `Expr` type extended by a constructor `Div` for representing a division operation. First of all, we change the semantic domain to be `Maybe Int` so that we have access to the constructor `Nothing` to represent an error (or undefined) situation. While this seems like an innocent change, it affects the definition of `sem` all over the place. Most annoyingly, we cannot use the results of `sem` as arguments to integer operations anymore, since `sem` returns values of type `Maybe Int` instead of `Int`. Consider the definition in the case of `Plus`.

```
sem (Plus e1 e2) = sem e1 + sem e2
```

This is a simple definition that captures the meaning of the `Plus` clearly. But since `sem e1` and `sem e2` can either return an integer (wrapped in a `Just` constructor) or undefined (represented by the `Nothing` constructor), we have four possible cases to deal with. Only in the case when both `sem e1` and `sem e2` return an integer, can we apply `+` and return a `Maybe Int` value built using `Just`. In all three other cases we have to return `Nothing`.

There are several ways to implement the definition, and it's instructive to take a closer look at the different options because it provides more practice in functional programming specifically and illustrates strategies for building type-based abstractions more generally. We will use these strategies again later, and they are good software engineering practices as well.

The first approach is to scrutinize the results of the two `sem` calls by distinguishing between the two cases of the `Maybe Int`. So far we have done the case analysis of data type values through pattern matching in equations. To do this as part of an expression we can employ the `case` construct of Haskell. An equation is basically only syntactic sugar for a `case` expression. Recall the pattern-matching definition of the function `sum` from page 34.

```
sum []     = 0
sum (x:xs) = x + sum xs
```

We can replace the two equation by introducing a variable for the list argument and scrutinizing the different cases using a `case` expression as follows.

```
sum l = case l of
          []   -> 0
          x:xs -> x + sum xs
```

We can use `case` expressions in a similar way to scrutinize the results of `sem e1` and `sem e2` and define `sem` for `Plus` as follows. Note that we compute and analyze `sem e2` only if `sem e1` is not `Nothing` because in the latter case we know that the overall result has to be `Nothing`.

```
sem (Plus e1 e2) = case sem e1 of
                     Just i -> case sem e2 of
                                 Just j -> Just (i+j)
                                 _      -> Nothing
                     _      -> Nothing
```

The definition shows that only if both `sem e1` and `sem e2` return an integer, can we apply `+` and return a new integer. While this definition is not difficult to understand, it is quite verbose: one line of code has ballooned into five. This is particularly annoying if we consider that we have to repeat the same code for other binary operations such as `Times` when the only thing that changes is the integer operation (replace `+` by `*` in the case of `Times`).

Since we only have to distinguish between two of the four possible cases, we can simplify the definition somewhat by scrutinizing the result of the pair `(sem e1,sem e2)` in the `case` expression, which leads to the following code.

```
sem (Plus e1 e2) = case (sem e1,sem e2) of
                     (Just i,Just j) -> Just (i+j)
                     _               -> Nothing
```

This is much better, but still not ideal, since we are forced to repeat the boilerplate code for the `case` expression. But since our metalanguage Haskell is a fully-fledged functional programming language, we can capture this programming pattern easily in a function definition. Remember that one of the features of functional programming is its ability to define customized control structures, and this particular way of computing with `Maybe` values occurs frequently enough that it deserves its own control structure. And as it turns out, a corresponding function already exists as a library function in Haskell. It is called `liftM2` and takes a binary function `f` and two `Maybe` values[3] `m1` and `m2` and applies `f` to the arguments of `m1` and `m2` if both of them are `Just` values. Otherwise, it returns `Nothing`.

```
liftM2 :: (a -> b -> c) -> Maybe a -> Maybe b -> Maybe c
liftM2 f (Just x) (Just y) = Just (f x y)
liftM2 _ _        _        = Nothing
```

From a denotational semantics point of view, `liftM2` ensures that the function `f` is applied only if both of its arguments are defined and otherwise propagates undefined. The type of the function also explains its name: It takes a function of two arguments and lifts it so that it can work with values lifted into the `Maybe` type.[4] With `liftM2` we can define the semantics for `Expr` with division as follows.

```
sem :: Expr -> Maybe Int
sem (Lit i)       = Just i
sem (Plus e1 e2)  = liftM2 (+) (sem e1) (sem e2)
sem (Times e1 e2) = liftM2 (*) (sem e1) (sem e2)
sem (Div e1 e2)   | sem e2==Just 0 = Nothing
                  | otherwise      = liftM2 div (sem e1) (sem e2)
```

In case you wonder: The recomputing of `sem e2` in the case for `Div` can be avoided by using a `let` expression.

[3] If you consult the definition of `liftM2` in the library `Control.Monad`, you will notice that the function is actually more general, but that is not important here.

[4] As you may have guessed, there is also a function `liftM` for lifting unary functions, `liftM3` for lifting ternary functions, etc. Also, the `M` stand for `Monad` and not `Maybe`, but again, this is not important here.

```
sem (Div e1 e2) = let d=sem e2 in
                      if d==Just 0 then Nothing
                                   else liftM2 div (sem e1) d
```

But since efficiency is not a concern in the context of defining semantics, the previous definition is absolutely acceptable and even preferable because it is simpler.

Note that instead of using a separate data type such as `Maybe`, errors can also be represented more directly if the semantic domain is already given by a data type (as discussed in Section 4.2.3). In that case, we can extend the domain by a constructor for for representing the undefined/error case, or we may even decide to have multiple constructors to distinguish between different error situations.

Exercise 4.4

(a) Consider again the `Move` language introduced in Exercise 4.3.

$m_1, m_2 \in move ::=$ `go up` n | `go right` n | m_1`;`m_2

Adapt the semantics so that the arguments of `go up` moves must not be larger than 9.

(b) Instead of just returning an undefined value, we can also provide an error message that explains the reason for failure. To this end, we can use the following data type as a substitute for `Maybe`.

```
data Result a = OK a | Error String
```

Change the semantics definitions for `Expr` and `Move` to use `Error` instead of `Maybe` for the semantic domain.

4.2.2 Product Domains

Product domains correspond to the cartesian product of sets. They can appear in two different situations.

First, a semantic domain is given as the cartesian product of simpler domains. Again, the `Move` language introduced in Exercise 4.3 provides an example: A position is given by a pair of integers, that is, an element of the type `(Int,Int)`, which correspond to the cartesian product of `Int` with itself. More generally, any aggregation of semantic information into tuples asks for a corresponding semantic product domain, which are represented by Haskell tuple types.

Second, if we have two different semantics for a language, we can combine them into one semantics that produces pairs of values, taken from the product domain of the individ-

ual domains. To illustrate this idea, let's assume we have defined a so-called *count semantics* that determines for an expression how many operations need to be evaluated for determining its value: No operation is needed for evaluating constants, and any expression using a binary operation requires 1 step plus the steps needed for evaluating the arguments.

```
type Count = Int

semCount :: Expr -> Count
semCount (Lit i)       = 0
semCount (Plus e1 e2)  = 1 + semCount e1 + semCount e2
semCount (Times e1 e2) = 1 + semCount e1 + semCount e2
```

This count semantics could be regarded as a simple cost model for expressions. If we want to create a new semantics that combines the value semantics and the cost semantics, we need to combine the domains and the semantic functions. The semantic domain for the combined semantics would be `(Int,Count)`, and we could define a new semantic functions, say `semVC :: Expr -> (Int,Count)`, by giving new equations that combine the expressions from both definitions. But we could also use the function `&&&` (called *fanout*) that applies two functions to a value and returns the pair of results combine the two semantics. It is defined as follows.[5]

```
(&&&) :: (a -> b) -> (a -> c) -> (a -> (b,c))
(f &&& g) x = (f x,g x)
```

With the help of `&&&` we can compose the combined semantics simply as follows.

```
semVC :: Expr -> (Int,Count)
semVC = sem &&& semCount
```

4.2.3 Union Domains

When the constructs of a language can produce values of different types, such as integers, booleans, or strings, the semantic domain must be able to represent values of these types. The obvious approach in Haskell is to define a data type with a constructor for each different value domain.

To consider a concrete example, let's extend the `Expr` language with an operation to compare values for equality, which returns boolean values, as well as an operation for negating logical values.

```
data Expr = Lit Int | Plus Expr Expr | Equal Expr Expr | Not Expr
```

[5] This function can be imported from the library library `Control.Arrow`.

Since an arbitrary expressions may denote either an integer or a boolean, the semantic domain must be a data type with constructors for integers and booleans. In a first attempt, we can use the following definition.

```
data Val = I Int | B Bool
```

We can now tag integers with `I` and booleans with `B` and thus join both types `Int` and `Bool` in the union type `Val`. Since the integers and booleans are tagged by the respective constructors `I` and `B`, a value of type `Val` is always different from an `Int` or `Bool` value.

If we now tried to define the semantic function with type `sem :: Expr -> Val`, we would soon run into a problem defining the semantics for some of the expressions. For example, what value should the expression `Not (Lit 3)` denote?

With multiple types we have suddenly introduced the possibility of *type errors* in expressions, and the semantics should reflect the fact that the meaning of type-incorrect expressions is undefined. Therefore, we either have to change the semantic domain to `Maybe Val`, or we integrate the undefined value directly into the union type `Val` through a separate constructor, as shown in the following definition.

```
data Val = I Int | B Bool | Undefined
```

When adopting this extended version of `Val`, the definition of the semantic function has to make sure that the arguments of operations are of the correct type; only then can a result be computed. Otherwise, the value `Undefined` must be returned. This situation is similar to the one in Section 4.2.1 where we had to deal with potential division-by-zero errors, except that now every operation can potentially introduce an error.

As before, we can employ different strategies for defining the semantic function. Instead of nesting `case` expressions, we can again group the semantics of arguments of binary operations into pairs for pattern matching, which is more concise, but still somewhat repetitive.

```
sem :: Expr -> Val
sem (Lit i)       = I i
sem (Plus e1 e2)  = case (sem e1,sem e2) of
                      (I i,I j) -> I (i+j)
                      _         -> Undefined
sem (Equal e1 e2) = case (sem e1,sem e2) of
                      (I i,I j) -> B (i==j)
                      _         -> Undefined
sem (Not e)       = case sem e of
                      B b -> B (not b)
                      _   -> Undefined
```

The repeated pattern of testing for a particular subtype of `Val` (that is, checking for the `I` and `B` constructors) is striking and calls for refactoring. In this example we can introduce different auxiliary functions that depend on the type of the function to be applied. For example, the function `liftI2` lifts a binary integer function to the `Val` type by ensuring through pattern matching against the `I` constructor that both arguments are integers and injecting the integer result again with the `I` constructor back into the `Val` type. If one of the arguments is not an integer, the result is `Undefined`. Thus the `liftI2` function effectively realizes the dynamic type checking[6] for binary integer functions; it is similar to the function `liftM2`, which we have used in Section 4.2.1.

```
liftI2 :: (Int -> Int -> Int) -> Val -> Val -> Val
liftI2 f (I x) (I y) = I (f x y)
liftI2 _ _     _     = Undefined
```

We can define similar functions for differently typed binary functions (see Exercise 4.5**(a)**). With the help of these auxiliary lifting/type checking functions we can define the semantic function more concisely.

```
sem :: Expr -> Val
sem (Lit i)       = I i
sem (Plus e1 e2)  = liftI2  (+)  (sem e1) (sem e2)
sem (Equal e1 e2) = liftI2B (==) (sem e1) (sem e2)
sem (Not e)       = liftB   not  (sem e)
```

Exercise 4.5

(a) Implement the the following type checking functions.

```
liftI2B :: (Int -> Int -> Bool) -> Val -> Val -> Val
liftB   :: (Bool -> Bool)       -> Val -> Val
```

(b) Extend the semantics for `Expr` so that two boolean values can also successfully compared by `Equal`. Note that you can't use the `liftI2B` function for this. But you can go back to using the `case` expression approach and provide two success cases (one for two integers and one for two booleans).

4.2.4 Domains for Modeling Stateful Computation

The essential idea behind denotational semantics is that a program denotes a value from a suitably chosen semantic domain, where the semantic function defines this mapping from

[6] A topic we will address in more detail in Unit 5.

programs to semantic values precisely. How do imperative languages fit into this picture?

The essence of an imperative program is to manipulate an underlying state and not to produce a value like functional programs. By accumulating the effects of individual statements on an initial state into a final state, the meaning of an imperative program can be described as a function that maps input states to output states.

To illustrate this basic idea, let's consider a very simple imperative language with operations to store and retrieve integers. The abstract syntax consists of two parts: (1) a language of expressions and (2) a language of statements.

```
type Name = String
data Expr = Lit Int | Plus Expr Expr | Var Name
data Stmt = Assign Name Expr | Seq Stmt Stmt
```

The `Var` constructor of the `Expr` type represents the use of a variable name in an expression. The following program (given in some fictitious concrete syntax) would map an empty start state into a state in which `x` is bound to `3` and `y` is bound to `5`.

```
x := 3;
y := x+2
```

This program can be represented in our abstract syntax as a value of type `Stmt` as follows.

```
Seq (Assign "x" (Lit 3))
    (Assign "y" (Plus (Var "x") (Lit 2))
```

To define the semantics `sem :: Stmt -> D` we next need to identify the semantic domain `D`. Since the semantics of an imperative program can be described as a mapping from a start to a result state, the appropriate type for `D` would be a function type `State -> State` where `State` is a mapping from names (that is, strings) to integers.

```
type State = Map Name Int
type D = State -> State
```

For simplicity we employ the predefined[7] type `Map` to implement mappings, which provides, in particular, the following definitions.[8]

```
empty           :: State
insert          :: Name -> Int -> State -> State
findWithDefault :: Int -> Name -> State -> Int
```

[7] The definitions can be imported by adding `import Data.Map` at the top of the Haskell file.

[8] The types are specialized here for the use to represent `State`; `Map` is a polymorphic data type that can be instantiated to other types as well.

The value `empty` denotes an empty state, and `insert x i s` updates the state `s` so that name `x` is mapped to the integer value `i`. If an entry for `x` already exists in `s`, it will be updated to `i`, while a new entry is created in case `x` does not exist in `s`.

Note that like with any data structure in a functional language, the state value `s` is not changed at all, and instead a new state is created and returned as a value by `insert`. We can verify this fact easily in `ghci`.[9]

```
> let s = insert "x" 3 empty
> s
fromList [("x",3)]
> let t = insert "x" 5 s
> t
fromList [("x",5)]
> s
fromList [("x",3)]
```

As we can see, the update of `x` to `5` is only visible in the map `t`; the old map `s` is unchanged and still carries the old value for `x`.

The function `findWithDefault i x s` tries to look up the value of `x` in `s` and returns it if `x` is defined in `s`. Otherwise, the default value `i` is returned. Continuing the example:

```
> findWithDefault 0 "x" s
3
```

```
> findWithDefault 0 "y" s
0
```

The function `findWithDefault` allows us to map unsuccessful variable lookups to a default value instead of creating an undefined or error value, which facilitates working with a simpler semantic domain that doesn't have to account for errors.

When we try to define `sem`, we quickly notice that we actually need to define *two* semantic functions, one for statements, of course, but also one for expressions, since we need to determine the values denoted by expressions as part of defining the semantics of the assignment operation. What should the semantic domain for `Expr` be? One may be tempted to pick `Int`, but what would then be the semantics of an expression such as `Var "x"`? The value of `x` depends on the state, and it is a hallmark of imperative languages that the value of `x` can change over the course of a program execution. In other words, the semantics of a variable (lookup) is *not* an integer, but rather something like a "state-dependent integer."

The state dependence of variables (and thus expressions in general) can be expressed

[9] Maps are internally represented as balanced search trees. For printing a map, a linear, list-like representation is constructed. The `fromList` prefix can be read as, *The map you want to print is the same that you get when you apply* `fromList` *to the shown list.* In fact, `fromList` is a function in the `Data.Map` module that allows the construction of a map from a list of pairs, and evaluating `s == fromList [("x",3)]` actually yields `True`.

through a function type, that is, the semantic domain for expressions should be the function type `State -> Int`, and the semantic function for expressions then maps `Expr` values to values of this type.

```
type E = State -> Int

semE :: Expr -> E
```

Before we define `semE`, we should take a moment to understand the types that are at play here.

First, when we substitute the definition of `E` in the type for `semE`, we notice that `semE` looks like a function that takes two arguments.[10]

```
semE :: Expr -> State -> Int
```

Doesn't this violate the general schema for semantic functions `sem :: S -> D` that stipulates that a semantic function is always a unary function that maps ASTs of some type `S` to semantic values of type `D` (or `E` in this case)?

No, it doesn't. We can actually look at the type of `semE` in two ways. On the one hand, we can understand `semE` as a function that takes *two* arguments, an AST and a state, and returns an integer. On the other hand, we can view `semE` also as a higher-order function (see Section 2.7), that is, `semE` takes an AST and returns a function from states to integers, and `t` is this function type that represents the semantic domain for `Expr`.

For defining the equations of `semE`, the first view is often preferred, since it treats the state as an auxiliary argument that can be consulted to yield the denotation of variables. The definition of `semE` consists of three equations, one for each constructor of `Expr`. The definition in the "two-argument view" looks as follows.

```
semE :: Expr -> State -> Int
semE (Lit i)      _ = i
semE (Plus e1 e2) s = semE e1 s + semE e2 s
semE (Var x)      s = findWithDefault 0 x s
```

We can see that the state is not needed for determining the value of an integer constant, which always denotes itself. The value of a `Plus` expression is obtained by determining the values of its arguments where the current state is passed to the recursive `semE` calls. The third case finally makes use of the state when a variable's value needs to be found in the state argument `s`. As mentioned the auxiliary function `findWithDefault` has a default value

[10] Remember that since the function arrow type constructor is right-associateve, the type `Expr -> (State -> Int)` is the same as `Expr -> State -> Int`.

of `0` ready to be returned in case `x` is not defined in the state `s`.

The definition of `SemE` looks and reads like the definition of a binary function because in each equation `semE` has two arguments. But we can rewrite the equations using lambda abstractions (see Section 2.3.2) to illustrate that we are actually defining a unary function that returns function values.

```
semE (Lit i)      = \_ -> i
semE (Plus e1 e2) = \s -> semE e1 s + semE e2 s
semE (Var x)      = \s -> findWithDefault 0 x s
```

For example, the second equation can now be read as, *The semantics of a* `Plus` *expression is a function that maps a state* `s` *to an integer that is obtained by adding the denotations of the two arguments in the context of the state* `s`. The other two equations can be similarly understood.

With `semE` we can now define the semantics of programs. Again, the function domain `D` allows the reading of the definition in two different ways. Here are the equations for `sem` in the standard form that treats the state as a second argument.

```
sem :: Stmt -> State -> State
sem (Assign x e) s = insert x (semE e s) s
sem (Seq s1 s2)  s = sem s2 (sem s1 s)
```

The semantics of assigning an expression `e` to a variable `x` is obtained by first determining the value of `e` in the current state `s` using the semantic function `semE` and then binding the resulting integer to `x` in the state `s`. The meaning of a sequence of statements `s1` followed by `s2` is the state that is obtained by first determining the state (say, `s'`) that results from performing `s1` in the current state `s` and to then determine the meaning of `s2` in the state `s'`. Again, we could rewrite the equation using lambda abstraction to emphasize that the meaning of a statement is a function.

```
sem (Assign x e) = \s -> insert x (semE e s) s
sem (Seq s1 s2)  = \s -> sem s2 (sem s1 s)
```

In particular, the definition for `Seq` is instructive, since it reveals an interesting fundamental relationship between imperative and functional programming. If you look at the definition closely, you might recognize that the definition of `sem (Seq s1 s2)` is the composition of the functions `sem s1` and `sem s2` (see Section 2.7 for the definition of function composition). This means we can rewrite the definition as follows.

```
sem (Seq s1 s2) = sem s2 . sem s1
```

This equation says that the meaning of sequential composition of imperative programs is

given by the composition of their state-changes. This also shows the basic control structure of sequential composition in imperative languages corresponds to function composition in functional languages.

With `sem` and `semE` we can now determine the meaning of imperative programs such as `x := 3; y := x+2`.

```
> sem (Seq (Assign "x" (Lit 3)) (Assign "y" (Plus (Var "x") (Lit 2)))) empty
fromList [("x",3),("y",5)]
```

Exercise 4.6

The goal of this exercise is to define the semantics of a language for controlling a one-register machine. Programs in this language are given by element of type `Prog`, which is defined as follows.

```
data Op = LD Int | INC | DBL
type Prog = [Op]
```

The operation `LD` loads an integer into the register, `INC` increases the value in the register by 1, and `DBL` doubles the value in the register. For example, assuming that the register is initially set to 0, the program `[LD 2,INC,DBL]` would leave the number 6 in the register.

(a) Define the semantic domain `D`.

(b) Define the semantic function `sem :: Prog -> D`. Note that it might be helpful to define an auxiliary function `semOp :: Op -> D` for capturing the semantics of individual operations.

Assume the register machine is extended to work with 2 registers, and the language adds a corresponding register parameter to its operations.

```
data Reg = A | B
data Op = LD Reg Int | INC Reg | DBL Reg
type Prog = [Op]
```

(c) Define the semantic domain `D` for the two-register machine language.

(d) Define the semantic function `sem :: Prog -> D` for the two-register machine language.

So far we have looked at an absolute minimal version of an imperative language to illustrate the need for two semantic functions and, in particular, function domains. It is not difficult to extend the language in many different directions (multiple types and type checking, returning an error for undefined variables, etc.). Here we should contemplate at least one other extension, namely the addition of loops. To add, say, while loops, we need to add a constructor to `Stmt` with an `Expr` argument for representing the loop's condition and a `Stmt` argument for representing the loop's body.

```
data Stmt = ... | While Expr Stmt
```

Since our language lacks comparison operators and boolean values, we let any non-zero integer correspond to the boolean value `True`, that is, a while loop will execute its body as long as the condition doesn't evaluate to zero. With this convention we can extend the definition of `sem` as follows.

```
sem w@(While c b) s | semE c s==0 = s
                    | otherwise   = sem w (sem b s)
```

First we notice the new notation `w@(While e b)`, called an *as pattern*, which matches in the same way as `While e b` but produces, in addition to the bindings for `c` and `b`, a binding of the variable `w` to the complete `Stmt` value. This comes in handy in the `otherwise` case where it saves us from rebuilding the `While` statement from its components.

The definition can be best understood operationally. It says that when the condition `c` is false (that is, `c` denotes `0` in the current state `s`), the loop will not be executed and has no effect, that is, the current state `s` is the result. Otherwise, the body of the loop, `b`, is executed once, which produces a new intermediate state, say `s'`. Then the loop is executed again in this intermediate state `s'`, which will end the loop if `c` evaluates in `s'` to zero or continue it otherwise.

If we add a `Times` constructor to `Expr`, we can now represent a program for computing factorials in our language. Here is a function that creates a factorial program for any given number `m`. (The backquotes turn a prefix operation into an infix operator, which makes the representation more readable.)

```
fac :: Int -> Stmt
fac m = Assign "f" (Lit m) `Seq`
        Assign "n" (Lit (m-1)) `Seq`
        While (Var "n") (Assign "f" ((Var "f") `Times` n) `Seq`
                         Assign "n" ((Var "n") `Plus` (Lit (-1))))
```

The type of `fac` indicates that it is a *program generator* that creates for a given integer a program in the language `Stmt` which computes the factorial for that number. With `fac` can easily test our semantics on different factorial programs.

```
> sem (fac 3) empty
fromList [("f",6),("n",0)]
```

```
> sem (fac 6) empty
fromList [("f",720),("n",0)]
```

Exercise 4.7

Extend the imperative language by an `if-then-else` construct.

(a) Extend the abstract syntax of `Stmt`.

(b) Extend the definition of `sem`.

5 Types

Types can be in a real nuisance because type checkers often complain, and programming quickly turns into a frustrating endeavor when you can't make it past the type checker. A type checker sometimes seems like a very strict and unforgiving teacher that instead of encouraging you with a "Good Job!" or "This is a good start," often harshly simply tells you that your program is incorrect. But when the type checker tells you that your program is not correct, then your program is not correct.[1] Thus, while potentially quite annoying, the type checker is really your best friend in your struggle to writing correct programs—albeit maybe ill-mannered.

Types and type systems enhance programming languages and offer a number of concrete benefits for programmers.

- Types *summarize* a program on an abstract level and thus provide a *precise documentation* of programs.
- Type correctness means *partial correctness* of programs. In other words, a type checker delivers *partial correctness proofs* for programs.
- Type systems can *prevent runtime errors* and thus can save a lot of debugging effort.
- Type information can be exploited by compilers in the generation of *efficient code*.

In this unit we will learn about types and how they are part of programming languages. Specifically, we will address the following questions in Section 5.2.

[1] For a static type checker this is not always the case, as we will soon discuss. So your anger is actually justified in some cases!

- What is a type?
- What is a type system?
- What are typing rules?
- What is a type error?

Since type systems are usually defined through typing rules, we start this unit in Section 5.1 by learning about *inference rules*, which are the formal basis for expressing typing rules. How a type system described by typing rules can be turned into a type checking algorithm for spotting type errors will then be explained in Section 5.3, and in Section 5.4 we will discuss the notion of *type safety*.

Finally, we will look in Section 5.5 into the difference between two different approaches to type checking, *static typing* and *dynamic typing*, and point out the limitations and advantages of each approach. Unless explicitly stated otherwise, I will use the type names and rules as they exist in Haskell (ignoring overloading). Later in Section 5.5 I will also present some types and typing rules of Python.

5.1 Inference Rules

Inference rules are a very general formalism for defining reasoning systems as well as inductive sets and relations. In general, an inference rule relates logical statements called *judgments* and has the following form.

$$\frac{\mathcal{P}_1 \qquad \ldots \qquad \mathcal{P}_n}{\mathcal{C}}$$

The judgments $\mathcal{P}_1, \ldots, \mathcal{P}_n$ above the horizontal line are called *premises*, and the judgment $\mathcal{C}$ is called the *conclusion* of the rule. The meaning of such a rule is that the conclusion $\mathcal{C}$ is true if all premises $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are true.

Inference rules have a wide range of applications.[2] One of the most basic application is the formulation of logical reasoning rules. For example, the reasoning rule called *modus ponens* says that if A implies B and if A is true, then we can conclude that B is true. This rule can be expressed as an inference rule as follows.

$$\frac{A \Rightarrow B \qquad A}{B}$$

[2] In addition to describing typing rules we will encounter them again in Unit 8 as Prolog rules.

In this example the judgments are variables standing for arbitrary propositions such as A and B and propositional formulas such as $A \Rightarrow B$. Other reasoning rules can be defined in a similar way.

More generally, we can use inference rules to define arbitrary relations. The simplest case are unary relations, which are just plain sets, defined as subsets over some known set. Consider, for example, the set of positive even numbers. We can introduce a judgment $even(n)$ and define it through two rules.

$$\frac{}{even(0)} \qquad\qquad \frac{even(n)}{even(n+2)}$$

The first rule has no premises and establishes the base case that 0 is an even number. A rule without any premises is unconditionally true and is also called an *axiom*. The second rule says that we can get an even number by adding 2 to a number known to be even. This example illustrates how inference rules can inductively define infinite sets.

It is important to understand that a judgment is just a specific way of talking about a relation. For example, the *even* judgment defines a unary predicate on numbers and thus corresponds to a unary relation, which is simply a set. Therefore, any judgment such as $even(0)$ or $even(n+2)$ can always be written using set notation such as $0 \in Even$ or $n + 2 \in Even$ (where *Even* is the name for the set), and we can write rules as implications. For example, the second rule can also be written as follows.

$$n \in Even \implies n + 2 \in Even$$

As an example for a binary relation consider the judgment $fac(n, m)$ that says that m is the factorial of n. In Section 2.4 we have already defined a Haskell function for computing factorials, but we can also define this binary relation between numbers with inference rules.

$$\frac{}{fac(1,1)} \qquad\qquad \frac{fac(n-1,m)}{fac(n, n \cdot m)}$$

Again, we have an axiom defining a base case and an inductive rule that can be used to generate pairs of this relation building on prior, simpler examples. The second rule can be written in a variety of different ways. For example, we can use $n + 1$ in the conclusion

instead of $n - 1$ in the premise, and we can also express the multiplication for the result as a separate premise.

$$\frac{fac(n, m)}{fac(n + 1, (n + 1) \cdot m)} \qquad \frac{fac(n - 1, k) \qquad k \cdot n = m}{fac(n, m)}$$

Note, again, that the prefix notation is just one way of writing a judgment, which in this case is binary. We could alternatively write the judgments using set notation such as $(1, 1) \in Fac$ or $(n, n \cdot m) \in Fac$, and we could write rules as implications.

$$(n - 1, m) \in Fac \implies (n, n \cdot m) \in Fac$$

Or we could invent special syntax and write the rules in the following way.

$$\frac{n! = m}{(n + 1)! = (n + 1) \cdot m} \qquad \frac{(n - 1)! = k \qquad k \cdot n = m}{n! = m}$$

One may think that inference rules are just a different notation for writing function definitions, but that's not the case. In fact, inference rules are more general, since they can be used to define relations that are not functions. As a simple example, consider the less-than relation on integers, which can be defined with two rules.

$$\frac{}{n < n + 1} \qquad \frac{n < k \qquad k < m}{n < m}$$

The axiom[3] says that any number is smaller than its immediate successor, and the rule defines the transitive closure of the successor relation.

The generative view of inference rules (that is, inductive rules produce complex examples from simpler ones) indicates that inference rules can also be used for the same purpose as grammars, namely defining the syntax of languages. Consider again the simplified grammar for arithmetic expressions used in Sections 3.2, 3.3, and 4.1.

$$\begin{aligned} n \in num &::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \ldots \\ e_1, e_2 \in expr &::= n \mid e_1\texttt{+}e_2 \mid e_1\texttt{*}e_2 \end{aligned}$$

[3] This rule is also called an *axiom schema*, since it describes, through the use of the variable n, an infinite set of axioms, one for each number.

We can define the set of expression *expr* as a set membership judgment of the form $e \in expr$ by the following inference rules.[4]

$$\frac{n \in \mathbb{Z}}{n \in expr} \qquad \frac{e_1 \in expr \qquad e_2 \in expr}{e_1{+}e_2 \in expr} \qquad \frac{e_1 \in expr \qquad e_2 \in expr}{e_1{*}e_2 \in expr}$$

The first rule says that any integer (that is, any element of the set $\mathbb{Z}$) is an expression. The second and third rules says that if e_1 and e_2 are expressions, then so are, respectively, $e_1{+}e_2$ and $e_1{*}e_2$. The notation "$\in expr$" can be understood as a unary predicate of on (potential) sentences, written in postfix notation.

Since inference rules can be used to define arbitrary relations, it should not be too surprising that we can express even denotational semantics through inference rules. Again, consider the mathematical definition from Section 4.1.

$$\begin{aligned} [\![\cdot]\!] &: expr \to \mathbb{Z}_\perp \\ [\![n]\!] &= \underline{n} \\ [\![e_1{+}e_2]\!] &= [\![e_1]\!] + [\![e_2]\!] \\ [\![e_1{*}e_2]\!] &= [\![e_1]\!] \cdot [\![e_2]\!] \end{aligned}$$

To define the semantics using inference rules we first have to decide what form the judgments should have. We could adopt the notation from the semantic function definition and use judgments of the form $[\![e]\!] = n$, but that notation is a bit confusing, since $[\![\cdot]\!]$ denotes a function, but we want to define a relation between expressions and numbers. We can resort to using a name such as *sem* or *eval*, similar to the *even* and *fac* relations, but it is customary to express evaluation judgments using arrow symbols. Thus we use the judgment $e \Downarrow n$ to express that the expression e evaluates to the number n.

$$\frac{}{n \Downarrow \underline{n}} \qquad \frac{e_1 \Downarrow n \qquad e_2 \Downarrow m}{e_1{+}e_2 \Downarrow n + m} \qquad \frac{e_1 \Downarrow n \qquad e_2 \Downarrow m}{e_1{*}e_2 \Downarrow n \cdot m}$$

Note how we have to use the notation $\underline{n}$ again in the axiom to map a number symbol (which is an element of *expr*) to its corresponding integer (an element of $\mathbb{Z}$). You may wonder why we use $\underline{n}$ in the first rule but not in the other rules.

It can be a bit tricky to parse and understand the notation correctly. It is always a good idea to be clear about the types of the relationships being defined. In this example, the $\Downarrow$ binary relationship has the type $\Downarrow \subseteq expr \times \mathbb{Z}$. For the notation in the rules this means that when n occurs as part of the left argument of $\Downarrow$, it is a metavariable that ranges over

[4] We could also add a rule for a judgment $n \in num$, but it wouldn't add anything important.

syntactic *num* elements and thus has to be transformed into a value of $\mathbb{Z}$ when used also as part of the right argument. On the other hand, when used only as part of right arguments of $\Downarrow$, n is a variable that ranges over elements of $\mathbb{Z}$ and doesn't require any transformation.

Exercise 5.1

(a) Define the binary relation $fib(n, m)$, which says that m is the nth Fibonacci number, with inference rules.

(b) Consider again the grammar for statements (see page 55 and Exercise 3.4**(c)** on page 59).

stmt ::= `skip` | `while` *cond* `{` *stmt* `}` | *stmt*`;` *stmt*

Define the judgment $s \in stmt$ with inference rules. (You can make use of the judgment $c \in cond$ as one of the premises.)

5.2 Type Systems

A *type system* defines (A) the types of all basic programming constructs (such as values and functions) and (B) provides rules of what compositions of those constructs are valid and what the resulting types are.

To talk about the types of values we need a language of types, discussed in Section 5.2.1. Equipped with such a type language, we can then describe the rules of a type system with so-called *typing rules*, which are explained in Section 5.2.2. Typing rules describe a type system on a very high level. A *type checker* is an algorithm that implements these rules with the specific purpose of identifying situations in which values and operations are used inconsistently with regard to their types, a situation called a *type error*. The process of executing a type checker is called *type checking*. I will explain this process in Section 5.3.

5.2.1 The Language of Types

A *type* is a collection of programming language elements that share a common behavior. While the extent of types varies with different programming languages, types are typically associated with values, functions (or procedures/methods/...), and control structures as well as their compositions.

Types often have a name. This is, in particular, the case for atomic, non-constructed programming language elements. For example, the type of integers is a collection of numbers that can be added, multiplied, compared, etc.; it has the name `Int`. But types don't necessarily have a name. Consider, for example, the type of pairs of integers. This type can

be denoted by the *type expression* `(Int,Int)`, but it doesn't have to have its own name, even though we can define a name for it using a *type definition* such as the following.

```
type IntPair = (Int,Int)
```

Note that a type definition is something very different from a *data type definition* (introduced in Section 2.6), which introduces constructors and their types. A type definition does not introduce any new values; it only defines a new name, called *type synonym*, for an already existing type. The synonym and the type it stands for can be used interchangeably. Therefore, type definitions and type synonyms are a convenient programming language feature, but they are not an essential part of a type system.

Type expressions are built by applying *type constructors* to type names and other type expressions. Examples of type constructors are the list type constructor `[]` (Section 2.5), the `Maybe` type constructor (Section 4.2.1), and the function type constructor `->` (Section 2.3).

Type constructors are functions on types that can have different arities (that is, different numbers of parameters), just like operators and functions on values. For example, `[]` and `Maybe` are unary type constructors, that is, they take exactly a single type as an argument. Examples of type expressions built using these constructors are `[Int]`, `Maybe Bool`, `[Maybe Int]`, and `Maybe [String]`. In contrast `->` and `(,)` are a binary type constructors that take two arguments. Examples of type expressions built using these constructors are `Int -> Bool`, `(Int,Bool)`, and `Int -> (Int -> Int)`.

Exercise 5.2

(a) Give example values that have the following types.

- `[Maybe Bool]`
- `Maybe [Bool]`
- `(Int,[Bool])`
- `[(Int,Bool])`
- `Bool -> Int`
- `Int -> (Int,Int)`

(b) What are the types of the following expressions?

- `Just (Just True)`
- `3/0`
- `[not]`
- `(+3):[]`
- `tail [3]`
- `Just even`

A function definition has typically the following form: It consists of a name `f`, has one or more parameters (represented by variables `x`, `y`, etc.), and has an expression `e` that defines the value to be returned.

```
f x y = e
```

Viewing a type constructor as a function on types suggests a very similar pattern for a type constructor definition: It consists of a type name `T`, has one or more parameters (represented by type variables `a`, `b`, etc.), and has a type expression `U` that defines the type to be constructed.

```
type T a b = U
```

Here are two examples of type constructors and their use: `Pair` is a binary type constructor for building different pair types, and `Property` constructs function types that have `Bool` as a return type.

```
type Pair a b = (a,b)
type Property a = a -> Bool
```

A type expression that contains type variables is a *polymorphic type*, which means it actually represents many different (in fact, infinitely many) types. More specifically, this kind of polymorphism is called *parametric polymorphism*, since the polymorphism is expressed by type parameters, that is, type variables. We have seen several examples already in Sections 2.5 (cf. the types of `head` or `null`) and 2.7 (cf. the types of `map` or `foldr`).

As we can apply functions to values to return values, we can apply type constructors to denote new types, and as we can use functions in expressions to define new functions, we can use type constructors in type expressions to define new type constructors.

```
type Pos = Pair Int Int
type ListProperty a = Property [a]
```

We can then use these type constructors and types in the type signatures of the definition of values.

```
pos :: Pos
pos = (3,4)
```

```
isZero :: Property Int
isZero = (==0)
```

```
isOrigin :: Property Pos
isOrigin = (==(0,0))
```

Exercise 5.3

Try to predict the polymorphic types of the following expressions.

- `Just Nothing`
- `Just Just`
- `(Just,Nothing)`
- `([]:[])`
- `(:[])`
- `([]:)`
- `head . tail`
- `tail . head`
- `head . head`

Then use the `:t` command in `ghci` to see whether your prediction was correct. Here is an example:

```
:t not . head
not . head :: [Bool] -> Bool
```

5.2.2 Typing Rules

Typing rules express judgments about the types of expressions. It is also customary to say that typing rules *assign* types to expressions, but this has nothing to do with the assignment operation found in imperative languages.

In general, a typing rule assigns a type to an expression e based on the structure of e and on preconditions about the types of e's parts. If the expression e is an atomic value or operation, that is, if e cannot be decomposed into further parts, then the typing rule for e universally assigns it a specific type without any precondition. As mentioned, such rules are also called *axioms*. For example, the integer constant `3` has type `Int`,[5] and the function `even` has type `Int -> Bool`.

On the other hand, the type of a non-atomic expression generally depends on the type of its parts. For example, the rule for the type of a function application, that is, an expression of the form $f\ e$, is as follows.

if the function f has type T `->` U **and**
  the expression e has type T
then
  the application $f\ e$ is well defined and has type U

Here, the metavariables f and e range over expressions and T and U range over types.

We can write such rules in a more concise way with the help of inference rules. The judgments are of the form e `::` T where e is an expression of the language and T is a type,[6] for example, `3 :: Int` and `even :: Int -> Bool`. Thus the rule for function application (called App) can be rewritten with inference rules as follows.

$$\mathit{Bool}^{\text{App}}\ \frac{f :: T \texttt{->} U \qquad e :: T}{f\ e :: U}$$

With this rule we can check that the type of `even 3` is `Bool` by substituting for the metavariables f and e, T, and U as follows.

$$\frac{\texttt{even :: Int -> Bool} \qquad \texttt{3 :: Int}}{\texttt{even 3 :: Bool}}$$

[5] In this unit we assume that each symbol uniquely determines a value of a specific type. In particular, we disregard the overloading of symbols; cf. the discussion of overloading on page 15 in Section 2.2.

[6] Haskell uses the symbol `::` for the "has-type" relation, since the single colon `:` is reserved for the cons operation on lists. In most other languages the typing relation is expressed with a single colon.

The binary typing judgment $e :: T$ seems plain and simple, and it works well in many cases. However, it is somewhat limited, since it assumes that the type for each name and symbol is globally defined (by axioms). Specifically, it is impossible to define the type of `let` expressions with the judgment $e :: T$. Consider, for example, the following expressions, which are all well typed.

(a) `let x=3 in x+1`
(b) `let x=True in not x`
(c) `let x=True in if x then let x=3 in x+1 else 5`

The type of (a) is `Int`, while the type of (b) is `Bool`. To type the body of the `let` expressions, `x` has to be of type `Int` in (a) but needs to be of type `Bool` in (b). As expression (c) illustrates, it must be possible for one variable to have different types in different local contexts of an expression: Here `x` is of type `Bool` in in the outer `let` expression but of type `Int` in the inner `let` expression.

Therefore, a type system must be able to flexibly assign types for variables depending on the context, which can't be achieved by using a global assignment. The solution is to extend the typing judgment into a ternary relation by adding a set Γ of so-called *type assumptions* to keep track of locally introduced variable/type pairs. A type assumptions consists of a variable and its type and is written like a plain type judgment. For example, $\Gamma = \{$`x :: Int, f :: Int -> Bool`$\}$ is a set of two type assumptions. The typing judgment now becomes $\Gamma \vdash e :: T$, which says that expression e has type T in the context of the type assumptions Γ. When e contains variables, their types are determined by the assumptions in Γ. For example, we have the following typing judgments.

$\{$`x :: Int`$\} \vdash$ `x :: Int`
$\{$`x :: Int, y :: Int`$\} \vdash$ `x+y :: Int`
$\{$`x :: Bool, y :: Int`$\} \vdash$ `not x :: Bool`

As the last example shows, type assumptions don't have to be used. On the other hand, we can't determine a type for an expression containing a variable which doesn't have a type assumption in Γ.

$\{\} \vdash$ `x+1 :: TYPE ERROR`

The typing rule for variables simply looks up their type in the type environment.

$$\textsc{Var}\ \frac{x :: T \in \Gamma}{\Gamma \vdash x :: T}$$

This rule says that a variable has whatever type it has been assigned in the type environment.

To define the typing rule for `let` expressions, we need to be able to add or overwrite type assumptions in the type environment, which is done by simply adding an assumption separated by a comma. For example, Γ, `x :: Int` has the same assumptions as Γ, except for `x` whose type is now `Int`.

With the ability to (temporarily) change the assumptions in the type environment, we can formulate the typing rule for `let` expressions as follows.

$$\textsc{Let}\ \frac{\Gamma \vdash e_1 :: U \qquad \Gamma, x :: U \vdash e_2 :: T}{\Gamma \vdash \texttt{let}\ x\texttt{=}e_1\ \texttt{in}\ e_2 :: T}$$

This rule says that the type of a `let` expression is given by the type of its body e_2, assuming that the bound variable x has the type of the expression e_1. By substituting `x` for x, `3` for e_1, `x+1` for e_2, and `Int` for U and T we can now see how the type for the expressions (a) is obtained.

$$\frac{\{\} \vdash \texttt{3} :: \texttt{Int} \qquad \{\texttt{x} :: \texttt{Int}\} \vdash \texttt{x+1} :: \texttt{Int}}{\{\} \vdash \texttt{let x=3 in x+1} :: \texttt{Int}}$$

In a similar way, we obtain the result for (b).

$$\frac{\{\} \vdash \texttt{True} :: \texttt{Bool} \qquad \{\texttt{x} :: \texttt{Bool}\} \vdash \texttt{not x} :: \texttt{Bool}}{\{\} \vdash \texttt{let x=True in not x} :: \texttt{Bool}}$$

In both example we are using $\{\}$ for Γ, which means that the typing does not make any other assumptions than the one locally added for `x` in the second premise. However, we do rely on the following axioms for the types of `3`, `True`, `+`, and `not`.

$$\frac{}{\Gamma \vdash \texttt{3} :: \texttt{Int}} \qquad \frac{}{\Gamma \vdash \texttt{True} :: \texttt{Bool}}$$

$$\frac{}{\Gamma \vdash \texttt{+} :: \texttt{Int -> Int -> Int}} \qquad \frac{}{\Gamma \vdash \texttt{not} :: \texttt{Bool -> Bool}}$$

Instead of using such axioms, we could also add type assumptions about predefined values and functions to a "standard" type environment Γ_0 to be used in all typing rules.

$$\Gamma_0 = \{\ldots, \texttt{3} :: \texttt{Int}, \texttt{+} :: \texttt{Int -> Int -> Int}, \ldots, \texttt{True} :: \texttt{Bool}, \texttt{not} :: \texttt{Bool -> Bool}, \ldots\}$$

Both examples use a non-atomic judgment as the second premise, namely $\{\texttt{x :: Int}\} \vdash \texttt{x+1 :: Int}$ in the rule instance for (a), and $\{\texttt{x :: Bool}\} \vdash \texttt{not x :: Bool}$ in the rule instance for (b). These judgments have to be justified by applying corresponding typing rules. In the case of (b) this is a simple application of the APP rule, for which in turn we need an application of the VAR rule to determine the type of `x`. For (a) we could use an additional application rule for binary operations (see Exercise 8.5). But if we consider `x+1` as syntactic sugar for `(+) x 1`,[7] then we can also derive the typing judgment for `x+1` in two steps. First, we show with rule APP that `+` partially applied to `x` has the type `Int -> Int`.

$$\frac{\{\texttt{x :: Int}\} \vdash \texttt{(+) :: Int -> Int -> Int} \qquad \{\texttt{x :: Int}\} \vdash \texttt{x :: Int}}{\{\texttt{x :: Int}\} \vdash \texttt{(+) x :: Int -> Int}}$$

In a second step, we can use this result in another application of APP.

$$\frac{\{\texttt{x :: Int}\} \vdash \texttt{(+) x :: Int -> Int} \qquad \{\texttt{x :: Int}\} \vdash \texttt{1 :: Int}}{\{\texttt{x :: Int}\} \vdash \texttt{(+) x 1 :: Int}}$$

You may have noticed that we could substitute the occurrence of a judgment $\mathcal{J}$ in a premise of a rule by a rule that has $\mathcal{J}$ as its conclusion and thus build whole trees of judgments (which are also called *derivations* or *proof trees*). However, we will not look into this aspect here.

Exercise 5.4

(a) Write a typing rule for applying arbitrary binary (infix) operations such as `+`. It might be a good idea to first create the rule for `+` and then generalize it to more general types.

(b) Write a typing rule for the cons operations (`:`) on lists and the empty list `[]`.

(c) Use the rules from part (b) to show that the expression `[3,4]` has type `[Int]`.

(d) Pick an appropriate type environment to formulate a typing judgment for the expression `xs ++ ys`. Then use the rule from part (a) to show that the judgment is true.

Finally, how are type errors and type-incorrect expressions dealt with by typing rules? The simple answer is that they aren't. Any expression for which it is impossible to derive a type with the typing rules is considered to be type incorrect. Typical examples of type errors are application of functions to values of the wrong type as in `not 3` or undefined

[7] Remember that the parentheses turn an infix operation into a prefix operation, cf. Section 2.3.2.

variables. But how can we be sure that an expression cannot be typed? For this we need to turn the descriptive typing rules into an algorithm for checking expressions and programs.

5.3 Type Checking

A *type checker* is an algorithm that uses *type declarations* of values and functions in a program to ensure that all functions and values are used according to the typing rules of the underlying programming language. A *type inference algorithm* does basically the same thing, but it generally doesn't need type declarations and instead automatically infers the most general types for functions and values.

We can operationalize typing rules, which are declarative statements, into a type checking algorithm in a systematic way by first identifying dedicated inputs and outputs of the relation and then transforming rules into function equations.

I'll explain this idea using the factorial rules as an example. First, $fac(n, m)$ is a binary relation, which means we want to turn it into a function $fac : \mathbb{N} \to \mathbb{N}$ that takes one argument and returns one result. Second, we probably want the first component to be the argument and the second component the result: Since $fac(n, m)$ says that m is the factorial of n, we want to derive a function that computes m given n.[8] Third, we turn each rule into a function equation by expressing the result component of the relations (here m) in terms of the relations and conditions in the premises of the rule. For an axiom the function definition is trivial, since the result does not depend on any premises. In our example, we can simply turn $fac(1, 1)$ into $fac(1) = 1$. The inductive rule is more interesting, since it has two premises and a recursive occurrence of the judgment.

$$\frac{fac(n-1,k) \qquad k \cdot n = m}{fac(n,m)}$$

We first turn the conclusion $fac(n, m)$ into the function definition $fac(n) = m$. Next we substitute the result variable m by its definition $k \cdot n$ given in the second premise. But now we need to substitute the variable k by its definition. Since k is the result of the judgment

[8] Note that we could, in principle, also do the opposite, that is, given a number m compute the number n of which m is a factorial. This would be some kind of "inverse factorial" function, and it would be a partial function that is undefined for many arguments m.

$fac(n-1,k)$, we can replace it by the corresponding function call $fac(n-1)$. Altogether we therefore obtain the following two equations.

$$fac(1) = 1$$
$$fac(n) = fac(n-1) \cdot n$$

Now let's try the same transformation with the typing rules. The first step is to identify the input and output of the typing judgment $\Gamma \vdash e :: T$. Since we want the type checker to determine the types of expressions, the third component of the relations should be the result, and consequently the first two components should be the arguments. As with denotational semantics, we want to define the type checker as a Haskell function so that we can experiment with examples. This means we need type definitions for the argument and result types. Specifically, we need a type for the abstract syntax of the language and a type for the type expressions to be returned by the type checker. We start with the expression language used in Section 4.2.3, which had expressions of two different types.

```
type Name = String
data Expr = Lit Int | Plus Expr Expr | Equal Expr Expr | Not Expr
          | Let Name Expr Expr | Var Name
```

To illustrate the use of the type environment in type checking, I have added two constructors for representing `let` expressions and variable references. With these we can represent an expression such as `let x=3 in x+1` as follows.

```
Let "x" (Lit 3) (Plus (Var "x") (Lit 1))
```

The type language in this example is quite simple: We only have expressions of type `Int` and `Bool` and no function types or type constructors yet. But we do need to represent the case when the type checker encounters a type error. As in Section 4.2.3, we have two possibilities for representing errors: We can define a type `Type` with only constructors for the two types and then use `Maybe Type` as the result type for the type checker, or we integrate the error into the type definition for `Type`. While the latter approach may lead to simpler definitions in some parts, it is also less clear. In particular, we have to store types in type environments, and being able to store the type "type error" for a variable seems wrong.

```
data Type = TyInt | TyBool
```

In addition to `Expr` and `Type`, we also need a Haskell type definition for representing the type environment Γ. Since a type environment is de facto a map of variable names to types, we can represent them using the type `Map` (much like what we did in Section 4.2.4 where we

Type System Concept	Haskell Representation
Syntax of object language	Data type (e.g., `Expr`)
Syntax of types/type language	Data type (e.g., `Type`)
Type	Constructor (e.g., `TyInt` of type `Type`)
Type Error	`Nothing`
Typing rule	Function equation
Type checker	Function (e.g. `tc`)

Table 5.1: Representing type system concepts using Haskell as a metalanguage. Note that a *type* (name) in the object language is represented by a *value* (that is, constructor) in the metalanguage, and the set of all object language types (that is, the *type language* as described in Section 5.2.1) is represented as a *type* in the metalanguage (see also Table 4.1 on page 75).

represented state as a mapping of variables to values).[9]

```
type TyEnv = M.Map Name Type
```

With these representations we can define the type signature for the type checking function. Note that we use the `Maybe` type constructor to distinguish between proper types of expressions and type errors (represented by the constructor `Nothing`). This needs to happen whenever turning a relation into a function results in a partial function.

```
tc :: TyEnv -> Expr -> Maybe Type
```

Since any type T is always represented in this approach by a value `Just T` (where `T` is the constructor of type `Type` to represent T), we introduce the following two abbreviations, which help simplify the following definitions.[10]

```
tInt  = Just TyInt
tBool = Just TyBool
```

Now we can translate typing rules into equations for the type checking function `tc`. The first two cases are simple: An integer always has type `Int`, and a variable has whatever type is stored for it in the type environment, where we can look it up using the `lookup` function.

[9] The relevant definitions can be imported by adding `import Data.Map (Map,empty,insert,lookup)` at the top of the Haskell file. However, since we need the function `lookup` from `Map`, we have to hide the definition of `lookup` on lists by also including `import Prelude hiding (lookup)`. An arguably cleaner alternative is to import the names from `Map` *qualified*, that is, we simply use `import qualified Data.Map as M` and then use all functions with an `M.` prefix in their name. Then there is no need for hiding `lookup`. The following code examples are based on this latter approach.

[10] A summary of how types and values in object- and metalanguage are related is provided in Table 5.1.

```
tc _ (Lit i) = tInt
tc g (Var x) = M.lookup x g
```

If a variable's type is undefined, that is, if no type is stored in the type environment, the type checker has encountered a type error and must return the constructor `Nothing`. This is achieved directly by the `lookup` function, which has itself the following type (in this usage context).

```
lookup :: Name -> Map Name Type -> Maybe Type
```

Next we define the equation for the `Not` operation. The typing rule expresses the condition that the argument of `Not` must have type `Bool` by a single premise.

$$\textsc{Not}\ \frac{\Gamma \vdash e :: \texttt{Bool}}{\Gamma \vdash \texttt{not}\ e :: \texttt{Bool}}$$

The sole purpose of the premise is to pose a constraint on the conclusion; the result type of a `Not` expression is always `Bool`. Therefore, we can express the resulting value for `tc` using a conditional.

```
tc g (Not e) = if tc g e == tBool then tBool else Nothing
```

We can express this equation more succinctly using a pattern guard for the condition and delegate the type error case as a catch-all equation at the very end of the definition of `tc` (see Figure 5.1).

```
tc g (Not e) | tc g e == tBool = tBool
```

We have a very similar situation for binary operations such as `Plus`: the typing rule does not "compute" a result type in the premises; it only expresses constraints on the conclusion.

$$\textsc{Plus}\ \frac{\Gamma \vdash e_1 :: \texttt{Int} \qquad \Gamma \vdash e_2 :: \texttt{Int}}{\Gamma \vdash e_1 + e_2 :: \texttt{Int}}$$

Correspondingly, the equation for `tc` needs two conditions. Since we have the full expressive power of Haskell available, we can introduce some auxiliary definitions to avoid redundancy in the definition. For example, we can use the function `both` to apply the type checker to both argument expressions "in parallel," returning a pair of results, and we can create a value for a pair of `Int` types to which we can compare the results to.

```
both :: (a -> b) -> (a,a) -> (b,b)
both f (x,y) = (f x,f y)

tc2 :: TyEnv -> (Expr,Expr) -> (Maybe Type,Maybe Type)
tc2 g = both (tc g)

tInt2  = (tInt,tInt)
tBool2 = (tBool,tBool)
```

With `tc2` and `tInt2` we can express the PLUS rule using an equation that combines both premises in one condition.

```
tc g (Plus e1 e2) | tc2 g (e1,e2) == tInt2 = tInt
```

The typing rule for `Equal` is quite similar to that of `Plus`, except that we may want to allow the comparison of integers as well as booleans, which means that we can define `Equal` to have *two* types. We can express the types by giving two separate rules.

$$\textsc{Equal}_1\quad \frac{\Gamma \vdash e_1 :: \texttt{Int} \qquad \Gamma \vdash e_2 :: \texttt{Int}}{\Gamma \vdash e_1 == e_2 :: \texttt{Bool}} \qquad \textsc{Equal}_2\quad \frac{\Gamma \vdash e_1 :: \texttt{Bool} \qquad \Gamma \vdash e_2 :: \texttt{Bool}}{\Gamma \vdash e_1 == e_2 :: \texttt{Bool}}$$

To derive the type of a concrete expression we can pick whichever rule is applicable depending on the context, that is, depending on whether the arguments have type `Int` or `Bool`. We can translate the rules into one equation that has two pattern guards, which means that in an application of the `tc` we would first check for `Int` arguments, and if that fails, `Bool` arguments.

```
tc g (Equal e1 e2) | tc2 g (e1,e2) == tInt2  = tBool
                   | tc2 g (e1,e2) == tBool2 = tBool
```

Finally, the LET rule (see page 178) requires in the first premise to check the type of the definition e_1 and use that type U as an assumption for the bound variable x when determining the type of the body of the `let` expression e_2. We turn this rule into a function equation where we examine the result of type checking `e1` and proceed with checking `e2` only if `e1` is well typed. Otherwise, we return a type error (that is, `Nothing`).

```
tc g (Let x e1 e2) = case tc g e1 of
                       Just t -> tc (M.insert x t g) e2
                       _      -> Nothing
```

The complete definition of the function `tc` is shown in Figure 5.1.

```
tc :: TyEnv -> Expr -> Maybe Type
tc _ (Lit i)       = tInt
tc g (Var x)       = M.lookup x g
tc g (Plus e1 e2)  | tc2 g (e1,e2) == tInt2  = tInt
tc g (Equal e1 e2) | tc2 g (e1,e2) == tInt2  = tBool
                   | tc2 g (e1,e2) == tBool2 = tBool
tc g (Not e)       | tc g e == tBool         = tBool
tc g (Let x e1 e2) = case tc g e1 of
                       Just t -> tc (M.insert x t g) e2
                       _      -> Nothing
tc _ _             = Nothing
```

Figure 5.1 Complete definition of the type checker.

Exercise 5.5

The goal of this exercise it to extend the `Expr` language with non-nested pairs of integers and extend the definition of the type checker accordingly.

(a) Extend the abstract syntax of the `Expr` language with an operation `Pair` for constructing non-nested pairs of integers, an operation `Fst`, which selects the first component of an integer pair, and `Swap`, which exchanges both components of an integer pair.

(b) Extend the data type `Type` with a constructor to represent an integer pair type.

(c) Define the typing rules for the operations `Pair`, `Fst`, and `Swap`.

(d) Expand the definition of the type checker `tc` by equations for the operations `Pair`, `Fst`, and `Swap`.

Exercise 5.6

This exercise builds on Exercise 5.5. Now we want to extend the `Expr` language with *nested* pairs of any types and extend the definition of the type checker accordingly. The abstract syntax `Expr` can remain unchanged.

(a) Change the data type `Type` so that the constructor for pair types can now represents arbitrary pair types (including nested pairs).

(b) Redefine the typing rules for the operations `Pair`, `Fst`, and `Swap`.

(c) Change the definition of the type checker `tc` by equations for the operations `Pair`, `Fst`, and `Swap` to account for nested pairs.

5.4 Type Safety

After having spent so much effort in designing typing rules and transforming them into a type checking algorithm, the questions arise, *Why do we need all of this*? and, *What are the benefits of a type system and its "enforcer" the type checker*? The most important feature of a type system (and its implementation in a type checker) is *type safety*, which is a property of a programming language that guarantees the absence of a large class of programming errors.

Consider, for example, the expression `not 3`. What should the result of this expression be? In Section 4.2.3 we have defined the semantics to be undefined.

```
sem :: Expr -> Val
...
sem (Not e) = case sem e of
                B b -> B (not b)
                _   -> Undefined
```

This definition is complicated by the fact that that the type of the semantics of `e` must be checked before the operation `not` can be applied. The same is actually true for all other operations: For every application we must ensure that its arguments have the correct type. Obviously, such repeated tests are quite inefficient.

If we knew in advance that every operation is applied only to arguments of the correct type, we wouldn't need any of these tests and could generate more efficient code. A type checker provides exactly such an assurance.

A program that passes the type checker won't apply operations to arguments of the wrong type. We can therefore simplify the implementation of an interpreter or compiler accordingly and get rid of all such tests. Here is a excerpt of a simplified version of `sem` from Section 4.2.3. This definition assumes that all arguments have the correct type. Note that we can therefore remove the `Undefined` constructor from the data type `Val`.

```
data Val = I Int | B Bool

eval :: Expr -> Val
...
eval (Plus e1 e2) = case (eval e1,eval e2) of (I i,I j) -> I (i+j)
eval (Not e)      = case eval e of B b -> B (not b)
```

Not that this implementation is by itself *not* safe, that is, if you try to evaluate an expression with a type error, the evaluation in `ghci` will abort with a runtime error.

```
> eval (Not (Lit 3))
*** Exception: ... Non-exhaustive patterns in case
```

This is to be expected, since `Lit 3` evaluates to `I 3`, but `eval (Not e)` does not deal with the case when `e` evaluates to an integer. However, we can define a safe version of `eval` that uses `tc` to filter out expressions containing type errors.

```
evalSafe :: Expr -> Maybe Val
evalSafe e = case tc M.empty e of
               Just _ -> Just (eval e)
               _      -> Nothing
```

With `evalSafe` we apply `eval` to an expression `e` only if the type checker `tc` has approved it, that is, if `tc` returns a proper type. Otherwise, we return `Nothing`.

```
> evalSafe (Not (Lit 1))
Nothing
```

```
> evalSafe (Not (Equal (Lit 1) (Lit 2)))
Just (B True)
```

Note how `Nothing` is used for two different kind of errors: The `Nothing` that may result from `tc` (which would be matched by the underscore in the last line) represents a type error whereas the `Nothing` that is returned in that line represents an undefined value to indicate that the expression can't be evaluated.

We can simplify the implementation even further by using a single type as a representation for all values. This could save us the inspection of tags extraction of values from union types in patterns. We can use integers in this case and represent `False` by `0` and `True` by `1`.

```
eval :: Expr -> Int
...
eval (Plus e1 e2) = eval e1 + eval e2
eval (Not e)      = if eval e==0 then 1 else 0
```

In the definition of `evalSafe` we only have to change the return type to `Maybe Int`. Type checking works as before, but the returned values are, of course, only integers.

```
> evalSafe (Not (Lit 1))
Nothing
```

```
> evalSafe (Not (Equal (Lit 1) (Lit 2)))
Just 1
```

The last implementation looks so simple. Why can't we just use it directly, without the type checker? Because the representation can lead to unexpected results: The definition of `eval` for `Not` considers de facto all non-zero integers as representations of `True`, since applying `Not` to any non-zero number yields `0`, which represents `False`. But now we are in the strange situation while in most cases `True + True` equals `True`, this is not always the case.

TYPE SAFETY

Type correctness is a property of individual programs. Each program can be deemed type correct or type incorrect by a type checker. In contrast, *type safety* is a property of a programming language and its type system: If the type system detects all type errors, then it is type safe. In a type-safe programming language no type errors go undetected, whereas unsafe languages generally allow operations to be performed on values for which they are not defined, leading to unpredictable results.

Note that type safety is independent of the time when types are checked, that is, whether a programming language uses static or dynamic typing is not relevant for its type safety. Specifically, dynamically typed languages can be type safe, and statically typed languages are not necessarily type safe.

Also note that most non-toy programming language that are considered type safe should probably better be called "mostly" type safe unless this property has been formally established. In particular, specific language implementations may contain "extra features" that allow type errors to escape the type checker.

	(Mostly) Type Safe	**Unsafe**
Statically Typed	Haskell, Java	C, C++
Dynamically Typed	Lisp, Python	Javascript

Consider, for example, `3` and `-3`. Thus this representation is inconsistent and makes reasoning about programs difficult.

Another example is this: We would expect `Not (Not e)` to be equal to `e`, but `Not (Not (Lit 2))` evaluates to `1` and not `2`. The concept of type safety is summarized in the box TYPE SAFETY. The most important take-away is that type-safe languages detect all type errors in programs and so can avoid a large number of program execution failures as well as semantic errors.

5.5 Static and Dynamic Typing

The most obvious difference between static and dynamic typing is the time when types are checked:

- *Static typing*: Types are checked *before* a program is run.
- *Dynamic typing*: Types are checked *while* a program is run.

If this were all, this section would be pretty short. However, the time of when typing rules are checked has some important implications and means different advantages for the two approaches. Consider the following expression. What is the resulting value, and what is its type?

```
if 1<2 then 3 else "Hello"
```

It depends on the language. In Haskell we get a type error, since the two branches of the conditional have different types. Therefore, the expression won't be evaluated and doesn't produce a result. In Python, on the other hand, the expression can be evaluated without a problem.[11]

```
>>> 3 if 1<2 else "Hello"
3
```

Correspondingly, the type of the Haskell expression is undefined whereas the type of the Python expression is `<type 'int'>` (which corresponds to `Int`).

```
>>> type(3 if 1<2 else "Hello")
<type 'int'>
```

The Haskell type checker requires both branches of a conditional expression to have the same type, while Python doesn't have that restriction. Haskell refuses to evaluate a perfectly fine expression that doesn't lead to any runtime error. Why is that? To see the problem that the Haskell type checker faces, consider the following function definition.

```
f x = if test x then x+1 else "Hello"
```

The argument type of `f` must be `Int` (because the expression `x+1` requires `x` to be of type `Int`), and the result of the function depends on the result of the condition `test x`: If it evaluates to `True`, the result type of `f` is `Int`, otherwise it is `String`. Thus to decide the return type of `f` we need to evaluate the expression `test x`, but since `test` can be an arbitrarily complicated function, this effectively amounts to executing a program, which would turn the Haskell static type checker into a dynamic type checker.

A specific, thorny problem with trying to determine the outcome of conditions to decide between two competing types for a conditional is that the evaluation of the condition might not terminate. And since we require a static type checker to terminate, the prospect of potential nontermination is unacceptable.

[11] I'm using the standard Python prompt `>>>` to distinguish interactions with the Python interpreter from those with `ghci`. Moreover, I'm using the Python single-line syntax for conditionals that inverts the usual order of the then/true-branch expression and the condition.

	Advantages	Disadvantages
Static Typing	smaller & faster code early error detection (saves debugging)	rejects some o.k. programs
Dynamic Typing	fewer programming restrictions faster compilation	slower execution more bugs in released programs

Table 5.2: Advantages and disadvantages of static and dynamic typing.

Static typing avoids this problem by requiring that both branches have the same type because in that case it doesn't matter which branch is taken, and we don't need to evaluate the condition. The typing rule for the conditional expresses this constraint by using the same metavariable T in the typing judgment for both branches.

$$\textsc{Cond}\ \frac{\Gamma \vdash e :: \texttt{Bool} \qquad \Gamma \vdash e_1 :: T \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash \texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 :: T}$$

Due to this rule, static type systems generally only *approximate* the types of conditional expressions. In particular, the COND rule cannot be used to assign a type to the examples shown above, which causes the type checker to report an error. As we have seen, this behavior is sometimes overly restrictive, since the rejected expression may very well evaluate without any problems. Thus while static type checking seems like a great idea, it also has its drawbacks. A summary comparison of the advantages and disadvantages of both approaches is given in Table 5.2.

Interestingly, while most typing rules have an equivalent representation in both the static type checker `tc` as well as in the semantics that employs dynamic type checks, this is not the case for the COND rule. This should not be too surprising, since the COND rule specifically expresses a constraint for static typing.

A technical comparison of static and dynamic typing is given in Figure 5.2 where we contrast excerpts from a dynamically typed interpreter `dyn`[12] with corresponding parts of a statically typed interpreter `stat`, which employs the already shown static type checker `tc` and an untyped evaluator `eval`, described earlier in Section 5.4.

We can now examine the different behavior of the dynamically and statically typed interpreters. First, we can observe that `dyn` works fine with conditionals using differently type branches that are rejected by `stat`.

[12] This is basically the semantic function `sem` described in Section 4.2.3.

```
data Val = I Int
         | B Bool
         | Undefined
```

```
tc :: Expr -> Maybe Type
...
tc (If e e1 e2) | tc e == tBool && tc e1==tc e2 = tc e1
```

```
eval :: Expr -> Val
...
eval (If e e1 e2) =
    case eval e of
        B True -> eval e1
        _      -> eval e2

stat :: Expr -> Maybe Val
stat e = case tc e of
           Just _ -> Just (eval e)
           _      -> Nothing
```

```
dyn :: Expr -> Val
...
dyn (If e e1 e2) =
    case dyn e of
        B True  -> dyn e1
        B False -> dyn e2
        _       -> Undefined
```

Figure 5.2 Comparing static and dynamic typing: `eval` does not check for type errors and relies on `tc` to ensure the first argument of `If` is of type `Bool`. In contrast, `dyn` has a third case for when a dynamic type error occurs. Moreover, `dyn` evaluates `e1` or `e2` irrespective of their types, whereas `stat` makes sure through `tc` that `e1` and `e2` have the same type before `eval` can evaluate.

```
> one  = Lit 1
> true = Equal one one
```

```
> stat (If true one true)
Nothing
```

```
> dyn (If true one true)
I 1
```

On the other hand, `stat` immediately rejects programs with obvious type errors, whereas `dyn` happily accepts them to only abort with a runtime error later.

```
> stat (Plus one true)
Nothing
```

```
> dyn (Plus one true)
Undefined
```

The output reveals the difference in behavior: `Undefined` is a value that represents a runtime error, whereas `Nothing` represents a type error, which shows that `stat` never calls `eval` with the type-incorrect expression. Specifically, `stat` will never produce a result `Just Undefined`; such a value would point to an incorrectness in the type checker, which would have ok'ed an expression that nevertheless evaluates to a runtime error.

In summary, types are a powerful abstraction mechanism that is successfully employed as precise program documentation as well as effective guidance for avoiding errors in pro-

Exercise 5.7

This exercise test your understanding of static typing and the difference between static and dynamic typing. The expressions are given in some fictitious notation that is close, but not identical to Haskell syntax. For each of the following expressions, answer the following three questions.

(1) Is the expression well typed under static typing, and if so what is the type of the expression?
(2) What is the type or behavior of the expression under dynamic typing? If the answer depends on the type and/or value of the used variables, explain the dependency.
(3) What type(s) must the variables in the expression have to make the expression well typed/not lead to a runtime error?

For example, for the expression `if x=3 then x+1 else not x` we get:

(1) Type Error
(2) `Int` if `x=3`, otherwise Type Error
(3) `x :: Int`

(a) `if x<2 then even x else x`

(b) `if head x then x else tail x`

(c) `if x then x+1 else x-1`

(d) `if False then "Hello" else x`

(e) `if x then x+1 else x`

(f) `if even x then x else even (x+1)`

(g) `if f(3) then 3 else f(True)`

grams. While static type checking does not protect against *all* runtime errors (for example, type-safe languages still can abort with division-by-zero errors or stack overflows), it makes programs overall significantly more reliable. Whether the benefits of static typing outweigh the restrictions imposed by the typing rules on programs will probably remain a matter of discussion and disagreement among programmers for the foreseeable future.

6 Scope

Names are one of the most fundamental language concepts: It seems anything that exists—and even things that don't exist—can be given a name to refer to it. A name also seems to be a fairly simple concept: A name is a word[1] that stands for something (an object, a person, and idea, a set, etc.), and we use names successfully all the time. It is therefore not surprising that names play an important part in programming languages as well. Names are used for variables, parameters, functions, types, and, depending on the language, many other programming abstractions (objects, modules, constructors, rules, etc.).

However, names are not as simple a concept as it may seem. Shakespeare's Juliet was on to something when she said:

> *What's in a name? That which we call a rose*
> *By any other name would smell as sweet.*

Juliet challenges the assumptions that names have intrinsic meaning, suggesting that names are rather arbitrary, a fact well known to every programmer.

The use of names is often complicated by the existence of so-called *synonyms* (when multiple names refer to one and the same thing) and *homonyms* (when one name refers to two or more different things). Synonyms can pose problems specifically in imperative languages when one data item can be accessed—and modified—through different names.

[1] One could widen the concept to also include symbols and other visual, non-textual representations, but that wouldn't change much in what we discuss here.

This situation is a root cause for side effects, which make the reasoning about imperative programs difficult and also complicates optimization. Homonyms pose a different kind of challenge, one that universally applies to most programming languages, and that is the question, *What does a name at a specific location in a program refer to*? In the following we will consider the case of names for values, but most of the results for named values apply to other names as well. We also use the common term *variable* for value names.

To motivate the contents of this unit, let us start with some general observations. First, names have to have to be introduced through a mechanism that associates a name `x` with some value v. Such a name/value is called a *binding* and is written as `x=`v. Bindings are created, for example, through an assignment operation or a `let` expression, but they also result whenever a function is applied to an argument.

Second, since one and the same name can refer to different values at different places in a program, we need a way to identify and talk about places in a program. The notion of *blocks* explained in Section 6.1 serves this purpose.

And third, since even at one specific location a name can refer to different values at different times (think about a function parameter), we also need a way to track the dynamic behavior of programs, at least as far as the dynamic behavior of bindings is concerned. The notions of a *runtime stack* and *activation record* are introduced in in Section 6.2 to explain the dynamic behavior of bindings.

6.1 The Landscape of Programs: Blocks and Scope

A program is a static description of computation. To define the visibility of names, we can partition a program into separate areas. When viewing a program as a tree (that is, an AST), a location or point in a program corresponds to a node in a tree. The region associated with any node n in a tree T is the set of n's descendants in T, for which we write $T[n]$. Thus the region for any syntactically correct part `p` of a program is given by $T[n]$ where T is the syntax tree of the program and n is the node representing `p` in T.

A *block* is a subtree of an AST which has itself two subtrees: one (called the *declaration*) that introduces one or more names, and one (called the *body*) that represents the part of the program where the names are available for use. If n is the root of a block, we write $D[n]$ for the region that contains the declarations and $B[n]$ for the region of the body. With this definition we obviously have $T[n] = D[n] \cup B[n]$. (In the following we sometimes equate a block with its root node, that is, when we talk about "block n" we really mean the whole region $T[n]$.) For example, for the expression in Figure 6.1 we have $D[1] = \{2..4\}$ and

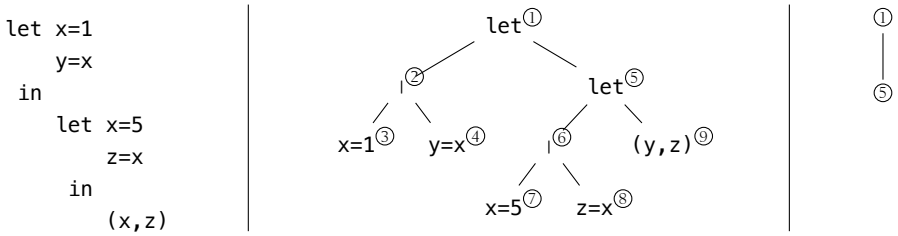


Figure 6.1 A nested `let` expression (left) and its corresponding (compressed)[3] AST (middle). Individual variable definitions and the innermost `let` body are shown as expressions instead of trees. The block structure that represents the nesting of blocks is shown as a tree on the right.

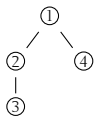
$B[1] = \{5..9\}$.[2] The example expression actually contains another block at node 5, for which we have $D[5] = \{6..8\}$ and $B[5] = \{9\}$.

Two different blocks in a program, say $T[n]$ and $T[m]$, can stand in one of exactly two relationships to one another: They are either be disjoint, that is, $T[n] \cap T[m] = \varnothing$, or one is included in the body of the other, that is, $T[n] \subseteq B[m] \vee T[m] \subseteq B[n]$. We can use block inclusion to define a partial order on AST nodes as follows.

$$n \sqsubset m :\Longleftrightarrow T[m] \subseteq B[n]$$

The relationship $n \sqsubset m$ can be read as "block n includes block m" or "block m is nested inside of block n." In the example, block 1 includes block 5, that is, $1 \sqsubset 5$, since $T[5] \subseteq B[1]$. In general, on block can include several distinct blocks, and the inclusion of blocks is transitive, that is, one block can be included in a block that is itself included in another block, etc. Consider the following example.

```
> let x=1 in (x,let y=2 in let z=3 in (y,z),let y=4 in y)
(1,(2,3),4)
```



If we identify, for convenience, each block with the number defined in its declaration, we see that $1 \sqsubset 2$, $1 \sqsubset 3$, $2 \sqsubset 3$, and $1 \sqsubset 4$. In particular, block 1 includes two non-overlapping blocks 2 and 4, and block 1 includes block 3 transitively.

This example suggests that the block structure of a program can be represented as a

[2] We abbreviate sequences of consecutive nodes as intervals and write, for example, $\{2..4\}$ for $\{2, 3, 4\}$ and $\{2..4, 6, 8..9\}$ for $\{2, 3, 4, 6, 8, 9\}$.

[3] For the purpose of discussing bindings and scope we employ an abbreviating notation of ASTs in which we allow to represent whole subtrees as syntactic expressions.

tree where the complete program is the block at the root. This tree determines for each node n a sequence of blocks from the root to the block that contains n, called the node's *block path* and denoted by $P[n]$. In Figure 6.1 we have $P[9] = 1, 5$ and $P[4] = 1$.

Each variable declaration with root n has a *scope*, written as $S[n]$, which is the set of nodes in which this variable is accessible (for reading and writing). The scope includes the body of the block, except those nodes in which the declaration is hidden by a nested declaration for the same variable. Depending on the programming language, the scope of a variable may also includes the definition part of the following declarations and even the declaration region itself. The expression in Figure 6.1 has variable declarations at nodes 3, 4, 7, and 8. As the use of `x` in node 4 indicates, the scope in a functional language like Haskell *does* include later declarations of the same block. Therefore, the scopes of the four variable declarations seem to be as follows.

$$S[3] = \{4..6\} \qquad S[7] = \{8..9\}$$
$$S[4] = \{5..9\} \qquad S[8] = \{9\}$$

We can make two important observations.

First, $S[3]$ does *not* include nodes $\{7..9\}$, because the declaration in node 7 is for the same name and thus *shadows* the declaration from node 4. In this example the shadowing simply "shortens" the scope of the declaration at node 3. In general, shadowing can "poke holes" into what otherwise is a contiguous scope region, and this can happen multiple times. Consider, for example, the following expression.

```
> let x=1 in (x,let x=2 in x,x,let x=3 in x,x)
(1,2,1,3,1)
```

It is instructive to visualize the scope with the help of the abstract syntax tree for this expression.

Exercise 6.1

Draw the abstract syntax tree for the following expressions, and identify the sets D, B, and S. Also, draw the tree representing the block structure.

(a) `let x=1 in (x,let x=2 in x,x,let x=3 in x,x)`

(b) `let x=1 in (x,let x=2 in (x,let x=3 in x,x),x)`

Second, the shown scopes do not include the region of their own declaration, that is, $n \notin S[n]$. However, in some languages, including Haskell, `let` expressions are recursive in

the sense that in a definition `let x=`e `in` e' the expression e may contain a reference to `x`. Consider, for example, the following definition of the factorial function.

```
> let fac = \x -> if x==1 then 1 else x*fac(x-1)
> fac 6
720
```

Thus the scopes of the four variable declarations that use recursive `let` expressions is actually as follows.

$$S[3] = \{3..6\} \qquad S[7] = \{7..9\}$$
$$S[4] = \{4..9\} \qquad S[8] = \{8..9\}$$

Next we need to distinguish between two principally different occurrences of a variable, that is, its *definition* and *use*. In general, each variable can have zero or more definitions and uses. The example in Figure 6.1 contains the three variables `x`, `y`, and `z`, which have the following definitions (d) and uses (u).

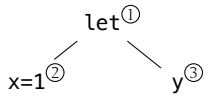
$$d[\mathtt{x}] = \{3, 7\} \qquad u[\mathtt{x}] = \{4, 8\}$$
$$d[\mathtt{y}] = \{4\} \qquad u[\mathtt{y}] = \{9\}$$
$$d[\mathtt{z}] = \{8\} \qquad u[\mathtt{z}] = \{9\}$$

For determining the meaning of an expression we need to figure out for each use of a variable which of its definitions is relevant and should be consulted to retrieve the variable's value. We write $R[x, n]$ for the location of the definition that is referenced by the use of a variable x at location n. In the example in Figure 6.1 we have the following reference information.

$$R[\mathtt{x}, 4] = 3 \qquad R[\mathtt{x}, 8] = 7 \qquad R[\mathtt{y}, 9] = 4 \qquad R[\mathtt{z}, 9] = 8$$

We can see that in this example each variable use has a definition and that each definition is used exactly once. But this is not always the case. In the following example the definition of `x` is not used at all, and `y` doesn't reference any definition, that is, $R[\mathtt{y}, 3] = \bot$.

```
let x=1 in y
```



```
{ int x;
  int y;
  x := 1;
  y := x;
  { int x;
    int z;
    x := 5;
    z := x;
    print y+z;
  }
}
```

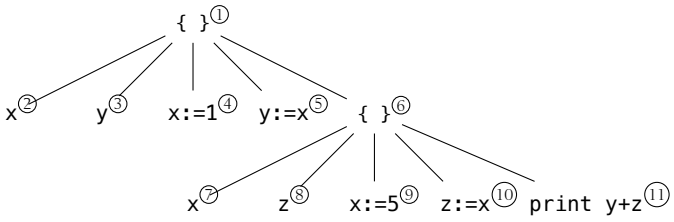


Figure 6.2 A nested block in an imperative programming language and its corresponding (compressed) AST.

In general, we can observe the following relationships between scope, definition, and use for defined variables.

$$u[x] \subseteq \cup_{n \in d[x]} S[n] \qquad \wedge \qquad n \in u[x] \implies R[x, n] \in d[x]$$

The concepts of block, declaration, body, definition, use, and reference hold for other language constructs as well. For example, assuming the definition of a function `f` by `f x = b`, the application `f e` forms a block in which the function parameter `x` and the argument `e` is the definition and `b` is the body. Moreover, the concepts are relevant for any language that provides some kind of block-based structuring mechanism. For example, the imperative program shown in Figure 6.2 has essentially the same block structure with the same definition and uses as the `let` expression from Figure 6.1.

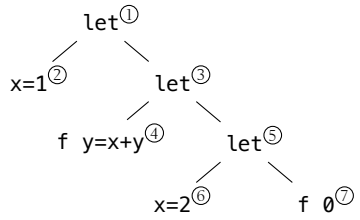
Exercise 6.2

Give the definitions for D, B, S, d, u, and R for the AST shown in Figure 6.2.

Finally, we can distinguish between *local* and *non-local* uses of a variable: When a variable is used in the same block where it is declared, its use is local. Otherwise, it is non-local, which means the declaration of the variable occurs in a block that contains the block where the use occurs. With the previous definitions we can make this idea more precise. Remember that $P[n]$ yields the block path for node n. For convenience we write $P^*[n]$ for the last element of $P[n]$, which give the innermost block that contains n, and we write $P^+[n]$ for $P[n]$ *without* its last element, which gives the list of all enclosing blocks. The use of variable

```
nonlocal =
    let x=1     in
    let f y=x+y in
    let x=2     in
        f 0

> nonlocal
1
```



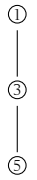


Figure 6.3 A non-local variable use in a function. Under *static scoping* the declaration visible at the function definition is relevant.

x at location n is:

- *local* iff $d[x] \cap D[P^*[n]] \neq \varnothing$.
- *non-local* iff $d[x] \cap D[P^*[n]] = \varnothing \wedge \exists k \in P^+[n].d[x] \cap D[k] \neq \varnothing$.

In Figure 6.1 the uses of `x` in nodes 4 and 8 and `z` in node 9 are local whereas the use of `y` in node 9 is non-local.

To determine the semantics of an expression or program we need to determine a value for each variable use. Since one variable can have multiple different definitions, the question is which one is relevant. For local uses of variables it seems obvious to pick the value as given by the corresponding local declaration of the same block. However, for non-local uses the situation is sometimes not so obvious. In simple cases, the answer seems to be clear. For example, in the following expression the use of `x` is non-local, and while we have two declarations of enclosing blocks to choose from, it seems natural to choose the latest definition from the closest enclosing block because this is the one that is visible, since it shadows the other definition.

```
> let x=1 in let x=2 in let y=3 in x
2
```

We can find the relevant declaration by searching the block path $P[n]$ from the last element (which is the local block) to the front (which is the outermost block), which gives us the declaration that is visible for that use.

For `let` expressions the case seems to be clear. But other forms of bindings can be more complicated. Consider the example in Figure 6.3. Here the situation is similar, but also somewhat different. First, we can observe two declarations for `x` and one non-local use of `x` in the body of the function `f`. The evaluation of `f 0` requires the value of `x`, which is obtained from block 1 and not from block 5. This behavior can indeed be predicted by the block structure: We have three scopes of definitions, one for `f`: $S[4] = \{4..7\}$, and two for

`x`: $S[2] = \{2..5\}$ and $S[6] = \{6..7\}$. Since the use of `x` in node 4 is only in the scope $S[2]$, this is the definition to be used.

So far so good. But we can notice that the evaluation of `f 0` also requires the value of `y`, and there is no block immediately discernible in the program where `y` is defined. So what's going on here? Indeed the binding for `y` is created dynamically at runtime. It cannot be directly inferred from the program.[4]

Thus the static block structure of a program is not enough to keep track of the creation and visibility of bindings. We need something else.

6.2 The Runtime Stack

Consider the latest version of the `Expr` language with `let` expression (Section 5.3, page 106). To simplify the following discussion, we omit the `Equal` and `Not` constructors.

```
type Name = String
data Expr = Lit Int | Plus Expr Expr | Let Name Expr Expr | Var Name
```

As we have already discussed in Section 4.2.4, a function that determines a value for an expression needs as an additional argument a mapping of variable names to values. Otherwise, we couldn't say what the result of an expression such as `Var "x"` is. Instead of an abstract mapping type, however, we use here a simple list of name/value pairs to keep track of bindings that are created when evaluating `let` expressions. This list is effectively used as a stack: We add elements only at the front of the list (which represents the top of the stack), and we look up variable bindings also by searching the stack from the top to bottom. This stack is called the *runtime stack*, since its existence is tied to the time of a program's evaluation.

The use of a stack causes newer declarations from nested blocks to effectively hide potential earlier declarations of the same variable and ensures that references to a variable that has multiple bindings will always be resolved in favor of the latest definition.

A very simple interpreter for `Expr` that uses a runtime stack for keeping bindings is shown in Figure 6.4. To better focus the discussion on bindings and what happens to the runtime stack, we omit all error handling from the code. In particular, that means the `eval` will fail with a runtime error in `ghci` when it tries to find an unbound variable.[5]

The equation for the `Let` constructor shows how the runtime stack evolves: The result

[4] In this simple example it actually could be inferred, but consider the slight modification of the example where $B[5]$ is `f (if test x then 0 else 9)`. Here we don't know what value is bound to `y`, since it depends on the result of `test x`, which will only be known at runtime.

[5] This happens in the function `find`, which is performing a simple list search for its argument in the stack.

```
type Stack = [(Name,Int)]

eval :: Stack -> Expr -> Int
eval _ (Lit i)       = i
eval s (Plus e1 e2)  = eval s e1 + eval s e2
eval s (Let x e1 e2) = eval ((x,eval s e1):s) e2
eval s (Var x)       = find x s
```

```
find :: Name -> Stack -> Int
find x ((y,i):s) =
     if x==y then i
             else find x s
```

Figure 6.4 An evaluator for the `Expr` language that uses a runtime stack for keeping track of bindings. Each entry on the stack is a single name/value binding.

of the defining expression `e1` (computed by `eval e1 s`) is pushed, paired with the variable name `x`, onto the current stack `s`, and the body `e2` of the `let` expression is evaluated in the context of this expanded stack.

It seems that the stack only grows larger. Does it ever shrink? Yes. Consider what happens when we evaluate the expression `(let x=1 in x+2)+4` in the context of an empty stack. The evaluation of `x+2` happens in the expanded stack `[x=1]`, but the second argument `4` of the addition, as well as the addition itself, is evaluated in the original, empty stack. Two traces of the computation are shown in Figure 6.5. We use a slightly extended version of the trace notation introduced in Section 2.2.2 where, in addition to the expression being evaluated, we also add the following representations.

(a) We show the current version of the stack.
(b) We show the results of the evaluation of subexpressions.
(c) We indent evaluations of subexpressions.

The detailed trace notation on the right of Figure 6.5 reads as follows. We start in the first line by showing the expression to be evaluated plus the empty stack in which it is evaluated (separated by a horizontal bar), which leads to (indicated by ⇒) the evaluation of the defining expression `1` in the context of the same stack. Since this expression is part of the `let` expression, its evaluation is indented. The third line shows the result of the evaluation, which is produced by the first equation of `eval`.

The fourth line shows the evaluation of `x+2`, the body of the `let` expression, which happens in the stack that is extended by the binding `x=1`. Since this expression is a subexpression of the `let` expression, its evaluation is also indented. Since the expression is an application of `+`, it requires evaluation of its subexpressions first, which happens in lines 5 through 8 in the same way as explained before, now indented two levels. The result `3` is shown in line 9, again indented only one level, aligning with the expression from line 4 for which it is the result. We indicate this connection of a result to its originating expression

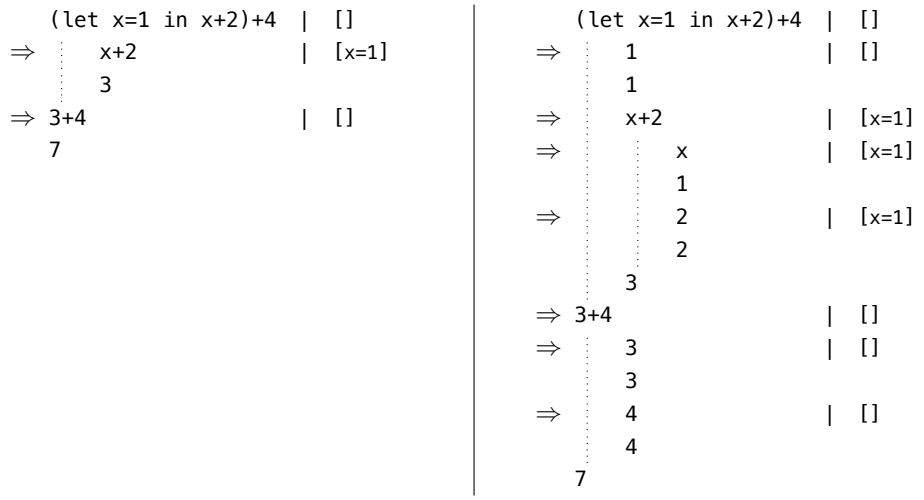


Figure 6.5 Two traces of the evaluation of `(let x=1 in x+2)+4`. On the left: An abbreviated trace that omits evaluation steps for variables and constants. On the right: A complete version that includes all steps.

by a dotted line.

At this point the evaluation of the `let` expression is complete, and the next step in line 10 is the evaluation of the top-level addition, in which the `let` expression is replaced by its result. The evaluation happens with the original empty stack because the extended stack was only passed to, and thus was visible in, the evaluation of the `let` body.[6] The evaluation then continues and ends with the final result.

We can observe that the stack in the function `eval` from Figure 6.4 stores single name/value bindings. How can we deal with blocks that contain multiple declarations, such as the `let` expression from Figure 6.3? The representation for `Stack` was chosen to keep the implementation of `eval` simple (it is actually not difficult to extend it to work with groups of bindings, but that implementation is more verbose while not contributing much to the understanding of scope).

On the one hand, the implementation doesn't actually prevent us from using groups of binding in stacks in the trace notation. Such a group of bindings is called an *activation record*. (The term *stack frame* is also used.) We write $\langle x_1{=}v_1, \ldots, x_n{=}v_n \rangle$ to denote an activation record with multiple bindings. Using this notation, the (abbreviated) trace for

[6] This is because the list cons operation is not an imperative update but creates a new list, cf. Section 2.6.

the `let` expression from Figure 6.3 looks as follows.[7] Note that we sometimes abbreviate traces further by not showing redundant results. In this example, the result of each `let` expressions is identical to the result of the innermost body.

```
   let x=1;y=x in let x=5;z=x in (x,z)  |  []
⇒      let x=5;z=x in (x,z)            |  [⟨x=1, y=1⟩]
⇒          (x,z)                       |  [⟨x=5, z=5⟩,⟨x=1, y=1⟩]
           (5,5)
```

On the other hand, we can always represent a `let` expression with multiple declarations by nested `let` expressions that each have only one declaration. The same applies to blocks in imperative languages. For the previous example, this looks as follows.

```
   let x=1 in let y=x in let x=5 in let z=x in (x,z)  |  []
⇒      let y=x in let x=5 in let z=x in (x,z)         |  [x=1]
⇒          let x=5 in let z=x in (x,z)               |  [y=1,x=1]
⇒              let z=x in (x,z)                      |  [x=5,y=1,x=1]
⇒                  (x,z)                             |  [z=5,x=5,y=1,x=1]
                   (5,5)
```

The splitting of declarations into nested `let` expressions changes the order in which some of the declarations appear on the stack. However, this does not lead to unintended shadowing and a different behavior, since all multiple declarations in one block must be for different variables.

The activation record notation sometimes leads to more succinct traces, and we use it whenever convenient. Otherwise, it is fine to assume and work with the simpler one-declaration-per block model.

Exercise 6.3

Create evaluation traces for the following expressions.

(a) `let x=1 in (x,let x=2 in x)`

(b) `let x=1 in (x,let x=2 in (x,let x=3 in x,x),x)`

The example in Figure 6.5 shows the effect of block structure during program execution on the runtime stack. Specifically, *entering* a block has the effect of *pushing* an activation record (or single binding in the simplified model) onto the stack, and *leaving* a block

[7] The use of the semicolon to connect multiple bindings for a `let` expression in a single line is actually valid Haskell syntax.

`{ int x;`		
`  int y;`	`[⟨x=?, y=?⟩]`	*enter block ⇒ push*
`  x := 1;`	`[⟨x=1, y=?⟩]`	*local update*
`  { int x;`	`[⟨x=?⟩,⟨x=1, y=?⟩]`	*enter block ⇒ push*
`    x := 5;`	`[⟨x=5⟩,⟨x=1, y=?⟩]`	*local update*
`    y := x;`	`[⟨x=5⟩,⟨x=1, y=5⟩]`	*local lookup & non-local update*
`  };`	`[⟨x=1, y=5⟩]`	*leave block ⇒ pop*
`  { int z;`	`[⟨z=?⟩,⟨x=1, y=5⟩]`	*enter block ⇒ push*
`    y := x;`	`[⟨z=?⟩,⟨x=1, y=1⟩]`	*non-local lookup & update*
`  };`	`[⟨x=1, y=1⟩]`	*leave block ⇒ pop*
`};`	`[]`	*leave ⇒ pop*

Figure 6.6 Execution trace for nested blocks in an imperative language.

causes the top element to be *popped off* the stack.

The trace notation works in a similar way for imperative languages. Figure 6.6 shows a slight variation of the example from Figure 6.2 to illustrate the push/pop behavior of the stack and local vs. non-local variable access.

With the runtime stack we are now in the position to trace examples in which the block structure changes dynamically during runtime. Consider again the example from Figure 6.3. In addition to the three nested blocks that result from the nested `let` expression, a fourth block is given by the body of the function `f`. This block is entered when the function is called with the argument `0`. As with the static `let` blocks, entering the function body causes a new activation record with the single binding `y=0` to be pushed onto the runtime stack. The following trace captures the evaluation up to the point where the interpreter is ready to evaluate the body of the function, `x+y`.

	`let x=1 in let f y=x+y in let x=2 in f 0`	`|`	`[]`
⇒	`  let f y=x+y in let x=2 in f 0`	`|`	`[x=1]`
⇒	`    let x=2 in f 0`	`|`	`[f=\y->x+y,x=1]`
⇒	`      f 0`	`|`	`[x=2,f=\y->x+y,x=1]`
⇒	`        x+y`	`|`	`[y=0,x=2,f=\y->x+y,x=1]`

But now we encounter a problem: Searching for the bindings of `y` and `x` starting from the top of the stack would retrieve the binding `x=2` and produce the result `2` instead of `1`. What has gone wrong here?

6.3 Static vs. Dynamic Scoping

A closer look at the runtime stack for the previous example helps explain the problem: First, we can see that the declaration `x=1` is hidden by the declaration `x=2`. This happened because `f` was called after the declaration `x=2`, and thus the non-local access to `x` in the block `x+y` created dynamically by the function call "sees" this later declaration.

In other words, the declaration `x=2` is the one that's visible *dynamically*, when the program is executed, whereas prior to execution, that is, *statically*, when looking at the program text, the declaration `x=1` is visible when the function `f` is declared.

A language definition (or implementor) has a choice: Either select the dynamically visible binding (in this case `x=2`) or the statically visible binding (in this case `x=1`). The former strategy is called *dynamic scoping*, and the latter strategy is called *static scoping*. In other words, in a programming language with static scoping (semantics) a non-local name refers to the declaration that is visible (that is, "in scope") where the function is defined. In contrast, in a programming language with dynamic scoping (semantics) a non-local name refers to the declaration that is visible (that is, "in scope") where the function is used.

To continue the example trace from the previous section, we can now easily produce the result under dynamic scoping, using the nearest definition for `x` and return `2`, but how can we create the result for static scoping? It seems we somehow should "skip over" and ignore the second declaration in the runtime stack to find the statically visible one. Essentially, we should start searching the stack not from the top, but from the place where `f` is defined, that is, at the activation record that contains the declaration of `f`.

This behavior can be implemented in different ways. One elegant method that can be easily realized in our metalanguage Haskell is the use of a so-called *closure*, which is a piece of code paired with an environment that provides definitions for unbound (that is, non-local) variables. The idea of a closure (as the name suggests) is to "close" the definition of a function by closing the holes that non-local variables leave open. By storing the current definitions for non-local variables, a closure prevents dynamic changes of these variable values through other declarations, which becomes possible when the code of the function is executed in different contexts.

To illustrate how static scoping can be implemented, we extend the definition of `eval` from Figure 6.4. The result is shown in Figure 6.7. First, we need to add two constructors to the `Expr` data type to represent anonymous functions (`Fun`) and function application (`App`). Then we have to use a `Val` data type as a result type for `eval`, since we need to distinguish between integers (`I`) and closures (`C`), which represent function values. A closure stores the parameter and body of the function (through the `Name` and `Expr` arguments) plus the stack that exists at the time the function is defined. These two cases of the `Val` type

```
data Expr = Lit Int | Plus Expr Expr | Let Name Expr Expr | Var Name
          | Fun Name Expr | App Expr Expr

data Val = I Int | C Name Expr Stack

type Stack = [(Name,Val)]

eval :: Stack -> Expr -> Val
eval _ (Lit i)       = I i
eval s (Plus e1 e2)  = case (eval s e1,eval s e2) of
                         (I i,I j) -> I (i+j)
eval s (Var x)       = find x s
eval s (Let x e1 e2) = eval ((x,eval s e1):s) e2
eval s (Fun x e)     = C x e s
eval s (App f e)     = case eval s f of
                         C x e' s' -> eval ((x,eval s e):s') e'
```

Figure 6.7 An evaluator for the `Expr` language that implements static scoping.

are scrutinized in `eval` in the second and last equation to ensure that the arguments for `+` are integers and the expression applied to another expression (constructor `I`) evaluates to a function value (that is, a closure).

To understand the use of closures, remember that the definition of a function happens in a `Let` expression. For example, the following part of the definition:

```
let f y = x+y in ...
```

is represented in the abstract syntax as:

```
Let "f" (Fun "y" (Plus x y)) ...
```

The fourth equation for `eval` shows that this definition leads to the following evaluation (where `s` is the current stack `[("x",I 1)]`):

```
eval (("f",eval s (Fun "y" (Plus x y))):s) ...
```

The fifth equation reveals that `eval s (Fun "y" (Plus x y))` results in the following closure value. We use the name `savedStack` for the current stack `[("x",I 1)]` to distinguish it from later occurrences of the name `s`.

```
C "y" (Plus x y) savedStack
```

The closure "freezes" the current values of all non-local variables and attaches the frozen

package to the code. Later when the code is activated, the variable definitions will be "thawed" to come back to life.

This closure is used as a binding for `f`, which is pushed onto `savedStack`, and the evaluation continues with this extended stack.

When the evaluation later encounters the expression `f 0` (represented in abstract syntax as `App (Var "f") (Lit 0)`), the stack `s` at that point contains the binding for `f` and also the second binding for `x`.

```
[("x",I 2), ("f",C "y" (Plus x y) savedStack), ("x",I 1)]
```

The definition for `eval` requires the evaluation `eval s (Var "f")`, which will retrieve with the help of `find` (see Figure 6.4) the closure value, which still contains the saved stack with the frozen variable values. Evaluation continues with the following computation (where `I 0` is the result of `eval s (Lit 0)`).

```
  eval (("y",I 0):savedStack) (Plus x y)
= eval [("y",I 0),("x",I 1)] (Plus x y)
```

By using the saved stack, the evaluation basically skips over the later definition for `x` and uses the statically visible earlier one.

We can also illustrate the static scoping evaluation using the trace notation.

```
   let x=1 in let f y=x+y in let x=2 in f 0  |  []
⇒      let f y=x+y in let x=2 in f 0         |  [x=1]
⇒          let x=2 in f 0                    |  [f=(\y->x+y,[x=1]),x=1]
⇒              f 0                           |  [x=2,f=(\y->x+y,[x=1]),x=1]
⇒                  x+y                       |  [y=0,x=1]
                   2
```

Most programming languages have adopted static scoping because it is easier to reason about: Under static scoping non-local variables have a fixed reference that can be understood by looking at the program text, whereas under dynamic scoping the value referred to by a variable can change dynamically and is unpredictable. Some older versions of Lisp have adopted dynamic scoping, and Perl lets the programmer choose between static and dynamic scoping.

Exercise 6.4

Consider the following program.

```
 1 { int x;
 2   x := 2;
 3   { int f(int y) {
 4       x := x*y;
 5       return (x+1);
 6     };
 7     { int x;
 8       x := 4;
 9       x := f(x-1);
10     };
11   };
12 }
```

Show the runtime stacks that result immediately after the statements on lines 8, 4, and 9 have been executed.

(a) Show the stacks using dynamic scoping.

(b) Show the stacks using static scoping.

Exercise 6.5

Show the development of the runtime stack for following program using (a) dynamic and (b) static scoping. What is the final value of z?

```
{ int y;
  int z;
  y := 1;
  z := 0;
  { int f(int x){return y+x};
    { int g(int y){return f(2)};
      z := g(3);
    };
  };
}
```

Exercise 6.6

Show the development of the runtime stack for the following program using (a) dynamic and (b) static scoping. What is the final value of z?

```
{ int z;
  z := 0;
  { int f(int x){return x+1};
    { int g(int y){return f(y)};
      { int f(int x){return x-1};
        z := g(3);
      };
    };
  };
}
```

7 Parameter Passing

Parameter passing is a core component of computing. A function definition (or any similar programming language abstraction for representing algorithms) provides the static description of (many different but similar) computations. But computation doesn't happen yet. Only when a function is applied to arguments does the dynamic semantics turn the function loose and creates a computation. During the computation the arguments are referred to by the parameters in the function's body. While it may seem obvious at first what happens when functions are called, there are several questions that a language definition has to answer to define a precise semantics.

For example, what kind of programming elements can be passed as arguments to parameters? Only values, or also expressions? What about variables? If we pass a variable, do we copy its value, or do we use the parameter as an alias? If we pass an expression, do we evaluate the expression once and pass the resulting value, or do we keep the expression and evaluate it every time the parameter is referenced (which makes a difference when the expression contains variables whose value could change)? And what is the effect of assignments to parameters? Can they change parameters temporarily? Do they change the arguments? The purpose of this unit is to answer these questions.

To better understand and compare the different forms of parameter passing, it is helpful to describe, on a generic level, what happens when a function call is executed. The general situation is as follows. A function application has the following form.

```
f e
```

Evaluation of `f e`

1	{Val | Evaluate `e`}
2	Create binding {Val | `x=v`}{Ref+ValRes | `x=y`}{Name+Need | `x=e`}
3	Evaluate `d`, where:
4	{Need | After first reference to `x`, replace `x=e` by `x=v`}
5	{ValRes | Use `x=w` to update `y=w`}
6	Remove binding `x=*`

Figure 7.1 A generic parameter passing schema for the evaluation of `f e`, which assumes that the argument `e` evaluates to `v`, that is, `eval e = v`. All references to bindings refer to bindings on the runtime stack. Steps 1 and 2 happen before control is transferred for executing the function code, which is described by steps 3 and 4. Steps 5 and 6 occur at the end of the function call. The notation {Par | *s*} means that step *s* happens only for the parameter passing scheme Par.

Here `f` is the name of a function that is defined by a single equation as follows.[1]

```
f x = d
```

The name `x` is `f`'s *parameter*,[2] and `d` is `f`'s *definition* or *body*. We also distinguish three different kinds of bindings that can appear on the runtime stack.

Value bindings	`x=v`
Variable bindings (or references)	`x=y`
Expressions bindings	`x=e`

Figure 7.1 shows a general schema of how `f e` is evaluated. Everything that happens can be explained by the effect it has on the runtime stack, and we can group the activities into three phases, depending *when* they happen, that is, *before* (steps 1 and 2), *during* (steps 3 and 4), and *at the end* (steps 5 and 6) of the evaluation of `d`.

The evaluation steps depend on a so-called *parameter passing schema* and also whether `e` is a variable `y` or an expression. We consider the following five parameter passing schemas.

[1] A definition that uses multiple equations, as is possible in Haskell, can always be expressed as a single equation and a corresponding `case` expression (see Section 4.2.1), and a function with multiple parameters can always be defined in curried form with one parameter and a function of the remaining parameters as a result (see Section 2.3.2). This pattern also covers functions in imperative programming languages where definitions have the form `f(x){ ...; return e}` (or something similar).

[2] The parameter `x` is also sometimes called the *formal parameter*, and the argument `e` is also sometimes called the *actual parameter*. Since these two names can be easily confused, and since "parameter" and "argument" are two perfectly good names, we'll stick to these.

- Call-By-Value (Val): Pass value of argument expression
- Call-By-Reference (Ref): Pass pointer to argument variable
- Call-By-Value-Result (ValRes): Pass copy of argument variable
- Call-By-Name (Name): Pass expression to be evaluated on every reference
- Call-By-Need (Need): Pass expression, and remember its value after first use

In the following sections we discuss each parameter passing schema in more detail. Unless explicitly state otherwise, we assume *static scoping*.

7.1 Call-By-Value

Call-by-value is the simplest, and probably most intuitive, parameter passing schema. Many, if not most, programming languages employ call-by-value. Under call-by-value, first the argument expression `e` is evaluated, and then the resulting value `v` is passed to the function, that is, a binding `x=v` is created on the runtime stack, which makes the argument value accessible to the function through the parameter name `x`. As an example, consider the following function definition.

```
f x = x+x
```

Here is a trace for the evaluation of the function call `f (3+4)` that illustrates call-by-value parameter passing.

```
   1+f (3+4)  |  []
⇒      3+4    |  []      Evaluate argument expression
       7
⇒      x+x    |  [x=7]   Evaluate function body with argument value bound to parameter
       14     |  [x=7]
⇒ 1+14        |  []
  15
```

In a functional language the evaluation of a function application `f e` with `f x = d` is equivalent to the evaluation of the expression `let x=e in d`. Under call-by-value the `let` expression simplifies to `let x=v in d` where `v = eval e`. One can also understand call-by-value also via textual substitution, that is, substitute the value `v` of the argument expression for all references to the parameter `x` in the body of the function definition (but not for other locally defined variables `x`) and then execute the function body.

In addition to references, an imperative language may or may not allow assignments to a call-by-value parameter. Even if this is allowed, the final value of the parameter is lost

Call-by-Value

1	Evaluate `e` to `v`
2	Create binding `x=v`
3	Evaluate `d`
6	Remove binding `x=u`

Figure 7.2 Call-by-value parameter passing schema. Steps 4 and 5 of the generic schema are not relevant. The parameter may have been a assigned a different value (`u`), but it is forgotten at the end.

once the activation record for the function call is popped off the runtime stack. A function call can have no effect to the call site through a call-by-value parameter. The call-by-value parameter passing schema is summarized in Figure 7.2 as an instance of the generic schema from Figure 7.1.

An example of an imperative program with a call-by-value function call can be found in Figure 7.3. We are using a few additional notational conventions in the trace presentation.

- We don't show the code of functions on the stack. In most implementations the code is not stored on the stack anyway. Instead we show an entry `f=·` to indicate the existence of the binding and its nesting position with respect to other bindings. An implementation would also store a pointer (called an *access link*) to the place on the stack the statically scoped bindings for the non-local variables in `f` can be found. In most cases these are simply the entries on the stack to the right of `f`.
- The notation `>> f(y)` indicates the jump to the code of the function `f`, which is called with the argument `y`.
- Evaluations of functions are indented, which is particularly helpful when tracing multiple nested function calls.
- The notation `<<v` on the stack indicates the value `v` that is returned from the function. This value will effectively be substituted for the function call at the call site.
- The notation `<<` indicates the jump back from the function call.

Most of the steps should be clear from the explanations and previous examples. Step 1 of the general schema is trivial in this example, since the argument is simply a variable. The reason for choosing the example in this way is to be able to use one program with different parameter passing schemas to get a better sense of their differences. Since call-by-reference as well as call-by-value-result both require the argument to be a variable, a variable had to be used.

```
 1  { int z;
 2    int y;
 3    y := 5;
 4    { int f(int x){
 5        x := x+1;
 6        y := x-4;
 7        x := x+1;
 8        return x;
 9      };
10      z := f(y)+y;
11    };
12  }
```

```
 1  [z=?]
 2  [y=?,z=?]
 3  [y=5,z=?]
 4  [f=·,y=5,z=?]
10  >> f(y)
       [x=5,f=·,y=5,z=?]
    5  [x=6,f=·,y=5,z=?]
    6  [x=6,f=·,y=2,z=?]
    7  [x=7,f=·,y=2,z=?]
    8  [<<7,x=7,f=·,y=2,z=?]
    <<
10  [f=·,y=2,z=9]
11  [y=2,z=9]
12  []
```

Figure 7.3 Example trace of an imperative program showing a *call-by-value* function call. The code is shown on the left, and the trace of the runtime stack on the right shows the stack after each line of code is executed. The changes in each stack version from the previous one are shown in color.

Note that the jump from line 4 to line 10 (in the fifth line of the trace) is due to the fact that the function definition extends from line 4 to line 9. In line 10, the assignment requires the computation of the result of `f(y)+y`, triggering the function call `f(y)`, which causes the binding `x=5` to be created on the stack. Next the code of the function body starting in line 5 is executed, and the corresponding changes to the runtime stack are shown indented. The `return` statement in line 8 has the effect of pushing the value of `x` onto the stack, where it is available (for a brief moment) after the return from the function call in line 10 to complete the computation of `f(y)+y`, which therefore results in `7+2` = `9`. After the return value has been consumed it will be removed together with `f`'s activation record (`x=7`) from the runtime stack.

We will use the same program again in Sections 7.2 and 7.3 to illustrate how call-by-value, call-by-reference, and call-by-value-result compare.

7.2 Call-By-Reference

Call-by-reference parameter passing is used in imperative languages to allow assignments in a function to affect (potentially many different) external variables. It is useful for performing systematic modifications (captured in a function) of large data structures without having to copy the data structure. Instead, the function is effectively handed a pointer to the data structure, and the update operations on a call-by-reference parameter directly

Call-by-Reference

2	Create binding `x=y`
3	Evaluate `d`
6	Remove binding `x=y`

Figure 7.4 Call-by-reference parameter passing schema. Steps 1, 4, and 5 of the generic schema are not relevant. All assignments to `x` are effectively redirected to the passed variable `y`.

modify the referenced data.

Only variables can be arguments for a call-by-reference parameter, and the parameter effectively acts as an alias for the argument variable: Any operation on `x` is de facto performed on `y`. Where call-by-value can be understood by *substituting a value* for all references of the parameter in the function body, one can understand call-by-reference as *renaming the parameter* to its argument variable name. The call-by-reference parameter passing schema is summarized in Figure 7.4 as an instance of the generic schema from Figure 7.1.

As an example we consider again the program from Figure 7.3 and show Figure 7.5 how it leads to a different trace under call-by-reference. The notation `x=y` for the variable binding suggests that `x` itself cannot be changed and that it acts solely as an indirection to the variable `y`.

```
 1  { int z;
 2    int y;
 3    y := 5;
 4    { int f(int x){
 5        x := x+1;
 6        y := x-4;
 7        x := x+1;
 8        return x;
 9      };
10      z := f(y)+y;
11    };
12  }
```

```
 1  [z=?]
 2  [y=?,z=?]
 3  [y=5,z=?]
 4  [f=·,y=5,z=?]
10  >> f(y)
       [x=y,f=·,y=5,z=?]
    5  [x=y,f=·,y=6,z=?]
    6  [x=y,f=·,y=2,z=?]
    7  [x=y,f=·,y=3,z=?]
    8  [<<3,x=7,f=·,y=3,z=?]
    <<
10  [f=·,y=3,z=6]
11  [y=3,z=6]
12  []
```

Figure 7.5 Example trace of the imperative program from Figure 7.3 showing a *call-by-reference* function call. The variable binding `x=y` amounts to storing in `x` a pointer to `y` with the effect that any assignments and references to `x` are redirected to the passed parameter `y`.

Call-by-Value-Result

2	Create binding `x=u` by copying the existing binding `y=u`
3	Evaluate `d`
5	Use `x=w` to update `y=w`
6	Remove binding `x=w`

Figure 7.6 Call-by-value-result parameter passing schema (also called Copy-in-copy-out). Steps 1 and 4 of the generic schema are not relevant. A local copy of the passed variable is created, and all assignments apply to this copy. During step 3, the variable `y` remains unaffected. After executing the function code, the final value of `x` is copied back to `y`.

7.3 Call-By-Value-Result

Call-by-value-result combines aspects of call-by-value and call-by-reference: Like call-by-reference, only variables can be passed as arguments to a call-by-value-result parameter, but unlike call-by-reference and like call-by-value, the value of the argument variable is copied to create a local binding for the functions call. Also like call-by-value, all assignments made to the parameter affect the parameter directly and *not* the variable passed as argument as in call-by-reference. In step 5 at the end of the function call, the final value of `x` is copied back to `y` is unique to call-by-value-result.

The two steps at the beginning and end of the function call of copying the value of `y` into the function and then back out give this parameter passing schema its alternative name *copy-in-copy-out*.

As can be seen in Figure 7.7, this leads to different computations than with call-by-value or call-by-reference. Under call-by-value the value of `z` after the function call is `9`, whereas under call-by-reference it is `6`. Under call-by-value-result we get `14`.

There is a further semantic subtlety to be considered here. Figure 7.7 shows an additional version of the runtime stack after line 9, which illustrates that the value of `y` is `7` when we compute the sum of `f(y)` and `y`. This is because `f(y)` is evaluated *before* `y`, which means the effect of `f` on `y` takes place before `y` is evaluated. If the order of the summands in the assignment in line 10 were reversed and we had to compute `z := y+f(y)` instead, the

```
 1  { int z;
 2    int y;
 3    y := 5;
 4    { int f(int x){
 5        x := x+1;
 6        y := x-4;
 7        x := x+1;
 8        return x;
 9      };
10      z := f(y)+y;
11    };
12  }
```

```
 1  [z=?]
 2  [y=?,z=?]
 3  [y=5,z=?]
 4  [f=·,y=5,z=?]
10  >> f(y)
       [x=5,f=·,y=5,z=?]
    5  [x=6,f=·,y=5,z=?]
    6  [x=6,f=·,y=2,z=?]
    7  [x=7,f=·,y=2,z=?]
    8  [<<7,x=7,f=·,y=2,z=?]
    9  [<<7,x=7,f=·,y=7,z=?]
    <<
10  [f=·,y=7,z=14]
11  [y=7,z=14]
12  []
```

Figure 7.7 Example trace of the imperative program from Figure 7.3 showing a *call-by-value-result* function call.

value of `y` used would be `5`, and the result for `z` would be `12`.

Exercise 7.1

Show the development of the runtime stack for following program using (a) call-by-value, (b) call-by-reference, and (c) call-by-value-result. What is the final value of `z` in each case?

```
 1  { int x;
 2    int z;
 3    int f(int y){
 4      x := y-4;
 5      y := y+1;
 6      return y;
 7    };
 8    y := 5;
 9    z := x+f(x);
10  }
```

7.4 Call-By-Name

Call-by-value and call-by-reference are two common parameter passing schemas that are used in most imperative programming languages. Call-by-value-result is a combination of the two that was introduced in Ada to better control the behavior of variables when used

Call-by-Name

2	Create binding `x=e`
3	Evaluate `d`
6	Remove binding `x=e`

Figure 7.8 Call-by-name parameter passing schema. Steps 1, 4, and 5 of the generic schema are not relevant. Instead of a value or a reference, an expression is stored on the runtime stack. This expression will be evaluated whenever the parameter `x` is accessed during step 3. Since `e` may contain variables whose value can change between different accesses to `x`, the value of `x` may change as well. This fact can be exploited to simulate function parameters through expressions with variables.

in multi-processor environments or with remote procedure calls.

In contrast, call-by-name and call-by-need have their origins in the foundations of functional programming, in particular, in the operational semantics of lambda calculus. Call-by-name was also available in Algol 68.

The main idea behind call-by-name is *not* to immediately evaluate the expression `e` that is passed as an argument, but to evaluate it whenever the parameter `x` is accessed in the body of the function. Call-by-name is similar to call-by-value in that the value of the argument expression is used whenever the parameter is accessed. However, there are two important differences.

- Assignments to call-by-name parameters are *not allowed*.
- The number of times the argument expression `e` is evaluated is, in general, not known in advance (for example, when the parameter `x` is accessed within a conditional or loop).

The latter fact has two significant implications. First, since `e` may be evaluated never, once, or more than once, call-by-name can be more efficient, as efficient, or less efficient than call-by-value.

Second, in situations when `e` is evaluated more than once, it might result in different values when variables that occur in `e` change their value in between. This can, of course, only happen in an imperative language with assignment operations and side effects. This behavior can be exploited to simulate the effect of higher-order functions. For example, passing an expression `e` that contains the variable `y` amounts to passing a function `\y->e`.

In a functional language (or any language without assignments and side effects), call-by-name behaves mostly like call-by-value if we disregard the efficiency aspect mentioned above. One exception is that call-by-name can terminate with a value in some situations when call-by-value either doesn't terminate or yields a runtime error. Consider the follow-

```
1 { int y;
2   y := 4;
3   { int f(int x){
4       y := 2*x;
5       return (y+x);
6     };
7     y := f(y+3);
8   };
9 }
```

```
1 [y=?]
2 [y=4]
3 [f=·,y=4]
7 >> f(y+3)
     [x=y+3,f=·,y=4]
  4  [x=y+3,f=·,y=14]
  5  [<<31,x=y+3,f=·,y=14]
  <<
7 [f=·,y=31]
8 [y=31]
9 []
```

Figure 7.9 Example trace of an imperative program illustrating a function call with *call-by-name* parameter passing. The argument expression `y+3` remains on the stack unevaluated the whole time the function is active. The two accesses to the parameter `x` yield two different results (`4` and `14`, respectively), since the variable `y` is changed in between.

ing two function definitions.

```
forever x = forever x
ignore x = 42
```

The function `forever` won't terminate for any argument. Under call-by-value, `ignore (div 1 0)` will cause a runtime error, and `ignore (forever "young")` will not terminate. Under call-by-name both expressions evaluate without problem because the argument expression is never evaluated.

```
> ignore (forever "young")
42
> ignore (div 1 0)
42
```

Figure 7.9 shows an example trace of a function call with call-by-name parameter passing. Since the argument expression `y+3` contains a variable whose value changes during the evaluation of the function, the two references to the function parameter `x` produce different values.

7.5 Call-By-Need

Call-by-need is almost identical to call-by-name with one important difference: After the first access to the parameter `x`, its binding to `e` is replaced with the value of `e` (as it is de-

Call-by-Need

2	Create binding `x=e`
3 4	Evaluate `d`, where: After first reference to `x`, replace `x=e` by `x=v`
6	Remove binding `x=*`

Figure 7.10 Call-by-need parameter passing schema. Steps 1 and 5 of the generic schema are not relevant. Instead of a value or a reference, an expression is stored on the runtime stack. This expression will be evaluated when the parameter `x` is accessed during step 3 for the first time, after which `e` is replaced by its value.

termined at that time). Call-by-need can be thought of as an optimized version of call-by-name (since the argument expression is not evaluated repeatedly) or a "delayed" version of call-by-value (since the argument expression is only evaluated if needed).

The difference between call-by-need and call-by-name is visible only in imperative languages where the values of variables referenced by the argument expression can change between successive accesses. The different behavior is illustrated in Figure 7.11 which shows the trace for example from Figure 7.9 with call-by-need parameter passing. The change to variable `y` has no effect, since the result of `y+3` is remembered after the first evaluation.

```
1 { int y;
2   y := 4;
3   { int f(int x){
4       y := 2*x;
5       return (y+x);
6     };
7     y := f(y+3);
8   };
9 }
```

```
1 [y=?]
2 [y=4]
3 [f=·,y=4]
7 >> f(y+3)
      [x=y+3,f=·,y=4]
   4  [x=7,f=·,y=14]
   5  [<<21,x=7,f=·,y=14]
   <<
7  [f=·,y=21]
8  [y=21]
9  []
```

Figure 7.11 Example trace of an imperative program illustrating a function call with *call-by-need* parameter passing. The argument expression `y+3` remains on the stack unevaluated until the parameter `x` is accessesed. At that point the expression is replaced by its value.

Exercise 7.2

Show the development of the runtime stack for following program using (a) call-by-name and (b) call-by-need. What is the final value of `a` in each case?

```
1 { int a;
2   a := 3;
3   int f(int b){
4     a := 2*b;
5     return (a+b);
6   };
7   a := f(a-1);
8 }
```

7.6 Summary

A high-level comparison of the different parameter passing schemas is presented in Figure 7.12. The table uses six aspects of parameter passing to highlight the commonalities and differences between the different approaches.

	Call-by-...				
	Value	*Reference*	*Value-Result*	*Name*	*Need*
Restriction on argument kind		only variables			
Restriction on parameter use				no assignments	
Direction of value flow	in	in & out	in & out	in	in
Stored on the runtime stack	value	pointer	value	expression	expression, then value
Parameter access	lookup	dereference	lookup	evaluation	evaluation, then lookup
At the end of function call			copy parameter to argument		

Figure 7.12 Comparison of parameter passing schemas.

8 Prolog

In Unit 2 we have already seen that computation doesn't have to be understood as a sequence of state transformations. Specifically, functional programming regards computation as a sequence of *expression simplifications*. The transformation of an expression is similar to imperative programming in that it happens linearly, step-by-step (although it is possible to simplify non-overlapping subexpressions in parallel). However, a crucial difference is that a functional program transforms an *expression* whereas an imperative programs transforms a *state*.

Logic programming in general, and Prolog in particular, provides yet another view on computation. First, in addition to representing data as (trees of) values (which is similar to functional programming), Prolog also supports the explicit representation of relationships among values. Second, running a Prolog program means to construct an answer to a query about the relationships by searching for facts and linking them together using rules.

But note that Prolog is untyped and that it is a first-order language, that is, we can only formulate queries about specific relationships and not express queries that determine, for example, what relationships hold between objects.

Table 8.1 extends Table 2.1 from Section 2 and gives a high-level overview of the differences between logic, functional, and imperative languages with regard to how they realize algorithmic concepts (see also Section 1.5).

In essence, *writing* a Prolog program is declaring facts and defining rules about objects and their relationships, and *running* a Prolog program is asking questions about the relationships defined in the program..

Concept	Prolog	Haskell	C, Java, etc.
Algorithm	Set of rules	Function	Program
Instruction	Term	Expression	Statement
Input	Value Argument	Function Argument	`read` Statement
Output	Variable Binding	Function Result	`print` Statement
Iteration	Recursion	Recursion	Loop
Step	Term unification	Expression simplification	State update
Computation	Tree of terms	Sequence of steps	

Table 8.1: How algorithmic concepts are realized in Prolog compared to functional and imperative programming languages (an extension of Table 2.1).

8.1 Getting Started

As with Haskell, Prolog programming consists of two separate activities that happen in two different environments: First, *writing* Prolog programs is done with an editor and involves editing fact and rule definitions in a file (that must have the extension `.pl`). In contrast, *executing* a Prolog program happens in a Prolog interpreter where definitions are loaded from Prolog files to set the context for formulating and answering queries.

A small example Prolog program is shown in Figure 8.1, which defines *facts* about the hobbies of two people as well as a *rule* that classifies people based on their hobby. A fact defines a single *relationship* between a tuple of objects. For example, the fact in the first line states that a relationship called `hobby` exists between the objects `alice` and `reading`. A *relation* is a set of relationships, which can be defined by a collection of individual facts or by rules. The three facts together constitute a so-called *predicate* definition. Sometimes the terms "relation" and "predicate" are used synonymously, but they are different things: A predicate is a syntactic Prolog object whose semantic value is a relation. Moreover, a predicate is used like a function,[1] that is, it is applied to a tuple of arguments and is evaluated to a boolean value that says whether or not the tuple is an element of the relationship denoted by the predicate.

The rule definition in the last line defines another predicate, `runner`, that denotes the unary relation of all people who have running as their hobby. (We will look at the details of rule definitions a little later.) In addition to the predicate definitions, Figure 8.1 also shows the relations denoted by the predicates.

[1] For each set S, the corresponding predicate $P_S(x) :\Leftrightarrow x \in S$ is called the *characteristic function* of S. In Prolog, a predicate is the characteristic function of a relation.

```
hobby(alice,reading).
hobby(alice,running).
hobby(bob,running).

runner(P) :- hobby(P,running).
```

hobby	
alice	reading
alice	running
bob	running

runner
alice
bob

Figure 8.1 On the left: The file `hobby.pl`, which contains facts that define a binary predicate `hobby` and a rule that defines a unary predicate `runner`. In the middle: The relation denoted by the `hobby` predicate. On the right: The relation denoted by the `runner` predicate.

In the following I will be using the SWI-Prolog implementation, which can be downloaded from www.swi-prolog.org/download/stable. After starting the interpreter,[2] we are presented with a message (shortened here), followed by an input prompt.

```
Welcome to SWI-Prolog (threaded, 64 bits, version 8.0.3)
...

?-
```

To answer queries, we first have to load the definitions from the Prolog file, which is done by entering the file name (without the `.pl` extension) in square brackets, followed by a period.

```
?- [hobby].
true.
```

Loading a file is a goal like any other in Prolog, and thus the response `true` means that the goal of loading the file could be achieved.

Similar to `ghci` (and other interpreter-based language systems), a Prolog program loaded into `swipl` can be used to perform a number of different computations, which means to repeatedly formulate *queries*. A query extracts information from the relations defined by the program. For example, we can check for specific elements by stating relationships as hypotheses that are then either confirmed or rejected by the interpreter.

```
?- hobby(alice,running).
true.
```

```
?- runner(alice).
true.
```

```
?- hobby(bob,reading).
false.
```

As expected, the first two hypotheses are confirmed while that third one is rejected because no fact mentions that Bob has reading as a hobby, and there is no rule for deriving that

[2] On Linux and Unix systems, including Mac OS, this is done by entering the command `swipl` in a terminal window.

relationship either.

In addition to letting Prolog confirm specific relationships (which quickly gets kind of boring), we can also use variables to pose more interesting queries. For example, we can ask who has running as a hobby by using a variable[3] for the first component of the relationship.

```
?- hobby(P,running).
P = alice
```

Prolog responds by generating variable bindings that satisfy the given goal. In this example we are presented with the binding `P=alice`, after which the interpreter waits for the user to respond (indicated by a (blinking) cursor at the end of the line). If we are satisfied with the answer, we can hit Return, in which case the interpreter stops looking for more answers.

```
?- hobby(P,running).
P = alice.
```

Alternatively, we can type a semicolon if we would like to see more solutions for `P`, in which case another solution is produced.

```
?- hobby(P,running).
P = alice ;
P = bob.
```

Since Prolog can tell that in this example there can be no more solutions, it stops automatically.

Looking at the program `hobby.pl` we can notice that the goal `hobby(P,running)` is actually the same as the definition of the predicate `runner`. Thus, we could have used the goal `runner(P)` instead to compute the same results.

```
?- runner(P).
P = alice ;
P = bob.
```

Rules are the main programming abstraction in Prolog and are often used to abbreviate more complex goals that are needed repeatedly. To leave the Prolog interpreter, enter:

```
?- halt.
```

The take-away message of this very brief introduction is this: Prolog programs consist of predicate definitions (denoting relations) that can be loaded into the Prolog interpreter.

[3] Note that variables start with an uppercase letter.

$$
\begin{aligned}
\mathit{program} &::= \mathit{rule}^{*} \\
\mathit{rule} &::= \mathit{structure}\,[\texttt{:-}\,\mathit{structure}^{+}]\texttt{.} \\
\mathit{structure} &::= \mathit{atom}\texttt{(}\mathit{term}\texttt{,}\ldots\texttt{,}\mathit{term}\texttt{)} \\
\mathit{term} &::= \mathit{structure} \mid \mathit{atom} \mid \mathit{var} \mid \mathit{int}
\end{aligned}
$$

Figure 8.2 Simplified Prolog syntax. The building blocks of a program are so-called *structures* or *compound terms*, which are atoms applied to tuples of (general) terms where a term can be an atom (a name that starts with a lowercase letter), a variable (a name that starts with an uppercase letter), an integer, or itself a structure. A list of structures constitutes a *rule*, and a Prolog program consists of a set of rules. Rules have a *head* (left of `:-`) and an optional *body*. A rule without a body is called a *fact*.

The definitions set the context for answering queries from within the interpreter. Queries can be distinguished form definitions by the preceding `?-` prompt, and the interpreter responds either with `true` or `false` (followed by a period) or with a variable binding as a solution. A sequence of responses is indicated by a `;` after each individual solution.

Exercise 8.1

Alice eats pizza and salad, Bob only likes burgers, and Carol's meals consist of salad.

(a) Capture these facts by defining a binary predicate `diet`.

(b) Write a goal to determine whether Bob eats salad.

(c) Write a goal that finds all people who eat salad.

(d) Write a goal that finds everything Alice eats.

8.2 Predicates and Goals

A Prolog program consists of predicate definitions, which are used to represent data as well as algorithms. This conceptual simplicity can sometimes lead to confusion, which is compounded by the fact that the Prolog syntax relies on a single concept of so-called *compound terms* (or *structures*) to represent predicates as well as complex objects (that is, data structures). A simplified version of the core Prolog syntax is shown in Figure 8.2.

The terminology can be quite confusing because the same concept often has different names, depending on whether we refer to its specific Prolog syntax, its semantics, or its use. For example, `runner(P)` is, syntactically, a *compound term*. In the file `hobby.pl` it appears as the *head* of a rule. When entered in the Prolog interpreter, it is a *goal* or *query*, and its

PROLOG TERMINOLOGY

When talking about Prolog programming, it is important to understand the difference between the closely related terms *relation*, *relationship*, and *predicate*.

Relation: A set of tuples that all have the same number of components, which defines the *arity* of the relation. A set of pairs is called a *binary relation*, and a set of triples is called a *ternary relation*. A special case is a set of single values, which is called a *unary relation*. A relation is represented in Prolog by a *predicate*.

Relationship: A tuple that belongs to a relation. A tuple may belong to different relations. For example, `(bob,car)` is a tuple that can be an element of the relation `owns`, indicating that Bob owns a car, as well as an element of the relation `sell`, indicating that he wants to sell his car. The two relationships are distinguished by their names as given in the corresponding predicate definitions, such as `owns(bob,car)` and `sell(bob,car)`. The tuple components are also called the *arguments* of the predicate.

Predicate: A collection of Prolog facts or rules that share a common name. The Prolog name of a predicate is called a *functor*.

Goal: A (compound) term used as a query in the Prolog interpreter. If the goal can be satisfied, Prolog will respond with one or more answers.

Clause: A fact or a rule.

semantics is a *relation*. Similarly, `hobby(bob,running)` is again a compound term. In the file `hobby.pl` it appears as a *fact*, and its semantics is a single relationship. When used in the Prolog interpreter, it is also a *goal* or *query*.

Therefore, it is important to be precise in the use of terminology when talking about the different concepts. While most of the definitions are straightforward, note that we *don't* use the words "relation" and "relationship" synonymously. As already indicated in Section 8.1, we the term *relation* refers to a set while the term *relationship* refers to a tuple that is an element of a relation. As summary of some important terms is given in the box PROLOG TERMINOLOGY.

8.2.1 Predicates

In Prolog, a predicate can be defined by *facts* and *rules* (to be discussed in Section 8.3). A fact defines a single relationship by applying (or attaching) a name of a relationship to a tuple of arguments for which the relationship is said to hold.

Notably, all the names in the definition of `hobby` start with a lowercase letter. These names are called *atoms*, and they can represent objects (such as `alice` or `reading`) and *functors* when they are used as a name for a predicate (such as `hobby` or `runner`). On the other

hand, names that start with an uppercase letter are *variables*. An example is the variable `P` used in the definition of the predicate `runner`. Like in other programming languages, variables in Prolog are placeholders for values, and the use of a variable in a term allows this term to be instantiated to different terms by substituting objects (or terms) for the variable. For example, by substituting `alice` for `P` in `hobby(P,running)`, we obtain the term `hobby(alice,running)`, which happens to be a fact in the program `hobby.pl`.

8.2.2 Goals

Any term can be used as a query (or goal), but goals with variables typically represent more interesting queries. When Prolog answers a query such as `hobby(P,running)`, it tries to find substitutions (or bindings) for the variables in the goal that lead to relationships that are defined in the program (through facts or rules). In this case, `hobby(alice,running)` is a fact in the program, and thus the binding `P=alice` is a solution for the goal.

Of course, variables can be used for any argument of a predicate. So we can ask what the hobbies of Bob are by formulating the following goal.

```
?- hobby(bob,H).
H = running.
```

Prolog is able to figure out that there is only one solution and presents the corresponding binding. Can we also use variables for both arguments? Absolutely!

```
?- hobby(P,H).
P = alice,
H = reading ;
P = alice,
H = running ;
P = bob,
H = running.
```

First, we can observe that Prolog delivers three answers and that each answer consists of two bindings. Not surprisingly, by calling the predicate with variables for all of its arguments we have produced—interactively—the relation denoted by the predicate.

The interactive scanning of predicates might work well for small examples, but it can become annoying pretty quickly. Prolog provides several predicates that allow the collection of multiple answers in a list. For example, the `findall` predicate takes three arguments: (1) the variable, say `X`, whose solutions we want to collect, (2) the goal to be satisfied (which should contain `X`), and (3) a variable to collect all the solutions computed for `X`. With `findall` we can gather all names and hobbies from the `hobby` predicate as follows.

```
?- findall(P,hobby(P,H),L).
L = [alice, alice, bob].

?- findall(H,hobby(P,H),L).
L = [reading, running, running].
```

By supplying a term with multiple variables as a first argument, we can also produce a list of relationships.

```
findall(hasHobby(P,H),hobby(P,H),L).
L = [hasHobby(alice,reading), hasHobby(alice,running), hasHobby(bob,running)].
```

Note that Prolog is a so-called *first-order* language, which means that variables can only be used for arguments of predicates and not for predicates themselves. Thus, we cannot express queries about what relationships hold between objects. For example, the following query is *not allowed* in Prolog.

```
R(alice,running).   -- Invalid goal.
```

The simple `hobby` predicate can also help illustrate an important difference between logic and functional programming. Consider how we could represent the facts about hobbies in Haskell. Since different persons can have different numbers of hobbies, we can't define a function of type `Person -> Hobby`. Instead, to capture the variability in the number of relationships, we need to employ a list (or some other data structure). For example, we could define the following function.

```
hobbies :: Person -> [Hobby]
hobbies Alice = [Reading,Running]
hobbies Bob   = [Running]
```

A significant difference between logic and functional (and imperative) programming becomes apparent when we consider how we can use function and predicate definitions for computations. In both functional and imperative languages, we apply functions to arguments to produce results. For example, we can use the function `hobbies` to determine the hobbies of Alice as follows.

```
> hobbies Alice
[Reading,Running]
```

However, functions can only be used in one direction, that is, we *cannot* use the inverse of the function and compute the persons who have a specific hobby. Since relations are not directed, Prolog can use predicates in either direction, and we can easily find out who has running as a hobby.

```
?- hobby(P,running).
P = alice ;
P = bob.
```

We will see later (for example, in Section 8.7) that the relational nature of logic programming is quite powerful and expressive.

8.2.3 Repeated Variables (aka Non-Linear Patterns)

As in any programming language, the actual choice of names as variables does not matter, and we could just as well choose `X` and `Y`, `Person` and `Hobby`, etc. However, what happens if we chose the *same* variable for two arguments in a rule or goal? In most programming languages, this would not be allowed. In contrast, Prolog does allows it. In the hobby example, such a query won't produce any result.

```
?- hobby(X,X).
false.
```

This query asks for the reflexive subset of the `hobby` relation, which is empty. Or viewed, as a graph with objects as nodes and relationships as edges, the query asks for all loops in the graph. Only homogeneous binary relations, that is, binary relations over one set can be reflexive, but since `hobby` relates people and activities, it can't contain any reflexive relationships.

As an example of a relation that admits reflexivity, consider the definition of the predicate `trust` with facts about who trust in the judgment of whom.

```
trust(bob,alice).
trust(alice,alice).
```

The query `trust(P,P)` asks for people who have trust in themselves. In other words, the query finds people with self confidence.

```
?- trust(P,P).
P = alice.
```

Reflexivity can also apply to projection on two columns of a relation with more than two columns. As a simple example consider a predicate `river` that describes a ternary relation with facts about the states of their source and mouth.

```
river(mississippi,mn,la).
river(riogrande,co,tx).
river(willamette,or,or).
```

Here we could be interested in rivers that do not cross state lines,[4] which we can list with the following query.

```
?- river(R,S,S).
R = willamette,
S = or.
```

The interesting point about both of these examples is that the use of the same variable at different places in a goal is not only perfectly fine but also provides an expressive query mechanism.

8.2.4 Conjunction

One way of creating more complex goals is to compose queries with logical connectives such as *and* and *or*. For example, to find out whether both Alice and Bob could go out together for a run, we would like to know whether they both have running as a hobby. We can create a corresponding query by joining two individual `hobby` queries using a comma that works as a logical *and*.

```
?- hobby(alice,running), hobby(bob,running).
true
```

The result itself is not surprising. What may come as a surprise, however, is that the `swipl` interpreter waits for user input. If we enter `;` to request further solutions, we are told that there are none.

```
?- hobby(alice,running), hobby(bob,running).
true;
false.
```

Now one might wonder what other answers could there possibly be. The answer is that there might exist other facts to satisfy the goal, and maybe the user is interested in those. In this case, this is unlikely, since the goal doesn't contain any variables and therefore the answer doesn't contain any bindings for which alternatives could be of interest. However, even in this case it is in principle possible to have alternative evidence for the goal. For example, assume that we duplicate the fact `hobby(bob,running)`. In this case the query evaluation looks indeed slightly different.

[4] This geometric interpretation is not quite correct, because a river could temporarily enter another state and then come back.

```
?- hobby(alice,running), hobby(bob,running).
true ;
true ;
false.
```

Here the two (identical) facts act as two pieces of evidence for the goal. Again, since no bindings are generated, this is not useful in this case.

8.2.5 Expressing Joins

We can obtain more interesting queries using conjunction to connect information from different relations. To illustrate this and some related aspects, let's add the following facts to the `hobby.pl` program.

```
sport(running).
sport(swimming).
```

Suppose now we would like to know who has a sport as a hobby. We can envision two strategies for generating answers. First, we could find out which hobbies are sports by consulting the `sport` predicate, and then finding for each sport facts in the `hobby` relation. Concretely, we can formulate the following Prolog goal.[5]

```
?- sport(H), hobby(P,H).
H = running,     |    H = running,
P = alice ;      |    P = bob
```

We could also go through the facts of the `hobby` relation and keep only those for which the hobby is a sport. While the query results in the same bindings, the order of variables is reversed.

```
?- hobby(P,H), sport(H).
P = alice,       |    P = bob,
H = running ;    |    H = running
```

This kind of query, which combines tuples from two (or more) relations based on the equality of values the columns of the relations, is called an *equi-join*. The equality of values is expressed using the same variable in the two different predicates. The name "equi" indicates that the values of the relations are required to be equal. More general forms of

[5] From now on, I will omit `;`-followed-by-`false.` responses (indicated by ending the last response *without* a period) to save some space.

joins obtain when using other relationships between the values. An example follows after the next exercise.

Exercise 8.2

(a) Define a unary predicate `healthy` through facts about healthy food.

(b) Write a query that finds people who eat (at least some) healthy food.

(c) What is missing in the query from (b) to find people that *only* eat healthy food?

We can also join a relation with itself. Consider, for example, the task to formulate a query to find out who has more than one hobby. To this end, we can join the `hobby` relation with itself based on the name column. We also have to use two different variables for the activity column. But this is not enough, because without further constraint, different variables can be bound to the same value. Therefore, we have to explicitly force the two hobby values to be different using the `\=` operator.

```
?- hobby(P,H1), hobby(P,H2), H1 \= H2.
P = alice,          |    P = alice,
H1 = reading,       |    H1 = running,
H2 = running ;      |    H2 = reading
```

As expected, we get only Alice as a result, since she is the only one with two hobbies. However, this result is presented twice, and it's instructive to understand why. To construct responses to queries, Prolog systematically traverses the fact (and rules) of the current program to identify candidates for results that are then subjected to potential constraints in the query.

Exercise 8.3

(a) Write a goal that finds pairs of people who share the same hobby.

(b) Write a goal that finds people who eat more than one thing.

(c) Write a goal that finds people with a healthy life style, that is, people who eat something healthy and have a sport as a hobby.

8.2.6 A Simple Operational Evaluation Model for Prolog

As an operational model for how Prolog answers queries, one can imagine that a series of nested `for` loops is created, one for each predicate that appears in the goal. Each loop ranges over possible relationships for the predicate and creates bindings for the variables used.

In the two-hobbies query the first (outer) loop starts by creating the bindings `P=alice` and `H1=reading` for the first fact. Since the second (inner) loop reuses the variable `P`, it is constrained by the value and can only range over facts that match the current value for `P`. Therefore, the loop only creates bindings for the single variable `H2`. Starting at the top, this yields `H2=reading`. With the binding for the three variables we now can test the condition `H1 \= H2`, which is not true in this case. Therefore, the current binding is rejected.

In the next step, the inner loop advances one fact and creates the binding `H2=running`. Since now `H1 \= H2` is true, the three bindings are reported as a result. Then the inner loop advances to the third fact, which is immediately rejected because the name `bob` doesn't match the current value of `P`.

Since the inner loop is completed, it is reset. and the outer loop advances to the second fact, where it creates the bindings `P=alice` and `H1=running`. The inner loop restarts with the first fact, again creating the binding `H2=reading`. Since `H1 \= H2` is true, the three bindings are reported as the second result. The only difference is that the values of `H1` and `H2` are swapped. After that, it's easy to see that none of the remaining cases yields any more results.

Sometimes the results produced by Prolog can be surprising, in particular, when expectations are guided by structural assumptions rooted in the application domain that are not exactly reflected in the formulation of predicate definitions or queries. In this example, we were, strictly speaking, looking for people who have a *set*, or *unordered pair*, of two (or more) hobbies, but the query was searching for an *ordered pair*.

Exercise 8.4

Suppose we omit the inequality constraint from the two-hobbies query, which leads to the following goal.

```
?- hobby(P,H1), hobby(P,H2).
```

How many results does Prolog produce and what bindings?

The example has shown that using the same variable in different predicates implicitly expresses an equality predicate whereas using different variables doesn't express any constraint on the generated bindings. Note that we can always express equality constraints

explicitly, that is, we can express the two-hobby query also as follows.

```
?- hobby(P1,H1), hobby(P2,H2), P1 = P2, H1 \= H2.
```

However, queries and programs are generally more readable when using equal variable names to express equality.

8.3 Rules

We can define predicates by explicitly listing facts for all individual relationships. With rules we can define predicates that are derived from other predicates. We have already seen how to derive relations with queries. Rules allow us to define an interface for queries, that is, we can provide them with names and potential parameters.

As summarized in Figure 8.2, a rule consists of two parts that are syntactically separated by the `:-` symbol (called the *neck* operator in the SWI-Prolog glossary):

- A *head* that defines the name and parameters of the predicate.
- A *body* given by a query that defines the resulting relation.

Our first example rule was for the predicate `runner`, shown in Figure 8.1. The head of that rule is `runner(P)` where `runner` is the name of the predicate and `P` is a parameter. The body of the rule is given by the term `hobby(P,running)`, which defines a query on the `hobby` relation that depends on the value of the parameter `P`.

In Section 8.2.5 we have discussed the query for determining people who have a sport as a hobby. We can define the following predicate based on this query.

```
sportHobby(P) :- hobby(P,H), sport(H).
```

The body of this rule consists of a conjunction of terms that express a join. A rule can be understood as an "if-then" rule that is read from right to left with subgoals connected by an "and," that is, the above rule can be read as:

if `P` has a hobby `H` **and**
  `H` is a sport
then
  `P` has a sport hobby

We have seen these rules before in Section 5.2.2 where we used the visual notation of in-

ference rules from Section 5.1 to present them. Prolog rules can be viewed as inference rules in the same way, that is, we can write the rule also in the following way.

$$\frac{\texttt{hobby(P,H)} \qquad \texttt{sport(H)}}{\texttt{sportHobby(P)}}$$

This view is helpful, in particular, for understanding how Prolog tries to satisfy goals by constructing a proof tree, which we will discuss in Section 8.4.1.

We can use the newly defined predicate `sportHobby` in basically two ways. First, we can supply an atom as an argument and check whether the expressed fact is true.

```
?- sportHobby(alice).
true.
```

Second, we can use the predicate to find all people with a sport as a hobby by using a variable as an argument. Note that the name of this variable can be arbitrary. In particular, it doesn't have to be the name of the parameter.

```
?- sportHobby(Anyone).
Anyone = alice ;
Anyone = bob.
```

It is instructive to compare the result with the results from Section 8.2.5. The `sportHobby` predicate produces only bindings for the names of people, whereas the plain query also produced bindings for the hobbies. Thus, rules are not just convenient abbreviations for complex or repeatedly used goals, the head of a rule also provides control over the amount of information that is revealed from the query in the body.

A related, intriguing aspect of the `sportHobby` rule is that the variable `H` in the body is *unbound* (or *free*). Usually, unbound variables are considered a programming mistake and are reported as type errors. Not so in Prolog (which is untyped): Here unbound variables are programming features that can be understood as an instruction for Prolog to find a suitable binding that can satisfy the goal(s) that use the free variable.[6] And, as indicated in Section 8.2.6, Prolog's underlying search mechanism can generate such bindings.

If we want to show the hobby in addition to the person, we could easily do this by adding `H` as a parameter to the head.

```
sportHobby(P,H) :- hobby(P,H), sport(H).
```

With this definition the goal `sportHobby(P,H)` produces the same results as the query in

[6] From a logics perspective, unbound variables are existentially quantified.

```
hobby(alice,reading).
hobby(alice,running).
hobby(bob,running).
hobby(carol,reading).
hobby(carol,chess).
```

```
diet(alice,pizza).
diet(alice,salad).
diet(bob,burger).
diet(carol,salad).
```

```
sport(running).
sport(swimming).

veggie(salad).
```

Figure 8.3 Extended Prolog program `hobby.pl`.

Section 8.2.5.

To support some more interesting queries, we add a few facts to the `hobby` relation and add a new predicate about people's diets, see Figure 8.3.

Suppose we want to find out who has a healthy life style, which we assume is having a sport as a hobby or vegetables in one's diet. We can define a corresponding predicate `healthy` by giving *two* rules, one for each condition.

```
healthy(P) :- diet(P,F), veggie(F).
healthy(P) :- sportHobby(P).
```

The second rule makes use of the `sportHobby` predicate defined earlier. As with auxiliary functions in Haskell, it is often a good idea to build an application gradually by identifying several relevant logical properties and capturing those in predicates. Similar to `sportHobby`, we could also have defined and used a predicate `veggieDiet` for the goal of the first rule. In any case, with this definition we find that everyone is healthy.

```
?- healthy(P).
P = alice ;
P = carol ;
P = alice ;
P = bob
```

Notice that Alice occurs twice in the result, since she satisfies both rules for being healthy: she eats veggies and has a sport hobby. However, the reasons for being healthy are not reported in the results, because the head of `healthy` only includes one parameter for persons. We can, of course, change this by adding another parameter, one for hobby in the first rule, and one for food in the second rule, but we then have to replace the reference to `sportHobby` by its definition.

```
healthy(P,F) :- diet(P,F), veggie(F).
healthy(P,H) :- hobby(P,H), sport(H).
```

If we run the query again with this extended definition, the bindings include the reason for being healthy.

```
?- healthy(P,Why).
P = alice,          |  P = carol,          |  P = alice,          |  P = bob,
Why = salad ;       |  Why = salad ;       |  Why = running ;     |  Why = running
```

Multiple rules, like multiple facts, define alternative ways to satisfy a predicate, that is, multiple rules effectively realize an *or* operation (or disjunction) on the conditions expressed by each body, whereas the terms in each body of a rule are connected by an *and*.

Exercise 8.5

(a) Define a unary predicate `brainy`, similar to `sport`, that classifies hobbies that are intellectual.

(b) Define a binary predicate `buddies`, that identifies pairs of people who have the same hobby. *Note*: Don't forget to add a constraint (as the last term) that prevents people being reported as their own buddies.

(c) How many results do you expect do get for the goal `buddies(P1,P2)` when using only the initial small set of hobby facts?

```
hobby(alice,reading).
hobby(alice,running).
hobby(bob,running).
```

Did you predict correctly? Explain the results.

(d) How many results do you get when you remove the inequality constraint from your rule definition? Test your prediction, and explain the result.

The fact that Alice is reported twice suggests the definition of another predicate `veryHealthy`, which is true if your diet includes vegetables and you have a sport hobby.

```
veryHealthy(P) :- diet(P,F), veggie(F), sportHobby(P).
```

As expected, according to this definition only Alice is very healthy.

```
?- veryHealthy(P).
P = alice
```

Spoiler alert! If you haven't worked on Exercise 8.5 on page 165, do it before continuing to read. A typical definition for the predicate `buddies` looks as follows.

```
buddies(P1,P2) :- hobby(P1,H), hobby(P2,H), P1 \= P2.
```

With this definition we might expect to get only one result. However, as discussed at the

end of Section 8.2.5, we get *two* results, since Prolog doesn't know that the `buddies` relation is symmetric and can't exploit this fact.

```
?- buddies(P1,P2).
P1 = alice,      |    P1 = bob,
P2 = bob ;       |    P2 = alice
```

We can avoid such duplicate results for symmetric relations by strenghtening the constraint on result pairs. Specifically, we can exploit the built-in ordering of atoms (implemented in the `@<` operator) and only return ordered pairs.

```
buddies(P1,P2) :- hobby(P1,H), hobby(P2,H), P1 @< P2.
```

With this definition we get only one result.

```
?- buddies(P1,P2).
P1 = alice,
P2 = bob
```

Exercise 8.6

For this exercise, we use the extended fact base from Figure 8.3.

(a) Define a predicate `foodies` that finds people who like to eat the same food. Since `foodies` describes a symmetric relation, use the `@<` operator to exclude unnecessary duplicates.

(b) Define the predicate `friends` that holds for two people if they are buddies or foodies.

(c) The use of the `@<` operator in both `buddies` and `foodies` avoids returning duplicate pairs. Explain why the goal `friends(P1,P2)` still returns one pair of people as a result twice.

Since facts are rules without a body, we can use rules and facts together to define predicates. For example, to add the information that Bob and Carol are friends, even though they have no hobby or food interest in common, we can simply add a correpsonding fact to the predicate definition.

```
friends(P1,P2) :- ...
friends(carol,bob).
```

Finally, we should mention that predicates can be overloaded, that is, we can use the same name for predicates with different numbers of parameters. For example, in addition to the unary `sport` predicate, we can also have a binary predicate of the same name that says how many people one needs to play the sport.

```
inside(bathroom,house).
inside(kitchen,house).
inside(fridge,kitchen).
inside(milk,fridge).

in(X,Y) :- inside(X,Y).
in(X,Z) :- inside(X,Y), in(Y,Z).
```

Figure 8.4 Prolog program `house.pl`.

```
sport(running,1).
sport(tennis,2).
sport(soccer,22).
```

Therefore, predicates have to be distinguished by name and arity. In Prolog we refer to a specific overloaded predicate by adding the arity to the predicate name. For example, `sport/1` refers to the unary `sport` predicate, whereas `sport/2` refers to the binary one.

One can debate whether it is a good idea to overload predicate names. In any case, the possibility of overloading presents a potential source for programming errors. For example, if you supply the wrong number of arguments for a predicate, this will generally not produce an error message, but Prolog may assume that you refer to a different predicate, which, if not defined, will simply lead to failure in satisfying goals.

8.4 Recursion

The factorial example in Section 5.2.2 illustrates that inference rules can be recursive. Similarly, we can use recursion in Prolog rules to define predicates. We will come back to the factorial example later in Section 8.8 when we discuss the special requirements for arithmetic in Prolog. For now, consider as an example facts about spatial containment shown in Figure 8.4, defined through the predicates `inside` and `in`. While `inside/2` is solely defined through facts, `in/2` is defined by two rules, one of which is recursive, and computes the reflexive, transitive closure of `inside`.

With these definitions, we can try to find out whether we have milk in the house by formulating the following goal.

```
?- in(milk,house).
true
```

We may ask ourselves now *why* this it true and how Prolog can figure it out. For this goal, which doesn't involve variables, we can construct an answer by repeatedly replacing goals with new goals, as required by rules, until we are left with a set of (true) `inside` facts. We can describe this process through a trace that shows the step-by-step expansion of subgoals.

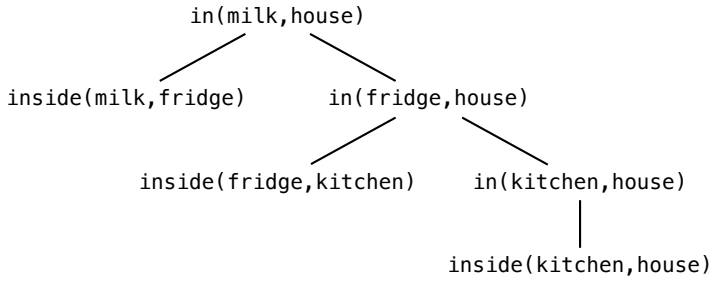
For each step we show on the right which rule was applied.

```
   in(milk,house)
⇒ inside(milk,fridge), in(fridge,house)                              | in2
⇒ inside(milk,fridge), inside(fridge,kitchen), in(kitchen,house)     | in2
⇒ inside(milk,fridge), inside(fridge,kitchen), inside(kitchen,house) | in1
```

The final entry in the trace, which only consists of facts from the program, provides a justification for the truth of the goal: Milk is in the house, because milk is inside the fridge, the fridge is inside the kitchen, and the kitchen is inside the house.

8.4.1 Trees as Computation Traces

We can give a more visual representation of the justification in the form of a tree which has the goal as a root and in which all leaves are facts. The internal nodes are subgoals that are defined by rules and require the truth of their subgoals.



The structure of the tree reflects the rules of the program that were used to answer the goal. Specifically, if the rule `H :- B1, ..., Bk` was used in the computation to justify the term T in the tree (using a binding σ as a substitution with $\sigma(\texttt{H}) = T$), then T has the children $T_1, \ldots, T_k$ in the tree where T_i is an instance of the term `Bi` of the rule's body, that is, $\sigma(\texttt{Bi}) = T_i$, (for $1 \leq i \leq k$).

For example, the subgoal `in(fridge,house)` is an instance of the rule head `in(X,Z)` with $\sigma = \langle$`X=fridge`, `Y=kitchen`, `Z=house`$\rangle$. Using the second[7] rule for the predicate `in` to satisfy this subgoal, leads to the two children $\sigma($`inside(X,Y)`$) =$ `inside(fridge,kitchen)` and $\sigma($`in(Y,Z)`$) =$ `in(kitchen,house)` in the tree. While it is clear that the bindings for `X` and `Z` are created by the arguments that are passed to the subgoal `in(fridge,house)`, it is not so clear where the binding for `Y` comes from. Since `Y` is a free variable in the second rule for `in`,

[7] We can't use the first rule, since we don't have a fact `inside(fridge,house)` in our program.

its binding is determined by Prolog's built-in search mechanism, which we'll discuss soon.

Using the inference rule notation, the same tree can also be written as a so-called *proof tree*, which can show the rules that were used at each branch.

$$\dfrac{\texttt{inside(milk,fridge)} \qquad \dfrac{\texttt{inside(fridge,kitchen)} \qquad \dfrac{\texttt{inside(kitchen,house)}}{\texttt{in(kitchen,house)}}\,\text{in}_1}{\texttt{in(fridge,house)}}\,\text{in}_2}{\texttt{in(milk,house)}}\,\text{in}_2$$

Both representations show the dynamic nature of rules in generating new goals during query evaluation. We will look into how the Prolog evaluation mechanism in more detail in the next section.

8.4.2 Left Recursion

The possibility for recursively defined queries distinguishes Prolog from most database query languages (such as SQL) and turns is into a fully fledged, powerful programming language—with all the dangers that come with the territory. An important caveat about recursion in Prolog is this.

Rules should not be left recursive.

The term *left recursion* describes the situation when the recursive use of a predicate appears as the first term in the body of a rule, in a form that matches the head of the rule. For example, suppose we change the order of subgoals in the second rule for `in` as follows. (And we also exchange the order of the two rules.)

```
in(X,Z) :- in(Y,Z), inside(X,Y).
in(X,Y) :- inside(X,Y).
```

Now the evaluation of the goal `in(milk,house)` goes into an infinite loop, and after some time the interpreter runs out of memory and stops with a runtime error. This is because in order to satisfy `in` with the first rule (which is tried first), another subgoal for `in` is generated that Prolog then tries to satisfy by using the first rule for `in`, etc.

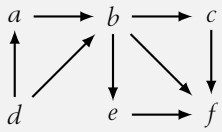
If we change the order of rules back to the original but keep the left recursion, the effect is a bit more subtle: Prolog produces the result `true`, but if we then ask for more results, we'll end up in an infinite loop anyway. For queries involving variables the situation is very similar.

The bottom line is that you should be aware of left recursion and avoid using it. In fact,

it is always a good strategy to put fact subgoals first in the body of rules.

Exercise 8.7

Consider the following directed, unlabeled graph in which nodes are represented by letters.



(a) Define the graph as a binary predicate `edge`.

(b) Define a binary predicate `path` so that `path(X,Y)` is true when there is a path from `X` to `Y` in the graph.

(c) How many results does the goal `path(a,f)` produce? Explain why.

(d) Write a goal that finds all nodes that are reachable from node *b*.

(e) Define a unary predicate `cycleAt` that yields `true` if the graph contains contains a cycle that starts and ends with at node `X`.

(f) A *join* is a constallation of three nodes such that two are predecessors of the third. Define a predicate `join`. How many joins does the graph contain?

8.5 Prolog's Search Mechanism

In the previous section we have seen how the result for a Prolog query can be represented in a tree that captures a trace of the rules that were involved in generating the result. But how does Prolog actually find the results? To describe the method in general, note that a goal can be satisfied by a fact or a rule. In the first case, the goal is satisfied and the search and computation has come to a successful end. If instead Prolog finds a rle whose head matches the goal, the terms of the body of this rule are added as additional subgoals (instantiated by applying the potential binding that result from matching the head of the rule).

In terms of our tree model, the new subgoals are added as children of the currently processed goal. Then Prolog continues trying to satisfy a new subgoal (which has to be a leaf in the tree because goals in internal nodes are already being processed).

This process is repeated until either all leaf nodes could be matched to facts and thus are satisfied (in which case all internal nodes and the goal in the root of the tree are also satisfied) or we encounter a leaf that cannot be satisfied. In the latter case, we remove that

goal and continue the search for an alternative fact or rule that can be used. If none can be found, we remove the parent goal and continue to look for an alternative for that one, etc. This process of removing intermediate goals is called *backtracking*. If backtracking proceeds all the way to the original goal in the root, and we run out of alternatives for that goal, the process fails, and the goal cannot be satisfied.

The search process can be described more precisely as a step-by-step transformation of a search state. This state consists of:

- A search tree with n nodes, each representing a goal.
- A pointer to a current goal in the tree.
- n pointers to facts and rules in the program, one for each goal.
- Sets of bindings (or substitutions), one set for each node.

In the following we will illustrate Prolog's search process by showing how the result for the example query `in(milk,house)` is built. But before we do that, we have to explain how bindings are constructed when a rule is matched against a goal.

8.5.1 Unification

When a goal is entered, Prolog scans the current program for rules whose head match the goal. More precisely, Prolog actually finds rules whose head can be *unified* with the goal. What does that mean?

We have seen examples of pattern matching in Haskell where a term built by data constructors is matched against a pattern when a function is applied to it. When a pattern matches a term, bindings for the variables in the pattern are generated. For example, the list `[2,3,4]` successfully matches the pattern `x:xs` and creates the binding ⟨`x=2`, `xs=[3,4]`⟩. In Prolog we can compare terms and patterns directly in the interpreter. Lists are written similar to Haskell lists, but patterns have a slightly different syntax, in particular, the pattern `x:xs` would be written in Prolog as `[X|XS]` (we will discuss this in more detail in Section 8.7). Therefore, we can express the pattern matching example as follows.

```
?- [2,3,4] = [X|XS].
X = 2,
XS = [3, 4].
```

The list `[2,3,4]` doesn't match the patterns `[]` or `1:xs` (since the first element in the list, `2`, doesn't match the constant `1` in the pattern) and thus doesn't generate any binding. We can directly verify this in Prolog.

```
?- [2,3,4] = [].
false.

?- [2,3,4] = [1|XS].
false.
```

Pattern matching always matches a term with only values against a pattern, which is a term that contains values and variables.

Unification is a generalization of pattern matching which takes two patterns and tries to find a substitution for variables that, when applied to both patterns, makes them equal. For example, we can ask whether the two lists `[X,3,4]` and `[1|XS]` can be made equal.

```
?- [X,3,4] = [1|XS].
X = 1,
XS = [3, 4].
```

We observe that, unlike in Haskell, in Prolog we can have variables in both terms, and Prolog finds a substitution if it exists. This process is called *unification*, and the resulting substitution that makes the two terms equal is called a *unifier*.

Now how about the two lists `[X,Y,4]` and `[1|XS]`? Can we make them also equal?

```
?- [X,Y,4] = [1|XS].
X = 1,
XS = [Y, 4].
```

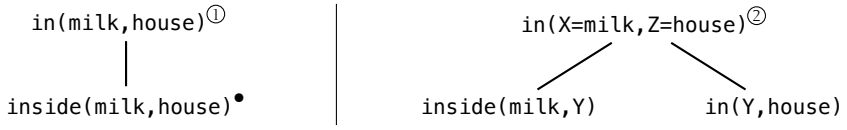
Yes, and we can see that (1) no substitution for `Y` is generated and (2) the substitution term for `XS` contains itself a variable. This is because any other solution (such as substituting `3` for `Y`) is less general and could be obtained as a special case from the more general one.

In fact, the unification algorithm employed by Prolog is always able to compute what is called the *most general unifier* (or *mgu*), which is a substitution from which all other valid substitutions can be obtained by adding further substitutions.

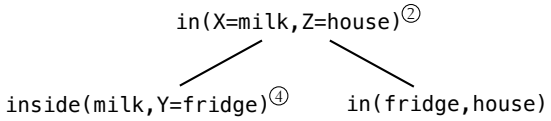
8.5.2 Scan, Expand, and Backtrack

With unification as the basic tool for finding rules that can satisfy a goal, we can now illustrate the Prolog search process in more detail. We represent the state of the search as a tree where each node is annotated either with a circled number ⓝ indicating the position of the pointer in the program or with a bullet •, indicating that all rules have been tested and that the goal cannot be satisfied, which means the goal has to be retracted in the next step. We also sometimes include the names of bound variables in goals for easier tracking. The current goal is always the leftmost leaf that contains an unsatisfied goal.

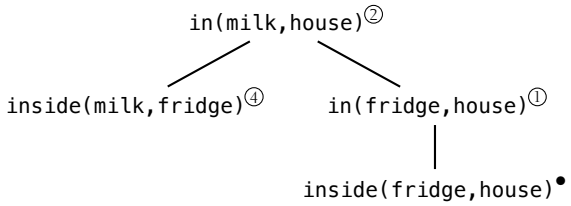
When we try to satisfy `in(milk,house)`, we first try the first rule for `in`, which leads to the subgoal `inside(milk,house)`, which cannot be satisfied by any of the `inside` facts. Thus, we are stuck at this point and have to retract this goal, which also means to advance the pointer for the `in` predicate to the next rule, which causes the creation of two new subgoals as children.



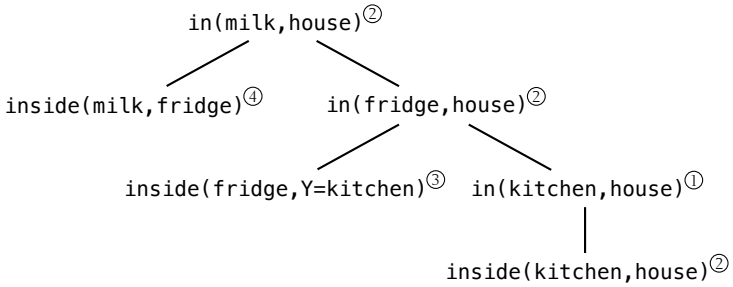
Next we need to satisfy the `inside(milk,Y)` goal. Scanning the `inside` facts we find a match for the fourth fact.



This creates the binding `Y=fridge` and instantiates `inside(fridge,house)` as the next subgoal to be satisfied. Starting with the first rule for `in` creates `inside(fridge,house)` as a new subgoal, which, however, cannot be satisifed.



Therefore, we have to backtrack again, retracting the `inside` goal as well as advancing the `in` goal to the second rule, which creates the two new subgoals `inside(fridge,kitchen)` and `in(kitchen,house)`. We find the third `inside` fact is a match for the first subgoal and start with the first rule for `in`, which creates `inside(kitchen,house)` as a new subgoal, which can finally be satisifed by the second `inside` fact.



At this point, all goals are satisfied, and the result can be reported. In this example, it is only the token `true`. For goals that contain variables, the generated bindings will be reported as well. For example, if we are wondering where the milk is, we could ask:

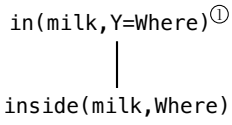
```
in(milk,Where).
```

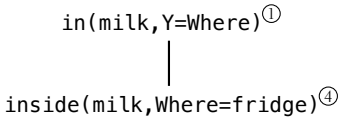
A difference to the previous goal is that we have passed the variable `Where` as an argument, which generates a binding of one variable to another one. This binding effectively establishes the two variables as synonyms for whatever value will be bound to either one. We can observe this behavior directly in the Prolog interpreter.

```
?- Y=Where, Y=fridge.
Y = Where, Where = fridge.
```

```
?- Y=Where, Where=fridge.
Y = Where, Where = fridge.
```

Therefore, we can consider `Y` and `Where` as aliases. Otherwise, the query is processed in much the same way as the previous one: The goal matches the head of the first `in` rule and generates the subgoal `inside(milk,Where)`, which can then be satisfied by the fourth `inside` fact.

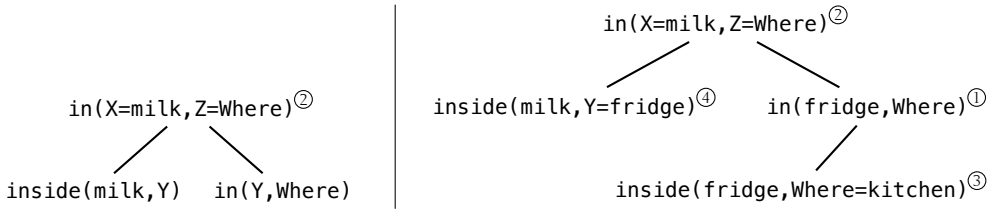




At this point, the binding that was generated for the variable provided in the goal will be reported as a result.

```
?- in(milk,Where).
Where = fridge
```

Rejecting the result by entering `;` and asking for a different solution, forces Prolog to backtrack. The effect is the removal of the `inside` goal as well as the binding for `Where`. Advancing the `in` goal to the second rule creates the two new subgoals `inside(milk,Y)` and `in(Y,Where)`. The first subgoal can again be satisfied by the fourth `inside` fact, which instantiates the second subgoal through the produced binding for `Y` and generates another `inside` subgoal by following the first `in` rule.



The third `inside` fact satisfies the last goal and produces the new binding for `Where` that is reported as a result.

```
?- in(milk,Where).
Where = fridge ;
Where = kitchen
```

The final result `house` can be produced in a similar way.

Exercise 8.8

Draw a sequence of trees that explain the computation of the results for the following queries.

(a) `in(What,kitchen)`

(b) `path(b,f)` (see Exercise 8.7)

8.6 Structures

A structure, or compound term, can represent a fact or be part of a rule, in which case it is interpreted as part of the definition of a relation. But structures can also represent complex data, such as person records, representations of geometric objects, trees, or arithmetic expressions.

```
person(name(jon,doe),dob(1,1,2000))

line((1,1),(3,4))
```

```
node(3,leaf(1),node(5,leaf(4),leaf(7)))

+(2,*(3,4))
```

Note that none of these terms is a fact, and they do *not* represent relationships but simply structured objects.[8] The functors (`person`, `name`, etc.) play a role that is similar to that of Haskell's data constructors. An important difference (apart from starting with a lowercase letter) is that functors are *untyped* and can be applied to arbitrary arguments. Therefore, they don't require introduction through a type definition either.

To be a part of a computation, compound terms will have to occur as part of predicates. The main use of compound terms is to provide a structured representation that can be exploited by queries. Assume, for example, that we have a program that represents facts about objects on a map.

```
map(road, line((1,1),(3,4))).
map(blvd, line((2,3),(2,4))).
map(river,line((1,2),(1,4))).
map(city, point(3,4)).
```

Now we could search the map, for example, for vertical objects with the following query.

```
?- map(Object,line((X,Y1),(X,Y2))).
Object = blvd,          |    Object = river,
X = 2,                  |    X = 1,
Y1 = 3,                 |    Y1 = 2,
Y2 = 4 ;                |    Y2 = 4.
```

The use of the functor `line` acts as a filter that picks out only facts with `line` objects as second arguments, which is similar to pattern matching against constructors in Haskell. Another example is a query to find line objects that end in a position where a point object can be found.

```
?- map(Line,line(_,(X,Y))), map(Point,point(X,Y)).
Line = road,
X = 3,
Y = 4,
Point = city
```

Note the use of the underscore, which is an anonymous variable, much like in Haskell, for which no bindings are generated. Here we have used it to simply ignore the coordinates of the line's first point.

[8] They could be turned into facts, though. For example, placing "`person(name(jon,doe),dob(1,1,2000)).`" (note the period at the end) into a Prolog program defines a binary predicate `person`; the functor `person` then becomes the name of a predicate (while the functors `name` and `dob` still only denote a term).

A more interesting application is to use terms as an abstract syntax representation, like we used data types in Haskell. Concretely, suppose we want to represent the same language as we did in Section 5.3 (see the data type definition on page 106) and implement a type checker for it. Consider, for example, the following program.

```
let x=3 in x+1
```

In Haskell we use constructors to represent it as follows.

```
Let "x" (Lit 3) (Plus (Var "x") (Lit 1))
```

In Prolog, we can represent it as a compound term in much the same way. The only differences are that we are using functors (that start with lowercase letters) instead of constructors and parentheses and commas to separate arguments.

```
let("x",lit(3),plus(var("x"),lit(1)))
```

While this representation works, it is not very readable. The simple and general structure of Prolog terms allows us to represent the syntax in a much more direct and less convoluted way. First, we can use atoms instead of strings to represent variables names. Second, since we can distinguish atoms from integers, we don't need to tag the latter with `lit`. Third, since arithmetic expressions (see Section 8.8) are represented in Prolog also as terms with symbolic operators as functors (that is, `x+1` is just syntactic sugar for `+(x,1)`), we can use them directly in the abstract syntax. Thus, we can represent the expression much more concisely with the following Prolog term.

```
let(x,3,x+1)
```

The types for our language can be represented by the two atoms `int` and `bool`.

As explained in Section 5.2.2, the definition of the typing rules rely on a ternary typing relation $\Gamma \vdash e :: T$ that says the type of an expression e is T under the type assumptions Γ.

In Prolog we can represent a type assumption as a list of variable/type pairs. For the typing rules we need only two operations on such lists: (1) extending a list (temporarily) by a new assumption, and (2) looking up the type of a variable. The latter can be achieved by the predicate `lookup/3` where `lookup(X,G,T)` holds if `G` contains the pair `(X,T)`. (We will define `lookup` in Section 8.7.) We have encountered the notation for extending a list with new elements already in Section 8.5.1: To extend `G` with a pair `(X,T)` we write `[(X,T)|G]`.

To implement the type system all we need to do now is to translate the visual inference rules into Prolog syntax. To that end we can represent the typing relation $\Gamma \vdash e :: T$ by a predicate `type/3` so that `type(G,E,T)` holds if `E` has type `T` given the assumptions `G`.

Recall the general form of inference rules from Section 5.1.

$$\frac{\mathcal{P}_1 \qquad \ldots \qquad \mathcal{P}_n}{\mathcal{C}}$$

Such an inference rule can be directly translated into a Prolog rule where the conclusion is represented by the head and the premises are combined using conjunction to define the body.

$\mathcal{C}$ `:-` $\mathcal{P}_1$`, ...,` $\mathcal{P}_n$

Consider again the typing rule for `let` expressions, which has two premises.

$$\textsc{Let}\ \frac{\Gamma \vdash e_1 :: U \qquad \Gamma, x :: U \vdash e_2 :: T}{\Gamma \vdash \texttt{let}\ x{=}e_1\ \texttt{in}\ e_2 :: T}$$

Using the `type/3` predicate, this rule therefore translates into the following Prolog rule.

```
type(G,let(X,E1,E2),T) :- type(G,E1,U), type([(X,U)|G],E2,T).
```

The complete type system is shown in Figure 8.5. Note that the last rule is for explicitly assigning a type error to type-incorrect expressions. We don't necessarily need such a rule, because if we omit it, type-incorrect expressions will simply fail the type checker. For example, with the last rule we get the following.

```
?- type([],not(2),T).
T = error.
```

In contrast, without the last rule the typing predicate simply fails for the goal.

```
?- type([],not(2),T).
false.
```

It is generally a good programming practice to distinguish error values of the application (in this case type errors) from runtime errors that result from failure to apply rules.

Compared with the Haskell implementation (see Figure 5.1), the Prolog implementation is shorter and provides a more direct realization of the typing rules. The Haskell implementation is more verbose for two main reasons. First, Haskell uses a typed representation of the abstract syntax, whereas in Prolog we can reuse much of Prolog's own syntax (that is, the syntax of the metalanguage) as the syntax for the object language. Second, in the Haskell implementation type errors are distinguished from proper types (`Just`

```
type(G,E1 + E2,int)    :- type(G,E1,int), type(G,E2,int).
type(G,E1 = E2,bool)   :- type(G,E1,T), type(G,E2,T).
type(G,not(E),bool)    :- type(G,E,bool).
type(G,let(X,E1,E2),T) :- type(G,E1,U), type([(X,U)|G],E2,T).
type(G,X,T)            :- lookup(X,G,T).
type(_,I,int)          :- integer(I).
type(_,_,error).
```

Figure 8.5 The type system from Figure 5.1 expressed in Prolog. The typing rules for the judgment $\Gamma \vdash e :: T$ are implemented by Prolog rules defining the predicate `type(G,E,T)`. The built-in predicate `integer/1` is true for integer objects only.

vs. `Nothing`), which causes some overhead in managing the typing of subexpressions. This could be avoided to some degree by representing a type error by a constructor `Error` that is part of the type `Type`, but the resulting code would be more intricate and arguably harder to follow.

In contrast, while the Prolog implementation is very succinct and is very close to the formal definition by inference rules, it can behave in unexpected ways and is also prone to errors. Consider the following example.

```
?- type([],let(x,1,let(x,2=3,x)),T).
T = bool
```

As expected, the result type is `bool`, the type of the `x` defined in the inner `let` expression. But curiously, Prolog offers us to enter `;` with the prospect of another result, and if we take up that offer, we find, surprisingly, the following answer.

```
?- type([],let(x,1,let(x,2=3,x)),T).
T = bool ;
T = int.
```

That seems to suggest that `x` can have both types `bool` *and* `int`, but that should not be the case according to the typing rules, since the binding for the inner `let` expression hides the outer one. The reason for this behavior is that backtracking allows Prolog to find alternative answers (in this case through the hidden binding in the type assumption). But we can get conflicting answers in even the most basic cases, as the following example illustrates.

```
?- type([],2,T).
T = int ;
T = error.
```

This seems to say that the type of `2` is `int`, which is correct, but also that it is a type error,

which is definitely not the case. In this example, it is easy to see what's going on: First, Prolog identifies the second-to-last rule for `type` to show that the type of `2` is `int`. But when forced by `;` to backtrack, the search advances to the next rule, which succeeds and yields `error` as another solution. Backtracking allows the catch-all case to be reached, even when another case was executed already.

This behavior can be prevented by using `!`, the so-called *cut*, which is a predicate that succeeds, but prevents backtracking when reached. Concretely, if we change the second-to-last rule as follows, the last rule cannot be reached after the `int` rule has fired.

```
type(_,I,int) :- integer(I), !.
```

The result is as expected.

```
?- type([],2,T).
T = int.
```

We will discuss the cut in Section 8.9.

And to illustrate how the lack of typing in Prolog facilitates semantic errors, consider the following example.

```
?- type([],x+2,T).
T = error.
```

Just as we expect, `x+2` is type incorrect, since `x` is undefined. But suppose we mistyped `int` in the body of the first rule and used `Int` instead, like so.

```
type(G,E1 + E2,int) :- type(G,E1,Int), type(G,E2,Int).
```

Now we can observe the following strange behavior.

```
?- type([],x+2,T).
T = int
```

Why is `x+2` all of a sudden considered to be of type `int`? Since `Int` is a Prolog variable, the first subgoal of the changed rule, `type([],x,Int)`, will succeed, creating the binding `Int=error`, since it can be satisfied with the last rule. The second subgoal then instantiates to `type([],2,error)`, which also succeeds due to the last rule, which means the query `type([],x+2,T)` succeeds binding `int` to `T`.

Exercise 8.9

Why doesn't adding a cut to the second-to-last rule change this behavior? Shouldn't in that case the second subgoal be typed to be `int` and the last rule not be reached? Apparently, this reasoning is wrong; the last rule is reached despite a cut. Why is that?

Note that the typo in the program does *not* lead to a compiler or even runtime error but to a semantic error that might be very difficult to find. These kind of programming mistakes are caught by type systems and could not happen in Haskell, which illustrates the trade-off between succinctness and programming convenience on the one hand versus more correctness guarantees on the other.

Exercise 8.10

This exercise is a continuation of Exercise 5.5. Extend the Prolog program in Figure 8.5 by rules for typing (potentially nested) pairs of expressions. Add rules for the construction of pairs (such as `(2,2=3)`) and the operations `swap` and `fst`. Here are several test cases.

```
type([],(2,2=3),(int,bool)).
type([],let(x,1,let(y,2=3,(x,y))),(int,bool)).
type([],swap(swap((2,2=3))),(int,bool)).
type([],fst(swap((2,2=3))),bool).
type([],swap(fst(((2,2=3),3))),(bool,int)).
```

8.7 Lists

Like in Haskell, lists are probably the most important and widely used data structures in Prolog as well. Moreover, as Haskell lists are just ordinary data types with special syntax, Prolog lists are ordinary terms with special syntax.

As in Haskell, the empty list in Prolog is represented by the atom `[]`. The role of the cons constructor `:` is played in Prolog generally by the dot functor `.`, however, SWI-Prolog uses the functor `'[|]'`,[9] so that the inductive representation of the list `[3,5]` is written in Haskell, Prolog, and SWI-Prolog, respectively as follows.

```
> 3:5:[]
[3,5]
```

```
?- L = .(3,.(5,[])).
L = [3, 5].
```

```
?- L = '[|]'(3,'[|]'(5,[])).
L = [3, 5].
```

Luckily, we almost never construct or deconstruct lists using the dot functor (or SWI-Prolog's version), since Prolog provides a generalized form of cons pattern that is expressive and convenient to use. The syntax has a list of elements separated by the bar symbol `|` from the rest list.

list ::= `[`*term* `,` ... `,` *term* `|` *term*`]`

To illustrate this notation, here are several alternatives to denote the list `[3,4,5]`.

[9] For an explanation of this design, see https://www.swi-prolog.org/pldoc/man?section=ext-lists.

```
[3,4,5]  |  [3|[4,5]]  |  [3,4|[5]]  |  [3,4,5|[]]
```

Note that `[3,4|5]` is a valid Prolog term, but it is different from `[3,4,5]`. In fact, it is not a proper list, since it doesn't end with an empty list.

Since Prolog is untyped, lists don't have to be homogeneous, and we can form lists of atoms and numbers as well as nested lists.

```
[4,privet_drive]  |  [1,(2,2),[1,(2,2)]]  |  [[],1,[[]],2,[[[]]],3]
```

Exercise 8.11

Consider the following fact.

```
book([george,orwell],[nineteen,eighty,four]).
```

Create goals that produce exactly the following results. Note that you can use the wildcard symbol _ to match terms without producing a binding.

(a)
```
Author = orwell,
Book = [nineteen, eighty, four].
```

(b)
```
Century = nineteen.
```

(c)
```
Year = [eighty, four].
```

(d)
```
X = nineteen,
Y = eighty,
Z = four.
```

Note: Try to express each query *twice*: using a list pattern with and without a bar. For one query this does *not* work. Which one?

One of the most basic predicates on lists is `member/2`, which takes an element and a list and returns `true` if the element is contained in the list. The predicate is defined as follows.

```
member(X,[X|_]).
member(X,[_|XS]) :- member(X,XS).
```

The first rule expresses the fact that the first element in a list is a member of that list, and the second rule says that an element can be also a member of a list if it is a member of the list's tail. Note that we don't need a base case for empty lists, because `member` should always fail for empty lists, which is the behavior when no rule is found.

We can use the `member` predicate in *three* different ways. First of all, we can of course test the membership of specific list elements.

```
?- member(3,[2,3,4,3]).
true ;
true.
```

The response shows that the goal can be satisfied *twice*, which is not surprising, because `3` occurs twice in the list, that is, when forcing Prolog to retract the first answer and continue to search for another evidence, it will find it in the last list element. When we formulate a `member` goal with an element that occurs only once in the list, only one result is produced.

```
?- member(4,[2,3,4,3]).
true ;
false.
```

Exercise 8.12

Change the definition of the predicate `member` so that it only produces at most answer.

Hint: Think about a condition that constrains the second rule: When should it *not* be used?

So far we have used Prolog almost exclusively to derive information from relations represented by predicates. The `member` predicate is different in that it computes information for input data. It is therefore the right time to highlight an important relationship between predicates and their corresponding functions in imperative or functional languages:

> A function of n arguments with a result type that is *not* `Bool` is represented in Prolog by a predicate of of $n + 1$ arguments.
>
> A function of n arguments with result type `Bool` is represented in Prolog by a predicate of of n arguments.

The reason for the second special case is that a `Bool` result type of a function is not needed for a corresponding Prolog predicate, because it is captured by success and failure of the predicate. This is why the function `succ :: Int -> Int` would be represented by a binary predicate `succ/2`, whereas a function `even :: Int -> Bool` would be represented by a unary predicate `even/1`. And it explains why `member` is a binary predicate, whereas the corresponding Haskell function has two arguments and a `Bool` result.

The shown usage scenario for `member`, providing two arguments and getting a boolean as a result, is the only one that Haskell offers through the function `elem`. In contrast, Prolog offers two more uses. For example, we can employ a variable as the first argument to enumerate elements from the list. To make the example a bit more interesting, let's add a condition to the elements.

```
?- member(X,[2,3,4,3]), X>2.
X = 3 ;
X = 4 ;
X = 3.
```

As illustrated in Section 8.2.2, instead of interactively enumerating the results, we can use `findall` to collect all results in a list.

```
?- findall(X, (member(X,[2,3,4,3]),X>2), L).
L = [3,4,3].
```

Finally, we can also employ a variable for the list argument (and provide an example element that should be in the list). This will lead to the following results.[10]

```
?- member(3,L).
L = [3|A] ;
L = [A,3|B] ;
L = [A,B,3|C]
```

To understand the results, we first have to understand the query, which asks, *What list is* `3` *a member of*? Prolog's first answer is a pattern that matches any non-empty list with `3` as its first element and an arbitrary tail. Prolog's second answer matches any list of two or more elements with `3` as its second element and an arbitrary first element and tail, and the third answer describes lists of at least three elements that have `3` as their third element.

How does Prolog produce these answers? The search starts by matching the goal `member(3,L)` against the first rule for `member`, which produces the binding ⟨`X=3`, `L=[3|A]`⟩ where the binding for `X` is reused in the one created for `L`. Also, Prolog always generates fresh variable names as a representation for the underscore (in this instance, `A`). With the substitutions for `X` and `L`, the goal can be satisfied using the first rule, which produces the first answer. This state of the search is captured by the search tree consisting of just one node shown in Figure 8.6 on the left.

When we force to retract the result, Prolog tries the second rule, which creates the bindings ⟨`X=3`, `L=[A|XS]`⟩ in the head of the rule and then uses it to create the subgoal `member(3,XS)` as required by the body of the rule. This subgoal can be satisfied by the first rule for `member`, which is solved like before and produces the additional bindings ⟨`X=3`, `XS=[3|B]`⟩. If we substitute the thus generated binding for `XS` in the binding for `L`, we obtain the second answer. This state of the search is represented by the search tree in the middle of Figure 8.6.

Note that each recursive instance of a predicate that is created as a subgoal has its own

[10] Instead of `A`, `B`, and `C`, `swipl` will generate numbered variable names that start with an underscore.

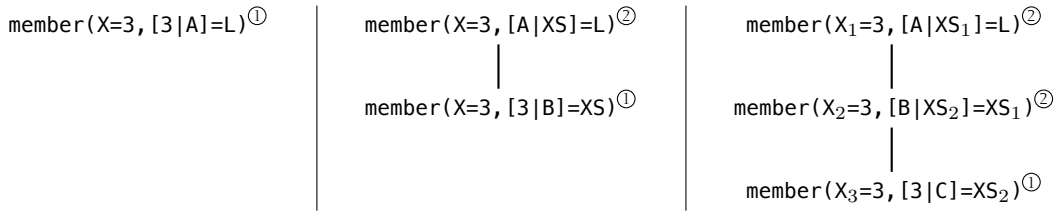


Figure 8.6 Search trees for the first three solutions for the goal `member(3,L)`.

set of variable bindings. In this second step we actually have two bindings for `X`, which we don't have to distinguish, since they don't differ. To see that is generally *does* matter to distinguish the variable bindings in different rule instances, consider how the solution for the third answer is generated. In this case backtracking revokes the latest instance of the first rule and tries to solve the subgoal `member(3,XS)` using the second rule, which, in particular, passes `XS` as the second argument to be unified with the term `[_,XS]`. But this latter variable `XS` refers to a different tail than the previous one. Therefore, to distinguish the different variable instances, we index them with the level of the search tree from which they originate, as shown in the search tree in Figure 8.6 on the right. The next result is computed in the same way as before, except with a new set of bindings. In particular, we now match $\texttt{XS}_1$ against $\texttt{[B|XS}_2\texttt{]}$ in the head of the second rule and then match $\texttt{XS}_2$ against `[3|C]` in the body. Altogether this produces three bindings for `X` plus the bindings $\langle \texttt{L=[A|XS}_1\texttt{]}, \texttt{XS}_1\texttt{=[B|XS}_2\texttt{]}, \texttt{XS}_2\texttt{=[3|C]}\rangle$. Substituting the binding for $\texttt{XS}_2$ in the binding for $\texttt{XS}_1$ produces $\texttt{XS}_1\texttt{=[B,3|C]}$ (note that in this notation the bar has to "shift over" the single element `3` to properly separate the single elements from the list), and then substituting this binding in the binding for `L` yields `L=[A,B,3|C]` as the third result.

Exercise 8.13

Consider the following goal.

```
member(3,L), member(4,L).
```

Predict the first three answers. Then test your prediction, and explain the output produced by Prolog.

In Section 8.6 we have used the predicate `lookup/3` in the definition of the typing rules. The definition of `lookup` is very similar to that of `member`. The main difference is that the list to be searched must contain pairs, and the condition for a successful lookup is that the searched element matches the first component of the pair, in which case the second

component is "returned" as the result value.

```
lookup(X,[(X,T)|_],T).
lookup(X,[_|G],T) :- lookup(X,G,T).
```

I have put "returned" in quotes because, as we have just seen with `member`, Prolog predicates do not generally have dedicated input and output parameters, that is, we can very well use `lookup` with a result value as input. For example, we can use `lookup` not only to find values using keys but also ask, *What key do I have to use to find* `3` *in the list* `[(a,2),(b,3)]`?

```
?- lookup(X,[(a,2),(b,3)],3).
X = b
```

And we can even leave the list unspecified.

```
?- lookup(X,L,3).
L = [(X,3)|A] ;
L = [A,(X,3)|B] ;
L = [A,B,(X,3)|C].
```

While this use case is somewhat contrived, it shows that Prolog predicates generally can produce many answers if parameters are underspecified, which can be a feature, but can also sometimes produce unwanted or even incorrect results (as discussed in Section 8.6).

Exercise 8.14

Define predicate `del/3`, such that `del(X,L1,L2)` holds when `L2` is equal to the list `L1` with the first occurrence of `X` removed.

Consider *all* results produced for the goal `del(a,[a,b],L)`. If your predicate definition leads to more than one result, explain why, and then add a suitable condition so that only one result is produced.

Another frequently used list predicate that can be elegantly used in a relational way is `append/3`, defined as follows.

```
append([],L,L).
append([X|L1],L2,[X|L3]) :- append(L1,L2,L3).
```

The first rule says that an empty list appended to any list `L` is the same as `L`. The second rule says that when we append a non-empty with first element `X` to another list, the resulting list starts with `X`. Moreover the tail of the result list is obtained by appending the tail of the first list to the second list.

Again, we can use `append` in different ways. The simplest case is the functional use for appending two lists.

```
?- append([2,3],[a,[c],d],L).
L = [2,3,a,[c],d].
```

More interestingly, we can use the predicate to solve list concatenation equations. For example, we can ask, *What list do I have to append to* `[2,3]` *to get the list* `[2,3,a,[c],d]`?

```
?- append([2,3],L,[2,3,a,[c],d]).
L = [a,[c],d].
```

This works similarly for the first parameter. Providing a value for the third ("result") parameter and using a variable for an "argument" parameter effectively allows us to compute inverse functions. And we get those for free!

But there is more. For example, by leaving both argument parameters as variables, we can ask for all decompositions of a list.[11]

```
?- findall(X+Y,append(X,Y,[3,4,5]),L).
L = [[]+[3,4,5], [3]+[4,5], [3,4]+[5], [3,4,5]+[]].
```

We can observe here a fundamental strength of logic programming: Whereas programs in imperative and functional programming languages are primarily about deriving output from input, logic programming can additionally use the same programs to generate data in a systematic way as input for further computation. This aspect shows up especially in applications of Prolog to solve puzzles. It can also be exploited in simpler scenarios, as illustrated by the next exercise.

Exercise 8.15

Define a predicate `sublist/2`, such that `sublist(S,L)` holds if the list `S` is a sublist of `L`, which is the case when `L` could be obtained by adding a prefix and suffix to `S`.

Hint: `S` is a sublist of `L` if `L` can be split into `Prefix` and `Rest`, where `Rest` can be split into `S` and `Suffix`. Use `append` to express the decomposition conditions.

8.8 Numbers and Arithmetic

Numbers and numerical expressions enjoy special treatment in Prolog. We have already seen in Section 8.6 that an expression such as `2+3*4` is simply a term (whose functors are operators). This is easy to verify.

[11] Note that I am using `+` here simply as a functor to build terms of two lists that are a bit easier to read than pairs. Other symbols that Prolog recognoizes as operators such as `-` or `^` work as well.

```
?- 2+3*4 = +(2,*(3,4)).
true.
```

While we can directly evaluate such an expression in Haskell, trying to evaluate it in `swipl` leads to an error that complains that `+` is not a defined predicate.

```
?- 2+3*4.
ERROR: Undefined procedure: (+)/2
```

To evaluate numerical expressions, Prolog provides a built-in predicate `is/2`, which is typically used in infix notation and whose first argument is a variable that will be bound to the result of evaluating the second argument.

```
?- X is 2+3*4.
X = 14.
```

Note that to succeed the second argument must be instantiated, that is, if it contains variables, they must be bound to an numerical term.

```
?- Y=3*4, X is 2+Y.
Y = 3*4,
X = 14.
```

```
?- Y is 3*4, X is 2+Y.
Y = 12,
X = 14.
```

```
?- Z=3, Y=Z*4, X is 2+Y.
Z = 3,
Y = 3*4,
X = 14.
```

Note that without the binding for `Y` as a first subgoal, the goal would fail. Moreover, the binding for `Y` must exist *before* it is used in the second argument of `is`. The variations of the example also show the crucial difference between `=` and `is`, namely `=` unifies terms (recall Section 8.5.1) while `is` evaluates expressions.

With `is` we can now implement predicates that perform arithmetic computations, for example, for computing squares.

```
sqr(X,Y) :- Y is X*X.
```

Note that the use of `is` in the definition of `sqr` restricts its use: We can supply only instantiated terms as the first argument. It is possible to supply a number for `Y` in which case `is` checks whether its second argument evaluates to that number, but a term passed as an argument for `Y` will not be evaluated.

```
?- sqr(3,Y).
Y = 9.
```

```
?- sqr(3+1,Y).
Y = 16.
```

```
?- sqr(3,9).
true.
```

```
?- sqr(3,8+1).
false.
```

In particular, since `is` works only in one direction and cannot be used to invert computations, we *cannot* us it to solve equations.

```
?- sqr(X,9).
ERROR: Arguments are not sufficiently instantiated
```

We are now ready to define a predicate for computing factorial numbers. Recall the inference rules from Section 5.1.

$$\frac{}{fac(1,1)} \qquad \frac{fac(n-1,m)}{fac(n,n\cdot m)}$$

While Prolog rules are very similar to inference rules, we have to be careful when dealing with arithmetic and ensure that arguments to `is` are sufficiently instantiated.

```
fac(1,1).
fac(N,F) :- K is N-1, fac(K,M), F is N*M.
```

Note that the order of the sobgoals in the second rule is fixed by the dependency of the involved evaluations. The definition works as expected, but, again, the predicate can be used in goals only in one direction.

```
?- fac(4,F).
F = 24 .
```

```
?- fac(X,24).
ERROR: Arguments are not sufficiently instantiated
```

Exercise 8.16

Explain what is wrong with each of the following definitions. First, try to predict what happens when you try to evaluate the goal `fac(3,F)`. Then explain how exactly they go wrong.

(a) `fac(N,F) :- fac(N-1,M), F is N*M.`

(b) `fac(N,N*M) :- K is N-1, fac(K,M).`

(c) `fac(N,N*M) :- fac(N-1,M).`

Exercise 8.17

Define a predicate `length/2`, such that `N` in `length(L,N)` is equal to the length of the list `L`.

8.9 The Cut

We have already briefly encountered the *cut* `!` in Section 8.6 where we have used it to fine-tune a predicate definition to produce *fewer* results. In general, Prolog's ability to satisfy a

goal repeatedly, by using different rules, is a powerful feature that nicely supports the programming of search problems. This behavior is implemented using a general backtracking mechanism that allows Prolog to systematically iterate over all alternatives of a predicate definition.

The cut provides a mechanism to modify this standard search algorithm. Concretely, the cut prevents backtracking and thus the re-satisfying of a subgoal. As an example, consider the following definition of a predicate `p` that is true for the first three natural numbers. The evaluation, especially, the re-satisfying of goals, should not be surprising: After rejecting the first solution `X=1`, Prolog presents the second `X=2`, and then the third `X=3` one. If we add the cut as a subgoal to the first rule, Prolog cannot retract the goal that came before, which means only one answer is computed for the goal `p(X)`. If we add the cut to the second rule, Prolog can retract a goal satisfied by the first but not second rule.

```
p(X) :- X=1.
p(X) :- X=2.
p(X) :- X=3.

?- findall(X,p(X),L).
L = [1,2,3].
```

```
p(X) :- X=1, !.
p(X) :- X=2.
p(X) :- X=3.

?- findall(X,p(X),L).
L = [1].
```

```
p(X) :- X=1.
p(X) :- X=2, !.
p(X) :- X=3.

?- findall(X,p(X),L).
L = [1,2].
```

The example is somewhat contrived, but it illustrates how the cut works. However, the effect of using a cut is not always so obvious. For example, with the definition of `member` shown earlier in Section 8.7, the goal `member(3,[3,3,3,3])` can be satisfied four times. If we add a cut to the first rule, we again only get one solutions. This version is more efficient than the more general definition for simply determining list membership, since it terminates computation as as soon as a solution is found.[12] This version, however, also generates only one solution for the goal `member(3,L)`.

When we add the cut to the second rule, goals containing `member` can be satisfied twice, similar to what happened with `p` earlier.

```
?- member(3,[3,3,3,3]).
true ;
true.
```

```
?- member(3,L).
L = [3|A] ;
L = [A,3|B].
```

But what happens with a goal such as `member(3,L), member(4,L).`?

[12] This predicate is predefined in SWI-Prolog and is called `memberchk`.

```
?- member(3,L), member(4,L).
L = [3,4|A] ;
L = [4,3|A] ;
L = [A,3,4|B].
```

This behavior is not all that obvious and shows that reasoning about goals that employ predicates whose definition use a cut requires careful thinking about Prolog's execution mechanism.

The cut can also easily lead to inconsistent predicate definitions. Consider, for example, the following attempt to define `xor` on the numbers `0` and `1`.

```
xor(X,X) :- !, false.
xor(0,1).
xor(1,0).
```

Clearly, the goal `xor(0,0)` should fail, and `xor(0,1)` should succeed. And they do, but the queries `xor(0,X)` or `xor(X,0)` both fail, which is inconsistent with the two facts.

```
?- xor(0,0).
false.
```

```
?- xor(0,1).
true.
```

```
?- xor(0,X).
false.
```

```
?- xor(X,0).
false.
```

Exercise 8.18

Find *two* different ways to fix the definition of predicate `xor` (whithout eliminating the cut).

The cut is used for efficiency and to make computations of Prolog programs (more) deterministic. While powerful and in certain situations indispensable, the cut is ultimately a non-logical, imperative feature that is not easy to use. The cut leaves a somewhat bitter aftertaste, since it abandons the otherwise elegant logic computation model. The bottom

line is that the cut should be used very carefully and deliberately.

Exercise 8.19

Consider the following predicate definition.

```
q(a).
q(b) :- !.
q(c).
```

What answers does Prolog produce for the following queries?

(a) `q(X).`

(b) `q(X), q(Y).`

(c) `q(X), !, q(Y).`

8.10 Negation

Negation is a critical operation in logic, and it can be employed to make Prolog predicate definitions more precise and eliminate unnecessary duplicates.

Consider again the predicate `friends` from Exercise 8.6, which says that two people are friends if they are buddies or foodies. A straightforward definition of the predicate provides the two alternatives as separate rules.

```
friends(P1,P2) :- buddies(P1,P2).
friends(P1,P2) :- foodies(P1,P2).
```

There is nothing wrong with this definition per se, but it might not be ideal, because it can provide the same solution several times. In fact, with the facts from Figure 8.3, Alice and Carol are mentioned twice as a result.[13]

```
?- findall(P1+P2,friends(P1,P2),L).
L = [alice+carol, alice+bob, alice+carol].
```

Maybe we can avoid the duplication by using the cut? We could add a cut at the end of the first rule with the goal of preventing a pair that has been produced using the `buddies` predicate to be again produced by `foodies`. However, if we try this, we may be surprised about the result.

[13] Remember that + is a functor that can be used to build terms. Sometimes the use of infix operators such as + leads to easier to read goals and results.

```
friends(P1,P2) :- buddies(P1,P2), !.
friends(P1,P2) :- foodies(P1,P2).

?- findall(P1+P2,friends(P1,P2),L).
L = [alice+carol].
```

While the duplicate was eliminated, we have also lost, incorrectly, the pair Alice and Bob (proving again that the cut is a tricky programming construct).

A correct way of defining the `friends` predicate is to spell out the three different, non-overlapping cases, which requires the use of `not` to express the requirement that a specific relationship does *not* hold.

```
friends(P1,P2) :- buddies(P1,P2), not(foodies(P1,P2)).
friends(P1,P2) :- foodies(P1,P2), not(buddies(P1,P2)).
friends(P1,P2) :- foodies(P1,P2), buddies(P1,P2).
```

With this definition each pair of friends is produced by a separate case, and we obtain the desired result.

```
?- findall(P1+P2,friendsB(P1,P2),L).
L = [alice+bob, alice+carol].
```

The negation predicate `not` is actually defined in terms of the cut. The trick is to say that if a predicate `P` succeeds, then `not(P)` should definitely fail. Otherwise, that is, if `P` fails and with it the first rule of `not(P)`, then `not(P)` should succeed. The cut in the first rule prevents `not(P)` from succeeding through the second rule when `P` is true.

```
not(P) :- P, !, fail.
not(P).
```

This implementation of negation through failure implies what is called the *closed-world assumption*, which means that Prolog considers as true only relationships that can be derived from the predicates defined in the currently active program and correspondingly considers everything else to be false. Consider, for example, the evaluation of the following query.

```
?- sport(football).
false.
```

The negative answer should be always qualified and implicitly prefixed by "according to the known facts."

While `not` is a convenient programming tool, it also has to be deployed carefully. We'll consider two examples to illustrate this point. For example, we could be tempted to try to solve the barber's paradox with Prolog. The paradox is given by the following riddle.

> *In a town, the barber (who is male) shaves all males who do not shave themselves. Does the barber shave himself?*

Here is a straightforward representation of the information through two predicates `shaves` and `male`.

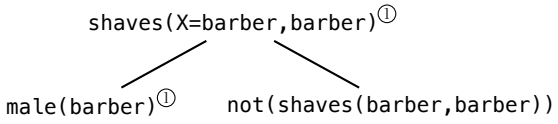
```
shaves(barber,X) :- male(X), not(shaves(X,X)).

male(barber).
```

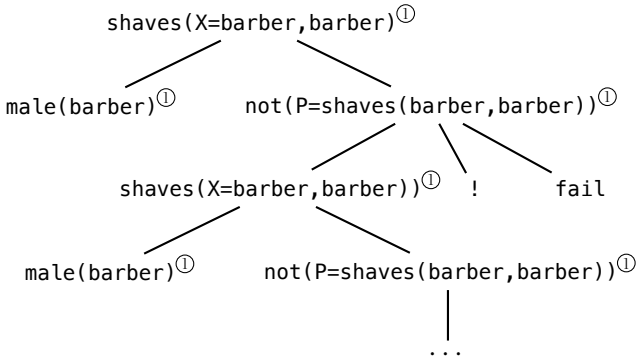
A solution would be given by an answer to the following goal.

```
?- shaves(barber,barber).
```

But as you can imagine, Prolog is unable to come up with an answer. It will enter an infinite recursion that will cause the interpreter to eventually run out of stack space. We can employ our tree model to understand how the goal satisfaction unfolds. First, the goal `shaves(barber,barber)` leads to the subgoal `not(shaves(barber,barber))`.



Second, the first rule for `not` applies, which creates three new subgoals, the first of which is the original goal—a case of left recursion.



Since Prolog satisfies subgoals from left to right, the recursion continues, and Prolog tries to solve `shaves(barber,barber)` again. Thus, we will never reach the cut or `fail`, and the goal

cannot be satisfied. This example is an instance of recursion without a base case, which has no well-defined semantics. The implication for philosophical puzzles in general is that we can't compute our way out of paradoxes.

A final problem with the treatment of negation in Prolog is the interaction with the quantification of variables. To illustrate the issue, suppose we want to find out what hobbies people have that are not considered a sport. We can express this query with the following goal.

```
?- hobby(_,H), not(sport(H)).
H = reading ;
H = reading ;
H = chess.
```

The result is as expected. But consider what happens if we reorder the goals to ask basically the same question, namely, *Which of the non-sports are hobbies that people have*?

```
?- not(sport(H)), hobby(_,H).
false.
```

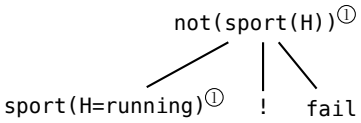
This result is surprising. Why do we not get any result? The comma represents the logical *and* operation, which is commutative. Therefore, we might expect the same result as before.

The different behavior is a consequence of the fact that free variables are exitentially quantified in Prolog. For example, the query `hobby(_,H)` is interpreted as ∃`H.hobby(_,H)`. The bindings for `H` are generated by Prolog's search mechanism with values taken from the predicate `hobby`. The additional condition `not(sport(H))` then acts as a filter on the generated bindings, and the logical formula becomes the following.

$$\exists \texttt{H.hobby(\_,H)} \wedge \texttt{not(sport(H))}$$

The crucial aspect is that at the time `not` is evaluated, the referenced free variable `H` is already bound.

When we change the order of the goals, Prolog first tries to satisfy `not(sport(H))`, which, following the first rule for `not`, leads to the three subgoals `sport(H)`, the cut, and `fail`. The first subgoal succeeds (with the binding `H=running`), which means that the cut succeeds. This leads to `fail`, which causes the failure of `not(sport(H))` and the whole goal.



The problem is that since `H` is unbound when `not` is evaluated, it causes `H` to be bound to a value for which `sport` succeeds, which then leads `not` to fail. It is similar to the following query.

```
?- not(sport(running)), hobby(_,running).
false.
```

By choosing `running` as an argument for `sport`, `not` and the whole query is bound to fail. If we magically could guess a value for which `sport` is false but that is part of `hobby`, the goals could succeed, as in the following example.

```
?- not(sport(chess)), hobby(_,chess).
true.
```

But that is not how Prolog works. By filling the unbound variable `H` with values from `sport`, Prolog effectively treats the query with the existential quantifier inside of the `not` predicate.

$$\texttt{not}(\exists\, \texttt{H.sport(H)}) \wedge \texttt{hobby(\_,H)}$$

The first conjunct is clearly false, since there exists an `H` such that `sport(H)` is true. Since $\neg(\exists x.P(x)) \equiv \forall x.\neg P(x)$, we can also move the quantifier to the outside, and then the query logically amounts to the following goal.

$$\forall\, \texttt{H.not(sport(H))} \wedge \texttt{hobby(\_,H)}$$

Again the first conjunct is obviously not true, since not all values are not a sport.

The bottom line is that, like the cut, negation has to be employed carefully, and it is a good strategy to reorder subgoals so that `not` is only applied to goals whose variables are already bound.