Zeroth-order Frank-Wolfe Variants for Matrix Completion

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2. Algorithms
3. Low Rank Matrix Completion
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The project

- Low rank matrix completion for recommender systems
- Zeroth-order variants of the Frank-Wolfe method

Frank-Wolfe

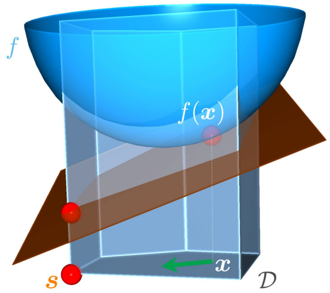
- Marguerite Frank and Philip Wolfe (1956)
- Iterative first-order optimization algorithm for problems with linear constraints:
$$\min_{x \in C} f(x)$$
- $f : \mathbb{R}^n \to \mathbb{R}$ convex and LCG with constant $L > 0$
- $C \subseteq \mathbb{R}^n$ convex and compact with diameter D

Idea

- Find a descent direction on the linear approximation of f in x_k:

$$\min_{x \in C} f(x_k) + \nabla f(x_k)^\top (x - x_k)$$

- Update the iterate
- No need to use projection



Pseudo-code

Algorithm 1 `Frank-Wolfe method`

```
Choose a point $x_1 \in C$
For $k = 1, \dots$
    Set $\hat{x}_k = \operatorname*{Argmin}_{x \in C} \nabla f(x_k)^\top (x - x_k)$
    If $\hat{x}_k$ satisfies some specific condition, then STOP
    Set $x_{k+1} = x_k + \alpha_k(\hat{x}_k - x_k)$, with $\alpha_k \in (0, 1]$
    suitably chosen stepsize
End for
```

Zeroth order

- Real-world application:
  - No access to the gradient
  - Slow to compute
- Zeroth order algorithms:
  - Use only function queries to approximate the gradient
  - Maintain good convergence rate, sometimes even better

Our work

- Convex case:
    - Deterministic Zeroth-order Frank-Wolfe (DZFW)
    - Stochastic Gradient Free Frank-Wolfe (3 variants)
    - First Order Frank-Wolfe (FOFW)
- Non-Convex case:
    - Stochastic Gradient Free Frank-Wolfe (SGFFW - I-RDSA)
    - Faster Zeroth-order Frank Wolfe (FZFW)
    - Faster Zeroth-order Conditional Gradient Sliding (FZCGS)

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SGFFW - Stochastic Gradient Free Frank-Wolfe

- Described by Kar, Sahu and Zaheer in *Towards Gradient Free and Projection Free Stochastic Optimization*
- Every iteration involves m oracle calls
- The estimation of the gradient keeps trace of information coming from previous estimations

SGFFW algorithm

Algorithm 1 Stochastic Gradient Free Frank-Wolfe (SGFFW – I-RDSA)

Require: Input, Loss Function $F(x)$, Convex Set C, number of directions m, sequences $\gamma = \frac{2}{T^{3/4}}$,

$$(p_t, c_t) = \left(\frac{4}{1 + \frac{d}{m}^{1/3}(t+8)^{2/3}}, \frac{2\sqrt{m}}{d^{3/2}(t+8)^{1/3}} \right)$$

Output: $\mathbf{x}_T$

1: Initialize $\mathbf{x}_0 \in C$

2: **for do**

3: Compute
Sample $\{\mathbf{z}_{i,t}\}_{i=1}^{m} \sim \mathcal{N}(0, \mathbf{I}_d)$,
$\mathbf{g}(\mathbf{x}_t, \mathbf{y}) = \sum_{i=1}^{d} \frac{F(\mathbf{x}_t + c_t \mathbf{z}_t; \mathbf{y}) - F(\mathbf{x}_t, \mathbf{y})}{c_t} \mathbf{z}_{i,t}$

4: Compute $\mathbf{d}_t = (1 - \rho_t)\mathbf{d}_{t-1} + \rho_t \mathbf{g}(\mathbf{x}_t, \mathbf{y})$

5: Compute $\mathbf{v}_t = \mathrm{argmin}_{s \in C} \langle \mathbf{s}, \mathbf{d}_t \rangle$,

6: Compute $\mathbf{x}_{t+1} = (1 - \gamma)\mathbf{x}_t + \gamma \mathbf{v}_t$

7: **end for**

FZFW - Faster Zeroth-Order Frank-Wolfe Method

- Described by H. Huang and H. Gao in *Can Stochastic Zeroth-Order Frank-Wolfe Method Converge Faster for Non-Convex Problems?*
- Computes the estimation of the gradient every q iterations over all the components
- The remaining $q-1$ times the estimation is computed over a small number of components

FZFW Algorithm

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5: Compute $\mathbf{v}_t = \text{argmin}_{s \in C} \langle \mathbf{s}, \mathbf{d}_t \rangle$,
6: Compute $\mathbf{x}_{t+1} = (1 - \gamma)\mathbf{x}_t + \gamma \mathbf{v}_t$
7: **end for**

FZCGS - Faster Zeroth-Order Conditional Gradient Method

Variant of FZFW with Conditional Gradient Sliding

Algorithm 3 $\mathbf{u}^+ = \text{condg}(\mathbf{g}, \mathbf{u}, \gamma, \eta)$ (Qu et al., 2017)

1: $\mathbf{u}_1 = \mathbf{u}$, $t = 1$

2: $\mathbf{v}_t$ be an optimal solution for

$$V_{\mathbf{g},\mathbf{u},\gamma}(\mathbf{u}_t) = \max_{\mathbf{x}\in\Omega}\langle \mathbf{g} + \frac{1}{\gamma}(\mathbf{u}_t - \mathbf{u}), \mathbf{u}_t - \mathbf{x}\rangle$$

3: If $V_{\mathbf{g},\mathbf{u},\gamma}(\mathbf{u}_t) \leq \eta$, return $\mathbf{u}^+ = \mathbf{u}_t$.

4: Set $\mathbf{u}_{t+1} = (1 - \alpha_t)\mathbf{u}_t + \alpha_t \mathbf{v}_t$ where $\alpha_t = \min\{1, \frac{\langle \frac{1}{\gamma}(\mathbf{u}-\mathbf{u}_t)-\mathbf{g}, \mathbf{v}_t-\mathbf{u}_t\rangle}{\frac{1}{\gamma}\|\mathbf{v}_t-\mathbf{u}_t\|^2}\}$.

5: Set $t \leftarrow t + 1$ and goto step 2.

Convergence results

Algorithm	Number of oracle calls
SGFFW	$\mathcal{O}\left(\frac{d^{4/3}}{\epsilon^4}\right)$
FZFW	$\mathcal{O}\left(\frac{n^{1/2}d}{\epsilon^2}\right)$
FZCGS	$\mathcal{O}\left(\frac{n^{1/2}d}{\epsilon}\right)$

Table 1: Order of oracle calls to reach an ϵ-stationary point

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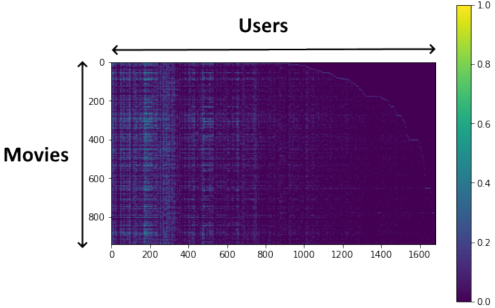
MovieLens100k

- Movie ratings from 1 to 5
- Original dataset: list of tuples *[user,movie,rating]*
- Reshaped into a 2d matrix:



Low Rank Matrix Completion

- Most of movies are not rated
- Objective: infer the ratings in order to provide recommendation

$$\min_{X \in \mathbb{R}^{m \times n}} \frac{1}{2} \sum_{(i,j) \in \mathcal{O}} \left(X_{ij} - Y_{ij} \right)^2 \tag{1}$$

- Keep only meaningful recommendation
  - Constraint on the rank

$$\text{subject to} \quad rank(X) \leq R$$

- Rephrase the constraint with the nuclear norm

$$\|X\|_* \leq \delta$$

Low Rank Matrix Completion - 2

- (1) is the convex formulation of the problem
- Very simple gradient

$$\nabla f(X_{\mathcal{O}}) = (X - Y)_{\mathcal{O}}$$

- Main focus: robust objective function (non convex)

$$\min_{X \in \mathbb{R}^{m \times n}} \sum_{(i,j) \in \mathcal{O}} \left(1 - \exp\left\{ - \frac{(x_{ij} - y_{ij})^2}{\sigma^2} \right\}\right)$$
$$\text{subject to} \quad \|X\|_* \leq \delta$$

- Gradient not easy to calculate: Zeroth-Order methods

Linear Subproblem

$$\arg\min\left\{\mathrm{Tr}(\nabla f(X_{\mathcal{O}}) \cdot X) : \|X\|_* \leq \delta\right\}$$

- Solve the FW subproblem
  - Perform SVD decomposition on $-\nabla f(X_{\mathcal{O}})$
  - Obtain u_1 and v_1
  - Return $\delta u_1 v_1^T$
- At every iteration of the main FW routine
- At the beginning after random initialization

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Code example

```python
def DZFW(Y, O, constr):
  """
  Deterministic Zeroth-order FW

  - Y: starting matrix
  - O: binary mask of observed values
  - constr: numerical value for the constraint on the nuclear norm
  """

  d0, d1 = Y.shape
  X0 = np.random.normal(0,1, (d0,d1)) # random initialization
  X0 = subproblem(X0,constr) # bring the initial iterate into the feasible set
  xs = [X0] # list that accumulates iterates

  for t in range(1, 100):
    stepsize = 1/(t+5)
    oracle = DZFW_oracle_call(Y, O, xs[-1], constr, stepsize) # oracle call
    x = (1 - stepsize) * xs[-1] + stepsize * (oracle - xs[-1])
    xs.append(x)
  return xs
```

- Python implementation for all the algorithms
- *NumPy* library for fast mathematical calculations

Execution time

Algorithm	CPU time (seconds)
FOFW	7.79
DZFW	9.10
SGFFW – KWSA	8.93
SGFFW – RDSA	23.51
SGFFW – I-RDSA	268.15

Table 2: CPU time for convex setting

Algorithm	CPU time (seconds)
SGFFW – I-RDSA	719.37
FZFW	26.48
FZCGS	15.49

Table 3: CPU time for non convex setting

Vectorization

```
for i in range(m):
  z_i = np.random.normal(0,1,(d0,d1))
  grad += (F(X + c_t * z_i) - F(X)) * z_i / c_t
return 1/m * grad
```

- *SGFFW – I-RDSA* algorithm requires `m` evaluations of gradient
- Unlike others, impossible to optimize with code vectorization
- Very slow

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Convergence - Convex case

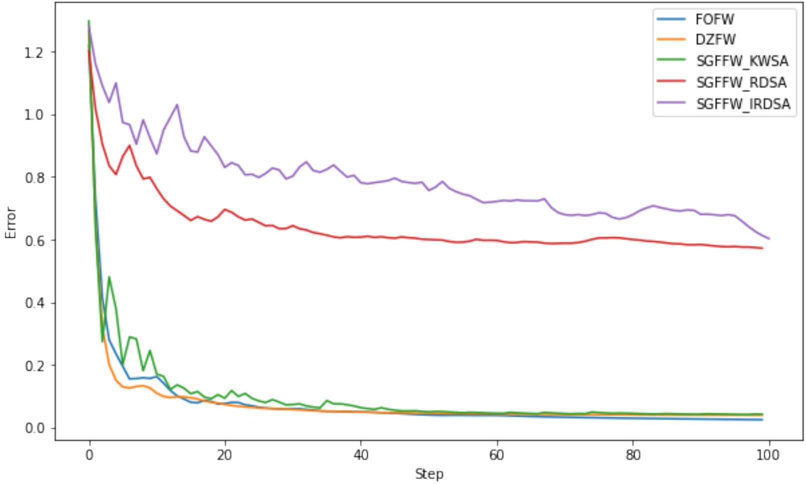


Figure 2: Convergence of methods for the convex case

Convergence - Non-convex case

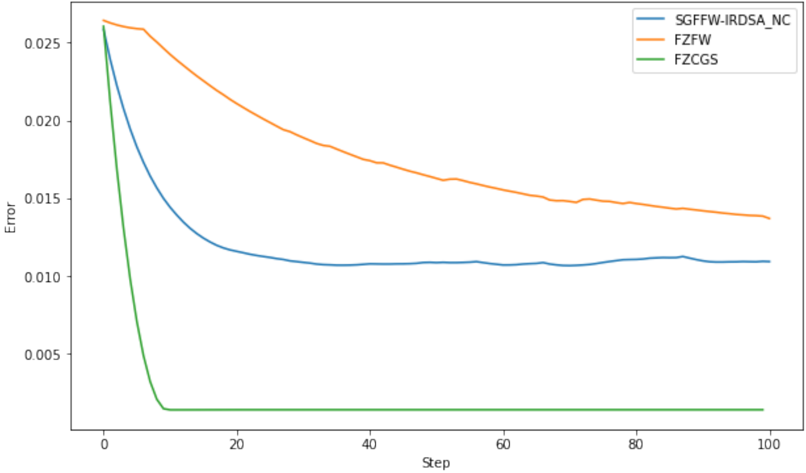
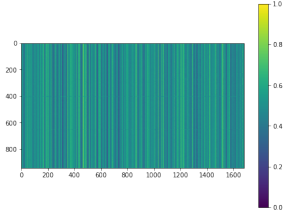
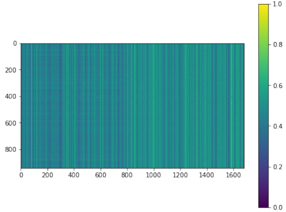


Figure 3: Convergence of methods for the non-convex case

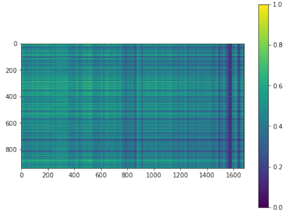
Generated matrices - Convex case



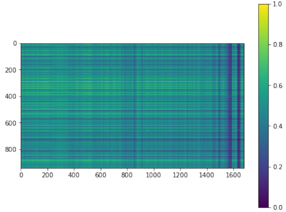
(a) SGFFW - RDSA



(b) SGFFW - I-RDSA

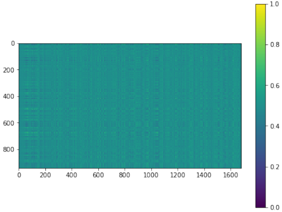


(c) SGFFW - KWSA

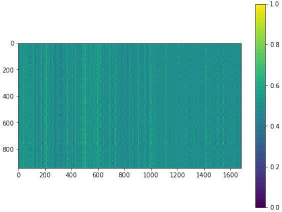


(d) Deterministic 0^{th} order

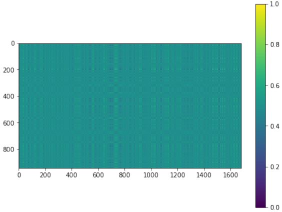
Generated matrices - Non-convex case



(a) SGFFW - I-RDSA



(b) FZFW



(c) FZCGS

Conclusions

- The plots of algorithms' convergence largely agree with those of (Sahu, Zaheer, and Kar 2019) and (Gao and Huang 2020)
- Experimental results have confirmed the effectiveness of the implemented methods for this low-rank matrix completion task

Gao, Hongchang and Heng Huang (2020). "Can Stochastic Zeroth-Order Frank-Wolfe Method Converge Faster for Non-Convex Problems?" In: *ICML'20*.

Sahu, Anit Kumar, Manzil Zaheer, and Soummya Kar (2019). "Towards Gradient Free and Projection Free Stochastic Optimization". In: *ArXiv* abs/1810.03233.