Chapter 8. Linear and Time-Invariant Systems

8.1 Input-Output Systems

The focus of this chapter are "operators", that is "objects" that take one signal as input and return another signal as output. Unless differently stated, the signals are assumed to be defined over the whole real axis, that is $t \in \mathbb{R}$. Therefore, a generic operator Ω would act as follows:

$$\Omega\left\{s(t)\right\} = w(t) \quad t \in \mathbb{R}$$

Eq. 8-1

These operators will be called “input-output systems”.

There are many different classes of systems, but we shall begin by looking at the distinction between two broad, mutually exclusive classes: linear and non-linear systems.

8.1.1 Linearity

A system Ω is said to be “linear” if it satisfies the following relationship:

$$\Omega\left\{\alpha_1 x_1(t)+\alpha_2 x_2(t)\right\}=\alpha_1\,\Omega\left\{x_1(t)\right\}+\alpha_2\,\Omega\left\{x_2(t)\right\}$$

$$\forall\alpha_1,\alpha_2\in\mathbb{R},\mathbb{C},\quad\forall x_1(t),x_2(t)$$

Eq. 8-2

Note that if this condition is satisfied, also the following condition is satisfied:

$$\Omega\left\{\sum_{n=1}^{N}\alpha_n s_n(t)\right\}=\sum_{n=1}^{N}\alpha_n\,\Omega\{s_n(t)\}$$

$$\forall\alpha_1,\alpha_2,....,\alpha_n\in\mathbb{R},\mathbb{C},\quad\forall s_1(t),s_2(t),....,s_n(t)$$

Eq. 8-3

In fact, the two can be considered equivalent, but of course Eq. 8-2 is simpler.

The condition of linearity is called, in other contexts, the “superposition of effects” rule. In words, it simply states that the system responds to the sum of two or more inputs by producing an output which is the sum of the outputs that would be generated by the individual inputs alone.

8.1.1.1 sufficient condition for non-linearity

A sufficient condition for a system to be non-linear is the following:

$$\Omega\left\{0(t)\right\} = v(t) \neq 0(t)$$

Proof

Let us assume that $\Omega\left\{0(t)\right\} = v(t) \neq 0(t)$. Let us then check whether Ω can be a linear operator.

Linearity, by definition, implies that it must be:

$$\Omega\left\{0(t) + 0(t)\right\} = \Omega\left\{0(t)\right\} + \Omega\left\{0(t)\right\}$$

Eq. 8-4

We first look at the left-hand side and execute the sum in the argument of the operator. We get: $\Omega\left\{0(t)+0(t)\right\}=\Omega\left\{0(t)\right\}$. The result, by assumption, is $\Omega\left\{0(t)\right\}=v(t)$, so we can establish that **the left-hand side of Eq. 8-4 is**:

$$\Omega\left\{0(t)+0(t)\right\}=v(t)$$

Eq. 8-5

We now look at the **right-hand side of Eq. 8-4** which can be directly evaluated as:

$$\Omega\left\{0(t)\right\}+\Omega\left\{0(t)\right\}=v(t)+v(t)=2v(t)$$

Eq. 8-6

However, we now see that Eq. 8-4 should be equal to both $v(t)$ (from Eq. 8-5) and to $2v(t)$ (from Eq. 8-6). This is absurd and therefore Ω being linear and $\Omega\{0(t)\} \neq 0(t)$ cannot be both true. We can then conclude that, if $\Omega\{0(t)\} \neq 0(t)$, then the system is certainly not linear, i.e., it is non-linear.

Note however that if $\Omega\{0(t)\} = 0(t)$, *nothing can be said*. The system could be either linear or non-linear. So, if $\Omega\{0(t)\} = 0(t)$, then only the full condition Eq. 8-3 should be checked to establish linearity or non-linearity.

Example 1

Is this system linear or non-linear?

$$\Omega\{s(t)\} = \alpha \cdot s(t) + \beta \cdot 1(t), \qquad \alpha, \beta \neq 0$$

We try the fast check:

$$\Omega\{0(t)\} = \alpha \cdot 0(t) + \beta \cdot 1(t)$$
$$= \beta \cdot 1(t) \neq 0$$

So, the system is *non-linear*.

$\boxed{\text{On your own:}}$ check that the system is non-linear using the method based on the definition.

Example 2

Is this system linear or non-linear?

$$\Omega\{s(t)\} = s^2(t)$$

We first look at the response to $0(t)$:

$$\Omega\{0(t)\} = 0^2(t) = 0(t)$$

Since we get $0(t)$, then nothing can be said.

Let us then check linearity using the sum of just two inputs:

$$\Omega\left\{s_1(t)+s_2(t)\right\}=\left[s_1(t)+s_2(t)\right]^2=s_1^2(t)+s_2^2(t)+2s_1(t)\cdot s_2(t)$$
$$\Omega\left\{s_1(t)\right\}+\Omega\left\{s_2(t)\right\}=s_1^2(t)+s_2^2(t)$$

The two results are, in general, different, so:

$$\Omega\left\{s_1(t)+s_2(t)\right\}\neq\Omega\left\{s_1(t)\right\}+\Omega\left\{s_2(t)\right\}$$

which means that Ω is non-linear.

Example 2

Is this system linear or non-linear?

$$\Omega\left\{s(t)\right\}=\int_{t-T}^{t}s(\theta)d\theta$$

We directly run the full linearity check. For Ω to be linear it must be, from Eq. 8-3:

$$\Omega\left\{\sum_{n=1}^{N}\alpha_n s_n(t)\right\}=\sum_{n=1}^{N}\alpha_n\Omega\{s_n(t)\}$$

We then evaluate the two sides of the above equation:

$$(1)\quad \Omega\left\{\sum_{n=1}^{N}\alpha_n s_n(t)\right\}=\int_{t-T}^{t}\sum_{n=1}^{N}\alpha_n s_n(\theta)d\theta$$

$$(2)\quad \sum_{n=1}^{N}\alpha_n\Omega\{s_n(t)\}=\sum_{n=1}^{N}\alpha_n\int_{t-T}^{t}s_n(\theta)d\theta$$

Are the two end results equal?

$$\int_{t-T}^{t} \sum_{n=1}^{N} \alpha_n s_n(\theta) d\theta \overset{?}{=} \sum_{n=1}^{N} \alpha_n \int_{t-T}^{t} s_n(\theta) d\theta$$

Yes, they are because, due to well-known properties of the integral operator, the summation in the left-hand side can be taken out of the integral. Doing so, the right-hand side is obtained. As a result, Ω is linear.

$\boxed{\text{On your own:}}$ show that:

$\Omega\{s(t)\} = \left|s\left(t\right)\right|^2$ is non-linear

$\Omega\{s(t)\} = s(t-T)$ is linear

$\Omega\{s(t)\} = \alpha\, s(t)$ is linear

$\Omega\{s(t)\} = \dfrac{ds(t)}{dt}$ is linear

$$\Omega\{s(t)\} = 2s(t) + \mathrm{u}(t)$$ is non-linear

$$\Omega\{s(t)\} = \log\{s(t)\}$$ is non-linear

$$\Omega\{s(t)\} = \int_{-\infty}^{t} s(\theta)d\theta$$ is linear

8.2 Systems with "Memory"

A system Ω is *memoryless* if its output signal at time t_0 depends only on the input signal at the same time t_0. This must be true $\forall t_0 \in \mathbb{R}$.

Otherwise, the system is said to have *memory*.

Note that memory and linearity are completely different concepts. There may be any combinations of memory and linearity characteristics, such as linear or non-linear systems with or without memory.

For instance,

$$y(t) = \Omega\left\{x(t)\right\} = x^2(t)$$

is non-linear and is also memoryless, because:

$$y(t)\Big|_{t=t_0} = y(t_0) = x^2(t_0)$$

Another example is the system: $y(t) = \alpha\, x(t)$. Such system is linear and memoryless.

Let us now consider:

$$y(t) = \Omega\{x(t)\} = \int_{t-T}^{t} x(\theta)\, d\theta$$

By looking at the output at a fixed time t_0, we get:

$$y(t_0) = \int_{t_0-T}^{t_0} x(\theta)\, d\theta = \int_{t_0-T}^{t_0} x(t)\, dt$$

Note that in the right-hand side we have replaced the integration variable θ with the customary time-variable t because there is no longer any problem of ambiguity, having set the integration interval to $[t_0 - T, t_0]$, which does not contain "t" anymore. The use of the more familiar t as integration variable may help comprehension.

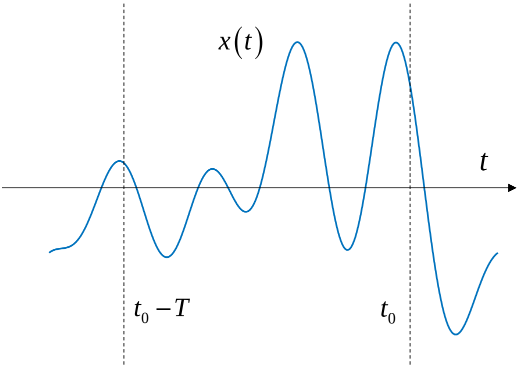


Fig. 8-1: integration window for $x(t)$: all values of $x(t)$ for $t \in [t_0 - T, t_0]$ contribute to the integral result $y(t_0)$

Clearly, the output $y(t_0)$ is formed by all values of the input $x(t)$ for $t \in [t_0 - T, t_0]$, and not only by the value of $x(t)$ for $t = t_0$.

The system:

$$y(t) = \Omega\left\{x(t)\right\} = \int_{t-T}^{t} x(\theta)d\theta$$

is therefore a system *with memory*. Incidentally, we have already shown that it is linear, so Ω is *linear with memory*.

A peculiar system is the *delay* system:

$$y(t) = \Omega\left\{x(t)\right\} = x(t-T)$$

This system is linear and *has memory*: in fact, the output at time t_0 does not depend on the input at time t_0, but depends on the input at a previous time, exactly T seconds before:

$$y(t_0) = x(t_0 - T)$$

One could actually picture this system as a "memory circuit" that retains the value of the input for T seconds before putting it out.

On your own: prove that the following system is non-linear with memory:

$$y(t) = \int_{t-T}^{t} x^2(\theta)d\theta$$

8.3 Time Invariance

Yet another important distinction among systems is time-invariance. This feature, too, combines in all possible ways with linearity/non-linearity and memoryless/with-memory.

Simply put, time-invariance means that the way the system operates on the input signals does not change over time. In other words, the system "definition" is time-independent.

An example of a time-invariant system is an LCR electronic circuit with constant and fixed values of the inductance L, capacity C and resistance R elements. Conversely, an electronic circuit whose values of either L, C or R elements change over time, is clearly time-variant.

A formal definition of time-invariance is as follows.

Given:

$$y(t) = \Omega\left\{x(t)\right\} \quad and \quad z(t) = \Omega\left\{x(t-T)\right\}$$

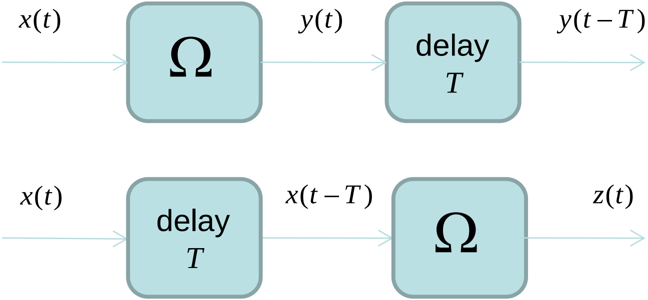
then the system Ω *is time-invariant if and only if:*

$$z(t) = y(t-T)$$
$$\forall t, \quad \forall x\left(t\right)$$

Eq. 8-7

The formulas tell us that a time-invariant system has the following behavior. If the same input is applied both now and tomorrow, the output that we get now and the one we get tomorrow are identical, except for the obvious one-day time-shift.

Another way of providing the definition is through block diagrams.



If the outputs of the two diagrams are the same, i.e., if:

$$z(t) = y(t-T)\,,$$

then the system is time-invariant.

Finally, another formal definition can be given in terms of "operator commutativity". We first assign the delay operator its own symbol:

$$T\left\{x(t)\right\} \triangleq x(t-T)$$

This done, time invariance can be quite compactly stated as:

$$T\left\{\Omega\left\{x(t)\right\}\right\} = \Omega\left\{T\left\{x(t)\right\}\right\}$$

Eq. 8-8

In operator jargon, this can be phrased as follows: a system Ω is time-invariant if it *commutes* with the delay operator, for any input signal $x(t)$. Commuting means that the order of the operators can be exchanged without the result being affected, as in Eq. 8-8.

Examples

Show that this operator is time-invariant:

$$\Omega\left\{x\left(t\right)\right\} = x^2\left(t\right)$$

We use the operator definition given in Eq. 8-8. We first apply the delay operator to the output of the system.

$$T\left\{\Omega\left\{x\left(t\right)\right\}\right\} = T\left\{x^2\left(t\right)\right\} = x^2\left(t-T\right)$$

Then we apply the delay to the input and then Ω:

$$\Omega\left\{T\left\{x\left(t\right)\right\}\right\} = \Omega\left\{x\left(t-T\right)\right\} = x^2\left(t-T\right)$$

The results are identical, so Ω is time-invariant.

Is the following system time-invariant?

$$\Omega\left\{x(t)\right\} = x(t)\cdot u(t)$$

We apply the delay to the system output:

$$T\left\{\Omega\left\{x(t)\right\}\right\} = T\left\{x(t)\cdot u(t)\right\} = x(t-T)\cdot u(t-T)$$

Then we apply it first to the input and afterwards apply Ω :

$$\Omega\left\{T\left\{x(t)\right\}\right\} = \Omega\left\{x(t-T)\right\} = x(t-T)\cdot u(t)$$

So, we have:

$$\Omega\left\{T\left\{x(t)\right\}\right\} = x(t-T)\cdot u(t)$$

$$T\left\{\Omega\left\{x(t)\right\}\right\} = x(t-T)\cdot u(t-T)$$

The two results are *different*. The system in fact changes its behavior over time. This could be more clearly seen if one thinks of an input $x(t)$ whose support is all in $t<0$. Then the output is $\Omega\{x(t)\}=0(t)$. Instead, if the input has support all in $t>0$, then the output is: $\Omega\{x(t)\}=x(t)$.

The two behaviors are *strikingly different* and show the system to really change its response over time.

Find out whether this system is time-invariant:

$$\Omega\{x(t)\}=\int_{t-t_0}^{t} x(\theta)\,d\theta$$

At first sight it may appear to be non-invariant, due to the explicit presence of the variable t in its definition. However, a more careful check leads to a different conclusion. We first delay the overall output:

$$T\left\{\Omega\left\{x(t)\right\}\right\}=T\left\{\int_{t-t_0}^{t} x(\theta)\,d\theta\right\}=\int_{t-t_0-T}^{t-T} x(\theta)\,d\theta$$

Note that the delay operator T affects the time-variable t but does not and cannot affect the time variable θ, which is the integration variable and is, as usual, totally "sealed" within the integral and cannot be affected by "outer" operators.

We now invert the operator order. We first delay the input and then apply Ω:

$$\Omega\left\{T\left\{x(t)\right\}\right\}=\Omega\left\{x(t-T)\right\}=\int_{t-t_0}^{t} x(\theta-T)\,d\theta=$$

Note that it is important to be careful in inserting the delayed input into Ω (details given during lecture).

The two results appear different:

$$T\left\{\Omega\left\{x(t)\right\}\right\} = \int_{t-t_0-T}^{t-T} x(\theta)\,d\theta$$

$$\Omega\left\{T\left\{x(t)\right\}\right\} = \int_{t-t_0}^{t} x(\theta - T)\,d\theta$$

However, by changing the integration variable in the lower integral:

$$\rho = \theta - T \quad d\theta = d\rho$$

$$\rho_{ll} = \theta_{ll} - T = t - t_0 - T$$

$$\rho_{ul} = \theta_{ul} - T = t - T$$

where the subscript *"ul"* and *"ll"* mean integration range *upper limit* and *lower limit*, respectively, we get:

$$\Omega\left\{T\left\{x(t)\right\}\right\} = \int_{t-t_0-T}^{t-T} x(\rho)\,d\rho$$

which is clearly completely equivalent to what we have previously found:

$$T\left\{\Omega\left\{x(t)\right\}\right\} = \int_{t-t_0-T}^{t-T} x(\theta)\,d\theta$$

Therefore, though not obviously so, this system is in fact time-invariant.

Find out whether this system is time-invariant:

$$\Omega\left\{x(t)\right\} = \int_{t-t_0}^{t} x(\theta)\cdot\cos(2\pi f_1\theta)\cdot d\theta$$

As usual:

$$T\left\{\Omega\left\{x(t)\right\}\right\} = \int_{t-T-t_0}^{t-T} x(\theta)\cdot\cos(2\pi f_1\theta)\cdot d\theta$$

$$\Omega\left\{T\left\{x(t)\right\}\right\} = \Omega\left\{x(t-T)\right\} = \int_{t-t_0}^{t} x(\theta - T)\cdot\cos(2\pi f_1\theta)\cdot d\theta =$$

Changing variable in the last integral:

$$\rho = \theta - T \quad d\theta = d\rho$$
$$\rho_{ll} = \theta_{ll} - T = t - t_0 - T$$
$$\rho_{ul} = \theta_{ul} - T = t - T$$

we get:

$$\Omega\left\{T\left\{x(t)\right\}\right\} = \int_{t-t_0-T}^{t-T} x(\rho)\cdot\cos\left[2\pi f_1(\rho + T)\right]\cdot d\rho$$

which is similar but *different* than:

$$T\left\{\Omega\left\{x(t)\right\}\right\}=\int_{t-T-t_0}^{t-T} x(\theta)\cdot\cos(2\pi f_1\theta)\cdot d\theta$$

So: $T\left\{\Omega\left\{x(t)\right\}\right\}\neq\Omega\left\{T\left\{x(t)\right\}\right\}$ and *the system is not time-invariant.*

8.4 Linear and Time-Invariant Systems

This is a most important class of systems. They have both the property of linearity and the property of time-invariance. They are called linear time-invariant systems: LTI. We will write them using the generic symbol L:

$$L\left\{x(t)\right\}=y(t)$$

The combined properties of linearity and time-invariance allow to describe such systems in an extremely powerful unified way.

8.4.1 The impulse response

We start off by recalling this important formula:

$$\int_{-\infty}^{\infty} x(\theta)\cdot\delta(t-\theta)\,d\theta = x(t)$$

Eq. 8-9

The result is $x(t)$ because the argument of δ vanishes for $t=\theta$. So, following the rules of integration with δ's, the result of this integral is the integrand function (without the delta) with θ replaced by t.

We can now re-write the input $x(t)$ to a generic LTI system using its equivalent expression Eq. 8-9:

$$L\{x(t)\} = L\left\{ \int_{-\infty}^{\infty} x(\theta) \cdot \delta(t-\theta) \, d\theta \right\}$$

We then invoke the first key property of LTI systems: linearity. Since linear operators commute, L and the integral inside the argument of L can be swapped:

$$L\{x(t)\} = \int_{-\infty}^{\infty} L\{x(\theta) \cdot \delta(t-\theta)\} \, d\theta$$

Note that this is possible because L acts on the time variable t and not on the integration variable θ. Also, anything that depends on just θ is seen by L as a constant and can be pulled out of L:

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(\theta) \cdot L\{\delta(t-\theta)\} d\theta$$

Eq. 8-10

We now see that L operates on a well-defined input, $\delta(t-\theta)$. We then *define*:

$$h(t) \triangleq L\{\delta(t)\}$$

Eq. 8-11

and call the signal $h(t)$ the *impulse-response* of the system L. The name is justified by the fact that, according to the definition, $h(t)$ represents the output (or "response") of the system L to an input "pulse", or "impulse" $\delta(t)$.

We now invoke the second key property of all LTI systems, that is time-invariance. Due to such property, we can write that, given:

$$L\{\delta(t)\} = h(t)$$

then:

$$L\{\delta(t-\theta)\} = h(t-\theta)$$

The factor $L\left\{\delta\left(t-\theta\right)\right\}$ is the one that appears in the integral in Eq. 8-10 and we can substitute $h\left(t-\theta\right)$ for it:

$$L\left\{x\left(t\right)\right\}=\int_{-\infty}^{\infty}x\left(\theta\right)\cdot L\left\{\delta\left(t-\theta\right)\right\}d\theta=\int_{-\infty}^{\infty}x\left(\theta\right)\cdot h\left(t-\theta\right)d\theta$$

The right-hand side is given a name of its own: it is called a **convolution product**. It can be written, as shorthand, as $x(t)*h(t)$. We can then write:

$$L\left\{x\left(t\right)\right\}=\int_{-\infty}^{\infty}x\left(\theta\right)\cdot h\left(t-\theta\right)d\theta=x(t)*h(t)$$

Eq. 8-12

This result is one of the most important and powerful results of systems theory. It essentially states that, independently of their definition, which can be very different, *all* LTI systems are fully and completely characterized by their impulse response $h(t)$.

In addition, their output $y(t)$ to *any* input $x(t)$ can be computed using a unified formula:

$$y(t) = L\{x(t)\} = x(t) * h(t) .$$

Eq. 8-13

Note that we have been able to get to Eq. 8-13 only because we invoked both linearity and time invariance. If either property does not hold, Eq. 8-13 is not verified.

8.4.1.1 commutativity of the convolution product

Given:

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\theta)\, x_2(t-\theta)\, d\theta$$

then changing:

$$\begin{matrix} t-\theta=\rho \\ -d\theta = d\rho \end{matrix} \quad \text{which implies} \Rightarrow \quad \begin{matrix} \theta = t-\rho \\ d\theta = -d\rho \end{matrix}$$

we get:

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\theta)\, x_2(t-\theta)\, d\theta =$$

$$\int_{+\infty}^{-\infty} x_1(t-\rho)\, x_2(\rho)(-d\rho) = -\int_{-\infty}^{+\infty} x_1(t-\rho)\, x_2(\rho)(-d\rho)$$

$$= \int_{-\infty}^{+\infty} x_2(\rho)\, x_1(t-\rho)\, d\rho = x_2(t) * x_1(t)$$

So, in fact there are two equivalent forms of the same convolution product:

$$\int_{-\infty}^{+\infty} x_1(\theta)\, x_2(t-\theta)\, d\theta = \int_{-\infty}^{+\infty} x_2(\theta)\, x_1(t-\theta)\, d\theta$$

8.4.1.2 Examples

We have seen that the system:

$$L\left\{x(t)\right\} = \alpha \cdot x(t)$$

Eq. 8-14

is LTI. So, it must be possible to express it as a convolution product. We only need to find its impulse response $h(t)$.

By definition:

$$h(t) = L\left\{\delta(t)\right\}$$

From the system definition we can immediately find:

$$L\left\{\delta(t)\right\} = \alpha \cdot \delta(t)$$

And from theory we know that:

$$L\{x(t)\} = x(t) * h(t) = x(t) * \alpha \cdot \delta(t)$$

In this simple example it easy to see that in fact:

$$x(t) * \alpha \cdot \delta(t) = \alpha \cdot x(t)$$

which coincides with the system description given by Eq. 8-14.

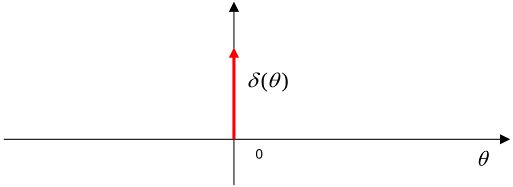
A less straightforward example is given by the LTI system:

$$L\{x(t)\} = \int_{t-T}^{t} x(\theta)\, d\theta$$

Again, we want to find its $h(t)$. We proceed by definition:

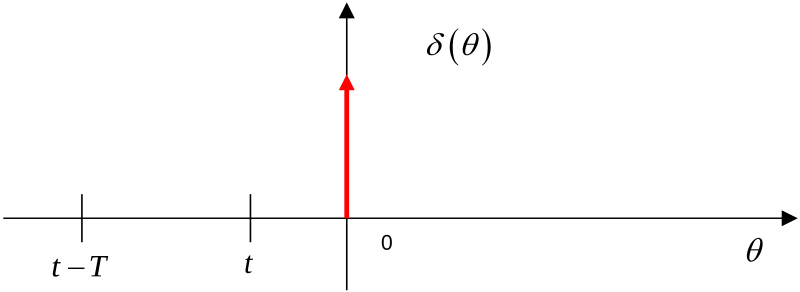
$$h(t) = L\{\delta(t)\} = \int_{t-T}^{t} \delta(\theta)\, d\theta$$

Let's plot its integrand function, vs. the integration variable θ:



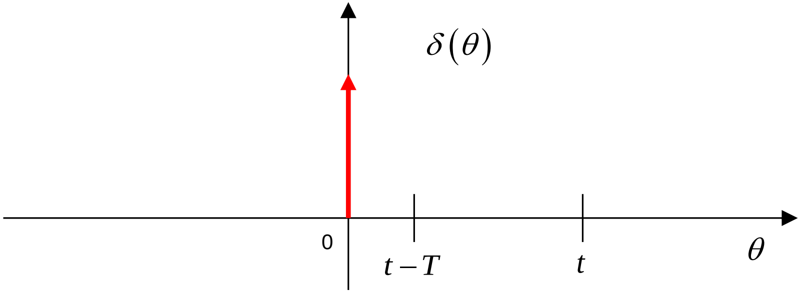
What changes vs. time t are the integration limits over θ. We can plot such limits, $[t-T, t]$, over the same graph.

→ we assume $t < 0$:



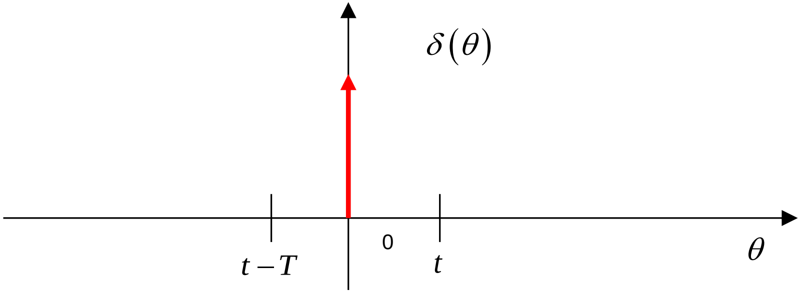
We clearly get $h(t) = 0$ because the δ is outside of the integration range.

→ we then assume $t > T$:



Again, we get $h(t)=0$ because the δ is outside of the integration range.

→ we finally assume $0<t<T$:

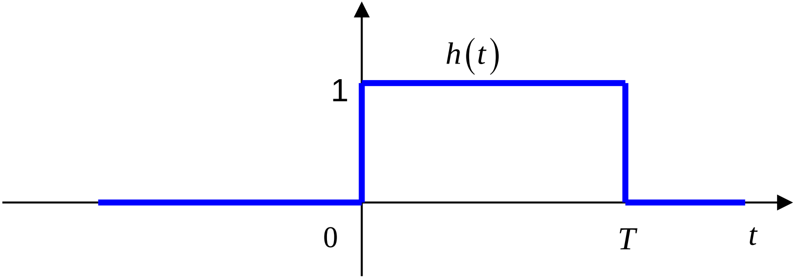


Now the δ distribution is inside of the integration range $[t-T,t]$ so that the result is $h(t)=1$.

Piecing everything together, we have:

$$h(t)=\begin{cases} 0 & t<0 \\ 1 & 0<t<T \\ 0 & t>T \end{cases}$$

We can now plot the overall impulse response $h(t)$:



Analytically we have:

$$h(t) = \Pi_T\left(t - \frac{T}{2}\right) = \pi_T(t)$$

and so:

$$L\left\{x(t)\right\} = x(t) * h(t) = x(t) * \Pi_T\left(t - \frac{T}{2}\right) = x(t) * \pi_T(t)$$

$$= \int_{-\infty}^{+\infty} \pi_T(\theta) \cdot x(t-\theta)\, d\theta = \int_{-\infty}^{+\infty} x(\theta) \cdot \pi_T(t-\theta)\, d\theta$$

$\boxed{\text{On your own,}}$ find the impulse response of the LTI systems:

$$L\left\{x(t)\right\} = T\left\{x(t)\right\} = x(t-T)$$

$$L\left\{x(t)\right\} = x(t+T)$$

Answer:

$$h(t) = \delta(t-T)$$

$$h(t) = \delta(t+T)$$

$$L\left\{x(t)\right\} = \int_{-\infty}^{t} x(\theta)\, d\theta$$

Answer: $h(t) = \mathrm{u}(t)$

$$L\left\{x(t)\right\} = \int_{-\infty}^{t-T} x(\theta)\, d\theta$$

Answer: $h(t) = \mathrm{u}(t-T)$

$$L\left\{x(t)\right\} = \int_{t}^{t+T} x(\theta)\, d\theta$$

Answer: $h(t) = \pi_T(-t)$

8.4.2 Causality

All LTI systems whose $h(t)$ has support different than just $t = 0$ have *memory*. You can easily prove this on your own.

The convolution product, however, does not pose any limitation to such memory being towards the *past* or towards the *future*. If memory is in the past, what is output by the system "now" depends on *past* values of the input. If memory is in the future, what is output by the system "now" depends on *future* values of the input.

Clearly, the fact that the future can influence the present is mathematically possible but not physically possible. All physical systems are *causal*, that is, their present output is only influenced by what happened in the past.

We will now find specific conditions on $h(t)$ for the system to be causal. Starting from the convolution formula:

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\theta) \cdot x(t-\theta) d\theta$$

we break up the integration range into two parts:

$$y(t) = \int_{-\infty}^{0^-} x(t-\theta) \cdot h(\theta) d\theta + \int_{0}^{+\infty} x(t-\theta) \cdot h(\theta) d\theta$$

Note the 0^- in the upper integration limit of the first integral. This simply means "up to but excluding 0". The value 0 is included in the next integral. In other words, the first integral covers $\theta \in [-\infty, 0[$, that is the point 0 is excluded, and the second integral covers $\theta \in [0,\infty]$. This way, when putting together the integrals, we get exactly: $[-\infty,0[\ \cup\ [0,\infty] = \mathbb{R}$. We'll come back on this shortly.

For convenience, we change the sign of the integration variable inside the first integral:

$$y(t) = \int_{\infty}^{0^-} x(t+\theta)\cdot h(-\theta)(-d\theta) + \int_{0}^{+\infty} x(t-\theta)\cdot h(\theta)d\theta =$$

$$= \int_{0^+}^{+\infty} x(t+\theta)\cdot h(-\theta)d\theta + \int_{0}^{+\infty} x(t-\theta)\cdot h(\theta)d\theta$$

We can discuss the causality of a system by looking at its output at time t_0. From the above formula, the output value $y(t_0)$ is made up of two contributions:

$$y(t_0) = \int_{0^+}^{+\infty} x(t_0+\theta)\cdot h(-\theta)\,d\theta + \int_{0}^{+\infty} x(t_0-\theta)\cdot h(\theta)\,d\theta$$

$$= y'(t_0) + y''(t_0)$$

namely:

$$y'(t_0) = \int_{0^+}^{+\infty} x(t_0+\theta)\cdot h(-\theta)\,d\theta$$

$$y''(t_0) = \int_{0}^{+\infty} x(t_0-\theta)\cdot h(\theta)\,d\theta$$

In both integrals θ is positive. As a result, $x(t_0-\theta)$ in $y''(t_0)$ looks at values of the input $x(t)$ which occurred *before* t_0 *or at most exactly at* t_0, i.e., <u>in the past</u>.

Instead, $x(t_0+\theta)$ in $y'(t_0)$ looks at values of the input $x(t)$ which will occur *after* t_0, i.e., in the future.

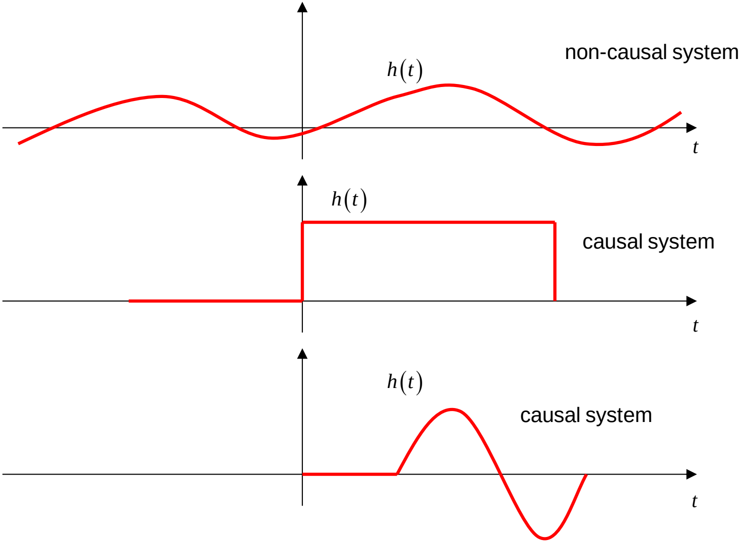
We can therefore conclude that:

$$y'(t_0)=\int_{0^+}^{+\infty} x(t_0+\theta)\cdot h(-\theta)\,d\theta$$ → non-causal contribution

$$y''(t_0)=\int_{0}^{+\infty} x(t_0-\theta)\cdot h(\theta)\,d\theta$$ → causal contribution

If we want to restrict to systems that are **causal**, then for all such systems **it must be** $y'(t_0) = 0, \forall t_0$**.** The only way to ensure this, is by imposing that $h(-\theta) = 0$ for $\theta \in]0, \infty]$, that is $h(t) = 0(t)$ for $t < 0$. To summarize:

an LTI system is causal if and only if: $h(t) = 0(t)$ *for* $t < 0$


non-causal system
$h(t)$
t
$h(t)$
causal system
t
$h(t)$
causal system
t

Note that for a causal system the convolution integral then could be simplified to:

$$y(t) = \int_0^{\infty} x(t-\theta)h(\theta)\,d\theta$$

On your own, say which one of the following LTI system is causal and which one is not:

(1) $L\{x(t)\} = x(t-T)$; (2) $L\{x(t)\} = x(t+T)$

(3) $L\{x(t)\} = \int_{-\infty}^{t} x(\theta)d\theta$; (4) $L\{x(t)\} = \int_{t}^{t+T} x(\theta)d\theta$

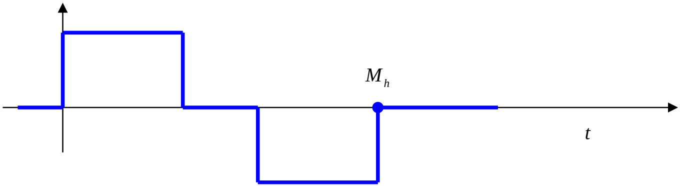
Answer:

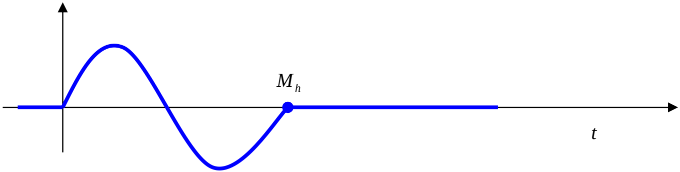
(1) and (3) are causal, (2) and (4) are not.

8.4.3 Memory duration

The concept of memory duration (or just "memory") can be defined for all systems, including non-causal systems. However, here we address *causal* systems only. Remember that causal systems are such that: $h(t)=0(t),\ t<0$.

Let us consider the following $h(t)$'s:





The time-instant marked as M_h in the plots is found as the largest value of t such that $h(t) \neq 0$. An alternative mathematical definition can be given as follows:

$$M_h = \min\left\{t_M \,\middle|\, h(t) = 0, \forall t > t_M\right\}$$

which can be understood as follows: ideally, one should start looking at $t = +\infty$, and then move backward along the time axis till a value of $h(t) \neq 0$ is found. The first value of t for which $h(t) \neq 0$ is M_h .

Let us now look at the causal part of the convolution product:

$$y(t) = \int_0^{\infty} x(t-\theta)h(\theta)d\theta$$

Assuming we know M_h, we clearly can write:

$$y(t) = \int_0^{M_h} x(t-\theta)h(\theta)d\theta$$

If we now want to evaluate $y(t)$ at an arbitrary t_0, we get:

$$y(t_0) = \int_0^{M_h} x(t_0-\theta)h(\theta)d\theta$$

From the last equation it is evident that the output of the system "now" (at time t_0) is found by integrating the input from "now", t_0, backward to $t_0 - M_h$.

This means that the system "remembers" and takes into account the values taken on by the input for a total of exactly M_h seconds in the past. This is why M_h is called the *"memory duration"* or simply the *"memory"* of the system.

Note: as we know, the convolution integral can be re-written with the operands swapped inside. What is the version of the convolution integral with swapped operands, when causality and memory are taken into account? It can be shown that we have:

$$y(t) = \int_0^{M_h} h(\theta)x(t-\theta)d\theta = \int_{t-M_h}^{t} x(\theta)h(t-\theta)d\theta$$

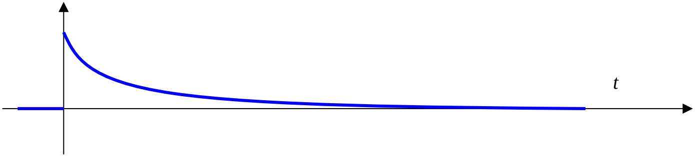
This result is easy to prove by using the property shown in Sect. 8.4.5.2.

8.4.3.1 Infinite memory

Memory can be infinite. Let us consider for instance:

$$h(t) = e^{-at} \cdot u(t), \text{ with } \; a > 0$$

The plot of $h(t)$ is:



Though $h(t)$ decays fast for $t \to \infty$, it is never really 0. So, according to the definition:

$$M_h = \min\left\{ t \,\middle|\, h(\tau) = 0(\tau), \forall \tau > t \right\}$$

Eq. 8-15

the result that is found is: $M_h = \infty$. That is, the output "now" is the consequence of everything that's happened since $t = -\infty$.

Although past events are increasingly less important as we look further back into the past, due to the decay of the exponential, mathematically they do contribute to the output which is seen "now".

$\boxed{\text{On your own:}}$ what is the memory of the following impulse responses:

$h(t) = \delta(t - T)$ (the delay system)

$h(t) = u(t)$ (the integrator from $-\infty$ system)

Answers:

$M_h = T$

$M_h = \infty$

8.4.4 Extension of a signal

In this and the following sections we will make use of the concept of the *extension* of a signal. This concept has substantial similarity to the concept of *support*, with some differences.

These results and the ones derived from them apply to **piecewise continuous signals**. Since all practical signals are of this kind, we will not discuss this condition further.

The extension of a signal $s(t)$ is an interval

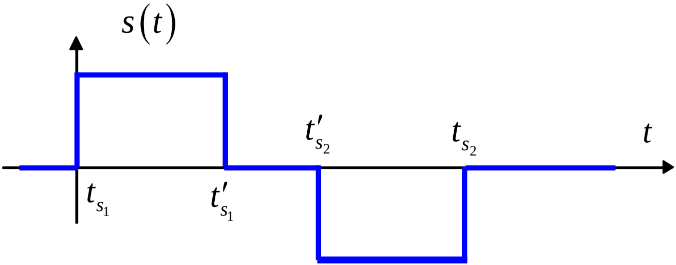
$$\mathrm{ext}\{s(t)\} = [t_{s1}, t_{s2}]$$

whose upper and lower limit are defined as follows:

$$t_{s_1} = \max\left\{ t \,\middle|\, s(\tau) = 0(\tau), \forall \tau < t \right\}$$
$$t_{s_2} = \min\left\{ t \,\middle|\, s(\tau) = 0(\tau), \forall \tau > t \right\}$$

In colloquial terms, these two formulas can be understood as follows. You should think of placing yourself at $t = -\infty$ and then ideally moving towards increasing values of time, till the first time-value for which $s(t) \neq 0$ is encountered. That particular time-value is called t_{s_1}.

Then you should think of placing yourself at $t = +\infty$ and then ideally moving towards decreasing values of time, till the first time-value for which $s(t) \neq 0$ is encountered. That particular time-value is called t_{s_2}.



In the plot above, t_{s_1} and t_{s_2} are shown. Note that the support of the signal would instead be:

$$\mathrm{supp}\{s(t)\} = [t_{s1}, t'_{s1}] \cup [t'_{s2}, t_{s2}]$$

because the support is the collection of all times when $s(t) \neq 0$. Clearly, during the interval $[t'_{s1}, t'_{s2}]$, $s(t) = 0$, so this interval cannot be in the support. It is instead included in the "extension".

8.4.5 Extension of the output

In this section we assess the **extension of the output** $y(t)$ of an LTI system, given a certain input $x(t)$ and given the system impulse response $h(t)$.

Regarding the input $x(t)$, we assume:

$$\text{ext}\{x(t)\} = [t_{x1}, t_{x2}]$$

Also, regarding the impulse response $h(t)$ we assume:

$$\text{ext}\{h(t)\} = [t_{h1}, t_{h2}]$$

Optional: derivation of the result.

We know that the output of the LTI system is found as:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\theta)h(\theta)d\theta$$

Due to the presence of $h(\theta)$ in the integrand function, clearly the integration range becomes limited to the extension of $h(\theta)$, $\left[t_{h1}, t_{h2}\right]$:

$$y(t) = \int_{t_{h1}}^{t_{h2}} x(t-\theta)h(\theta)d\theta$$

Key to finding the result is to analyze the extension of the function $x(t-\theta)$ vs. the integration variable θ. We start from our assumption regarding the extension of $x(t)$:

$$\text{ext}\{x(t)\} = [t_{x1}, t_{x2}]$$

That is, the *argument* of $x(t)$, in this case simply t, is *within the "extension" of* $x(t)$ *if*:

$$t_{x1} < t < t_{x2} \ .$$

But the *argument* of the function $x(t-\theta)$ is $t-\theta$, and we are interested in finding the extension of $x(t-\theta)$ vs. the variable θ.

So we first remark that *the argument, in this case* $t-\theta$*, is within the "extension" of* $x(t-\theta)$ *if*:

$$t_{x1} < t-\theta < t_{x2}$$

Using the standard rules for inequality manipulation we then get:

$$t_{x1} - t < -\theta < t_{x2} - t$$
$$t - t_{x1} > \theta > t - t_{x2}$$
$$t - t_{x2} < \theta < t - t_{x1}$$

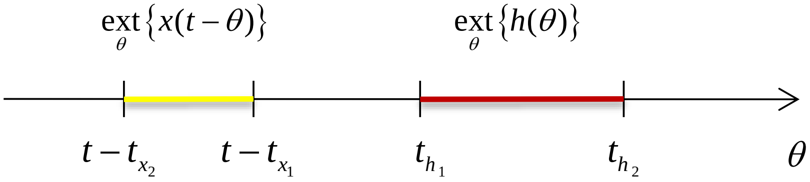
So we have:

$$\underset{\theta}{\mathrm{ext}}\left\{x(t-\theta)\right\} = \left[t - t_{x2}, t - t_{x1}\right]$$

Eq. 8-16

Note that the extension of $x(t-\theta)$ vs. the integration variable θ is not fixed, but actually "shifts" according to the value of the parameter t.

The overall integration picture is then shown graphically as:



The convolution integral giving $y(t)$ is therefore certainly zero when the two intervals above are *not* superimposed, as it is for instance in figure. Whether they are superimposed or not, this clearly depends on t.

Mathematically:

$$y(t)=0 \quad \leftrightarrow \quad \{t-t_{x1}<t_{h1}\} \cup \{t-t_{x2}>t_{h2}\}$$

where the symbol "$\cup$" is a "union" symbol in set theory, which can also be read as "or", in plain language.

By simple inequality manipulations we get:

$$y(t)=0 \quad \leftrightarrow \quad \{t<t_{h1}+t_{x1}\} \cup \{t>t_{h2}+t_{x2}\}$$

Re-casting this in terms of conventional intervals, the above equation simply means that $y(t)=0$ if $t \notin \left]t_{h1}+t_{x1},\, t_{h2}+t_{x2}\right[$.

End of optional material.

It is found that:

$$\operatorname{ext}\{y(t)\} = [t_{h1} + t_{x1}, t_{h2} + t_{x2}]$$

Eq. 8-17

Note that the above result is very simple and can easily be memorized: the lower and upper limit of the extension are the sum of the respective lower and upper limits of the extensions of $x(t)$ and of $h(t)$.

8.4.5.1 duration of the output

Given a signal whose extension is:

$$\operatorname{ext}\{x(t)\} = [t_{x1}, t_{x2}]$$

then we define the "duration" of $x(t)$ as:

$$\operatorname{dur}\left\{x(t)\right\} = t_{x2} - t_{x1}$$

We would like to compare the duration of a signal $x(t)$ used as input to an LTI system $h(t)$ with that of the output $y(t)$, given that:

$$\operatorname{ext}\left\{h(t)\right\} = \left[t_{h1}, t_{h2}\right] \quad \rightarrow \quad \operatorname{dur}\left\{h(t)\right\} = t_{h2} - t_{h1}$$

We know from Eq. 8-17 that:

$$\operatorname{ext}\left\{y(t)\right\} = \left[t_{h1} + t_{x1}, t_{h2} + t_{x2}\right]$$

Therefore:

$$\operatorname{dur}\{y(t)\} = \left(t_{h2} + t_{x2} \right) - \left(t_{h1} + t_{x1} \right) =$$
$$\left(t_{h2} - t_{h1} \right) + \left(t_{x2} - t_{x1} \right) = \operatorname{dur}\{x(t)\} + \operatorname{dur}\{h(t)\}$$

Eq. 8-18

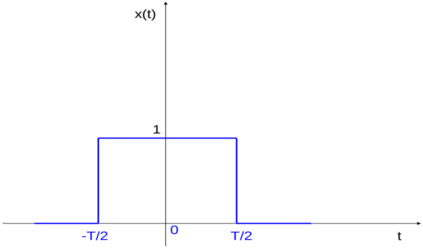
This result shows that the output $y(t)$ to a certain input $x(t)$ is actually "longer" in time than the input itself. Its "duration" in time is the sum of the duration of the input and of the duration of the impulse response $h(t)$.

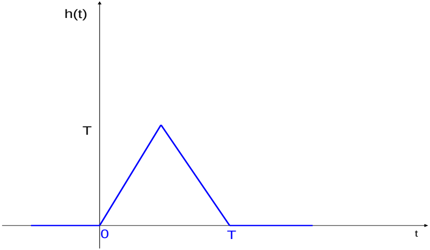
Example

Find the extension and duration of the output of an LTI system whose input is: $x(t) = \Pi_T(t)$, *and whose impulse response is:*

$$h(t) = \Lambda_{T/2}(t - T/2)$$

Find also the memory of the system.





Solution.

According to Eq. 8-17, the extension of $y(t)$ is:

$$\operatorname{ext}\{y(t)\} = \left[t_{x1} + t_{h1}, t_{x2} + t_{h2}\right] =$$
$$= \left[-\frac{T}{2} + 0, +\frac{T}{2} + T\right] = \left[-\frac{T}{2}, \frac{3T}{2}\right]$$

Note that the system is causal because $h(t) = 0, t < 0$. The output starts at $-T/2$ just because the input started at $-T/2$. The duration of the output is, according to Eq. 8-18:

$$\operatorname{dur}\{y(t)\} = \left(t_{h2} - t_{h1}\right) + \left(t_{x2} - t_{x1}\right) = \left(T - 0\right) + \left[\frac{T}{2} - \left(-\frac{T}{2}\right)\right] = 2T$$

The memory of the system, according to Eq. 8-15, is T .

Example

Compute the extension of the output of an LTI system whose input is: $x(t) = \Pi_T\left(t - T/2\right) = \pi_T\left(t\right)$ *and whose impulse response is:*

$$h(t) = te^{-t} \cdot \mathrm{u}(t)$$

Also find the duration of the output and the memory of the system.

<u>Solution.</u>

Again, we want to use Eq. 8-17:

$$\mathrm{ext}\left\{y(t)\right\} = \left[t_{h1} + t_{x1}, t_{h2} + t_{x2}\right]$$

From the definitions and plots of $x(t)$ and $h(t)$ we easily see that:

$$\mathrm{ext}\left\{x(t)\right\} = \left[0, T\right],$$

$$\operatorname{ext}\{h(t)\} = [0, +\infty].$$

As a result:

$$\operatorname{ext}\{y(t)\} = [0+0, T+\infty] = [0, \infty];$$

Finally, both the duration of $y(t)$ and M_h are equal to ∞.

8.4.5.2 more results on manipulating convolution integrands

In general, given a signal $x(t)$ and a specific time-instant t_0 where does the signal value $x(t_0)$ show up vs. the variable θ in the signal $x(t-\theta)$?

The answer is straightforward and is useful in a number of practical cases. Clearly, it is enough to equate the two arguments and then solve for θ :

$$x(t_0) = x(t-\theta) \quad \Leftrightarrow \quad t_0 = t-\theta \quad \Leftrightarrow \quad \theta = t - t_0$$

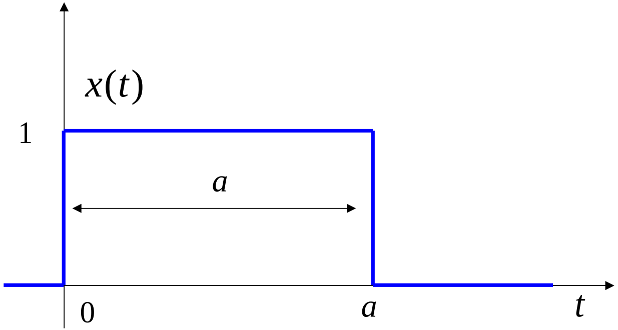
In other words, the signal value that shows up at time t_0 *now shows up at* $\theta = t - t_0$.

Check $\boxed{\text{on your own}}$ that this general result provides an alternative way to derive the formula:

$$\underset{\theta}{\text{ext}}\left\{x(t-\theta)\right\} = \left[t - t_{x2}, t - t_{x1}\right]$$

8.4.6 Evaluating convolution products

We will now go through the detailed calculations related to the evaluation of the response to the same simple rectangular input signal $x(t)$ of two different systems. The signal $x(t)$ is shown below:



Such signal can be written as:

$$x(t) = \Pi_a\left(t - \frac{a}{2}\right) = \pi_a(t)$$

where the rightmost side is just a shorthand that we will use to simplify notation.

Example 1

An LTI system has impulse response:

$$h(t) = \Pi_b\left(t - \frac{b}{2}\right) = \pi_b(t)$$

Find the output when the input signal is $x(t) = \pi_a(t)$, *with* $a > b$.

As for all LTI systems, the output is found by calculating the convolution product:

$$y(t) = x(t) * h(t)$$

Specifically, we have:

$$y(t)=\int_{-\infty}^{+\infty} h(\theta)x(t-\theta)d\theta=\int_{-\infty}^{+\infty} x(\theta)h(t-\theta)d\theta$$

The two forms above of the convolution product are completely equivalent. Here we use the latter. Substituting, we get:

$$y(t)=\int_{-\infty}^{+\infty} \pi_a(\theta)\pi_b(t-\theta)d\theta$$

The integration variable is θ and the integral must be performed over the whole of $\mathbb{R}$. Note that one of the two rectangular signals, $\pi_a(\theta)$, is "stationary" over θ.

The extension of $\pi_a(\theta)$ is simply:

$$\text{ext}\left\{\pi_a(\theta)\right\}=\left[0,a\right]$$

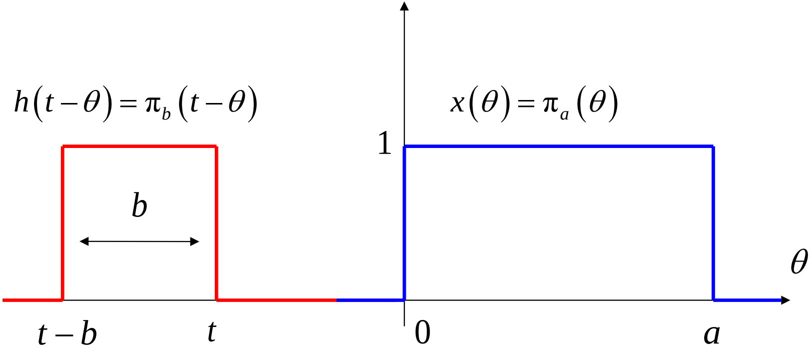
The other one, $\pi_b(t-\theta)$, shifts along the θ axis depending on the value of the parameter t. We recall the result given in Eq. 8-16: given a generic signal $s(t)$ with extension $\text{ext}\{s(t)\} = [t_{s1}, t_{s2}]$, then: $\underset{\theta}{\text{ext}}\{s(t-\theta)\} = [t - t_{s2}, t - t_{s1}]$.

Therefore:

$$\underset{\theta}{\text{ext}}\{\pi_b(t-\theta)\} = [t-b, t]$$

We can then graphically discuss the integral. We have different possible cases which will be dealt with one by one.

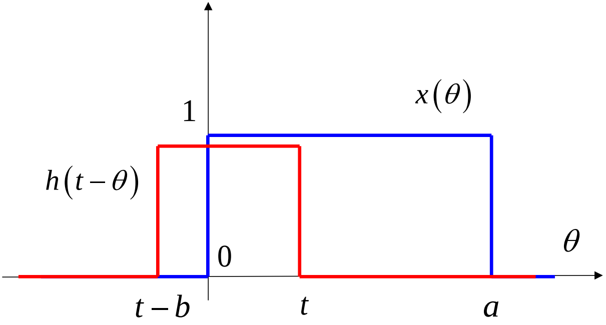
Case $t < 0$



The two rectangular functions have disjoint extensions. Their product is $0(t), \forall\theta \in \mathbb{R}$. As a result the integral is 0:

$$y(t) = 0, \quad t < 0$$

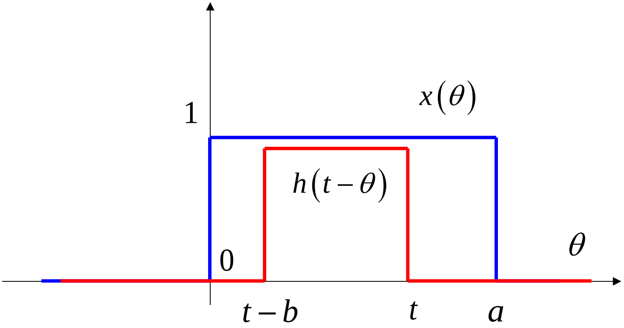
Case $t > 0, \quad t - b < 0$



The two rectangular functions have partially superimposed extensions. The resulting integral is:

$$y(t) = \int_0^t 1(\theta)\, d\theta = t, \quad 0 < t < b$$

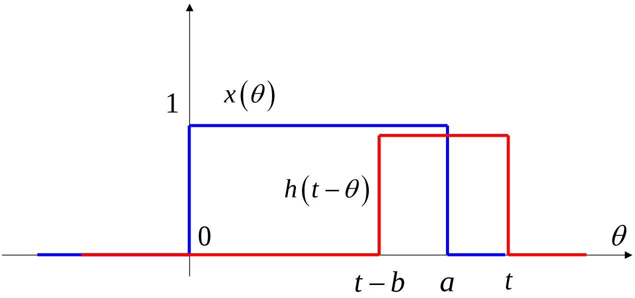
Case $t-b>0, \quad t<a$



The two rectangular functions have fully superimposed extensions. The resulting integral is:

$$y(t)=\int_{t-b}^{t} 1(\theta)\,d\theta=b, \quad b<t<a$$

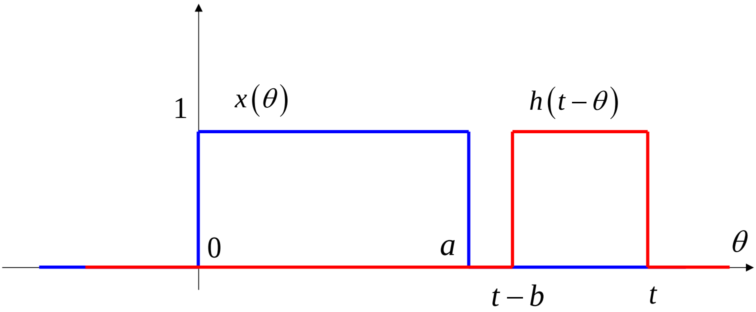
Case $t > a, \quad t - b < a$



The two rectangular functions have again partially superimposed extensions. The resulting integral is:

$$y(t) = \int_{t-b}^{a} 1(\theta)\, d\theta = \theta\Big|_{t-b}^{a} = a + b - t, \quad a < t < a + b$$

Case $t > a + b$



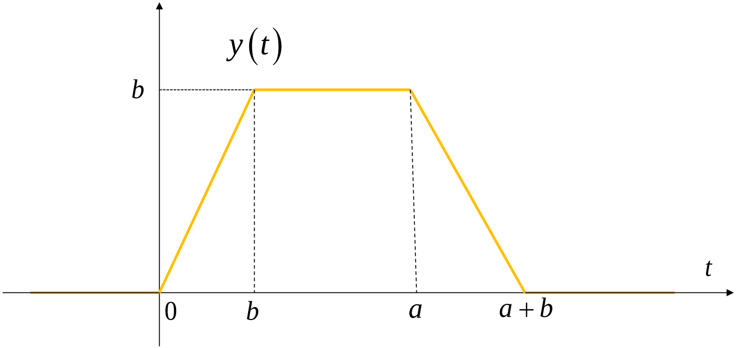
The two rectangular functions have completely disjoint extensions. Their product is $0(t), \forall \theta \in \mathbb{R}$. As a result the integral is 0:

$$y(t) = 0, \quad t > a + b$$

Pulling together all the partial results, we get:

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < b \\ b & b < t < a \\ a + b - t & a < t < a + b \\ 0 & t > a + b \end{cases}$$

The plot of the output is the following:



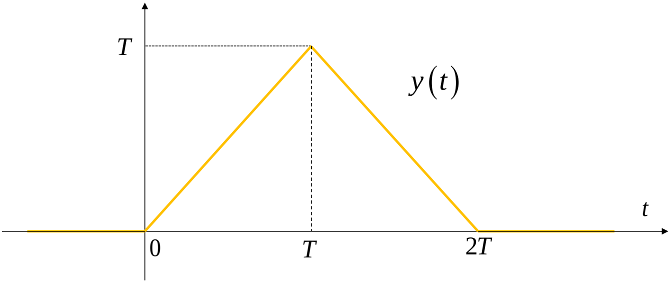
A few remarks: the extension of $y(t)$ is in fact:

$$\text{ext}\{y(t)\} = \left[t_{h1} + t_{x1}, t_{h2} + t_{x2}\right] = [0, a+b]$$

Also, the duration of $y(t)$ is $a+b$, that is, the sum of the duration of $x(t)$ and $h(t)$, as expected. Finally, it is interesting to remark that even though both $x(t)$ and $h(t)$ are discontinuous functions, $y(t) = x(t) * h(t)$ is continuous. This aspect will be commented on again later.

On your own: redo the above problem assuming that $b > a$

Notice that in the case $b = a = T$, then we have $y(t) = \pi_T(t) * \pi_T(t) = T \cdot \Lambda_T(t - T)$, as shown in the figure below:



$\boxed{\text{On your own:}}$ show that the following result holds:

$$y(t) = \Pi_T(t) * \Pi_T(t) = T \cdot \Lambda_T(t)$$

Example 2

An LTI system has impulse response:

$$h(t) = e^{-at} \cdot \mathrm{u}(t) \quad , \quad a > 0$$

Find the output when the input signal is $x(t) = \pi_T(t)$.

Again, we have:

$$y(t) = x(t) * h(t)$$

As before, there are two forms of the convolution product:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} h(\theta)x(t-\theta)d\theta = \int_{-\infty}^{+\infty} x(\theta)h(t-\theta)d\theta$$

We choose again to use the latter.

On your own, redo all the calculations using the other form.

The integration variable is θ and the integral must be performed over the whole of $\mathbb{R}$. The rectangular signal $\pi_T(\theta)$ does not shift according to t, that is, its position along the θ axis does not change. The extension of $\pi_T(\theta)$ is simply:

$$\text{ext}\left\{\pi_T(\theta)\right\} = \left[0, T\right]$$

The other signal, $h(t-\theta)$, shifts along the θ axis depending on the value of t. Recalling again Eq. 8-16, we have:

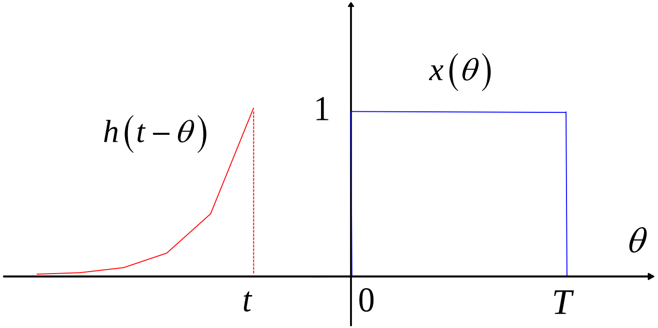
$$\underset{\theta}{\text{ext}}\left\{h(t-\theta)\right\} = \left[t - t_{h_2}, t - t_{h_1}\right]$$

Clearly: $\text{ext}\left\{h(t)\right\} = \text{ext}\left\{e^{-at} \cdot \text{u}(t)\right\} = \left[0, \infty\right]$ and therefore:

$$\underset{\theta}{\text{ext}}\left\{h(t-\theta)\right\} = \left[-\infty, t\right]$$

We can then graphically discuss the integral.

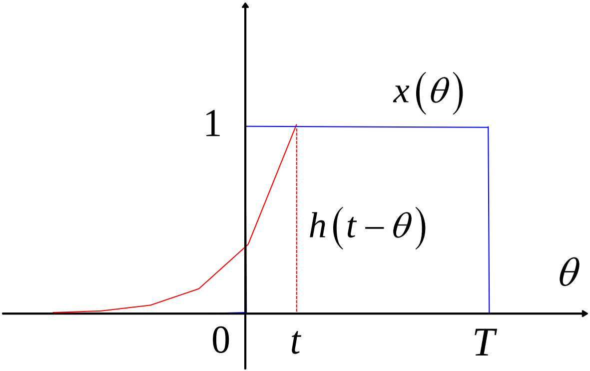
Case $t<0$



The two functions $h(t-\theta)$ and $x(\theta)$ have disjoint extensions. Their product is $0(t), \forall\theta\in\mathbb{R}$. As a result the integral is 0:

$$y(t)=0, \quad t<0$$

Case $t > 0, \quad t < T$



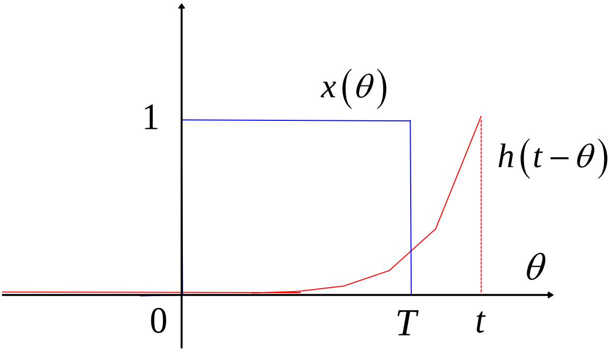
The two functions $h(t-\theta)$ and $x(\theta)$ have partially superimposed extensions.

The resulting integral is:

$$y(t)=\int_0^t e^{-a(t-\theta)}d\theta = e^{-at}\int_0^t e^{a\theta}d\theta = e^{-at}\left[\frac{e^{a\theta}}{a}\right]_0^t =$$

$$=\frac{e^{-at}}{a}\left(e^{at}-1\right)=\frac{1-e^{-at}}{a},\quad 0<t<T$$

Case $t>T$



The extension of $x(\theta)$ is completely superimposed to that of $h(t-\theta)$. The resulting integral is:

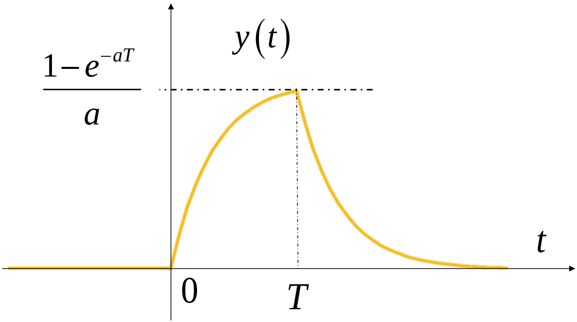
$$y(t)=\int_0^T e^{-at}e^{a\theta}d\theta = e^{-at}\left[\frac{e^{a\theta}}{a}\right]_0^T = \frac{e^{-at}}{a}\cdot\left(e^{aT}-1\right) = \frac{e^{-a(t-T)}-e^{-at}}{a} =$$

$$\frac{e^{-a(t-T)}-e^{-at}e^{aT}e^{-aT}}{a} = \frac{e^{-a(t-T)}-e^{-a(t-T)}e^{-aT}}{a} = \frac{1}{a}\left(1-e^{-aT}\right)\cdot e^{-a(t-T)}$$

Pulling together all the partial results, we get:

$$y(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{a}\left(1 - e^{-at}\right) & 0 < t < T \\ \dfrac{1}{a}\left(1 - e^{-aT}\right) \cdot e^{-a(t-T)} & t > T \end{cases}$$

and the graphical representation of $y(t)$ is:



A few remarks: the extension of $y(t)$ is in fact:

$$\text{ext}\{y(t)\} = \left[t_{h1} + t_{x1}, t_{h2} + t_{x2}\right] = [0, \infty]$$

Also, the duration of $y(t)$ is ∞. Finally, as in the previous example, both $x(t)$ and $h(t)$ are discontinuous functions, but $y(t) = x(t) * h(t)$ is continuous. This aspect will be commented on later.

$\boxed{\text{On your own:}}$ redo the above two examples of convolution product, using the alternate form of the convolution formula:

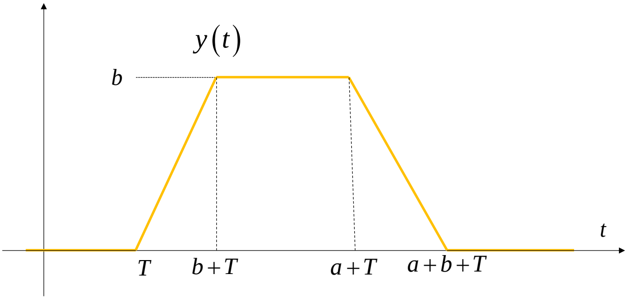
$$x(t) * h(t) = \int_{-\infty}^{+\infty} h(\theta) x(t - \theta) d\theta$$

and see if you can find the same result.

On your own:

An LTI system has impulse response $h(t)=\pi_b(t)$. It is given the input $x(t)=\pi_a(t-T)$, with $a>b$ and $T>0$. Find the output $y(t)$.

Answer:



Compare the above result with that of example 1. Could the result have been predicted, without re-doing all the calculations?

$\boxed{\text{On your own:}}$

An LTI system has impulse response $h(t) = \left(1 - t/b\right) \cdot \pi_b(t)$. It is given the input $x(t) = \pi_a(t) - \pi_a(t-a)$, with $a > b$. Find the output $y\left(t\right)$.

8.4.6.2 Convolutions of decreasing exponentials

Let us define:

$$w_n\left(t\right) = \frac{t^n}{n!} e^{-at} \cdot u\left(t\right)$$

One interesting feature of these functions is the following:

$$w_n\left(t\right) * w_m\left(t\right) = w_{n+m+1}\left(t\right)$$

Note that this is valid even for the special case $a = 0$.

Showing that this is true in frequency domain. From the table of transforms in Chapter 6, we know that:

$$W_n(f) = \frac{1}{(a + j2\pi f)^{n+1}}$$

Carrying out the convolution in frequency domain, we then get:

$$W_n(f) \cdot W_m(f) = \frac{1}{(a + j2\pi f)^{n+1}} \cdot \frac{1}{(a + j2\pi f)^{m+1}} = \frac{1}{(a + j2\pi f)^{n+m+2}} = W_{n+m+1}(f)$$

On your own:

Carry out the convolution product

$$w_0(t) * w_0(t)$$

both in frequency domain and in time domain and show that the result is $w_1(t)$.

Comment: calculations are very easy both in frequency and time domain.

On your own:

Redo the above exercise and show that the result holds even if $a = 0$.

8.5 LTI Systems and Fourier Transforms

We have seen that LTI systems can be given a unified description based on their impulse response and the convolution product. All this was done in time-domain. We will now move to frequency-domain. We will find out that LTI systems can be dealt with again in a unified way, which in most cases is actually easier and more powerful than the time-domain description.

8.5.1 The Transfer Function

We first recall that for all LTI systems:

$$y(t) = x(t) * h(t)$$

We then take the Fourier transform of both sides of this equation:

$$\mathrm{F}\left\{y(t)\right\} = Y(f) = \mathrm{F}\left\{x(t) * h(t)\right\} =$$

$$\mathrm{F}\left\{\int_{-\infty}^{\infty} x(\theta)h(t-\theta)d\theta\right\} = \int_{-\infty}^{+\infty} e^{-j2\pi f t} \int_{-\infty}^{\infty} x(\theta)h(t-\theta)d\theta\, dt$$

We then exchange the order of integration, by first integrating over t:

$$\mathrm{F}\left\{y(t)\right\} = \int_{-\infty}^{+\infty} x(\theta) \int_{-\infty}^{+\infty} h(t-\theta)e^{-j2\pi f t} dt\, d\theta$$

Eq. 8-19

Note that the inner integral is simply the Fourier transform of $h(t-\theta)$:

$$\int_{-\infty}^{+\infty} h(t-\theta)e^{-j2\pi f t}dt = \mathrm{F}\{h(t-\theta)\} = H(f)e^{-j2\pi f\theta}$$

Eq. 8-20

where $H(f) = \mathrm{F}\{h(t)\}$.

Substituting the right-hand side of Eq. 8-20 into Eq. 8-19, we get:

$$\mathrm{F}\{y(t)\} = \int_{-\infty}^{+\infty} x(\theta)H(f)e^{-j2\pi f\theta}d\theta =$$

$$= H(f)\int_{-\infty}^{+\infty} x(\theta)e^{-j2\pi f\theta}d\theta = H(f)X(f)$$

So, to summarize:

$$Y(f) = H(f)X(f)$$

The Fourier transform of the impulse response, $H(f)$, is clearly an extremely important quantity. It is called the *transfer function* of the system.

We now have two ways of dealing with any possible LTI system. A time-domain unified description and a frequency-domain unified description:

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f)$$

The fact that the latter involves only an ordinary product rather than a convolution product makes it quite powerful. On the other hand, it must be said that it requires that the Fourier transforms of both $h(t)$ and $x(t)$ be computed. Also, if the result is needed in time-domain, the inverse Fourier transform of $Y(f)$.

Nonetheless, the frequency-domain representation typically allows to gain excellent insight on how the system operates and therefore it is more commonly used than the other.

8.5.1.1 Example

From Sect. we saw that given:

$$x(t) = \pi_T(t)$$

$$h(t) = \pi_T(t)$$

then:

$$y(t) = x(t) * h(t)$$

$$= \pi_T(t) * \pi_T(t) = T \cdot \Lambda_T(t)$$

The calculation was done using the convolution product in time domain.

We now want to do it in frequency domain.

$$Y(f) = X(f) \cdot H(f)$$

We know that:

$$F\left\{x(t)\right\} = F\left\{h(t)\right\} = T \cdot \mathrm{Sinc}\left(T \cdot f\right)$$

so, we can directly write:

$$Y(f) = X(f) \cdot H(f) = T^2 \cdot \mathrm{Sinc}^2\left(T \cdot f\right)$$

From the table of transforms in Chapter 6, we know that:

$$F^{-1}\left\{T \cdot \mathrm{Sinc}^2\left(T \cdot f\right)\right\} = \Lambda_T\left(t\right)$$

so that:

$$y(t) = F^{-1}\left\{T^2 \cdot \mathrm{Sinc}^2\left(T \cdot f\right)\right\} = T \cdot \Lambda_T\left(t\right)$$

Therefore the two results, the one obtained in time-domain and the one obtained in frequency-domain, coincide.

8.5.1.2 Fourier transforms of convolution products

In the previous section we did not make any specific assumption on the signals $h(t)$ and $x(t)$. As a consequence, the above result on the Fourier transform of a convolution product is in fact quite general and constitutes a property of Fourier transforms. We can write:

$$s_1(t) * s_2(t) \quad \xleftrightarrow{\mathrm{F}} \quad S_1(f) \cdot S_2(f)$$

Eq. 8-21

It is also easy to prove the dual relationship, using similar methods:

$$s_1(t) \cdot s_2(t) \quad \xleftrightarrow{\mathrm{F}} \quad S_1(f) * S_2(f)$$

Note the special case of "squared" signals:

$$s^2(t) \xleftrightarrow{\mathrm{F}} S(f) * S(f)$$

Ordering of convolutions

By repeatedly using Eq. 8-21 above, it is easy to find that:

$$\mathrm{F}\left\{\left[s_1(t) * s_2(t)\right] * s_3(t)\right\} \xleftrightarrow{\mathrm{F}} S_1(f) \cdot S_2(f) \cdot S_3(f)$$

$$\mathrm{F}\left\{s_1(t) * \left[s_2(t) * s_3(t)\right]\right\} \xleftrightarrow{\mathrm{F}} S_1(f) \cdot S_2(f) \cdot S_3(f)$$

$$\mathrm{F}\left\{\left[s_1(t) * s_3(t)\right] * s_1(t)\right\} \xleftrightarrow{\mathrm{F}} S_1(f) \cdot S_2(f) \cdot S_3(f)$$

Specifically, one can reason this way. Let’s consider: $F\left\{\left[s_1(t) * s_2(t)\right] * s_3(t)\right\}$

This can be re-written as:

$$\mathrm{F}\left\{w(t) * s_3(t)\right\} \xleftrightarrow{\mathrm{F}} W(f) \cdot S_3(f)$$

having defined: $w(t) = [s_1(t) * s_2(t)]$.

But then:

$$W(f) = S_1(f) \cdot S_2(f)$$

and substituting we indeed obtain:

$$F\{w(t) * s_3(t)\} = F\{[s_1(t) * s_2(t)] * s_3(t)\} \quad \xleftrightarrow{F} \quad S_1(f) \cdot S_2(f) \cdot S_3(f)$$

A similar procedure can be used for $\mathrm{F}\{s_1(t) * [s_2(t) * s_3(t)]\}$ and for $\mathrm{F}\{[s_1(t) * s_3(t)] * s_1(t)\}$, always obtaining the same result: $S_1(f) \cdot S_2(f) \cdot S_3(f)$

This result shows that executing the convolution product in any order on the left-hand side leads to the same Fourier transform. However, since two signals that have the same Fourier transform are the same signal, this also means that:

$$\left[s_1(t) * s_2(t)\right] * s_3(t) = s_1(t) * \left[s_2(t) * s_3(t)\right] = \left[s_1(t) * s_3(t)\right] * s_2(t)$$

This shows the important result that ***the ordering of the convolution of any three signals does not change the result***.

As a result, there is no need to use brackets when indicating a cascade of three convolutional products. Specifically, it is enough to write:

$$w(t) = s_1(t) * s_2(t) * s_3(t)$$

and, regarding the Fourier transform:

$$\mathrm{F}\left\{s_1(t)*s_2(t)*s_3(t)\right\} \quad \xleftrightarrow{\mathrm{F}} \quad S_1(f)\cdot S_2(f)\cdot S_3(f)$$

This result can be easily generalized, simply by repeated application of Eq. 8-21, to a cascade of any number convolution products:

$$\begin{aligned}
&\mathrm{F}\left\{\left[\left[\left[\left[s_1(t)*s_2(t)\right]*s_3(t)\right]*s_4(t)\right]*\ldots\right]*s_N(t)\right\} = \\
&= \mathrm{F}\left\{s_1(t)*s_2(t)*s_3(t)*s_4(t)*\ldots*s_N(t)\right\} \\
&= S_1(f)\cdot S_2(f)\cdot S_3(f)\cdot S_4(f)\cdot\ldots\cdot S_N(f)
\end{aligned}$$

Eq. 8-22

These results will be discussed again in Sect. 8.5.3.1.

8.5.2 LTI systems response to sinusoidal signals

We now discuss how an LTI system responds in general to an input of the form:

$$x(t) = A \cdot e^{j\varphi} \cdot e^{j2\pi f_0 t}$$

where A is a constant that sets the "amplitude" of the oscillation and φ is a constant phase-shift.

This is a key result and students are supposed to know it well.

We know that:

$$Y(f) = X(f) \cdot H(f)$$

So we first take the Fourier transforms of the input:

$$X(f) = A \cdot e^{j\varphi} \cdot \delta\left(f - f_0\right)$$

We then have:

$$Y(f) = X(f) \cdot H(f) = A \cdot e^{j\varphi} \cdot \delta\left(f - f_0\right) \cdot H(f) =$$
$$A \cdot e^{j\varphi} \cdot H(f_0) \cdot \delta\left(f - f_0\right)$$

Going back to time-domain:

$$y(t) = \mathrm{F}^{-1}\left\{A \cdot e^{j\varphi} \cdot H(f) \cdot \delta\left(f - f_0\right)\right\}$$
$$= A \cdot e^{j\varphi} \cdot H(f_0) \cdot \mathrm{F}^{-1}\left\{\delta\left(f - f_0\right)\right\} =$$
$$= H(f_0) \cdot A \cdot e^{j2\pi f_0 t} e^{j\varphi} = H(f_0) \cdot x(t)$$

This is an extremely important result in the theory of LTI systems. Simply put, it says that given a pure "frequency tone" at the input, the output remains the same, except for a multiplicative constant. Such multiplicative constant is the transfer function, at the frequency of the input tone.

In operator writing, and omitting the multiplicative constants which do not change the result:

$$L\left\{e^{j2\pi f_0 t}\right\} = H(f_0)\cdot e^{j2\pi f_0 t}$$

Eq. 8-23

This result reveals important aspects of LTI systems. One aspect is the fact that an LTI system *cannot create new frequencies*. Given a frequency tone at the input, it is also found at the output, with the same frequency value. No other frequencies are created.

This is actually also clear from the formula:

$$Y(f) = X(f)\cdot H(f)$$

If $X(f)$ has certain non-zero frequency components, those will be found in $Y(f)$ multiplied by $H(f)$. If a certain frequency component is not present in the input, that is if $X(f_1)=0$, then f_1 cannot be present at the output, i.e., it will be $Y(f_1)=0$ The LTI system cannot "create it". Note that non-linear systems can instead create new frequencies.

We now extend the result to "real" tones, that is sines and cosines. In this context we will also make the assumption that $h(t) \in \mathbb{R}$.

For cosines:

$$x(t) = A \cdot \cos\left(2\pi f_0 t + \varphi\right)$$

$$X(f) = A \cdot \mathrm{F}\left\{\cos\left(2\pi f_0 t + \varphi\right)\right\} = A \cdot \mathrm{F}\left\{\frac{e^{j2\pi f_0 t} e^{j\varphi} + e^{-j2\pi f_0 t} e^{-j\varphi}}{2}\right\} =$$

$$= \frac{A}{2}\left[e^{j\varphi} \cdot \delta\left(f - f_0\right) + e^{-j\varphi} \cdot \delta\left(f + f_0\right)\right]$$

$$Y(f) = X(f) \cdot H(f) = \frac{A}{2}\left[e^{j\varphi} \cdot \delta\left(f - f_0\right) + e^{-j\varphi} \cdot \delta\left(f + f_0\right)\right] \cdot H(f) =$$

$$= \frac{A}{2}\left[e^{j\varphi} \cdot \delta\left(f - f_0\right) H(f_0) + e^{-j\varphi} \cdot \delta\left(f + f_0\right) H(-f_0)\right]$$

Using the polar form for the complex numbers:

$$H\left(f\right) = \left|H(f)\right| e^{j\varphi_H(f)}$$

$$H\left(-f\right) = \left|H(-f)\right| e^{j\varphi_H(-f)}$$

we can write:

$$Y(f)=\frac{A}{2}\left[e^{j\varphi}\cdot\delta\left(f-f_0\right)\left|H(f_0)\right|e^{j\varphi_H(f_0)}+e^{-j\varphi}\cdot\delta\left(f+f_0\right)\left|H(-f_0)\right|e^{j\varphi_H(-f_0)}\right]$$

However, since we have assumed $h(t)\in\mathbb{R}$, then we know that:

$$H(-f)=H^*(f)$$

or, using the polar form:

$$H\left(f\right)=\left|H(f)\right|e^{j\varphi_H(f)}$$

$$H\left(-f\right)=\left|H(f)\right|e^{-j\varphi_H(f)}$$

Taking this into account:

$$Y(f) =$$
$$= \frac{A}{2}\left[e^{j\varphi} \cdot \delta\left(f - f_0\right)\left|H(f_0)\right|e^{j\varphi_H(f_0)} + e^{-j\varphi} \cdot \delta\left(f + f_0\right)\left|H(f_0)\right|e^{-j\varphi_H(f_0)}\right]$$
$$= \frac{A}{2}\left|H(f_0)\right|\left[e^{j\left[\varphi+\varphi_H(f_0)\right]} \cdot \delta\left(f - f_0\right) + e^{-j\left[\varphi+\varphi_H(f_0)\right]} \cdot \delta\left(f + f_0\right)\right]$$

which contains the transform of a cosine. Going back to time-domain:

$$y(t) = A \cdot \left|H(f_0)\right| \cdot \cos\left(2\pi f_0 t + \varphi + \varphi_H(f_0)\right)$$

Eq. 8-24

Finally, for the real sine waveform, again assuming $h(t) \in \mathbb{R}$, we get with similar calculations:

$$x(t) = A \cdot \sin\left(2\pi f_0 t + \varphi\right)$$

$$X(f) = A \cdot \mathrm{F}\left\{\sin\left(2\pi f_0 t + \varphi\right)\right\} =$$

$$= A \cdot \mathrm{F}\left\{\frac{e^{j2\pi f_0 t} e^{j\varphi} - e^{-j2\pi f_0 t} e^{-j\varphi}}{2j}\right\} =$$

$$= \frac{A}{2j}\left[e^{j\varphi} \cdot \delta\left(f - f_0\right) - e^{-j\varphi} \cdot \delta\left(f + f_0\right)\right]$$

$$Y(f) = X(f) \cdot H(f) =$$

$$= \frac{A}{2j}\left|H(f_0)\right|\left[e^{j\left[\varphi + \varphi_H(f_0)\right]} \cdot \delta\left(f - f_0\right) - e^{-j\left[\varphi + \varphi_H(f_0)\right]} \cdot \delta\left(f + f_0\right)\right]$$

$$y(t) = A \cdot \left|H(f_0)\right| \cdot \sin\left(2\pi f_0 t + \varphi + \varphi_H(f_0)\right)$$

Eq. 8-25

$\boxed{\text{On your own:}}$ show that we could have found Eq. 8-25 from Eq. 8-24 simply by leveraging the trigonometric identity:

$$\cos\left(\alpha - \pi/2\right) = \sin\left(\alpha\right)$$

To summarize these important results, we can say that for all LTI systems:

$$L\left\{A \cdot e^{j\varphi} \cdot e^{j2\pi f_0 t}\right\} = H(f_0) \cdot A \cdot e^{j\varphi} \cdot e^{j2\pi f_0 t}$$

Eq. 8-26

For all *real* LTI systems, that is when $h(t) \in \mathbb{R}$:

$$L\left\{A \cdot \cos\left(2\pi f_0 t + \varphi\right)\right\} = A \cdot \left|H(f_0)\right| \cdot \cos\left(2\pi f_0 t + \varphi + \varphi_H(f_0)\right)$$

$$L\left\{A\cdot\sin\left(2\pi f_0 t+\varphi\right)\right\}=A\cdot\left|H(f_0)\right|\cdot\sin\left(2\pi f_0 t+\varphi+\varphi_H(f_0)\right)$$

8.5.2.1 pure tones as eigenfunctions of LTI systems

We have repeatedly pointed out that a signal can be seen as an element of an inner product (or Hilbert) space. We have also shown that the orthonormal set:

$$\left\{e^{j2\pi f_0 t}\right\}_{f_0=-\infty}^{+\infty}$$

is a basis for the Hilbert space $L^2_{\mathbb{R}}$, but can also be used to represent some finite average-power signals (such as periodic signals).

We now point out that in simple Hilbert spaces, such as $\mathbb{R}^3$, one can define linear transformations, typically in the form of matrices. Any vector $\overline{x}=\left[x_1,x_2,x_3\right]\in\mathbb{R}^3$ can be input to such linear transformation to produce an output vector:

$$\bar{y} = [y_1, y_2, y_3] \in \mathbb{R}^3$$

That is typically done by means of a row-by-column product. For instance, given the matrix:

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = [m_{nk}]$$

then:

$$\bar{y}^T = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{M} \cdot \bar{x}^T = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This can be re-written compactly as:

$$\left[y_n\right] = \sum_{k=1}^{3} m_{nk} \cdot x_k \qquad n = 1...3$$

This operation is rather complex and involves nine products and 6 sums. However, it is much simpler if the basis used to express vectors in $\mathbb{R}^3$ *coincides with the eigenvectors of the operator* $\mathbf{M}$.

Note that for this to be possible, it must be that $\mathbf{M}$ has 3 linearly independent eigenvectors. This can also be stated by saying that $\mathbf{M}$ must be *diagonalizable*. For the purpose of this section, we need actually assume an even stricter condition: we assume that $\mathbf{M}$ has 3 *orthogonal* eigenvectors (not just linearly independent). For this to be the case, $\mathbf{M}$ must be *Hermitian*. If so, the three orthogonal eigenvectors can be normalized to unit norm and can therefore form an *orthonormal basis* of $\mathbb{R}^3$.

Therefore, we assume that $\mathbf{M}$ is Hermitian and we call the orthonormal basis formed by its eigenvectors as: $\left\{\hat{u}_{\mathbf{M}_n}\right\}_{n=1}^{3}$. We then express all vectors according to the new orthonormal basis $\left\{\hat{u}_{\mathbf{M}_n}\right\}_{n=1}^{3}$. As a result, we can write:

$$\bar{y}^T = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{M} \cdot \bar{x}^T = \begin{bmatrix} \lambda_{M_1} & 0 & 0 \\ 0 & \lambda_{M_2} & 0 \\ 0 & 0 & \lambda_{M_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda_{M_1} \cdot x_1 \\ \lambda_{M_2} \cdot x_2 \\ \lambda_{M_3} \cdot x_3 \end{bmatrix}$$

which requires just three products and no sums. The numbers $\lambda_{M_1}, \lambda_{M_2}, \lambda_{M_3}$, are the so-called *eigenvalues* of $\mathbf{M}$. Specifically, they are those numbers that multiply an input eigenvector to form the respective output vector:

$$\mathbf{M}\cdot\hat{u}_{\mathbf{M}_1}=\lambda_{M_1}\cdot\hat{u}_{\mathbf{M}_1},\quad \mathbf{M}\cdot\hat{u}_{\mathbf{M}_2}=\lambda_{M_2}\cdot\hat{u}_{\mathbf{M}_2},\quad \mathbf{M}\cdot\hat{u}_{\mathbf{M}_3}=\lambda_{M_3}\cdot\hat{u}_{\mathbf{M}_3}$$

The reason why LTI systems are complicated to solve in time-domain and extremely easy to solve in frequency-domain, is the same reason why solving the linear transformation shown above in $\mathbb{R}^3$ is complicated in general but much easier in the basis of the eigenvectors of the matrix operator. That is:

the basis of the frequency domain:

$$\left\{e^{j2\pi f_0 t}\right\}_{f_0=-\infty}^{+\infty}$$

consists of the "eigenvectors", or eigenfunctions, of all possible LTI systems. In addition, we remark that they form an orthonormal basis.

The eigenvalues of the operator are found by using each of the eigenfunctions as input:

$$L\left\{ e^{j2\pi f t}\Big|_{f=f_0} \right\} = L\left\{ e^{j2\pi f_0 t} \right\} = H(f_0)\cdot e^{j2\pi f_0 t}$$

So:

the transfer function $H(f)$ *is the collection of all eigenvalues of the LTI operator* L *vs. all of the eigenfunctions* $\left\{ e^{j2\pi f_0 t} \right\}_{f_0=-\infty}^{+\infty}$.

When we take the Fourier transform of a signal $x(t)$, we express the signal in the basis of the eigenfunctions of the LTI systems. Indeed, as it happens in $\mathbb{R}^3$, to find the output to any input to any LTI system we just need to multiply the component vs. each eigenfunction times the eigenvalue of the corresponding eigenfunction, that is:

$$Y(f) = X(f) \cdot H(f)$$

This is the reason why the Fourier transform is so powerful in dealing with LTI systems: it is a change-of-basis operator that puts any signal into the basis of the eigenfunctions of the LTI system. It also directly finds the eigenvalues describing the operator as $H(f)$.

Advanced, optional

On your own, find out what happens for LTI operators having periodic signals as inputs and show that the resulting mathematical relations are identical to matrix operators over a vector space $\mathbb{C}^\infty$.

8.5.2.2 the "integrator" system

Given the following system, on your own:

$$\Omega\left\{x(t)\right\} = \int_{-\infty}^{t} x(\theta)d\theta$$

- prove that it is linear
- prove that it is time invariant
- find its impulse response $h(t)$
- show that $h(t)$ is causal and find its extension

The result for the impulse response is $h(t) = \mathrm{u}(t)$. We now remark that this operator is exactly the

$$\frac{d^{-1}}{dt^{-1}}$$

operator that we discussed when dealing with differentiability classes. It is also called the anti-derivative operator.

By definition, its transfer function is:

$$H(f) = \mathrm{F}\{h(t)\} = \mathrm{F}\{\mathrm{u}(t)\}$$

We already discussed this transform in Section 6.3. The result is therefore:

$$H(f) = \mathrm{F}\{h(t)\} = \mathrm{F}\{\mathrm{u}(t)\} = \frac{\delta(f)}{2} + \frac{1}{2j\pi f}$$

This result is often cited as a "property" of the Fourier transform. It is said that the:

the Fourier transform of the "integral" of a signal (or, more precisely, of its anti-derivative) is given by:

$$\mathrm{F}\left\{\frac{d^{-1}}{dt^{-1}}x(t)\right\} = \mathrm{F}\left\{x(t) * \mathrm{u}(t)\right\} = \mathrm{F}\left\{x(t)\right\} \cdot \mathrm{F}\left\{\mathrm{u}(t)\right\} =$$

$$= \left[\frac{\delta(f)}{2} + \frac{1}{2j\pi f}\right] X(f)$$

Eq. 8-27

8.5.3 Combining LTI systems

LTI systems can be cascaded, or linearly combined. An exceptionally important result is that any cascade, or linear combination, or mixture of the two, of any number of LTI systems, gives rise to one, equivalent, operator which is still an LTI system.

8.5.3.1 cascaded LTI systems

Given a generic cascade of LTI systems, the overall output of the cascade is given by:

$$y(t) = \Big[\Big[\big[\big[x(t) * h_1(t)\big] * h_2(t)\big] * h_3(t)\Big] .... * h_N(t)\Big] = \Omega\{x(t)\}$$

A first important result related to such cascade is that:

the overall resulting operator Ω *is still an LTI system.*

This appears obvious by looking at the cascade of LTI systems in frequency domain. We know that under Fourier transform the cascaded convolution products become:

$$Y(f) = X(f) \cdot H_1(f) \cdot H_2(f) \cdot H_3(f) \cdots H_N(f)$$

Clearly, we can define:

$$H_{eq}(f) = \prod_{n=1}^{N} H_n(f)$$

and then the overall cascade reduces to one equivalent LTI system:

$$Y(f) = H_{eq}(f) X(f)$$

Going back to time domain, there will then be an overall equivalent impulse response:

$$h_{eq}(t) = \mathrm{F}\left\{H_{eq}(f)\right\}$$

with an equivalent LTI system being simply:

$$y(t) = x(t) * h_{eq}(t)$$

Another extremely important property of cascaded LTI systems is that:

the overall equivalent LTI system is independent of the order in which the individual LTI systems are applied.

This is easily seen in frequency domain, where we can re-order the transfer function product, obviously without the result changing at all. For example, if we just rearrange the order of the top cascade to obtain the bottom one:

$$Y(f) = X(f) \cdot H_1(f) \cdot H_2(f) \cdot H_3(f) \cdots H_N(f)$$

$$Y(f) = X(f) \cdot H_3(f) \cdot H_5(f) \cdot H_2(f) \cdots H_k(f)$$

then the bottom cascade cannot but give the same result as the top cascade, provided that the same transfer functions appear in the two cascades, the same number of times.

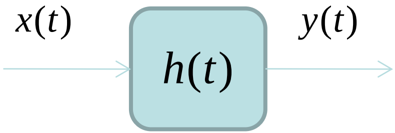
Also, if we take the inverse Fourier transform of the bottom cascade, according to the new ordering, we certainly still find the same $y(t)$, though it turns out to be generated by a different sequence of convolution products:

$$y(t) = \left[\left[\left[\left[x(t) * h_3(t)\right] * h_5(t)\right] * h_2(t)\right] \ldots * h_k(t)\right]$$

This result also proves a general property of the convolution product, i.e., that the convolution product is completely commutative. It also confirms the known general property of linear operators, i.e., that they fully commute with one another.

8.5.3.2 block diagrams of LTI systems

As already pointed out in Section 8.3, linear systems can be represented by means of block diagrams. The simplest such block diagram is:



which means that the input signal $x(t)$ goes through the LTI system, whose impulse response is $h(t)$, and that the output is $y(t)$.

Such block diagrams can be used to pictorially show the cascade of several LTI systems. For example:

which is equivalent to:

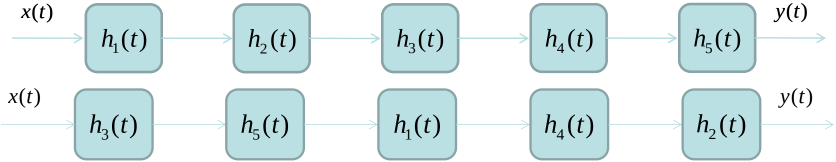
$$y(t) = \left[\left[\left[\left[x(t) * h_1(t)\right] * h_2(t)\right] * h_3(t)\right] * h_4(t)\right] * h_5(t)$$

Block diagrams can be "Fourier transformed", that is, converted to frequency domain. In essence, the topology of the block diagram remains the same. Simply, all signals are transformed into frequency domain:

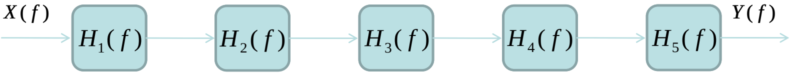
which is equivalent to:

$$Y(f) = X(f) \cdot H_1(f) \cdot H_2(f) \cdot H_3(f) \cdot H_4(f) \cdot H_5(f)$$

Commutativity can be easily visualized with block diagrams. In fact, the same exact overall equivalent LTI system as above is still obtained after changing the order of the component blocks, either in time or in frequency domain:



The two cascades above are equivalent,



and also the cascades above are.

8.5.3.3 linear combinations of LTI systems

Yet another important result, which can be very easily proved, is the following:

any linear combination of LTI systems is an LTI system

In math, if all the L_n operators are LTI systems, then L_{eq} given by:

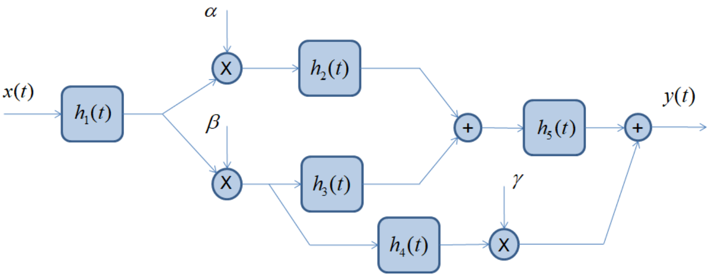
$$y(t) = \sum_{n=1}^{N} \alpha_n L_n \{x(t)\} = L_{eq}\{x(t)\}$$

is LTI too.

In addition, it is also easy to show that:

$$h_{eq}(t) = \sum_{n=1}^{N} a_n h_n(t) \qquad H_{eq}(t) = \sum_{n=1}^{N} a_n H_n(t)$$

Block diagrams can picture both cascades and linear combinations of LTI systems at the same time, by introducing the splitter, adder and multiplier symbols. The following example shows it:



In math, respecting the ordering:

$$y(t) = \gamma \cdot L_4 \left\{ \beta \cdot L_1 \left\{ x(t) \right\} \right\} + L_5 \left\{ L_2 \left\{ \alpha \cdot L_1 \left\{ x(t) \right\} \right\} \right\} + L_5 \left\{ L_3 \left\{ \beta \cdot L_1 \left\{ x(t) \right\} \right\} \right\} =$$
$$= L_{eq} \left\{ x(t) \right\} = x(t) * h_{eq}(t)$$

Replacing the operator notation with convolution products, and remembering that the ordering for convolution products is irrelevant, we can write:

$$y(t) = \gamma \cdot \beta \cdot x(t) * h_1(t) * h_4(t)$$
$$+\alpha \cdot x(t) * h_1(t) * h_2(t) * h_5(t)$$
$$+\beta \cdot x(t) * h_1(t) * h_3(t) * h_5(t)$$
$$= x(t) * \left[\gamma \cdot \beta \cdot h_1(t) * h_4(t) + \alpha \cdot h_1(t) * h_2(t) * h_5(t) + \beta \cdot h_1(t) * h_3(t) * h_5(t) \right]$$
$$= x(t) * h_{eq}(t)$$

The resulting system is still LTI, with impulse response $h_{eq}(t)$ given by:

$$h_{eq}(t) = \gamma \cdot \beta \cdot h_1(t) * h_4(t) + \alpha \cdot h_1(t) * h_2(t) * h_5(t) + \beta \cdot h_1(t) * h_3(t) * h_5(t)$$

On your own: redraw the above block diagram in frequency domain, and find the resulting $H_{eq}(f)$, both directly on the block diagram and Fourier-transforming $h_{eq}(t)$.

8.5.4 Stability of LTI systems

There are many notions of "stability" of LTI systems. Here we adopt the so-called BIBO stability criterion. To introduce it, we first need to define the concept of "bounded signal".

A signal $s(t)$ is said to be “bounded” if and only if:

$$|s(t)| < A; \quad t \in \mathbb{R}; \quad A < \infty .$$

The signal in figure is an example of a signal bounded over a certain interval. Note that the definition requires boundedness over the whole of $\mathbb{R}$.

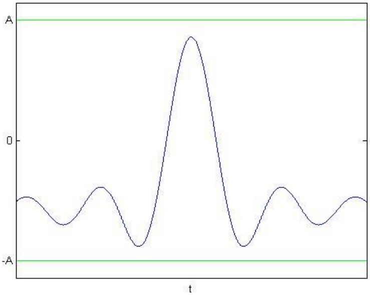


Figura 1 – Bounded signal over an interval

Definition

An LTI system is said to be bounded **"BIBO"** if, for any possible bounded input $|x(t)| < A$, the corresponding output is also bounded, that is $|y(t)| < B$.

In formulas:

$$\textit{given} \quad y(t) = L\{x(t)\} \quad ,$$

$$\textit{the LTI system } \; L \; \textit{ is said to be BIBO if}$$

$$\forall x(t) : |x(t)| < A \quad \Rightarrow \quad |y(t)| < B$$

$$t \in \mathbb{R} \quad 0 < A, B < \infty$$

8.5.4.2 BIBO stability condition in time-domain

There is a simple necessary and sufficient condition for a system to be BIBO.

An LTI system is BIBO if and only if:

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$$

Eq. 8-28

Optional

The **sufficiency** of such condition is easily proven as follows:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\theta)h(t-\theta)\,d\theta$$

$$\Rightarrow |y(t)| = \left|\int_{-\infty}^{+\infty} x(\theta)h(t-\theta)\,d\theta\right| \leq \int_{-\infty}^{+\infty} |x(\theta)h(t-\theta)|\,d\theta = \int_{-\infty}^{+\infty} |x(\theta)||h(t-\theta)|\,d\theta$$

Now we recall that by assumption $x(t)$ is bounded:

$$\left|y(t)\right| \le \int_{-\infty}^{+\infty} \left|x(\theta)\right| \left|h\left(t-\theta\right)\right| d\theta \le A \int_{-\infty}^{+\infty} \left|h\left(t-\theta\right)\right| d\theta = A \int_{-\infty}^{+\infty} \left|h(\rho)\right| d\rho$$

where in the last passage a change of variable $t-\theta=\rho$ was carried out.

Therefore, to ensure that $\left|y(t)\right| < B$, it is sufficient that $\int_{-\infty}^{+\infty} \left|h(\rho)\right| d\rho$ be bounded,

or, going back to the dummy variable t:

$$\int_{-\infty}^{+\infty} \left|h(t)\right| dt < C < \infty$$

Eq. 8-29

The **necessity** is also easily proven.

We assume that $\int_{-\infty}^{+\infty} |h(t)| dt \to \infty$.

Can we *always* pick a bounded signal $x(t)$ such that:

$$y(t) = \int_{-\infty}^{+\infty} h(\theta) x(t-\theta) d\theta \to \infty$$

at least for one time $t = t_0$?

Looking at the two integrals, the answer is immediate: we have to provide a bounded input $x(t)$ such that $h(\theta)x(t_0-\theta) = |h(\theta)|$.

To do so, we can simply pick the bounded signal: $x(t_0-\theta) = \mathrm{sign}(h(\theta))$

In fact:

$$h(\theta)\cdot x(t_0-\theta)=h(\theta)\cdot\operatorname{sign}\left(h(\theta)\right)=\left|h\left(\theta\right)\right|$$

If we use the input: $x(t_0-\theta)=\operatorname{sign}\left(h(\theta)\right)$, then:

$$y(t_0)=\int_{-\infty}^{+\infty}h(\theta)x(t_0-\theta)d\theta=\int_{-\infty}^{+\infty}\left|h(\theta)\right|d\theta\to\infty$$

which implies $\left|y(t_0)\right|\to\infty$, that is we get an unbounded output.

Since we can *always* pick such $x(t)$, and therefore we can always cause the output to be unbounded every time that $\int_{-\infty}^{+\infty}\left|h(t)\right|dt=\infty$, then it is clear that $\int_{-\infty}^{+\infty}\left|h(t)\right|dt<\infty$ is a necessary condition for BIBO stability.

End of optional material

8.5.4.3 BIBO stability necessary condition in frequency-domain

We can write:

$$\left|H(f)\right| = \left|\int_{-\infty}^{+\infty} h(t) e^{-j2\pi ft} dt\right| \le \int_{-\infty}^{+\infty} \left|h(t) e^{-j2\pi ft}\right| dt = \int_{-\infty}^{+\infty} \left|h(t)\right| \left|e^{-j2\pi ft}\right| dt = \int_{-\infty}^{+\infty} \left|h(t)\right| dt$$

If we impose that $\int_{-\infty}^{+\infty} \left|h(t)\right| dt < \infty$ then we have that it must be:

$$\left|H(f)\right| \le \int_{-\infty}^{+\infty} \left|h(t)\right| dt < \infty$$

So, the BIBO conditions implies the *necessity* that the Fourier Transform be bounded over all frequencies.

To discuss a frequency-domain *sufficiency* condition would require discussing the relationship between Fourier and Laplace transforms, which we are going to omit.

8.5.4.4 example

$$h(t) = u(t)e^{-at}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} u(t)e^{-at} dt = \int_{0}^{+\infty} e^{-at} dt = \frac{1}{a} < \infty$$

Eq. 8-30

The result complies with the BIBO stability constraint.

8.5.4.5 example

$$L\{x(t)\} = \int_{-\infty}^{t} x(\theta)d\theta = y(t) \rightarrow h(t) = L\{\delta(t)\} = \int_{-\infty}^{t} \delta(\theta)d\theta = u(t)$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} |u(t)|\, dt = \int_{0}^{+\infty} 1(t)\, dt$$

Eq. 8-31

The last integral diverges, so the system is not BIBO.

Note also that the frequency-domain condition $|H(f)| < \infty$ is violated, since

$$F\{u(t)\} = \frac{\delta(f)}{2} + \frac{1}{2j\pi f}$$

is certainly not bounded.

8.5.4.6 problem

$\boxed{\text{On your own:}}$ show that indeed the LTI system with impulse response $h(t) = u(t)$ is not BIBO, because it produces an unbounded output when the bounded input $x(t) = u(t)$ is applied to it.

Answer

$y(t) = x(t) * h(t) = u(t) * u(t) = t \cdot u(t)$ which is certainly not bounded.

8.5.4.7 physical realizability

An LTI system is said to be "physically realizable" if it is *causal* and *real*.

In some textbooks "physical realizability" is not related to $h(t) \in \mathbb{R}$, because it is possible to implement the *equivalent* of $h(t) \in \mathbb{C}$ by operating over the real and imaginary parts of $x(t)$, separately.

However, in this course, we assume by definition that “physical realizability” requires $h(t)$ being both causal and real.

8.6 LTI systems viewed as “filters”

LTI systems are often used to “process” signals, that is, to change their properties in both frequency and time domain. One typical use of LTI systems is as “filters”, especially in telecommunications, signal processing and automation systems.

In traditional lay terms, a “filter” (for instance an air filter, or a water filter, or a gasoline filter) is an object that lets through certain things and not others. That is exactly why some LTI systems have been called “filters”: typically, they let through certain “frequency bands” (i.e., certain frequency intervals) and reject other “frequency bands”.

It should also be noted, though, that the term “filter” has eventually been used to indicate essentially any LTI system used for signal processing, even when it does not just let through and reject frequency bands but rather does more complex things. In the following, however, we will focus on LTI systems that do in fact act as “filters”, by letting through only specific frequency bands.

8.6.1 The three most common “filters”

The three most common types of “filters” that let through a certain frequency band unaltered, while blocking all other frequencies, are:

- the bandpass filter
- the lowpass filter
- the highpass filter

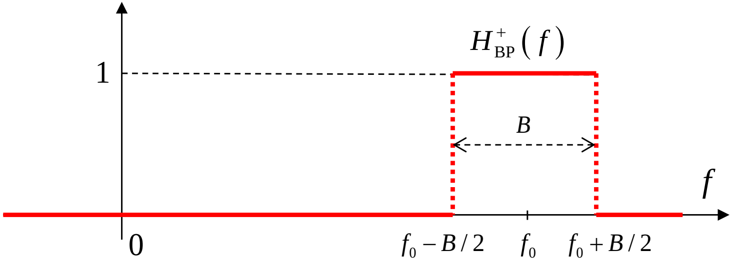
We will start by illustrating the “ideal” versions of all three such filters

8.6.1.1 the "ideal" bandpass filter

We start out with the mathematical expression of an *ideal bandpass filter for positive frequencies*:

$$H_{\mathrm{BP}}^{+}(f) = \Pi_B(f - f_0)$$

Graphically, this is what it looks like:



Clearly, when used as a transfer function, so that:

$$Y(f) = X(f) H_{\mathrm{BP}}^{+}(f)$$

such filter lets through only those frequency components of $X(f)$ that fall within the frequency band $\left[f_0 - B/2, f_0 + B/2\right]$, which is called the *passband*. Such frequency components go through unaltered, because within the passband they are multiplied times $1(f)$. Outside, they are completely blocked. The filter shown above is said to have *bandwidth* B, which corresponds to the width of the passband interval.

The filter $H_{\mathrm{BP}}^{+}(f)$ has the problem of *having a complex impulse response*, because it clearly does not comply with the symmetry rules that ensure that its corresponding $h_{\mathrm{BP}}^{+}(t)$ be real. In particular, it obviously fails the condition that the absolute value be even: $\left|H_{\mathrm{BP}}^{+}(f)\right| = \left|H_{\mathrm{BP}}^{+}(-f)\right|$

$\boxed{\text{On your own:}}$ calculate the impulse response corresponding to $H_{\mathrm{BP}}^{+}(f)$ and show that it is complex.

Answer: the result is $h_{\mathrm{BP}}^{+}(t) = B\,\mathrm{Sinc}(Bt)\cdot e^{j2\pi f_0 t}$, which is clearly complex.

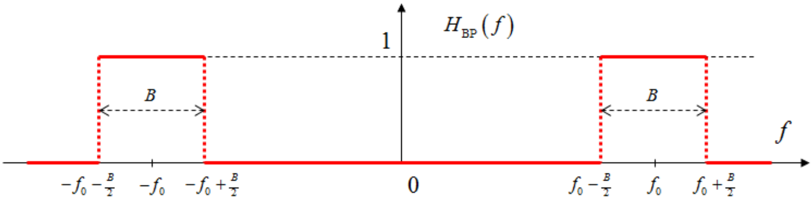
A more realistic, though still ideal, bandpass filter, is one whose impulse response is *certainly* real:

$$H_{\mathrm{BP}}(f) = \Pi_B(f + f_0) + \Pi_B(f - f_0)$$

$\boxed{\text{On your own:}}$ calculate the impulse response corresponding to $H_{\mathrm{BP}}(f)$ and show that it is real.

Answer: the result is $h_{\mathrm{BP}}(t) = 2B\,\mathrm{Sinc}(Bt)\cdot\cos(2\pi f_0 t)$, which is clearly real.

$H_{\mathrm{BP}}(f)$ *is still said to have bandwidth equal to* B, even though, if one counted also the negative frequencies, the total band "width" passed through would be $2B$ However, note that *in all fields of application of filters*, the conventional understanding is that **the bandwidth is computed over positive frequencies only**.



Although it has a real impulse response, this filter is still problematic in many respects. To mention one, $H_{\mathrm{BP}}(f)$ is *not physically realizable* because its impulse response is *non-causal*.

$\boxed{\text{On your own:}}$ show that $h_{\mathrm{BP}}(t)$ is non-causal.

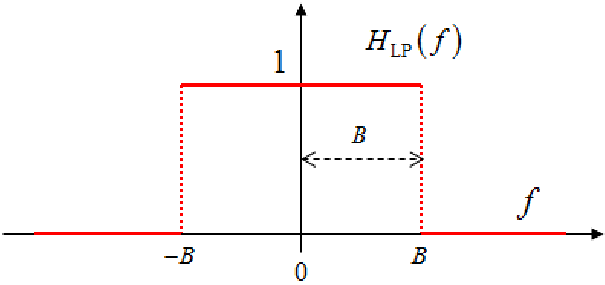
We will comment on how to deal with this problem later on.

8.6.1.2 the "ideal" lowpass filter

The ideal lowpass filter is simply:

$$H_{\mathrm{LP}}(f) = \Pi_{2B}(f)$$

whose graphical appearance is of course:



Note that the bandwidth B is again just the width of the passband for *positive frequencies* alone.

The corresponding impulse response is:

$$h_{\text{LP}}(t) = 2B\,\text{Sinc}(2Bt)$$

which clearly shows this filter to be non-causal as well. Again, this aspect will be dealt with later. The above impulse response can be easily found as follows. First, we know that:

$$F\left\{\Pi_{\Delta}\left(t\right)\right\}=\Delta\mathrm{Sinc}\left(\Delta\cdot f\right)$$

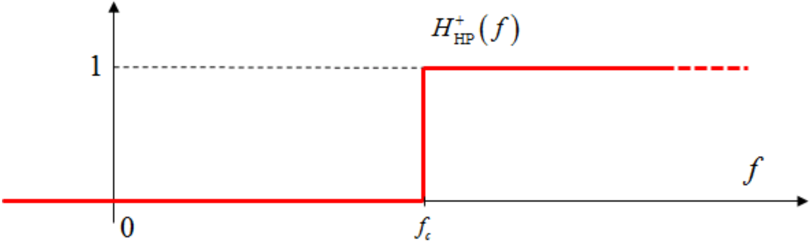
We then use the time-frequency symmetry property, which ensures that:

$$F\left\{\Delta\mathrm{Sinc}\left(\Delta\cdot t\right)\right\}=\Pi_{\Delta}\left(-f\right)=\Pi_{\Delta}\left(f\right)$$

Then it is enough to replace $\Delta = 2B$.

8.6.1.3 the "ideal" highpass filter

The ideal highpass filter shape (for positive frequencies) is as follows:



where f_c is the so-called cut-off frequency, to be intended as the frequency below which the filter blocks all signal frequency components. Note that the passband of this filter is ideally infinite, as there is no upper limit. Analytically:

$$H^+_{\mathrm{HP}}(f) = u(f - f_c)$$

This transfer function has a complex impulse response. To make it real, the negative symmetric frequency passband must be added:

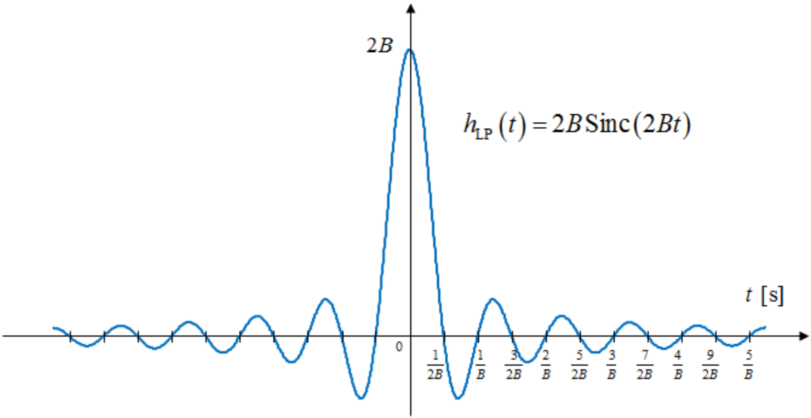
$$H_{HP}(f) = u(f - f_c) + u(-f - f_c)$$

8.6.1.4 from "ideal" to physically realizable filters

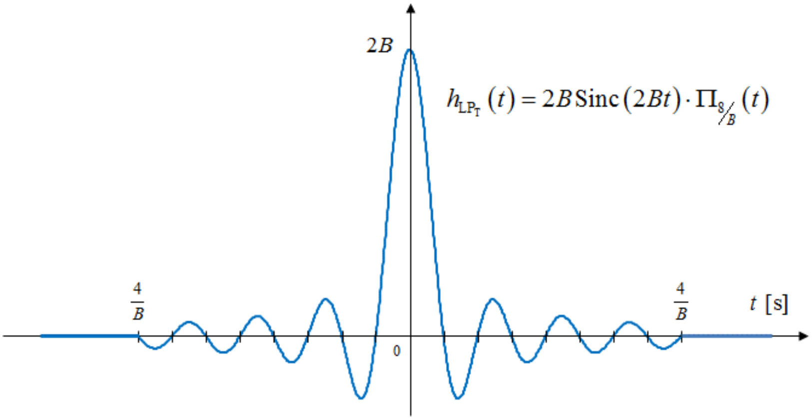
We deal with this matter focusing on the lowpass filter. As shown, the impulse response of the ideal lowpass filter is real:

$$h_{\mathrm{LP}}(t) = 2B\,\mathrm{Sinc}(2Bt)$$

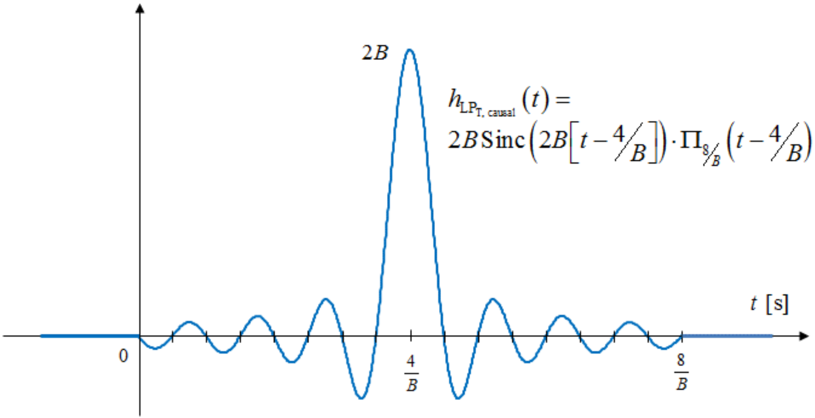
but clearly $h_{\mathrm{LP}}(t) \neq 0,$ for $t < 0.$ Hence, this is a non-causal filter, which of course is not physically realizable:



One possibility for overcoming the causality problem is that of first truncating the impulse response to some finite time-length, as shown below. In the next plot, $h_{LP}(t)$ is truncated over the interval $t \in [-4/B, 4/B]$:



Then, the overall impulse response is delayed so as to make it causal:



The resulting filter is therefore now physically realizable. However, we have performed two alterations: *truncation in time* and *delay*. Let us discuss the impact of the delay first.

Delaying the impulse response of a generic LTI system $y(t) = x(t) * h(t)$ means that we are using $h(t - t_d)$ instead of $h(t)$. We first remark that we can write:

$$h(t - t_d) = h(t) * \delta(t - t_d)$$

We then look at the output of the delayed filter to a generic input $x(t)$. We have:

$$x(t) * h(t - t_d) = x(t) * h(t) * \delta(t - t_d)$$

Since the convolution product is fully commutative (the order does not matter) we can first carry out the convolution $x(t) * \delta(t - t_d)$ obtaining:

$$x(t) * \delta(t - t_d) = x(t - t_d)$$

We then carry out the remaining convolution:

$$x(t-t_d)*h(t)=y(t-t_d)$$

where the result is due to time-invariance property of LTI systems. So indeed delaying the impulse response by t_d seconds simply delays the filter output by the same amount. In general, adding such delay is not a problem for practical applications.

We also remark that the impact of a time-delay on the transfer function is just a phase factor:

$$h(t) \;\to\; H(f)$$
$$h(t-t_d) \;\to\; H(f)\,e^{j2\pi f\, t_d}$$

Therefore, *the filter passband does not change at all*, since:

$$\left| H(f)\, e^{j2\pi f\, t_d} \right| = \left| H(f) \right|.$$

Truncation in time instead does have an impact on the filter shape in frequency.

The physically realizable $h_{\mathrm{LP}_{\mathrm{T,\,causal}}}(t)$ **does not exactly realize an ideal lowpass filter**, because of the truncation in time, but an approximation of it. The resulting $H_{\mathrm{LP}_{\mathrm{T,\,causal}}}(f)$ will never have exactly rectangular passbands. Of course, the approximation gets increasingly better, as *longer time-stretches* of the original ideal impulse response $h_{\mathrm{LP}}(t)$ are included in the truncated impulse response[1] $h_{\mathrm{LP}_{\mathrm{T,\,causal}}}(t)$.

---

[1] Optional for the interested students: note that $h_{\mathrm{LP}}(t)$ drops off only as $1/|t|$, because $H_{\mathrm{LP}}(f) \in C^{-1}$. So in order to get a good approximation of $H_{\mathrm{LP}}(f)$, the truncation of the impulse response $h_{\mathrm{LP}}(t)$ must be very "long" in time. In any case, the resulting $H_{\mathrm{LP}_{\mathrm{T,\,causal}}}(f)$ will suffer from some "ripples" about the leading and trailing edges of the passband, which get 'squeezed' horizontally towards the edges of the passband as the truncation gets longer, but whose amplitude does not decrease. These ripples are called the 'Gibbs phenomenon' or 'Gibbs effect'. The Gibbs effect can be avoided by using a filter whose impulse response drops off faster than $1/|t|$

In that case, as the truncation gets longer in time, the resulting $H_{\mathrm{LP}_{\mathrm{T,\,causal}}}(f)$ has "ripples" about the leading and trailing edge that get squeezed both horizontally and vertically.

If interested you can read more here: https://en.wikipedia.org/wiki/Gibbs_phenomenon.

$\boxed{\text{On your own:}}$ Write a computer program that truncates $h_{\mathrm{LP}}(t)$, delays it to make it causal and calculates its Fourier transform $H_{\mathrm{LP}_{\mathrm{T,\ causal}}}(f)$ numerically. Plot the results of $H_{\mathrm{LP}_{\mathrm{T,\ causal}}}(f)$ while making the time-truncation interval of $h_{\mathrm{LP}}(t)$ bigger. Discuss what you see.

8.6.1.5 more realistic lowpass filters

One important type of "lowpass filter" is the raised-cosine filter. The analytical expression of its transfer function is:

$$H(f)=\rho_{2B,\alpha}(f)=\begin{cases} 1, & |f|\le B(1-\alpha) \\ \dfrac{1}{2}\left[1+\cos\left(\dfrac{\pi}{2B\alpha}\left[|f|-B(1-\alpha)\right]\right)\right] & B(1-\alpha)<|f|<B(1+\alpha) \\ 0 & \text{elsewhere} \end{cases}$$

which results in the following plots, depending on the values of the roll-off parameter α :

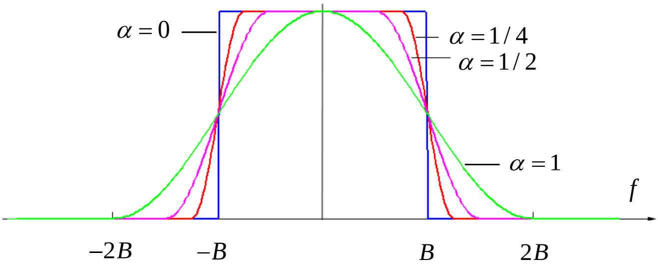


Fig. 8-2 Raised-cosine filter transfer functions.

For the case $\alpha = 0$ the filter coincides with an ideal lowpass filter, that is, it becomes a perfect rectangle $H(f) = \Pi_{2B}(f)$. In all other cases, the transfer function is smoother and in fact it is not just continuous, but also differentiable once, that is: $H(f) \in C^1$. As a result, for $\alpha > 0$, the resulting impulse response:

$$h(t) = 2B \cdot \mathrm{Sinc}\left(2B \cdot t\right) \cdot \frac{\cos\left(2\pi B\alpha t\right)}{1-16B^2\alpha^2 t^2}$$

Eq. 8-32

drops off as $1/|t|^3$, much faster than the impulse response of the ideal lowpass filter.

If we plot the impulse response by posing for convenience $T = 2B$, we get the following plot:

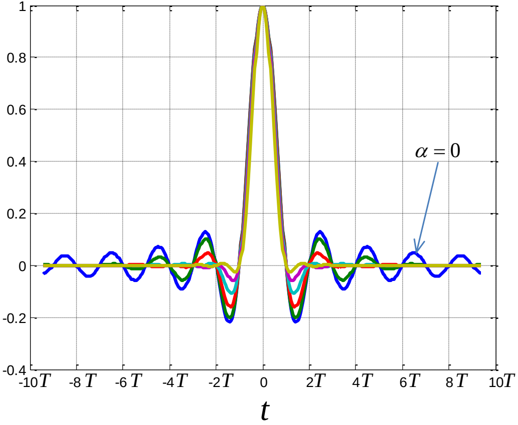


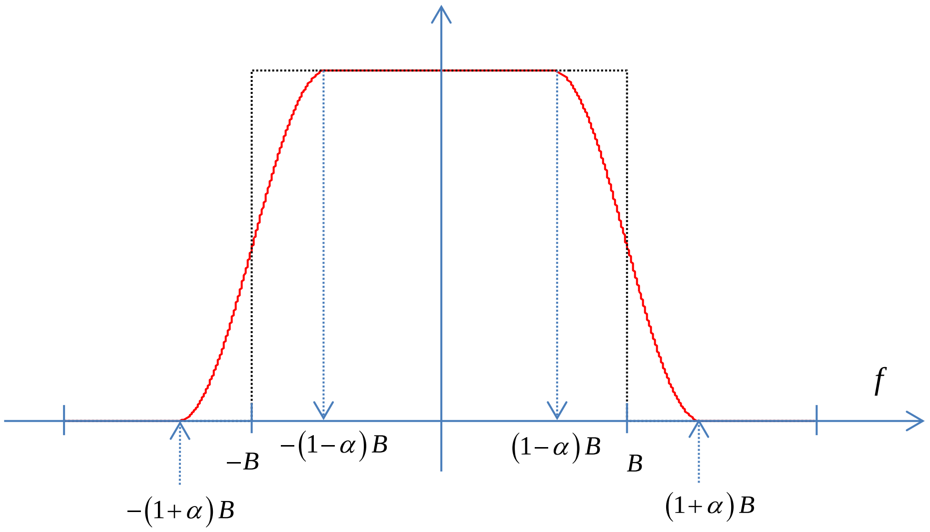
Fig. 8-3 Raised-cosine filter impulse responses for the following values of α: dark blue, 0; dark green, 0.2; red, 0.4; light blue, 0.6; purple (violet), 0.8; yellow, 1.

The plot clearly shows that, with the exception of course of the case $\alpha = 0$, the impulse response of the filter goes to zero very fast. Truncation and delay are still necessary to make the filter physically realizable, since the impulse response is non-causal and, mathematically, it never really goes to zero except at $|t| = \infty$. However, truncation and delay are much less problematic[2] than in the ideal lowpass filter case since, as shown, $h(t) = O\left(1/|t|^3\right)$. In other words, $h(t)$ very quickly goes down to virtually zero.

There is some disagreement on what the "bandwidth" of this lowpass filter is.

---

[2] Optional. Also, if $\alpha > 0$, the Gibbs phenomenon does not occur because the discontinuity in $H(f)$ is no longer present. Quick convergence to the ideal $H(f)$ is easily obtained. The Gibbs phenomenon is indeed caused by discontinuities in $H(f)$.


f
$-(1+\alpha)B$
$-B$
$-(1-\alpha)B$
$(1-\alpha)B$
B
$(1+\alpha)B$

Some indicate the bandwidth as the total positive frequency stretch till the filter transfer function $H(f)$ goes to zero. That depends on α and it would be:

$$\text{“bandwidth”} = B \cdot (1+\alpha)$$

Others indicate the bandwidth as the positive frequency stretch till $H(f)$ goes down to $1/2$. Then:

$$\text{“bandwidth”} = B$$

Still others consider the filter bandwidth as the positive frequency stretch for which $H(f) = 1(f)$, that is, over which the input signal spectrum is not at all altered. In this case the “bandwidth” is only:

$$\text{“bandwidth”} = B \cdot (1-\alpha)$$

Eq. 8-33

This last determination is probably the most correct, since it identifies the range of frequencies of the input signal that go through completely unchanged.

In practice, which definition is the most appropriate, it depends on the application. In the context of sampling and reconstruction, as we shall see, the definition Eq. 8-33 is the most appropriate one.

8.6.1.6 bandpass filters through frequency translation

Bandpass filters can be directly derived from lowpass filters by means of the frequency-shifting property.

Given *any* lowpass filter

$$h_{LP}(t) \overset{\mathrm{F}}{\longleftrightarrow} H_{LP}(f)$$

then the following filter:

$$h_{LP}(t)\cdot e^{j2\pi f_0 t} = h_{BP}(t) \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad H_{LP}(f - f_0) = H_{BP}(f)$$

has a "passband" at f_0, and so is a "bandpass" filter. To keep the filter real, one can do:

$$h_{LP}(t)\cdot \cos\left(2\pi f_0 t\right) = h_{BP}(t) \quad \overset{\mathrm{F}}{\longleftrightarrow}$$

$$\overset{\mathrm{F}}{\longleftrightarrow} \quad \frac{H_{LP}(f - f_0) + H_{LP}(f + f_0)}{2} = H_{BP}(f)$$

This filter has two passbands, located at $\pm f_0$.

8.6.2 LTI systems and periodic signals

We know from previous chapters that a periodic signal written in the form:

$$x(t) = \sum_{n=-\infty}^{\infty} q_x(t - nT_0)$$

has Fourier transform:

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} Q_x(nf_0)\delta\left(f - nf_0\right)$$

with $T_0 = 1/f_0$.

What happens when a periodic signal goes through an LTI system? The result is easily found:

$$Y(f) = X(f)H(f) = f_0 H(f) \sum_{n=-\infty}^{\infty} Q_x(nf_0)\delta(f - nf_0) =$$
$$= f_0 \sum_{n=-\infty}^{\infty} H(nf_0) Q_x(nf_0)\delta(f - nf_0)$$
$$= f_0 \sum_{n=-\infty}^{\infty} Q_y(nf_0)\delta(f - nf_0)$$

So, in essence, the spectrum of the output is still made up of spectral "lines". In fact, it is made up of the *same* spectral lines. Only their coefficients have changed. Therefore, we can actually say that:

the output $y(t)$ *of an LTI system, whose input is a periodic signal* $x(t)$ *of period* T_0 *, is still periodic of same period* T_0 *, or a submultiple of* T_0 *; its spectrum has the same deltas, with coefficients changed from* $Q_x(nf_0)$ *into:*

$$Q_y(nf_0) = H\left(nf_0\right)Q_x(nf_0)\,.$$

Note that $y(t)$ has in general the same period because the spectral lines in $Y\left(f\right)$ are the same as in $X\left(f\right)$, since LTI systems cannot add any new spectral lines.

However, note that if the LTI system transfer function is such that $H\left(nf_0\right) = 0$ for some values of n, certain spectral lines can be *canceled*. This may cause the signal $y(t)$ to be periodic not only of period T_0, but also of a sub-multiple of T_0. That is: $T_{0,y} = \dfrac{T_0}{k}$, for some $k \in \mathbb{N}$. An example of this is given by the problem shown below in Section 8.6.2.1, where the input signal has period T_0 and the output signal has period $T_{0,y} = T_0/5$.

An interesting result can also be given in time domain. The output $y(t)$ can always be written as:

$$y(t) = \sum_{n=-\infty}^{\infty} q_y(t - nT_0)$$

where $q_y(t) = q(t) * h(t)$. Prove this $\boxed{\text{on your own}}$ (easy).

8.6.2.1 example

Given the signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} e^{-a(t-nT_0)} \mathrm{u}(t - nT_0) \quad , \quad a \in \mathbb{R}^+$$

Eq. 8-34

and given the transfer function of an ideal "band-pass" filter:

$$H_{\mathrm{BP}}^{+}(f) = \Pi_{f_0}(f - f_1), \qquad f_1 = 5 \cdot f_0$$

Eq. 8-35

with $f_0 = 1/T_0$ *as usual, find:*

$$X(f), Y(f), y(t).$$

Solution

We first define:

$$q_x(t) = e^{-at}\mathrm{u}(t)$$

and remark that we can re-write:

$$x(t) = \sum_{n=-\infty}^{+\infty} q_x(t - nT_0)\,.$$

So, $x(t)$ is a periodic signal of period $T_0 = 1/f_0$. As a result, its spectrum is:

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} Q_x(nf_0)\delta\left(f - nf_0\right)$$

The coefficients $Q_x(nf_0)$ are found as follows. First:

$$Q_x(f) = \mathrm{F}\left\{e^{-at}\mathrm{u}(t)\right\} = \frac{1}{a + j2\pi f}$$

Eq. 8-36

then:

$$Q_x(nf_0) = \left.\frac{1}{a + j2\pi f}\right|_{f=nf_0} = \frac{1}{a + j2\pi nf_0}$$

Therefore, the Fourier transform of the periodic signal $x(t)$ is:

$$X(f) = \sum_{n=-\infty}^{\infty} \frac{f_0}{a + j2\pi nf_0} \delta\left(f - nf_0 \right)$$

Eq. 8-37

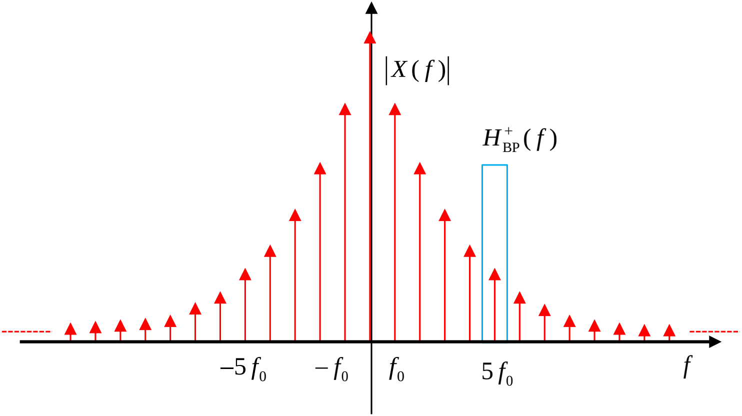


Fig. 8-4 – Absolute value of the Fourier transform of the input, $|X(f)|$, and transfer function $H^{+}_{\mathrm{BP}}(f)$.

Since the ideal band-pass filter given in Eq. 8-35 is centered at $f_1 = 5 \cdot f_0$ and has extension f_0, all spectral lines of the input are actually suppressed. The only delta that goes through the LTI system is the one centered at $5 \cdot f_0$. So, the spectrum of the output $Y\left(f\right)$ is:

$$Y(f) = X(f) H_{\mathrm{BP}}^{+}(f) = \frac{f_0}{a + j2\pi \cdot 5 f_0} \delta\left(f - 5 f_0\right)$$

Going back to time-domain, we finally get:

$$y(t) = \mathrm{F}^{-1}\left\{Y(f)\right\} = \frac{f_0}{a + j2\pi \cdot 5 f_0} e^{j2\pi\left(5 f_0\right) \cdot t}$$

Note that, as mentioned in the previous section, the period of the output signal, due to the canceling of most deltas, has changed to $T_y = T_0 / 5$.

$\boxed{\text{On your own:}}$ the output $y(t)$ of the LTI system in the previous example is complex. What is the reason? Could you have seen that without even calculating it?

$\boxed{\text{On your own:}}$ redo the above example, using now a different transfer function: $H(f) = \Pi_{f_0}(f - f_1) + \Pi_{f_0}(f + f_1)$, all other parameters and problem data unchanged. What happens to $y(t)$?

8.6.3 The smoothing effect of the convolution product

We first recall the following property of the Fourier transform:

Given a signal $s(t) \in C^n$*, its Fourier transform (if it exists) has an asymptotic behaviour for* $f \to \pm\infty$ *of the kind:*

$$S(f) = O\left(\frac{1}{|f|^{n+2}}\right)$$

Conversely, if a signal $s(t)$ *has a Fourier transform that is*

$$S(f) = O\left(\frac{1}{|f|^{n+2}}\right)$$

then such signal belongs to differentiability class C^n .

We now investigate what happens assuming the following. Let $x_1(t) \in C^{n_1}$ and $x_2(t) \in C^{n_2}$. Let $y(t)$ be:

$$y(t) = x_1(t) * x_2(t)$$

We want to find out what the differentiability class of $y(t)$ is.

First of all, we write down the system output in frequency domain, which is equivalent to:

$$Y(f) = X_1(f) X_2(f)$$

Then we remark that

$$X_1(f) = O\left(\frac{1}{|f|^{n_1+2}}\right) \qquad X_2(f) = O\left(\frac{1}{|f|^{n_2+2}}\right)$$

because $x_1(t) \in C^{n_1}$ and $x_2(t) \in C^{n_2}$. As a result, we have:

$$Y(f) = O\left(\frac{1}{|f|^{n_1+2}}\right) \cdot O\left(\frac{1}{|f|^{n_2+2}}\right) = O\left(\frac{1}{|f|^{n_1+n_2+4}}\right) = O\left(\frac{1}{|f|^{m+2}}\right)$$

By equating the exponents, we get: $(m+2) = (n_1 + n_2 + 4)$, that is:

$$m = n_1 + n_2 + 2$$

As a result, we can say:

$$y(t) \in C^m = C^{n_1+n_2+2}$$

In other words, the convolution product delivers a signal whose differentiability class order is the sum of the differentiability class orders of the two factor signals, incremented by two.

In particular, the convolution product of two discontinuous signal of class C^{-1}, such as rectangular signals, produces as a result a C^{0} signals, that is a signal that is continuous.

8.7 *Optional:* *An Intuitive View of the Convolution Product*

<u>This section is OPTIONAL</u> and is provided only to aid in the intuitive comprehension of how the convolution product operates.

We know that all LTI systems can be dealt with in a unified way by means of the convolution product. However, it may not be intuitive why the convolution product indeed models the response of such systems.

To provide an intuitive picture of why and how the convolution product actually operates, we need first to introduce an interesting result, which may be helpful not only in this context but in many other contexts.

8.7.1 Response to narrow rectangular signals

We assume to be dealing with an LTI system with impulse response $h(t)$ which is subjected to the input $x(t) = \Pi_T(t)$. We claim the following:

$$\begin{aligned} y(t) &= L\{x(t)\} = L\{\Pi_T(t)\} \\ &= \Pi_T(t) * h(t) \\ &\approx T \cdot h(t) \end{aligned}$$

where the approximation is valid
if T is "sufficiently small"

This interesting result can be justified in a number of ways. Here we do it in frequency domain. We rewrite the above expression as:

$$\begin{aligned} Y(f) &= F\{L\{x(t)\}\} \\ &= X(f)H(f) \\ &\approx T \cdot H(f) \end{aligned}$$

where the approximation is valid
if T is "sufficiently small"

Looking specifically at the approximate equality:

$$X(f)H(f) \approx T \cdot H(f)$$

it is clear that it holds true if $X(f) \approx 1(f) \cdot T$, at least over the range of f for which $H(f)$ is "significantly" different from zero. The exact transform of $x(t)$ is:

$$X(f) = T \,\mathrm{Sinc}(f\,T)$$

so it is not immediately evident that:

$$\mathrm{Sinc}(f\,T) \approx 1(f)$$

On the other hand, by reducing the value of T, the Sinc gets "wider" and essentially coincides with $1(f)$ over larger and larger intervals in f. Since T is arbitrary, such larger intervals can be made large enough to include the range of f for which $H(f)$ is "significantly" different from zero and then it can be claimed that indeed, if the input is $\Pi_T(t)$:

$$Y(f) \approx T \cdot H(f)$$
$$y(t) \approx T \cdot h(t)$$

8.7.2 Approximating a generic signal with narrow rectangular signals

We now claim that any signal $x(t)$ of practical interest can be approximated with arbitrarily good accuracy by the following expression:

$$x(t) \approx \sum_n x(t_n)\Pi_T(t - t_n) = x_{app}(t)$$

where $t_n - t_{n-1} = T$.

This approximation of $x(t)$ is *not* a canonical approximation because the coefficients $x(t_n)$ are not the inner product of the signal with respect to each rectangular function. Still, its accuracy can be discussed in terms of energy of the error and it can be easily shown that the energy of such error can be made arbitrarily small by decreasing T .

With the above results in mind, we can now deal with the response of a generic LTI system to a generic input $x(t)$.

8.7.3 The system response

We want to discuss the general case of:

$$y(t) = L\{x(t)\}$$

We start out by replacing $x(t)$ with its approximate expression:

$$\begin{aligned} y(t) &= L\{x(t)\} \approx L\{x_{app}(t)\} \\ &= L\left\{\sum_n x(t_n)\Pi_T(t-t_n)\right\} \\ &= \sum_n x(t_n) L\{\Pi_T(t-t_n)\} \\ &\approx \sum_n x(t_n) h(t-t_n) T \end{aligned}$$

all this being "accurate enough" for a "small enough" value of T .

This latter formula can be given a straightforward interpretation. We concentrate on a specific time t_{out} , at which we want to know the output of the system. In other words, we want to find $y(t_0)$. The formula then is:

$$y(t_{\text{out}}) \approx \sum_n x(t_n)h(t_{\text{out}} - t_n)T =$$

$$\dots\; x(t_1)h(t_{\text{out}} - t_1)T + x(t_2)h(t_{\text{out}} - t_2)T + x(t_3)h(t_{\text{out}} - t_3)T + \dots$$

By looking at just the first contribution shown on the bottom line above, we find that it consists of the coefficient marking how tall the rectangular input was at time t_1, that is $x(t_1)$, followed by the system response to such rectangle, that is $h(t_{\text{out}} - t_1)T$, evaluated at the time $t_{out} - t_1$ which represents exactly the amount of time elapsed between when the rectangular input was applied and when the output is looked at.

In fact, the above formula clearly shows the superposition of effects concept, whereby the output of the system at time t_{out} is the superposition of the response of the system to each one of the individual narrow rectangular signals that approximate $x(t)$.

Even though easy to understand, the formula:

$$y(t_{\text{out}}) \approx \sum_n x(t_n)\, h(t_{\text{out}} - t_n)\, T$$

Eq. 8-38

is an approximate one. However, we point out that by making T smaller, two things happen at the same time:

$$x_{app}(t) \to x(t)$$

$$L\left\{\Pi_T(t)\right\} \to T\, h(t)$$

so that the approximation errors that exist in Eq. 8-38 tend to go to zero. Also, by making T smaller, the summation, which is in the form of a Riemann sum, tends to become:

$$\sum_n x(t_n)h(t_{\text{out}}-t_n)T \to \int x(\theta)h(t_{\text{out}}-\theta)d\theta$$

So clearly the convolution product emerges as the “limit” of the above approximations, when they actually become exact. As a mind model, the convolution product can therefore be thought of as a mathematical object that performs the sum of infinitely many contributions to the output, each one caused by a very narrow “sliver” of input occurring at time θ, and whose effect is properly summed at the right time, that is after . $t_{\text{out}}-\theta$. seconds have elapsed.

End of optional material.

8.8 Problems

8.8.1 problem

Given an LTI system with impulse response:

$$h(t) = \left(1 - t / T\right) \cdot \pi_T\left(t\right)$$

find the output of the system, when the input $x(t) = \pi_T(t)$ is applied. Then draw the result. What differentiability class are the input, the impulse response and the output? Does the extension of the output match the expected value?

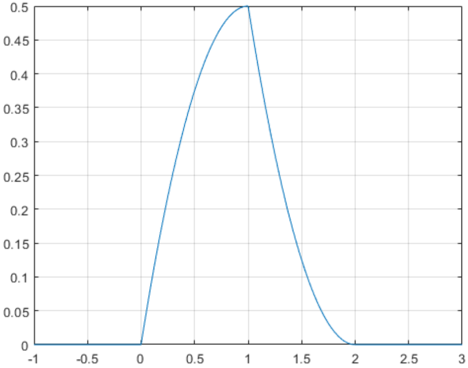
Answers

The output is given by:

$$y(t)=\left(t-\frac{t^2}{2T}\right)\cdot\pi_T(t)+\frac{(t-2T)^2}{2T}\pi_T(t-T)$$

The input is C^{-1}, the impulse response is C^{-1} and the output is, as expected, C^0. in other words, the output is continuous, but its derivative is discontinuous.

The extension of the output is $[0,2T]$ and it matches the expected value. Here below the plot of the output for $T=1$ (s).



The plot is generated by the following Matlab code, which uses the “Hppi” function that can be found in the “lowpass filters” folder.

```
T=1; t=-1:0.01:3; figure; plot(t, (t-(t.^2) /2 /T) .* Hppi(T,t) + ( (t-2*T).^2 ) /2 /T .*Hppi(T, t-T) ); grid on;
```

8.8.2 problem

Given an LTI system with impulse response:

$$h(t) = \Lambda\left(\frac{t-T}{T}\right) = \Lambda_T\left(t-T\right)$$

find the output of the system, when the following input is applied:

$$x(t) = \Pi_T\left(t-\frac{T}{2}\right) = \pi_T(t)$$

Then draw the result. What differentiability class are the input, the impulse response and the output? Does the extension of the output match the expected value?

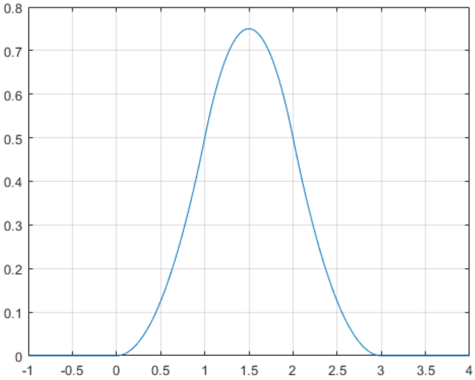
Answer:

$$y(t)=\frac{t^2}{2T}\pi_T(t)+\left[\frac{3T}{4}-\frac{1}{T}\left(t-\frac{3}{2}T\right)^2\right]\pi_T(t-T)$$

$$+\frac{(t-3T)^2}{2T}\pi_T(t-2T)$$

The input is C^{-1}, the impulse response is C^0 and the output is, as expected, C^1. in other words, the first derivative of the output is continuous, but its second derivative is discontinuous.

The extension of the output is $[0,3T]$ and it matches the expected value. Here below the plot of the output for $T=1$ (s).



The plot is generated by the following matlab code, which uses the “Hppi” function that can be found in the “lowpass filters” folder.

```
T=1; t=-1:0.01:4; figure; plot(t, (t.^2) /2 /T .* Hppi(T,t) + ( (t-3*T).^2 ) /2 /T .*Hppi(T, t-2*T) + Hppi(T, t-T) .* ( 3/4 * T - ( t - 3/2*T).^2 /T)); grid on;
```

8.8.2.1 problem

Given an LTI system with impulse response:

$$h(t) = \pi_{2T}(t)$$

find the output of the system, when the following input is applied:

$$x(t) = \cos\left(\frac{2\pi t}{T}\right) \cdot \pi_T(t)$$

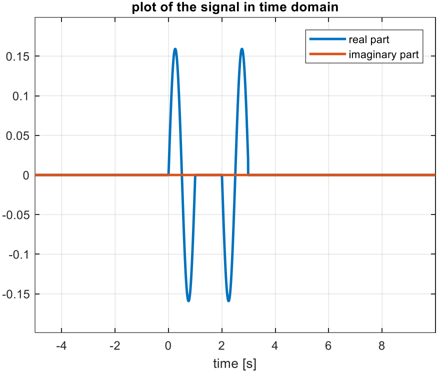
Then draw the result. What differentiability class are the input, the impulse response and the output?

Answer

$$y(t) = \frac{T}{2\pi}\pi_T(t)\sin\left(\frac{2\pi t}{T}\right) - \frac{T}{2\pi}\pi_T(t-2T)\sin\left(\frac{2\pi t}{T}\right)$$

The input is C^{-1}, the impulse response is C^{-1} and the output is, as expected, C^0. In other words, the output is continuous but its first derivative is discontinuous.

The extension of the output is $3T$ and it matches the expected value. Here below the plot of the output for $T = 1$ (s).



8.8.3 problem

Consider the following systems:

$$\Omega\{s(t)\} = |s(t)|^2$$

$$\Omega\left\{s(t)\right\} = s(t-T)$$

$$\Omega\left\{s(t)\right\} = \alpha\, s(t)$$

$$\Omega\left\{x\left(t\right)\right\} = x\left(t\right)\cdot u\left(t\right)$$

$$\Omega\left\{s(t)\right\} = s^2\left(t\right)$$

$$\Omega\left\{s(t)\right\} = \alpha\, s(t) + b1\left(t\right)$$

$$\Omega\left\{s(t)\right\} = s(t)\cdot e^{j2\pi f_o t}$$

$$\Omega\left\{s(t)\right\} = 2s(t) + \mathrm{u}(t)$$

$$\Omega\left\{s(t)\right\} = \sin\left(s(t)\right)$$

$$\Omega\left\{s(t)\right\} = \int_{-\infty}^{t} s(\theta)d\theta$$

$$\Omega\left\{s(t)\right\} = \int_{-\infty}^{t} s^2(\theta)d\theta$$

$$\Omega\left\{s(t)\right\} = \int_{t-T}^{t} s(\theta)d\theta$$

$$\Omega\left\{s(t)\right\} = \int_{t-T}^{t} s^2(\theta)d\theta$$

1. Which one of these systems is linear and which one is non-linear?
2. Which one of these systems has memory and which one is memoryless?
3. Which one of these systems is time-invariant and which one is time-variant?

8.8.4 problem

Consider the following systems:

$$\Omega\left\{x(t)\right\} = x(t)\cdot x(t-T)$$

$$\Omega\left\{s(t)\right\} = s(t) - s^2(t)$$

$$\Omega\left\{s(t)\right\} = \alpha\, s(t) + b1(t)$$

$$\Omega\left\{s(t)\right\} = \exp\left(s(t)\right)$$

$$\Omega\left\{s(t)\right\} = \int_{t-T}^{t} s(\theta)d\theta$$

$$\Omega\left\{x(t)\right\} = \int_{t-T}^{t} x(\theta)\cdot\cos(2\pi f\theta)\cdot d\theta$$

1. Which one of these systems is linear and which one is non-linear?

2. Which one of these systems has memory and which one is memoryless?
3. Which one of these systems is time-invariant and which one is time-variant?

8.8.5 problem

(I)

Prove that each and every single-input, single-output LTI system $y(t) = L\{x(t)\}$ can be mathematically described by means of the following formula:

$$y(t) = x(t) * h(t)$$

where $h(t)$ is the impulse response of the system.

Hint: use the property of $\delta(t)$: $x(t) * \delta(t) = x(t)$

(II)

Show why the convolution formula above does not hold if the system is linear, but not time-invariant.

(III)

Show why the convolution formula above does not hold if the system is time-invariant, but non-linear.

8.8.6 problem

Consider the following LTI systems:

$$L\{x(t)\} = \int_{t-T}^{t} x(\theta)\, d\theta$$

$$L\{x(t)\} = \int_{t-T}^{t+T} x(\theta)\, d\theta$$

$$L\{x(t)\} = \int_{-\infty}^{t} x(\theta)\,d\theta$$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(\theta)e^{-a(t-\theta)}u(t-\theta)\,d\theta$$

1. Find the impulse response of each one of them
2. Which one of these systems is causal?
3. Which one of these systems is BIBO?
4. Find the value of the memory of the systems (where applicable).
5. Calculate the response of each system to the input signal:

$$x(t) = -2\delta(t) + 3\delta(t - t_d)$$

8.8.7 problem

What is the condition of causality of an LTI system?

Write it down clearly and also <u>prove</u> it.

Then, consider the following systems and say which one of them is causal.

Is the system $h(t) = \Lambda_T(t)$ causal or non-causal?

Is the system $h(t) = e^{-at}u(t)$ causal or non-causal?

Is the system $h(t) = \Pi_T\left(t - \frac{T}{2}\right)$ causal or non-causal?

Is the system $h(t) = \delta\left(t - \frac{T}{2}\right)$ causal or non-causal?

Is the system $h(t) = \delta\left(t + \frac{T}{2}\right)$ causal or non-causal?

Find the value of the memory M_h of the causal systems above.

8.8.8 problem

Define the concept of "memory of a causal LTI system" as rigorously as you can.

Look at the following causal LTI systems and determine their memory M_h.

$$h(t) = \Lambda_T(t - T)$$

$$h(t) = e^{-at}u(t) \qquad a \in \mathbb{R}^+$$

$$h(t) = \Pi_T\left(t - \tfrac{T}{2}\right)$$

$$h(t) = \delta\left(t - \tfrac{T}{2}\right)$$

$$L\left\{x(t)\right\} = \int_{t-T}^{t} x(\theta)\,d\theta$$

$$L\left\{x(t)\right\} = \int_{t-2T}^{t-T} x(\theta)\,d\theta$$

$$L\left\{x(t)\right\} = \int_{-\infty}^{t} x(\theta)\,d\theta$$

8.8.9 problem

Let:

$$x(t) = \Pi_a\left(t - \frac{a}{2}\right) = \pi_a(t)$$

and

$$h(t) = \Pi_b\left(t - b\right) = \pi_b\left(t - \frac{b}{2}\right)$$

with $a < b$.

1. Is the LTI system whose impulse response is $h(t)$ causal? Is it stable BIBO? What is its memory M_h ?
2. Calculate the output of the LTI system whose impulse response is $h(t)$ when the input is $x(t)$

8.8.10 problem

Let $y(t) = x(t) * h(t)$.

1. Prove that: $Y(f) = X(f) \cdot H(f)$ where:

$$X(f) = \mathrm{F}\{x(t)\},\ Y(f) = \mathrm{F}\{y(t)\},\ H(f) = \mathrm{F}\{h(t)\}$$

2. Given $x(t) = A \cdot e^{j\varphi_0} \cdot e^{j2\pi f_0 t}$,

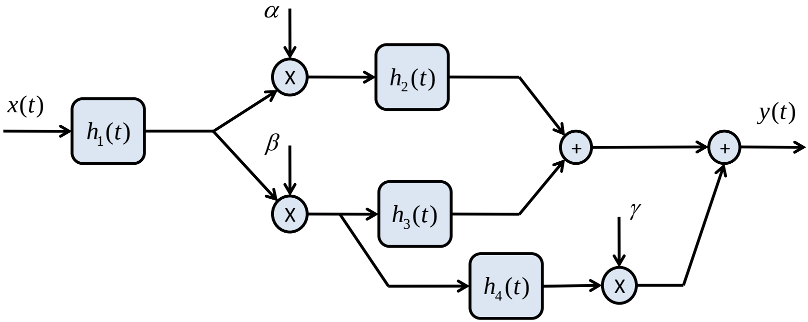
   calculate $y(t)$

3. Given $x(t) = A \cdot \cos\left(2\pi f_0 t + \varphi_0\right)$,

   calculate $y(t)$

8.8.11 problem

Consider the following block diagram:

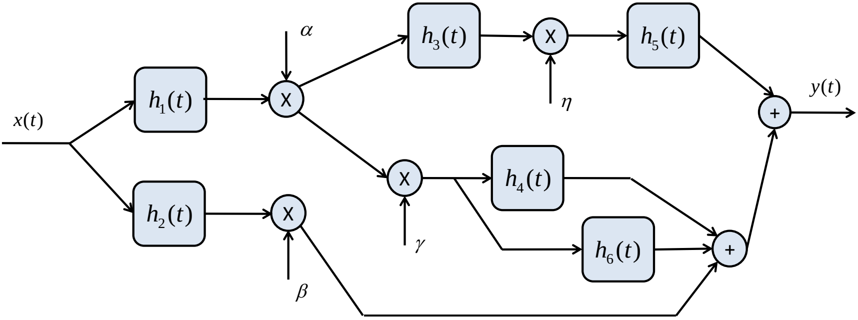


1. Is the overall equivalent system $y(t) = \Omega\{x(t)\}$ linear and time-invariant? Why?
2. If it is LTI, then find the overall equivalent impulse response $h_{eq}(t)$ and the overall equivalent transfer function $H_{eq}(f)$.

3. Calculate the response the system to the input signal: $$x(t) = -2\delta(t) + 3\delta(t - t_d)$$

8.8.12 problem

Consider the following block diagram:



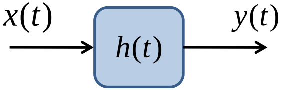
1. Is the overall equivalent system $y(t)=\Omega\{x(t)\}$ linear and time-invariant? Why?
2. If it is LTI, then find the overall equivalent impulse response $h_{eq}(t)$ and the overall equivalent transfer function $H_{eq}(f)$.
3. Calculate the response of the system to the input signal: $$x(t)=-\delta(t+t_0)+2\delta(t-t_0)$$

8.8.13 problem

Consider the signal:

$$x(t)=\sum_{n=-\infty}^{+\infty} e^{-(t-nT)}\mathrm{u}(t-nT)$$

Consider the LTI system:



where $H(f) = \Pi_{f_0}(f - 5f_0)$ and $f_0 = 1/T$.

1. Find the Fourier transform of $x(t)$ and draw it vs. frequency.
2. Calculate $Y(f)$
3. Calculate $y(t)$
4. Is $y(t)$ real or complex? Why?
5. Does the result of point (4) change if

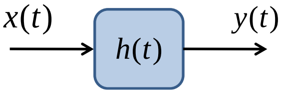
   $H(f) = \Pi_{f_0}(f - 5f_0) + \Pi_{f_0}(f + 5f_0)$? Why?

8.8.14 problem

Consider the signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} \Pi_{T_0/2}(t - nT_0)$$

Consider the LTI system:



where $H(f) = \Pi_{f_0}(f - 5f_0)$ and $f_0 = 1/T_0$.

1. Find the Fourier transform of $x(t)$ and draw it vs. frequency.
2. What type of 'filter' is $H(f)$? Draw the transfer function.

3. Calculate $Y(f)$
4. Calculate $y(t)$
5. Is $y(t)$ real or complex? Why?
6. Does the result of the previous point change if

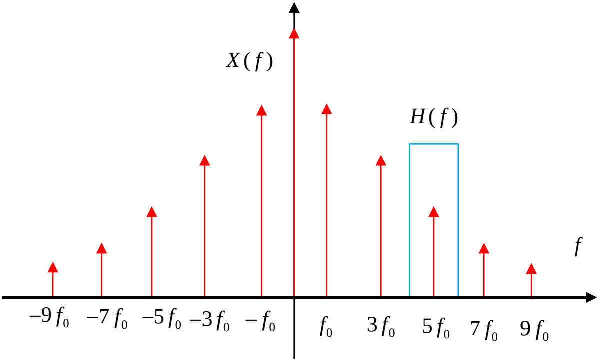
   $H(f) = \Pi_{f_0}(f - 5f_0) + \Pi_{f_0}(f + 5f_0)$? Why?

Answers

1. $$X(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \text{Sinc}\left(\frac{n}{2}\right)\delta\left(f - nf_0\right)$$

   For the plot, see below.

2. $H(f)$ is an ideal bandpass filter. It is shown below.



3. $Y(f) = \frac{1}{2}\mathrm{Sinc}\left(\frac{5}{2}\right)\delta\left(f - 5f_0\right)$

4. $y(t) = \frac{1}{2}\mathrm{Sinc}\left(\frac{5}{2}\right)e^{j2\pi\cdot(5f_0)\cdot t}$

5. $y(t)$ is complex. The reason is that its spectrum clearly does not comply with the symmetries expected for a real signal. This in turn is due to $H(f)$ having only one passband, for positive frequencies.
6. The result of the previous point changes because $H(f)$ now has two passbands, one for positive and one for negative frequencies. Re-doing the calculations, one gets:

$$Y(f) = \frac{1}{2}\mathrm{Sinc}\left(\frac{5}{2}\right)\left[\delta\left(f - 5f_0\right) + \delta\left(f + 5f_0\right)\right]$$

$$y(t) = \mathrm{Sinc}\left(\frac{5}{2}\right)\cos\left(2\pi\left[5f_0\right]t\right)$$

8.8.15 problem

Consider the signal:

$$x(t) = \sum_{n=-\infty}^{+\infty} \pi_{T_0/4}(t - nT_0)$$

Consider the LTI system:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

with transfer function:

$$H(f) = F\{h(t)\} = \text{Sinc}\left(\frac{f - 3f_0}{f_0}\right)$$

where $f_0 = 1/T$.

Do the following:

1. Find the Fourier transform of $x(t)$ and draw its absolute value vs. frequency.

2. Draw $H(f)$ vs. frequency.
3. Find $h(t)$ and draw it vs. time.
4. Find $y(t)$; do the calculation in frequency domain and then come back to time domain.

Solution

The signal $x(t)$ is periodic of period T_0. One possible truncated signal is $x_{T_0}(t) = \pi_{T_0/4}(t)$. The Fourier transform has general formula:

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} X_{T_o}(nf_0)\delta(f - nf_0)$$

The truncated signal transform $X_{T_o}(f)$ is:

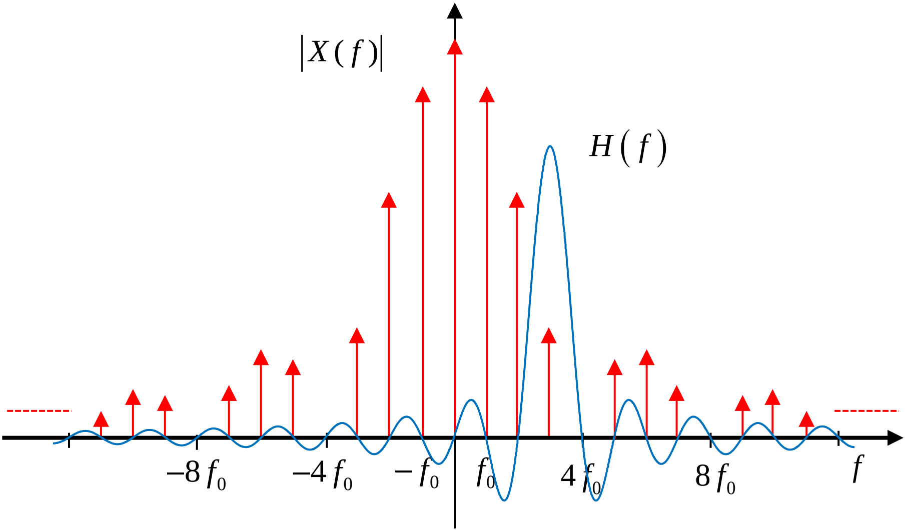
$$X_{T_o}(f) = F\left\{x_{T_0}(t)\right\} = F\left\{\pi_{T_0/4}(t)\right\} = \frac{T_0}{4}\mathrm{Sinc}\left(\frac{T_0}{4}f\right)e^{-j2\pi f\cdot(T_0/8)}$$

The Fourier transform of the periodic signal is then:

$$X(f) = f_0\sum_{n=-\infty}^{\infty}\frac{T_0}{4}\mathrm{Sinc}\left(\frac{T_0}{4}nf_0\right)e^{-j2\pi nf_0\cdot(T_0/8)}\delta(f-nf_0)$$

$$= \frac{1}{4}\sum_{n=-\infty}^{\infty}\mathrm{Sinc}\left(\frac{n}{4}\right)e^{-j(n\pi/4)}\delta(f-nf_0)$$

The plot of the absolute value of $X(f)$ is shown in the plot below in red, where also the plot of the transfer function $H(f)$ is shown (in blue).



The impulse response is:

$$h(t) = F^{-1}\{H(f)\} = F^{-1}\left\{\mathrm{Sinc}\left(\frac{f-3f_0}{f_0}\right)\right\} = F^{-1}\{\mathrm{Sinc}(T_0[f-3f_0])\}$$

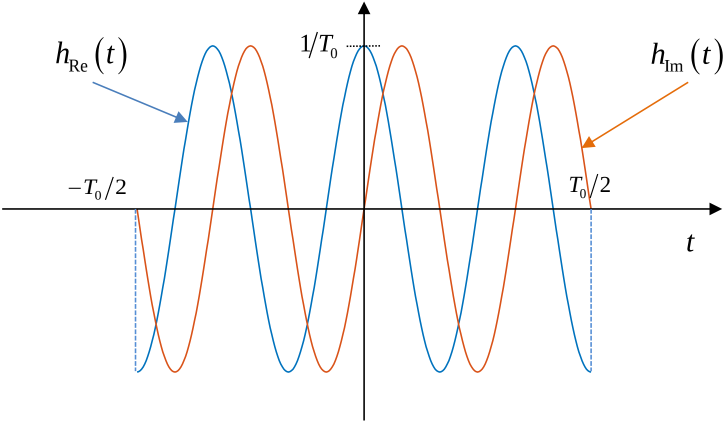
From the transform table we have:

$$F^{-1}\{\mathrm{Sinc}(T_0 f)\} = \frac{1}{T_0}\Pi_{T_0}(t)$$

The frequency shift by $3f_0$ can be accounted for using the frequency shift property of the Fourier transform. the result is:

$$F^{-1}\{\mathrm{Sinc}(T_0[f-3f_0])\} = h(t) = \frac{1}{T_0}\Pi_{T_0}(t)\cdot e^{j2\pi\cdot(3f_0)\cdot t}$$

The impulse response is therefore simply a rectangle of extension $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$, multiplied times a complex exponential oscillating at frequency $3f_0$. Being a complex impulse response, both the real and imaginary parts must be plotted. They are shown here below.



The output $y(t)$ can be found by operating in frequency domain. From the plot of $|X(f)|$ and $H(f)$ above, it is easy to see that all spectral lines of the input signal are brought to zero by the transfer function, with the exception of the one at frequency $3f_0$. As a result, one can immediately write:

$$Y(f) = X(f)H(f) = \frac{1}{4}\text{Sinc}\left(\frac{3}{4}\right)e^{-j(3\pi/4)}\delta(f-3f_0)$$

Taking the inverse Fourier transform:

$$y(t) = \frac{1}{4}\text{Sinc}\left(\frac{3}{4}\right)e^{-j(3\pi/4)}e^{-j2\pi(3f_0)\cdot t}$$

8.8.16 problem

Consider the signals:

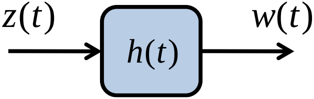
$$x(t) = e^{-(t/T)^2}$$

$$y(t) = \delta(t) - \delta(t - T)$$

$$z(t) = x(t) \cdot y(t)$$

The signal $z(t)$ is passed through an LTI system with impulse response:

$$h(t) = \Pi_T(t - T/2) = \pi_T(t)$$



Find the system output $w(t)$ and plot it.

Solution

First, we write down the input signal $z(t)$:

$$z(t) = x(t) \cdot y(t) = e^{-(t/T)^2} \cdot \left[\delta(t) - \delta(t-T) \right]$$
$$= e^{-(t/T)^2} \delta(t) - e^{-(t/T)^2} \delta(t-T) = \delta(t) - e^{-1} \delta(t-T)$$

The output of the system will be given by:

$$W(t) = z(t) * h(t) = \left[\delta(t) - e^{-1} \delta(t-T) \right] * \pi_T(t) = \pi_T(t) - e^{-1} \pi_T(t-T)$$

8.8.17 problem

Consider the signals:

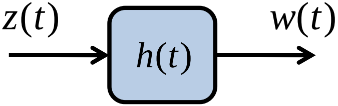
$$x(t) = \cos\left(\frac{2\pi}{T}t\right) \quad , \quad a \in \mathbb{R}^+$$

$$y(t) = \delta(t) + \delta(t - T/2)$$

$$z(t) = x(t) \cdot y(t)$$

The signal $z(t)$ is passed through an LTI system with impulse response:

$$h(t) = e^{-at}\mathrm{u}(t) \quad , \quad a \in \mathbb{R}^+$$



Find the system output $w(t)$ and plot it.