Chapter 9. Sampling and Reconstruction

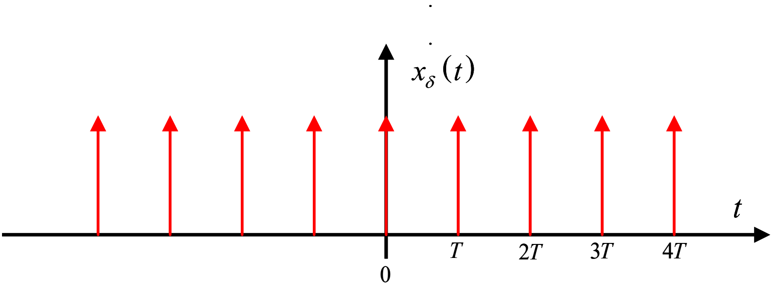
9.1 The Sampling Signal $x_\delta(t)$

We have already introduced in Chapter 6 the "train of delta's" signal:

$$x_\delta(t) = T \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

Eq. 9-1

Here we have slightly modified it by multiplying it times the normalization factor T, for reasons that will become clear later. The signal can be represented as follows:



We will soon need the Fourier transform of $x_\delta(t)$. In Chapter 6 we calculated the Fourier transform of a similar *train of deltas*:

$$F\left\{\sum_{n=-\infty}^{+\infty}\delta(t-nT_0)\right\}=f_0\sum_{n=-\infty}^{+\infty}\delta(f-f_0)$$

where $T_0=1/f_0$.

The interesting result is that the Fourier transform of a train of deltas in time is a train of deltas in frequency.

We can easily adapt the above result from Chapter 6 to obtain:

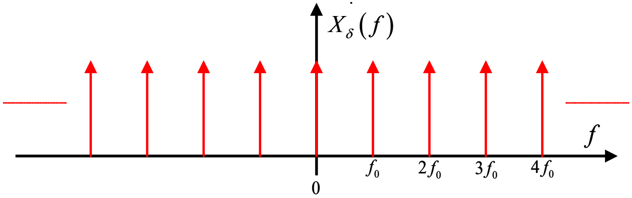
$$F\{x_\delta(t)\} = F\left\{T\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right\} = T\cdot F\left\{\sum_{n=-\infty}^{+\infty}\delta(t-nT)\right\}$$
$$= T\cdot f_0\sum_{n=-\infty}^{+\infty}\delta(f-nf_0) = \sum_{n=-\infty}^{+\infty}\delta(f-nf_0)$$

So, quite simply:

$$F\{x_\delta(t)\} = \sum_{n=-\infty}^{+\infty}\delta(f-nf_0)$$

with $T = 1/f_0$

So, the Fourier transform of the train of deltas $x_\delta(t)$ in time is a train of deltas in frequency, with period $f_0 = 1/T$:



9.2 Sampling a Signal

Given a generic signal $x(t)$, it is possible to extract its "samples" by multiplying it times the sampling signal $x_\delta(t)$. We have:

$$x(t) \cdot x_\delta(t) = x(t) \cdot T \sum_{n=-\infty}^{+\infty} \delta(t - nT) = T \sum_{n=-\infty}^{+\infty} x(t) \cdot \delta(t - nT)$$

Eq. 9-2

We then exploit the known "sampling property" of the delta:

$$x(t) \cdot \delta(t - t_0) = x(t_0) \cdot \delta(t - t_0)$$

In essence, in the above equation, of the overall signal $x(t)$ only one sample survives after multiplication times a delta: $x(t_0)$. Using this result, we can re-write Eq. 9-5:

$$x_s(t) = x(t) \cdot x_\delta(t) = T \sum_{n=-\infty}^{+\infty} x(nT) \cdot \delta(t - nT)$$

Eq. 9-3

where the subscript "*s*" stands for "sampled". Note that $x_s(t)$ only contains the samples $x(nT)$ of the original signal $x(t)$. All the other values of $x(t)$, at any other times, are, at this point, "lost". In other words, the $x(nT)$'s are all the residual information regarding $x(t)$ that is present in the sampled signal $x_s(t)$.

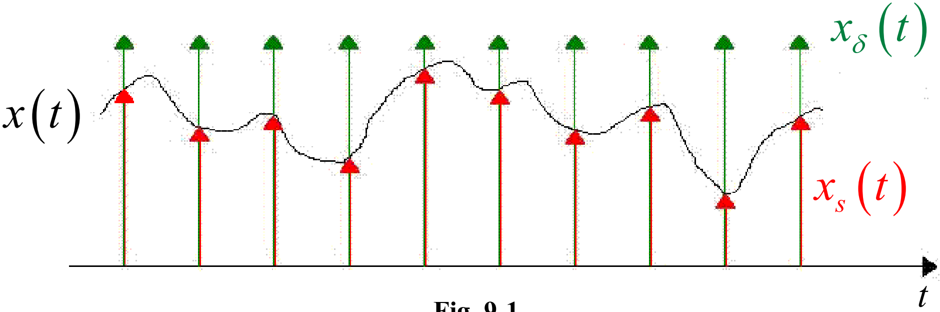


Fig. 9-1

Note also that the quantity $f_0 = 1/T$ is called the "sampling frequency". Physically, the number f_0 tells us how many samples of the signal $x(t)$ are taken each second. Conversely, T tells us how many seconds separate two successive signal samples.

9.3 Reconstructing a Signal from its Samples

We now want to find out whether it possible to reconstruct the signal $x(t)$ from the sampled signal $x_s(t)$. In principle, this appears to be difficult, because we have thrown away almost all of $x(t)$, except for relatively few samples, equally-spaced in time: $x(nT)$. So, looking at this problem from a time-domain perspective, it would seem as if only an approximation of $x(t)$ could be realistically obtained, but not the exact original $x(t)$.

However, we want to try in frequency domain and see if a different picture emerges there. First, we evaluate the Fourier transform of $x_s(t)$:

$$F\left\{x_s(t)\right\} = F\left\{x(t)\cdot x_\delta(t)\right\} = F\left\{x(t)\right\} * F\left\{x_\delta(t)\right\} = X(f) * F\left\{x_\delta(t)\right\}$$

We found in Section 9.1 that:

$$F\left\{x_{\delta}(t)\right\} = \sum_{n=-\infty}^{+\infty} \delta(f - f_0)$$

and so:

$$F\left\{x_s(t)\right\} = X(f) * F\left\{x_{\delta}(t)\right\} = X(f) * \sum_{n=-\infty}^{+\infty} \delta(f - nf_0) =$$
$$= \sum_{n=-\infty}^{+\infty} X(f) * \delta(f - nf_0)$$

Eq. 9-4

We now recall another important property of the delta:

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

which of course has its symmetric counterpart in frequency domain:

$$X(f) * \delta(f - f_0) = X(f - f_0)$$

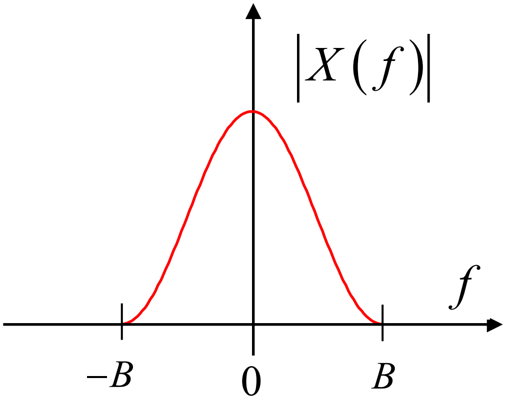
Eq. 9-5

Using Eq. 9-8 we can re-write Eq. 9-7 as:

$$X_s(f) = F\{x_s(t)\} = \sum_{n=-\infty}^{+\infty} X(f - nf_0)$$

Somewhat unexpectedly, in this formula of the spectrum of $x_s(t)$, the spectrum of the original signal $x(t)$ *does* appear. However, it appears in multiple copies, so it is not yet clear whether a single copy can be isolated and used to reconstruct $x(t)$

To better understand the outcome of the sampling process, we plot the sampled signal spectrum $X_s(f)$. We, first of all, assume that the extension of $X(f)$ is limited to $[-B, B]$:



We have two possible cases.

case of $f_0/2 < B$

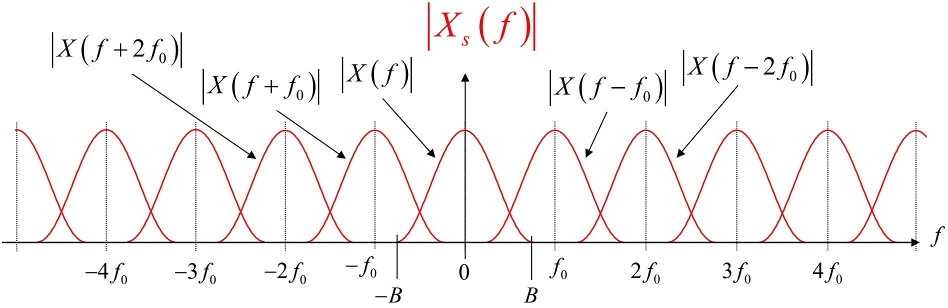


Fig. 9-2

In this case the spectral copies of $X(f)$, translated to multiples of f_0, clearly overlap.

case of $f_0/2 > B$

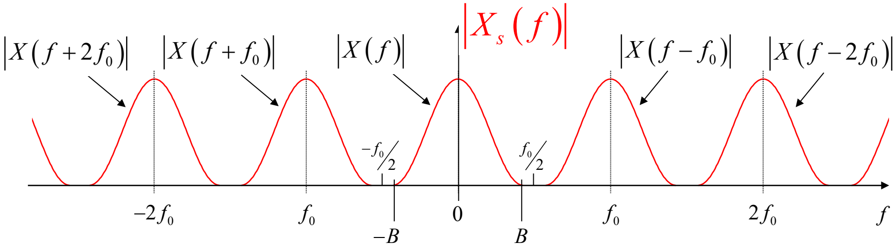


Fig. 9-3

In this case the spectral copies of $X(f)$, still translated to multiples of f_0, do not overlap because f_0 is large enough to keep them separate. So, one could think of using a suitable filter to recover one of the copies.

In fact, the easiest option is to use a low-pass filter to extract the fundamental copy, centered at the origin. Ideally, one could use an ideal (rectangular) low-pass filter, such as for instance:

$$H_R(f) = \Pi_{f_0}(f) \ ,$$

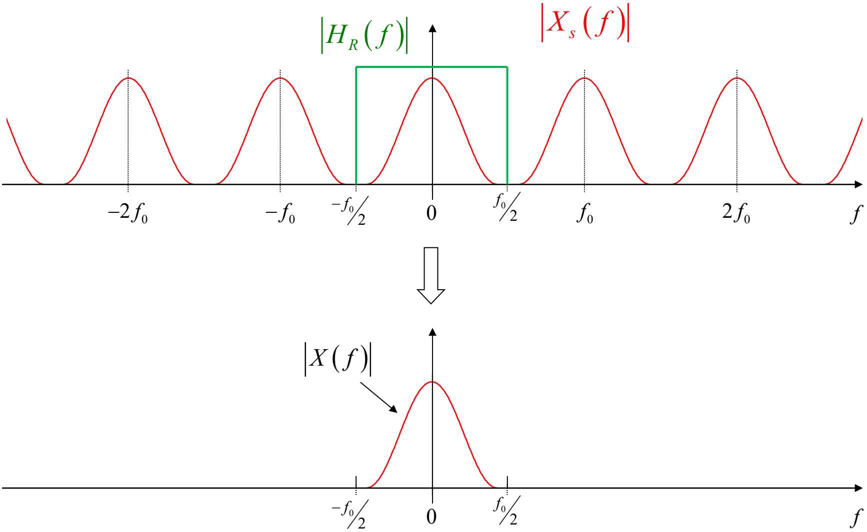
where the subscript "*R*" stands for "reconstruction". The surprising result is:

$$X_s(f)H_R(f) = X_s(f)\Pi_{f_0}(f) = X(f)$$

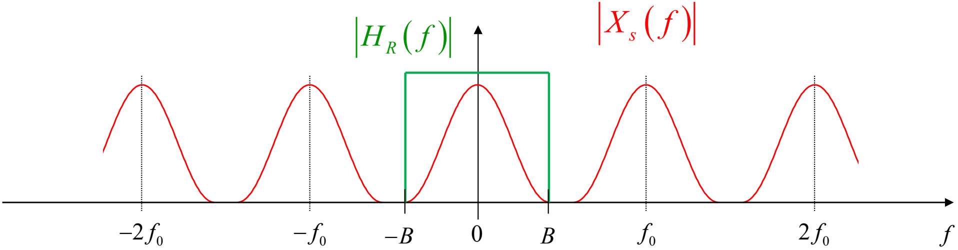
or, in time domain:

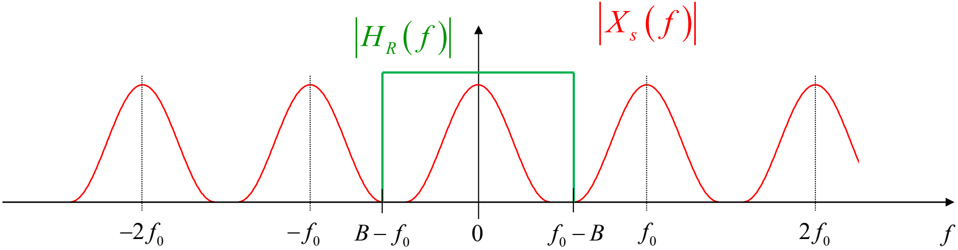
$$x_s(t) * h_R(t) = x(t)$$

In other words, the original signal $x(t)$ is fully recovered, as the frequency domain analysis shows, by proper low-pass filtering.

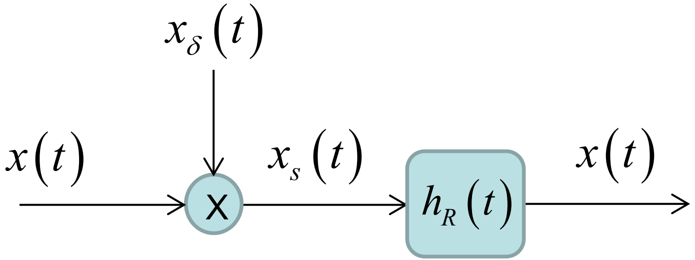


Note that in fact there is some freedom in choosing the ***bandwidth*** of the ideal low-pass filter. In fact, any value between B and $(f_0 - B)$ would still produce a perfectly recovered signal $x(t)$, as shown in figure.





Interestingly, we can draw a *block diagram* that is equivalent to the sampling and reconstruction process:

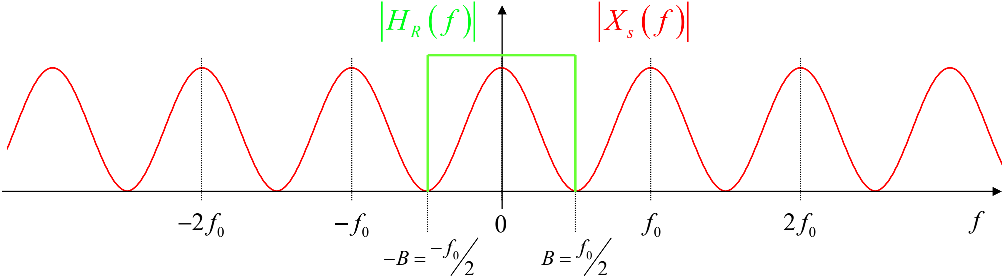


9.3.2 The Sampling Theorem (Nyquist Theorem)

Note that in the case $f_0/2 < B$ shown in Fig. 9-1, the reconstruction filtering would fail to recover $x(t)$ because the spectral copies are overlapped and there is no way to separate them out *a posteriori* by means of a filter, or by any other means.

Instead, if $f_0/2 > B$, as shown, it is possible to reconstruct the original signal with no loss of information, despite the fact that the sampling process saves only discrete-time samples of the original signal.

From Fig. 9-1 and Fig. 9-2 it is then clear that the limit condition for reconstruction to be possible is $f_0/2 = B$ or $f_0 = 2B$. Also, when $f_0 = 2B$, the support of the ideal low pass-filter must be equal to exactly f_0 for perfect reconstruction to take place.



This is the smallest possible f_0 and from this limiting condition the following theorem, called the *sampling theorem*, also known as Nyquist theorem, is derived:

given a signal $x(t)$ *whose spectrum* $X(f)$ *has extension* $[-B,B]$, *then it is possible to exactly reconstruct* $x(t)$ *from its samples* $x(nT)$ *provided that the sampling frequency* $f_0 = 1/T$ *is greater than or equal to* $2B$, *that is:* $f_0 \geq 2B$.

Note that the frequency $2B$ is sometimes called "Nyquist frequency". Similarly the condition $f_0 \geq 2B$ is called the "Nyquist condition".

9.3.3 Practical aspects of sampling and reconstruction

The sampling theorem provides the theoretical foundations for the transition from the "analog" to the "digital" world. In truth, there are a number of practical issues that need to be clarified.

The first aspect that needs to be clarified is that the "sampling signal" $x_\delta(t)$ is just a mathematical tool that we have used to model the extraction of the samples $x(nT)$ from $x(t)$. In practice, considering the sampling of an audio signal coming, for example, from a microphone, such signal is sampled by an electronic circuit that simply measures the voltage at its input at regular intervals of T seconds.

So, such circuit directly outputs $x(nT)$ without any associated $x_\delta(t)$ signal, which would anyway be impossible to generate in practice because of the deltas. The name of such circuit is ADC, or analog-to-digital converter.

To reconstruct the analog signal, theory says we should do the following:

$$x(t) = x_s(t) * h_R(t)$$

This procedure appears unrealistic, since it requires generating the sampled signal

$x_s(t)$, which contains deltas in time.

Even though one could try and approximate the deltas with short and tall pulses, *there is in fact no need to do so*. We can instead first *formally* execute the convolution product:

$$x(t) = x_s(t) * h_R(t) = \left[T \sum_{n=-\infty}^{+\infty} x(nT) \cdot \delta(t - nT) \right] * h_R(t)$$

Due to the linearity of the convolution product:

$$x(t) = x_s(t) * h_R(t) = T \sum_{n=-\infty}^{+\infty} x(nT) \cdot \left[\delta(t - nT) * h_R(t) \right]$$

so in the end:

$$x(t) = T \sum_{n=-\infty}^{+\infty} x(nT) \cdot h_R(t - nT)$$

This reconstruction formula has no deltas and so this major practical problem is solved.

We can also remark that of course the signal $x(t)$ will have a *limited extension*. As a result, *the summation index limits are finite*:

$$x(t) = T \sum_{n=0}^{N-1} x(nT) \cdot h_R(t - nT)$$

Eq. 9-6

All we need is some electronic circuit that at regular intervals outputs the sum of waveforms $h_R(t)$, scaled by $x(nT)$ and delayed as needed.

However, even though Eq. 9-9 is free of deltas, it still has a fundamental problem.

If we want to use an ideal (rectangular) low-pass filter as reconstruction filter, the corresponding impulse response is:

$$h_R(t) = \mathrm{F}^{-1}\left\{\Pi_{2B}(f)\right\} = \frac{\sin(2\pi Bt)}{\pi t} = 2B \cdot \mathrm{Sinc}\left(2Bt\right)$$

Unfortunately, $2B \cdot \mathrm{Sinc}\left(2Bt\right)$ has extension $\left[-\infty, \infty\right]$: it is clearly non-causal and therefore it is not *physically realizable*. Also, apart from causality, implementing long-lasting impulse responses, even when causal, is problematic anyway.

One pragmatic approach to this problem is that of using an approximation of $h_R\left(t\right)$ which could somehow be made causal. For instance, one could “cut out” a section of $h_R\left(t\right)$ with finite extension, and use it as an approximation of $h_R\left(t\right)$:

$$h_{R_{app}}(t) = 2B \cdot \mathrm{Sinc}\left(2B\left[t - t_d\right]\right) \cdot \Pi_{2t_d}(t - t_d)$$

The rectangular signal kills the tails of $h_R(t)$ outside of a time range whose duration is $2t_d$ seconds. The resulting approximated impulse response is then delayed by t_d to make it causal. Note that these aspects were discussed at length in Chapter 8 so please go back to that material for more details.

Note that attributing a delay t_d to the reconstructing filter has the only effect of causing the same delay to appear on the reconstructed signal $s(t) \to s(t - t_d)$ (prove it on your own).

This technique is effective and it is indeed possible to use it, but unfortunately $\mathrm{Sinc}\left(2B\left[t - t_d\right]\right)$ decays only as $1/|t|$, so the time-window $2t_d$ necessary to obtain a reasonably good approximation $h_{R_{app}}(t)$ of $h_R(t)$ is very large. This would make an actual implementation complex and expensive.

To solve this further problem, it is possible to use a reconstructing filter $H_R(f)$ whose impulse response decays faster in time than that of an ideal low-pass filter.

For instance, it is possible to use a raised-cosine low-pass filter such as the one shown in Chapter 8. The fact that the transfer function is continuous and differentiable (that is, C^1) ensures that its corresponding impulse response $h_R(t)$ decays as $1/|t|^3$, as shown in Chapter 8. As a result, the time-window $2t_d$ needed to obtain a good approximation of $h_R(t)$ is much shorter. In fact, it is even possible to use reconstruction filters with spectral profiles that are smoother than that of a raised-cosine. It is in fact possible to use $h_R(t) \in C^n$ with $n > 1$, so that

$h_R(t) = O\left(1/|t|^{n+2}\right)$.

The drawback of this strategy is that the raised-cosine (or smoother) filters are not perfectly rectangular at the sides and therefore, to ensure perfect reconstruction, it is necessary to use a larger sampling rate than the minimum. In other words: $f_0 > 2B$. This is because more spectral separation is necessary among the spectral copies of $X(f)$ to allocate the non-perfectly vertical sides of the reconstructing filter $H_R(f)$.

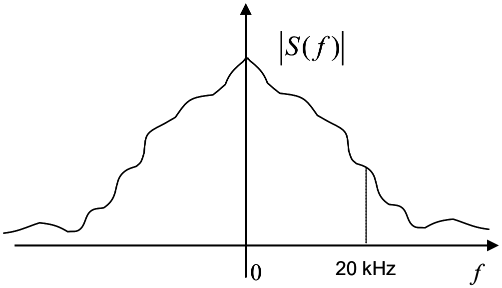
All of the above is essentially what is truly done in practice, either directly or in ways that are *equivalent* to this.

The circuit that implements conversion from digital to analog is generally called DAC (digital-to-analog converter).

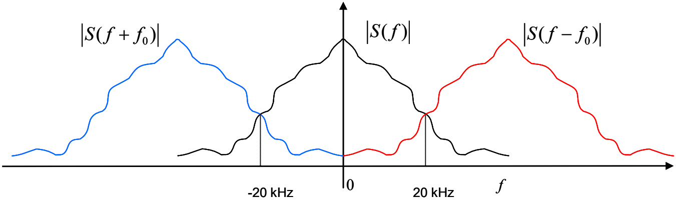
9.3.3.1 the problem of "aliasing"

A proper design of a sampling circuit must start from the *requirements* of the application. For instance, dealing with audio signals, it is well-known that the human hearing does not extend beyond 20 kHz. This would suggest that a proper f_0 for audio applications should be about 40 kHz.

However, the spectrum of music, as shown by actual measurements, extends well beyond 20 kHz: musical instruments emit "higher harmonics" up to 40-50 kHz (or even higher) as shown in the example below:



Sampling this signal with only f_0=40 kHz would lead to very substantial aliasing and the reconstructed signal would be corrupted by the tails of the adjacent spectral copies:



To fix this problem it would be necessary to significantly increase f_0, perhaps up to 100 kHz. However, increasing f_0 requires faster and more expensive electronics and, in addition, taking more samples per seconds also means *accumulating more data to store on disk.*

The solution to this problem is the use of a “pre-filter”, called *anti-aliasing* filter $H_A(f)$. The criterion is as follows: if the needed frequency range is only up to 20 kHz, then a low-pass anti-aliasing filter is inserted before the ADC to limit the bandwidth of the signal to $B = 20$ kHz:

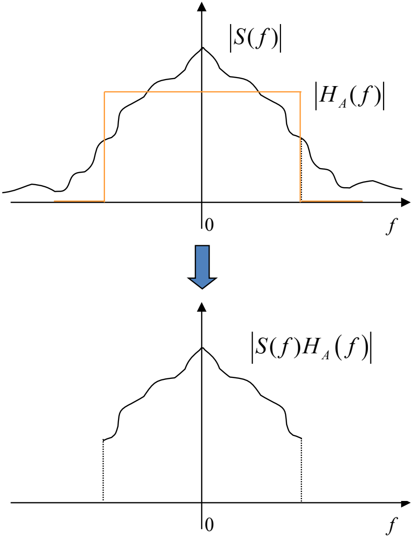
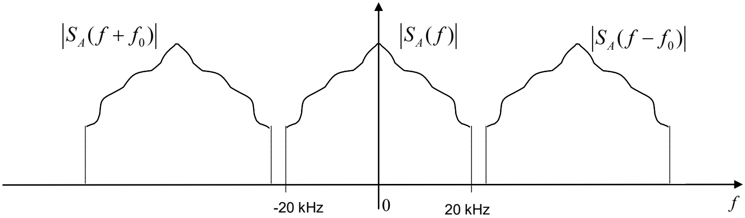


Fig. 9-4 Anti-aliasing filter and its effect.

The resulting signal:

$$S_A(f) = S(f)H_A(f)$$

is band-limited to 20 kHz and the sampling frequency can now be chosen close to the theoretical 40 kHz, without incurring aliasing problems:



Note that in practice the anti-aliasing filter cannot be an ideal low-pass filter as assumed in this section, for simplicity. Once again, the anti-aliasing filter will be a suitable approximation of an ideal low-pass filter.

9.3.4 Estimating the amount of stored data for a sampled signal

In a Chapter 5, we showed that a musical signal could be "digitized" and then "reconstructed" by means of its Fourier series coefficients.

We had then assumed a musical signal 200 [s] long, stereo, with Fourier coefficients stored up to ± 20 kHz. We had also assumed that real numbers would be represented using 16 bits (2 bytes). The total amount of necessary data was found to be 32 Mbytes. That method could be considered a frequency-domain digitizing approach, since the Fourier series coefficients represent the frequency components of the signal and are closely related to its Fourier transform.

We now want to see whether there is any difference in the needed amount of data storage when using time-domain sampling to digitize the signal.

First off, we assume we can use an anti-aliasing filter, so that the frequency extension of the musical signal does not exceed 20 kHz, as shown in Fig. 9-3. We then assume for now that we can use an ideal low-pass filter as reconstructing filter $H_R(f)$. As a result, we can then employ the minimum f_0 indicated by the sampling theorem: $f_0 = 2B = 40$ kHz.

Using this sampling frequency, each second 40 thousand real samples are taken, for each of the two signals making up the overall stereo musical signal. This means 80 thousand samples per second. Since the duration of the signal is 200 [s], the total amount of samples is 16 million. Each sample requires 2 bytes to be stored so, in the end, the total amount of data is 32 Mbytes.

This number is *identical* to the number we had obtained using the Fourier-coefficients method. This should not be surprising, because bits or bytes measure *information*. The identical result simply confirms that the amount of information contained in the signal is 32 Mbytes. We may use different methods to extract it, but that is how much information there is.

The fact that a frequency-domain method and a time-domain method yield the same result also confirms what we already know: *frequency and time-domain are fully equivalent ways to represent signals*.

However, the Fourier coefficients method is not used in practice because it would be more expensive and less practical to implement (though it would be perfectly *possible* to realize it). Note that certain special encoding methods, such as those used by MP3, bear a greater resemblance to the Fourier method than the time-domain method. On the other hand, MP3 is a highly lossy encoding standard, whereby a lot of information is thrown out. MP3 relies on psychoacoustics research to implement suitable strategies that make the loss of information relatively inaudible. Any comparison with the essentially lossless or low-loss systems introduced in this class is improper. We will not expand on this topic further.

The standard time-domain sampling method is the one in use in CDs and many other audio recording standards. Since ideal low-pass filters are not practical, smoother filters are used and, as a result, it must be $f_0 > 2B$. In fact, the audio CD standard, for instance, specifies $f_0 = 44.1$ kHz, which is more than $2B$, for the target $B = 20$ kHz. Each sample is then stored on the audio CD using 16 bits, exactly as we have assumed in our examples.

The 10% higher f_0 is a compromise to ease implementation. The advantage is simpler and cheaper electronics, the disadvantage is the need to store about 10% more information than strictly needed.

9.3.4.1 Problem

On your own:

A signal $x(t)$ is input to an ideal low-pass filter whose transfer function $H_1(f)$ is the following:

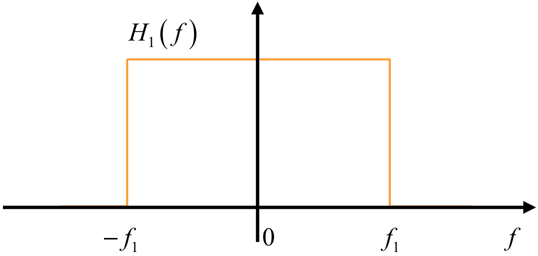


Fig. 9-5– $H_1(f)$

The output signal, called $v(t)$, goes into a sampling system that operates as follows:

$$w(t) = v(t) \cdot x_\delta(t) = v(t) \cdot \frac{1}{f_0} \sum_{n=-\infty}^{+\infty} \delta\left(t - \frac{n}{f_0}\right)$$

Eq. 9-7

The sampled signal is then fed to another ideal low-pass filter $H_2(f)$:

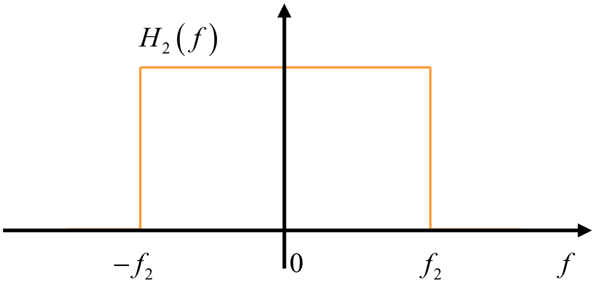


Fig. 9-6 $H_2(f)$

The output of the second ideal low-pass filter is called $y(t)$. A block diagram is provided below:

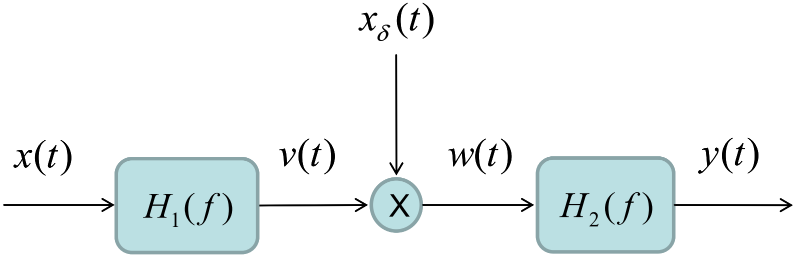


Fig. 9-7 – Block diagram representation of the system.

Find the relationship among f_0, f_1, f_2*, such that:* $y(t) \equiv v(t)$, $\forall x(t)$ and $\forall t$.

Solution:

The first filter, $H_1(f)$, guarantees that the signal $v(t)$ is band-limited, with extension: $\text{ext}\{V(f)\} = [-f_1, f_1]$.

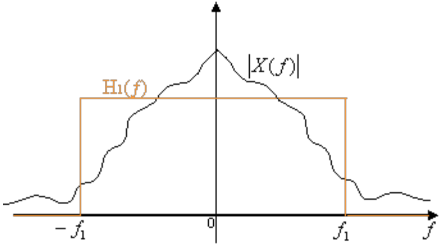


Fig. 9-8 Spectrum of the signal $x(t)$ and transfer function $H_1(f)$.

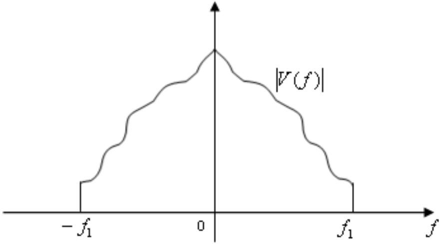


Fig. 9-9 – Spectrum of the signal $v(t)$ at the output of $H_1(f)$.

The sampling block multiplies $v(t)$ times a train of deltas in time, that is, times a "sampling signal" $x_\delta(t)$. In frequency domain, we have:

$$W(f) = \mathrm{F}\{w(t)\} = V(f) * \mathrm{F}\left\{\frac{1}{f_0}\sum_{n=-\infty}^{+\infty}\delta\left(t - \frac{n}{f_0}\right)\right\} =$$

$$= V(f) * \sum_{n=-\infty}^{+\infty}\delta(f - nf_0) = \sum_{n=-\infty}^{+\infty} V(f - nf_0)$$

As a result, the signal $W(f)$ is periodic in frequency, with period f_0, and is made up of frequency shifted copies of $V(f)$. So there are two cases:

1. $f_0 - f_1 < f_1$

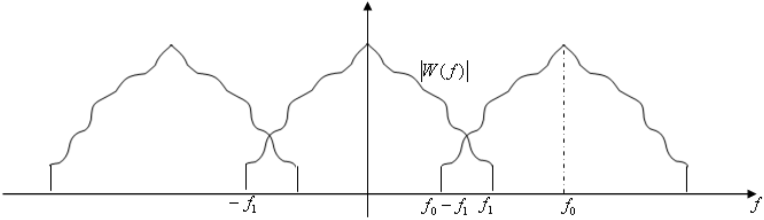


Fig. 9-10 – Spectrum of $W(f)$ when $f_0 - f_1 < f_1$

The copy of $V(f)$ located at the origin is corrupted by the tails of the successive copies $V(f - nf_0)$, with $n \neq 0$. In this case there seems to be no chance to ensure that $y(t) \equiv v(t)$, as requested by the problem.

2. $f_0 - f_1 > f_1$

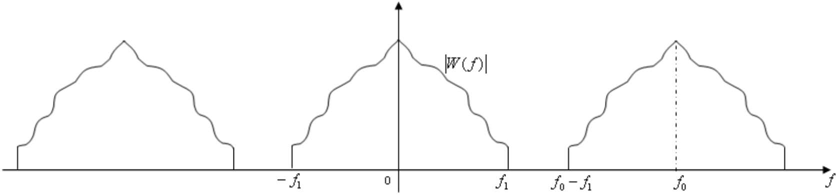


Fig. 9-11 – Spectrum of $W(f)$ when $f_0 - f_1 > f_1$

In this case at the origin there is a "clean" copy of $V(f)$. As a first result, then it clearly must be:

$$f_0 - f_1 > f_1$$

Eq. 9-8

We have now to find the possible values of f_2 that ensure $y(t) \equiv v(t)$. Looking at the figure below, we have the following cases:

1. $f_2 = f_2''$: the filter is too wide, a considerable portion of the adjacent copies of $V(f)$ are let through;
2. $f_2 = f_2'$: the filter is still too wide, some portion of the adjacent copies of $V(f)$ are let through;
3. $f_2 = f_2'''$: the filter is still too narrow, only the baseband copy of $V(f)$ goes through but some of it is missing;
4. $f_1 < f_2 < f_0 - f_1$: in this case only the baseband copy of $V(f)$ goes through and it is not distorted

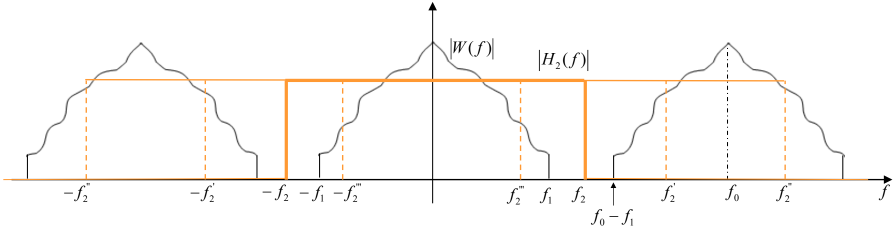


Fig. 9-12 – Reconstructing filter $H_2(f)$ applied to $W(f)$

In summary, to make sure that

$$y(t) \equiv v(t), \quad Y(f) \equiv V(f)$$

f_0, f_1, f_2 must be bound together by this relationship:

$$f_1 < f_2 < f_0 - f_1$$

Clearly, taking the limiting case of all equalities one finds:

$$f_0 = 2f_2 = 2f_1$$

9.3.4.2 Problem: using a raised cosine filter as reconstruction filter

Assume that the signal $s(t)$ is of bandwidth B and that it is sampled at $f_0 > 2B$ The signal is then reconstructed using a raised cosine filter. Find the parameters of the raised cosine filter, B_{FWHM} and the roll-off factor α, such that:

1) the reconstructed signal $s_R(t)$ is perfectly identical to $s(t)$
2) the flat-top part of the filter is as narrow as possible
3) α is the maximum possible

Solution

With reference to Fig. 9-12, the constraint that $s_R(t) = s(t)$ requires that the flat top of the filter transfer function, which extends to frequency $\frac{B_{\text{FWHM}}}{2}(1-\alpha)$, be at least as wide as the bandwidth of the signal B, or wider. Mathematically, this can be stated as:

$$\frac{B_{\text{FWHM}}}{2}(1-\alpha) \geq B$$

But requirement (2) asks for the flat top of the raised cosine filter to be the smallest possible, that is minimize $\frac{B_{\text{FWHM}}}{2}(1-\alpha)$. To do so, it is enough to take the above relation with the equal sign:

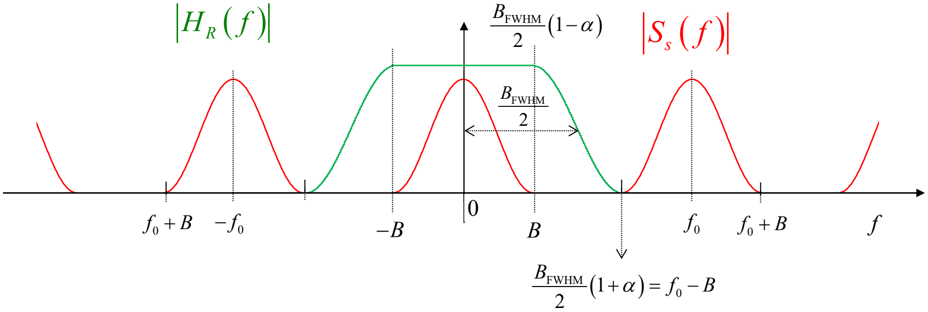


Fig. 9-13

$$\frac{B_{\text{FWHM}}}{2}(1-\alpha)=B$$

Then, to maximize α according to requirement (3), we need to extend the support of the raised-cosine filter as much as possible. This means that the support of the filter for positive frequencies must extend as far as right as the start of the first copy of the signal spectrum, that is, up to f_0-B. From this condition (see Fig. 9-12) we have:

$$\frac{B_{\text{FWHM}}}{2}(1+\alpha)=f_0-B$$

We now have two equations in the two unknowns α and B_{FWHM}. Solving for them we get:

$$B_{\text{FWHM}}=\frac{f_0}{2} \quad , \quad \alpha=1-\frac{2B}{f_0}$$

The resulting α is indeed the maximum possible under the given constraints.

9.4 Signal processing on sampled signals

As mentioned, the sampled signal is written:

$$x_s(t) = T \sum_{n=-\infty}^{+\infty} x(nT) \cdot \delta(t - nT)$$

We then found its spectrum by taking its Fourier transform:

$$X_s\left(f\right) = F\left\{s_s(t)\right\} = \sum_{n=-\infty}^{+\infty} X(f - nf_0)$$

We point out that the same result can be written differently:

$$X_s(f) = F\{x_s(t)\} = F\left\{T\sum_{n=-\infty}^{+\infty} x(nT)\cdot\delta(t-nT)\right\}$$
$$= T\sum_{n=-\infty}^{+\infty} x(nT)\cdot F\{\delta(t-nT)\}$$
$$= T\sum_{n=-\infty}^{+\infty} x(nT)\cdot e^{-j2\pi f\cdot nT}$$

The last line can clearly be viewed as a DTFT. So, we could write:

$$X_s(f) = \mathrm{DTFT}\{s[n]\}$$

where we have defined: $x[n] = T\cdot x(nT)$ to conform to the notation of sequences in discrete-time (see also for reference https://en.wikipedia.org/wiki/Discrete-time_Fourier_transform).

We then further point out that:

$$\text{DTFT}\{x[n]\} = X(z)_{z=e^{j2\pi fT}}$$

where $X(z) = \mathrm{Z}\{x[n]\}$ is the Z-transform of $x[n]$. So, overall, we have:

$$X_s(f) = F\{x_s(t)\} = \text{DTFT}\{x[n]\} = \mathrm{Z}\{x[n]\}_{z=e^{j2\pi fT}}$$

These formulas show that all these quantities are deeply connected and, in fact, essentially equivalent. Also, the sequence $x[n]$ can be altered using for instance a numerical filter defined or designed through a Z-transform of the transfer function. If the sequence is then converted back to analog, the spectral alterations due to such numerical filtering will carry over to the analog reconstructed signal.

We will not further comment on these aspects here, since this is done in depth in the discrete-time section of the course.

9.5 The Paradox of Time-Frequency Extensions [optional]

The sampling theorem appears to provide a clear solution to the problem of sampling and then reproducing a time-domain signal.

However, rigorously speaking, the sampling theorem presents an unsolved problem. In the following we will show what this problem is and provide some comments.

9.5.1 Fourier transforms of finite-extension signals

We remind the reader that a finite-extension signal $s(t)$ is such that:

$$\mathrm{ext}\{s(t)\} = [t_0, t_1], \quad |t_0|, |t_1| < \infty$$

Finite-extension can of course be defined in frequency domain as well:

$$\mathrm{ext}\{S(f)\} = [f_0, f_1], \quad |f_0|, |f_1| < \infty$$

An important theorem regarding Fourier transforms states that:

given a signal $s(t)$ and its Fourier transform $S(f)$, if either one of them has finite extension, then the other has infinite extension.

A rigorous proof of this theorem is somewhat complicated. We give here an easy, though not fully rigorous, proof.

We first consider a time-domain signal $s'(t)$. We make no assumption on the extension of such signal, but we derive from it a signal that is *certainly* finite-extension, by multiplying it times a rectangular signal:

$$s(t) = s'(t) \cdot \Pi_{2T}(t - t_d)$$

As a result, $s(t)$ has finite extension: $\left[t_d - T, t_d + T\right]$.

We now take the Fourier transform of $s(t)$:

$$\begin{aligned} S(f) &= \mathrm{F}\left\{s(t)\right\} \\ &= \mathrm{F}\left\{s'(t) \cdot \Pi_{2T}(t - t_d)\right\} \\ &= \mathrm{F}\left\{s'(t)\right\} * \mathrm{F}\left\{\Pi_{2T}(t - t_d)\right\} \\ &= 2T \cdot S'\left(f\right) * \left[\mathrm{Sinc}\left(2Tf\right) e^{j2\pi f t_d}\right] \end{aligned}$$

The rightmost side tells us that we have to carry out a convolution product in frequency domain. We first remark that:

$$\text{ext}\{\text{Sinc}(2fT)\} = [-\infty, \infty].$$

Instead, we know nothing regarding the extension of $S'(f)$, which we leave generic:

$$\text{ext}\{S'(f)\} = [f_0, f_1]$$

As for all convolution products, the extension of the result is given by the "sum" rule: the lower limit is the sum of the lower limits of the two convolved signals, and the upper limit is the sum the upper limits of the two convolved signals. Therefore:

$$\text{ext}\{S(f)\} = [f_0 - \infty, f_1 + \infty] = [-\infty, +\infty]$$

This result clearly shows that the extension of $S(f)$ is *always* going to be infinite.

The reason for this is that we made sure that the extension of $s(t)$ was finite, by multiplying $s'(t)$ times a rectangular signal of extension $\left[t_d - T, t_d + T\right]$.

This theorem, which can be more rigorously proved, means for instance that if a time-domain signal is time-limited, it must contain some contribution from all possible frequencies, because its spectrum has infinite frequency extension.

This conclusion is quite unsettling, because all practical time-domain signals are essentially finite-extension in time. For instance, when we start playing a song and then the song ends and perhaps we even switch off the player, that makes the time-domain musical signal finite-extension "for sure". This, however, appears to require that such signal contains components from extreme frequencies, in fact infinitely high, which is hard to consider possible in the physical world.

Note that the theorem also applies to signals that are finite-extension in frequency. Then, their time-domain counterpart is infinitely extended over time. This "dual" version of the time-frequency extension theorem makes the sampling theorem run into problems.

Specifically, one key assumption of the sampling theorem is that the spectrum of the time-domain signal to be sampled is finite-extension in frequency $[-B,+B]$. This is mathematically possible, but then the corresponding time-domain signal is for sure infinitely extended over time. Therefore, *there is no single "practical"* (that is finite-extension and finite-duration) *time-domain signal that satisfies the assumption of the sampling theorem*, because the spectrum of any time-limited signal is simply never band-limited.

Typically this paradox is ignored, by relying on the fact that physical signals have spectra that decay rather fast, so that one can always find a finite "bandwidth" $[-B,+B]$ which virtually contains "all" of the spectrum of a signal. However, the paradox remains: in theory, some aliasing will always occur.

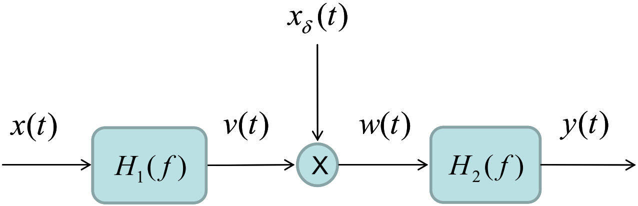
It is interesting to note that the time-frequency extension theorem can be given a more general form whereby even causality becomes problematic. We will however not discuss it here.

End of Optional Material

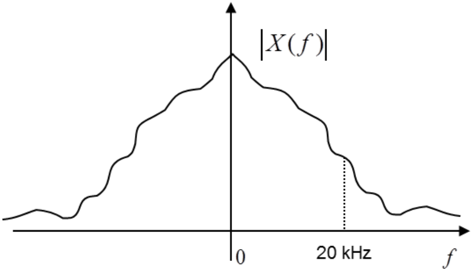
9.6 Problems

9.6.1

Consider the system:



where: $x_\delta(t) = T_s \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$ and $x(t) \in \mathbb{R}$ is an audio signal, whose Fourier Transform is:



The requested result is the following:

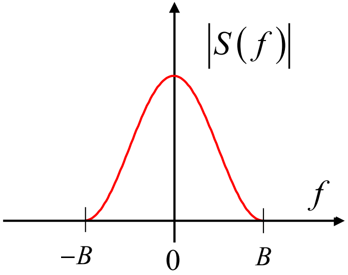
$$Y(f) = X(f) \quad , \quad f \in [-20 \text{ kHz}, 20 \text{ kHz}]$$

$$Y(f) = 0 \quad , \quad f \notin [-20 \text{ kHz}, 20 \text{ kHz}]$$

Specify the filters $H_1(f)$, $H_2(f)$, and the minimum value of the constant $f_s = 1/T_s$ such that the requested result is obtained. Prove mathematically that your proposed solution works.

9.6.2

Assume that the signal $s(t)$ has limited bandwidth B and its Fourier Transform is approximately as in figure:



Assume that the signal is sampled at frequency $f_0 = 4B$. It is then reconstructed using the following reconstruction filter impulse response:

$$h_R(t) = \frac{\pi_T(t)}{T}$$

where $T = 1/f_0$

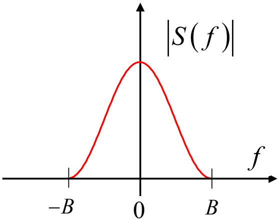
- Draw the Fourier transform of the sampled signal: $s_s(t) = s(t) \cdot x_\delta(t)$, where:

  $$x_\delta(t) = T \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

- Draw the reconstruction filter transfer function.
- Draw the Fourier transform of the reconstructed signal.
- Write the formula of the reconstructed signal.
- Draw a (qualitative) example of a possible $s(t)$ and the corresponding reconstructed signal in time. Is the reconstructed signal equal to $s(t)$?
- What happens (qualitatively) if the sampling frequency is pushed up to $f_0 = 16B$?

9.6.3

Assume that the signal $s(t)$ has limited bandwidth B and its Fourier Transform is approximately as in figure:



Assume that the signal is sampled at frequency $f_0 = 4B$. It is then reconstructed using the following reconstruction filter impulse response:

$$h_{\mathrm{R}}(t) = \frac{1}{T} \Lambda_T (t - T)$$

where $T = 1/f_0$

- Draw the Fourier transform of the sampled signal: $s_s(t) = s(t) \cdot x_\delta(t)$, where:

$$x_\delta(t) = T \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

- Draw the reconstruction filter transfer function.
- Draw the Fourier transform of the reconstructed signal.
- Is the reconstructed signal equal to $s(t)$?
- Draw a (qualitative) example of a possible $s(t)$ and the corresponding reconstructed signal in time. Is the reconstructed signal equal to $s(t)$?

What happens (qualitatively) if the sampling frequency is pushed up to $f_0 = 16B$?