

Chapter 2. Signals

2.1 *Continuous-time signals?*

A “continuous-time signal” $s(t)$ is a real or a complex function of the independent variable t , which is assumed to be “time”.

The domain of the independent variable t can be the whole of \mathbb{R} or it can be restricted to an interval over \mathbb{R} , i.e., it may be $t \in \mathbf{I} = [t_0, t_1]$. The range is typically all or a subset of \mathbb{R} or \mathbb{C} .

In other words, typically:

$$s(t) : \mathbb{R} \rightarrow \mathbb{R}$$

$$s(t) : \mathbb{R} \rightarrow \mathbb{C}$$

A signal $s(t)$ can be either discontinuous, with various kinds of discontinuities, continuous, or differentiable any number of times. In this course, we will use signals that may be any of the above.

2.2 *Sampled and quantized signals*

With the advent of computers and of “digital signal processing”, new types of signals have emerged. Specifically, the so-called “**sampled**” or “discrete-time signals” and the “**quantized**” or “discrete-value” signals.

Sampled signals will be dealt with extensively in the “discrete time” section of this course. In essence, a sampled signal is a sequence of numbers $s[n]$, with n an integer index, which is found by looking at a conventional signal $s(t)$ only at certain evenly-spaced points in time:

$$s[n] = s(nT)$$

where T is the “sampling time”.

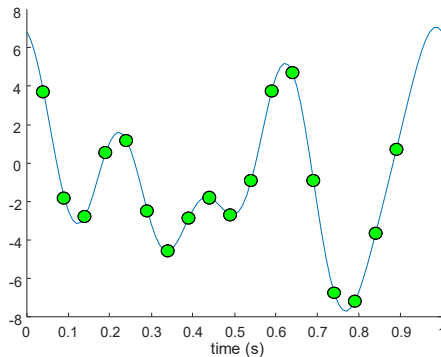


Fig. 2-1: example of a sampled signal.
The green dots are the “samples” extracted from
the signal represented as a solid blue line.

A quantized signal instead is a signal whose values on the vertical axis cannot be any real number but just a discrete set of values, typically equally spaced.

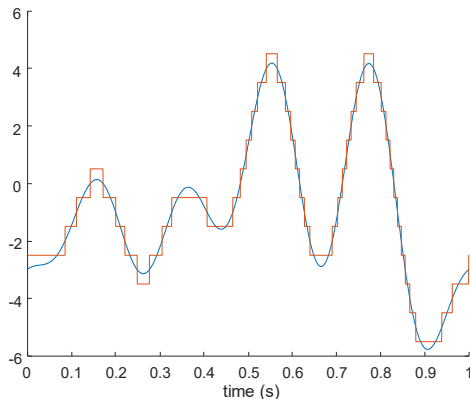


Fig. 2-2: example of a quantized signal.
The red solid line is the “quantized” version
of the signal represented as a solid blue line.

Mathematically, the quantized signal $s_q(t)$ derived from a conventional signal $s(t)$ could be written as:

$$s_q(t) = \text{round}\left(\frac{s(t)}{\Delta}\right) \cdot \Delta$$

where Δ is the step of the grid of possible values on the y -axis and “round” returns the nearest integer to the argument.

In many modern applications (such as video and audio) you can find signals that are both sampled and quantized. These aspects will be dealt with or commented on later in the course.

2.3 *What are signals used for?*

A signal $s(t)$ is typically used in engineering applications to represent the value over time of some physical quantity, such as a **current, voltage, electric field, sound pressure, temperature, a spatial coordinate, a force**, and so on. As a result, *signals are used in all engineering branches* and the theory of their properties and how to use or analyze them is fundamental to many, if not all, engineering sectors.

In many cases, a change in a physical quantity is caused on purpose. For instance, the value of a current can be varied to transmit information through a wire. In that case, the signal that describes such variation can be thought of as to be “carrying information”. As an example, all *transmission systems*, analog or digital, are based on this concept and are modeled using “signals”.

2.4 *Frequently-used signals*

In this section, several signals that will be used very frequently in this course are introduced. The independent variable t spans all of \mathbb{R} , unless otherwise pointed out. It is also implicitly assumed that t is indeed “time” and its unit is seconds, i.e. (s), unless otherwise indicated.

2.4.1 The “rectangular” signal “Heaviside Π ”

The “rectangular” signal is formally called “Heaviside Π ” and is defined as follows:

$$\Pi(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| > 1/2 \\ 1/2, & t = \pm 1/2 \end{cases} .$$

Note that this signal is discontinuous for two values of t , namely $t = \pm 1/2$, and is continuous everywhere else. The discontinuities are “jump”, or “step” discontinuities, also called “discontinuities of the first kind” (these denominations are all equivalent). We will use the term “*jump discontinuity*” as it seems to be the most common way of calling it.

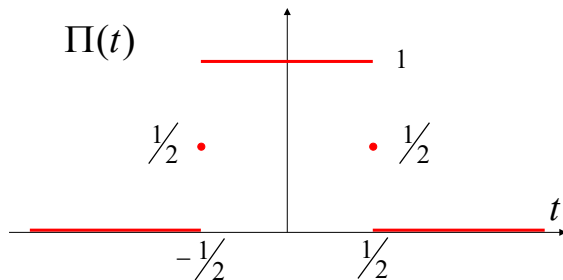


Fig. 2-3: The rectangular signal $\Pi(t)$

To obtain a rectangular signal whose “width” is a generic value T , then it is enough to divide the argument of Π by T :

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| < T/2 \\ 0, & |t| > T/2 \\ 1/2, & t = \pm T/2 \end{cases}$$

On your own: verify the above result. Make sure you understand why dividing the argument by T has the indicated effect.

However, since this signal will be used very often in this class, for convenience, the following intuitive shorthand is introduced:

$$\Pi_T(t) = \Pi\left(\frac{t}{T}\right)$$

In other words, when a subscript is present, such subscript identifies the “time-width” of the rectangle.

Also, we will adopt a formally *wrong* but graphically convenient way of plotting the rectangular function, different from Fig. 2-3:

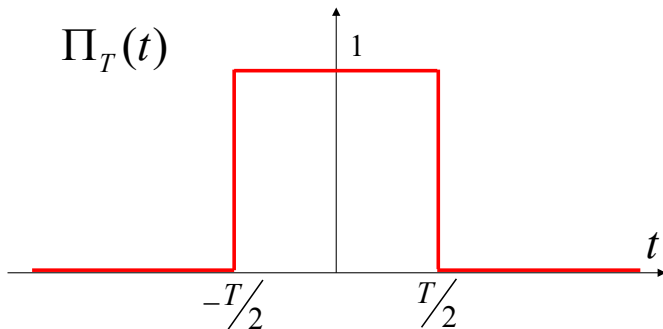


Fig. 2-4: Typical graphical representation of $\Pi(t)$

On your own: why is this representation “formally wrong” ?

Finally, we will also use for convenience a variant of the Heaviside Π , which is denoted by a lowercase π and is as follows:

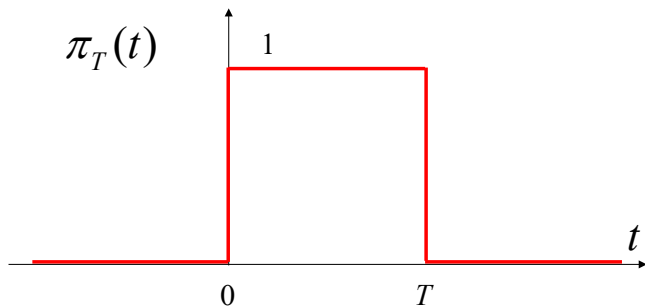


Fig. 2-5: The Heaviside $\pi(t)$ “starts” at time 0.

In formulas:

$$\pi_T(t) = \Pi_T(t - T/2)$$

2.4.1.1 The Heaviside $\Pi(t)$ in Matlab


Here below a possible implementation of the Heaviside $\Pi(t)$ in Matlab:

```
% This function implements a Heaviside Pi signal
%
% The input parameters are:
% t --> time, it can be an array
% T --> the duration of the signal (the width of the rectangle)
%
% The output parameter is:
% y --> it has the same size as t
%
function [y] = HPi(T,t)
    y=1*((t<T/2) & (t>-T/2))+1/2*((t==-T/2)+(t==T/2));
end
```

You can then plot the function, for instance:

```
figure;t=-2:0.01:2;plot(t,HPi(1,t),'r','linewidth',2); grid on; axis([-2,2,-0.2,1.4]);
```

On your own: modify the Matlab function `HPi(T,t)` so that it implements a Heaviside $\pi(t)$

 Tip: when you convert from $s(t)$ to $s(t/T)$ the plot of the signal remains the same, except the labels of the times on the horizontal axis get multiplied by T

2.4.2 The “triangular” signal “Heaviside Λ ”

The “triangular” signal is defined as follows:

$$\Lambda(t) = \begin{cases} 1+t & -1 < t \leq 0 \\ 1-t & 0 < t < 1 \\ 0 & |t| \geq 1 \end{cases} .$$

The shape of the signal is that of an isosceles triangle, of base length 2 and height 1. The vertex is located at $t = 0$. The triangular signal is continuous everywhere, but is not differentiable, because its first derivative is discontinuous at $t = -1, 0, 1$.

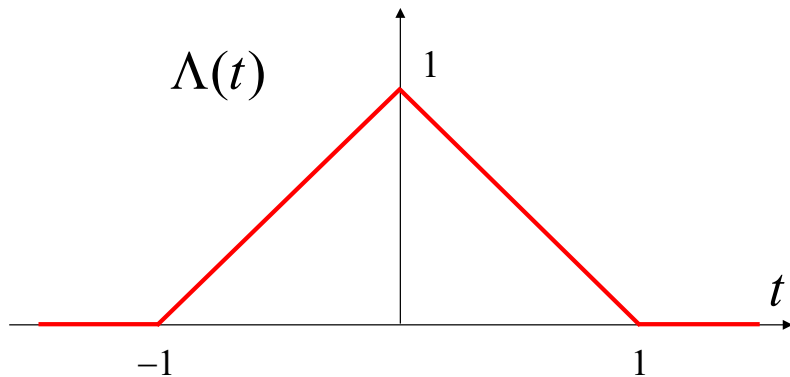


Figura 2.1: The triangular signal $\Lambda(t)$

Similar to the case of the rectangular function, in order to obtain a triangular signal with “base length” equal to $2T$, rather than just 2, it is enough to divide the argument by T :

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \frac{t}{T} + 1 & -T < t \leq 0 \\ 1 - \frac{t}{T} & 0 < t < T \\ 0 & |t| \geq T \end{cases}$$

Since this signal too will be used very often in this class, for convenience, the following shorthand is introduced:

$$\Lambda_T(t) = \Lambda\left(\frac{t}{T}\right)$$

In other words, when a subscript is present, **such subscript identifies the “half-time-width”** of the triangle.

Notice: it may seem strange that the subscript amounts to the full “base length” of the rectangle for the Heaviside Π and only half of the “base length” of the triangle for the Heaviside Λ .

However, notice that if you define the subscript as the signal time-width not at its “base” but **taken between the points where the signal value is $\frac{1}{2}$ of its maximum value**, then this definition is perfectly consistent for both the rectangular and triangular signals.

This type of “width” definition is used for various signals in different branches of Engineering and Physics and is called “**full-width half-maximum**”, or **FWHM** for short. We could therefore say that the subscript T of both $\Pi_T(t)$ and $\Lambda_T(t)$ is their “FWHM”.

2.4.2.1 The Heaviside $\Lambda(t)$ in Matlab

Here below a possible implementation of the Heaviside $\Lambda(t)$ in Matlab:

```
% This function implements a Heaviside Lambda signal
%
% The input parameters are:
% t --> time, it can be an array
% T --> the time-width at half height of the triangle
%
% The output parameter is:
% y --> it has the same size as t
%
```

```
function [y] = HLambda(T,t)
    y= (1-t/T).*(t>0 & t<T) +...
        (1+t/T).*(t>-T & t<0)+1*(t==0);
end
```

Example plot:

```
figure;plot(t,HLambda(1,t),'r','linewidth',2); grid on; axis([-2,2,-0.2,1.4])
```

2.4.3 The “unilateral step” signal

The unilateral step signal has value zero for $t < 0$ and value 1 for $t > 0$. It is jump-discontinuous at $t = 0$, where its value is formally defined to be $1/2$.

In the literature it is called many different names, such as “Heaviside” or “Heaviside Θ ”, and still others. In this class, for convenience, the simple notation $u(t)$ will be employed, and the denomination will be the longer, but easier to

understand, “unilateral step signal”, which is also in wide use. Its mathematical definition is:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

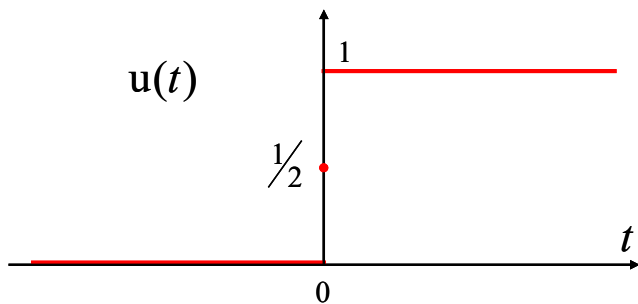


Fig. 2-6: Unilateral step signal, or “Heaviside Θ ” signal.

Similar to the case of the rectangular function, we will typically represent it graphically as follows:

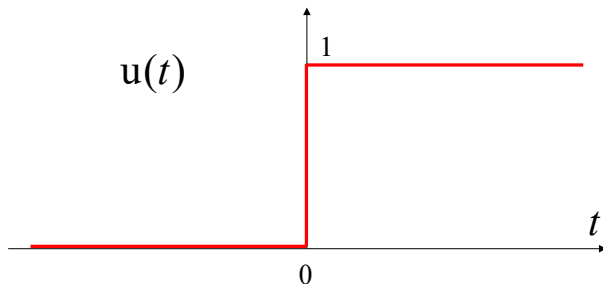


Fig. 2-7: Typical graphical representation of the unilateral step signal

2.4.3.1 The unilateral step function in Matlab

Here below a possible implementation of $u(t)$ in Matlab:

```
% This function implements a unilateral step function
%
% The input parameters are:
% t --> time, it can be an array
% T --> the duration of the signal (the width of the rectangle)
%
% The output parameter is:
% y --> it has the same size as t
%
function [y] = u(t)
    y=1*(t>0)+1/2*(t==0);
end
```

2.4.4 The “sign” or “signum” signal

This signal is either -1 or 1, depending on the sign of the argument being positive or negative. When the argument is zero, then the signal is also zero. Notice that there is a jump discontinuity at $t = 0$:

$$\text{sign}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

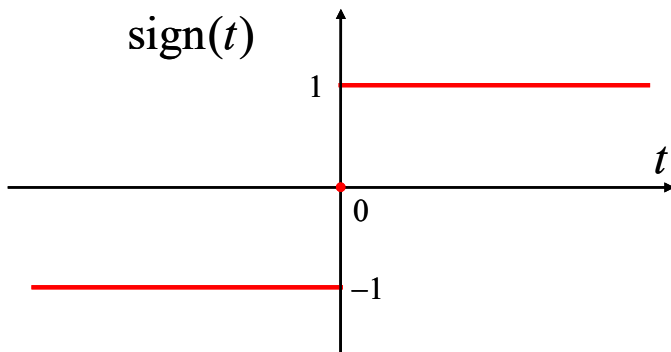


Fig. 2-8: The “sign” or “signum” signal.

Note that, even though the spelling is different, the English words “sign” and “sine” are pronounced the same. Whenever needed, the “sign” signal will therefore be called “signum” and pronounced accordingly, to distinguish it from the “sine” signal $\sin(t)$ (see below).

2.4.4.1 Matlab implementation of the sign function

Matlab has a built-in function “sign”, which accepts arrays as input.

2.4.5 The sine and cosine signals

The sine and cosine signals are among the most important signals that we use in this course. Their plots as a function of time are shown below but should be an already well-established part of each student’s basic mathematical knowledge. Typically, in this class, their argument is not simply t . Time is almost always multiplied times a constant, and in fact a very special one, called “**frequency**”. The concept of frequency is introduced in the next section.

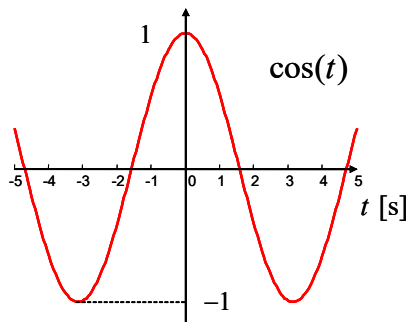
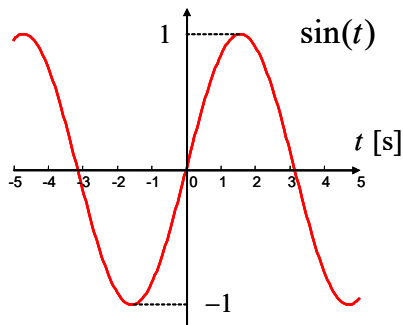


Fig. 2-9: The sine and cosine signals.

2.4.5.1 The concept of “frequency”

In Signal Analysis the sin and cos signals are typically expressed as: $\sin(2\pi f_0 t)$, $\cos(2\pi f_0 t)$, where the constant f_0 is called “frequency”. Note that

the subscript “0” is used only to remind the reader that, within $\sin(2\pi f_0 t)$ and $\cos(2\pi f_0 t)$, f_0 is a constant, whereas t is the independent variable.

The meaning of frequency can be explained in various equivalent ways, but it is fundamentally related to the fact that both \sin and \cos are “periodic” signals, that is, they “repeat” themselves at regular intervals. Specifically, they repeat themselves every time their argument adds or subtracts a full 2π :

$$\sin(x) = \sin(x + k \cdot 2\pi)$$

$$\cos(x) = \cos(x + k \cdot 2\pi)$$

where k can be any integer number, that is $\forall k \in \mathbb{Z}$.

Given this, it is easy to see that in $\sin(2\pi f_0 t)$, $\cos(2\pi f_0 t)$:

1. f_0 tells us how many repetitions of the \sin or \cos signals occur in one second (on your own prove it as an exercise); note that f_0 *needs not* be

an integer number because, for instance, we can have “2.75”, repetitions of the sin signal in one second, that is, $f_0 = 2.75 \notin \mathbb{Z}$;

2. $T_0 = 1/f_0$ is how long it takes (in time) to accumulate 2π in the argument, that is how long it takes before the sin or cos signals start repeating themselves (on your own prove it as an exercise). **Note that the quantity T_0 is called the “period”** of the sine or cosine function. Both functions can be written with their period made explicit (rather than their frequency) in their argument: $\sin\left(\frac{2\pi}{T_0}t\right), \cos\left(\frac{2\pi}{T_0}t\right)$.

Note that f_0 has dimensions of (s^{-1}) or (Hz), and T_0 of time (s). Also, the overall argument of the sin and cos signals should always be *dimensionless*, that is, it should be a pure number, without any physical dimensions. For instance, writing

$\sin(t)$ “assumes” that in fact we are looking at $\sin(2\pi f_0 t)$ with $f_0 = 1/2\pi$, so that the overall argument $(2\pi f_0 t)$ is a pure number.

Note finally that instead of f_0 you can find the symbol ω_0 in the argument of \sin and \cos functions, with the following equivalence: $\sin(2\pi f_0 t) = \sin(\omega_0 t)$. The relation between the two is therefore: $\omega_0 = 2\pi f_0$.

The use of ω_0 allows to avoid explicitly writing the 2π factor. However, even though it may be cumbersome to always write the 2π factor, in Signal Theory the use of ω_0 brings about notational problems which largely exceed the advantage of dropping the 2π . Therefore, in this class we will always use $2\pi f_0$.

An incredible array of physical phenomena shows a periodic or “oscillatory” behavior characterized by sin and cos functions. In fact, the sin and cos functions (or signals) are truly “fundamental” to many branches of science and engineering and frequency is an extremely important related concept.

2.4.5.2 Matlab implementation of sin and cos

Matlab has of course built-in sin and cos functions, which accept arrays as input, in which case they return arrays as outputs.

2.4.6 The complex exponential signal

As significant as sin and cos, the “complex exponential” is a very important function in this and many other courses. As is well-known, it can be defined in terms of a sin and a cos function, according to **Euler’s formula**:

$$e^{jx} = \exp(jx) = \cos(x) + j \sin(x), \quad x \in \mathbb{R}$$

Notice that throughout this class, the complex unity is always going to be written “ j ” and not “ i ”.

It is well-known that the complex number e^{jx} spans a full circle of radius one in the complex plane, for x spanning the interval $[0, 2\pi]$. In addition, e^{jx} is periodic in x , that is, it repeats itself every 2π . This property is obviously inherited from the cos and sin functions that make it up.

As a result, we can introduce a complex “rotating” signal which incorporates the concept of “frequency”, as for the cos and sin signals, as follows:

$$e^{j2\pi f_0 t} = \exp(j2\pi f_0 t) = \cos(2\pi f_0 t) + j \sin(2\pi f_0 t)$$

In the context of the complex exponential signal, the meaning of “frequency” can be viewed in yet another way:

- f_0 tells us how many “turns” the complex point $z = e^{j2\pi f_0 t}$ executes in one second over a circle of radius 1 on the complex plane (on your own prove it as an exercise); note that f_0 needs not be an integer number because, for instance, we can have “9.352” turns in one second.
- $T_0 = 1 / f_0$ is how long it takes (in time) to accumulate 2π in the argument, that is how long it takes to run one full circle on the complex plane (on your own prove it as an exercise).

The quantity T_0 is called the “period” of the exponential function, which could be equally well written with its period (rather than its frequency) made explicit in the argument:

$$z = e^{j\frac{2\pi}{T_0}t}.$$

The complex exponential is not less important than its components sin and cos. In fact, it could be considered as even more important in this course, as it allows to handle many calculations more effectively than using sin and cos separately.

2.4.6.1 Matlab implementation of the complex exp.

You can write a complex exponential in Matlab using the built-in function ‘exp’:

$$\exp(2\pi j f_0 t)$$

Note that the constants π and j are pre-defined (you do not have to initialize them). Note also that lately Matlab has started recommending the use of '1i' instead of 'j' as the imaginary unit. However, both forms are currently OK. If you use 'j', then make sure you are not using it for other purposes, such as the index of a 'for' loop, since this could obviously create problems.

You can produce the plot of Fig. 2-10 by entering the following Matlab statements:

```
t=0:0.01:10;z=exp(j*2*pi*t);figure;  
plot3(t,real(z),imag(z),'LineWidth',2);grid on;  
axis equal; axis([0,10,-1.2,1.2,-1.2,1.2,]);  
xlabel('time (s)'); ylabel('real part'); zlabel('imaginary part'); hold  
on;  
plot3(t,zeros(size(real(z))),zeros(size(imag(z))), 'k', 'LineWidth',1);  
hold off;
```

on your own: use Matlab's "grab and turn" capability to rotate the plot till you make just a sin function or a cosine function appear on screen.

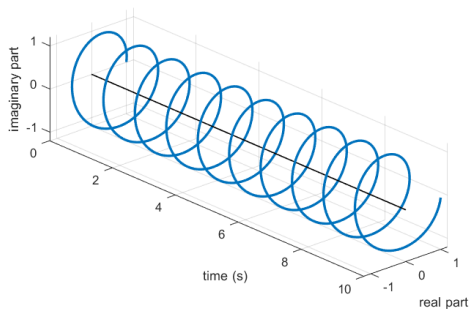


Fig. 2-10: 3D plot of a complex exponential of frequency $f_0 = 1$.

The black line is the time axis.

2.4.7 The “raised-cosine” signal

The raised-cosine signal is a rectangular signal with *smoothed-out* edges, so that *the discontinuities are eliminated*. It can therefore be thought of as a more “realistic” pulse than $\Pi_T(t)$, because the discontinuities have been removed.

The analytical formula is:

$$\rho_{T,\alpha}(t) = \begin{cases} 1, & |t| \leq \frac{1-\alpha}{2}T \\ \frac{1}{2} \left[1 + \cos \left(\frac{\pi}{\alpha T} \left[|t| - \frac{1-\alpha}{2}T \right] \right) \right] & \frac{1-\alpha}{2}T < |t| < \frac{1+\alpha}{2}T \\ 0 & \text{elsewhere} \end{cases}$$

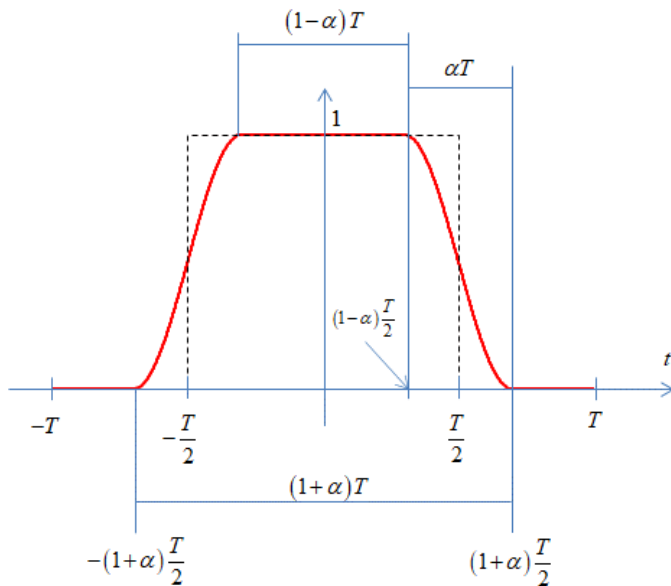


Fig. 2-11: plot of the raised-cosine signal. Here $\alpha = 0.4$

It appears somewhat complicated but the reason is that it pieces together a straight horizontal segment of time-length $(1 - \alpha)T$ placed at height 1, with two half-periods of a cosine, which make up the sides. These two cosine parts are “raised up” to connect to the flat-top segment on one end and to the horizontal axis on the other. From this feature, the name of the signal “raised-cosine”.

Note that the two key parameters are the time-width at half-height (or FWHM) T , similar to the Heaviside Pi signal and the Lambda signal, and the very important “roll-off factor” α . From the figure you can appreciate how α really shapes the signal.

The raised-cosine signal is very popular in telecommunications, where it is one of the most-used elementary signals that are employed to encode information. Below, a possible Matlab implementation:

```

function rho_t=rho(t,alpha,T0,center,root)

% t is the array of times at which the signal is evaluated
% alpha is the roll-off factor
% T0 is the width at 1/2 height
% center is the signal center time
% root at 1 indicates that a root-raised-cosine signal is returned
% root at 0 indicates that a raised-cosine signal is returned

f0=T0^-1;
passband=(1-alpha)/2/f0;
stopband=(1+alpha)/2/f0;

% initializing
rho_t=zeros(1,length(t),'double')+1i*zeros(1,length(t),'double');

for nch=1:length(center)

    tt=abs(t-center(nch));
    t=abs(t-center(nch))-passband;

    if alpha==0
        rho_t=(t<=0)+rho_t;
    else
        if root==1
            rho_t=...

```



```

        (t<=0)+...
        sqrt(1/2*(1+cos(pi*f0/alpha*(t)))).*(t>0).*(abs(tt)<=stopband))+rho_t;
    else
        rho_t=...
        (t<=0)+...
        (1/2*(1+cos(pi*f0/alpha*(t)))).*(t>0).*(abs(tt)<=stopband))+rho_t;
    end;
end;

end;

return

```

on your own: try to re-derive the analytical formula of the raised-cosine signal by just looking at its plot.

on your own: draw the signal $\rho_{T,\alpha}(t)$ with α equal to 1, 0 and $\frac{1}{2}$. What happens at the value 0 ? And at the value 1 ?

Verify your drawing by plotting the signal using Matlab code.



Tip: given a signal $s(t)$, the plot of the signal $s(|t|)$ is simply found by taking the plot of the signal $s(t)$ for positive times and flipping it about the y -axis, onto the negative times.

2.4.8 The “Sinc” signal

The “Sinc” signal is defined as follows¹:

$$\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$

Eq. 2-1

¹ *Warning*: an alternative definition of the Sinc function is also in wide use:

$$\text{Sinc}(t) = \frac{\sin(t)}{t}.$$

We will not use this definition. We will use instead Eq. 2-1.

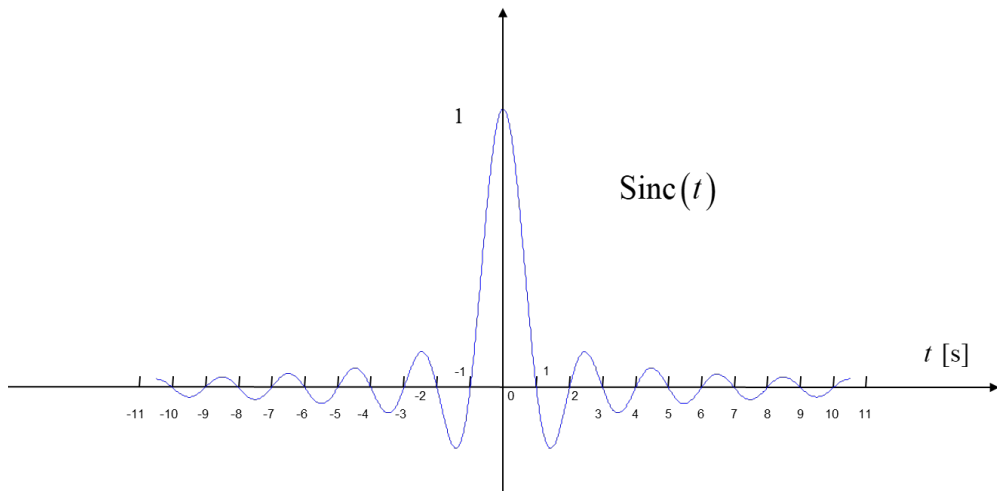


Fig. 2-12: Plot of the $\text{Sinc}(t)$ signal

Both the numerator and the denominator of the “Sinc” are continuous functions. However, for $t = 0$ an undetermined form develops of the $0/0$ type. This problem can be solved by assigning to $\text{Sinc}(0)$ the limit for $t \rightarrow 0$ of the Sinc itself. It is easy to see (prove it on your own; hint, use for instance de l'Hôpital rule, or expand $\sin(\pi t)$ in a Taylor series) that:

$$\lim_{t \rightarrow 0} \text{Sinc}(t) = \lim_{t \rightarrow 0} \frac{\sin(\pi t)}{\pi t} = 1$$

therefore the complete definition of the Sinc function is:

$$\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

With this definition, the Sinc function becomes both continuous and differentiable. In particular, it can be differentiated any number of times.

Note that the denominator is never zero (except for $t = 0$, which we have already discussed). Instead, the numerator goes to zero every time $\sin(\pi t) = 0$, that is for:

$$\pi t = k\pi \rightarrow t = k, \quad k \in \mathbb{Z}$$

In other words, $\text{Sinc}(t) = 0$ for all integer values of t (except $t = 0$).

In many Signal Analysis applications, the Sinc function argument is normalized as follows:

$$\text{Sinc}\left(\frac{t}{T}\right) = \frac{\sin\left(\pi \frac{t}{T}\right)}{\pi \frac{t}{T}}.$$

The resulting Sinc has nulls at all integer multiples of T , excluding zero. In other words:

$$\text{Sinc}\left(\frac{t}{T}\right) = 0, \quad t = \pm T, \pm 2T, \pm 3T, \dots$$

as shown in Fig. 2-13.

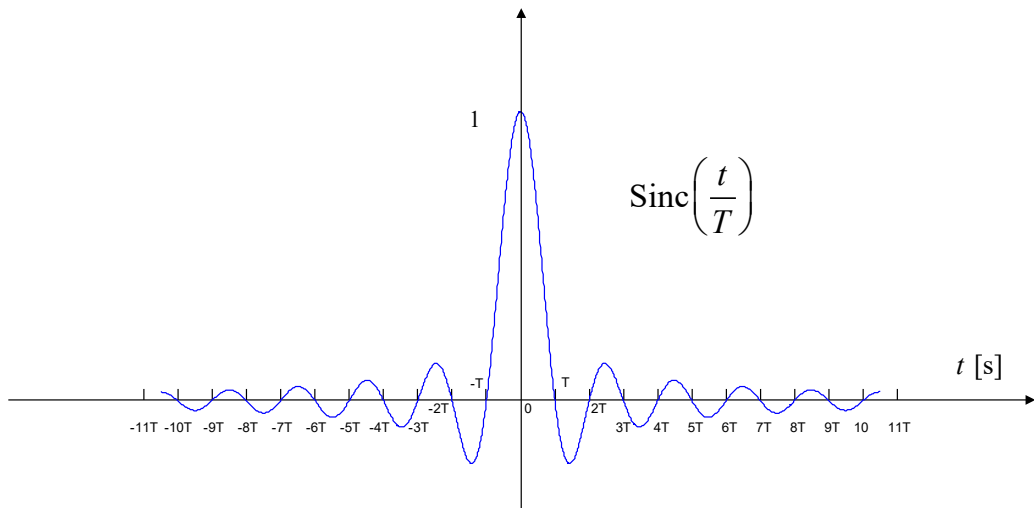


Fig. 2-13: Plot of the $\text{Sinc}\left(\frac{t}{T}\right)$ signal

An important property of the Sinc signal is the following (prove it on your own as an exercise):

$$\left| \text{Sinc}\left(\frac{t}{T}\right) \right| \leq \frac{T}{\pi} \cdot \left| \frac{1}{t} \right|$$

This means that the Sinc signal goes to zero as $t \rightarrow \pm\infty$, but it does so only as fast as $|T/\pi t|$.

This can also be seen in Fig. 2-14 below:

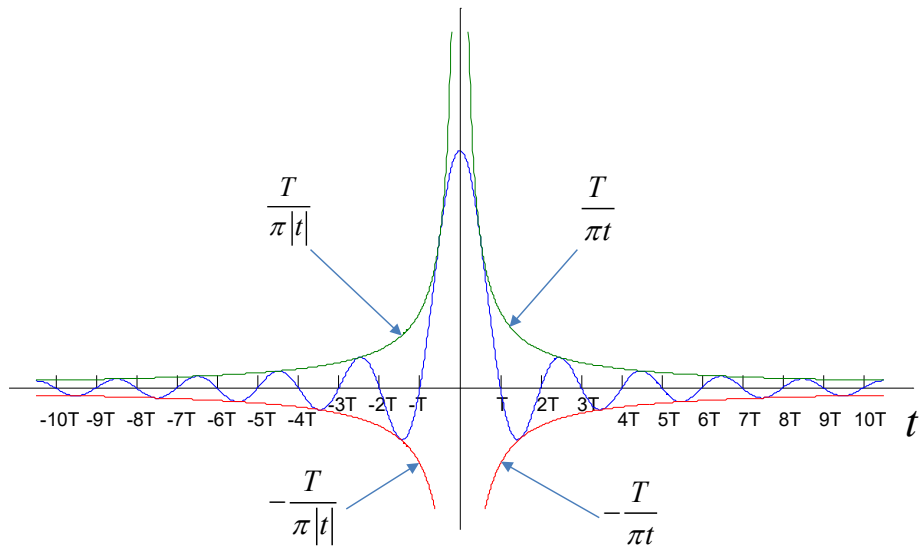


Fig. 2-14: Plot of the signal $\text{Sinc}\left(\frac{t}{T}\right)$ and of the functions $\pm|T/\pi t|$

Finally, the integral of the Sinc over the whole of \mathbb{R} is 1:

$$\int_{-\infty}^{\infty} \text{Sinc}(t) dt = 1$$

We will prove this result later on.

2.4.8.1 Matlab implementation of $\text{Sinc}(t)$

Matlab has a built-in implementation of $\text{Sinc}(t)$. Note that in Matlab the initial is lowercase: `sinc(t)`. The function accepts an array input.

2.4.9 The single-sided decreasing exponential signal

Another important signal is the so called single-sided or “unilateral” decreasing exponential signal. Its analytical expression is:

$$u(t) \cdot e^{-a \cdot t} \quad a \in \mathbb{R}, a > 0$$

Note that per se the exponential functions of a real negative argument would be non-zero over all \mathbb{R} , but the presence of the unilateral step signal makes this signal non-zero only for $t \geq 0$.

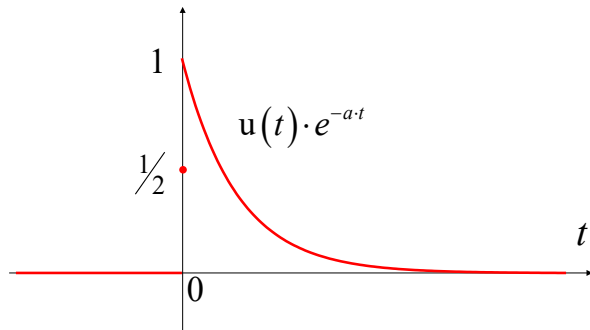


Fig. 2-15: the single-sided decreasing exponential

Note that this signal is jump-discontinuous at $t = 0$.

On your own Why is the value at $t = 0$ equal to $\frac{1}{2}$? Find out.

On your own Write a Matlab implementation of the single-sided decreasing exponential signal and plot it.

2.4.10 The Gaussian signal

Another important signal is the Gaussian signal. One possible definition is:

$$G(t) = e^{-\frac{t^2}{2T^2}} \quad T \in \mathbb{R}, T > 0$$

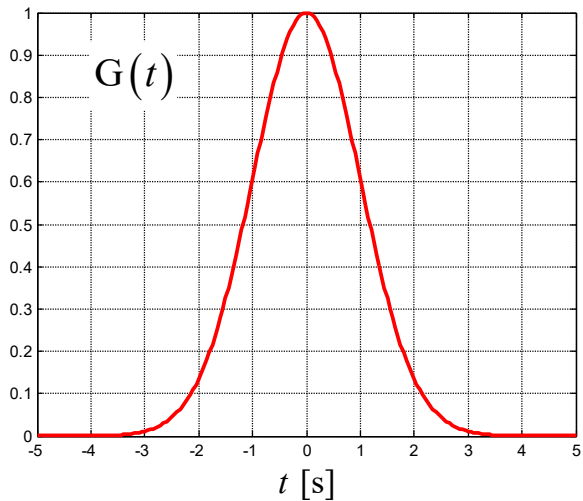


Fig. 2-16: a Gaussian signal $G(t)$ with $T = 1$

The Gaussian signal is mathematically non-zero over the whole of \mathbb{R} . However, it decreases very fast for increasing $|t|$.

$$|t| = 0 \Rightarrow G(t) = 1$$

$$|t| = T \Rightarrow G(t) \simeq 0.61$$

$$|t| = 2T \Rightarrow G(t) \simeq 0.135$$

$$|t| = 3T \Rightarrow G(t) \simeq 0.01111$$

$$|t| = 5T \Rightarrow G(t) \simeq 0.00000373$$

$$|t| = 10T \Rightarrow G(t) \simeq 0.0000000000000000000002$$

The Gaussian function is very important in Statistics as well. In that context it typically has a normalization constant in front of it. (On your own Can you remember what the normalization factor is, and why it is used?)

On your own (optional) write the Gaussian function so that its width-scaling parameter is its FWHM.

Solution: $G(t) = e^{-4 \cdot \log_e(2) \cdot \left(\frac{t}{T_{\text{FWHM}}}\right)^2}$

In fact, substituting $t = \pm T_{\text{FWHM}} / 2$, one gets: $e^{-4 \cdot \log_e(2) \cdot \left(\frac{\pm T_{\text{FWHM}}}{2 T_{\text{FWHM}}}\right)^2} = e^{-\log_e(2)} = 1/2$

Note also that the relationship between T and T_{FWHM} is:

$$T_{\text{FWHM}} = T \cdot 2 \sqrt{2 \log_e(2)} \approx T \cdot 2.355$$

2.4.11 The constant unit signal and the constant signal

A constant unit signal is a signal whose value is 1 for all times. We will write it as $1(t)$. From this elementary signal, arbitrarily valued constant signals can be derived by simply multiplying $1(t)$ times a number α :

$$\text{const}(\alpha, t) = \alpha \cdot 1(t)$$

2.4.12 The constant “zero” signal

A constant “zero” signal is a signal whose value is zero at all times. We will write it as $0(t)$. Note that even though it is often written in textbooks as just “0”, the signal $0(t)$ is quite a different object than the number “0”.

2.4.13 On your own

On your own, look at how all of the signals introduced so far behave for $t \rightarrow \pm\infty$. What classes of signals can you identify, according to their behavior at $t \rightarrow \pm\infty$?

Also, try and classify them in terms of their continuity/discontinuity and differentiability features.

All of these signals can be re-scaled in time and translated left and right. How do you do it? To practice, consider the following:

$$s(t), s(\alpha \cdot t), s(t / \alpha), s(t - t_0), s(t + t_0), s(\alpha[t - t_0]), s([t - t_0] / \alpha)$$

Take as $s(t)$ any of the signals previously introduced, such as $\Pi(t)$ or $\Lambda(t)$, or any other, and apply the above transformations. Draw the results.

If you have trouble doing it, refresh the material and do more exercises: these basic transformations are essential and are taken for granted as pre-established background.

Using *Matlab*, plot some of the elementary signals that we have seen. Try also to draw them using the translations and scaling proposed in the previous paragraphs.

Also, the concept of “adding signals” or “multiplying signals” must absolutely be clear in the mind of the students. If you have trouble understanding what it means to do:

$$s(t) = v(t) + w(t)$$

$$s(t) = v(t) \cdot w(t)$$

then you need to immediately revise your calculus background. Try to do the above using as $v(t)$ and $w(t)$ any of the previously introduced signals. Make sure you can easily draw the result on your own. If you are unsure, revise the theory. Then you can check the result using for instance the *plot* function of *Matlab*.

2.5 *Dirac's "delta"*

Dirac's "delta" writing " $\delta(t)$ " may lead to mistaking it for a conventional "signal", that is, a conventional function of time. However, Dirac's delta is not a "proper" function, because its value for $t = 0$ is undefined, even though informally

such value is said to be “infinity”. Note that, at all other times except 0, $\delta(t)$ is a conventional signal and in particular:

$$\delta(t) = 0(t) \quad \forall t \neq 0$$

As a result, $\delta(t)$ is typically represented graphically as follows:

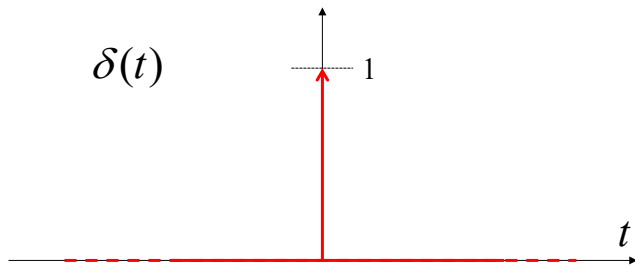


Fig. 2-17: formal graphical representation of $\delta(t)$

The arrow at the origin, of formal height equal to 1, is meant to clearly show that something peculiar happens at $t = 0$. Indeed, Dirac's delta is not a function, but a special object that acquires meaning only within an integral operator.

The actual “definition” of Dirac's delta can be written in terms of its essential integral property:

“ $\delta(t)$ ” is a mathematical object such that:

$$\int_{-\infty}^{+\infty} \delta(t) \cdot s(t) dt = s(0)$$

Eq. 2-2

Note that this formula requires² that $s(t)$ exists and is *continuous* at $t = 0$.

Note also that as a direct consequence of the above definition, by choosing $s(t) = 1(t)$ we have the special case:

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

Eq. 2-3

² If $s(t)$ was otherwise, Eq. 2-3 might not yield the same result or may not make sense at all. We will discuss a few special cases later on.

As a further detail, note that it is not strictly necessary that the integration range be the whole real axis. In fact it is enough that the *origin* is included:

$$\int_a^b \delta(t) \cdot s(t) dt = s(0) \quad a < 0 < b$$

Eq. 2-4

These integral definitions, however, do not provide much insight regarding the concept of delta. Rather, it is useful to resort to one of the many possible ways of arriving at the delta as a sort of “limit” (in a special way) of a sequence of ordinary signals. To show that, we need the following *result*.

Result:

if $s(t)$ is a continuous³ function then the following equality holds:

$$\lim_{T \rightarrow 0} \int_{-\infty}^{+\infty} \frac{\Pi_T(t)}{T} s(t) dt = s(0)$$

Eq. 2-5

We will take this results from granted, but for the interested students the following is a rigorous and simple proof that Eq. 2-5 holds.

START of OPTIONAL MATERIAL:

³ The result holds for less stringent requirements on $s(t)$ but for simplicity we assume that $s(t)$ is continuous everywhere.

Proof of the result:

We start out by writing down a well-known relationship, which is called the “mean value theorem for definite integrals”:

$$\int_{-T/2}^{+T/2} s(t) dt = T \cdot s(t_0) \quad , \quad t_0 \in \left[-\frac{T}{2}, \frac{T}{2} \right]$$

Eq. 2-6

In words, we can say that the integral above has value equal to the length of the integration interval T , times the value which the integrand function $s(t)$ takes on *at some point in time* t_0 within the integration interval $[-T/2, T/2]$

We then argue that⁴:

$$s(t_0) \leq \max_{t \in [-T/2, T/2]} \{s(t)\}$$

$$s(t_0) \geq \min_{t \in [-T/2, T/2]} \{s(t)\}$$

Eq. 2-7

Combining Eq. 2-6 and Eq. 2-7 it is possible to write:

⁴ Eq. 2-6 is intuitive but to prove it formally the “extreme value theorem” is required. It essentially states that a continuous function over an interval has both a maximum and a minimum and that it actually reaches both over the interval. Proof can be found for instance on Wikipedia.

$$T \cdot \min_{t \in [-T/2, T/2]} \{s(t)\} \leq \int_{-T/2}^{+T/2} s(t) dt \leq T \cdot \max_{t \in [-T/2, T/2]} \{s(t)\}$$

Eq. 2-8

We then remark that we can rewrite the center section of Eq. 2-8 as:

$$\int_{-T/2}^{+T/2} s(t) dt = \int_{-\infty}^{\infty} \Pi_T(t) \cdot s(t) dt$$

Eq. 2-9

(Why? It is easy to show it, do it on your own).

Substituting Eq. 2-9 into Eq. 2-8, we have:

$$T \cdot \min_{t \in [-T/2, T/2]} \{s(t)\} \leq \int_{-\infty}^{\infty} \Pi_T(t) \cdot s(t) dt \leq T \cdot \max_{t \in [-T/2, T/2]} \{s(t)\}$$

Eq. 2-10

We then divide all three members of the formula by T , which we can always do because T it is a positive non-zero constant:

$$\min_{t \in [-T/2, T/2]} \{s(t)\} \leq \int_{-\infty}^{\infty} \frac{\Pi_T(t)}{T} s(t) dt \leq \max_{t \in [-T/2, T/2]} \{s(t)\}$$

We now take the limit for $T \rightarrow 0$ of all sections of the above inequality:

$$\lim_{T \rightarrow 0} \left\{ \min_{t \in [-T/2, T/2]} \{s(t)\} \right\} \leq \lim_{T \rightarrow 0} \int_{-\infty}^{\infty} \frac{\Pi_T(t)}{T} s(t) dt \leq \lim_{T \rightarrow 0} \left\{ \max_{t \in [-T/2, T/2]} \{s(t)\} \right\}$$

Eq. 2-11

If we look at the leftmost and rightmost sides of the inequality, thanks to the assumption that $s(t)$ is continuous, it is clear that:

$$\lim_{T \rightarrow 0} \left\{ \min_{t \in [-T/2, T/2]} \{s(t)\} \right\} = s(0)$$

$$\lim_{T \rightarrow 0} \left\{ \max_{t \in [-T/2, T/2]} \{s(t)\} \right\} = s(0)$$

The middle member of the inequality Eq. 2-11 is then *guaranteed* to converge to the same value too⁵:

⁵ This last result is due to the so-called “squeeze theorem” or “pinch theorem”. Since it is a rather intuitive theorem, we will omit to prove it (you can look it up, though, on Wikipedia, if interested).

$$\lim_{T \rightarrow 0} \int_{-\infty}^{+\infty} \frac{\Pi_T(t)}{T} s(t) dt = s(0)$$

Eq. 2-12

Then Eq. 2-12 coincides with the result Eq. 2-5 which is therefore proved.

END of OPTIONAL MATERIAL

We now have the right “tool”, Eq. 2-5, for studying delta. We remark that Eq. 2-2 and Eq. 2-5 have the same right-hand side, $s(0)$. We can therefore claim that their left-hand sides must also coincide, from which:

$$\lim_{T \rightarrow 0} \int_{-\infty}^{+\infty} \frac{\Pi_T(t)}{T} s(t) dt = \int_{-\infty}^{+\infty} \delta(t) \cdot s(t) dt$$

Eq. 2-13

This equality is very important and shows that $\delta(t)$ is perhaps not such a mysterious object. Rather, it appears to be related to simple rectangular functions of the form:

$$\frac{\Pi_T(t)}{T}$$

Eq. 2-14

when $T \rightarrow 0$. In fact, looking at Eq. 2-13, one could be tempted to write:

$$\lim_{T \rightarrow 0} \frac{\Pi_T(t)}{T} = \delta(t)$$

Unfortunately, the above limit is meaningless (not properly defined) in the sense of conventional limits.

However, the identity:

$$\lim_{T \rightarrow 0} \int_{-\infty}^{+\infty} \frac{\Pi_T(t)}{T} \cdot s(t) dt = s(0) = \int_{-\infty}^{+\infty} \delta(t) \cdot s(t) dt$$

is instead a legitimate expression. In fact, based on it, a new form of “limit” can be defined, which has meaning in the framework of the so-called *theory of generalized functions or distributions*.

Dirac’s delta is not a function but, within this theory, a “generalized function” or “distribution”. In the framework of such “generalized functions”, one can write:

$$\lim_{\substack{\text{dist} \\ T \rightarrow 0}} \frac{\Pi_T(t)}{T} = \delta(t)$$

Eq. 2-15

where this new limit operator \lim_{dist} is said to be “in the sense of *distributions*”.

Apart from the mathematical details, what this limit operator really means, in lay terms, is that:

- “the $\frac{\Pi_T(t)}{T}$ function tends to acquire the same *integral properties* as Dirac’s delta, in the limit of $T \rightarrow 0$ ”.

Note that there are many signals that have this same property of “converging” to $\delta(t)$, besides $\Pi_T(t)/T$. For instance, the Gaussian signal or the triangular signal do too. In fact, it can be shown that:

$$\lim_{\substack{\text{dist} \\ T \rightarrow 0}} \exp(-t^2 / 2T^2) / \sqrt{2\pi T^2} = \delta(t) .$$

$$\lim_{\substack{\text{dist} \\ T \rightarrow 0}} \frac{\Lambda_T(t)}{T} = \delta(t)$$

Less obvious, but very important in practice, also the Sinc signal can be made to “generate” $\delta(t)$:

$$\lim_{\substack{\text{dist} \\ T \rightarrow 0}} \frac{1}{T} \text{Sinc}\left(\frac{t}{T}\right) = \lim_{\substack{\text{dist} \\ T \rightarrow 0}} \frac{\sin\left(\frac{\pi t}{T}\right)}{\pi t} = \delta(t)$$

Eq. 2-16

On your own: using a mathematical software like *Matlab*, plot all the functions whose “limit” is Dirac’s delta. What happens to their value at the origin when T

shrinks? What happens to the temporal “width” of these functions? Also use *Matlab* to calculate their integral, numerically. What do you observe?

2.5.1.1 computational rules

There are some useful immediate extensions of the basic integral property of Dirac’s delta. In particular:

$$\int_{-\infty}^{+\infty} \delta(\alpha[t-t_0]) \cdot s(t) dt = \frac{1}{|\alpha|} s(t_0)$$

Eq. 2-17

More in general, to be able to solve an integral involving a delta with an argument of the type as shown in Eq. 2-17, it is enough to:

- 1) find out what the integration variable is (it is the one appearing in the differential, in this case t);
- 2) find the value of the integration variable for which the argument of the delta is zero; in the example above $\alpha[t - t_0] = 0$ for $t = t_0$;
- 3) the result of the integral is the integrand function $s(t)$ evaluated at $t = t_0$, divided by $|\alpha|$

Eq. 2-17 can be easily proved by first changing integration variable into $\tau = \alpha[t - t_0]$ and then applying Eq. 2-2.

On your own: perform the calculation and prove the result Eq. 2-17. Then look below at the solution...

Solution

We look at two different cases. If $\alpha > 0$ we have:

$$t = \frac{\tau}{\alpha} + t_0 \quad \rightarrow \quad dt = \frac{d\tau}{\alpha}$$

Also, the lower integration limit formally becomes:

$$t_{\text{low}} = -\infty \quad \rightarrow \quad \tau_{\text{low}} = \alpha[-\infty - t_0] = -\infty - \alpha t_0 = -\infty$$

$$t_{\text{upp}} = +\infty \quad \rightarrow \quad \tau_{\text{low}} = \alpha[\infty - t_0] = \infty - \alpha t_0 = \infty$$

and therefore:

$$\int_{-\infty}^{+\infty} \delta(\alpha[t - t_0]) \cdot s(t) dt = \int_{-\infty}^{+\infty} \delta(\tau) \cdot s\left(\frac{\tau}{\alpha} + t_0\right) \frac{d\tau}{\alpha} = \frac{1}{\alpha} s(t_0)$$

If we now assume $\alpha > 0$, which we can write $-|\alpha|$, then:

$$t = -\frac{\tau}{|\alpha|} + t_0 \quad \rightarrow \quad dt = -\frac{d\tau}{|\alpha|}$$

Also, the lower integration limit formally becomes:

$$t_{\text{low}} = -\infty \quad \rightarrow \quad \tau_{\text{low}} = -|\alpha|[-\infty - t_0] = +\infty + |\alpha|t_0 = +\infty$$

$$t_{\text{upp}} = +\infty \quad \rightarrow \quad \tau_{\text{low}} = -|\alpha|[\infty - t_0] = -\infty - |\alpha|t_0 = -\infty$$

and therefore:

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(\alpha[t - t_0]) \cdot s(t) dt &= \int_{-\infty}^{+\infty} \delta(-|\alpha|[t - t_0]) \cdot s(t) dt = \\ &= -\int_{+\infty}^{-\infty} \delta(\tau) \cdot s\left(-\frac{\tau}{|\alpha|} + t_0\right) \frac{d\tau}{|\alpha|} = \int_{-\infty}^{+\infty} \delta(\tau) \cdot s\left(-\frac{\tau}{|\alpha|} + t_0\right) \frac{d\tau}{|\alpha|} = \frac{1}{|\alpha|} s(t_0) \end{aligned}$$

We now remark that also for $\alpha > 0$ we could write:

$$\frac{1}{\alpha} s(t_0) = \frac{1}{|\alpha|} s(t_0)$$

So it is possible to unify the results for $\alpha > 0$ and $\alpha < 0$ into one:

$$\int_{-\infty}^{+\infty} \delta(\alpha[t-t_0]) \cdot s(t) dt = \frac{1}{|\alpha|} s(t_0)$$

which is indeed Eq. 2-17.

2.5.1.2 delta as the “time-sampling” function

Notice the “time-sampling” property of the delta function:

$$\int_{-\infty}^{+\infty} \delta(t-t_0) \cdot s(t) dt = s(t_0)$$

In other words, by integrating a signal together with a delta, centered at a specific time instant t_0 , one can “extract” from the signal the value that the signal takes on at t_0 . That is, the signal $s(t)$ is “sampled” at the instant t_0 .

From the property above, **we can also derive the very important property:**

$$s(t) \cdot \delta(t - t_0) = s(t_0) \cdot \delta(t - t_0)$$

Eq. 2-18

Why? Can you prove this? Do it on your own.

(Remember that delta is defined in terms of its integral behavior so the above identity should be checked under integration... However, it can also be justified graphically, at an intuitive level.)

Dirac's delta has the following useful property:

$$\delta(t) = \delta(-t)$$

Prove it On your own: based for instance on Eq. 2-15.

START of OPTIONAL

2.5.2 Some integral transformation rules

Here are some basic integration variable transformations.

$$\int_{t_0}^{t_1} s(\alpha \cdot t) dt = \begin{cases} \frac{1}{|\alpha|} \int_{\alpha \cdot t_0}^{\alpha \cdot t_1} s(t) dt & \alpha > 0 \\ \frac{1}{|\alpha|} \int_{\alpha \cdot t_1}^{\alpha \cdot t_0} s(t) dt & \alpha < 0 \end{cases}$$

$$\int_{t_0}^{t_1} s(t/\alpha) dt = \begin{cases} |\alpha| \int_{t_0/\alpha}^{t_1/\alpha} s(t) dt & \alpha > 0 \\ |\alpha| \int_{t_1/\alpha}^{t_0/\alpha} s(t) dt & \alpha < 0 \end{cases}$$

$$\int_{t_0}^{t_1} s(t - t_d) dt = \int_{t_0 - t_d}^{t_1 - t_d} s(t) dt$$

$$\int_{t_0}^{t_1} s(t + t_d) dt = \int_{t_0 + t_d}^{t_1 + t_d} s(t) dt$$

$$\int_{t_0}^{t_1} s(\alpha \cdot [t - t_d]) dt = \begin{cases} \frac{1}{|\alpha|} \int_{\alpha \cdot [t_0 - t_d]}^{\alpha \cdot [t_1 - t_d]} s(t) dt & \alpha > 0 \\ \frac{1}{|\alpha|} \int_{\alpha \cdot [t_1 - t_d]}^{\alpha \cdot [t_0 - t_d]} s(t) dt & \alpha < 0 \end{cases}$$

$$\int_{t_0}^{t_1} s(\alpha \cdot [t - t_d]) g(t) dt = \begin{cases} \frac{1}{|\alpha|} \int_{\alpha \cdot [t_0 - t_d]}^{\alpha \cdot [t_1 - t_d]} s(t) g\left(\frac{\tau}{\alpha} + t_d\right) dt & \alpha > 0 \\ \frac{1}{|\alpha|} \int_{\alpha \cdot [t_1 - t_d]}^{\alpha \cdot [t_0 - t_d]} s(t) g\left(\frac{\tau}{\alpha} + t_d\right) dt & \alpha < 0 \end{cases}$$

They all derive from the *substitution rule of integration* which can be written in a rather general form as:

$$\int_{t_0}^{t_1} s(\varphi(t)) dt = \int_{\varphi(t_0)}^{\varphi(t_1)} s(\tau) \frac{1}{\varphi'(\varphi^{-1}(\tau))} d\tau$$

$$\text{where } \tau = \varphi(t), \quad t = \varphi^{-1}(\tau)$$

The relationship is easily proved as follows. Since $\tau = \varphi(t)$, then differentiating both sides of this relationship we get:

$$\tau = \varphi(t) \rightarrow d\tau = \varphi'(t) dt \rightarrow dt = \frac{d\tau}{\varphi'(t)} \rightarrow dt = \frac{d\tau}{\varphi'(\varphi^{-1}(\tau))}$$

Eq. 2-19

where $\varphi'(t) = \frac{d\varphi(t)}{dt}$.

Note also that this form of the substitution rule requires that the function φ be invertible, since $\varphi^{-1}(\tau)$ is explicitly required in the formula. Remember that being invertible is assured if φ is monotonic, either increasing or decreasing. More general rules can be written, but we will not need them for now.

Finally, an alternative equivalent form is found if the differential is treated differently:

$$\tau = \varphi(t) \rightarrow t = \varphi^{-1}(\tau) \rightarrow dt = \frac{d}{d\tau}(\varphi^{-1}(\tau)) \cdot d\tau$$

leading to the alternative formula:

$$\int_{t_0}^{t_1} s(\varphi(t)) dt = \int_{\varphi(t_0)}^{\varphi(t_1)} s(\tau) \frac{d}{d\tau}(\varphi^{-1}(\tau)) \cdot d\tau$$

$$\text{where } \tau = \varphi(t), \quad t = \varphi^{-1}(\tau)$$

Note that if two functions, say s and w , are multiplied in the integrand function, and the argument substitution is made with respect to one of them, say s , the argument of the other must be changed accordingly:

$$\int_{t_0}^{t_1} s(\varphi(t)) w(t) dt = \int_{\varphi(t_0)}^{\varphi(t_1)} s(\tau) \frac{w(\varphi^{-1}(\tau))}{\varphi'(\varphi^{-1}(\tau))} d\tau$$

$$\text{with } \varphi'(t) = \frac{d\varphi(t)}{dt}$$

On your own: try to prove one of the previous transformations using the substitution rule.

END of OPTIONAL

2.6 Questions

2.6.1

Consider the triangular signal $s(t) = \Lambda(t)$.

Plot the following signals:

$$s(t) = \Lambda(t - 2)$$

$$s(t) = \Lambda(t + 1)$$

$$s(t) = \Lambda\left(\frac{t}{2}\right)$$

$$s(t) = \Lambda(3t)$$

$$s(t) = \Lambda(2[t - 1])$$

$$s(t) = \Lambda(2[t+2])$$

$$s(t) = \Lambda\left(\frac{t}{2} - 1\right)$$

$$s(t) = \Lambda(2t+2)$$

2.6.2

Calculate:

$$\int_{-\infty}^{+\infty} \delta(t-t_0) \cdot s(t) dt$$

Answer: $s(t_0)$

Important Notice: The questions on the use of Dirac's delta are very important. Failure to show that one can handle integration or manipulation of delta may result by itself in a failed exam, given the importance of delta throughout the course.

2.6.3

Calculate:

$$\int_{-\infty}^{+\infty} \delta(t_0 - t) \cdot \Pi_T(t) dt$$

$$0 < t_0 < T / 2$$

Answer: 1

2.6.4

Calculate:

$$\int_{-\infty}^{+\infty} \delta\left(\frac{1}{2}[t - T_0]\right) \cdot \sin(2\pi t / T_0) dt$$

$$\int_{-\infty}^{+\infty} \delta\left(\frac{1}{2}[t - T_0]\right) \cdot \cos(2\pi t / T_0) dt$$

Answer: $2 \sin(2\pi) = 0$, $2 \cos(2\pi) = 2$

2.6.5

Verify the results of the following integrals involving delta.

$$\int_{-\infty}^{+\infty} \delta(t - t_0) \cdot s(2t) dt = s(2t_0)$$

$$\int_{-\infty}^{+\infty} \delta(-t) \cdot s(t) dt = s(0)$$

$$\int_{-\infty}^{+\infty} \delta(-t) \cdot s(t - t_1) dt = s(-t_1)$$

$$\int_{-\infty}^{+\infty} \delta(t_0 - t) \cdot s(t) dt = s(t_0)$$

$$\int_{-\infty}^{+\infty} \delta(2[t - t_0]) \cdot s(t) dt = \frac{1}{2} s(t_0)$$

2.6.6

Verify the results of the following integrals involving delta.

$$\int_{-\infty}^{+\infty} \delta(t_0 - t/2) \cdot s(t_1 - 4t) dt = 2s(t_1 - 8t_0)$$

$$\int_{-\infty}^{+\infty} \delta(t - t_0 + t_1) \cdot s(t) dt = s(t_0 - t_1)$$

$$\int_{-\infty}^{+\infty} \delta(t - t_0 + t_1) \cdot s(-t) dt = s(t_1 - t_0)$$

$$\int_{-\infty}^{+\infty} \delta(\alpha t - t_0 + \beta t_1) \cdot s(-t) dt = \frac{1}{|\alpha|} s\left(\frac{\beta t_1}{\alpha} - \frac{t_0}{\alpha}\right)$$

2.6.7

Can the following expression be simplified in any way?

$$s(t) = e^{-a \cdot t} \cdot \delta(t - t_0)$$

Answer: $s(t) = e^{-a \cdot t_0} \cdot \delta(t - t_0)$

2.6.8 optional

Read the chapter and then try to prove that:

$$\lim_{\substack{\text{dist} \\ T \rightarrow 0}} \frac{\Pi_T(t)}{T} = \delta(t)$$

2.6.9

Given the signal $s(t) = \sin(27.5\pi \cdot t)$, find its frequency and its period. How many maxima of the signal can be found over the interval $[0, 100]$ (s)? How long does it take for $s(t)$ to complete 32 full cycles?

Answer: The frequency is $f_0 = 13.75$ Hz, the period is

$$T = f_0^{-1} = 1/13.75 = 0.0727272...$$

Over $[0, 100]$ the signal goes through a number of periods which is:

$$N_p = 100 / T = 100 \cdot f_0 = 100 \cdot 13.75 = 1375$$

Every period has one maximum, therefore, the answer is 1375.

2.6.10 “**optional** challenge question”

Solve the following integral involving delta:

$$\int_{-\infty}^{+\infty} \delta(\sin(t)) \cdot s(t) dt$$

2.6.11 “**optional** challenge question”

Solve the following integral involving delta:

$$\int_{-\infty}^{+\infty} \delta(\sin(2\pi t)) \cdot s(t) dt$$

Is the result the same as the previous problem or not? Why?

2.6.12 “optional challenge question”

Given the integral:

$$\int_{-\infty}^{+\infty} \delta(g(t)) \cdot s(t) dt$$

where the function $g(t)$ has K distinct zeros, that is:

$$g(t) = 0 \quad , \quad t = t_k, \quad k = 1, \dots, K$$

assuming $g(t)$ is continuous at each one of its zeros, what is the result of the integral ?

Answer:

$$\int_{-\infty}^{+\infty} \delta(g(t)) \cdot s(t) dt = \sum_{k=1}^K \frac{s(t_k)}{|g'(t_k)|}$$

2.6.13

Re-derive the formula:

$$\int_{-\infty}^{+\infty} \delta(\alpha[t - t_0]) \cdot s(t) dt = \frac{1}{|\alpha|} s(t_0)$$

using the result of the previous exercise.

Chapter 3. Time-Average, Energy and Power

In this chapter we first introduce the important concept of “**time-average**” and then we provide the basic mathematical definitions of “**energy**” and “**power**” for signals.

We also show a few examples, some of which are related to actual physical phenomena.

Finally, we introduce a special signal classification based on “energy” and “average power”.

3.1 *Time Averages*

Let us define the **time-average operator** for a signal over a time interval $\mathbf{I} = [t_0, t_1]$ as follows⁶:

$$\langle s(t) \rangle_{[t_0, t_1]} = \langle s(t) \rangle_{\mathbf{I}} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} s(t) dt$$

Eq. 3-1

⁶ In this course, when specifying an interval as $\mathbf{I} = [t_0, t_1]$, the lower limit is always smaller than the upper limit, that is: $t_0 < t_1$. This also ensures that $t_1 - t_0 > 0$.

Note that the result of the time-average operator is *not* a signal but *just a number*.

Time-averages are of interest in many practical cases. For instance, when a “cause” and an “effect” are related in time through an integral, the time-average tells us what *constant* “cause” would generate the same “effect” as the actual time-varying “cause”.

The following problem will clarify this aspect.

3.1.1 Problem

There is a water tank which is filled by a faucet whose water output is irregular (not constant over time) and is represented by a function of time $s(t)$ (in ℓ/s , liters per second).

1. Assuming the tank was empty at time t_0 , we want to know how to compute the total water accumulated in the tank over the interval: $\mathbf{I} = [t_0, t_1]$.
2. We would like to know what *constant* water output, which we could write as: $z(t) = \alpha \cdot 1(t)$, would have the same effect (fill the water tank to the same level) as $s(t)$ over the same time interval \mathbf{I} .

Solution

Simple physical arguments suggest that the water accumulated in the tank at time t_1 would be:

$$W = \int_{t_0}^{t_1} s(t) dt$$

where W is liters, ℓ .

We now would like to know what **constant** water output would have the same effect. That is: assuming the faucet had a constant output α , in ℓ/s , what value should α need to have, in order for the faucet to output the same amount of water W over the same interval $\mathbf{I} = [t_0, t_1]$?

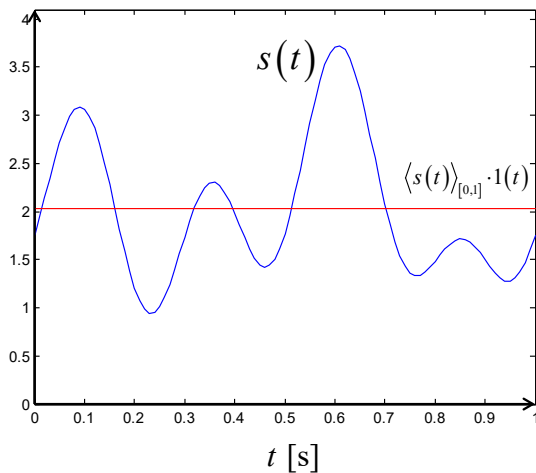


Fig. 3-1: plot of a signal $s(t)$ in \mathcal{A}_s over $t \in [0,1]$ and of the signal $\langle s(t) \rangle_{[0,1]} \cdot 1(t)$

To find it, we perform a direct calculation. We assume a constant water output $\alpha \cdot 1(t)$ over the interval $\mathbf{I} = [t_0, t_1]$. Then, the amount of water W' output by such a constant source would be:

$$W' = \int_{t_0}^{t_1} \alpha \cdot 1(t) dt = \alpha \cdot (t_1 - t_0)$$

We now recall the calculation for the water delivered by the faucet W :

$$W = \int_{t_0}^{t_1} s(t) dt = \int_{t_0}^{t_1} s(t) dt = \frac{1}{(t_1 - t_0)} \int_{t_0}^{t_1} s(t) dt \cdot (t_1 - t_0) = \langle s(t) \rangle_{[t_0, t_1]} \cdot (t_1 - t_0)$$

Eq. 3-2

and then we impose that $W' = W$, finding:

$$W' = W \quad \Rightarrow \quad \alpha \cdot (t_1 - t_0) = \left\langle s(t) \right\rangle_{[t_0, t_1]} \cdot (t_1 - t_0)$$

that is: $\alpha = \left\langle s(t) \right\rangle_{[t_0, t_1]}$

The answer is that the value of α should be equal to the time-averaged water output of $s(t)$ over $\mathbf{I} = [t_0, t_1]$.

In other words, a faucet with constant output:

$$z(t) = \alpha \cdot 1(t) = \left\langle s(t) \right\rangle_{[t_0, t_1]} \cdot 1(t)$$

would accumulate the same number of liters W in the tank as $s(t)$, over $\mathbf{I} = [t_0, t_1]$.

This example is also illustrated in Fig. 3-1, where the average water output of 2.03 ℓ/s (red plot) produces the same tank filling as the variable water output $s(t)$ (blue line) over the interval $\mathbf{I} = [0, 1]$.

In conclusion, we can say that:

the time-averaged value of a signal $s(t)$ over an interval \mathbf{I} is the value of a *constant* signal that has the same *integral* (or the same subtended area) as the original signal $s(t)$ over the same interval \mathbf{I} .

On your own: think of what happens when a current goes into a capacitor. Can you come up with a similar example as the above, in terms of current into a capacitor and a voltage at the terminals? Of course in this case you need to exploit the capacitance formula: $V = \frac{q}{C}$.

3.1.2 Linearity of the average operator

The time-average operator is *linear*. We will encounter this property many times in this course. It means that, given the sum of many signals $s_n(t)$, weighed through multiplicative constants a_n :

$$\sum_{n=1}^N a_n \cdot s_n(t)$$

then the average of such weighed sum of signals equals the weighed sum of the averages:

$$\left\langle \sum_{n=1}^N a_n \cdot s_n(t) \right\rangle_{[t_0, t_1]} = \sum_{n=1}^N a_n \cdot \langle s_n(t) \rangle_{[t_0, t_1]}$$

The reason for this property is that the integral operator enjoys such property itself and therefore the average operator, which is based on an integral, inherits it.

3.1.3 Time-average of a constant

Another straightforward property of the time-average is that the average of a constant signal is the constant signal itself (prove it on your own):

$$\langle \alpha \cdot 1(t) \rangle_{[t_0, t_1]} = \alpha \cdot 1(t)$$

3.1.4 Going to infinite intervals

Averages can be extended in time at will, over larger and larger intervals. However, if one wants to actually extend an average over the whole of \mathbb{R} , then a problem is incurred: the factor $1/(t_1 - t_0)$ goes to zero as $t_0 \rightarrow -\infty$ and $t_1 \rightarrow +\infty$.

So, it is necessary to introduce a limit operator:

$$\langle s(t) \rangle_{\mathbb{R}} = \lim_{\substack{t_0 \rightarrow -\infty \\ t_1 \rightarrow +\infty}} \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} s(t) dt$$

When averaging over the whole of \mathbb{R} it is convenient to define a single time parameter T , which can replace t_0, t_1 since there is no need to keep t_0, t_1 distinct as they respectively go to $\pm\infty$. Doing so we get:

$$\langle s(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t) dt$$

Note that there is no guarantee that such an average converges to a single and finite value. Convergence must be discussed case-by-case.

3.1.5 Exercises on your own

On your own: find the time-average of:

1. $\Pi(t)$
2. $\cos(2\pi t)$
3. $\Lambda(t)$

over the intervals $\mathbf{I} = [-1, 1]$ and $\mathbb{R} = [-\infty, \infty]$.

Results:

1. $\frac{1}{2}, 0$

2. $0, \lim_{T \rightarrow \infty} \text{Sinc}(T) = 0$

3. $\frac{1}{2}, 0$

Solutions

(1)

$$\langle \Pi(t) \rangle_{[-1,1]} = \frac{1}{(1 - (-1))} \int_{-1}^1 \Pi(t) dt = \frac{1}{2} \int_{-1/2}^{1/2} 1(t) dt = \frac{1}{2} [t]_{-1/2}^{1/2} = \frac{1}{2}$$

The solutions could also be found numerically by using Matlab.

In this case the code is simply:

`integral (@ (t) HPi (1, t) , -1, 1) / 2`

$$\langle \Pi(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Pi(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-1/2}^{1/2} 1(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} = 0$$

(2)

$$\langle \cos(2\pi t) \rangle_{[-1,1]} = \frac{1}{(1 - (-1))} \int_{-1}^1 \cos(2\pi t) dt =$$

$$= \frac{1}{2} \frac{1}{2\pi} [\sin(2\pi t)]_{-1}^1 = \frac{1}{4\pi} [\sin(2\pi) - \sin(-2\pi)] = 0$$

$$\begin{aligned}
\langle \cos(2\pi t) \rangle_{\mathbb{R}} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos(2\pi t) dt = \frac{1}{2\pi} \lim_{T \rightarrow \infty} \frac{1}{T} \left[\sin(2\pi t) \right]_{-T/2}^{T/2} \\
&= \frac{1}{2\pi} \lim_{T \rightarrow \infty} \frac{1}{T} \left[\sin(\pi T) - \sin(-\pi T) \right] = \lim_{T \rightarrow \infty} \frac{2 \sin(\pi T)}{2\pi T} = \lim_{T \rightarrow \infty} \frac{\sin(\pi T)}{\pi T} \\
&= \lim_{T \rightarrow \infty} \text{Sinc}(T) = 0
\end{aligned}$$

This result can also be found by symbolic integration in Matlab:

```
syms T t
int(1/T*cos(2*pi*t), -T/2, T/2)
```

which returns:

```
sin(pi*T) / (T*pi)
```

(3)

$$\langle \Lambda(t) \rangle_{[-1,1]} = \frac{1}{(1-(-1))} \int_{-1}^1 \Lambda(t) dt = \frac{1}{2} \int_{-1}^1 \Lambda(t) dt$$

Remembering the definition of $\Lambda(t)$:

$$\Lambda(t) = \begin{cases} 1+t & -1 < t \leq 0 \\ 1-t & 0 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$

we can then write:

$$\begin{aligned}\frac{1}{2}\int_{-1}^1 \Lambda(t)dt &= \frac{1}{2}\left[\int_{-1}^0 (1+t)dt + \int_0^1 (1-t)dt\right] = \frac{1}{2}\left\{\left[t + \frac{t^2}{2}\right]_{-1}^0 + \left[t - \frac{t^2}{2}\right]_0^1\right\} \\ &= \frac{1}{2}\left[\left(-(-1) - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{1}{2}\end{aligned}$$

The solutions could also be found numerically by using Matlab.
In this case the code is simply:

```
integral( @(t) HLambda(1,t) , -1, 1) / 2
```

Regarding:

$$\langle \Lambda(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Lambda(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-1}^1 \Lambda(t) dt$$

the integral $\int_{-1}^1 \Lambda(t) dt$ was already solved in the previous exercise, providing 1 as a result. So:

$$\langle \Lambda(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot 1 = 0$$

On your own: find the average $\langle \Lambda(t) \rangle_{[-1/2, 1/2]}$.

Answer: $\frac{3}{4}$

It can also be found in Matlab as:

```
integral( @ (t) HLambda(1,t) , -1/2, 1/2)
```

3.2 *Instantaneous and Time-Averaged Power*

Let us define the **instantaneous power** of a signal $s(t)$ as:

$$P_s(t) = |s(t)|^2$$

Although we introduce it as a mathematical definition, it is consistent with many physical situations. For instance, assuming $s(t)$ to be a current in Ampère (A), then the instantaneous power delivered by such a current into a resistor R would in fact be:

$$P_s(t) = |s(t)|^2 \cdot R$$

The constant resistance R is there to adjust dimensions and make the result appear as Watts (W), but the key fact is that power is physically proportional to the square of the current “signal”, and this is consistent with the mathematical definition that we have given.

Note that $P_s(t)$ is a signal itself, so we can take its time-average. By combining the definition of instantaneous power of a signal and that of time-average, we obtain the **time-averaged power** of a signal $s(t)$ over a certain interval $\mathbf{I} = [t_0, t_1]$ as:

$$\langle P_s(t) \rangle_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} P_s(t) dt = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |s(t)|^2 dt$$

Eq. 3-3

In the following we will use a specific notation to indicate this quantity: $\mathcal{P}_I \{s(t)\}$.

As shown in Sect. 3.1, time-averages can be extended to the whole of \mathbb{R} . This allows us to introduce the important concept of **time-averaged power of a signal $s(t)$ over all time**:

$$\mathcal{P}_{\mathbb{R}} \{s(t)\} = \langle P_s(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} P_s(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt$$

Again, there is no guarantee that this limit converges to a finite value.

3.2.1 Examples

Let us calculate the average power of the signal $\Pi_{T_0}(t)$ over the interval $\mathbf{I} = [-T/2, T/2]$, assuming $T > T_0$:

$$\begin{aligned}\mathcal{P}_{[-T/2, T/2]} \{ \Pi_{T_0}(t) \} &= \left\langle \left| \Pi_{T_0}(t) \right|^2 \right\rangle_{[-T/2, T/2]} \\&= \frac{1}{T/2 - (-T/2)} \int_{-T/2}^{T/2} \left| \Pi_{T_0}(t) \right|^2 dt \\&= \frac{1}{T} \int_{-T/2}^{T/2} \left| \Pi_{T_0}(t) \right|^2 dt = \frac{1}{T} \int_{-T_0/2}^{T_0/2} 1(t) dt = \frac{1}{T} [t]_{-T_0/2}^{T_0/2} = \frac{T_0}{T}\end{aligned}$$

Note that, if the observation interval time-length T grows, then the average power tends to decrease steadily. If the average power is assessed over the whole of \mathbb{R} , it actually goes to zero (do this calculation on your own).

\therefore

We now want to consider the signal $e^{-at}u(t)$, $a > 0$. We consider the interval $\mathbf{I} = [-T/2, T/2]$:

$$\begin{aligned}\mathcal{P}_{[-T/2, T/2]} \{e^{-at}u(t)\} &= \left\langle |e^{-at}u(t)|^2 \right\rangle_{[-T/2, T/2]} \\&= \frac{1}{T/2 - (-T/2)} \int_{-T/2}^{T/2} e^{-2at} u^2(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} e^{-2at} u(t) dt \\&= \frac{1}{T} \int_0^{T/2} e^{-2at} dt = -\frac{1}{2aT} [e^{-2at}]_0^{T/2} = -\frac{1}{2aT} (e^{-aT} - 1) = \frac{1}{2aT} (1 - e^{-aT})\end{aligned}$$

Note that, if the observation interval time-length T grows, then the average power tends to decrease steadily. If the average power is assessed over the whole of \mathbb{R} , it actually goes to zero.

\therefore

On your own consider the case $e^{at}u(t)$, $a > 0$. Plot the resulting signal. What happens to the average power when $\mathbf{I} = [-T/2, T/2]$ and when $\mathbf{I} = \mathbb{R}$?

Results: $\frac{1}{2aT}(e^{aT} - 1)$, ∞ .

\therefore

Let us now calculate the average power of the signal $\cos(2\pi f_0 t)$ over the interval $\mathbf{I} = [-T/2, T/2]$:

$$\begin{aligned}
\mathcal{P}_{[-T/2, T/2]} \{ \cos(2\pi f_0 t) \} &= \\
&= \left\langle |\cos(2\pi f_0 t)|^2 \right\rangle_{[-T/2, T/2]} = \frac{1}{T/2 - (-T/2)} \int_{-T/2}^{T/2} \cos^2(2\pi f_0 t) dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi f_0 t) \right] dt = \frac{1}{T} \int_{-T/2}^{T/2} \frac{1(t)}{2} dt + \frac{1}{T} \int_{-T/2}^{T/2} \frac{1}{2} \cos(4\pi f_0 t) dt \\
&= \frac{1}{2T} [t]_{-T/2}^{T/2} + \frac{1}{2} \frac{1}{4\pi f_0 T} [\sin(4\pi f_0 t)]_{-T/2}^{T/2} \\
&= \frac{1}{2T} \left(\frac{T}{2} - \frac{-T}{2} \right) + \frac{1}{2} \frac{1}{4\pi f_0 T} [\sin(2\pi f_0 T) - \sin(-2\pi f_0 T)] \\
&= \frac{1}{2} + \frac{1}{2} \cdot \frac{2 \cdot \sin(2\pi f_0 T)}{2 \cdot (2\pi f_0 T)} = \frac{1}{2} + \frac{1}{2} \frac{\sin(2\pi f_0 T)}{2\pi f_0 T} = \frac{1}{2} + \frac{1}{2} \text{Sinc}(2f_0 T)
\end{aligned}$$

Note that, if the observation interval time-length grows, then the average power tends to settle on the value $\frac{1}{2}$.

What happens then, when the average extends over the whole of \mathbb{R} ? That is, what is the result of:

$$\mathcal{P}_{\mathbb{R}} \left\{ \cos(2\pi f_0 t) \right\} \quad ?$$

From the definitions it is easy to see that we can write:

$$\mathcal{P}_{\mathbb{R}} \left\{ \cos(2\pi f_0 t) \right\} = \lim_{T \rightarrow \infty} \mathcal{P}_{[-T/2, T/2]} \left\{ \cos(2\pi f_0 t) \right\}$$

But we know from the previous example what the value of $\mathcal{P}_{[-T/2, T/2]} \left\{ \cos(2\pi f_0 t) \right\}$ is. So we can just substitute:

$$\begin{aligned}
\mathcal{P}_{\mathbb{R}} \{ \cos(2\pi f_0 t) \} &= \lim_{T \rightarrow \infty} \mathcal{P}_{[-T/2, T/2]} \{ \cos(2\pi f_0 t) \} \\
&= \lim_{T \rightarrow \infty} \left[\frac{1}{2} + \frac{1}{2} \text{Sinc}(2f_0 T) \right] = \frac{1}{2} \\
&\therefore
\end{aligned}$$

As a practical relevant example, assume that a current $i(t) = I_0 \cos(2\pi f_0 t)$ is injected by a current generator into a resistor of resistance R . What is the average power in Watts dissipated by the resistor over the time-interval $\mathbf{I} = [-T/2, T/2]$ and over the whole of \mathbb{R} ?

From Electronics, we know that the instantaneous power (in Watts) delivered onto a resistor by a current $i(t)$ is given by:

$$P(t) = i^2(t) \cdot R$$

The time-average of $P(t)$ in this example is then:

$$\begin{aligned} \langle P(t) \rangle_{[-T/2, T/2]} &= \langle i^2(t) \cdot R \rangle_{[-T/2, T/2]} = R \cdot \langle i^2(t) \rangle_{[-T/2, T/2]} \\ &= R \cdot \langle P_i(t) \rangle_{[-T/2, T/2]} = R \cdot \mathcal{P}_{[-T/2, T/2]} \{i(t)\} = R \cdot \left\langle \left[I_0 \cos(2\pi f_0 t) \right]^2 \right\rangle_{[-T/2, T/2]} \\ &= RI_0^2 \cdot \langle \cos^2(2\pi f_0 t) \rangle_{[-T/2, T/2]} = RI_0^2 \left[\frac{1}{2} + \frac{1}{2} \text{Sinc}(2f_0 T) \right] \end{aligned}$$

Letting $T \rightarrow \infty$ we also immediately find:

$$\langle P(t) \rangle_R = \lim_{T \rightarrow \infty} \langle P(t) \rangle_{[-T/2, T/2]} = \lim_{T \rightarrow \infty} RI_0^2 \left[\frac{1}{2} + \frac{1}{2} \text{Sinc}(2f_0 T) \right] = R \frac{I_0^2}{2}$$

Note how the mathematical concept of average power if the signal $i(t)$ matches the physical concept of actual average power in Watts, once the proper multiplying constant R is taken into account.

Based on the result over \mathbb{R} , why do you think the value $I_0 / \sqrt{2}$ is commonly called the current “**effective value**” in AC power grids, or also its **RMS** value? Note that RMS means “root mean square”, that is you first average the square of the current, then take the square root of the result. Also, compare Wikipedia’s article:

https://en.wikipedia.org/wiki/Root_mean_square

3.2.1.1 Further problems on your own

On your own: find the average power over the intervals $\mathbf{I} = [-1, 1]$, $\mathbf{I} = [0, 2]$, and then over \mathbb{R} of the signals:

- $\Lambda(t) \rightarrow \frac{1}{3}, \frac{1}{6}, 0$

Hint: remember the definition:

$$\Lambda(t) = \begin{cases} 1+t & -1 < t \leq 0 \\ 1-t & 0 < t < 1 \\ 0 & |t| \geq 1 \end{cases}$$

- $e^{j2\pi f_0 t} \rightarrow 1$, all three cases

- $a \cos(2\pi t) + b \cos(4\pi t) \rightarrow \frac{|a|^2 + |b|^2}{2}$, all three cases
- $\frac{1}{2} \cos(2\pi t) + \sin(6\pi t) \rightarrow \frac{5}{8}$, all three cases

Try and discuss the results. Remember that:

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2} \cos(2\alpha)$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$$

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2} \cos(2\alpha)$$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2} \sin(\alpha + \beta) + \frac{1}{2} \sin(\alpha - \beta)$$

and of course:

$$\cos\left(\alpha - \frac{\pi}{2}\right) = \sin(\alpha)$$

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos(\alpha)$$

On your own: Redo the previous problems numerically using Matlab and verify that all results coincide with the analytical ones. The code solving some of the problems is reported here below.

%

```
integral( @(t) HLambda(1,t).^2 , 0, 2)/2
```

%

```
integral( @(t) abs(exp(j*2*pi*t)).^2 , 0, 2 )/2
```

```

%
integral( @(t) (1/2*cos(2*pi*t) +...
    sin(6*pi*t)).^2, 0, 2 )/2

%
% Numerical values must be assigned to the
% constants a and b. For instance:
a=1/2;b=1/3;
integral( @(t) (a*cos(2*pi*t) +...
    b*cos(4*pi*t)).^2, 0, 2 )/2

%

```

3.3 *Energy*

In Physics, energy is the integral of instantaneous power over a certain time interval. In signal theory we adopt the same definition, so:

$$\mathcal{E}_{[t_0, t_1]} \{s(t)\} = \int_{t_0}^{t_1} P_s(t) dt = \int_{t_0}^{t_1} |s(t)|^2 dt$$

Eq. 3-4

where we define $\mathcal{E}_{[t_0, t_1]} \{s(t)\}$ as the “energy operator” extracting the energy of $s(t)$ over the interval $\mathbf{I} = [t_0, t_1]$.

We now recall the average power equation:

$$\langle P_s(t) \rangle_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} P_s(t) dt = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |s(t)|^2 dt$$

By comparing the average power and the energy equations it is immediately seen that the following relationship holds:

$$\mathcal{E}_{[t_0, t_1]} \{s(t)\} = \mathcal{P}_{[t_0, t_1]} \{s(t)\} \cdot (t_1 - t_0)$$

Eq. 3-5

In other words, *the energy delivered over an interval $\mathbf{I} = [t_0, t_1]$ is equal to the average power over the same interval, times the length (in time) of the interval.*

Note that this *agrees* with the concepts of *energy and power* from physics.

Also note that one can think of $P_s(t)$ as a “faucet” pouring a time-varying amount of Joules per second into an “energy tub” over the interval \mathbf{I} collecting W Joules. Similar to the case of Problem 3.1.1.1 about water flow, here the same amount of energy W is delivered over $\mathbf{I} = [t_0, t_1]$ either by the actual time-varying power $P_s(t)$ or by the *constant* power $\langle P_s(t) \rangle_{[t_0, t_1]} \cdot 1(t)$, written based on the time-average of $P_s(t)$. In formulas:

$$W = \int_{t_0}^{t_1} P_s(t) dt = \langle P_s(t) \rangle_{[t_0, t_1]} \cdot (t_1 - t_0)$$

which is the same as Eq. 3-2. There it is in terms of a water flow, here it is in terms of an energy flow (that is, a power).

∴

Energy can be computed over the whole of \mathbb{R} . In this case, differently from average power, we do not need the “limit” operator because we do not have the factor $1/T$ before the integral.

So, we simply extend the integration range to the whole of \mathbb{R} :

$$E_{\mathbb{R}} \{s(t)\} = \int_{-\infty}^{\infty} P_s(t) dt = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

Again, depending on the signal, this integral may or may not converge.

3.3.1 Examples

Let us compute the energy of the signal $\Pi_{T_0}(t)$ over the interval $\mathbf{I} = [-T/2, T/2]$, assuming $T > T_0$. Given Eq. 3-4 and the result from Sect. 3.2.1 we can immediately write:

$$\mathcal{E}_{[-T/2, T/2]} \{ \Pi_{T_0}(t) \} = \mathcal{P}_{[-T/2, T/2]} \{ \Pi_{T_0}(t) \} \cdot T = \frac{T_0}{T} \cdot T = T_0$$

Note that in this case extending the calculation to $\mathbf{I} = \mathbb{R}$ does not change the *energy* result at all:

$$\begin{aligned}\mathcal{E}_{\mathbb{R}}\left\{\Pi_{T_0}(t)\right\} &= \int_{-\infty}^{\infty} \left|\Pi_{T_0}(t)\right|^2 dt \\ &= \int_{-T_0/2}^{T_0/2} 1(t) dt = [t]_{-T_0/2}^{T_0/2} = T_0\end{aligned}$$

We now consider again the signal $e^{-at}u(t)$, $a \in \mathbb{R}$, $a > 0$. We compute its energy over the interval $\mathbf{I} = [-T/2, T/2]$. We could simply use Eq. 3-4 and the average power for the same signal and interval from Sect. 3.2.1. For convenience, we redo the calculation:

$$\begin{aligned}
E_{[-T/2, T/2]} \{e^{-at} u(t)\} &= \int_{-T/2}^{T/2} |e^{-at} u(t)|^2 dt \\
&= \int_0^{T/2} e^{-2at} dt = -\frac{1}{2a} [e^{-2at}]_0^{T/2} = -\frac{1}{2a} (e^{-aT} - 1) = \frac{1}{2a} (1 - e^{-aT})
\end{aligned}$$

Eq. 3-6

We now extend the calculation to the whole of \mathbb{R} . We can find the result directly:

$$\begin{aligned}
E_{\mathbb{R}} \{s(t)\} &= \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 dt \\
&= \int_0^{\infty} |e^{-at}|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{-1}{2a} e^{-2at} \Big|_0^{\infty} = \frac{1}{2a}
\end{aligned}$$

or we can use the result in Eq. 3-5 and let $T \rightarrow \infty$. The two results clearly coincide.

We then calculate the energy of the signal $\cos(2\pi f_0 t)$ over the interval $\mathbf{I} = [-T/2, T/2]$. We use Eq. 3-4 and the average power result from Sect. 3.2.1. We can then directly write:

$$E_{[-T/2, T/2]} \{ \cos(2\pi f_0 t) \} = \int_{-T/2}^{T/2} |\cos(2\pi f_0 t)|^2 dt = \frac{T}{2} [1 + \text{Sinc}(2f_0 T)]$$

Eq. 3-7

If we try to extend the energy calculation to $\mathbf{I} = \mathbb{R}$, we now run into a problem. Either trying to extend the result of Eq. 3-6 or trying to redo the calculation from

scratch, the result does not converge. In both cases it appears to indicate an “**infinite energy**”. This circumstance will be discussed in the Section 3.4.

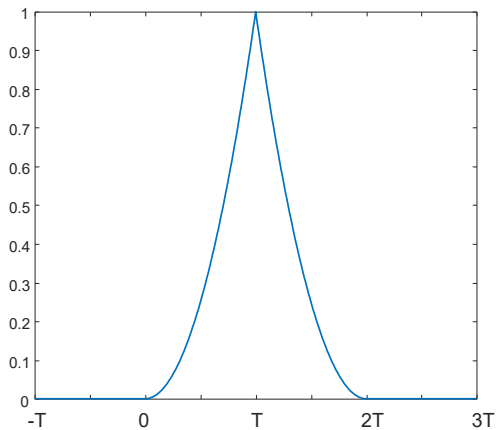
3.3.1.1 Problem

Given the signal:

$$s(t) = \Lambda\left(\frac{t-T}{T}\right)$$

1. find its instantaneous power $P_s(t)$
2. assuming $P_s(t)$ has dimension kW (kiloWatts), find the energy in kWh (kiloWatt-hour) delivered by $s(t)$ over the interval $\mathbf{I} = [0, 2T]$, where T is 2 hours

3. find its time-averaged power over the same interval.



Solution:

1. The instantaneous power is:

$$P_s(t) = \left| \Lambda \left(\frac{t-T}{T} \right) \right|^2$$

2. The energy delivered over the time-interval $\mathbf{I} = [0, 2T]$ is given by:

$$\begin{aligned} E_{[0,2T]} \left\{ \Lambda \left(\frac{t-T}{T} \right) \right\} &= \int_0^{2T} \left| \Lambda \left(\frac{t-T}{T} \right) \right|^2 dt = \int_0^{2T} \Lambda^2 \left(\frac{t-T}{T} \right) dt = \\ 2 \int_0^T \Lambda^2 \left(\frac{t-T}{T} \right) dt &= 2 \int_0^T \left(\frac{t}{T} \right)^2 dt = \frac{2}{T^2} \int_0^T t^2 dt = \frac{2}{T^2} \cdot \frac{t^3}{3} \Big|_0^T = \frac{2}{3} T \end{aligned}$$

So it is exactly $\frac{2}{3}T$ (kWh). For $T = 2$ hours, the energy is $\frac{4}{3}$ (kWh).

3. The resulting time-averaged power over the interval $\mathbf{I} = [0, 2T] = [0, 4 \text{ hours}]$ is $\frac{1}{3}$ kW.

On your own: Redo the calculation above using Matlab, by performing numerical integration. Use the Matlab command 'integral'. The solution is reported below

```
% executing these lines of code
% solves the problem
T=2;
integrand= @(t) HLambda(T,t-T).^2;
```

```
energy=integral(integrand,0,2*T)
time_averaged_power=energy/(2*T)
```

On your own: what can you say about the average power of a signal whose energy, computed over the whole of \mathbb{R} , is finite and non-zero? After thinking about it, read on and see if your thoughts agree with what is written next.

3.4 Classification of Signals Based on Average Power and Energy

Signals are divided into various classes depending on their energy and power properties. These properties will have a big impact on certain key aspects of signal analysis and representation that will be introduced in the next chapter.

This is the topic that we are going to deal with in the following slides.

3.4.1 Finite energy

3.4.1.1 finite energy over a finite interval

Finite energy signals over a finite interval $\mathbf{I} = [t_0, t_1]$ are all those signals for which:

$$\mathcal{E}_{[t_0, t_1]} \{s(t)\} = \int_{t_0}^{t_1} |s(t)|^2 dt < \infty$$

All *physical* signals must satisfy this condition.

As a *sufficient* condition for a signal to be finite-energy over a finite interval $\mathbf{I} = [t_0, t_1]$, it is enough that the signal be *bounded* over the interval, that is:

$$|s(t)|^2 < C, \quad \forall t \in \mathbf{I} = [t_0, t_1].$$

Eq. 3-8

where C is a positive and finite constant. Note that this condition must be satisfied at the extremes of the interval, too.

The same sufficient condition could also be written in terms of the instantaneous power of the same signal:

$$P_s(t) < \infty, \quad \forall t \in \mathbf{I} = [t_0, t_1].$$

This means that every signal whose instantaneous power is bounded at all times over $\mathbf{I} = [t_0, t_1]$ also has finite energy over the same interval.

Three of the signals that we looked at in the previous pages, that is: $\Pi_{T_0}(t)$; $e^{-at}u(t)$, ($a \in \mathbb{R}, a > 0$); $\cos(2\pi f_0 t)$, are finite-energy over a finite interval, as the results of the calculations in Sect. 3.3.1 show.

As a counter-example, there are of course signals that are not finite-energy, even over a finite interval. For instance, the signals $s(t) = t^{-1}$ or $s(t) = t^{-1/2}$ have “infinite” energy over the interval $\mathbf{I} = [0, 1]$. Prove it on your own.

On your own: prove the sufficient condition of Eq. 3-7.

On your own: try and find the mathematical expression of another signal for which $\mathcal{E}_{[t_0, t_1]} \{s(t)\}$ is not finite.

Optional: Condition Eq. 3-7 is sufficient, by it is not necessary. There are signals whose instantaneous power is not bounded, but still their energy is finite. For instance $s(t) = t^{-1/4}$ has finite energy over the interval $\mathbf{I} = [0, 1]$, and such *energy has value 2*. However, its instantaneous power is unbounded. Verify it **on your own.**

End of optional.

3.4.1.2 finite energy over the whole of \mathbb{R}

Finite energy signals over the whole of \mathbb{R} are all those signals for which:

$$E_{\mathbb{R}} \{s(t)\} = \int_{-\infty}^{+\infty} |s(t)|^2 dt < \infty$$

The three signals: $\Pi_{T_0}(t)$; $e^{-at}u(t)$, ($a \in \mathbb{R}, a > 0$); $\cos(2\pi f_0 t)$, have different behavior when their energy is computed over the whole of \mathbb{R} . Specifically: $\Pi_{T_0}(t)$ and $e^{-at}u(t)$, ($a \in \mathbb{R}, a > 0$) are **finite-energy over \mathbb{R}** .

Instead, $\cos(2\pi f_0 t)$ is **infinite-energy**, as the results of the calculations in Sect. 3.3.1 show, see Eq. 3-7.

On your own: which of the signals: $\Lambda(t)$, $e^{at}u(t)$, ($a \in \mathbb{R}, a > 0$), and $e^{j2\pi f_0 t}e^{-at}u(t)$ ($a \in \mathbb{R}, a > 0$) is finite-energy over \mathbb{R} ?

Solution: $\Lambda(t)$ and $e^{j2\pi f_0 t}e^{-at}u(t)$ ($a \in \mathbb{R}, a > 0$) are finite energy over \mathbb{R} . $e^{at}u(t)$, ($a \in \mathbb{R}, a > 0$) is not.

3.4.2 Finite average power

3.4.2.1 finite average power over a finite interval

Finite-average-power signals over a finite interval $\mathbf{I}=[t_0, t_1]$ are all those signals for which:

$$\mathcal{P}_{[t_0, t_1]} \{s(t)\} = \langle P_s(t) \rangle_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} P_s(t) dt = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |s(t)|^2 dt < \infty$$

Note that time-averaged power and energy are related through Eq. 3-4, which can be re-written as:

$$\mathcal{P}_{[t_0, t_1]} \{s(t)\} = \frac{\mathcal{E}_{[t_0, t_1]} \{s(t)\}}{(t_1 - t_0)}$$

Eq. 3-9

Since, given a finite interval $\mathbf{I} = [t_0, t_1]$, clearly: $0 < (t_1 - t_0) < \infty$, then it turns out that a signal $s(t)$ that has finite energy over an interval $\mathbf{I} = [t_0, t_1]$ also has finite average power over the same interval, and vice-versa. In other words, ***all signals that have finite energy over $\mathbf{I} = [t_0, t_1]$ also have finite average power over $\mathbf{I} = [t_0, t_1]$ (and vice-versa).***

Put it another way, the set of all signals that have finite energy over $\mathbf{I} = [t_0, t_1]$ coincides with the set of all signals that have finite average power over the same interval. This situation is depicted in Fig. 3-2:

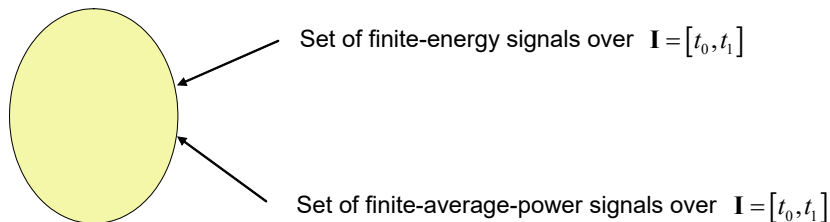


Fig. 3-2

On your own: look at Sects. 3.2.1 and 3.3.1 and check that indeed $\Pi_{T_0}(t)$, $e^{-at}u(t)$ with $(a \in \mathbb{R}, a > 0)$, and $\cos(2\pi f_0 t)$ are both finite-energy and finite average-power over any finite interval $\mathbf{I} = [t_0, t_1]$.

3.4.2.2 finite average power over the whole of \mathbb{R}

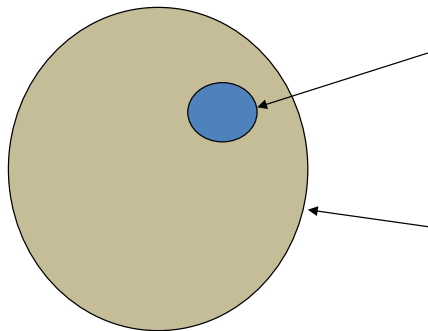
A *finite average power signal over the whole of \mathbb{R}* is such that:

$$\mathcal{P}_{\mathbb{R}} \{s(t)\} = \langle P_s(t) \rangle_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} P_s(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt < \infty$$

In the previous section (Section 3.4.2.1), we have seen that the set of finite energy signals and the set of finite average power signals coincide, as long as the interval $\mathbf{I} = [t_0, t_1]$ is finite. However, when the time-interval extends to all of \mathbb{R} , the situation is different.

In particular, those signal that have *non-zero finite average power* over \mathbb{R} , that is $0 < \mathcal{P}_{\mathbb{R}} < \infty$, *must have infinite energy* $E_{\mathbb{R}} \rightarrow \infty$.

This is enough to prove that the two sets (finite average power and finite energy) do not coincide over \mathbb{R} . This situation is depicted in Fig. 3-3, where the set of finite energy signals \mathbf{E} no longer coincides with the set of finite average power signals \mathbf{P} .



E: set of finite-energy signals over \mathbb{R}
 They all have zero average power over \mathbb{R}

The two sets DO NOT coincide

Fig. 3-3

Another interesting result is that all finite-energy signals (signals in \mathbf{E}) have *zero average power over \mathbb{R}* . This is obvious from Eq. 3-8, reproduced here for convenience:

$$\mathcal{P}_{[t_0, t_1]} \{s(t)\} = \frac{E_{[t_0, t_1]} \{s(t)\}}{(t_1 - t_0)}$$

To see it, it is enough to let the interval $\mathbf{I} = [t_0, t_1]$ extend to \mathbb{R} and discuss the right-hand side: the numerator is finite over \mathbb{R} because we assume the signal to be finite-energy, while the denominator grows to infinity. The result is zero.

On your own: look at the results of Sects. 3.2.1 and 3.3.1 and check that:

$$\Pi_{T_0}(t) \in \mathbf{E}$$

$$e^{-at}u(t) \in \mathbf{E} \quad (a \in \mathbb{R}, a > 0)$$

$$\cos(2\pi f_0 t) \in \mathbf{P}$$

$$\exp(j2\pi f_0 t) \in \mathbf{P}$$

Verify also that the average power over \mathbb{R} of the first two signals is zero.

On your own: when both power and energy are calculated over \mathbb{R} , which of the above sets **E** and **P** do the following signals belong to?

$$\Lambda(t)$$

$$e^{at}u(t) \quad (a \in \mathbb{R}, a > 0)$$

$$e^{j2\pi f_0 t} e^{-at}u(t) \quad (a \in \mathbb{R}, a > 0)$$

$$\text{sign}(t)$$

Results: respectively, **E**, neither **E** nor **P** (the signal is infinite average power), **E**, **P**.

On your own: In your opinion, what is the difference between finite-energy and finite-power signals (over \mathbb{R}), in simple words?

3.4.3 Summary

A schematic summary of what just shown is as follows

Finite Intervals:

$$I = [t_0, t_1] \quad , \quad \mathcal{P}_{[t_0, t_1]} \{s(t)\} = \frac{\mathcal{E}_{[t_0, t_1]} \{s(t)\}}{(t_1 - t_0)}$$

$$\mathcal{E}_I = 0 \quad \rightarrow \quad \mathcal{P}_I = 0$$

$$0 < \mathcal{E}_I < \infty \quad \rightarrow \quad 0 < \mathcal{P}_I < \infty$$

$$\mathcal{E}_I = \infty \quad \rightarrow \quad \mathcal{P}_I = \infty$$

The whole of \mathbb{R} : $\mathcal{P}_{\mathbb{R}} = \lim_{T \rightarrow \infty} \frac{E_T}{T}$

$$E_{\mathbb{R}} = 0 \rightarrow \mathcal{P}_{\mathbb{R}} = 0$$

$$0 < E_{\mathbb{R}} < \infty \rightarrow \mathcal{P}_{\mathbb{R}} = 0$$

$$0 < \mathcal{P}_{\mathbb{R}} < \infty \rightarrow E_{\mathbb{R}} = \infty \quad \therefore$$

$$\text{optional} \left\{ \begin{array}{l} E_{\mathbb{R}} = \infty, E_T = o(T) \xrightarrow{T \rightarrow \infty} \mathcal{P}_{\mathbb{R}} = 0 \\ E_{\mathbb{R}} = \infty, E_T = O(T) \xrightarrow{T \rightarrow \infty} 0 < \mathcal{P}_{\mathbb{R}} < \infty \\ E_{\mathbb{R}} = \infty, E_T = O(T^{1+\varepsilon}) \xrightarrow{T \rightarrow \infty, \varepsilon > 0} \mathcal{P}_{\mathbb{R}} = \infty \end{array} \right.$$

3.4.3.1 **Optional:** Behavior of finite-average-power signals over finite time intervals

One result that will be useful in certain calculations that will be carried out in future chapters is the following:

finite-average-power signals (over \mathbb{R}) are finite-energy over any finite interval.

In other words, given $s(t)$ such that $\mathcal{P}_{\mathbb{R}}\{s(t)\} < \infty$, then:

$$\mathcal{E}_{[t_0, t_1]} \{s(t)\} = \int_{t_0}^{t_1} |s(t)|^2 dt < \infty \quad , \quad \forall |t_0|, |t_1| < \infty$$

The reason is that if the above was not true, the right-hand-side of the average-power calculation would diverge for some finite value of T . Instead, by definition of limit, the right hand side in:

$$\mathcal{P}_{\mathbb{R}} \{s(t)\} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt < \infty$$

must be finite for all finite values of T , for the limit to exist.

A completely equivalent result, which will be used in later chapters, has to do with the so-called “truncated signal”. A “truncated signal” $x_{[t_0, t_1]}(t)$ derived from a signal $x(t)$ is defined as follows:

$$x_{[t_0, t_1]}(t) = \begin{cases} x(t) & t \in [t_0, t_1] \\ 0(t) & t \notin [t_0, t_1] \end{cases} \quad |t_0|, |t_1| < \infty$$

The same signal can also be written as:

$$x_{[t_0, t_1]}(t) = x(t) \cdot \Pi_T(t - t_d)$$

$$T = t_1 - t_0 \quad t_d = \frac{t_1 + t_0}{2} \quad |t_0|, |t_1| < \infty$$

Then, the following holds: *given a finite-average-power signals (over \mathbb{R}) any truncated signal derived from it is a finite-energy signal over \mathbb{R} .*

In other words, $x_{[t_0, t_1]}(t)$ is finite-energy over \mathbb{R} , for any value of t_0 and t_1 .

end of optional \therefore

3.5 *Questions*

3.5.1

The speed of a car is given by the following “signal”:

$$v(t) = 100 \cdot \Lambda(t-1)$$

where the units of $v(t)$ are km/h and t is in hours.

1. What is the average speed of the car over the interval $\mathbf{I} = [0, 2]$ hours?

And over the interval $\mathbf{I}' = [0, 10]$ hours?

2. What is the distance travelled D by the car over the interval $\mathbf{I}=[0,2]$?
And over the interval $\mathbf{I}'=[0,10]$?
3. How does distance relate to time-averaged speed?
4. Defining $a(t)$ as the car acceleration along the direction of motion, i.e.,
 $a(t) = \frac{dv}{dt}$, what is the average acceleration of the car over the same intervals mentioned above?

Results

$$\langle v(t) \rangle_{[0,2]} = 50 \text{ km/h}$$

$$\langle v(t) \rangle_{[0,10]} = 10 \text{ km/h}$$

The distance travelled D over $\mathbf{I}=[0,2]$ and $\mathbf{I}'=[0,10]$ is 100 km, in both cases. In both cases it corresponds to the time-averaged speed, times the time-length of the corresponding interval, that is:

$$D = \langle v(t) \rangle_{[0,2]} \cdot 2 = \langle v(t) \rangle_{[0,10]} \cdot 10 = 100 \text{ km}$$

The average acceleration is zero in both cases:

$$\langle a(t) \rangle_{[0,2]} = \langle a(t) \rangle_{[0,10]} = 0$$

The reason why the average acceleration is zero is because the car is stationary both at the start and at the end of the considered time intervals. So the average effect on speed is zero, because speed is zero.

Solution

1. The average speed is, by definition of time-average:

$$\langle v(t) \rangle_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} v(t) dt$$

Substituting the speed function and considering $\mathbf{I} = [0, 2]$, we get:

$$\langle 100 \cdot \Lambda(t-1) \rangle_{[0, 2]} = \frac{1}{2-0} \int_0^2 100 \cdot \Lambda(t-1) dt = 50 \int_0^2 \Lambda(t-1) dt$$

By visual inspection, we see that:

$$\int_0^2 \Lambda(t-1) dt = 2 \int_0^1 \Lambda(t) dt = 2 \int_0^1 t dt = 2 \left[\frac{t^2}{2} \right]_0^1 = \left[t^2 \right]_0^1 = 1$$

Substituting, we get:

$$\langle 100 \cdot \Lambda(t-1) \rangle_{[0,2]} = 50 \int_0^2 \Lambda(t-1) dt = 50 \text{ km/h}$$

When the interval is $\mathbf{I}' = [0,10]$, we get:

$$\langle 100 \cdot \Lambda(t-1) \rangle_{[0,10]} = \frac{1}{10-0} \int_0^{10} 100 \cdot \Lambda(t-1) dt = 10 \int_0^{10} \Lambda(t-1) dt = 10 \int_0^2 \Lambda(t-1) dt$$

We now remark that:

$$\int_0^{10} \Lambda(t-1) dt = \int_0^2 \Lambda(t-1) dt = 1$$

which is obvious given that $\Lambda(t-1) = 0$ for $t \notin [0,2]$ (draw a picture to see it).

So we can write directly:

$$\langle 100 \cdot \Lambda(t-1) \rangle_{[0,10]} = 10 \int_0^{10} \Lambda(t-1) dt = 10 \text{ km/h}$$

2. The distance travelled, from Physics, is simply:

$$D = \int_{t_0}^{t_1} v(t) dt$$

So, considering either interval, one gets the same result:

$$D = \int_0^2 100 \cdot \Lambda(t-1) dt = \int_0^{10} 100 \cdot \Lambda(t-1) dt = 100 \text{ km}$$

3. If we look at the distance formula and the time-average formula, we see that they relate through the time-length of the interval, as follows:

$$D = \int_{t_0}^{t_1} v(t) dt = \langle v(t) \rangle_{[t_0, t_1]} (t_1 - t_0)$$

We can also say that if the car travelled at a constant speed $v'(t)$ equal to the average speed over the given time-intervals, that is $v'(t) = \langle v(t) \rangle_{[t_0, t_1]} \cdot 1(t)$, the travelled distance would not change:

$$D' = \int_{t_0}^{t_1} v'(t) dt = \int_{t_0}^{t_1} \langle v(t) \rangle_{[t_0, t_1]} \cdot 1(t) dt = \langle v(t) \rangle_{[t_0, t_1]} (t_1 - t_0) = D = \int_{t_0}^{t_1} v(t) dt$$

that is:

$$\int_{t_0}^{t_1} v'(t) dt = \int_{t_0}^{t_1} v(t) dt$$

4. The acceleration of the car is:

$$\begin{aligned}
 a(t) &= \frac{dv(t)}{dt} = \frac{d[100\Lambda(t-1)]}{dt} = \\
 &= 100 \left[\Pi\left(t - \frac{1}{2}\right) - \Pi\left(t - \frac{3}{2}\right) \right] = 100 [\pi(t) - \pi(t-1)]
 \end{aligned}$$

where we used the result:

$$\frac{d\Lambda(t)}{dt} = \Pi\left(t + \frac{1}{2}\right) - \Pi\left(t - \frac{1}{2}\right) = \pi(t+1) - \pi(t)$$

(prove it on your own either graphically or analytically).

The average acceleration is then:

$$\langle a(t) \rangle_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} a(t) dt$$

and considering $\mathbf{I} = [0, 2]$ we then have:

$$\begin{aligned}\langle a(t) \rangle_{[0,2]} &= \frac{1}{2-0} \int_0^2 a(t) dt = \frac{100}{2} \int_0^2 [\pi(t) - \pi(t-1)] dt \\ &= 50 \left[\int_0^2 \pi(t) dt - \int_0^2 \pi(t-1) dt \right]\end{aligned}$$

But:

$$\int_0^2 \pi(t) dt = \int_0^1 1(t) dt = 1$$

$$\int_0^2 \pi(t-1) dt = \int_1^2 1(t) dt = 1$$

$$\text{So } \langle a(t) \rangle_{[0,2]} = 0$$

$$\text{Similar calculations lead also to: } \langle a(t) \rangle_{[0,10]} = 0$$

3.5.2

A Space-X Falcon-9 rocket first-stage burns from launch $t = 0$ to first-stage cut-off, which occurs at $t = 160$ s. It imparts to the rocket an acceleration of 10 m/s^2 at $t = 0$, which then goes up linearly in time, reaching 30 m/s^2 at first-stage cut-off.

- Draw a “signal” representing the rocket acceleration over the interval $\mathbf{I} = [0, 160] \text{ s}$
- Calculate analytically and draw a “signal” representing the rocket speed over the interval $\mathbf{I} = [0, 160] \text{ s}$
- Calculate analytically and draw a “signal” representing the rocket altitude over the interval $\mathbf{I} = [0, 160] \text{ s}$. (Assume that the rocket goes straight up.)

- Find the average acceleration and the average speed of the rocket over the interval $[0,160]$ s
- Can you solve the problem in Matlab? Can you plot the diagram of altitude vs. time?

[Disclaimer: numbers are arbitrary...]

The following Matlab code contains all the **answers** in analytic form and also calculates them by numerical integration.

```
clear all; close all;           % initial clean up
dt=0.1;                         % time-step for numerical integration (s)
t=0:dt:160;                    % time-array for numerical integration (s)
a=10+t/8;                      % acceleration (m/s^2) vs time (s)
figure;plot(t,a);              % plotting acceleration (m/s^2) vs time (s)
title('acceleration vs time');
ylabel('m/s^2');xlabel('t');grid on;
pause;                         % waiting for keystroke to go on
%
for n=1:numel(t)               % numerical integration to obtain speed
```

```

if n==1
    v(1)=0;
else
    % trapezoidal integration rule
    v(n)=v(n-1)+1/2*(a(n-1)+a(n))*dt;
end;
end;
figure;plot(t,v);hold on; % plotting speed (m/s) vs time (s)
title('speed vs time');
ylabel('m/s');xlabel('t');grid on;
pause; % waiting for keystroke to go on
%
va=10*t+t.^2/16; % analytical formula for speed
plot(t,va,'r--','LineWidth',2);hold off; % plotting it too, for comparison,
% red dashed
pause; % waiting for keystroke to go on
%
for n=1:numel(t) % numerical integration to obtain height
    if n==1
        h(1)=0;
    else
        % trapezoidal integration rule
        h(n)=h(n-1)+1/2*(v(n-1)+v(n))*dt;
    end;
end;
figure;plot(t,h);hold on; % plotting height (m) vs time (s)
title('altitude vs time');
ylabel('m');xlabel('t');grid on;
pause; % waiting for keystroke to go on
%
ha=5*t.^2+t.^3/48; % analytical formula for height
plot(t,ha,'r--','LineWidth',2);hold off; % plotting it too, for comparison,

```

```

                                % red dashed
pause;                          % waiting for keystroke to go on
%
% time-averaged acceleration (m/s^2)
display(['time-averaged acceleration (m/s^2): ', num2str(1/160*integral(@(t) 10+t/8, 0,
160))])
% time-averaged speed (m/s)
display(['time-averaged speed (m/s): ', num2str(1/160*integral(@(t) 10*t+t.^2/16, 0, 160))])
% this can also be done analytically,
% with the following exact results:
% average acceleration 20 (m/s^2)
% average speed (4000/3) (m/s)

```

3.5.3

Consider the signal $s(t) = \frac{1}{3} \sin^2(10\pi \cdot t)$.

- Find its time-average over the interval $[-5, 5]$ and then over the whole of \mathbb{R} .

- Find its time-averaged power over the interval $[-5, 5]$ and then over the whole of \mathbb{R} .

Hint:

Use the trigonometric formulas for squares, repeatedly if needed:

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Answers:

1/6 , 1/6

1/24, 1/24

Matlab code for the finite-interval cases:

```
% time-average  
1/10*integral(@(t) 1/3*(sin(10*pi*t).^2) , -5, 5)
```

```
% time-averaged power  
1/10*integral(@(t) (1/3*(sin(10*pi*t).^2)).^2 , -5, 5)
```

You can also use symbolics in Matlab. Check it out:

```
sims t;  
1/10*int(1/3*(sin(10*pi*t).^2), -5, 5)
```

3.5.4

Consider the signal $s(t) = \cos(4\pi f_0 t) \cdot \Pi_{T_0}(t)$, with $f_0 = 1/T_0$. Find its energy and average power over the interval $[-T/2, T/2]$, assuming $T \geq T_0$. Then find its energy and average power over the whole of \mathbb{R} . Discuss what class of signals $s(t)$ belongs to, regarding energy and average power, assuming first a finite time-interval and then the whole of \mathbb{R} .

Answers:

$$\mathcal{E}_{\mathbf{I}} \{s(t)\} = \frac{T_0}{2}, \quad P_{\mathbf{I}} \{s(t)\} = \frac{T_0}{2T}$$

$$\mathcal{E}_{\mathbb{R}} \{s(t)\} = \frac{T_0}{2}, \quad P_{\mathbb{R}} \{s(t)\} = 0$$

The signal $s(t)$ is finite-energy and finite time-averaged power both over $[-T/2, T/2]$ and \mathbb{R} .

3.5.5

Consider the signal $u(t)$. Find its energy and average power over the interval $[-T/2, T/2]$. Then find its energy and average power over the whole of \mathbb{R} . Discuss what class of signals $u(t)$ belongs to, regarding energy and average power, assuming first a finite time-interval and then the whole of \mathbb{R} .

Answers:

$$\mathcal{E}_I \{u(t)\} = \frac{T}{2}, \quad P_I \{u(t)\} = \frac{1}{2}$$

$$\mathcal{E}_{\mathbb{R}} \{u(t)\} = \infty, \quad P_{\mathbb{R}} \{u(t)\} = \frac{1}{2}$$

The signal $u(t)$ is finite-energy and finite time-averaged power over $[-T/2, T/2]$. It is infinite-energy and finite time-averaged power over \mathbb{R} .

3.5.6

The speed of a car is described by the following signal $v(t)$:

$$v(t) = 50 \cdot \Lambda_2(t - 2)$$

where the units of $v(t)$ are km/h and t is in hours. Answer the following questions.

- What is the average speed of the car over the interval $\mathbf{I} = [0, 4]$ hours?
- What is the average speed of the car over the interval $\mathbf{I} = [0, 12]$ hours?
- What is the average acceleration of the car over the interval $\mathbf{I} = [0, 4]$?

Reminder: acceleration is $a(t) = \frac{dv}{dt}$.

- What is the average kinetic energy of the car over the interval $\mathbf{I} = [0, 4]$ knowing that the car mass is 1000 kg? Reminder: kinetic energy is

$$e_{\text{kin}}(t) = \frac{1}{2} m \cdot v^2(t) \text{ .}$$

3.5.7

Consider the signals:

$$s(t) = \frac{1}{3} \sin(\pi \cdot t)$$

$$q(t) = \frac{1}{2} \sin^2(2\pi \cdot t)$$

$$w(t) = 2u(t)e^{-at} \quad , \quad a \in \mathbb{R}$$

1. Find their average value over the interval $[-5, 5]$
2. Find their average value over the whole of \mathbb{R} .
3. Discuss the result of the average of the signal $w(t)$ over the whole of \mathbb{R} , when $a < 0$, $a > 0$, or $a = 0$.

3.5.8

Consider the signal $s(t) = \cos(2\pi f_0 t) \cdot u(t)$.

1. Find its energy and average power over the interval $[-T/2, T/2]$.
2. Find its energy and average power over the interval $[0, T]$.
3. Find its energy over the whole of \mathbb{R} .
4. Find its average power over the whole of \mathbb{R} .
5. What class of signals does $s(t)$ belong to, regarding energy and average power?

3.5.9

Consider the signal $s(t) = \cos(4\pi f_0 t) \cdot \Pi_{T_0}(t)$, with $f_0 = 1/T_0$.

Consider the time-interval $\mathbf{I} = [-T/2, T/2]$, with $T \geq T_0$.

1. Draw the signal $s(t)$ over the interval \mathbf{I} .
2. Find the energy and average power of $s(t)$ over the interval \mathbf{I} .
3. What class of signals does $s(t)$ belong to, regarding energy and average power, over the interval \mathbf{I} ?
4. Find the energy and average power of $s(t)$, if time spans the whole of \mathbb{R} .

5. What class of signals does $s(t)$ belong to, regarding energy and average power, if time spans the whole of \mathbb{R} ?

3.5.10

Consider the signals:

$$q(t) = \sqrt{2} \cdot e^{j10\pi \cdot t}$$

$$s(t) = \frac{1}{3} \sin(10\pi \cdot t)$$

$$w(t) = \frac{1}{2} u(t) e^{-at} \quad , \quad a > 0$$

1. Find their energy over the interval $[-5, 5]$, and then over the whole of \mathbb{R} .
2. Find their average power over the interval $[-5, 5]$, and then over the whole of \mathbb{R} .
3. Based on the above results, what class of signals do $q(t)$, $s(t)$ and $w(t)$ belong to, regarding energy and average power, if time spans the whole of \mathbb{R} ?

3.5.11

Find the average power over the interval $\mathbf{I} = [-1, 1]$ and over \mathbb{R} of the signals:

- $\Lambda(t)$
- $e^{j2\pi f_0 t}$
- $\cos(2\pi t) + \cos(4\pi t)$

Based on the above results, what class of signals do these signals belong to, regarding energy and average power, if time spans the whole of \mathbb{R} ?

3.5.12 **optional** “Challenge question”

An electrical resistance is used to warm up 100 liters of water in an electric boiler. The electrical power is supplied to the resistance in pulses, as follows:

$$P(t) = P_{\text{peak}} \cdot \sum_{n=0}^N \pi_{\tau}(t - n \cdot T)$$

We assume that such power is completely and immediately transmitted to the water.

Knowing that the thermal capacity of water is about 4000 Joules per liter per degree centigrade [$J/(l \cdot C^\circ)$], and that $T = 1$ seconds, $\tau = 0.1$ seconds, $N = 1000$ and $P_{\text{peak}} = 1$ kW, find:

- the *average* power dissipated by the resistance over the time interval $[0, N \cdot T]$
- the water temperature at time $N \cdot T$
- the signal representing the temperature of the water over the time interval $[0, 5T]$, assuming that the water temperature is zero C° at $t = 0$ (a plot is sufficient)

- the *average* derivative of the water temperature over the time $[0, N \cdot T]$
- the value of a constant dissipated power that would provide the same water temperature rise over the same amount of time