12 Random Processes

12.1 Introduction

Random processes (RPs) are essentially random signals.

More specifically, similar to the case of RVs, a random process consists of a random experiment. Differently from a RV, the result of the random experiment is not a single number, but a complete signal. In other words, every time the random experiment is executed, a whole signal is obtained.

Note that the result itself of the random experiment is no longer "random". When a random experiment is performed for a RV, the result is a completely defined number: a die is cast and the result is, say, "5", a fully defined number. With RPs, when the random experiment is performed, the result is a fully defined signal, such as "$\cos(2\pi t)$".

The deterministic signals that are the results of the random experiment have a specific name: they are called ***realizations*** of the RP. The concept of realization of a RP is of the greatest importance and it is required the all students know what it means.

A straightforward, elementary example of a RP which will help clarify the above concepts is the following:

$$X(t) = \sum_{n=0}^{2} \alpha_n \, \pi_T(t - nT)$$

Eq. 12-1

where the α_n are statistically independent RVs, that can take on the values 0 and 1 with equal probability $1/2$. Notice the *capitalized letter* X to denote a RP.

In the case of RVs, a symbol such as ξ was *not* a number, but a complex object encompassing a space of events (i.e., the set of all possible results $\mathbb{R}$) and an associated random experiment. Likewise, in the case of RPs, $X(t)$ is *not* a signal, but a complex object consisting of a set of events, i.e., the set of all possible signals (or *realizations*), together with an associated random experiment that generates them.

In the example Eq. 12-1, all the possible realizations associated to $X(t)$ are the following 8 signals:

$$
\begin{aligned}
X\left(t;s_1\right) &= 0(t) \\
X\left(t;s_2\right) &= \pi_T\left(t\right) \\
X\left(t;s_3\right) &= \pi_T\left(t-T\right) \\
X\left(t;s_4\right) &= \pi_T\left(t-2T\right) \\
X\left(t;s_5\right) &= \pi_T\left(t\right)+\pi_T\left(t-T\right) \\
X\left(t;s_6\right) &= \pi_T\left(t\right)+\pi_T\left(t-2T\right) \\
X\left(t;s_7\right) &= \pi_T\left(t-T\right)+\pi_T\left(t-2T\right) \\
X\left(t;s_8\right) &= \pi_T\left(t\right)+\pi_T\left(t-T\right)+\pi_T\left(t-2T\right)
\end{aligned}
$$

Eq. 12-2

In other words, the *space of events* Ω, that is, the set of all possible results of the random experiment, which we call S for a RP, contains exactly 8 realizations. We labeled them $s_1 \ldots s_8$ and used this label explicitly to mark each realization.

Formally:

$$S = \left\{ X\left(t; s_n\right) \right\}_{n=1}^{8}$$

where each one of the realizations $X\left(t; s_n\right)$ is a completely defined deterministic signal, as shown in Eq. 12-2.

Each one of the realizations has a probability attached to it, called $P\left(X\left(t; s_n\right)\right)$. The event space S together with the associated set of probabilities P_S:

$$P_S = \left\{ P\left(X\left(t; s_n\right)\right) \right\}_{n=1}^{8}$$

completely define the random experiment and, therefore, the RP.

In this simple example, the probabilities in P_S are induced by the statistical properties of the RVs $\alpha_0, \alpha_1, \alpha_2$. In fact, the random experiment for this RP is equivalent to performing three random experiments, one for each of the RVs $\alpha_0, \alpha_1, \alpha_2$. Clearly, each resulting $X(t; s_n)$ has the same probability $P(s_n) = 1/8$.

Finally, note that in this example the $X(t; s_n)$ are finite-energy signals and the RP is made up of realizations that are all finite-energy signals. However, in general, the realizations of a generic RP can be any type of signal, including finite-average power signals.

For example, Eq. 12-1 can be made to generate a process whose realizations are finite-average-power signals, by extending the summation to infinity. We then get an instance of a so-called *data transmission RP*:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \pi_T(t - nT)$$

Eq. 12-3

where the event space is:

$$S = \left\{ X(t; s_n) \right\}_{n=-\infty}^{\infty}$$

In this case, each $X(t; s_n)$ is a finite-average-power signal.

Incidentally, a process like Eq. 12-1 is a more realistic model of a physical data transmission RP than Eq. 12-3, because in reality all data transmission signals are finite-extension (they have a start-time and an end-time). However, in the analysis of transmission systems, the idealized model of Eq. 12-3 is more often used, because it allows to eliminate some effects of the "borders" (beginning and end) of the signal that make statistical calculations more difficult. The infinite-extension

data transmission RP is therefore a convenient *tool* for the analysis or data transmission processes although, in itself, it contains unrealistic aspects (much as Dirac's delta is a convenient idealization of a very narrow pulse, though it clearly cannot exist in practice).

The examples of Eq. 12-1 and Eq. 12-3 belong to a subset of all possible RP, whereby one can actually provide a semi-deterministic structure to the process. In other words, part of the defining formulas are deterministic functions. However, in general, processes cannot be written down in terms of deterministic functions, not even partially. For instance, most noise processes are denoted simply by a symbol $X(t)$ because it is not possible to provide any further structure to them. Their description is only statistical.

In the next section we will formalize the distinction between these two main classes of RPs.

12.2 Quasi-Defined Random Processes

A quasi-defined random process is such that:

$$X(t) = g(t; \xi_1, \xi_2, ..., \xi_n, ...)$$

where $g(t; x_1, x_2, ..., x_n, ...)$ *is a deterministic function of both time and of a set of parameters* $\{x_n\}$*, and the* ξ_n*'s are RVs.*

The definition makes no assumption as to the statistical dependence or independence of the ξ_n. To provide a full statistical characterization of such process it is necessary, in general, to have knowledge of the overall joint pdf:

$$f_{\xi_1, \xi_2, ..., \xi_n, ...}(x_1, x_2, ..., x_n, ...)$$

12.2.1.1 Examples

Other simple examples of quasi-defined processes, besides the data transmission process shown in the previous section, are:

$$X(t) = \sin(2\pi\ f_0\, t + \xi),$$
$$X(t) = \sin(2\pi\, \xi\, t),$$
$$X(t) = \Pi(t - \xi),$$
$$X(t) = \Pi_{\xi}(t).$$

where, in all cases, the RV is only one, i.e., ξ. The number of RV's involved can however be any. In this example it is three:

$$X(t) = \xi \sin(2\pi\, \psi\, t + \varphi)$$

namely the amplitude ξ, the frequency ψ and the phase φ of a sine function. Likewise in this other example, the RP is generated by three RVs:

$$X(t) = \rho \cdot \pi_{\xi}(t - \eta)$$

where ξ sets the duration, η sets the delay and ρ sets the amplitude.

An interesting example of a quasi-defined RP is the *random periodic process*:

$$X(t) = \sqrt{f_0} \sum_{n=-\infty}^{+\infty} \xi_n e^{j2\pi n f_0 t}$$

Eq. 12-4

This process has the form of a Fourier series and therefore it generates a periodic signal with period $T = 1/f_0$. Every time the random experiment is carried out, the ξ_n's generate a different set of coefficients for the Fourier series and, therefore, a

different periodic realization $X\left(t;s_k\right)$.

Note that the coefficient of a Fourier series are, in general, complex, so the ξ_n's should be understood as *complex* random variables. However, to avoid dealing with complex RV's, it is possible to define:

$$\xi_n = \rho_n + j\gamma_n$$

where both ρ_n and γ_n are real RV's.

Note also that it is possible to constrain $X\left(t\right) \in \mathbb{R}$, according to the properties of Fourier series, simply by imposing that $\xi_{-n} = \xi_n^* = \rho_n - j\gamma_n$. In fact, with the constraint $\xi_{-n} = \xi_n^*$, a *real* process similar to Eq. 12-4 can be re-written as:

$$
\begin{aligned}
X(t) &= \sum_{n=-\infty}^{\infty} \xi_n e^{j2\pi n f_0 t} \\
&= \sqrt{f_0}\left[\xi_0 + \sum_{n=-\infty}^{-1} \xi_n e^{j2\pi n f_0 t} + \sum_{n=1}^{\infty} \xi_n e^{j2\pi n f_0 t} \right] \\
&= \sqrt{f_0}\left[\xi_0 + \sum_{n=1}^{\infty} \xi_{-n} e^{-j2\pi n f_0 t} + \sum_{n=1}^{\infty} \xi_n e^{j2\pi n f_0 t} \right] \\
&= \sqrt{f_0}\left[\xi_0 + \sum_{n=1}^{\infty} \left(\xi_n^* e^{-j2\pi n f_0 t} + \xi_n e^{j2\pi n f_0 t} \right) \right] \\
&= \sqrt{f_0}\xi_0 + 2\sqrt{f_0} \sum_{n=1}^{\infty} \mathrm{Re}\left\{ \xi_n e^{j2\pi n f_0 t} \right\}
\end{aligned}
$$

All the processes that cannot be given a description in the form of a quasi-defined process make up the remainder of all possible processes, which we call “generic” processes.

For quasi-defined processes, their statistical characterization can be given in terms of the joint pdf of the RVs appearing in their formulas.

For generic processes, we clearly need a different approach, which is introduced in the next section. Such different approach can in fact be used for any RP and so it could also be used as an alternative statistical description for quasi-defined processes as well.

12.2.2 Statistical description of generic processes

As mentioned, a RP can be viewed as a collection of realizations, with an associated random experiment producing them. It is however quite difficult to statistically describe a RP in terms of its realizations.

Instead, one can think of characterizing the RP by "sampling" it. Essentially, rather than looking at entire realizations, one just concentrates on a single instant in

time $t = t_o$ and takes a "sample" of the process there. Formally, a quantity $X(t_0)$ is found. The nature of this quantity is clarified if we assume to repeatedly carry out the random experiment. This situation is depicted in Fig. 12-1, where a vertical line is drawn at $t = t_o$. Every time the experiment is run, a different realization $X(t; s_n)$ is generated. At $t = t_o$ it takes on a value $X(t_0; s_n)$ which, in general, will be a different *number* for each realization.

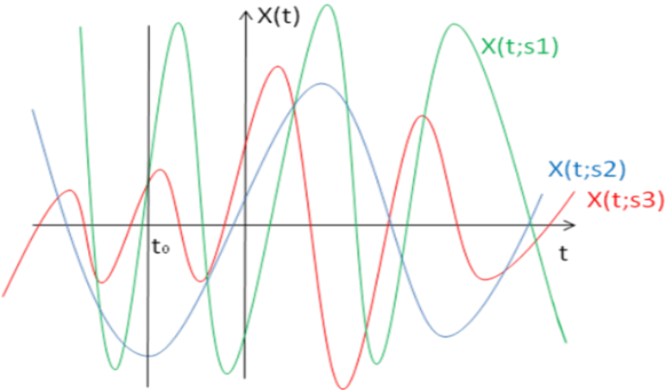


Fig. 12-1: Three realizations of a generic random process

So, in practice, the object "$X(t_0)$" generates a different number every time the random experiment is run; but this is exactly the definition of a RV and therefore we have found this important result:

given a random process $X(t)$, then $X(t_0)$ is a random variable.

This key result suggests a totally different way of defining a random process, which completely disregards the concept of "signals" or "realizations":

a random process is a set of random variables identified through a continuous index " t *".*

In other words, we could write: $X(t) \equiv \{\xi_t\}$, where $\{\xi_t\}$ is a set of RV's identified through the index t. Every time we choose a specific value t_o for the index t, we are then looking at the single RV $\xi_{t_0} = X(t_0)$.

This clearly shows the double-sided nature of a RP $X(t)$:

- a collection of realizations with an associated random experiment generating them
- a collection of RVs with a continuous index "t" to identify them.

The concept of $X(t)$ as a collection of RV is of extreme importance to obtain a general formal statistical description of a RP. Since $X(t_0)$ is a RV, then we can directly borrow all the statistical tools of RVs to describe it. In particular, a single RV is fully characterized by its pdf. Therefore, given the RV:

$$\xi_{t_0} = X(t_0)$$

the knowledge of its pdf $f_{\xi_{t_0}}(z)$ would be enough to characterize it. Changing time, we get another RV: , which, in general, may have a different pdf $f_{\xi_{t_1}}(z)$. And so

on, for all times. If we know the pdf at all times, we can leave t generic and write:

$$\xi_t = X(t) \quad \rightarrow \quad f_{\xi_t}(z)$$

This notation for such pdf is cumbersome and is typically replaced by the following:

$$f_{\xi_t}(z) \quad \Rightarrow \quad f_X(z;t)$$

Eq. 12-5

where the subscript X refers to the process and time is explicitly shown as a parameter. In addition, the dummy variable that labels all possible events in the space of events $\mathbb{R}$ is typically chosen so that it coincides with the letter used for the RP, but lowercase: $f_X(x;t)$.

12.2.3 Problems

12.2.3.1 problem 1

Consider the following pdf of a random process:

$$f_X(x;t) = \frac{1}{|t|}\Pi_{|t|}(x)$$

It is a uniform distribution over the interval:

$$\left[-|t|/2\,,\,|t|/2\right]$$

So, it clearly changes over time (it gets wider), so that $X(t_1)$ has a different pdf than $X(t_2)$, for $|t_1| \neq |t_2|$.

On your own, do the following.

- Draw $f_X(x;t)$ for $t=1$, $t=-2$, $t=0$. For $t=0$ in particular, what happens?
- Draw a few realizations which are compatible with the process pdf $f_X(x;t)$.

12.2.3.2 problem 2

Let us then go back to the example of the quasi-defined process of data transmission:

$$X(t)=\sum_{n=-\infty}^{\infty}\alpha_n\,\pi_T(t-nT)$$

where the α_n's are discrete, statistically independent RV's, that can take values 0 and 1 with equal probability $\frac{1}{2}$.

Done in class, find the process pdf $f_X(x;t)$.

Result:

$f_X(x;t) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1) \quad , \quad t \neq nT$

$f_X(x;t) = \frac{1}{4}\delta(x) + \frac{1}{2}\delta(x-1/2) + \frac{1}{4}\delta(x-1) \quad , \quad t = nT$

12.2.3.3 problem 3

Let the RP $X(t)$ be defined as:

$$X(t) = \pi_{\xi}(t).$$

Let's assume that ξ is uniformly distributed over the interval $[0,T]$, that is: $f_{\xi}(z) = \frac{1}{T}\pi_T(z)$. We want to find the process pdf $f_X(x;t)$.

First off, we ask ourselves what values can be observed when sampling the process at a specific time t_s. It is immediately evident that $X(t_s) = \pi_\xi(t_s)$ can only be 0, 1, or ½, since the generic signal $\pi_a(t)$ can only take on these three values. Therefore, the pdf of the process will have the form:

$$f_X(x;t_s) = p_0\delta(x) + p_{1/2}\delta(x-1/2) + p_1\delta(x-1)$$

where p_0, $p_{1/2}$ and p_1 depend on the parameter t_s. We then need to find the values of these three quantities, which represent the probabilities of observing the values reported as subscripts. Note that of course: $p_0 + p_{1/2} + p_1 = 1$.

Considering $p_{1/2}$, we have:

$$p_{1/2} = P\left(\left\{X(t_s) = 1/2\right\}\right) = P\left(\left\{\pi_\xi(t_s) = 1/2\right\}\right)$$

The event $\left\{\pi_{\xi}\left(t_s\right)=1/2\right\}$ occurs in two specific cases:

1) if $t_s=0$ it occurs for *any* value of ξ, that is, it occurs with probability 1
2) if $\xi=t_s$; however, the corresponding $p_{1/2}=P\left(\left\{\xi=t_s\right\}\right)$ is:

$$P\left(\left\{\xi=t_s\right\}\right)=\int_{t_s}^{t_s} f_{\xi}\left(z\right)dz=0 \text{ , for any value of } t_s$$

So, we can conclude that the only time at which $p_{1/2}$ is not zero is for $t_s=0$, in which case its value is 1 and therefore:

$$f_X\left(x;t_s\right)=\delta\left(x-1/2\right) \quad , \quad t_s=0$$

At all other times, that is for $t_s\neq 0$, $p_{1/2}=0$ and therefore the pdf will be of the form:

$$f_X(x;t_s) = p_0\delta(x) + p_1\delta(x-1) \quad , \quad t_s \neq 0$$

We then look at:

$$p_1 = P\left(\left\{X(t_s) = 1\right\}\right) = P\left(\left\{\pi_\xi(t_s) = 1\right\}\right), \; t_s \neq 0$$

The function $\pi_\xi(t_s)$ is 1 if $t_s \in]0, \xi[$. Since ξ can only be positive, all negative values of t_s fall outside of the interval $]0, \xi[$. Also, all values of t_s that are $\geq T$ fall outside of $]0, \xi[$ because the maximum possible value of ξ is indeed T .

As a result, $p_1 = 0$, for both $t_s < 0$ and for $t_s \geq T$. Simultaneously, it must be $p_0 = 1$, and therefore:

$$f_X(x;t_s) = \delta(x) \quad t_s < 0 \, , t_s \geq T$$

To summarize, so far, we have fully characterized $f_X(x;t_s)$ for $t_s \le 0$ and $t_s \ge 1$ We still need to characterize it when $t_s \in]0,T[$.

Considering p_1 for $t_s \in]0,T[$, we have:

$$p_1 = P\left(\left\{X(t_s)=1\right\}\right) = P\left(\left\{\pi_\xi(t_s)=1\right\}\right) = P\left(\left\{\xi > t_s\right\}\right)$$
$$= \int_{t_s}^{\infty} f_\xi(z)\,dz = \frac{1}{T}\int_{t_s}^{T} \pi_T(z)\,dz = \frac{T-t_s}{T} = 1-\frac{t_s}{T} \quad , \quad t_s \in]0,T[$$

As a result, we also have:

$$p_0 = 1-p_1 = 1-\left(1-\frac{t_s}{T}\right) = \frac{t_s}{T} \quad , \quad t_s \in]0,T[$$

This fully characterizes the pdf $f_X(x;t_s)$ for all observation times t_s. We now

remark that t_s is a completely generic time-instant, so we can simply call it t. Pulling all results together, we have:

$$f_X(x;t)=\begin{cases}\delta(x) & t<0\\ \delta(x-1/2) & t=0\\ \frac{t}{T}\delta(x)+\left(1-\frac{t}{T}\right)\delta(x-1) & t\in\left]0,T\right[\\ \delta(x) & t\geq T\end{cases}$$

On your own: plot $p_0(t)$ and $p_1(t)$. Do the plots make sense given your intuition of what the process $X(t)$ is like?

12.2.3.4 problem 4

On your own, find the pdf $f_X(x;t)$ of the following quasi-defined process:

$$X(t) = \Pi_{\xi}(t)$$

Assume that ξ is uniformly distributed over the interval $[0,2]$ that is $f_{\xi}(z) = \frac{1}{2}\pi_2(z)$.

***Answer*:** $f_X(x;t) = \left[1-\Lambda(t)\right]\delta(x) + \Lambda(t)\delta(x-1)$

Hint: proceed in a manner similar to the previous problem.

12.2.3.5 problem 5

Find the process pdf $f_X(x;t)$ for the following quasi-defined process:

$$X(t) = \pi_T(t-\xi)$$

assuming that ξ is uniformly distributed over the interval $[0,T]$.

Solution:

First, the pdf of ξ is: $f_{\xi}(z) = \frac{1}{T}\Pi_T\left(z - \frac{T}{2}\right) = \frac{1}{T}\pi_T(z)$.

Then we remark that, given any possible specific choice of the observation time t_s, the value of $X(t_s) = \pi_T(t_s - \xi)$ can only be either:

$$1 \text{ if } \quad (t_s - \xi) \in \text{ext}\{\pi_T\} = [0, T]$$
$$0 \text{ if } \quad (t_s - \xi) \notin \text{ext}\{\pi_T\} = [0, T]$$

So $X(t_s)$ is a discrete RV which can only take on the values 0 and 1. Its pdf will therefore be of the form:

$$f_X(x; t_s) = p_0\delta(x) + p_1\delta(x-1)$$

Eq. 12-6

The numbers p_0 and p_1 are the probabilities of $X(t_s)$ taking on the values 0 and 1. Hence:

$$p_1 = P\left(\left\{\left(t_s - \xi\right) \in [0,T]\right\}\right)$$
$$p_0 = P\left(\left\{\left(t_s - \xi\right) \notin [0,T]\right\}\right)$$

Considering the first one, it can be re-written as:

$$p_1 = P\left(\left\{\left(t_s - \xi\right) \in [0,T]\right\}\right) = P\left(\left\{0 < t_s - \xi < T\right\}\right)$$
$$= P\left(\left\{t_s - T < \xi < t_s\right\}\right) = P\left(\left\{\xi \in [t_s - T, t_s]\right\}\right)$$

The probability of a RV falling within an interval is immediately found as the integral of its density over the interval:

$$p_1 = \int_{t_s - T}^{t_s} f_\xi(z)\,dz = \frac{1}{T}\int_{t_s - T}^{t_s} \pi_T(z)\,dz = \Lambda_T(t_s - T)$$

where the final result is found through standard techniques (plot the integrand function and the integration range vs. the integration variable z and discuss their superposition as a function of the parameter t_s).

Having found p_1, the value of p_0 can be found directly by remarking that it must be: $p_0 = 1 - p_1$, because the pdf Eq. 12-6 must sum to 1. Therefore:

$$p_0 = 1 - p_1 = 1 - \Lambda_T(t_s - T)$$

The resulting pdf of $X(t_s)$ is then:

$$f_X(x;t_s) = \left[1 - \Lambda_T(t_s - T)\right]\delta(x) + \Lambda_T(t_s - T)\,\delta(x-1)$$

We then recognize that this expression is valid for any generic time t and therefore we have characterized $f_X(x;t)$.

On your own: redo the above problem assuming that ξ is uniformly distributed over the interval $[0, 2T]$.

12.2.4 Is the single-time pdf enough?

Even though $f_X(x;t)$ provides a statistical description of the behavior of each *instant* in time of $X(t)$, one by one, as isolated RVs, this unfortunately falls short of fully characterizing the overall process $X(t)$. A striking illustration of this is provided by the following example.

12.2.4.1 Example

Let $X_{\mathrm{SI}}(t)$ and $X_{\mathrm{CP}}(t)$ be two RPs. Let us assume that:

$$f_{X_{\mathrm{SI}}}(x;t) = f_{X_{\mathrm{CP}}}(x;t) = f_X(x),$$

that is, the RV's extracted from either $X_{\mathrm{SI}}(t)$ or $X_{\mathrm{CP}}(t)$ at any time have the same pdf $f_X(x)$.

If this information was enough to describe a RP, we could then conclude that $X_{\mathrm{SI}}(t)$ and $X_{\mathrm{CP}}(t)$ are (statistically) identical, i.e., they are the same process.

This information is however *not* enough. To show this, we further assume that:

- $X_{\mathrm{SI}}(t)$: the RV's corresponding to any two different times are all

statistically independent (from which the subscript SI) of one another;

- $X_{\mathrm{CP}}(t)$: the RV's corresponding to any two different times are completely dependent and in fact take on the exact same value: $X_{\mathrm{CP}}(t_1) = X_{\mathrm{CP}}(t_2),\ \forall t_1, t_2 \in \mathbb{R}$.
-

Note that these further assumptions *do not change* the process pdf's, which remain identical for $X_{\mathrm{SI}}(t)$ and $X_{\mathrm{CP}}(t)$. However, the two processes turn out to be dramatically different.

In fact, according to these assumptions, $X_{\mathrm{CP}}(t)$ can be written as: $X_{\mathrm{CP}}(t) = \xi \cdot 1(t)$ In other words, this process turns out to be a quasi-defined process. Moreover, each realization is just a *constant* for $-\infty < t < \infty$ (from which the subscript CP that stands for "constant process"). Its constant value changes depending on the result of the random experiment:

$$X_{\mathrm{CP}}(t;s_k) = x_k \cdot 1(t)$$

where x_k is one of the possible values that the RV ξ can take on (specifically, the value corresponding to the realization whose label is s_k).

The nature of $X_{\mathrm{SI}}(t)$ is instead completely different. It can be shown that the realizations $X_{\mathrm{SI}}(t;s_k)$ of $X_{\mathrm{SI}}(t)$ are very strange functions of time: they are *everywhere discontinuous*. It is in fact *impossible to draw them* and, ultimately, they do not correspond to any physical signal. *[Discussion of why, done in class.]*

This example shows with force that just the knowledge of $f_X(x;t)$ is not enough to fully characterize a process. Looking at this example one would be tempted to guess that perhaps the knowledge of the *correlation coefficient, or the covariance,* between RVs extracted at different times would be useful. However, though useful, in general it could still not be enough, as discussed in the next section.

12.2.5 Higher order pdf's of a random process

The pdf of a single time-sample of a process $X(t)$*:*

$$f_X(x;t)$$

is called first-order probability density function of the process $X(t)$.

This pdf characterizes a single "sample" of the RP, taken at a certain time t.

If instead we look at *two* time-samples of $X(t)$, taken at times t_1 and t_2, we actually extract two RVs:

$$X(t_1) = \xi_{t_1} \quad , \quad X(t_2) = \xi_{t_2}$$

We can then jointly characterize them through their joint pdf $f_{\xi_{t_1}\xi_{t_2}}(x_1, x_2)$. As it was done for the first-order pdf, a change of notation is convenient and we define

the following quantity:

the second-order probability density function of the process $X(t)$ *is defined as:*

$$f_X(x_1, x_2; t_1, t_2) \triangleq f_{\xi_{t_1}\xi_{t_2}}(x_1, x_2)$$

Eq. 12-7

Further extensions are straightforward. Extracting three RVs out of the process, at times t_1, t_2, and t_3:

$$X(t_1) = \xi_{t_1}, \; X(t_2) = \xi_{t_2}, \; X(t_3) = \xi_{t_3}$$

then the third-order probability density function of $X(t)$ is defined as:

$$f_X(x_1, x_2, x_3; t_1, t_2, t_3) \triangleq f_{\xi_{t_1}\xi_{t_2}\xi_{t_3}}(x_1, x_2, x_3)$$

In general, the *n*-th order pdf of the process will be written as:

$$f_X(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n)$$

Eq. 12-8

which represents the joint pdf of the *n* RVs extracted from the process by sampling it at time-instants $t_1, t_2, ..., t_n$.

The following important results holds:

full statistical characterization of a generic random process is achieved only through the knowledge of the joint pdf's of the process ***for all orders****.*

This may be easy or difficult depending on the nature of the process. For instance, for the processes of the example given above 12.2.4.1, we do have a full statistical

characterization of both of them because we can readily write down the analytical expression of their joint pdf's for any order. However, in general, it may be very difficult to achieve such complete statistical description.

One notable and very important exception is that of Jointly Gaussian processes. We will address them in Sect. 12.3.6.

12.2.5.1 Example

We want to discuss the second-order pdf's of the two processes $X_{\mathrm{SI}}(t)$ and $X_{\mathrm{CP}}(t)$ of example 12.2.4.1. They actually are:

$$f_{X_{\mathrm{SI}}}(x_1, x_2; t_1, t_2) = f_X(x_1)\, f_X(x_2)$$

Eq. 12-9

$$f_{X_{\mathrm{CP}}}(x_1, x_2; t_1, t_2) = f_X(x_1)\delta(x_2 - x_1) = f_X(x_2)\delta(x_1 - x_2)$$

Eq. 12-10

The first result is obvious, given the assumption of statistical independence between samples taken at any two times.

To derive the second result, one needs to remember that for any pair of generic RVs the following result holds:

$$f_{\xi_1\xi_2}(x_1, x_2) = f_{\xi_1|\xi_2}(x_1 \mid x_2)\, f_{\xi_2}(x_2) = f_{\xi_2|\xi_1}(x_2 \mid x_1)\, f_{\xi_1}(x_1)$$

Using these formulas for the process $X_{\mathrm{CP}}(t)$ we get:

$$f_{X_{\mathrm{CP}}}(x_1, x_2; t_1, t_2) = f_{X_{\mathrm{CP}}}(x_1; t_1)\, f_{X_{\mathrm{CP}}(t_2)|X_{\mathrm{CP}}(t_1)}(x_2 \mid x_1; t_2, t_1)$$

Eq. 12-11

where $f_{X_{\mathrm{CP}}(t_2)|X_{\mathrm{CP}}(t_1)}(x_2 \mid x_1; t_2, t_1)$ is the conditional pdf of the random variable $X_{\mathrm{CP}}(t_2)$ given that (conditioned on, assuming that) the value of the RV $X_{\mathrm{CP}}(t_1)$ has

been found to be equal to x_1. The notation for this pdf is somewhat redundant (the two times t_1, t_2 appear twice in the formula) but we did it intentionally to make it clear that it is a conditional pdf and what it is conditioned on.

Regarding $f_{X_{\mathrm{CP}}}(x_1; t_1)$, by assumption it is:

$$f_{X_{\mathrm{CP}}}(x_1; t_1) = f_X(x_1)$$

Regarding $f_{X_{\mathrm{CP}}(t_2)|X_{\mathrm{CP}}(t_1)}(x_2 \mid x_1; t_2, t_1)$, we have to ask ourselves what the value of a realization could be at time t_2, given that its value at time t_1 was x_1. Being this the "constant" process, the value at t_2 is certainly (with probability 1) the same as that at time t_1, that is, x_1. The pdf will therefore be that of the single discrete value x_1, found with probability 1. We will simply have:

$$f_{X_{\mathrm{CP}}(t_2)|X_{\mathrm{CP}}(t_1)}(x_2 \mid x_1; t_2, t_1) = \delta(x_2 - x_1)$$

from which Eq. 12-10 is immediately found.

12.2.6 Moments and joint moments of a random process

Another extremely important class of processes for which it is possible to achieve full statistical characterization is that of Gaussian processes, which we will introduce later on.

In general, however, it may be quite difficult or practically impossible to obtain a full statistical characterization of a RP. On the other hand, significant information regarding the behavior of a process can be obtained from *certain low-order joint moments*, which we introduce in the next sections.

We recall the definition of the expectation of a RV:

$$E_{\xi}\{\xi\} \triangleq \int_{-\infty}^{\infty} x\, f_{\xi}(x)dx$$

Eq. 12-12

When we sample a RP at $t = t_o$ we get a RV and so we can apply the same definition to the RV $X(t_0)$:

$$E_X\{X(t_0)\} = \int_{-\infty}^{\infty} x\, f_X(x;t_0)dx$$

Eq. 12-13

Since this definition is valid for any single time $t_0 \in \mathbb{R}$, we can actually leave the time-instant generic and define the *mean* or *expectation of a process* as:

$$\mu_x(t) \triangleq E_X\{X(t)\} = \int_{-\infty}^{\infty} x\, f_X(x;t)dx$$

Eq. 12-14

The quantity $\mu_x(t)$ is in fact the collection of the expectations of $X(t)$ at all times t. In other words, $\mu_x(t)$ is a function of time which provides us with the statistical average of $X(t)$ at any time t.

A similar definition can be given regarding the variance of a RP. We first recall that, given a RV, the definition of its variance is:

$$\sigma_\xi^2 \triangleq E_\xi\left\{\left(\xi - \mu_\xi\right)^2\right\} = E\{\xi^2\} - \mu_\xi{}^2 =$$

$$= \int_{-\infty}^{\infty} (x-\mu_\xi)^2\, f_\xi(x)\, dx = \int_{-\infty}^{\infty} x^2\, f_\xi(x)\, dx - \mu_\xi{}^2$$

Eq. 12-15

Adjusting this definition to suit a process, we get:

$$\sigma_X^2(t) = E_X\left\{\left[X(t) - \mu_X(t)\right]^2\right\} = E_X\{X^2(t)\} - \mu_X^2(t) =$$

$$= \int_{-\infty}^{\infty}\left[x - \mu_X(t)\right]^2 f_X(x;t)\,dx = \int_{-\infty}^{\infty} x^2 f_X(x;t)\,dx - \mu_X^2(t)$$

Eq. 12-16

Expectations can also be taken jointly over RVs extracted at different times. Two of the most important statistical quantities regarding RPs are found by computing the order (1,1) joint central and non-central moments of a pair of RVs taken from a RP at two different times.

12.2.6.1 Autocorrelation and autocovariance

We assume to sample a process $X(t)$ at two time-instants t_1, t_2, thus extracting

the RVs $X(t_1), X(t_2)$. Then, by definition, their joint *non-central* moment of order (1,1) is given by:

$$E_X\left\{X(t_1)X(t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2$$

This quantity is so important that it is given a specific symbol and name.

The autocorrelation function of a RP $X(t)$ *is defined as:*

$$R_X(t_1, t_2) \triangleq E_X\left\{X(t_1)X(t_2)\right\}$$

Eq. 12-17

Note also that: $R_X(t_1, t_1) = E_X\left\{X^2(t_1)\right\}$, so $R_X(t_1, t_2)$ contains the mean square

value of the process.

Also, the joint *central* moment of order (1,1) of the two RVs $X(t_1)$, $X(t_2)$, called autocovariance, is given by:

$$E_X\left\{\left[X(t_1)-\mu_X(t_1)\right]\left[X(t_2)-\mu_X(t_2)\right]\right\} =$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x_1-\mu_X(t_1)\right]\left[x_2-\mu_X(t_2)\right] f_X(x_1,x_2;t_1,t_2)dx_1dx_2$$

Eq. 12-18

This quantity too is so important that it is given a specific symbol and name.
The autocovariance or covariance function of a RP $X(t)$ is defined as:

$$K_X(t_1,t_2) \triangleq E_X\left\{\left[X(t_1)-\mu_X(t_1)\right]\left[X(t_2)-\mu_X(t_2)\right]\right\}$$

Eq. 12-19

It is immediately seen that:

$$K_X\left(t_1,t_2\right)=R_X\left(t_1,t_2\right)-\mu_X\left(t_1\right)\mu_X\left(t_2\right)$$

Eq. 12-20

In fact:

$$\begin{aligned}
&E_X\left\{\left[X\left(t_1\right)-\mu_X\left(t_1\right)\right]\left[X\left(t_2\right)-\mu_X\left(t_2\right)\right]\right\}=\\
&=E_X\left\{\left[X\left(t_1\right)X\left(t_2\right)-\mu_X\left(t_1\right)X\left(t_2\right)-\mu_X\left(t_2\right)X\left(t_1\right)+\mu_X\left(t_1\right)\mu_X\left(t_2\right)\right]\right\}=\\
&=E_X\left\{X\left(t_1\right)X\left(t_2\right)\right\}-\mu_X\left(t_1\right)E_X\left\{X\left(t_2\right)\right\}-\mu_X\left(t_2\right)E_X\left\{X\left(t_1\right)\right\}+\mu_X\left(t_1\right)\mu_X\left(t_2\right)=\\
&=E_X\left\{X\left(t_1\right)X\left(t_2\right)\right\}-\mu_X\left(t_1\right)\mu_X\left(t_2\right)-\mu_X\left(t_2\right)\mu_X\left(t_1\right)+\mu_X\left(t_1\right)\mu_X\left(t_2\right)=\\
&=R_X\left(t_1,t_2\right)-\mu_X\left(t_1\right)\mu_X\left(t_2\right)
\end{aligned}$$

Note also that: $K_X\left(t_1,t_1\right)=E_X\left\{X^2\left(t_1\right)-{\mu_X}^2\left(t_1\right)\right\}=\sigma_X^2\left(t_1\right)$, so $K_X\left(t_1,t_2\right)$ contains the variance of the process.

Finally, it is also possible to define the *correlation coefficient* of a process. We first recall that for two RVs ξ, η it is:

$$\rho_{\xi\eta} = \frac{E_{\xi\eta}\{\xi\eta\} - \mu_\xi \mu_\eta}{\sigma_\xi \sigma_\eta}$$

Therefore, extracting two RVs from a process $X(t)$, it is:

$$\rho_X(t_1, t_2) = \frac{E_X\{X(t_1)X(t_2)\} - \mu_X(t_1)\mu_X(t_2)}{\sigma_X(t_1)\sigma_X(t_2)}$$

Eq. 12-21

But the numerator coincides with the definition of the auto-covariance so, more compactly:

$$\rho_X(t_1,t_2)=\frac{K_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)}$$

Eq. 12-22

12.2.7 Expectations of generic functions of RPs

We have so far looked at moments of RVs extracted from a process. It is possible to extend the expectation to any function of one or more RVs extracted from the process.

For a RV, we have from **Error! Reference source not found.**:

$$E_\xi\{g(\xi)\}=\int_{-\infty}^{\infty} g(x)\,f_\xi(x)dx$$

So, for a RV extracted from a process $X(t)$:

$$E_X\left\{g\left(X(t)\right)\right\} = \int_{-\infty}^{\infty} g(x)\, f_X(x;t)dx$$

Eq. 12-23

For two RVs we have from **Error! Reference source not found.**:

$$E\left\{g(\xi,\eta)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{\xi,\eta}(x,y)dxdy$$

So, for two RVs extracted from a process $X(t)$:

$$E\left\{g\left(X(t_1),X(t_2)\right)\right\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x_1,x_2)\,f_X(x_1,x_2;t_1,t_2)dx_1dx_2$$

Eq. 12-24

12.2.8 Examples

We now apply some of the statistical tools introduced in the previous sections to some examples.

12.2.8.1 The ramdom-delay rectangle

We look again at problem 12.2.3.5. We found that the pdf was:

$$f_X(x;t)=\left[1-\Lambda_T\left(t-T\right)\right]\delta\left(x\right)+\Lambda_T\left(t-T\right)\delta\left(x-1\right)$$

We can now then find the mean value of the process. By definition:

$$E\left\{X\left(t\right)\right\}=\mu_X\left(t\right)=\int_{-\infty}^{\infty}x\cdot f_X\left(x;t\right)dx$$

$$=\int_{-\infty}^{\infty}x\cdot\left\{\left[1-\Lambda_T\left(t-T\right)\right]\delta\left(x\right)+\Lambda_T\left(t-T\right)\delta\left(x-1\right)\right\}dx$$

$$=\left[1-\Lambda_T\left(t-T\right)\right]\int_{-\infty}^{\infty}x\cdot\delta\left(x\right)dx+\Lambda_T\left(t-T\right)\int_{-\infty}^{\infty}x\cdot\delta\left(x-1\right)dx=\Lambda_T\left(t-T\right)$$

To find the variance, we need the MSV:

$$\text{MSV}\left\{X\left(t\right)\right\}=E\left\{X^2\left(t\right)\right\}=\int_{-\infty}^{\infty}x^2\cdot f_X\left(x;t\right)dx$$

$$=\int_{-\infty}^{\infty}x^2\cdot\left\{\left[1-\Lambda_T\left(t-T\right)\right]\delta\left(x\right)+\Lambda_T\left(t-T\right)\delta\left(x-1\right)\right\}dx$$

$$=\left[1-\Lambda_T\left(t-T\right)\right]\int_{-\infty}^{\infty}x^2\cdot\delta\left(x\right)dx+\Lambda_T\left(t-T\right)\int_{-\infty}^{\infty}x^2\cdot\delta\left(x-1\right)dx=\Lambda_T\left(t-T\right)$$

From the above results:

$$\sigma_X^2(t) = \mathrm{MSV}\{X(t)\} - \mu_X^2(t) = E\{X^2(t)\} - \mu_X^2(t)$$
$$= \Lambda_T(t-T) - \Lambda_T^2(t-T)$$

On your own: plot the variance and try to explain its values along the time axis. In particular: why is the variance equal to zero at $t = T$? Why is it maximum at $t = T/2$ and $t = 3T/2$?

12.2.8.2 the random amplitude carrier

We now introduce the *random amplitude radio carrier process*:

$$X(t) = \xi \cdot \cos(2\pi f_0 t + \varphi_0)$$

Eq. 12-25

It is a quasi-defined process that contains the RV ξ. This RV sets the amplitude of

the cosine signal that it multiplies.

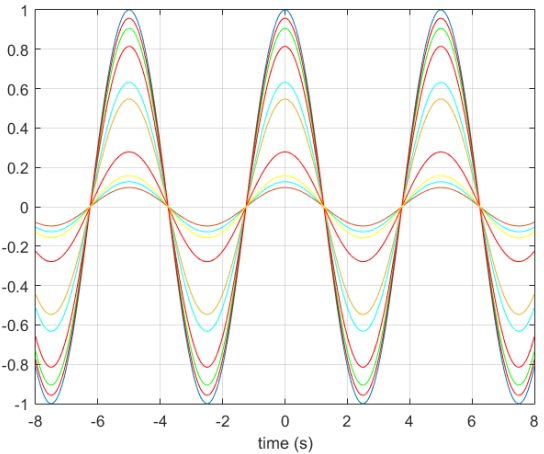


Fig. 12-2: a few realizations of $X(t) = \xi \cdot \cos(2\pi f_0 t + \phi_0)$, assuming $f_0 = 1/5$, $\varphi_0 = 0$ and $A = 1$

The quantities f_0 and φ_0 are deterministic parameters. Each realization is

therefore a fully defined signal:

$$X(t;s_n) = x_n \cos(2\pi f_0 t + \phi_0)$$

where x_n is the result of the random experiment over ξ. We then assume that ξ is a uniformly-distributed RV, in the range $[0, A]$, that is: $f_\xi(z) = \frac{1}{A}\Pi_A\left(z - \frac{A}{2}\right)$.

We want to evaluate the *mean, autocorrelation* and *auto-covariance* of this process. According to the definition of mean:

$$\mu_X(t) = E_X\{X(t)\} = \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx$$

However, we do not have the first-order pdf $f_X(x;t)$. We can then try using some of the properties of the expectation operator:

$$E_X\left\{X(t)\right\} = E_\xi\left\{\xi\cos(2\pi f_0 t + \varphi_0)\right\} =$$
$$= \cos(2\pi f_0 t + \varphi_0) E_\xi\left\{\xi\right\} = \mu_\xi \cos(2\pi f_0 t + \varphi_0)$$

As for μ_ξ, we have:

$$\mu_\xi = E_\xi\left\{\xi\right\} = \int_{-\infty}^{\infty} z\, f_\xi(z)\, dz = \tfrac{1}{A}\int_{-\infty}^{\infty} z\, \Pi_A\left(z - \tfrac{A}{2}\right) dz =$$

$$= \frac{1}{A}\int_0^A z\, dz = \left[\frac{z^2}{2A}\right]_0^A = \frac{A}{2}$$

Therefore:

$$\mu_X(t) = \tfrac{A}{2}\cos(2\pi f_0 t + \varphi_0)$$

Eq. 12-26

Regarding the autocorrelation of $X(t)$, we should evaluate:

$$R_X(t_1,t_2) = E_X\left\{X(t_1)X(t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1 x_2; t_1, t_2)dx_1 dx_2$$

However, we do not have the second-order pdf: $f_X(x_1 x_2; t_1, t_2)$. We then try using the properties of the expectation operator:

$$E_X\left\{X(t_1)X(t_2)\right\} = E_\xi\left\{\left[\xi\cos(2\pi f_0 t_1 + \varphi_0)\right]\left[\xi\cos(2\pi f_0 t_2 + \varphi_0)\right]\right\} =$$
$$= \cos(2\pi f_0 t_1 + \varphi_0)\cos(2\pi f_0 t_2 + \varphi_0)E_\xi\left\{\xi^2\right\}$$

The mean-square value of ξ is:

$$E_\xi\left\{\xi^2\right\} = \int_{-\infty}^{\infty} z^2\, f_\xi(z)\, dz = \frac{1}{A}\int_{-\infty}^{\infty} z^2\, \Pi_A\left(z - \frac{A}{2}\right) dz =$$

$$= \frac{1}{A}\int_0^A z^2\, dz = \left[\frac{z^3}{3A}\right]_0^A = \frac{A^2}{3}$$

So, in the end:

$$R_X(t_1, t_2) = \frac{A^2}{3}\cos(2\pi f_0 t_1 + \varphi_0)\cos(2\pi f_0 t_2 + \varphi_0)$$

Eq. 12-27

The autocovariance can be obtained from the autocorrelation as shown in Eq. 12-20:

$$
\begin{aligned}
K_X(t_1,t_2) &= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \\
&= \left(E_\xi\left\{\xi^2\right\} - \mu_\xi^2\right)\cos(2\pi f_0 t_1 + \varphi_0)\cos(2\pi f_0 t_2 + \varphi_0) \\
&= \sigma_\xi^2 \cos(2\pi f_0 t_1 + \varphi_0)\cos(2\pi f_0 t_2 + \varphi_0) = \\
&= \tfrac{A^2}{12}\cos(2\pi f_0 t_1 + \varphi_0)\cos(2\pi f_0 t_2 + \varphi_0)
\end{aligned}
$$

Eq. 12-28

where we have used the result that the variance of a uniformly-distributed RV is equal to the width of its rectangular pdf squared, divided by 12 (see Sect.**Error! Reference source not found.**).

From this formula we can directly find the variance:

$$\sigma_X^2(t) = K_X(t,t) = \tfrac{A^2}{12}\cos^2(2\pi f_0 t + \varphi_0)$$

Finally, the correlation coefficient is:

$$\rho_X(t_1,t_2)=\frac{K_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)}=\frac{K_X(t_1,t_2)}{\sqrt{K_X(t_1,t_1)}\sqrt{K_X(t_2,t_2)}}$$

Substituting, we find:

$$\rho_X(t_1,t_2)=\frac{\frac{A^2}{12}\cos(2\pi f_0 t_1+\varphi_0)\cos(2\pi f_0 t_2+\varphi_0)}{\sqrt{\frac{A^2}{12}\cos^2(2\pi f_0 t_1+\varphi_0)}\sqrt{\frac{A^2}{12}\cos^2(2\pi f_0 t_2+\varphi_0)}}$$

$$=\frac{\cos(2\pi f_0 t_1+\varphi_0)\cos(2\pi f_0 t_2+\varphi_0)}{\left|\cos(2\pi f_0 t_1+\varphi_0)\cos(2\pi f_0 t_2+\varphi_0)\right|}$$

$$=\operatorname{sign}\left(\cos(2\pi f_0 t_1+\varphi_0)\cos(2\pi f_0 t_2+\varphi_0)\right)$$

The above formula shows that the correlation coefficient can only be +1 or -1. On your own, can you explain why?

On your own, show that the first-order pdf of the process is the following:

$$f_X(x;t) = \frac{1}{|a|}\Pi_{|a|}\left(x - \tfrac{a}{2}\right)$$

$$a = A\cos(2\pi f_0 t + \varphi_0)$$

Or equivalently:

$$f_X(x;t) = \frac{1}{A\left|\cos(2\pi f_0 t + \varphi_0)\right|} \cdot \pi_A\left(\frac{x}{\cos(2\pi f_0 t + \varphi_0)}\right)$$

Hint: use the result on the transformation of RV shown in Sect. **Error! Reference source not found.**.

12.2.8.3 the random amplitude and random phase carrier

We look at a similar process to the previous one, except both the amplitude ξ and the phase φ are now random variables:

$$X(t) = \xi \cdot \cos(2\pi f_0 t + \varphi)$$

Eq. 12-29

The amplitude has the same pdf as before. In addition the phase is uniformly distributed over $[0, 2\pi]$. Its pdf is:

$$f_\varphi(y) = \frac{1}{2\pi} \Pi_{2\pi}(y - \pi)$$

Eq. 12-30

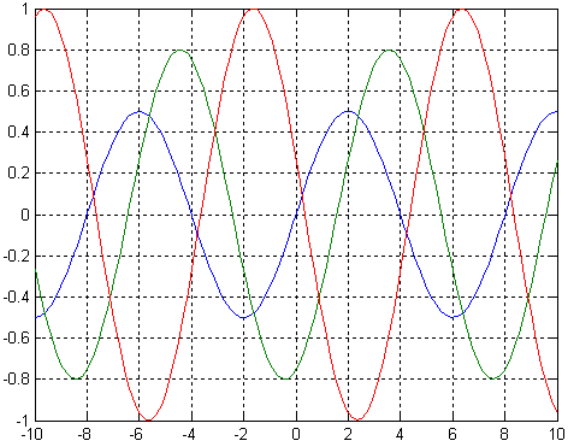


Fig. 12-3: A few realizations of the process $X(t) = \xi \cdot \cos(2\pi f_0 t + \varphi)$ assuming assuming $f_0 = 1/8$, $A = 1$ and φ uniformly distributed between 0 and 2π

In addition, we assume that the amplitude and the phase are statistically independent RVs.

According to the definition, the mean of the process is:

$$\mu_X(t) = E_X\left\{X(t)\right\} = \int_{-\infty}^{\infty} x\, f_X(x;t)\,dx$$

Once again we do not know the first-order pdf of the whole process, i.e., $f_X(x;t)$, and therefore we rearrange the expectation:

$$E_X\left\{X(t)\right\} = E_{\xi\varphi}\left\{\xi\cos(2\pi f_0 t + \varphi)\right\}$$

The function to be averaged is factorizable, that is:

$$\xi \cos(2\pi f_0 t + \varphi) = g(\xi) g(\varphi)$$

In addition, we know that ξ and φ are statistically independent. Therefore, according to the properties of the expectation operator, we have:

$$E_X \left\{ X(t) \right\} = E_{\xi\varphi} \left\{ \xi \cos(2\pi f_0 t + \varphi) \right\} =$$
$$= E_\xi \left\{ \xi \right\} E_\varphi \left\{ \cos(2\pi f_0 t + \varphi) \right\}$$

We have already calculated the average $E_\xi \left\{ \xi \right\} = \mu_\xi = \frac{A}{2}$. As for the other average:

$$E_\varphi\left\{\cos(2\pi f_0 t+\varphi)\right\}=\int_{-\infty}^{\infty}\cos(2\pi f_0 t+y)\,f_\varphi(y)dy=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(2\pi f_0 t+y)\Pi_{2\pi}(y-\pi)dy=$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}\cos(2\pi f_0 t+y)dy=\frac{1}{2\pi}\left[\sin(2\pi f_0 t+y)\right]_0^{2\pi}=$$

$$=\frac{1}{2\pi}\left[\sin(2\pi f_0 t+2\pi)-\sin(2\pi f_0 t)\right]=$$

$$=\frac{1}{2\pi}\left[\sin(2\pi f_0 t)-\sin(2\pi f_0 t)\right]=0$$

So we have:

$$\mu_X(t)=0$$

Eq. 12-31

We compute now the autocorrelation. By definition:

$$R_X(t_1,t_2) = E_X\left\{X(t_1), X(t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1 x_2; t_1, t_2) dx_1 dx_2$$

but we do not have $f_X(x_1 x_2; t_1, t_2)$. So:

$$E_X\left\{X(t_1), X(t_2)\right\} = E_{\xi\varphi}\left\{\left[\xi\cos(2\pi f_0 t_1 + \varphi)\right]\left[\xi\cos(2\pi f_0 t_2 + \varphi)\right]\right\} =$$
$$= E_{\xi\varphi}\left\{\xi^2\cos(2\pi f_0 t_1 + \varphi)\cos(2\pi f_0 t_2 + \varphi)\right\}$$

Due again to the independence of ξ and φ, and due to the factorization of the function within the expectation, we can write:

$$E_X\left\{X(t_1), X(t_2)\right\} = E_\xi\left\{\xi^2\right\} E_\varphi\left\{\cos(2\pi f_0 t_1 + \varphi)\cos(2\pi f_0 t_2 + \varphi)\right\}$$

Remembering that:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

we can re-write the formula as:

$$\begin{aligned} & E_X\left\{X(t_1), X(t_2)\right\} = \\ & = \tfrac{A^2}{3} E_\varphi\left\{\tfrac{1}{2}\cos\left(2\pi f_0\left[t_1+t_2\right]+2\varphi\right)+\tfrac{1}{2}\cos\left(2\pi f_0\left[t_1-t_2\right]\right)\right\} \\ & = \tfrac{A^2}{6} E_\varphi\left\{\cos\left(2\pi f_0\left[t_1+t_2\right]+2\varphi\right)\right\}+\tfrac{A^2}{6}\cos\left(2\pi f_0\left[t_1-t_2\right]\right) \end{aligned}$$

where the second term within the expectation was taken out of it because it no longer depends on φ.

We now have to evaluate the expectation:

$$E_\varphi\left\{\cos\left(2\pi f_0\left[t_1+t_2\right]+2\varphi\right)\right\}=\int_{-\infty}^{\infty}\cos\left(2\pi f_0\left[t_1+t_2\right]+2y\right)f_\varphi(y)dy=$$

$$=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(2\pi f_0\left[t_1+t_2\right]+2y\right)\Pi_{2\pi}(y-\pi)dy=$$

$$=\tfrac{1}{2\pi}\int_{0}^{2\pi}\cos\left(2\pi f_0\left[t_1+t_2\right]+2y\right)dy=\tfrac{1}{4\pi}\left[\sin\left(2\pi f_0\left[t_1+t_2\right]+2y\right)\right]_0^{2\pi}=$$

$$=\tfrac{1}{4\pi}\left[\sin\left(2\pi f_0\left[t_1+t_2\right]+4\pi\right)-\sin\left(2\pi f_0\left[t_1+t_2\right]\right)\right]=$$

$$=\tfrac{1}{4\pi}\left[\sin\left(2\pi f_0\left[t_1+t_2\right]\right)-\sin\left(2\pi f_0\left[t_1+t_2\right]\right)\right]=0$$

In conclusion, the autocorrelation of $X(t)$ is:

$$R_X(t_1,t_2)=\tfrac{A^2}{6}\cos\left(2\pi f_0\left[t_1-t_2\right]\right)$$

Eq. 12-32

Finally, we want to calculate the autocovariance. We know that:

$$K_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

but in this case $\mu_X(t) = 0$, so:

$$K_X(t_1,t_2) = R_X(t_1,t_2) = \frac{A^2}{6}\cos\left(2\pi f_0\left[t_1 - t_2\right]\right)$$

Eq. 12-33

The variance is then:

$$\sigma_X^2(t) = K_X(t,t) = \frac{A^2}{6}$$

Finally, the process correlation coefficient is:

$$\rho_X(t_1,t_2)=\frac{K_X(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)}$$
$$=\frac{\frac{A^2}{6}\cos\left(2\pi f_0\left[t_1-t_2\right]\right)}{\frac{A^2}{6}}=\cos\left(2\pi f_0\left[t_1-t_2\right]\right)$$

Note that indeed $\left|\rho_X(t_1,t_2)\right|\le 1$ as expected of a correlation coefficient.

Regarding the first-order probability density function, while it is possible to obtain it, it turns out to be a lot trickier than in the case of a fixed (non random) phase and we omit the calculation.

12.2.8.4 discussion of the results

These first two examples have quite different features, induced by the fact that the phase in one case is a constant and in the other case is random. This discussion

was done in class. Try on your own to redo it yourself.

12.2.8.5 the antipodal data transmission process

One of the many possible instances of a so-called data transmission process is as follows:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \pi_T(t - nT)$$

where the α_n's are statistically independent RVs that can take on the values (-1) and (+1) with equal probability. In other words, the pdf of each one of the α_n's is:

$$f_{\alpha_n}(z) = \tfrac{1}{2}\delta(z-1) + \tfrac{1}{2}\delta(z+1)$$

Clearly, $X(t)$ is a quasi-defined process. We want to evaluate its mean,

autocorrelation, autocovariance and autocorrelation coefficient and pdf.

We start from the mean. As for the previous example, we do not have the process pdf. So:

$$\mu_X(t) = E_X\left\{X(t)\right\} = E\left\{\sum_{n=-\infty}^{\infty} \alpha_n \, \pi_T(t-nT)\right\} =$$

$$= \sum_{n=-\infty}^{\infty} \pi_T(t-nT)\, E_{a_n}\left\{\alpha_n\right\}$$

The expectation $E_{\alpha_n}\left\{\alpha_n\right\}$ can be calculated directly:

$$E_{\alpha_n}\{\alpha_n\} = \int_{-\infty}^{\infty} z\, f_{\alpha_n}(z)dz =$$

$$= \int_{-\infty}^{\infty} z\left[\tfrac{1}{2}\delta(z-1) + \tfrac{1}{2}\delta(z+1)\right]dz = +\tfrac{1}{2} - \tfrac{1}{2} = 0$$

Therefore:

$$\mu_X(t) = 0$$

We then look at the autocorrelation:

$$R_X(t_1, t_2) = E_X\left\{X(t_1)\,X(t_2)\right\} =$$

$$= E\left\{\sum_{m=-\infty}^{\infty} \alpha_m\, \pi_T(t_1 - mT) \sum_{n=-\infty}^{\infty} \alpha_n\, \pi_T(t_2 - nT)\right\} =$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \pi_T(t_1 - mT)\,\pi_T(t_2 - nT) E_{\alpha_m\alpha_n}\{\alpha_m\alpha_n\}$$

The expectation $E_{\alpha_m\alpha_n}\{\alpha_m\alpha_n\}$ needs to be discussed. For $m \neq n$ we have the expectation of a joint moment of two independent RV which therefore factorizes:

$$E_{\alpha_m\alpha_n}\{\alpha_m\alpha_n\} = E_{\alpha_m}\{\alpha_m\}E_{\alpha_n}\{\alpha_n\} = 0 \cdot 0 = 0$$

For $m = n$ we have instead:

$$E_{\alpha_m\alpha_n}\{\alpha_m\alpha_n\} = E_{\alpha_n}\{\alpha_n^2\}$$

However, $\alpha_n^2 = 1$ so, in fact, α_n^2 is no longer a random quantity. Rather, it has become a constant. So: $E_{\alpha_n}\{\alpha_n^2\} = 1$. Pulling together the two cases, we have:

$$E_{\alpha_m \alpha_n}\{\alpha_m \alpha_n\} = \delta_{mn}$$

As a result:

$$\begin{aligned} R_X(t_1, t_2) &= \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \pi_T(t_1 - mT)\, \pi_T(t_2 - nT) \cdot \delta_{mn} = \\ &= \sum_{n=-\infty}^{\infty} \pi_T(t_1 - nT)\, \pi_T(t_2 - nT) \end{aligned}$$

Since $\mu_X(t) = 0$, then:

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sum_{n=-\infty}^{\infty} \pi_T(t_1 - nT)\, \pi_T(t_2 - nT)$$

Note also that:

$$\sigma_X^2(t) = K_X(t,t) = \sum_{n=-\infty}^{\infty} \pi_T(t-nT)\,\pi_T(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} \pi_T^2(t-nT) = \begin{cases} 1(t) & , \quad t \neq kT \\ 1/2 & , \quad t = kT \end{cases}$$

The plot of the autocorrelation and autocovariance is shown in the next page, assuming $T = 1$.

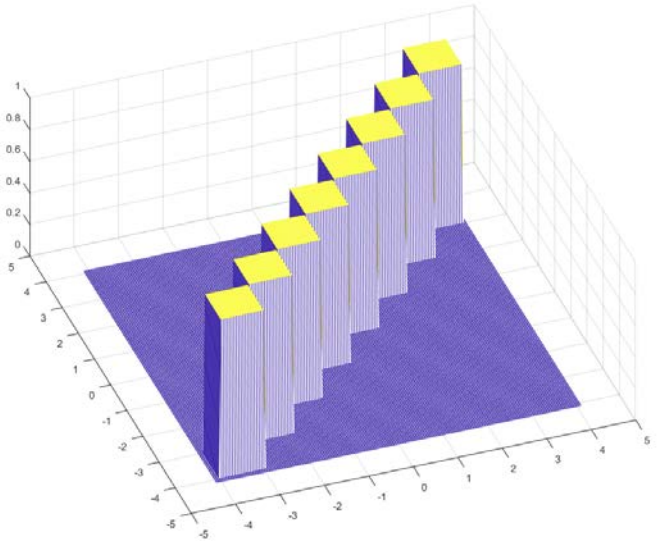


Fig. 12-4

Regarding its autocorrelation coefficient, we have:

$$\rho_X(t_1,t_2)=\frac{K_X(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}}=$$

$$=\frac{K_X(t_1,t_2)}{\sqrt{K_X(t_1,t_1)K_X(t_2,t_2)}}=\frac{\sum_{n=-\infty}^{\infty}\pi_T(t_1-nT)\pi_T(t_2-nT)}{\sqrt{\sum_{m=-\infty}^{\infty}\pi_T^2(t_1-mT)\sum_{n=-\infty}^{\infty}\pi_T^2(t_2-nT)}}$$

Notice that the denominator is almost everywhere equal to 1, except for the points $t_1=mT$ and $t_2=nT$. If plotted, $\rho_X(t_1,t_2)$ would look essentially identical to $K_X(t_1,t_2)$, since the difference is localized to specific isolated points.

On your own: Find the first-order pdf.

Result: the first order pdf can be easily found following the same procedure shown in class and used in problem 12.2.3.2. The result is:

$$f_X(x;t) = \tfrac{1}{2}\delta(x+1) + \tfrac{1}{2}\delta(x-1) \quad , \quad t \neq nT$$
$$f_X(x;t) = \tfrac{1}{4}\delta(x+1) + \tfrac{1}{2}\delta(x) + \tfrac{1}{4}\delta(x-1) \quad , \quad t = nT$$

On your own: check that mean value and variance, calculated directly using the first-order pdf, correspond to the already found results.

On your own: try to find the second-order pdf of the process. To keep complexity low, just look at times that do not coincide with multiples of T. You can then redo the calculations for the autocorrelation and autocovariance using the second-order pdf and check that they coincide with the results above.

12.2.8.6 the unipolar data transmission process

On your own: We assume the same RP as in 12.2.8.5 with the only difference that now the α_n's can take on the values 0 and 1 with equal probability, rather than +1 and -1. They are still are statistically independent.

- Find:

$\mu_X(t)$, $\sigma_X^2(t)$, $R_X(t_1,t_2)$, $K_X(t_1,t_2)$, $\rho_X(t_1,t_2)$, $f_X(x;t)$

- Why is the the correlation coefficient identical to the case 12.2.8.5 ?
- *Hint: notice that:* $\sum_{n=-\infty}^{\infty} \pi_T(t-nT) = 1(t)$.

Anwers:

$$\mu_X(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \pi_T(t-nT) = 1/2$$

$$R_X(t_1,t_2)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\pi_T(t_1-nT)\,\pi_T(t_2-nT)$$
$$+\frac{1}{4}\sum_{m=-\infty}^{\infty}\sum_{\substack{n=-\infty\\ m\neq n}}^{\infty}\pi_T(t_1-mT)\,\pi_T(t_2-nT)$$

$$K_X(t_1,t_2)=\frac{1}{4}\sum_{n=-\infty}^{\infty}\pi_T(t_1-nT)\,\pi_T(t_2-nT)$$

$$\rho_X(t_1,t_2)=\frac{\displaystyle\sum_{n=-\infty}^{\infty}\pi_T(t_1-nT)\,\pi_T(t_2-nT)}{\sqrt{\displaystyle\sum_{m=-\infty}^{\infty}\pi_T^2(t_1-mT)\sum_{n=-\infty}^{\infty}\pi_T^2(t_2-nT)}}$$

The first-order pdf was already found in problem 12.2.3.2. It was:

$$f_X(x;t) = \tfrac{1}{2}\delta(x) + \tfrac{1}{2}\delta(x-1) \quad , \quad t \neq nT$$
$$f_X(x;t) = \tfrac{1}{4}\delta(x) + \tfrac{1}{2}\delta(x-1/2) + \tfrac{1}{4}\delta(x-1) \quad , \quad t = nT$$

On your own: check that the mean value $\mu_X(t)$ calculated directly based on the first-order pdf yields the same results as **Error! Reference source not found.**. Same for the variance.

12.2.8.7 triangular pulse data transmission process

On your own: redo all the calculations of 12.2.8.6, assuming now that the α_n's are statistically independent RVs that can take on the values (-1) and (+1) with equal probability and that the data transmission process is now defined as:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \Lambda_T(t - nT)$$

Answers:

$$\mu_X(t) = 0$$

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sum_{n=-\infty}^{\infty} \Lambda_T(t_1 - nT)\Lambda_T(t_2 - nT)$$

12.2.8.8 the antipodal data transmission process with random delay

The process:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \pi_T(t - nT)$$

is already a good model for data transmission but it is too idealized in one respect: all realizations are synchronous, in the sense that the "time-grid" to which all elementary signals $\pi_T(t-nT)$ are aligned is the same for all realizations. A more realistic model of data transmission would assume *random* time-grid alignment.

We therefore re-define the data transmission process as:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \pi_T(t-nT-\theta)$$

where θ is a random delay, uniformly distributed over the interval $[0,T]$. Also, θ is independent of the α_n's. We will now redo the calculations for mean, autocorrelation and autocovariance, and discuss the difference with the case where there is no delay.

For the mean:

$$\mu_X(t) = E_X\left\{X(t)\right\} = E\left\{\sum_{n=-\infty}^{\infty} \alpha_n \pi_T\left(t - nT - \theta\right)\right\} =$$

$$= \sum_{n=-\infty}^{\infty} E_\theta\left\{\pi_T\left(t - nT - \theta\right)\right\} E_{a_n}\left\{\alpha_n\right\} = \sum_{n=-\infty}^{\infty} E_\theta\left\{\pi_T\left(t - nT - \theta\right)\right\} \cdot 0 = 0$$

The fact that $E_{\alpha_n}\left\{\alpha_n\right\} = 0$, brings the mean to zero in this case too. We now look at the autocorrelation:

$$R_X(t_1,t_2) = E_X\left\{X(t_1)X(t_2)\right\} =$$

$$= E\left\{\sum_{m=-\infty}^{\infty}\alpha_m\,\pi_T(t_1 - mT - \theta)\sum_{n=-\infty}^{\infty}\alpha_n\,\pi_T(t_2 - nT - \theta)\right\} =$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}E_\theta\left\{\pi_T(t_1 - mT - \theta)\pi_T(t_2 - nT - \theta)\right\}E_{\alpha_m\alpha_n}\left\{\alpha_m\alpha_n\right\}$$

We have already found that: $E_{\alpha_m\alpha_n}\left\{\alpha_m\alpha_n\right\} = \delta_{mn}$, so:

$$R_X(t_1,t_2) =$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E_\theta \left\{ \pi_T(t_1 - mT - \theta)\, \pi_T(t_2 - nT - \theta) \right\} \delta_{mn} =$$

$$= \sum_{n=-\infty}^{\infty} E_\theta \left\{ \pi_T(t_1 - nT - \theta)\, \pi_T(t_2 - nT - \theta) \right\} =$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \pi_T(t_1 - nT - z)\, \pi_T(t_2 - nT - z)\, f_\theta(z) dz$$

The pdf of θ is:

$$f_\theta(z) = \tfrac{1}{T} \Pi_T \left(z - \tfrac{T}{2} \right)$$

Then:

$$R_X(t_1,t_2) =$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \pi_T\left(t_1 - nT - z\right)\pi_T\left(t_2 - nT - z\right)\Pi_T\left(z - \tfrac{T}{2}\right)dz =$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{0}^{T} \pi_T\left(t_1 - nT - z\right)\pi_T\left(t_2 - nT - z\right)dz$$

We first remark that we can express the second time-variable as $t_2 = t_1 - \tau$ with no loss of generality. We get:

$$R_X(t_1,t_1-\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{0}^{T} \pi_T\left(t_1 - nT - z\right)\pi_T\left(t_1 - \tau - nT - z\right)dz$$

We then change integration variable as follows:

$$r = t_1 - nT - z \quad \rightarrow \quad z = t_1 - nT - r$$

$$dr = -dz$$

$$r_{\text{low}} = t_1 - nT - z_{\text{low}} = t_1 - nT$$

$$r_{\text{upp}} = t_1 - nT - z_{\text{upp}} = t_1 - nT - T = t_1 - (n+1)T$$

Therefore:

$$R_X(t_1, t_1 - \tau) =$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_0^T \pi_T\left(t_1 - nT - z\right) \pi_T\left(t_1 - \tau - nT - z\right) dz$$

$$= -\frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{t_1 - nT}^{t_1 - (n+1)T} \pi_T\left(r\right) \pi_T\left(r - \tau\right) dr$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{t_1 - (n+1)T}^{t_1 - nT} \pi_T\left(r\right) \pi_T\left(r - \tau\right) dr$$

Eq. 12-34

We then remark that the integration range of the integral of index n is actually adjacent to that of index $n+1$. This is true for all n. In fact, the union of all the integration ranges gives exactly:

$$\bigcup_{n=-\infty}^{\infty}\left[(t_1-nT)-T\,,(t_1-nT)\right]=\left[-\infty,\infty\right]$$

So, in practice, the summation of the infinitely many integrals shown in Eq. 12-34 is equivalent to a single integral over $\left[-\infty,\infty\right]$:

$$R_X(t_1,t_1-\tau)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{t_1-(n+1)T}^{t_1-nT}\pi_T(r)\,\pi_T(r-\tau)\,dr=$$

$$=\frac{1}{T}\int_{-\infty}^{\infty}\pi_T(r)\,\pi_T(r-\tau)\,dr$$

But the last integral is an autocorrelation integral, in the sense of deterministic

signals. We have performed this autocorrelation calculation before, in Chapter 10, and the result is a triangular function as shown below:

$$R_X(t_1, t_1-\tau) = \frac{1}{T}\int_{-\infty}^{\infty} \pi_T(r)\,\pi_T(r-\tau)\,dr =$$
$$= \frac{1}{T} R_{\pi_T}(\tau) = \frac{1}{T}\cdot T\ \Lambda_T(\tau) = \Lambda_T(\tau) = \Lambda_T(t_1 - t_2)$$

In summary:

$$R_X(t_1, t_2) = \Lambda_T(\tau) = \Lambda_T(t_1 - t_2)$$

Since we also found that $\mu_X = 0$, then we can also write:

$$K_X(t_1, t_2) = \Lambda_T(\tau) = \Lambda_T(t_1 - t_2)$$

from which we can also find the variance: $\sigma_X^2(t) = K_X(t,t) = 1(t)$.

The autocorrelation and autocovariance can be drawn vs. t_1 and t_2, as in Fig. 12-4 for the previous data-transmission process. The result is shown belowfor $T = 1$:

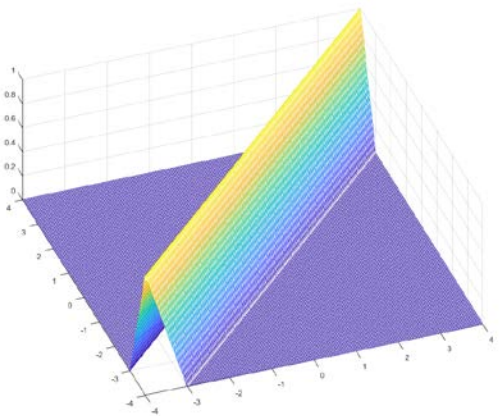


Fig. 12-5

However, it can also be plotted more simply directly vs. τ, still retaining all the information provided by the function. Below, the plot for $T = 1$:

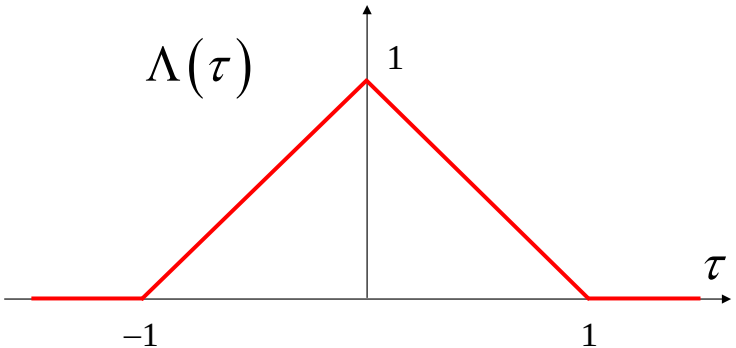


Fig. 12-6

On your own: Find the first-order pdf.

Result: the first order pdf can be easily found following a similar procedure to the

one shown in class and used in problem 12.2.3.3. The result is:

$$f_X(x;t) = \tfrac{1}{2}\delta\left(x+1\right) + \tfrac{1}{2}\delta\left(x-1\right)$$

Note that all time dependence has disappeared. The reason is that sampling the value 0 is also possible, but its probability turns out to be zero and therefore it does not show up in the pdf. A similar situation was found in problem 12.2.3.3, where the sampled value 1/2 was possible but occurred with zero probability.

12.2.8.9 another example of the antipodal data transmission process with random delay

On your own: Redo the above exercise, assuming now that the pdf of θ is:

$$f_\theta(z) = \tfrac{1}{2T}\Pi_{2T}\left(z-T\right)$$

Result: there is no change in the result. In fact, it can be shown that nothing

changes assuming $f_\theta(z) = \frac{1}{NT}\Pi_{NT}\left(z - \frac{N}{2}T\right)$ for any integer $N \geq 1$.

12.2.8.10 unipolar data transmission process with random delay

On your own: redo all calculations of 12.2.8.8, assuming now that the α_n's are statistically independent RVs that can take on the values 0 and 1 with equal probability.

Results:

$$\mu_X(t) = 1/2$$

$$R_X(t_1, t_2) = \frac{1}{4}\Lambda_T(t_1 - t_2) + \frac{1}{4}$$

$$K_X(t_1, t_2) = \frac{1}{4}\Lambda_T(t_1 - t_2)$$

$$f_X(x;t) = \tfrac{1}{2}\delta(x) + \tfrac{1}{2}\delta(x-1)$$

(Hint: use the variable substitution $\rho = t - nT - z$ in the expectation integrals.)

12.2.8.11 Optional: the random periodic process

A further interesting case is that of the random periodic process:

$$X(t) = \sqrt{f_0} \sum_{n=-\infty}^{+\infty} \xi_n e^{j2\pi n f_0 t}$$

In general, the coefficients of a Fourier series are complex. So ξ_n is a complex RV. We can however split it up into its real and imaginary part:

$$\xi_n = \rho_n + j\gamma_n$$

We then assume that for $n > 0$ the ρ_n's and γ_n's are all statistically independent Gaussian-distributed RVs, with zero mean and variance σ_n^2. Specifically:

$$f_{\rho_n}(z) = f_{\gamma_n}(z) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left\{-\frac{z^2}{2\sigma_n^2}\right\}, \quad n > 0$$

In addition, we assume that for $n \neq 0$ we have:

$$\rho_{-n} = \rho_n \quad , \quad \gamma_{-n} = -\gamma_n$$

Eq. 12-35

This assumption ensures that:

$$\xi_{-n} = \xi_n^* .$$

This, in turn, guarantees that *all* realizations of $X(t)$ are *real* signals: $X(t; s_k) \in \mathbb{R}$. Note also that for $n = 0$ the constraint $\gamma_{-n} = -\gamma_n$ has the only solution $\gamma_0 = 0$. So

for $n = 0$:

$$\xi_0 = \rho_0, \quad \gamma_0 = 0$$

where ρ_0, is Gaussian, independent of all other RVs, zero-mean and with variance $2\sigma_0^2$. Note that the factor "2" in front of σ_0^2 is intentional and is imposed for convenience.

We then proceed to compute the average of the process. We have:

$$\begin{aligned}\mu_X(t) &= E_X\left\{X(t)\right\} \\ &= E\left\{\sqrt{f_0}\sum_{n=-\infty}^{+\infty}\xi_n e^{j2\pi n f_0 t}\right\} = \sqrt{f_0}\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_0 t} E_{\xi_n}\left\{\xi_n\right\}\end{aligned}$$

The average is:

$$E_{\xi_n}\left\{\xi_n\right\} = E_{\rho_n\gamma_n}\left\{\rho_n + j\gamma_n\right\} = E_{\rho_n}\left\{\rho_n\right\} + jE_{\gamma_n}\left\{\gamma_n\right\} = 0 + 0 = 0$$

Therefore the mean of the process is $\mu_X(t) = 0$.

We then evaluate the autocorrelation.

Note that when dealing with complex processes, that is processes whose realizations are complex signals, then the following definition of the autocorrelation should be used: $R_X(t_1, t_2) = E_X\left\{X\left(t_1\right)X^*\left(t_2\right)\right\}$. However, in this example we have made sure that $X\left(t\right)$ is a real process, so we can drop the "complex conjugate" and write:

$$R_X(t_1,t_2) = E_X\left\{X(t_1)X(t_2)\right\} =$$

$$= E\left\{\sqrt{f_0}\sum_{n=-\infty}^{+\infty}\xi_n e^{j2\pi n f_0 t_1}\sqrt{f_0}\sum_{m=-\infty}^{+\infty}\xi_m e^{j2\pi m f_0 t_2}\right\} =$$

$$= f_0\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} e^{j2\pi n f_0 t_1} e^{j2\pi m f_0 t_2} E_{\xi_n\xi_m}\left\{\xi_n\xi_m\right\}$$

We first carry out the calculation for the expectation $E_{\xi_n\xi_m}\left\{\xi_n\xi_m\right\}$ when $n \neq m$ and $n \neq -m$. We have:

$$E_{\xi_n\xi_m}\left\{\xi_n\xi_m\right\}_{n\neq m, n\neq -m,} = E_{\xi_n}\left\{\xi_n\right\}E_{\xi_m}\left\{\xi_m\right\} = 0\cdot 0 = 0$$

We then look at the case $n = m$, excluding for now $n = m = 0$

$$E_{\xi_n \xi_m}\left\{\xi_n \xi_m\right\}_{n=m,n\neq 0} = E_{\xi_n}\left\{\xi_n \xi_n\right\} = E_{\xi_n}\left\{\xi_n^2\right\} = E\left\{\left(\rho_n + j\gamma_n\right)^2\right\}$$

$$= E_{\rho_n}\left\{\rho_n^2\right\} - E_{\gamma_n}\left\{\gamma_n^2\right\} + jE_{\rho_n}\left\{\rho_n\right\}E_{\gamma_n}\left\{\gamma_n\right\} + jE_{\gamma_n}\left\{\gamma_n\right\}E_{\rho_n}\left\{\rho_n\right\} =$$

$$= \sigma_n^2 - \sigma_n^2 + j0\cdot 0 + j0\cdot 0 = 0$$

We then concentrate on the case $n = -m$, excluding again $n = m = 0$:

$$E_{\xi_n \xi_m}\left\{\xi_n \xi_m\right\}_{n=-m,n\neq 0} = E_{\xi_n \xi_{-n}}\left\{\xi_n \xi_{-n}\right\} = E_{\xi_n}\left\{\xi_n \xi_n^*\right\} = E_{\xi_n}\left\{\left|\xi_n\right|^2\right\} =$$

$$= E\left\{\rho_n^2 + \gamma_n^2\right\} = E_{\rho_n}\left\{\rho_n^2\right\} + E_{\gamma_n}\left\{\gamma_n^2\right\} = \sigma_n^2 + \sigma_n^2 = 2\sigma_n^2$$

The only case left is $n = m = 0$:

$$E_{\xi_n}\left\{\xi_0^2\right\} = E_{\rho_0}\left\{\rho_0^2\right\} = 2\sigma_0^2$$

Pulling all cases together we can claim that:

$$E_{\xi_n \xi_m}\left\{\xi_n \xi_m\right\} = 2\sigma_n^2 \cdot \delta_{n(-m)}$$

Therefore:

$$R_X(t_1,t_2) = f_0 \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{j2\pi n f_0 t_1} e^{j2\pi m f_0 t_2} E_{\xi_n \xi_m}\left\{\xi_n \xi_m\right\} =$$

$$= 2 f_0 \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{j2\pi n f_0 t_1} e^{j2\pi m f_0 t_2} \sigma_n^2 \delta_{n(-m)} =$$

$$= 2 f_0 \sum_{n=-\infty}^{+\infty} e^{j2\pi n f_0 t_1} e^{-j2\pi n f_0 t_2} \sigma_n^2 = 2 f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 e^{j2\pi n f_0 [t_1 - t_2]}$$

To sum up:

$$\mu_X(t) = 0$$

$$R_X(t_1, t_2) = K_X(t_1, t_2) = 2 f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 e^{j 2\pi n f_0 [t_1 - t_2]}$$

$$\sigma_X^2(t) = K_X(t, t) = 2 f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2$$

Eq. 12-36

End of optional material.

12.2.9 Joint moments of two or more random processes

As an extension to what we have seen above for a single process, it is possible to address joint moments of arbitrary functions of time-samples taken from not just one, but from two or more RP's.

For instance, recalling once more the joint expectation of a generic function of two RVs:

$$E_{\xi\eta}\left\{g(\xi,\eta)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{\xi,\eta}(x,y)dxdy$$

we could assume $\xi = X(t_1)$ and $\eta = Y(t_2)$, so that we would have:

$$E_{XY}\left\{g\left(X(t_1),Y(t_2)\right)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y;t_1,t_2)dxdy$$

Eq. 12-37

Note the need for a joint pdf of the two processes $f_{XY}(x,y;t_1,t_2)$, which is simply the joint pdf of the two RV's $\xi = X(t_1)$ and $\eta = Y(t_2)$:

$$f_{XY}(x,y;t_1,t_2) = f_{X(t_1)Y(t_2)}(x,y) = f_{\xi\eta}(x,y)$$

All the different notations above are in fact equivalent, though the one that we use is the first.

As usual, the functions of greatest interest are the simplest moments. The *joint non-central cross-moment of order* (1,1) *of the processes* $X(t)$ and $Y(t)$ is written as:

$$E_{XY}\left\{X(t_1)Y(t_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y;t_1,t_2)dxdy$$

This quantity is given a specific symbol and name:

the cross-correlation function of the RP's $X(t)$ *and* $Y(t)$ *is defined as:*

$$R_{XY}(t_1,t_2) \triangleq E_{XY}\left\{X(t_1)Y(t_2)\right\}$$

Eq. 12-38

The *joint central cross-moment of order* (1,1) *of the processes* $X(t)$ and $Y(t)$ is

also used:

$$E_{XY}\left\{\left[X(t_1)-\mu_X(t_1)\right]\left[Y(t_2)-\mu_Y(t_2)\right]\right\}=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x-\mu_X(t_1)\right]\left[y-\mu_y(t_2)\right]f_{XY}(x,y;t_1,t_2)dxdy$$

This quantity too is given a specific symbol and name:

The cross-covariance of the RP's $X(t)$ and $Y(t)$ is defined as:

$$K_{XY}(t_1,t_2)\triangleq E_{XY}\left\{\left[X(t_1)-\mu_X(t_1)\right]\left[Y(t_2)-\mu_Y(t_2)\right]\right\}$$

Eq. 12-39

Once again, between cross-correlation and cross-covariance there is a close relation:

$$K_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2)$$

Eq. 12-40

In fact:

$$\begin{aligned}
&E_{XY}\left\{\left[X(t_1)-\mu_X(t_1)\right]\left[Y(t_2)-\mu_Y(t_2)\right]\right\} = \\
&= E_{XY}\left\{\left[X(t_1)Y(t_2)-\mu_X(t_1)Y(t_2)-\mu_Y(t_2)X(t_1)+\mu_X(t_1)\mu_Y(t_2)\right]\right\} = \\
&= E_{XY}\left\{X(t_1)Y(t_2)\right\}-\mu_X(t_1)E_Y\left\{X(t_2)\right\}-\mu_Y(t_2)E_X\left\{X(t_1)\right\}+\mu_X(t_1)\mu_Y(t_2) = \\
&= E_{XY}\left\{X(t_1)Y(t_2)\right\}-\mu_X(t_1)\mu_Y(t_2)-\mu_Y(t_2)\mu_X(t_1)+\mu_X(t_1)\mu_Y(t_2) = \\
&= R_{XY}(t_1,t_2)-\mu_X(t_1)\mu_Y(t_2)
\end{aligned}$$

Finally, it is also possible to define the *cross-correlation coefficient* of two

processes:

$$\rho_{XY}(t_1,t_2)=\frac{E_X\left\{X(t_1)Y(t_2)\right\}-\mu_X(t_1)\mu_Y(t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}$$

Eq. 12-41

But the numerator coincides with the definition of the cross-covariance so, more compactly:

$$\rho_{XY}(t_1,t_2)=\frac{K_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}$$

Eq. 12-42

It is then possible to define moments and expectations of functions involving three or more processes. This extension is straightforward. On the other hand, such quantities are less frequently used, so they will not be addressed here.

12.3 Stationarity

12.3.1 Stationarity for expectations

A random process $X(t)$ is said to be stationary for a certain statistical quantity if such statistical quantity does not depend on "absolute time". Some definitions will clarify the concept.

We start out by defining stationarity for expectations.

A RP $X(t)$ ***is said to be stationary:***

- ***for the mean***, if $E\{X(t)\} = \mu_X(t) = \mu_X$;

- ***for the mean square value***, if $E\{X^2(t)\} = a$, where "a" is a constant;

- ***for the variance***, if $\sigma_X^2(t) = E\left\{X^2(t)\right\} - \mu_X^2(t) = \sigma_X^2$;

- ***for any generic moment or function of one sample of the process***

if $E\left\{g\left(X(t)\right)\right\} = a$, where "$a$" is a constant;

Stationarity is a straightforward concept, as shown above, if only a single "time" is referenced in the expectation. When two times are referenced, such as in autocorrelations, the definition is somewhat less obvious.

A RP $X(t)$ ***is said to be stationary:***

- ***for the autocorrelation*** if $R_X(t_1, t_2) = R_X(t_1 - t_2)$

- ***for the autocovariance*** if $K_X(t_1, t_2) = K_X(t_1 - t_2)$

- ***for any generic moment or function involving two samples of the process***

taken at times t_1, t_2, if

$$E\left\{g\left(X(t_1), X(t_2)\right)\right\} = q(t_1 - t_2)$$

The meaning of stationarity is therefore not "time independence": requiring for instance that an autocorrelation function should be "time independent" would make little sense. Rather, for two-times statistical quantities, stationarity means that the statistical quantity does not depend on absolute time, but on the *time-distance* between the two samples: $t_1 - t_2$.

This concept can perhaps be better understood by writing a *generic* (non-stationary) autocorrelation as follows:

$$R_X(t_1, t_2) = R_X(t_1, t_1 - \tau)$$

In the right-hand side we have simply formally substituted $t_2 = t_1 - \tau$. Clearly,

$\tau = t_1 - t_2$ is the time-distance between t_1 and t_2.

This form $R_X(t_1, t_1 - \tau)$ explicitly shows *absolute time* as t_1 and *time difference* as τ. In general, an autocorrelation depends on both absolute time as t_1 and time difference τ. However, if it turns out that $R_X(t_1, t_1 - \tau)$ does not really depend on t_1 but only on τ, *then the autocorrelation is stationary*. A fully equivalent definition of stationarity for the autocorrelation is therefore the following:

a RP $X(t)$ *is said to be stationary for the autocorrelation* if

$$R_X(t_1, t_2) = R_X(\tau)$$

Eq. 12-43

with $\tau = t_1 - t_2$. This same reasoning can be applied to the autocovariance, correlation coefficient and to any other function of two time-samples of the process.

12.3.1.1 Examples

The autocorrelation of the random amplitude carrier was found to be:

$$R_X(t_1,t_2)=\frac{A^2}{3}\cos(2\pi f_0 t_1+\varphi_0)\cos(2\pi f_0 t_2+\varphi_0)$$

It can actually be rearranged in this form:

$$R_X(t_1,t_2)=\frac{A^2}{6}\left[\cos\left(2\pi f_0\left[t_1-t_2\right]\right)+\cos\left(2\pi f_0\left[t_1+t_2\right]+2\varphi_0\right)\right]$$

The first term within the square brackets depends on the time difference $\left(t_1-t_2\right)$ but the second term depends on $\left(t_1+t_2\right)$ and therefore it depends on absolute times. This process is therefore non-stationary for the autocorrelation.

The random amplitude and random phase carrier has instead autocorrelation:

$$R_X(t_1,t_2)=\frac{A^2}{6}\cos\left(2\pi f_0\left[t_1-t_2\right]\right)$$

This autocorrelation manifestly depends only on the time-difference t_1-t_2, We can in fact re-write it as:

$$R_X(\tau)=\frac{A^2}{6}\cos\left(2\pi f_0\tau\right)$$

without any loss of information about this quantity.

The data transmission process of Section 12.2.8.5 was found to have autocorrelation:

$$R_X(t_1,t_2)=\sum_{n=-\infty}^{\infty}\pi_T\left(t_1-nT\right)\pi_T\left(t_2-nT\right)$$

This function clearly does not depend only on the time-difference between t_1 and t_2, but individually on both times. Therefore this process is non-stationary for the autocorrelation.

However, when a random delay was added to the data-transmission process, in Section 12.2.8.8, then the resulting autocorrelation was:

$$R_X(t_1,t_2)=\Lambda_T\left(t_1-t_2\right)=\Lambda_T\left(\tau\right)$$

which clearly depends only on the time-difference $t_1-t_2=\tau$ and is therefore stationary.

Optional: As a last example, the random periodic signal autocorrelation was:

$$R_X(t_1,t_2)=2f_0\sum_{n=-\infty}^{+\infty}\sigma_n^2 e^{j2\pi nf_0[t_1-t_2]}$$

In this case too, stationarity is obvious. We can in fact re-write this autocorrelation as:

$$R_X(\tau) = 2 f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 e^{j 2\pi n f_0 \tau}$$

End of optional material.

12.3.1.2 stationarity for three or more time samples

A further conceptual hurdle presents itself when stationarity must be defined for quantities that depend not just on two, but on three or more time-instants.

The underlying idea is the same: the quantity is stationary if it does not depend on absolute times but only on time differences among the samples of the process.

It is then useful to look for a formal and general definition of stationarity of a generic statistical quantity that depends on N samples of a process.

The expectation of any function of N *time samples of a process* $X(t)$*, taken at times* $t_1, t_2, ..., t_N$*, is said to be stationary if such expectation depends only on the* $N-1$ *time differences* $(t_1 - t_2), (t_1 - t_3), ..., (t_1 - t_N)$ *.*

More formally, if:

$$E_X\left\{g\left(X(t_1), X(t_2), ..., X(t_N)\right)\right\} = q(t_1 - t_2, t_1 - t_3, ..., t_1 - t_N),$$

then $E_X\left\{g\left(X(t_1), X(t_2), ..., X(t_N)\right)\right\}$ *is said to be stationary.*

Note that we could equivalently write that the expectation is stationary if:

$$E_X\left\{g\left(X(t_1), X(t_2), ..., X(t_N)\right)\right\} = q(\tau_1, \tau_2, ..., \tau_{N-1})$$

where $\tau_k = t_1 - t_{k+1}$.

12.3.2 Stationarity of pdf's

We have so far looked at stationarity for expectations. Also pdf's can be stationary:

a RP $X(t)$ *is said to be first-order stationary if*

$$f_X(x;t) = f_X(x);$$

First order stationarity obviously means that the statistical distribution of the RV corresponding to any time-sample of the process is the same.

Extension to second-order pdf's follows the same reasoning as shown above for

the autocorrelation:

a RP $X(t)$ *is said to be second-order stationary if*

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 - t_2) = f_X(x_1, x_2; \tau);$$

In other words, a second-order stationary process has a joint pdf of two RVs extracted from it that does not depend on when exactly these samples are taken, but rather on how far apart in time they are. One could also say that the joint pdf that is found today for two samples taken τ_0 seconds apart is the same as we would find between two samples taken τ_0 seconds apart tomorrow, or a year from now.

Further extensions are now straightforward:

a RP $X(t)$ *is said to be N-th-order stationary if*

$$
\begin{aligned}
&f_X(x_1, x_2, ..., x_N; t_1, t_2, ..., t_N) = \\
&= f_X(x_1, x_2, ..., x_N; t_1 - t_2, t_1 - t_3, ..., t_1 - t_N) = \\
&= f_X(x_1, x_2, ..., x_N; \tau_1, \tau_2, ..., \tau_{N-1})
\end{aligned}
$$

where $\tau_k = t_1 - t_{k+1}$

An important result is the following:

given an N*-th* ***order stationary process, such process is also stationary of order*** $N-1,\ N-2,$ ***…,2,1.***

$(N-1)$*-th* N*-th* N*-th* $(N-1)$*-th* Another important implication, which is straightforward to prove, is that:

if a RP $X(t)$ *is* N*-th order stationary, then any expectation of any function of* N

time samples of the process taken at times $t_1, t_2, ..., t_N$, *is stationary too, that is, it depends only on the* $N-1$ *time differences* $(t_1 - t_2), (t_1 - t_3), ..., (t_1 - t_N)$.

Also, any expectation of any function of $N-1$, $N-2$, …,*2,1, time samples of the process is stationary as well.*

This result means, for instance, that ***second-order stationarity implies that autocorrelation, autocovariance and correlation coefficient of the process are stationary*****.** It also implies that the mean and variance are stationary as well.

12.3.3 Strong-Sense Stationarity

What is the most stationary a process can get? This definition answers this question:

given a RP $X(t)$, *such process is said to be* ***strong-sense stationary (SSS)*** *if it is*

N-th order stationary, for any arbitrarily large N.

In other words, if it can be proved that the *N-th* order pdf of a process is stationary for any N, then the process is SSS.

12.3.4 Weak-Sense Stationarity

A much weaker concept of stationarity is that of weak-sense stationarity. The definition is the following:

*given a RP $X(t)$, such process is said to be **weak-sense stationary (WSS)** if it is stationary for the mean and for the autocorrelation.*

In other words, to ascertain weak-sense stationarity it is enough to prove that:

$$E\{X(t)\} = \mu_X(t) = \mu_X \quad \text{and} \quad R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

Note that since:

$$K_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_1)$$

then, if the process is WSS the autocovariance can be written as:

$$K_X(t_1,t_2) = R_X(\tau) - \mu_X^2$$

so we see that the autocovariance is stationary too and we can write:

$$K_X(t_1,t_2) = K_X(\tau)$$

However “weak” the WSS stationarity may be, it already establishes certain important features for the process, as we shall see later.

12.3.5 Stationarity implications

It is easy to see that second-order stationarity implies WSS. This is because the autocorrelation integral is based on the second-order pdf. If the second-order pdf depends only on τ, the autocorrelation then does too. In addition, the mean of the process is found using the first-order pdf, but second-order stationarity implies first order stationarity, so the first-order pdf is stationary and therefore the mean turns out to be time-independent as well.

Note though that the converse is not, in general, true: *WSS does not, in general, imply second-order stationarity*. So:

$$\left\{\begin{matrix}\text{second order}\\ \text{stationarity}\end{matrix}\right\} \Rightarrow \{\text{WSS}\}$$

$$\{\text{WSS}\} \not\Rightarrow \left\{\begin{matrix}\text{second order}\\ \text{stationarity}\end{matrix}\right\}$$

The only case in which WSS does imply second order stationarity (and, in fact, all order stationarity, that is SSS) is when the process is jointly Gaussian, as pointed out in the next section.

12.3.6 Jointly Gaussian processes and stationarity

First, we recall that the pdf of N jointly Gaussian RVs is:

$$f_{\xi_1\xi_2\ldots\xi_N}(x_1, x_2, \ldots, x_N) = f_{\boldsymbol{\xi}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\sqrt{\det\{\boldsymbol{\Sigma}_{\boldsymbol{\xi}}\}}} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}_{\boldsymbol{\xi}}\right)^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_{\boldsymbol{\xi}}^{-1} \cdot \left(\mathbf{x}-\boldsymbol{\mu}_{\boldsymbol{\xi}}\right)\right\}$$

Eq. 12-44

where:

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \boldsymbol{\mu}_{\xi} = \begin{bmatrix} \mu_{\xi_1} \\ \mu_{\xi_2} \\ \vdots \\ \mu_{\xi_N} \end{bmatrix} \qquad \boldsymbol{\Sigma}_{\xi} = \begin{bmatrix} \sigma^2_{\xi_1} & \sigma_{\xi_1\xi_2} & \cdots & \sigma_{\xi_1\xi_N} \\ \sigma_{\xi_2\xi_1} & \sigma^2_{\xi_2} & & \\ \vdots & & \ddots & \\ \sigma_{\xi_N\xi_1} & & & \sigma^2_{\xi_N} \end{bmatrix}$$

To adapt it to the context of random processes, we simply have to redefine the symbols according to the following correspondence

$$\xi_1 \to X(t_1) \quad , \quad \xi_2 \to X(t_2) \quad , \quad \xi_3 \to X(t_3) \quad , \quad \ldots \quad , \quad \xi_N \to X(t_N)$$

Then, we can re-write the following quantities accordingly:

$$\boldsymbol{\xi} \to X \quad , \qquad \boldsymbol{\mu}_{\xi} = \begin{bmatrix} \mu_{\xi_1} \\ \mu_{\xi_2} \\ \vdots \\ \mu_{\xi_N} \end{bmatrix} \quad \to \quad \boldsymbol{\mu}_X(\mathbf{t}) = \begin{bmatrix} \mu_X(t_1) \\ \mu_X(t_2) \\ \vdots \\ \mu_X(t_N) \end{bmatrix} \qquad \text{with} \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix}$$

and also:

$$\boldsymbol{\Sigma}_{\boldsymbol{\xi}} = \begin{bmatrix} \sigma^2_{\xi_1} & \sigma_{\xi_1\xi_2} & \cdots & \sigma_{\xi_1\xi_N} \\ \sigma_{\xi_2\xi_1} & \sigma^2_{\xi_2} & & \\ \vdots & & \ddots & \\ \sigma_{\xi_N\xi_1} & & & \sigma^2_{\xi_N} \end{bmatrix} \quad \to \quad \boldsymbol{\Sigma}_X(\mathbf{t}) = \begin{bmatrix} K_x(t_1,t_1) & K_x(t_1,t_2) & \cdots & K_x(t_1,t_N) \\ K_x(t_2,t_1) & K_x(t_2,t_2) & & \\ \vdots & & \ddots & \\ K_x(t_N,t_1) & & & K_x(t_N,t_N) \end{bmatrix}$$

The correspondence is obvious because:

$$\sigma_{\xi_1\xi_2} = E\left\{\left(\xi_1 - \mu_{\xi_1}\right)\left(\xi_2 - \mu_{\xi_2}\right)\right\} =$$
$$= E\left\{\left[X(t_1) - \mu_X(t_1)\right]\left[X(t_2) - \mu_X(t_2)\right]\right\} = K_X(t_1, t_2)$$

It can be seen from the above formulas that the array of the means is found from the process mean function $\mu_X(t)$. Also, the covariance matrix depends uniquely on the process autocovariance function $K_X(t_1, t_2)$.

We can then write the overall *N*-th order pdf as:

$$f_\xi(\mathbf{x}) \quad \rightarrow \quad f_X(\mathbf{x};\mathbf{t}) = f_X(x_1, x_2, ..., x_N; t_1, t_2, ..., t_N)$$

where:

$$f_X(\mathbf{x};\mathbf{t}) = f_X(x_1, x_2, ..., x_N; t_1, t_2, ..., t_N)$$

$$= \frac{1}{(2\pi)^{N/2}\sqrt{\det\{\boldsymbol{\Sigma}_X(\mathbf{t})\}}} \exp\left\{-\frac{1}{2}\left[\mathbf{x}-\boldsymbol{\mu}_X(\mathbf{t})\right]^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_X^{-1} \cdot \left[\mathbf{x}-\boldsymbol{\mu}_X(\mathbf{t})\right]\right\}$$

The final appearance is very similar to that of the joint Gaussian pdf of *N* random variables, Eq. 12-44. Note also the very important fact that:

a Jointly Gaussian RP is fully statistically characterized by just two statistical quantities, which are the low-order moments of the process $\mu_X(t)$ ***and*** $K_x(t_1, t_2)$

or equivalently by $\mu_X(t)$ and $R_x(t_1, t_2)$. No other quantity is needed to fully characterize the joint pdf of the process, to any order.

12.3.6.1 jointly-Gaussian WSS processes

A key property of jointly-Gaussian RP's is the following:

if a jointly-Gaussian RP is WSS, then it is SSS too

In other words, for jointly Gaussian processes:

$$WSS \rightarrow SSS$$

The reason for this important feature of Gaussian processes is the following.

If the process is WSS, $\mu_X(t) = \mu_X$. In addition, the autocovariance depends only on time differences: $K_X(t_1, t_2) = K_X(t_1 - t_2) = K_X(\tau)$.

Therefore:

$$\boldsymbol{\mu}_X(\mathbf{t}) = \begin{bmatrix} \mu_X(t_1) \\ \mu_X(t_2) \\ \vdots \\ \mu_X(t_N) \end{bmatrix} \quad \rightarrow \quad \boldsymbol{\mu}_X = \begin{bmatrix} \mu_X \\ \mu_X \\ \vdots \\ \mu_X \end{bmatrix}$$

In addition:

$$\boldsymbol{\Sigma}_X(\mathbf{t}) = \begin{bmatrix} K_X(t_1,t_1) & K_X(t_1,t_2) & \dots & K_X(t_1,t_N) \\ K_X(t_2,t_1) & K_X(t_2,t_2) & & \\ \vdots & & \ddots & \\ K_X(t_N,t_1) & & & K_X(t_N,t_N) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} K_X(0) & K_X(t_1-t_2) & \dots & K_X(t_1-t_N) \\ K_X(t_2-t_1) & K_X(0) & & \\ \vdots & & \ddots & \\ K_X(t_N-t_1) & & & K_X(0) \end{bmatrix} =$$

$$\boldsymbol{\Sigma}_X(\boldsymbol{\tau}) = \begin{bmatrix} \sigma_X^2 & K_X(\tau_1) & \dots & K_X(-\tau_{N-1}) \\ K_X(-\tau_1) & \sigma_X^2 & & \\ \vdots & & \ddots & \\ K_X(-\tau_{N-1}) & & & \sigma_X^2 \end{bmatrix}$$

That is, $\boldsymbol{\mu}_X$ does not depend on time and $\boldsymbol{\Sigma}_X$ depends only on the time-differences $\boldsymbol{\tau}$. As a result, **the overall pdf clearly depends only on time differences:**

$$f_X(\mathbf{x};\mathbf{t}) = f_X(\mathbf{x};\boldsymbol{\tau}) =$$
$$= \frac{1}{(2\pi)^{N/2}\sqrt{\det\{\boldsymbol{\Sigma}_X(\boldsymbol{\tau})\}}} \exp\left\{-\frac{1}{2}\left[\mathbf{x}-\boldsymbol{\mu}_X\right]^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_X^{-1}(\boldsymbol{\tau}) \cdot \left[\mathbf{x}-\boldsymbol{\mu}_X\right]\right\}$$

Therefore, we can conclude that, indeed, ***a jointly Gaussian WSS process is SSS,* that is, it is stationary for any order.**

12.3.7 Properties of WSS processes

In the following, we assume WSS. A large number of very significant processes have these features and in practice the results for this class of processes are of extreme practical importance.

12.3.7.1 properties of autocorrelations of WSS processes

Stationary autocorrelations have some interesting properties, listed in the following.

12.3.7.2 value at the origin

We have:

$$R_X(0) = E\left\{X^2(t)\right\} = \sigma_X^2 + \mu_X^2 \geq 0$$

12.3.7.3 even symmetry

Given a <u>real</u> WSS process $X(t)$, then:

$$R_X(\tau) = R_X(-\tau)$$

Proof: optional:

We prove this property as follows. First, since the process is WSS by assumption, we can write:

$$E\left\{X(t)X(t-\tau)\right\} = R_X(\tau)$$

The parameter t is therefore *irrelevant* for the result. We can arbitrarily assign any value to it. For instance, we can assign to it the value $t = t_1$, with the result still being $R_X(\tau)$:

$$E\left\{X(t)X(t-\tau)\right\}_{t=t_1} = E\left\{X(t_1)X(t_1-\tau)\right\} = R_X(\tau)$$

We can equally legitimately assign to it the value $t = t_1 + \tau$ and the result must still be $R_X(\tau)$:

$$E\{X(t)X(t-\tau)\}_{t=t_1+\tau} = E\{X(t_1+\tau)X(t_1)\} = R_X(\tau)$$

However, $E\{X(t_1+\tau)X(t_1)\}$ is clearly the same as $R_X(-\tau) = E\{X(t_1)X(t_1+\tau)\}$. Therefore, pulling together the sequence of generated equalities:

$$R_X(\tau) = E\{X(t_1)X(t_1-\tau)\} = E\{X(t_1)X(t_1+\tau)\} = R_X(-\tau)$$

End of optional material

So we can conclude:

$$R_X(\tau) = R_X(-\tau)$$

In other words, *a stationary autocorrelation is an even function.*

12.3.7.4 boundedness

Given a real WSS process $X(t) \in \mathbb{R}$, then:

$$R_X(\tau) \in \left[-R_X(0), R_X(0)\right],$$

which can also be written as:

$$\left|R_X(\tau)\right| \le R_X(0) = \mathrm{MSV}\left\{X(t)\right\}$$

In other words, $R_X(\tau)$ is a *bounded* function, with bounding value $R_X(0)$ or, equivalently, the mean square value of the process.

Proof: optional

Given a WSS process $X(t)$, we define a "service" process as the following

process:

$$Y(t) = X(t) + X(t-\tau)$$

We then compute:

$$\begin{aligned} E_Y\left\{Y^2(t)\right\} &= E_X\left\{X^2(t) + 2X(t)X(t-\tau) + X^2(t-\tau)\right\} = \\ &= E\left\{X^2(t)\right\} + 2E\left\{X(t)X(t-\tau)\right\} + E\left\{X^2(t-\tau)\right\} = \\ &= R_X(0) + 2R_X(\tau) + R_X(0) = 2\big(R_X(0) + R_X(\tau)\big). \end{aligned}$$

Note now that the mean square value of $Y(t)$ is certainly a positive number:

$$E_Y\left\{Y^2(t)\right\} \geq 0$$

Therefore:

$$E_Y\left\{Y^2(t)\right\} = 2R_X(0) + 2R_X(\tau) \geq 0$$

that is:

$$R_X(\tau) \geq -R_X(0)$$

Eq. 12-45

We now define another "service" process:

$$Z(t) = X(t) - X(t-\tau)$$

We then compute:

$$\begin{aligned} E_Y\left\{Z^2(t)\right\} &= E_X\left\{X^2(t) - 2X(t)X(t-\tau) + X^2(t-\tau)\right\} = \\ &= E\left\{X^2(t)\right\} - 2E\left\{X(t)X(t-\tau)\right\} + E\left\{X^2(t-\tau)\right\} = \\ &= R_X(0) - 2R_X(\tau) + R_X(0) = 2\left(R_X(0) - R_X(\tau)\right) \geq 0 \end{aligned}$$

In short:

$$R_X(0) - R_X(\tau) \geq 0$$

from which

$$R_X(\tau) \leq R_X(0)$$

Eq. 12-46

Finally, we have already shown that $R_X(0) \geq 0$. Therefore, looking at Eq. 12-45 and Eq. 12-46, one can say that $R_X(\tau)$ is always greater than the negative number $-R_X(0)$ and is always smaller than the positive number $R_X(0)$. That is:

$$R_X(\tau) \in \left[-R_X(0), R_X(0)\right],$$

or equivalently:

$$\left|R_X(\tau)\right| \leq R_X(0)$$

End of optional material.

12.3.7.5 continuity

The following result holds:

if $R_X(\tau)$ *is continuous at* $\tau = 0$, *then* $R_X(\tau)$ *is everywhere continuous.*

Another property links the continuity of $R_X(\tau)$ with the continuity of the process realizations:

if $R_X(\tau)$ *is continuous, then the process realizations are either continuous or have at most a countable number of "jump" discontinuities.*

This result can be tested against the knowledge that we have of the quasi-defined processes that we have analyzed. For instance, the random carrier with random

phase is a WSS process. We found that its autocorrelation has formula:

$$R_X(t_1, t_2) = R_X(\tau) = \frac{A^2}{6}\cos(2\pi f_0 \tau)$$

$R_X(\tau)$ is clearly continuous at the origin and everywhere and, in fact, each realization is everywhere continuous too.

Another example is the data transmission signal with random delay. It has realizations that are discontinuous at every bit transition, because of the discontinuity of the rectangular signals used to encode the bits.

However, the number of such discontinuities is however: that is, they can be counted using integers. The data transmission signal with random delay autocorrelation function in fact has autocorrelation:

$$R_X(\tau) = \Lambda_T(\tau)$$

which is continuous at the origin and everywhere.

The autocorrelation function of the process $X_{\mathrm{SI}}(t)$ from Section 12.2.4.1 is instead highly pathological. The assumption there is that any two samples of the process are statistically independent. The process is stationary WSS and in fact it is also SSS (you can easily prove it on your own), so its autocorrelation is stationary as well. If we compute it, though, we find:

$$\begin{aligned}
&R_{X_{\mathrm{SI}}}(\tau) = E_{X_{\mathrm{SI}}}\left\{X_{\mathrm{SI}}(t)X_{\mathrm{SI}}(t-\tau)\right\} = \\
&= E_{X_{\mathrm{SI}}}\left\{X_{\mathrm{SI}}(t)\right\}E_{X_{\mathrm{SI}}}\left\{X_{\mathrm{SI}}(t-\tau)\right\} = \mu_{X_{\mathrm{SI}}}^2, \quad \tau \neq 0 \\
&R_{X_{\mathrm{SI}}}(0) = E_{X_{\mathrm{SI}}}\left\{X_{\mathrm{SI}}(t)X_{\mathrm{SI}}(t)\right\} = \sigma_{X_{\mathrm{SI}}}^2 + \mu_{X_{\mathrm{SI}}}^2
\end{aligned}$$

So, this autocorrelation is discontinuous at the origin and in fact the process itself has realizations that are discontinuous everywhere. The number of discontinuities is not countable, but uncountable (there are as many as the real numbers).

In a somewhat simplistic, non-rigorous manner, we could also state the following heuristic criterion:

if $R_X(\tau)$ ***is continuous, then the process realizations "can be drawn with paper and pencil"***

which means that they are "reasonable" signals, that can be plotted, rather than highly pathological ones, such as everywhere-discontinuous functions of time.

Optional: for the interested reader, the above condition ensures that the process realizations are not too pathological because it ensures the so-called "mean-square continuity". In detail, what is ensured is the following:

$$\lim_{\tau \to 0} E\left\{ \left[X(t) - X(t-\tau) \right]^2 \right\} = 0$$

See https://en.wikipedia.org/wiki/Continuous_stochastic_process for more information on the topic.

End of optional.

12.3.7.6 Optional: everywhere positive Fourier transform

The following property also holds for statistical autocorrelation, similar to what happens for deterministic-signals autocorrelations:

the Fourier transform of any stationary autocorrelation function is everywhere positive

$$F\left\{ R_X(\tau) \right\} \geq 0 \qquad \forall f \in \mathbb{R}$$

This result will be justified later on, in Section 12.5.2.

End of optional material

12.3.7.7 stationary autocovariance and correlation coefficients

The autocovariance of a stationary process has the same continuity properties as the autocorrelation function, because it is simply rescaled through the relationship:

$$K_X(\tau) = R_X(\tau) - \mu_X^2$$

Note that of course:

$$K_X(0) = \sigma_X^2$$

The correlation coefficient takes on a simple form:

$$\rho_X(\tau) = \frac{R_X(\tau) - \mu_X^2}{\sigma_X^2} = \frac{K_X(\tau)}{\sigma_X^2}$$

Eq. 12-47

It has the same continuity properties as $R_X(\tau)$ and its bounds are as usual:

$$\rho_X(\tau) \in [-1,1]$$

Eq. 12-48

Note that it *always* is: $\rho_X(0) = 1$.

Putting together Eq. 12-47 and Eq. 12-48, we also find immediately:

$$\left|K_X(\tau)\right| \leq \sigma_X^2$$

that is, the autocovariance of a WSS process is bounded between the values $\pm\sigma_X^2$.

The property of the realizations being plottable with pen and paper can be re-phrased in terms of the autocorrelation coefficient as follows:

a process realizations "can be drawn with paper and pencil" if $\lim_{\tau\to 0}\rho_X(\tau)=1$

In fact, if we consider the highly pathological process $X_{\mathrm{SI}}(t)$ from Section 12.2.4.1, then we can easily find that its correlation coefficient is discontinuous at the origin:

$$\rho_{X_{\mathrm{SI}}}(0)=1$$
$$\rho_{X_{\mathrm{SI}}}(\tau)=0 \quad , \quad \tau\neq 0$$

In particular, $\lim_{\tau\to 0}\rho_X(\tau)=0$ and not 1, and in fact the realizations of the process

"cannot be drawn with paper and pencil".

12.3.7.8 stationarity for cross-correlation, cross-covariance and cross-correlation coefficient

Also cross-process expectations, such as the cross-correlation, cross-covariance and cross-correlation coefficients of two processes, can be stationary. Specifically, if

$$R_{XY}\left(t_1,t_2\right)=E_{XY}\left\{X(t_1),Y(t_2)\right\}=R_{XY}\left(\tau\right)$$

$$K_{XY}\left(t_1,t_2\right)=R_{XY}(t_1,t_2)-\mu_X(t_1)\mu_Y(t_2)=K_{XY}\left(\tau\right)$$

$$\rho_{XY}(t_1,t_2)=\frac{K_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}=\frac{K_{XY}(\tau)}{\sigma_X\sigma_Y}=\rho_{XY}(\tau)$$

then such quantities are said to be stationary.

12.4 Stochastic Processes and LTI Systems

In practical systems, such as transmission and control systems, random signals go through LTI systems, such as electronic circuits, filters etc.

It is therefore essential to be able to predict what happens in this kind of situation.

12.4.1 "Filtering" a random process

We want to study the following situation:

$$X(t) * h(t)$$

where $X(t)$ is a RP. One way of looking at this processing is the following: for

each realization of $X(t)$, we compute the corresponding LTI system output, which we call $Y(t;s_k)$:

$$Y(t;s_k) = X(t;s_k) * h(t)$$

Eq. 12-49

In this formula, all signals are deterministic. We could then collect the outputs $Y(t;s_k)$ and form a RP out of them, which we could call $Y(t)$. The probabilistic characterization of $Y(t)$ would be inherited somehow from that of $X(t)$, by the correspondence of the realizations obtained through Eq. 12-49.

This procedure makes sense but it may not always work. The reason is that, depending on the features of the realizations of $X(t)$ and that of the system $h(t)$, for certain realizations the convolution integral might not be possible to calculate or may not converge:

$$X(t;s_k) * h(t) \Rightarrow \text{no convergence}$$

In this case, it is not possible to build a properly defined output process $Y(t)$.

Instead, if there is convergence for all realizations, that is Eq. 12-49 always provides an output signal for each input realization, we can then write:

$$Y(t) = X(t) * h(t)$$

where $Y(t)$ is a properly defined new RP.

Optional:

There are rather mild ***sufficient*** conditions (but not necessary: we shall see exceptions like the WGN process) under which this happens. They are the following:

- The LTI system is stable BIBO;
- the process $X(t)$ is such that $|\mu_X(t)| < \infty$ and $|\sigma_X^2(t)| < \infty$, for all $t \in [-\infty, \infty]$
- the process realizations have at most a countable number of "jump" discontinuities (such as, for instance, the data transmission process of infinite duration)

End of optional material

Having established that the output $Y(t)$ is still a RP, it is in general not easy to find its joint pdf's, even when these are completely available for the input process. In some cases it may not be possible at all to do it analytically.

There are however some powerful results for the more important fundamental expectations, such as the mean and autocorrelation.

12.4.1.1 mean of a filtered process

Given a process $X(t)$ and an LTI system $h(t)$, assuming the process

$Y(t) = X(t) * h(t)$ *is properly defined, then*

$$\mu_Y(t) = \mu_X(t) * h(t)$$

In addition, if $X(t)$ *is WSS, then*

$$\mu_Y = \mu_X H(0)$$

Eq. 12-50

The proof is as follows:

$$\mu_Y(t) = E\{X(t) * h(t)\} = E_X\left\{\int_{-\infty}^{+\infty} X(\tau)h(t-\tau)d\tau\right\} =$$
$$= \int_{-\infty}^{+\infty} E\{X(t)\}h(t-\tau)d\tau = \int_{-\infty}^{+\infty} \mu_X(t)h(t-\tau)d\tau =$$
$$= \mu_X(t) * h(t).$$

If the process is WSS, then $\mu_X(t) = \mu_X$ and:

$$\mu_Y(t) = \int_{-\infty}^{+\infty} \mu_X(t) h(t-\tau) d\tau = \mu_X \int_{-\infty}^{+\infty} h(t-\tau) d\tau =$$
$$= \mu_X \int_{-\infty}^{+\infty} h(\theta) d\theta = \mu_X \int_{-\infty}^{+\infty} h(\theta) e^{-j2\pi f\theta} d\theta \Big|_{f=0} =$$
$$= \mu_X \mathrm{F}\{h(\theta)\}_{f=0} = \mu_X H(0)$$

12.4.1.2 autocorrelation of a filtered process

We now look at autocorrelation, assuming again that we are able to make sure that $Y(t) = X(t) * h(t)$ is still a process. We have:

$$R_Y(t_1,t_2) = E\{Y(t_1)Y(t_2)\} =$$
$$E_X\left\{\int_{-\infty}^{+\infty} h(\theta_1)X(t_1-\theta_1)d\theta_1 \int_{-\infty}^{+\infty} h(\theta_2)X(t_2-\theta_2)d\theta_2\right\} =$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)E\{X(t_1-\theta_1)X(t_2-\theta_2)\}d\theta_1 d\theta_2 =$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)R_X(t_1-\theta_1,t_2-\theta_2)d\theta_1 d\theta_2$$

Eq. 12-51

If the input process is non-stationary for the autocorrelation, there is nothing else we can do to further simplify this formula.

In case $X(t)$ is WSS we can proceed further, to obtain one of the most powerful results of the combined theory of WSS processes and LTI systems. Together with the result for the mean, we can state that:

given a WSS process $X(t)$ *and an LTI system* $h(t)$*, assuming the process* $Y(t) = X(t) * h(t)$ *is properly defined, then* $Y(t)$ *is WSS as well, with:*

$$\mu_Y = \mu_X \cdot H(0)$$
$$R_Y(\tau) = R_X(\tau) * R_h(\tau)$$

To show it, we start from Eq. 12-51. We assume that $X(t)$ is WSS. We then remark that, since we now know that the autocorrelation only depends on the difference of the two arguments, we can write:

$$R_X(t_1 - \theta_1, t_2 - \theta_2) = R_X\left(\left[t_1 - \theta_1\right] - \left[t_2 - \theta_2\right]\right)$$
$$= R_X\left(t_1 - t_2 - \left[\theta_1 - \theta_2\right]\right) = R_X\left(\tau - \left[\theta_1 - \theta_2\right]\right)$$

having replaced: $\tau = t_1 - t_2$. We then get:

$$R_Y(t_1,t_2) = E\left\{Y(t_1)Y(t_2)\right\} = \\ = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)R_X\left(\tau - \left[\theta_1 - \theta_2\right]\right)d\theta_1 d\theta_2$$

Eq. 12-52

The right-hand side clearly depends only on the time-lag τ and not on the individual absolute times t_1, t_2. This is already an important result, because it shows that forcedly the left-hand side, $R_Y(t_1,t_2)$, must also depend only on τ, that is:

$$R_Y(t_1,t_2) = R_Y(\tau)$$

We carry out the rest of the derivation in *frequency domain*. We first Fourier-transform $R_Y(\tau)$, of course vs. τ as time-variable. We have:

$$F\{R_Y(\tau)\} = F\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)R_X\left(\tau-[\theta_1-\theta_2]\right)d\theta_1 d\theta_2\right\}$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)F\left\{R_X\left(\tau-[\theta_1-\theta_2]\right)\right\}d\theta_1 d\theta_2$$

The Fourier transform filters through the integrals. It also lets out $h(\theta_1), h(\theta_2)$ because they do not depend on τ. It stops at $R_X\left(\tau-[\theta_1-\theta_2]\right)$ since this function does depend on τ. Calling:

$$F\{R_X(\tau)\} = G_X(f)$$

and then applying the delay property, we get:

$$F\left\{R_X\left(\tau-[\theta_1-\theta_2]\right)\right\} = G_X(f)e^{-j2\pi(\theta_1-\theta_2)} = G_X(f)e^{-j2\pi\theta_1}e^{+j2\pi\theta_2}$$

Substituting the above expression inside the integrals, we get:

$$F\{R_Y(\tau)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)G_X(f)e^{-j2\pi\theta_1}e^{j2\pi\theta_2}d\theta_1 d\theta_2$$
$$= G_X(f)\int_{-\infty}^{+\infty} h(\theta_1)e^{-j2\pi\theta_1}d\theta_1 \int_{-\infty}^{+\infty} h(\theta_2)e^{j2\pi\theta_2}d\theta_2$$
$$= G_X(f)H(f)H^*(f) = G_X(f)|H(f)|^2$$

We now come back to the time domain by taking the inverse Fourier transform of both sides. We get a very simple and extremely powerful result:

$$R_Y(\tau) = R_X(\tau) * F^{-1}\left\{|H(f)|^2\right\} = R_X(\tau) * R_h(\tau)$$

Eq. 12-53

having defined:

$$R_h(\tau) = F^{-1}\left\{|H(f)|^2\right\}$$

Note that from Chapter 10 we learnt that:

$$R_h(\tau) = \int_{-\infty}^{\infty} h(t) h^*(t-\tau)\, dt = F^{-1}\left\{ \left| H(f) \right|^2 \right\}$$

is the so-called "autocorrelation" for deterministic signals.

Therefore, the autocorrelation function of a WSS process at the output of a LTI system is simply the convolution of the *statistical* autocorrelation of input process with the autocorrelation (*in the sense of deterministic signals*) of the LTI system impulse response:

$$R_Y(\tau) = R_X(\tau) * R_h(\tau)$$

The above result also shows that:

assuming that a WSS process goes through an LTI system, the output process

is WSS as well

In fact, we have:

$$\mu_Y(t) = \mu_X H(0) = \mu_Y$$

$$R_Y(t_1, t_2) = R_X(\tau) * R_h(\tau) = R_Y(\tau)$$

which means that a stationary mean and autocorrelation at the input remain stationary at the output. This result can actually be generalized (see below).

12.4.1.3 stationarity and LTI systems

A very general and quite important result on stationarity is:

given a process $X(t)$ *and an LTI system* $h(t)$*, then the process* $Y(t) = X(t) * h(t)$ *is at least as stationary as* $X(t)$ *and possibly more.*

For instance, if $X(t)$ is second-order stationary, also $Y(t)$ is second-order stationary (or possibly greater stationarity, but not less). If $X(t)$ is stationary only for certain moments, then $Y(t)$ is stationary for the same moments (and possibly others).

The underlying reason for this preservation of stationarity is the *time-invariance* of LTI systems.

12.4.2 Example of a non-WSS process through an LTI system

One non-WSS process that we have considered is the data transmission process of the form:

$$X(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, \pi_T(t - nT)$$

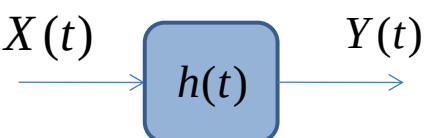
where the α_n are statistically independent RVs that can take on the values +1 and -1 with equal probability ½ each. We found (see 12.2.8.5) that

$$\mu_X = 0$$

$$R_X(t_1, t_2) = K_X(t_1, t_2) = \sum_{n=-\infty}^{\infty} \pi_T(t_1 - nT)\,\pi_T(t_2 - nT)$$

Clearly, the autocorrelation and autocovariance are not stationary.

We now assume that $X(t)$ is passed through an LTI system with impulse response $h(t) = \pi_T(t)/T$:



Since this is a quasi-defined process, the realizations of $X(t)$ are known in analytical form. Therefore, we can attempt to find explicitly the correspondence between input and output realizations.

We can also check that indeed for each input realization there is an output one, and hence $Y(t)$ is a legitimate random process.

In fact, we can formally write for a single realization:

$$Y(t;s_n)=\left[\sum_{n=-\infty}^{\infty} a_n\ \pi_T(t-nT)\right]*h(t)$$

$$=\sum_{n=-\infty}^{\infty} a_n\left[\pi_T(t-nT)*h(t)\right]=\sum_{n=-\infty}^{\infty}\frac{a_n}{T}\left[\pi_T(t-nT)*\pi_T(t)\right]$$

$$=\sum_{n=-\infty}^{\infty} a_n\ \Lambda_T(t-[n+1]\cdot T)$$

where the a_n are the actual values (either 1 or -1) taken on by each RV α_n when the "random experiment" is carried out and the realization is generated.

Looking at the formula, there appears to be no analytical difficulty in filtering any possible input realizations. So, we can actually re-instate the α_n and simply conclude that the output process is:

$$Y(t)=\sum_{n=-\infty}^{\infty}\alpha_n\ \Lambda_T(t-[n+1]\cdot T)$$

We then want to characterize the mean value. The calculation can be done as it was done in 12.2.8.5. But we know that $\mu_X = 0,$ i.e., we can write:

$$\mu_Y = \mu_X \cdot H\left(0\right) = 0 \cdot H\left(0\right) = 0$$

Regarding the autocorrelation, since it not stationary, we are forced to use the formula Eq. 12-51:

$$R_Y(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\theta_1) h(\theta_2) R_X(t_1 - \theta_1, t_2 - \theta_2) d\theta_1 d\theta_2$$

We have:

$$R_Y(t_1,t_2) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)h(\theta_2)\sum_{n=-\infty}^{\infty}\pi_T\left(t_1-\theta_1-nT\right)\pi_T\left(t_2-\theta_2-nT\right)d\theta_1 d\theta_2$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{+\infty} h(\theta_1)\pi_T\left(t_1-\theta_1-nT\right)d\theta_1 \int_{-\infty}^{+\infty} h(\theta_2)\pi_T\left(t_2-\theta_2-nT\right)d\theta_2$$

It is interesting to see that the integral completely separates in this example (it is not however a general property). So, they can be executed independently. They are both of the type:

$$\int_{-\infty}^{+\infty} h(\theta)\,\pi_T\left(t-\theta-nT\right)d\theta = \frac{1}{T}\int_{-\infty}^{+\infty}\pi_T(\theta)\,\pi_T\left(t-\theta-nT\right)d\theta$$

$$= \frac{1}{T}\pi_T\left(t\right) * \pi_T\left(t-nT\right) = \frac{1}{T}\pi_T\left(t\right) * \pi_T\left(t\right) * \delta\left(t-nT\right)$$

$$= \Lambda_T\left(t-T\right) * \delta\left(t-nT\right) = \Lambda_T\left(t-\left[n+1\right]T\right)$$

So in the end:

$$R_Y(t_1,t_2)=\sum_{n=-\infty}^{\infty}\Lambda_T\left(t_1-[n+1]T\right)\Lambda_T\left(t_2-[n+1]T\right)$$

It is interesting to compare the plots of $R_X(t_1,t_2)$ in Fig. 12-4 and $R_Y(t_1,t_2)$ in Fig. 12-7 below. On your own try to make sense of the differences between the two.

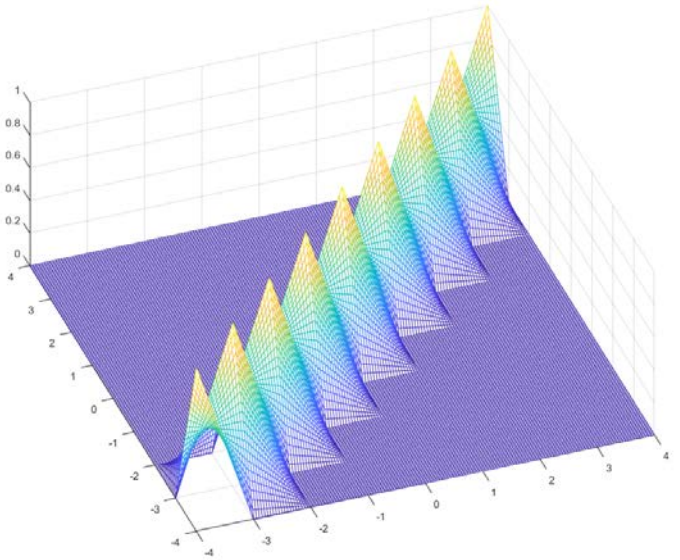


Fig. 12-7

12.4.3 LTI systems and Gaussian processes

Jointly Gaussian processes are important as they typically are good models for noise in electronic circuits. Quite commonly such noise goes through LTI systems, typically the electronic circuits themselves.

It is therefore important to extract all the possible statistical information present in the output process $Y(t) = X(t) * h(t)$, when the input process $X(t)$ is indeed jointly Gaussian.

We therefore provide without proof the following quite general and very significant result:

given a jointly Gaussian process $X(t)$ ***and an LTI system*** $h(t)$, ***assuming the process*** $Y(t) = X(t) * h(t)$ ***is properly defined, then*** $Y(t)$ ***is jointly Gaussian too.***

Note that this property of preserving the same general form of the statistical distribution is ***unique*** to the jointly Gaussian processes. If $X(t)$ has any other statistical distribution, then the statistical distribution of $Y(t)$ is *different* and also typically quite difficult to estimate.

The underlying reason for this preservation of the Gaussian distribution is the *linearity* of LTI systems (together with the property of the sum of joint Gaussian RVs always producing joint Gaussian RVs, even when correlated among themselves).

We can now pull together a number of results obtained in the previous sections. We can state the following ***important set of properties for filtered Gaussian processes***.

Assuming as usual that the output process $Y(t)$ is properly defined, then:

given a process $X(t)$ and an LTI system $h(t)$, if $X(t)$ is WSS and jointly Gaussian, with autocorrelation $R_X(\tau)$ and mean μ_X, then:

- *$X(t)$ is SSS*
- *$Y(t)$ is jointly Gaussian and SSS*
- *we can characterize the jointly Gaussian pdf of $Y(t)$ to any order just by means of the two quantities:*

$$\mu_Y = \mu_X \cdot H(0)$$
$$R_Y(\tau) = R_X(\tau) * R_h(\tau)$$

Therefore, if we know the statistical features of a jointly-Gaussian WSS (that is, SSS) input process to an LTI system, *we can easily find all the statistical features of the output* WSS (and also SSS) jointly-Gaussian process too.

12.4.4 White Gaussian Noise (WGN)

We introduce a *very special* Gaussian process, called **White Gaussian Noise (WGN)** and typically represented with the symbol $N(t)$.

We must say from the start that this process is non-physical. Nonetheless, it is universally used to model noise in electrical, optical and various other physical systems, since it effectively approximates such physical noise and it is easy to mathematically describe and handle.

Its role is similar to that of the delta signal, which is clearly a non-physical signal and yet is a very effective tool, which is used to model certain situations and to carry out certain calculations effectively.

By definition, $N(t)$ has the following properties:

- it is jointly Gaussian
- it is WSS and therefore SSS
- it is zero-mean: $\mu_N = 0$
- its auto-correlation and autocovariance functions are given by:

$$R_N(\tau) = K_N(\tau) = \frac{N_0}{2}\delta(\tau)$$

Eq. 12-54

A quick look at $R_N(\tau)$ immediately tells us that the realizations of this process are highly pathological. Not only the autocorrelation function is *discontinuous*, but it even *diverges* at the origin. So, there we will not be able to draw the realizations at all.

Also, $K_N(\tau) = 0(\tau)$ for all values of τ (except $\tau = 0$). Therefore, any two samples of $N(t)$ are statistically independent, no matter how close in time they are taken.

This property was also shared by the process $X_{\mathrm{SI}}(t)$ of Section 12.2.4.1, whose autocorrelation was also discontinuous at the origin. Differently from $X_{\mathrm{SI}}(t)$, however, the RV's extracted from $N(t)$ have *infinite* variance. In fact, it is:

$$\sigma_N^2 = R_N(\tau)\Big|_{\tau=0} = \frac{N_0}{2}\delta(\tau)\Big|_{\tau=0} \to \infty$$

Though it may seem difficult that such a strange process has any meaning at all, in fact it does, and we shall later better understand why.

12.4.4.1 filtering a WGN process

If we "filter" a WGN process, that is, we put it through an LTI system, we formally have:

$$Y(t) = N(t) * h(t)$$

and also assuming as a hypothesis that the output $Y(t)$ is still a proper process, then, due to the properties of jointly Gaussian processes going through LTI systems, $Y(t)$ has the following noteworthy features:

- *it is jointly Gaussian*
- *it is SSS*
- *it is zero-mean:* $\mu_Y = 0$
- *its auto-correlation function is given by:*

$$R_Y(\tau) = \frac{N_0}{2} R_h(\tau)$$

The result about the autocorrelation is due to the fact that:

$$R_Y(\tau) = R_N(\tau) * R_h(\tau) = \frac{N_0}{2} \delta(\tau) * R_h(\tau) = \frac{N_0}{2} R_h(\tau)$$

The output process then looks a lot less pathological than the input one. Note that, assuming $h(t)$ is at least C^{-1}, then $R_h(\tau)$ is at least C^0, i.e., a continuous function. As a result, the autocorrelation of $Y(t)$ is continuous too and its realizations are not "pathological" (it should be possible to *draw* them, unlike the process $X_{\mathrm{SI}}(t)$ of Section 12.2.4.1) or WGN itself.

Note also that by putting WGN through an LTI system we obtain a process whose

statistical autocorrelation is assigned by the deterministic autocorrelation of the "filter". This is a quite powerful result of the theory of filtered processes.

Finally, one may wonder why $N(t)$ is called "white" noise. This aspect will become clear after spectral analysis of RP's has been introduced.

12.5 Spectral Properties of Random Processes

Random processes can be analyzed in frequency domain, too. One could think of taking the Fourier transform of each realization and then studying the statistical properties of such transforms. Although this is in principle possible, it is not often done, for practical reasons. In addition, finding the transform of each realization is typically easy or possible only for quasi-defined processes.

In the context of this course, we will limit the scope of our spectral analysis: *we*

make the key assumption that the process realizations must be finite-average-power signals.

If so, a suitable spectral characterization quantity is the *average power spectral density* of the process. This quantity estimates how much power is carried by each frequency, averaged across all possible realizations. Even though more sophisticated spectral analyses could be performed, the process average power spectral density is typically enough in most pratical cases.

We will also show that if the process is at least WSS, i.e., its statistical mean and autocorrelation are stationary, then its realizations are typically finite-average power signals and its average power spectral density is quite easy to obtain.

The circumstamce that the realizations of WSS processes are typically finite-average-power signals is discussed below.

12.5.1 Nature of the realizations of WSS processes

A WSS process is characterized by having a constant variance. This aspect alone shows that this process is "sustained" over the whole of $\mathbb{R}$, that is, there is no such thing as a time t_0 past which all the realizations go identically to zero, or even just all of them tend to vanish, for time that proceeds towards infinity. If they did, then the process variance would have to go to zero as well, contradicting the WSS assumption that the variance is constant in time.

Therefore, in general, the realizations of WSS processes are signals that *persist* in time, for all times. A *persistent* signal, i.e., a non-vanishing signal, is typically a finite-average power signal, as we have seen in previous chapters. Therefore, ***a WSS process has realizations that are in general finite-average-power signals.***

As a result, these processes should be looked at in terms *power quantities*, such as their *average power* and *average power spectral density*. This is the approach

that we are going to follow.

12.5.2 Average power of a process

In the context of *deterministic signals*, we defined the quantity: $P_x(t) = x^2(t)$ as the "instantaneous power" of the signal. The time-averaged signal power was then given by:

$$P\{x(t)\} = < P_x(t) >_{\mathbb{R}} = < x^2(t) >_{\mathbb{R}}$$

Eq. 12-55

where $P_x(t)$ is the instantaneous power of the signal $x(t)$.

Of course, for deterministic signals, the average was a *time-average*, since there is nothing statistical or "random" in the signal. However, we would like to use this definition of average power for processes too, because it makes physical sense. In the following, we will always call it *time-averaged power*, to ensure that it is not confused with statistical averages.

To use such definition in the context of random processes, we must consider each individual process realization $X(t;s_n)$. Since each realization is a *deterministic signal*, we can compute its time-averaged power using Eq. 12-55, by just replacing $x(t)$ with $X(t;s_n)$:

$$P\{X(t;s_n)\} = < X^2(t;s_n) >_{\mathbb{R}}$$

The problem with this procedure is that, in general, we could get a different time-averaged power for each realization of the process. In other words, if we do not specify the realization, but simply write:

$$P\{X(t)\} = < X^2(t) >_{\mathbb{R}}$$

then the quantity that we have written, $P\{X(t)\}$, is not a number, but a *random*

variable, say ξ, whose value depends on which realization the random experiment produces:

$$P\{X(t)\} = \xi$$

But ot course we would like to get a single defined value, not a random variable. To solve this problem, we could perform another average, a statistical one, over all possible realizations s_n, by formally *also* taking a *statistical average after the time-average*:

$$P_X = E\{\xi\} = \underset{X}{E}\left\{P\{X(t)\}\right\} = \underset{X}{E}\left\{< X^2(t) >_{\mathbb{R}}\right\}$$

Eq. 12-56

It turns out that *this formula is the correct one*, for finding a proper average power P_X of *any* process, including non-WSS processes. In essence, it is a double-average, consisting of a cascaded time and statistical average.

Unfortunately, Eq. 12-56 is somewhat cumbersome and difficult to use in many cases, due to the two nested operators. The question is then: is there a simpler equivalent formula that allows to achieve the same result P_X ?

Eq. 12-56 shows two nested operators, one is a statistical average and the other is a time-average. They are both linear and operate over different variables (one over time t and the other over the process X or equivalently over the formal variable s labeling each realization).

As a result, they can be swapped (see for instance [1]):

$$P_X = < E\left\{ X^2(t) \right\} >_{\mathbb{R}} = < \mathrm{msv}_X(t) >_{\mathbb{R}}$$

Eq. 12-57

This change of order already helps, because we often can compute the mean-

square value of the process, $\text{msv}_X(t)$ relatively easily. We then need to average it over time.

However, if we can claim that the process is *stationary for the mean-square-value*, then $\text{msv}_X(t)$ is time-independent, i.e., a constant msv_X. The outer time-average has no effect on a constant and the result is the constant itself. We would simply have:

$$P_X = E\left\{X^2(t)\right\} = \text{msv}_X$$

In particular, WSS processes are stationary for the mean-square-value (it is easy to show it, prove it on your own), which in turn coincides with the autocorrelation at $\tau = 0$. So, for WSS processes we can write:

$$P_X = E\left\{X^2(t)\right\} = R_X(0) = \text{msv}_X$$

Eq. 12-58

In other words, for WSS processes the average power of the process coincides with their (stationary) mean square value or, equivalently, with the value of the autocorrelation at the origin.

If the process is non-stationary for the mean-square-value, either Eq. 12-56 or Eq. 12-57 should be used.

12.5.3 Power spectral density of process

For the power spectral density (or power spectrum) of a process, we follow the same reasoning that we used for average power in the previous section.

First off, we concentrate on a specific realization of the process, which is a fully

deterministic signal: $X(t;s_n)$. For such signal, the power spectral density is the Fourier transform of its *deterministic signal* time-autocorrelation. Since we are assuming that the realizations are finite-average-power signals, we have to use the following definition of time-autocorrelation:

$$\Phi_X(\tau;s_n) = < X(t;s_n)X(t-\tau;s_n) >_{\mathbb{R}}$$

Then, the corresponding power spectral density is:

$$G_X(f;s_n) = F\left\{\Phi_X(\tau;s_n)\right\} = F\left\{< X(t;s_n)X(t-\tau;s_n) >_{\mathbb{R}}\right\}$$

Apart from notation, the above formulas are not new: they are those introduced in Chapter 10 for non-random signals.

If instead of one specific realization, we applied the above formula to the whole

process, that is to the whole set of realizations, we could write:

$$G_X(f) = F\left\{\Phi_X(\tau)\right\} = F\left\{< X(t)X(t-\tau) >_{\mathbb{R}}\right\}$$

Eq. 12-59

The difference vs. the previous formula is that the result, instead of being a deterministic function, is in fact a random process. In particular, $\Phi_X(\tau)$ is a random process in the time-variable τ and $G_X(f)$ is a random process in frequency. They are not single functions, but a collection of possible realizations of either the time-autocorrelation or the power spectral density, possibly all different, depending on the realization of the process $X(t)$ that generates them.

This is typically too complicated a description. What we would typically like to see is the *statistically averaged* time-autocorrelation and a *statistically averaged* power spectral density of the realizations of the process.

To formally obtain them, we can use the expectation operator. We start out with Φ:

$$E\left\{\Phi_X(\tau)\right\} = E\left\{< X(t)X(t-\tau) >_{\mathbb{R}}\right\}$$

We now concentrate on the right-hand side. There are actually two cascaded operators: statistical average and time-average. They are both linear and they operate on independent quantities. We therefore argue that they can be swapped. In particular, we let the statistical average migrate inside:

$$E\left\{\Phi_X(\tau)\right\} =< E\left\{X(t)X(t-\tau)\right\} >_{\mathbb{R}}$$

The inner expectation is of course the *statistical autocorrelation* of $X(t)$ and therefore we can write:

$$E\left\{\Phi_X(\tau)\right\} =< R_X(t, t-\tau) >_{\mathbb{R}}$$

Eq. 12-60

If the RP was non-stationary for the autocorrelation, we would have to perform the time-average over t, to obtain the final result. However, we are assuming WSS processes and therefore we can write:

$$E\left\{\Phi_X\left(\tau\right)\right\} =< R_X\left(\tau\right) >_{\mathbb{R}}$$

We then argue that $R_X\left(\tau\right)$ does not depend on t and therefore the time-average has no effect on it. We can then write:

$$E\left\{\Phi_X\left(\tau\right)\right\} = R_X\left(\tau\right)$$

Eq. 12-61

This result is of great importance, as it relates a rather complex quantity $E\{\Phi_X(\tau)\}$ to one of the key statistical parameters of a WSS process, its stationary statistical autocorrelation.

Having obtained $E\{\Phi_X(\tau)\}$, we now want to find $E\{G_X(f)\}$. Eq. 12-59 told us that:

$$G_X(f) = F\{\Phi_X(\tau)\}$$

Taking the statistical expectation of both sides:

$$E\{G_X(f)\} = E\{F\{\Phi_X(\tau)\}\}$$

In the right-hand side we have once again two linear operators operating on different quantities. We swap them and we can write:

$$E\left\{G_X\left(f\right)\right\} = F\left\{E\left\{\Phi_X(\tau)\right\}\right\}$$

The using Eq. 12-60 we write:

$$E\left\{G_X\left(f\right)\right\} = F\left\{E\left\{\Phi_X(\tau)\right\}\right\} = F\left\{< R_X(t,t-\tau) >_{\mathbb{R}}\right\}$$

If the process is non stationary for the statistical autocorrelation R_X, then this is the final formula, which requires time-averaging over t. However, for WSS processes we can write:

$$F\left\{< R_X\left(t,t-\tau\right) >_{\mathbb{R}}\right\} = F\left\{< R_X\left(\tau\right) >_{\mathbb{R}}\right\} = F\left\{R_X\left(\tau\right)\right\}$$

Substituting this formula into the previous one, we obtain one of the most

important and powerful results of the theory of RP, the so-called **Wiener-Kintchin-Einstein formula**.

$$E\{G_X(f)\} = F\{R_X(\tau)\}$$

Eq. 12-62

This formula easily relates a very complex quantity, the average process PSD, to a simple stationary second-order moment of the process.

12.5.3.1 IMPORTANT: notation details

For notational ease, the average power spectral density $E\{G_X(f)\}$ will actually be denoted simply as $G_X(f)$. In other words, ***when we write*** $G_X(f)$ ***we mean the statistically-averaged power-spectral density*** $E\{G_X(f)\}$.

As a consequence, we can formally re-write the Wiener-Kintchin-Einstein

formula Eq. 12-62 for WSS processes more simply as:

$$G_X(f) = F\left\{R_X(\tau)\right\}$$

Eq. 12-63

Also, for *non-stationary processes*, the formula reads:

$$G_X(f) = F\left\{< R_X(t, t-\tau) >_{\mathbb{R}}\right\}$$

Similarly, the statistical average of the time-autocorrelation functions generated by all realizations, $E\left\{\Phi_X(\tau)\right\}$ will actually be denoted simply as $\Phi_X(\tau)$. ***In other words, when we write*** $\Phi_X(\tau)$ ***we always mean thestatistically-averaged time-autocorrelation*** $E\left\{\Phi_X(\tau)\right\}$.

Therefore, we can re-write Eq. 12-62 for a WSS process as:

$$\Phi_X(\tau) = R_X(\tau)$$

Eq. 12-64

For *non-stationary processes*, the formula is:

$$\Phi_X(\tau) = < R_X(t, t-\tau) >_{\mathbb{R}}$$

Eq. 12-65

12.5.3.2 properties of $G_X(f)$

The properties of $G_X(f)$ are similar to those of the power spectral density of a deterministic (non-random) signal. We first remark that, inverting Eq. 12-63 we get:

$$R_X(\tau) = F^{-1}\{G_X(f)\}$$

Then, if we impose $\tau = 0$, we have:

$$R_X(0) = F^{-1}\{G_X(f)\}_{\tau=0} = \int_{-\infty}^{\infty} G_X(f) e^{j2\pi f\tau} df \Big|_{\tau=0} = \int_{-\infty}^{\infty} G_X(f) df$$

But it is also:

$$R_X(0) = E\{X^2(t)\} = P_X$$

so the rightmost sides of the last two equations must be equal too:

$$P_X = \int_{-\infty}^{\infty} G_X(f)df$$

Eq. 12-66

From Eq. 12-67, by differentiation, we can then write:

$$\frac{dP_X}{df} = G_X(f)$$

In other words, we directly show that $G_X(f)$ is *the spectral density of the average power* P_X.

This also means that, if we are interested about how much contribution to the total average power P_X is provided (on average) by the process realizations, over the frequency range $[f_1, f_2]$, we just need to integrate $G_X(f)$ over that frequency

range:

$$P_X\Big|_{[f_1,f_2]} = \int_{f_1}^{f_2} G_X(f)df$$

12.5.3.3 power spectral density of WSS filtered processes

We have seen that for WSS processes:

$$R_Y(\tau) = R_X(\tau) * R_h(\tau)$$

We can then compute the power spectral density of the output and find:

$$G_Y(f) = \mathrm{F}\{R_Y(\tau)\} = \mathrm{F}\{R_X(\tau) * R_h(\tau)\} = G_X(f)\left|H(f)\right|^2$$

So, in essence:

$$G_Y(f) = G_X(f)\left|H(f)\right|^2$$

Eq. 12-67

This formula is formally identical to the one providing the power spectral density of the output of an LTI system, given the power spectral density of the input, for fully deterministic signals.

Eq. 12-68 means that the average power (or mean square value) of the output process $Y(t)$ is now given by:

$$P_Y = E_Y\left\{Y^2(t)\right\} = \int_{-\infty}^{\infty} G_Y(f)df = \int_{-\infty}^{\infty} G_X(f)\left|H(f)\right|^2 df$$

Eq. 12-68

So, the LTI system *physically* affects the process power spectral density

"frequencies" and it does it in the same way it affects them when the input is deterministic.

12.5.3.4 spectral analysis of WGN

Let us consider again the WGN process. It is a WSS process, so that its power spectral density is:

$$G_N(f) = \mathrm{F}\{R_N(\tau)\} = \mathrm{F}\left\{\frac{N_0}{2}\delta(\tau)\right\} = \frac{N_0}{2}\cdot 1(f)$$

Note the remarkable feature that ***the WGN process has a completely flat power spectral density***. The amount of power per unit frequency carried by this process is constant, across all frequencies.

Of course, this leads to paradoxes. In particular, the process average power is infinite:

$$P_N = \int_{-\infty}^{\infty} G_N(f)\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} 1(f)\,df = "\infty"$$

However, this should be of no concern, in the sense that the WGN is simply a special "tool" in RP theory, similar to how $\delta(t)$ is a special tool for dealing with defined signals.

If WGN goes through an LTI system, we can apply the standard formulas and conclude that:

$$G_Y(f) = G_N(f)\left|H(f)\right|^2 = \tfrac{N_0}{2}\left|H(f)\right|^2$$
$$R_Y(\tau) = R_N(\tau) * R_h(\tau) = \tfrac{N_0}{2} R_h(\tau)$$

that is, the output process $Y(t)$ has the same power spectral density as the transfer

function absolute value squared of the LTI system, times the constant $N_0/2$. It also has the a statistical autocorrelation $R_Y(\tau)$ which is identical to the time-average autocorrelation of the (deterministic) LTI system impulse response $R_h(\tau)$, times the constant $N_0/2$.

12.5.3.5 why "white"?

We can now comment on why WGN is called "white". First, we point out that the electric field constituting visible light is a Gaussian random process. It then turns out that the power spectral density of *white light* is essentially flat over the range of frequencies that are visible to the human eye.

By analogy, any RP whose power spectral density is flat in frequency has been called by scientists and mathematicians "white". Note that when noise is *not* flat in frequency, it is called "colored", similar to visible light whose power spectral density is non-flat, which appears "colored" to the eye.

Note that in all physical cases these “white” random processes do not not really have a flat PSD over *all* frequencies. As shown, this produces an infinite-power process (WGN) which is a good mathematical tool but makes no physical sense. The physical “white” processes are processes whose PSD is flat over large intervals of frequencies, but of course not over all frequencies.

Optional

Frequency-flat noise is a very accurate model of electronic noise, from DC up to about 100 GHz. The actual power spectral density of electrical noise measured at the terminals of a physical resistor at a temperature of T_0 Kelvin is in fact:

$$G_N(f) = \frac{R\,hf}{e^{\frac{hf}{k_B T^0}} - 1} \underset{f<100\ \text{GHz}}{\approx} R\,k_B T_0 \cdot 1(f)$$

Eq. 12-69

where k_B is Boltzmann's constant, h is Planck's constant and R is the resistor's resistance value. That is, it appears as white noise with $N_0/2 = R\,k_B T_0$, for $f < 100$ GHz (at room temperature). Above 100 GHz such noise spectrum starts to taper off.

On your own: Plot the above formula and verify that it is flat till about 100 GHz, where it starts decreasing. At what frequency is it ½ of the value that it has in the flat region?

The almost totality of electronic devices operate below 100 GHz, and this means that they are affected by the frequency components of electrical noise which lie below 100 GHz, where such noise is flat. This makes WGN a quite convenient model, for electrical noise, as it is flat too. Of course, WGN beyond 100 GHz, whci is unrealistic for electrical noise, but whatever equipment is assumed to create or be affected by it, would automatically cut-off the unrealistic higher frequencies because of its intrinsic bandwidth limitations.

End of Optional

12.5.4 Examples

12.5.4.1 power spectral density of the random carrier

We want to find out what the power spectral density of the random amplitude and random phase carrier process of Section 12.2.8.3 is. The mathematical expression of this process is:

$$X(t) = \xi \cdot \cos(2\pi f_0 t + \varphi)$$

where the phase is uniformly-distributed over $[0, 2\pi]$ and the amplitude ξ is uniformly-distributed over $[0, A]$. We have found the following results:

$$E_X\{X(t)\} = \mu_X = 0$$

$$R_X(\tau) = K_X(\tau) = \frac{A^2}{6}\cos(2\pi f_0 \tau)$$

So the process is WSS. Its power spectral density is then simply:

$$G_X(f) = \mathrm{F}\{R_X(\tau)\} = \mathrm{F}\left\{\frac{A^2}{6}\cos(2\pi f_0 \tau)\right\} =$$

$$= \frac{A^2}{12}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

In this simple case it is also interesting to see that we can actually calculate the power spectral density of each realization and then take the statistical average afterwards, obtaining the same result, as expected.

The individual k-th realization can be written as:

$$X(t; s_k) = \xi^{(k)} \cdot \cos\left(2\pi f_0 t + \varphi^{(k)}\right)$$

where $\xi^{(k)}$ and $\varphi^{(k)}$ are no longer RV but the specific constant values that are

generated by the random experiment. As a result $X(t;s_k)$ is a fully deterministic signal. We then remark that it is a *periodic* signal and for periodic signals there is an easy way to calculate the power spectral density. First, we need to obtain the Fourier transform, which is simply:

$$X(f;s_k) = F\{X(t;s_k)\} = \frac{\xi^{(k)}}{2} e^{j\varphi^{(k)}} \delta(f - f_0) + \frac{\xi^{(k)}}{2} e^{-j\varphi^{(k)}} \delta(f + f_0)$$

Then, the corresponding PSD has just the same deltas, but the coefficients multiplying them are absolute-value-squared, yielding:

$$G_X(f;s_k) = \frac{\left|\xi^{(k)}\right|^2}{4} \delta(f - f_0) + \frac{\left|\xi^{(k)}\right|^2}{4} \delta(f + f_0)$$

Finally, we want to average over all possible realizations:

$$G_X(f) = E_X\{G_X(f;s)\}$$
$$= \frac{E_\xi\{|\xi|^2\}}{4}\delta(f-f_0) + \frac{E_\xi\{|\xi|^2\}}{4}\delta(f+f_0)$$
$$= \frac{A^2}{12}\delta(f-f_0) + \frac{A^2}{12}\delta(f+f_0)$$

which exactly coincides with the previous result found using the Wiener-Kintchin-Einstein formula.

12.5.4.2 power spectral density of the data transmission process

We recall the data transmission process with random delay:

$$X(t) = \sum_{n=-\infty}^{\infty} a_n \pi_T(t - nT - \theta)$$

where θ is a random delay, uniformly distributed over the interval $[0,T]$. Also, θ is independent of the a_n's.

The result we found were:

$$\mu_X(t) = 0$$

$$R_X(\tau) = \frac{1}{T} \mathrm{R}_{\pi_T}(\tau) = \Lambda_T(\tau)$$

$$K_X(\tau) = \Lambda_T(\tau)$$

The process is therefore WSS. We can directly find $G_X(f)$ as:

$$G_X(f) = \mathrm{F}\{R_X(\tau)\} = \mathrm{F}\{\Lambda_T(\tau)\} = T\,\mathrm{Sinc}^2\{fT\}$$

12.5.4.3 Optional: power spectral density of the periodic random process

The periodic random process $X(t) = \sqrt{f_0} \sum_{n=-\infty}^{+\infty} \xi_n e^{j2\pi n f_0 t}$ introduced in Section 12.2.8.11 was found to be WSS, because it turned out that:

$$\mu_X(t) = 0$$

$$R_X(\tau) = 2 f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 e^{j2\pi n f_0 \tau}$$

Its power spectral density is:

$$G_X(f) = F\{R_X(\tau)\} = F\left\{ 2f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 e^{j2\pi n f_0 \tau} \right\}$$

$$= 2f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 F\left\{ e^{j2\pi n f_0 \tau} \right\} = 2f_0 \sum_{n=-\infty}^{+\infty} \sigma_n^2 \, \delta(f - n f_0)$$

Therefore, this process has a line-spectrum, whose deltas are multiplied times the variance of the corresponding zero-mean random variable ξ_n's.

End of optional material

12.6 Optional: Ergodicity of a random process

Note: the entire section 12.6, with all of its subsections, is optional.

A random process is said to be ***ergodic*** *for a specific average if the statistical*

average and the temporal average coincide for each and all single realizations of the process.

For instance, the customary statistical mean of the process is:

$$E\{X(t)\} = \int_{-\infty}^{+\infty} y\, f_X(y)dy$$

Its corresponding temporal average is:

$$\langle X(t;s_0)\rangle_{\mathbb{R}} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} X(t;s_0)dt$$

where the subscript s_0 means that we are looking at one specific realization. Then, we can say that the process is *ergodic for the mean* if:

$$E\{X(t)\} = \langle X(t;s_0)\rangle_{\mathbb{R}} \quad \forall s_0$$

Eq. 12-70

Note that the right-hand side cannot be a function of time, because time is averaged out. So, for ergodicity to be verified, the result must be a number. Therefore ergodicity for the mean intrinsically requires that the process be stationary for the statistical mean:

$$E\{X(t)\} = \langle X(t;s_0)\rangle_{\mathbb{R}} = \mu_X$$

Similarly, a process is *ergodic for the autocorrelation* if:

$$E\{X(t)X(t-\tau)\} = \langle X(t;s_0)X(t-\tau;s_0)\rangle_{\mathbb{R}} \quad \forall s_0$$

Eq. 12-71

where:

$$\left\langle X(t;s_0)X(t-\tau;s_0)\right\rangle_{\mathbb{R}} =$$

$$\Phi_X(\tau;s_0) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} X(t;s_0)X^*(t-\tau;s_0)dt$$

is the autocorrelation of the deterministic signal $X(t;s_0)$. Notice that $\left\langle X(t;s_0)X(t-\tau;s_0)\right\rangle_{\mathbb{R}}$ can only be a function of τ and not t because time is used for time-averaging. As a result, also $E\{X(t)X(t-\tau)\}$ is a function of τ and not t. Therefore, ergodicity for the autocorrelation requires that the process be stationary for the (statistical) autocorrelation, which we can then write $R_X(\tau)$.

In short, Eq. 12-72 tells us that, if the process is *ergodic for the autocorrelation*, then:

$$R_X(\tau) = \Phi_X(\tau; s_0) \quad \forall s_0$$

Eq. 12-72

This relation indicates that the right-hand side does not really depend on s_0, because the left hand side of the equation does not depend on s_0 either. In other words $\Phi_X(\tau; s_0)$, which is the autocorrelation (in the sense of deterministic signals) of the single realization $X(t; s_0)$, and which would in general depend on the actual realization, is independent of the chosen realization. It yields in fact the same result $R_X(\tau)$, whatever the process realization used to compute it.

This result is very important in practical applications because it implies that the autocorrelation of the entire process $R_X(\tau)$ can be estimated by performing a time-average over any single realization $X(t; s_0)$. It is not important which specific

realization of the process is used, any of the possible realizations $X(t;s_n)$ is adequate for estimating $R_X(\tau)$. All other realizations would provide the same result.

Note that if the process was not ergodic, one could still use the time-averaged autocorrelation to estimate $R_X(\tau)$, but the procedure would *also* require averaging the result *over all possible realizations*. In formulas:

$$R_X(\tau) = E_s\left\{\Phi_X(\tau;s)\right\}$$

Ergodicity has a direct impact on the practical estimation of the power spectral density (PSD) of a process, too. As already pointed out, the PSD of a process is:

$$G_X(f) = \mathrm{F}\left\{R_X(\tau)\right\}$$

but if a RP $X(t)$ is ergodic for the autocorrelation, then:

$$G_X(f) = F\left\{R_X(\tau)\right\} = F\left\{\Phi_X(\tau; s_0)\right\} \quad \forall s_0$$

So it is enough to observe a single realization $X(t; s_0)$, compute its autocorrelation $\Phi_X(\tau; s_0)$ and then Fourier-transform it to obtain the PSD of all the process. In other words, the PSD of each single realization is the same.

If the process $X(t)$ was not ergodic, then in general each individual $G_X(f; s_0)$ would be different. To find the average PSD of the overall process, one would have to perform or estimate a statistical average of all the possible PSDs of all the possible realizations:

$$G_X(f) = E_s\left\{G_X(f; s)\right\} = E_s\left\{\mathrm{F}\left\{\Phi_X(\tau; s)\right\}\right\}$$

Finally, note that ergodicity for the autocorrelation also implies:

$$E\left\{X^2(t)\right\} = R_X(0) = \Phi_X(0;s_0) = \mathcal{P}\left\{X(t;s_0)\right\} \quad \forall s_0$$

which shows that the average power of each realization of the process is the same and actually coincides with the statistical mean square value.

Ergodicity for the autocorrelation is clearly key to a number of useful results. We can now discuss under what circumstances a RP is ergodic for the autocorrelation.

12.6.1.1 Ergodicity for jointly Gaussian processes

We provide without proof the following condition which allows to assess the ergodicity of a jointly Gaussian RP.

Given a jointly Gaussian WSS (SSS) RP $X(t)$*, the process is ergodic for all possible expectations of the form:*

$$E_X\left\{g\left(X(t_1), X(t_2), ..., X(t_N)\right)\right\}$$

if the following condition is satisfied:

$$\int_{-\infty}^{+\infty} \left|K_X(\tau)\right| d\tau < \infty$$

Eq. 12-73

If so, the jointly Gaussian process is then simply said to be "ergodic".

For jointly Gaussian processes then, a very simple condition establishes a very broad ergodicity. Together with the powerful results obtained for jointly Gaussian processes going through LTI systems, this makes the statistical handling of such processes quite effective in a large number of practical situations.

Notice the meaning of Eq. 12-74: for the integral to converge, it must be:

$$\left|K_X(\tau)\right| \to o\left(\frac{1}{|\tau|}\right) \quad , \quad |\tau| \to \infty$$

Such vanishing of the autocovariance means that two samples taken from the process at a sufficiently large time-distance, that is for a sufficiently large $|\tau|$, are uncorrelated. For jointly Gaussian processes, uncorrelation also means statistical independence. In practice, ergodicity requires that when taking two samples of the process at sufficiently distant times, then such samples are statistically independent.

With some intuition, it is possible to see that, under such condition, time-averages and statistical averages must coincide. Specifically, as the time-average progresses, the realization loses correlation with its past, after every large-enough time-interval. It is as if after every large-enough time-interval a new and independent realization is considered. Integrating over time then goes through many "independent" sections of the realization and acts as if averaging was being done over many independent results of the random experiment.

Formal proofs of this are however complicated and will not be dealt with here.

Note that if the process is not jointly Gaussian, condition Eq. 12-74 still has meaning, though much more limited:

Given a WSS RP $X(t)$*, the process is ergodic for the mean:*

$$E_X\left\{X(t)\right\}$$

if the following condition is satisfied:

$$\int_{-\infty}^{+\infty}\left|K_X(\tau)\right|d\tau<\infty$$

12.6.1.2 ergodicity of WGN

Let us consider again the WGN process. We can now add one more feature to it: it is ergodic. In fact:

$$\int_{-\infty}^{+\infty} \left| K_N(\tau) \right| d\tau = \frac{N_0}{2} \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = \frac{N_0}{2} < \infty$$

12.6.1.3 ergodicity of filtered WGN

Given a WGN process $N(t)$ transiting through an LTI system and producing an output $Y(t)$, we know that:

$$K_Y(\tau) = K_N(\tau) * R_h(\tau) = \frac{N_0}{2} R_h(\tau)$$

As a result, the output process $Y(t)$ is ergodic provided that:

$$\int_{-\infty}^{+\infty} \left| R_h(\tau) \right| d\tau < \infty$$

A sufficient condition for this to happen is that the LTI system is BIBO. Can you prove this on your own?

As a result, we can conclude that **all BIBO-filtered WGN processes are ergodic**.

12.6.1.4 ergodicity of certain quasi-defined processes

On your own, check the ergodicity for the mean and autocorrelation of the random-amplitude and random-phase carrier.

Show that these processes are ergodic for the mean but *not for the autocorrelation*. Show also that the autocorrelation calculated operating over a single realization, considered as a deterministic signal, and then averaged over all

possible realizations, provides the same result as the statistical autocorrelation.

Compute their power spectral densities as the average of the PSD's of each single realization and show that the results coincide with the Fourier transform of the autocorrelation.

End of optional material

12.7 Optional: Can the statistical features of RPs be measured?

One interesting topic related to RPs is whether it is possible to characterize them experimentally.

In Chapter 11, we claimed that Borel's law of large numbers ensured the possibility of characterizing a RV by repeatedly executing the random experiment and then suitably processing the results.

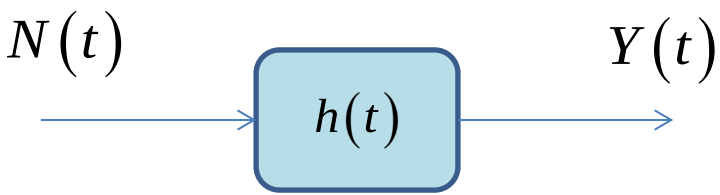
For processes, ideally, the same is true. However, as soon as one starts looking into this matter, one immediately realizes that there are daunting hurdles. For one thing, in order to apply Borel's law, we need ***many results*** of the random experiment. If the RP is for instance the output noise voltage at the terminals of an electronic circuit, clearly a single circuit produces a single realization. Since we need very many realizations, we ideally would have to build very many identical copies of the same circuit. This is of course quite impractical.

However, at least in principle, it is *physically possible* that we build hundreds or thousands of identical circuits. But then another, worse, problem presents itself. Each realization is typically a signal spanning $t \in \left[-\infty, \infty\right]$. If so, measuring even a single realization in its entirety is *physically impossible*.

These constraints make many RPs theoretically impossible to characterize experimentally, even for simple quantities such as, for instance, their variance

$\sigma^2(t)$.

Some processes do, however, lend themselves to relatively easy experimental characterization. One very important class of such processes includes all jointly Gaussian processes that can be modeled as a WGN process that has gone through a **BIBO** LTI system:



with $h(t)$ BIBO stable. These processes have the very important feature of being ***ergodic*** (see Sect. 12.6.1.3). When a process is ergodic, *statistical averages coincide with time-averages.*

For example, if we wanted to characterize the variance of $Y(t)$, we would know

that the following equality holds:

$$\sigma_Y^2 = E\left\{Y^2(t)\right\} = < Y^2(t, s_0) >_{\mathbb{R}}$$

where $Y(t, s_0)$ is one *single realization* of the process. This circumstance drastically simplifies the problem, since now we only need *one circuit*, not many, because we only need to observe *one realization*, not many.

Unfortunately, we have only solved part the problem. We still have the daunting problem that $< Y^2(t, s_0) >_{\mathbb{R}}$ is a time-average over the whole of $\mathbb{R}$, that is over $t \in [-\infty, \infty]$, and of course we cannot wait an infinite amount of time to carry out our desired time-average measurement.

However, the same class of processes has also the very important feature that their correlation coefficient, which is specifically given by (see Problem 12.8.3):

$$\rho_Y(\tau) = R_h(\tau) \Big/ \left\| h(t) \right\|^2$$

tends to go to zero for large values of τ. This behavior of $\rho_Y(\tau)$ is due to the BIBO condition on $h(t)$, because it makes $R_h(\tau) \to 0$ for large-enough τ.

In practical terms, the fact that $\rho_Y(\tau) \approx 0, \quad |\tau| > \tau_{\text{decorr}}$, means that after τ_{decorr} seconds the realization has become statistically independent of itself[1]. If we keep on measuring beyond τ_{decorr} seconds, then it is as if we were observing another, independent, realization. This can be shown to cause the convergence of the time-average measurement to the desired statistical average, according to Borel's law.

[1] Notice that the ergodicity of the process and the feature that $\rho_Y(\tau) \approx 0, \quad |\tau| > \tau_{\text{decorr}}$ are strictly related, although we will not expand here on this topic.

Specifically:

$$\sigma_Y^2 = E\left\{Y^2(t)\right\} = < Y^2(t, s_0) >_{\mathbb{R}} \approx < Y^2(t, s_0) >_{N \cdot \tau_{\text{decorr}}}$$

Note that for reliable estimation, we need to observe over many multiples of τ_{decorr}, that is use N very large.

In practice, τ_{decorr} cannot be too long for a practical measurement to be feasible. If τ_{decorr} is ms or μs, as it is the case in many electronic devices and circuits, then in a relatively short time the convergence of the time-average to the corresponding statistical average can be obtained.

On the contrary, even when ergodicity theoretically holds, if τ_{decorr} was, say, years, then experimental characterization would clearly be impossible.

References

[1] Roy M. Howard “A Signal Theoretic Introduction to Random Processes,” John Wiley & Sons, New York, 2016. See p. 295 in particular.

12.8 Problems

12.8.1

Let the RP $X(t)$ be defined as:

$$X(t) = \xi \cdot \pi_T(t - \eta T).$$

where η is a discrete RV that can take on the values 0,1,2,3 with probability 1/6,1/3,1/3,1/6, respectively, and ξ is a discrete RV that can take on the values -1 and 1 with probability ½ each. Do the following:

1. On your own: draw at least three different realizations of the process.
2. Find the process pdf $f_X(x;t)$. Neglect for simplicity the time-instants $t = 0, T, 2T, 3T, 4T$.
3. Find the mean value of the process.

4. Find the autocorrelation and autocovariance of the process.
5. Find the correlation coefficient of the process.
6. On your own: draw all of the above quantities.

Solution:

To find the pdf of the process, we first of all look at the possible values that can be observed when sampling the process at a given time t_s, excluding the time-instants $t = 0, T, 2T, 3T, 4T$. These values can be easily found to be only 0, 1, -1. As a result, the pdf will have the general form:

$$f_X(x;t_s) = p_0(t_s)\cdot\delta(x) + p_1(t_s)\cdot\delta(x-1) + p_{-1}(t_s)\cdot\delta(x+1)$$

We first look at the probability of seeing exactly zero:

$$p_0(t_s) = P\left(\left\{X(t_s) = 0\right\}\right) = P\left(\left\{\xi \cdot \pi_T(t_s - \eta T) = 0\right\}\right)$$

This probability is 1 (i.e., the value 0 is seen with probability 1) when $t_s < 0$ and when $t_s > 4T$, because in these time regions $\pi_T(t_s - \eta T) = 0,$ for any possible value of η:

$$p_0(t_s) = P\left(\left\{X(t_s) = 0\right\}\right) = 1, \quad t_s < 0, \quad t_s > 4T$$

Since $p_0(t_s) = 1$, all other probabilities must of course be zero over the same intervals, so we can immediately write:

$$p_1(t_s) = p_{-1}(t_s) = 0, \quad t_s < 0, \quad t_s > 4T$$

We then look at $p_0(t_s)$ when $t_s \in [0, 4T] \setminus \{0, T, 2T, 3T, 4T\}$. Over the given time

interval, whether we see $X(t_s)=0$ or not, it depends only on the random variable η. In particular, we have:

$$t_s \in]0,T[, \quad p_0(t_s) = P\left(\{\pi_T(t_s-\eta T)=0\}\right) = P\left(\{\eta \neq 0\}\right)$$

$$= P\left(\{\eta=1\}\cup\{\eta=2\}\cup\{\eta=3\}\right) = \frac{1}{3}+\frac{1}{3}+\frac{1}{6}=\frac{5}{6}$$

$$t_s \in]T,2T[, \quad p_0(t_s) = P\left(\{\pi_T(t_s-\eta T)=0\}\right) = P\left(\{\eta \neq 1\}\right)$$

$$= P\left(\{\eta=0\}\cup\{\eta=2\}\cup\{\eta=3\}\right) = \frac{1}{6}+\frac{1}{3}+\frac{1}{6}=\frac{2}{3}$$

$$t_s \in]2T,3T[, \quad p_0(t_s) = P\left(\{\pi_T(t_s-\eta T)=0\}\right) = P\left(\{\eta \neq 2\}\right)$$

$$= P\left(\{\eta=0\}\cup\{\eta=1\}\cup\{\eta=3\}\right) = \frac{1}{6}+\frac{1}{3}+\frac{1}{6}=\frac{2}{3}$$

$$t_s \in]3T,4T[, \quad p_0(t_s) = P\left(\{\pi_T(t_s-\eta T)=0\}\right) = P\left(\{\eta \neq 3\}\right)$$

$$= P\left(\{\eta=0\}\cup\{\eta=1\}\cup\{\eta=2\}\right) = \frac{1}{6}+\frac{1}{3}+\frac{1}{3}=\frac{5}{6}$$

As for the probability of seeing 1 and -1, we know that in each one of the subintervals listed above it must be: $p_1(t_s)+p_{-1}(t_s)=1-p_0(t_s)$

We also know that $p_1(t_s)=p_{-1}(t_s)$ because seeing one or the other (given that we do not see 0) only depends on the value of ξ, which is either 1 or -1 with equal probability. Putting together the relations:

$$\begin{cases} p_1(t_s)+p_{-1}(t_s)=1-p_0(t_s) \\ p_1(t_s)=p_{-1}(t_s) \end{cases}$$

with the results for $p_0(t_s)$, we find:

$$t_s \in]0,T[, \quad p_1(t_s)=p_{-1}(t_s)=\frac{1-p_0(t_s)}{2}=\frac{1}{12}$$

$$t_s \in]T,2T[, \quad p_1(t_s)=p_{-1}(t_s)=\frac{1-p_0(t_s)}{2}=\frac{1}{6}$$

$$t_s \in]2T,3T[, \quad p_1(t_s)=p_{-1}(t_s)=\frac{1-p_0(t_s)}{2}=\frac{1}{6}$$

$$t_s \in \left]3T, 4T\right[, \quad p_1(t_s) = p_{-1}(t_s) = \frac{1 - p_0(t_s)}{2} = \frac{1}{12}$$

All these results can be collected together in the following formula:

$$t \neq 0, T, 2T, 3T, 4T$$

$$f_X(x;t) = p_0(t) \cdot \delta(x) + p_1(t) \cdot \delta(x-1) + p_{-1}(t) \cdot \delta(x+1)$$

$$\begin{cases} p_1(t) = p_{-1}(t) = \frac{1}{12}\pi_T(t) + \frac{1}{6}\pi_{2T}(t-T) + \frac{1}{12}\pi_T(t-3T) \\ p_0(t) = u(-t) + \frac{5}{6}\pi_T(t) + \frac{2}{3}\pi_{2T}(t-T) + \frac{5}{6}\pi_T(t-3T) + u(t-4T) \end{cases}$$

The mean value of the process is:

$$\mu_X(t) = E\{X(t)\} = E\{\xi \cdot \pi_T(t - \eta T)\} = E\{\xi\} \cdot E\{\pi_T(t - \eta T)\}$$

But of course $E\{\xi\} = 0$ so:

$$\mu_X(t) = 0(t)$$

Note that this result clearly holds for all times, including $t = 0, T, 2T, 3T, 4T$.

As for the autocorrelation, we have:

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = E\{\xi \cdot \pi_T(t_1 - \eta T) \cdot \xi \cdot \pi_T(t_2 - \eta T)\}$$

$$= E\{\xi^2\} E\{\pi_T(t_1 - \eta T) \cdot \pi_T(t_2 - \eta T)\} = E\{\pi_T(t_1 - \eta T) \cdot \pi_T(t_2 - \eta T)\}$$

where we used the result $E\{\xi^2\} = E\{1\} = 1$. We then have to evaluate:

$$E\{\pi_T(t_1 - \eta T) \cdot \pi_T(t_2 - \eta T)\} = \int_{-\infty}^{\infty} \pi_T(t_1 - zT) \cdot \pi_T(t_2 - zT) \cdot f_\eta(z)\, dz$$

But:

$$f_\eta(z) = \frac{1}{6}\delta(z) + \frac{1}{3}\delta(z-1) + \frac{1}{3}\delta(z-2) + \frac{1}{6}\delta(z-3)$$

Integrating, the following result is obtained:

$$R_X(t_1,t_2)=E\{X(t_1)X(t_2)\}=E\{\pi_T(t_1-\eta T)\cdot\pi_T(t_2-\eta T)\}$$

$$=\int_{-\infty}^{\infty}\pi_T(t_1-zT)\cdot\pi_T(t_2-zT)\cdot\left[\frac{1}{6}\delta(z)+\frac{1}{3}\delta(z-1)+\frac{1}{3}\delta(z-2)+\frac{1}{6}\delta(z-3)\right]dz$$

$$=\frac{1}{6}\pi_T(t_1)\cdot\pi_T(t_2)+\frac{1}{3}\pi_T(t_1-T)\cdot\pi_T(t_2-T)$$

$$+\frac{1}{3}\pi_T(t_1-2T)\cdot\pi_T(t_2-2T)+\frac{1}{6}\pi_T(t_1-3T)\cdot\pi_T(t_2-3T)$$

The autocovariance is the same as the autocorrelation, since the mean value is zero:

$$R_X(t_1,t_2)=K_X(t_1,t_2)$$

The variance is:

$$\sigma_X^2(t)=K_X(t,t)=\frac{1}{6}\pi_T^2(t)+\frac{1}{3}\pi_T^2(t-T)+\frac{1}{3}\pi_T^2(t-2T)+\frac{1}{6}\pi_T^2(t-3T)$$

Finally, the correlation coefficient is:

$$\rho_X(t_1,t_2)=\frac{K_X(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)}\sqrt{\sigma_X^2(t_2)}}=$$

$$\frac{\frac{1}{6}\pi_T(t_1)\cdot\pi_T(t_2)+\frac{1}{3}\pi_T(t_1-T)\cdot\pi_T(t_2-T)+\frac{1}{3}\pi_T(t_1-2T)\cdot\pi_T(t_2-2T)+\frac{1}{6}\pi_T(t_1-3T)\cdot\pi_T(t_2-3T)}{\sqrt{\frac{1}{6}\pi_T^2(t_1)+\frac{1}{3}\pi_T^2(t_1-T)+\frac{1}{3}\pi_T^2(t_1-2T)+\frac{1}{6}\pi_T^2(t_1-3T)}\sqrt{\frac{1}{6}\pi_T^2(t_2)+\frac{1}{3}\pi_T^2(t_2-T)+\frac{1}{3}\pi_T^2(t_2-2T)+\frac{1}{6}\pi_T^2(t_2-3T)}}$$

The formula appears complicated, but it reduces to 1 every time t_1 and t_2 are such that:

$$t_1,t_2\in\left]0,T\right[\quad,\quad t_1,t_2\in\left]T,2T\right[\quad,\quad t_1,t_2\in\left]2T,3T\right[\quad,\quad t_1,t_2\in\left]3T,4T\right[$$

Can you explain why?

12.8.2

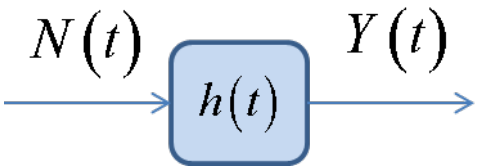
Let's look at a RP of the type $X_{\mathrm{CP}}(t)=\xi\cdot 1(t)$.

Let us assume that the RV ξ is Gaussian-distributed with variance σ_ξ^2 and mean value μ_ξ. Do the following:

- find the mean value and variance of $X_{\mathrm{CP}}(t)$
- find the autocorrelation, autocovariance and correlation coefficient of $X_{\mathrm{CP}}(t)$
- is the distribution of the process Jointly Gaussian?
- is the process stationary? To what extent?

12.8.3

Look at the block diagram below.



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signal $h(t)$ is the impulse response of an LTI system.

Answer the following questions:

1. What type of random process is $Y(t)$?
2. Is it stationary? If it is, what form of stationarity does it have?

3. What are its mean and variance?
4. What are its autocorrelation, autocovariance and correlation coefficient?
5. What is its average power spectral density?
6. Write down the second-order probability density function of the process.
7. Calculate explicitly the results of the points 3,4,5,6 assuming that $h(t) = e^{-at}u(t)$, $a > 0$. Also answer the question: can the realizations be represented with "pen and paper"?
8. Calculate explicitly the results of the points 3,4,5,6 assuming that $h(t) = \pi_T(t)$.Also answer the question: can the realizations be represented with "pen and paper"?
9. Are there any values of τ such that the RVs $Y(t_0)$ and $Y(t_0 - \tau)$ are statistically independent, assuming either $h(t) = e^{-at}u(t)$, $a > 0$, or $h(t) = \pi_T(t)$?
10. Optional: Is the process ergodic when $h(t) = \pi_T(t)$? For what moments

of the process? Could you find its statistical autocorrelation having only one realization at your disposal? How?

Solution to question (1)

The RP $Y(t)$ is jointly Gaussian. This is because $N(t)$ is jointly Gaussian and because passing a jointly Gaussian process through an LTI system produces a jointly Gaussian output process.

Solution to question (2)

The input process $N(t)$ is SSS. Systems of the LTI type preserve stationarity. Therefore the output process $Y(t)$ is also SSS.

Solution to question (3)

Given that $N(t)$ is stationary, the mean of the output is given by: $\mu_Y = H(0)\mu_N$.

However, $\mu_N = 0$ by assumption, so $\mu_Y = 0$ as well. As for its variance, we can find it as: $\sigma_Y^2 = K_Y(0)$. The autocovariance $K_Y(\tau)$ will be calculated in the next answer and so will the variance.

Solution to question (4)

For the autocorrelation we can use the general formula for the autocorrelation of an output process, given an (at least) WSS input process, which in this case says:

$$R_Y(\tau) = R_N(\tau) * R_h(\tau)$$

Since $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$, then we have:

$$R_Y(\tau) = \frac{N_0}{2} R_h(\tau)$$

In general, for an at least WSS process: $K_Y(\tau) = R_Y(\tau) - \mu_Y^2$. However, $\mu_Y = 0$, so in this case: $K_Y(\tau) = R_Y(\tau)$.

The variance of the process is:

$$\sigma_Y^2 = K_Y(0) = R_Y(0) = \frac{N_0}{2} R_h(0) = \frac{N_0}{2} \mathcal{E}\{h(t)\} = \frac{N_0}{2} \|h(t)\|^2$$

Finally, its correlation coefficient is:

$$\rho_Y(\tau) = K_Y(\tau) / K_Y(0) = K_Y(\tau) / \sigma_Y^2 = R_h(\tau) / \|h(t)\|^2$$

Solution to question (5)

For an at least WSS process, we have the general formula:

$$G_Y(f) = F\{R_Y(\tau)\}$$

In this case:

$$G_Y(f) = F\left\{\frac{N_0}{2} R_h(\tau)\right\} = \frac{N_0}{2} |H(f)|^2$$

The same result can also be found by remembering that the average power spectral density of a process $Y(t)$ at the output of an LTI system whose input is

an at least WSS process $X(t)$ is:

$$G_Y(f) = G_X(f)\left|H(f)\right|^2$$

In this case the input process has: $G_N(f) = \frac{N_0}{2}\cdot 1(f)$ and therefore, again:

$$G_Y(f) = G_N(f)\cdot\left|H(f)\right|^2 = \frac{N_0}{2}\left|H(f)\right|^2$$

Solution to question (6)

The output process is jointly Gaussian, SSS and zero-mean. This means that its 2nd order pdf, i.e. the joint pdf of the random variables $Y(t)$ and $Y(t-\tau)$, can be written as:

$$f_Y(\mathbf{y},\tau)=\frac{1}{2\pi\sqrt{\det\{\mathbf{\Sigma}_Y\}}}\exp\left(-\frac{1}{2}\,\mathbf{y}^{\mathrm{T}}\cdot\mathbf{\Sigma}_Y^{-1}\cdot\mathbf{y}\right)$$

where:

$$\mathbf{y}=\begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$\mathbf{\Sigma}_Y=\begin{bmatrix}\sigma_Y^2 & K_Y(\tau)\\ K_Y(\tau) & \sigma_Y^2\end{bmatrix}=\frac{N_0}{2}\begin{bmatrix}\|h(t)\|^2 & R_h(\tau)\\ R_h(\tau) & \|h(t)\|^2\end{bmatrix}$$

$$=\frac{N_0}{2}\|h(t)\|^2\begin{bmatrix}1 & \rho_h(\tau)\\ \rho_h(\tau) & 1\end{bmatrix}=\sigma_Y^2\begin{bmatrix}1 & \rho_h(\tau)\\ \rho_h(\tau) & 1\end{bmatrix}$$

Solution to question (7)

Assuming $h(t)=e^{-at}u(t)$, then we have:

$$\mu_Y=0$$

$$\sigma_Y^2=\frac{N_0}{2}\int_0^\infty e^{-2at}dt=\frac{N_0}{4a}$$

$$K_Y(\tau) = R_Y(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)\,dt = \frac{N_0}{4a}e^{-a|\tau|}$$

$$\rho_Y(\tau) = K_Y(\tau)/K_Y(0) = e^{-a|\tau|}$$

$$G_Y(f) = F\{R_Y(\tau)\} = F\left\{\frac{N_0}{4a}e^{-a|\tau|}\right\} = \frac{N_0}{2}\frac{1}{a^2+4\pi^2 f^2}$$

Regarding the second-order pdf, the relevant quantity is the autocovariance matrix. In this case it is:

$$\mathbf{\Sigma}_Y = \frac{N_0}{4a}\begin{bmatrix} 1 & e^{-a|\tau|} \\ e^{-a|\tau|} & 1 \end{bmatrix}$$

Looking at the correlation coefficient, we see that:

$$\lim_{\tau\to 0}\rho_Y(\tau) = \lim_{\tau\to 0}e^{-a|\tau|} = 1$$

This is enough to say that the realizations are “reasonably” well-behaved and that they can be represented with “pen and paper”.

Solution to question (8)

Assuming $h(t) = \pi_T(t)$, then we have:

$$\mu_Y = 0$$

$$\sigma_Y^2 = \frac{N_0}{2}\int_0^\infty \left|\pi_T(t)\right|^2 dt = \frac{N_0}{2}T$$

$$K_Y(\tau) = R_Y(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} \pi_T(t)\,\pi_T(t-\tau)\,dt = \frac{N_0}{2}T\Lambda_T(\tau)$$

$$\rho_Y(\tau) = K_Y(\tau)/K_Y(0) = \Lambda_T(\tau)$$

$$G_Y(f) = F\{R_Y(\tau)\} = F\left\{\frac{N_0 T}{2}\Lambda_T(\tau)\right\} = \frac{N_0 T^2}{2}\mathrm{Sinc}^2(fT)$$

Regarding the second-order pdf, the relevant quantity is the autocovariance matrix. In this case it is:

$$\mathbf{\Sigma}_Y = \frac{N_0 T}{2}\begin{bmatrix} 1 & \Lambda_T(\tau) \\ \Lambda_T(\tau) & 1 \end{bmatrix}$$

Looking at the correlation coefficient, we see that:

$\lim_{\tau\to 0} \rho_Y(\tau) = \lim_{\tau\to 0} \Lambda_T(\tau) = 1$

This is enough to say that the realizations are "reasonably" well-behaved and that they can be represented with "pen and paper".

Solution to question (9)

The two RVs $Y(t_0)$ and $Y(t_0-\tau)$ are obtained by sampling the jointly-Gaussian RP $Y(t)$, and therefore they are themselves jointly Gaussian. Two jointly Gaussian RVs are statistically independent if their correlation coefficient is 0. The correlation coefficient of $Y(t_0)$ and $Y(t_0-\tau)$ is simply $\rho_Y(\tau)$, so they are statistically independent if $\rho_Y(\tau) = 0$.

In the case $h(t)=\pi_T(t)$, then $\rho_Y(\tau)=\Lambda_T(\tau)$ and so it turns out that: $\rho_Y(\tau)=0, \quad |\tau|\geq T$. In other words, sampling the process T or more seconds apart, returns two independent RVs.

In the case $h(t)=e^{-at}u(t)$, $a>0$, then $\rho_Y(\tau)=e^{-a|\tau|}$ and this function never goes to exactly zero. Strictly speaking, therefore, two samples taken from the process, however far apart they are taken in time, are never completely independent. In practical terms, though, when the correlation coefficient has gone down enough from its maximum possible value of 1, for instance 10^{-3}, then one can assume that for all practical purposes the two RVs are indeed independent. Assuming that in general one wants $|\rho_Y(\tau)|\leq 10^{-N}$, the condition is then: $|\tau|\geq \frac{N}{a}\log_e(10)$.

Solution to question (10)

$Y(t)$ is an SSS jointly Gaussian process. Theory says that it is ergodic for all of its moments if: $\int_{-\infty}^{+\infty}\left|K_Y(\tau)\right|d\tau<\infty$. In this case, we have:

$$\int_{-\infty}^{+\infty}\left|K_Y(\tau)\right|d\tau=\frac{N_0}{2}\int_{-\infty}^{+\infty}\left|R_h(\tau)\right|d\tau$$

so the condition is actually on the autocorrelation of the impulse response of the LTI system. It can be shown that if the system is BIBO, then it follows that $\int_{-\infty}^{+\infty}\left|R_h(\tau)\right|d\tau<\infty$. Since the $h(t)=\pi_T(t)$ is indeed BIBO, then we can conclude that $\int_{-\infty}^{+\infty}\left|K_Y(\tau)\right|d\tau<\infty$. As a result, $Y(t)$ is ergodic for all of its moments.

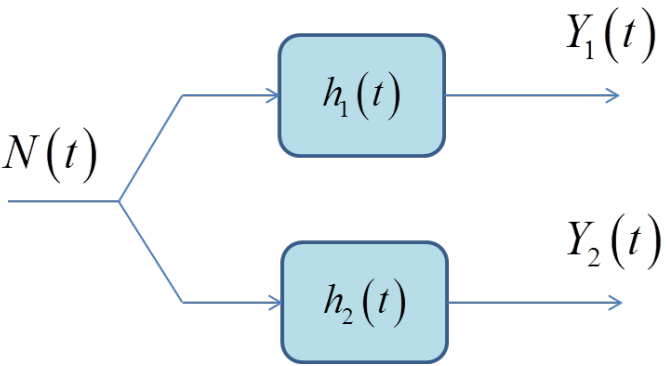
We can also directly verify the condition by first remarking that: $R_h(\tau)=T\Lambda_T(t)$ and then:

$$\int_{-\infty}^{+\infty} \left| K_Y(\tau) \right| d\tau = \frac{N_0}{2} \int_{-\infty}^{+\infty} \left| R_h(\tau) \right| d\tau$$

$$= \frac{N_0}{2} T \int_{-\infty}^{+\infty} \Lambda_T(t)\, d\tau = \frac{N_0}{2} T^2 < \infty$$

12.8.4

Look at the block diagram below.



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signals $h_1(t)$ and $h_2(t)$ are the impulse responses of two LTI systems.

1. Find the cross-correlation, cross-covariance and cross-correlation-coefficient of the RPs $Y_1(t)$ and $Y_2(t)$.
2. Focusing on the two RVs $Y_1(t_0)$ and $Y_2(t_0)$, find the condition on $h_1(t)$ and $h_2(t)$ that ensures that they are statistically independent.
3. Assume $h_1(t) = e^{-at}u(t)$, $a > 0$ and $h_2(t) = \pi_T(t)$. What is the correlation coefficient between $Y_1(t_0)$ and $Y_2(t_0)$?
4. Assume $h_1(t) = \pi_{T/2}(t)$ and $h_2(t) = \pi_T(t)$. What is the correlation coefficient between $Y_1(t_0)$ and $Y_2(t_0)$?

Solution to question (1)

The cross-correlation between $Y_1(t)$ and $Y_2(t)$ is:

$$R_{Y_1Y_2}(t_1,t_2) = E\{Y_1(t_1)Y_2(t_2)\}$$

$$= E\left\{\int_{-\infty}^{\infty} h_1(\theta_1)N(t_1-\theta_1)\,d\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)N(t_2-\theta_2)\,d\theta_2\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)\,E\{N(t_1-\theta_1)N(t_2-\theta_2)\}\,d\theta_1 d\theta_2$$

We concentrate on the average:

$$E\{N(t_1-\theta_1)N(t_2-\theta_2)\} =$$

$$= R_N(t_1-t_2-[\theta_1-\theta_2])$$

$$= R_N(\tau-[\theta_1-\theta_2]) = \frac{N_0}{2}\delta(\tau-[\theta_1-\theta_2])$$

Substituting back into the integral, we get:

$$R_{Y_1Y_2}(t_1,t_2) = R_{Y_1Y_2}(\tau)$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\theta_1) h_2(\theta_2) \delta(\tau - \theta_1 + \theta_2) d\theta_1 d\theta_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h_2(\theta_1 - \tau) h_1(\theta_1) d\theta_1 = \frac{N_0}{2} R_{h_1h_2}(\tau)$$

where the last line was found by integrating over θ_1.

The cross-covariance coincides with the cross-correlation because the mean values of $Y_1(t)$ and $Y_2(t)$ are zero:

$$R_{Y_1Y_2}(\tau) = K_{Y_1Y_2}(\tau)$$

As for the cross-correlation coefficient, its general formula is:

$$\rho_{Y_1Y_2}(\tau) = \frac{K_{Y_1Y_2}(\tau)}{\sigma_{Y_1}\sigma_{Y_2}}$$

Regarding the standard deviations, we have:

$$\sigma_{Y_1} = \sqrt{\sigma_{Y_1}^2} = \sqrt{\frac{N_0}{2}\mathcal{E}\left\{h_1(t)\right\}} = \sqrt{\frac{N_0}{2}}\left\|h_1(t)\right\|$$

$$\sigma_{Y_2} = \sqrt{\sigma_{Y_2}^2} = \sqrt{\frac{N_0}{2}\mathcal{E}\left\{h_2(t)\right\}} = \sqrt{\frac{N_0}{2}}\left\|h_2(t)\right\|$$

So, in conclusion:

$$\rho_{Y_1Y_2}(\tau) = \frac{K_{Y_1Y_2}(\tau)}{\frac{N_0}{2}\sqrt{\mathcal{E}\left\{h_1(t)\right\}\mathcal{E}\left\{h_2(t)\right\}}} = \frac{K_{Y_1Y_2}(\tau)}{\frac{N_0}{2}\left\|h_1(t)\right\|\cdot\left\|h_2(t)\right\|}$$

Solution to question (2)

If we focus on the two processes at the same time t_0, then statistical independence is ensured by the condition:

$$\rho_{Y_1Y_2}(0) = 0$$

Note that this condition is sufficient for statistical independence only because the two processes are jointly Gaussian. We have:

$$\rho_{Y_1Y_2}(0) = \frac{K_{Y_1Y_2}(0)}{\frac{N_0}{2}\|h_1(t)\| \cdot \|h_2(t)\|}$$

Obviously, the denominator cannot be zero. Therefore, the condition $\rho_{Y_1Y_2}(0) = 0$ is the same as $K_{Y_1Y_2}(0) = 0$. In turn, this means:

$$K_{Y_1Y_2}(\tau) = \frac{N_0}{2} R_{h_1h_2}(0) = 0$$

and further in turn this means: $R_{h_1h_2}(0) = 0$. This condition is the same as requiring that $(h_1(t), h_2(t)) = 0$, so in other words, the two jointly Gaussian RVs $Y_1(t_0)$ and $Y_2(t_0)$ are statistically independent provided that $h_1(t), h_2(t)$ are orthogonal.

Solution to question (3)

The correlation coefficient formula is, as shown above:

$$\rho_{Y_1Y_2}(0) = \frac{K_{Y_1Y_2}(0)}{\frac{N_0}{2}\left\|h_1(t)\right\| \cdot \left\|h_2(t)\right\|}$$

We start calculating from the denominator. The energy of the two impulse responses are:

$$\left\|h_1(t)\right\|^2 = \frac{1}{2a}$$

$$\left\|h_2(t)\right\|^2 = T$$

Regarding the numerator, $K_{Y_1Y_2}(\tau) = \frac{N_0}{2} R_{h_1h_2}(0)$, so we need the knowledge of:

$$R_{h_1h_2}(0) = \left(h_1(t), h_2(t)\right) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot \pi_T(t)\, dt = \int_0^T e^{-at} dt = \frac{1 - e^{-aT}}{a}$$

So, the result is:

$$\rho_{Y_1Y_2}(0)=\frac{\frac{N_0}{2}\frac{1-e^{-aT}}{a}}{\frac{N_0}{2}\sqrt{\frac{T}{2a}}}=\left(1-e^{-aT}\right)\sqrt{\frac{2}{aT}}$$

It is interesting to verify that $\left|\rho_{Y_1Y_2}(0)\right|\le 1$. Actually, from the formula above it is obvious that $\rho_{Y_1Y_2}(0)>0$, so we only need to check $\rho_{Y_1Y_2}(0)\le 1$. In practice, we need to find the maximum of $\rho_{Y_1Y_2}(0)$ as a function of its parameters, which are a and T. In fact, the always appear as a product aT so we can study $\rho_{Y_1Y_2}(0)$ as a function of aT.

However, a closed-form solution for the maximum of $\rho_{Y_1Y_2}(0)$ as a function of aT cannot be found. It can be estimated numerically and it turns out to be $\rho_{Y_1Y_2}(0)\approx 0.9025$, for $aT\approx 1.256$. Hence, the condition $\rho_{Y_1Y_2}(0)\le 1$ is satisfied, as expected. The maximum correlation between the two output processes, when

sampled at the same time, can be as high as 0.9025, but not higher.

Solution to question (4)

The formula to be used is the same:

$$\rho_{Y_1Y_2}(0) = \frac{K_{Y_1Y_2}(0)}{\frac{N_0}{2}\left\|h_1(t)\right\| \cdot \left\|h_2(t)\right\|}$$

We now have:

$$\left\|h_1(t)\right\|^2 = \frac{T}{2}$$

$$\left\|h_2(t)\right\|^2 = T$$

As for the numerator, again we need:

$$R_{h_1 h_2}(0) = \left(h_1(t), h_2(t)\right) = \int_{-\infty}^{\infty} \pi_{T/2}(t) \cdot \pi_T(t)\, dt = \int_0^{T/2} 1(t)\, dt = \frac{T}{2}$$

So, the result is:

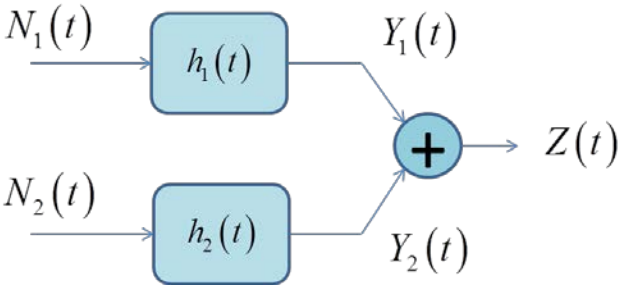
$$\rho_{Y_1 Y_2}(0) = \frac{\frac{N_0}{2}\frac{T}{2}}{\sqrt{\frac{N_0}{2}\frac{T}{2} \cdot \frac{N_0}{2} T}} = \frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

In this example it is easy to see that if the two filters had the same time-length, that is, if they were identical, then the correlation coefficient would be 1. This makes sense, because in that case we would have $Y_1(t) = Y_2(t)$.

The more the time-length of the two filters differs, instead, the smaller the correlation coefficient is (show it on your own). This also makes sense because the more different the two filters are, the more different $Y_1(t)$ and $Y_2(t)$ are.

12.8.5

Look at the block diagram below.



The signals $N_1(t)$ and $N_2(t)$ are two WGN process, independent of each other, with average power spectral density $G_{N_1}(f)=\frac{N_1}{2}$ and $G_{N_2}(f)=\frac{N_2}{2}$. The signals $h_1(t)$ and $h_2(t)$ are the impulse responses of two LTI systems.

1. What type of random process is $Z(t)$?
2. Is it stationary? If it is, what form of stationarity does it have?
3. What are its mean and variance?
4. What are its autocorrelation, autocovariance and autocorrelation coefficient?
5. What is its average power spectral density?
6. Write down the second-order probability density function of the process.
7. Calculate explicitly the results of the points 2,3,4,5 assuming that $h_1(t) = e^{-at}u(t)$, $a > 0$ and $h_2(t) = \pi_T(t)$.
8. Optional: Is the process ergodic for $h_1(t)$ and $h_2(t)$ as defined above? For what moments of the process? Could you find its statistical autocorrelation having only one realization at your disposal? How?

Solution to question (1) and (2)

The processes $Y_1(t)$ and $Y_2(t)$ both originate from two independent WGN processes $N_1(t)$, and $N_2(t)$ then going through the LTI systems $h_1(t)$ and $h_2(t)$, respectively. As such, $Y_1(t)$ and $Y_2(t)$ are (see Problem 12.8.3) Jointly Gaussian, SSS and zero-mean.

The process $Z(t) = Y_1(t) + Y_2(t)$ is the sum of two jointly Gaussian processes. The sum of two or more JG RPs is always a jointly Gaussian process (both when they are statistically dependent and when they are independent). This property derives from the general statistical theory result that the sum of two Gaussian random variables always produces a Gaussian RV, whether they are SI or dependent (see Chapter 11).

The sum operator is LTI, so stationarity is preserved too: since both $Y_1(t)$ and $Y_2(t)$ are SSS, so is $Z(t)$. This fact will also be verified after calculating the mean and autocorrelation of $Z(t)$, which will turn out to be stationary (see below).

Therefore, $Z(t)$ is WSS and, being JG, also SSS.

Solution to question (3)

Regarding the mean:

$$\mu_Z = \mathrm{E}\{Z(t)\} = \mathrm{E}\{Y_1(t) + Y_2(t)\} = \mathrm{E}\{Y_1(t)\} + \mathrm{E}\{Y_2(t)\} = 0 + 0 = 0$$

Regarding the variance, it can be derived from the autocovariance, so we will find it in the next answer.

Solution to question (4)

The autocorrelation is:

$$R_Z(t_1,t_2)=\mathrm{E}\{Z(t_1)Z(t_2)\}=\mathrm{E}\{[Y_1(t_1)+Y_2(t_1)][Y_1(t_2)+Y_2(t_2)]\}$$
$$=\mathrm{E}\{Y_1(t_1)Y_1(t_2)\}+\mathrm{E}\{Y_2(t_1)Y_2(t_2)\}$$
$$+\mathrm{E}\{Y_2(t_1)Y_1(t_2)\}+\mathrm{E}\{Y_1(t_1)Y_2(t_2)\}$$
$$=R_{Y_1}(t_1-t_2)+R_{Y_2}(t_1-t_2)+\mathrm{E}\{Y_2(t_1)Y_1(t_2)\}+\mathrm{E}\{Y_1(t_1)Y_2(t_2)\}$$

Regarding the two autocorrelations, from Problem 12.8.3 we know that:

$$R_{Y_1}(t_1,t_2)=R_{Y_1}(\tau)=\frac{N_0}{2}R_{h_1}(\tau)$$

$$R_{Y_2}(t_1,t_2)=R_{Y_2}(\tau)=\frac{N_0}{2}R_{h_2}(\tau)$$

We then need to find out about the terms $\mathrm{E}\{Y_2(t_1)Y_1(t_2)\}$ and $\mathrm{E}\{Y_1(t_1)Y_2(t_2)\}$. These terms are technically *cross-correlations*, of the two processes $Y_1(t)$ and $Y_2(t)$. So we can name them accordingly (although this is not strictly necessary;

we do it for convenience):

$$\mathrm{E}\left\{Y_1(t_1)Y_2(t_2)\right\} = R_{Y_1Y_2}(t_1,t_2)$$

$$\mathrm{E}\left\{Y_2(t_1)Y_1(t_2)\right\} = R_{Y_2Y_1}(t_1,t_2)$$

We start with the former:

$$R_{Y_1Y_2}(t_1,t_2) = \mathrm{E}\left\{Y_1(t_1)Y_2(t_2)\right\} =$$

$$= \mathrm{E}\left\{\int_{-\infty}^{\infty} h_1(\theta_1)N_1(t_1-\theta_1)d\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)N_2(t_2-\theta_2)d\theta_2\right\}$$

$$= \mathrm{E}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)N_1(t_1-\theta_1)N_2(t_2-\theta_2)d\theta_2 d\theta_1\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)\mathrm{E}\left\{N_1(t_1-\theta_1)N_2(t_2-\theta_2)\right\}d\theta_2 d\theta_1$$

Clearly, due to the SI of $N_1(t)$ with $N_2(t)$:

$$\mathrm{E}\left\{N_1\left(t_1-\theta_1\right)N_2\left(t_2-\theta_2\right)\right\}=\mathrm{E}\left\{N_1\left(t_1-\theta_1\right)\right\}\mathrm{E}\left\{N_2\left(t_2-\theta_2\right)\right\}=0\cdot 0=0$$

and therefore:

$$\mathrm{E}\left\{Y_1\left(t_1\right)Y_2\left(t_2\right)\right\}=0$$

With a completely analogous derivation we can get:

$$\mathrm{E}\left\{Y_2\left(t_1\right)Y_1\left(t_2\right)\right\}=0$$

Therefore, in summary:

$$R_Z\left(t_1,t_2\right)=R_Z\left(\tau\right)=\frac{N_0}{2}\left[R_{h_1}\left(\tau\right)+R_{h_2}\left(\tau\right)\right]$$

which is clearly stationary. So this together with the result $\mu_Z=0$ confirms that the process is WSS and, being JG, it is also SSS.

The autocovariance coincides with the autocorrelation, due to $\mu_Z=0$:

$$K_Z\left(\tau\right)=R_Z\left(\tau\right)=\frac{N_0}{2}\left[R_{h_1}\left(\tau\right)+R_{h_2}\left(\tau\right)\right]$$

The variance of the process is

$$\sigma_Z^2 = K_Z(0) = \frac{N_0}{2}\left[R_{h_1}(0) + R_{h_2}(0)\right] = \frac{N_0}{2}\left[\|h_1\|^2 + \|h_1\|^2\right]$$

where $\|h_1\|^2$ and $\|h_2\|^2$ are the norm squared (or energy) of the LTI systems impulse responses.

The correlation coefficient of the process $Z(t)$ then comes to:

$$\rho_Z(\tau) = \frac{K_Z(\tau)}{K_Z(0)} = \frac{\frac{N_0}{2}\left[R_{h_1}(\tau) + R_{h_2}(\tau)\right]}{\frac{N_0}{2}\left[\|h_1\|^2 + \|h_1\|^2\right]} = \frac{R_{h_1}(\tau) + R_{h_2}(\tau)}{\|h_1\|^2 + \|h_1\|^2}$$

Note that clearly $|\rho_Z(\tau)| \le 1$ because from Chapter 10 we know that: $R_{h_1}(\tau) \le \|h_1\|^2$, $R_{h_2}(\tau) \le \|h_2\|^2$.

Solution to question (5)

The power spectral density of the process can be found as either:

$$G_Z(f) = \mathrm{F}\{R_Z(\tau)\} =$$
$$= \frac{N_0}{2}\mathrm{F}\{R_{h_1}(\tau) + R_{h_2}(\tau)\} = \frac{N_0}{2}\left[|H_1(f)|^2 + |H_2(f)|^2\right]$$

Note that this turns out to coincide with:

$$G_Z(f) = G_{Y_1}(f) + G_{Y_2}(f) = G_{N_1}(f)|H_1(f)|^2 + G_{N_2}(f)|H_2(f)|^2$$

The result that the power spectral density of a process $Z(t)$ which is the sum of two processes $Y_1(t) + Y_2(t)$, turns out to be the sum of the power spectral densities of $Y_1(t)$ and $Y_2(t)$, *is not a general result*. In this case it holds because the two processes $Y_1(t)$ and $Y_2(t)$ are *statistically independent*. Othwerwise, there may be additional terms beyond $G_{Y_1}(f)$ and $G_{Y_2}(f)$.

Solution to question (6)

The process $Z(t)$ is JG, SSS and zero-mean. This means that its 2nd order pdf, i.e. the joint pdf of the random variables $Z(t)$ and $Z(t-\tau)$, can be written as:

$$f_Y(\mathbf{z},\tau)=\frac{1}{2\pi\sqrt{\det\{\mathbf{\Sigma}_Z\}}}\exp\left(-\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\cdot\mathbf{\Sigma}_Z^{-1}\cdot\mathbf{z}\right)$$

where:

$$\mathbf{z}=\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$

$$\mathbf{\Sigma}_Z=\begin{bmatrix}\sigma_Z^2 & K_Z(\tau)\\ K_Z(\tau) & \sigma_Z^2\end{bmatrix}=\frac{N_0}{2}\begin{bmatrix}\|h_1(t)\|^2+\|h_2(t)\|^2 & R_{h_1}(\tau)+R_{h_2}(\tau)\\ R_{h_1}(\tau)+R_{h_2}(\tau) & \|h_1(t)\|^2+\|h_2(t)\|^2\end{bmatrix}$$

$$=\frac{N_0}{2}\left[\|h_1(t)\|^2+\|h_2(t)\|^2\right]\begin{bmatrix}1 & \rho_Z(\tau)\\ \rho_Z(\tau) & 1\end{bmatrix}=\sigma_Z^2\cdot\begin{bmatrix}1 & \rho_Z(\tau)\\ \rho_Z(\tau) & 1\end{bmatrix}$$

Solution to question (7)

To recalculate all of the previous results for the specific case:

$h_1(t) = e^{-at}u(t),\ a > 0$ and $h_2(t) = \pi_T(t)$,

we only need to evaluate explicitly the following four quantities:

$$R_{h_1}(\tau),\ R_{h_2}(\tau),\ \left\|h_1(t)\right\|^2,\ \left\|h_2(t)\right\|^2,\ \left|H_1(f)\right|^2,\ \left|H_2(f)\right|^2.$$

They were already calculated in previous problems:

$$R_{h_1}(\tau) = \frac{e^{-a|\tau|}}{2a}, \qquad \left\|h_1(t)\right\|^2 = R_{h_1}(0) = \frac{1}{2a}$$

$$R_{h_2}(\tau) = T\Lambda_T(\tau), \qquad \left\|h_2(t)\right\|^2 = R_{h_2}(0) = T$$

$$\left|H_1(f)\right|^2 = \frac{1}{a^2 + 4\pi^2 f^2}$$

$$\left|H_2(f)\right|^2 = T^2\mathrm{Sinc}^2(fT)$$

These quantities then can just be plugged into the formulas for the previous asnwers. For instance, for the autocovariance of $Z(t)$ we have:

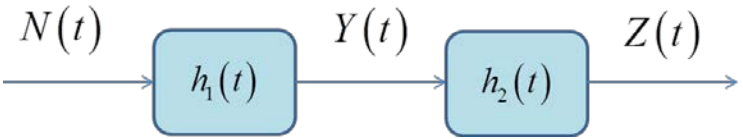
$$K_Z(\tau) = R_Z(\tau) = \frac{N_0}{2}\left[R_{h_1}(\tau) + R_{h_2}(\tau)\right] =$$

$$= \frac{N_0}{2}\left[\frac{e^{-a|\tau|}}{2a} + T\Lambda_T(\tau)\right]$$

and so on for the other quantities.

12.8.6

Look at the block diagram below.



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signals $h_1(t)$ and $h_2(t)$ are the impulse responses of two LTI systems.

1. Find $\mathrm{E}\{Z(t)Y(t)\}$, which can also be called the *cross-correlation* of the processes $Z(t)$ and $Y(t)$, or $R_{ZY}(\tau)$.

2. Calculate point (1) explicitly assuming that $h_1(t) = h_2(t) = e^{-at}u(t)$, $a > 0$.

Solution to question (1)

We directly compute:

$$R_{ZY}(t_1, t_2) = \mathrm{E}\{Z(t_1)Y(t_2)\} = \mathrm{E}\left\{\int_{-\infty}^{\infty} h_2(\theta)Y(t_1-\theta)\,d\theta \cdot Y(t_2)\right\}$$

$$= \mathrm{E}\left\{\int_{-\infty}^{\infty} h_2(\theta)Y(t_1-\theta)Y(t_2)\,d\theta\right\} = \int_{-\infty}^{\infty} h_2(\theta)\mathrm{E}\{Y(t_1-\theta)Y(t_2)\}\,d\theta$$

The expectation in the integral $\mathrm{E}\{Y(t_1-\theta)Y(t_2)\}$ is the autocorrelation of $Y(t)$. Such autocorrelation is of course stationary, because $Y(t)$ is produced by the WGN process $N(t)$. Therefore we have:

$$\mathrm{E}\left\{Y\left(t_1-\theta\right)Y\left(t_2\right)\right\}=R_Y\left(t_1-\theta,t_2\right)=R_Y\left(t_1-\theta-t_2\right)=R_Y\left(\tau-\theta\right)$$

Substituting into the the integral we have:

$$R_{ZY}\left(t_1,t_2\right)=R_{ZY}\left(\tau\right)=\int_{-\infty}^{\infty}h_2\left(\theta\right)R_Y\left(\tau-\theta\right)d\theta=R_Y\left(\tau\right)*h_2\left(\tau\right)$$

which is a very interesting result of its own. It shows that the cross-correlation between output and input of an LTI system can be calculated as the convolution of the input process with the impulse response of the system, if the input is stationary for the autocorrelation. Note that the resulting cross-correlation is also stationary.

Finally, from previous problems we know that $Y\left(t\right)$ has autocorrelation:

$$R_Y\left(\tau\right)=\frac{N_0}{2}R_{h_1}\left(\tau\right)$$

so:

$$R_{ZY}\left(\tau\right)=\frac{N_0}{2}R_{h_1}\left(\tau\right)*h_2\left(\tau\right)$$

Solution to question (2)

Formally the calculation becomes:

$$R_{ZY}(\tau) = \frac{N_0}{2} R_{h_1}(\tau) * h_2(\tau)$$

$$= \frac{N_0}{2} \frac{e^{-a|\tau|}}{2a} * e^{-at}u(t) = \frac{N_0}{4a}\left[e^{-a\tau}u(\tau) + e^{a\tau}u(-\tau)\right] * e^{-a\tau}u(\tau)$$

$$= \frac{N_0}{4a}\left[e^{-a\tau}u(\tau) * e^{-a\tau}u(\tau) + e^{a\tau}u(-\tau) * e^{-a\tau}u(\tau)\right]$$

Calculations are easier in frequency domain, so we transform all functions:

$$\mathrm{F}\left\{R_{ZY}(\tau)\right\} =$$

$$= \frac{N_0}{4a}\left[\mathrm{F}\left\{e^{-a\tau}u(\tau) * e^{-a\tau}u(\tau)\right\} + \mathrm{F}\left\{e^{a\tau}u(-\tau) * e^{-a\tau}u(\tau)\right\}\right]$$

$$= \frac{N_0}{4a} \frac{1}{(a + j2\pi f)^2} + \frac{N_0}{4a} \frac{1}{a^2 + 4\pi^2 f^2}$$

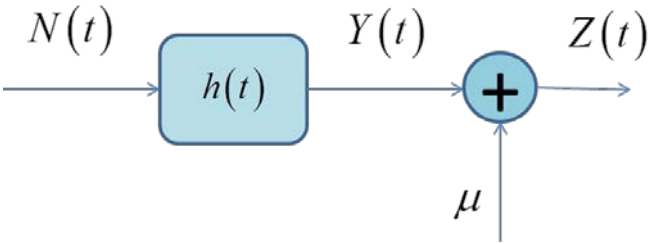
Finally, going back to τ :

$$R_{ZY}(\tau) = \frac{N_0}{4a}\left[\tau e^{-a\tau}u(\tau) + \frac{1}{2a}e^{-a|\tau|}\right]$$

Note that cross-correlations have no symmetry limitations, as opposed to auto-correlations. The above cross-correlation is in fact neither even nor odd.

12.8.7

Look at the block diagram below.



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signal $h(t)$ is the impulse response of an LTI system. The symbol μ indicates a real constant.

Answer the following questions:

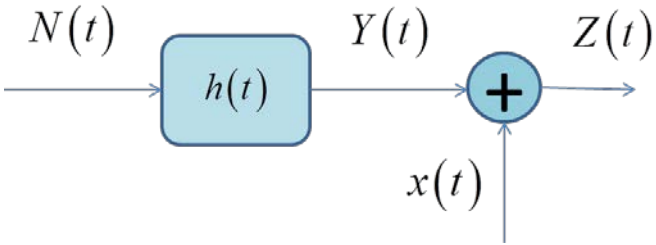
1. What type of random process is $Z(t)$?
2. Is it stationary? If it is, what form of stationarity does it have?
3. What are its mean and variance?
4. What are its autocorrelation and autocovariance?
5. What is its average power spectral density?
6. Write down the second-order probability density function of the process.
7. Calculate explicitly the results of the points 2,3,4,5 assuming that $h(t) = e^{-at}u(t)$, $a > 0$.
8. Calculate explicitly the results of the points 2,3,4,5 assuming that $h(t) = \pi_T(t)$.
9. Optional: Is the process ergodic when $h(t) = \pi_T(t)$? For what moments of the process? Could you find its statistical autocorrelation having only one realization at your disposal? How?

Solution to questions

This problem is a simplified version of the next problem. All solutions can be derived from the next problem by assigning $x(t) = \mu \cdot 1(t)$ in the solutions of the next problem.

12.8.8

Look at the block diagram below.



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signal $h(t)$ is the impulse response of a BIBO LTI system. The signal $x(t)$ is a real non-random finite-average-power signal.

1. What type of random process is $Z(t)$?
2. What is its mean?
3. What are its autocorrelation, autocovariance and variance?
4. Is $Z(t)$ it stationary? If it is, what form of stationarity does it have?
5. What is its average power spectral density $G_Z(f)$?
6. Assume that: $h(t)=t\cdot e^{-t}u(t)$ and that $x(t)=\sum_{n=-\infty}^{\infty}\pi_{1/2}(t-n)$. Calculate explicitly $G_Z(f)$.

Solution to question (1)

The RP $Y(t)$ is jointly Gaussian. This is because $N(t)$ is jointly Gaussian and because passing a jointly Gaussian process through an LTI system produces a jointly Gaussian output process. Regarding $Z(t)$, it is $Z(t)=Z(t)+x(t)$. Adding the deterministic signal only changes the mean value (see answer to next question),

but not the type of statistical distribution, which remains JG (more on this later).

Solution to question (2)

Given that $N(t)$ is stationary, the mean of $Y(t)$ is given by: $\mu_Y = H(0)\mu_N$. However, $\mu_N = 0$ by assumption, so $\mu_Y = 0$ as well. We then have:

$$\mu_z(t) = E\{Z(t)\} = E\{Y(t) + x(t)\} = E\{Y(t)\} + x(t) = x(t)$$

So the process $Z(t)$ is non-stationary for the mean and such mean coincides with $x(t)$.

Solution to question (3)

The statistical autocorrelation of the process is:

$$R_Z(t,t-\tau) = E\{Z(t)Z(t-\tau)\} =$$
$$= E\{Y(t)Y(t-\tau) + x(t)x(t-\tau) + Y(t)x(t-\tau) + x(t)Y(t-\tau)\}$$
$$= R_Y(\tau) + x(t)x(t-\tau)$$

where we have used the obvious results:

$$E\{Y(t)x(t-\tau)\} = E\{Y(t)\}x(t-\tau) = 0 \cdot x(t-\tau) = 0$$
$$E\{x(t)Y(t-\tau)\} = E\{Y(t-\tau)\}x(t) = 0 \cdot x(t) = 0$$

The autocovariance is:

$$K_Z(t,t-\tau) = E\{[Z(t)-\mu_z(t)][Z(t-\tau)-\mu_z(t-\tau)]\} =$$
$$= E\{Y(t)Y(t-\tau)\} = R_Y(\tau) = K_Y(\tau) = \frac{N_0}{2}R_h(\tau)$$

So in the end:

$$K_Z(\tau) = K_Y(\tau) = \frac{N_0}{2}R_h(\tau)$$

Notice the interesting result that, even though the mean and the autocorrelation of

$Z(t)$ are made non-stationary by adding $x(t)$ to the process, the autocovariance of $Z(t)$ remains stationary and it is identical to that of $Y(t)$.

Regarding the variance, its value is also unchanged vs. $Y(t)$:

$$\sigma_Z^2 = \sigma_Y^2 = K_Z(0) = \frac{N_0}{2}\|h\|^2$$

Solution to question (4)

$Z(t)$ has some stationarity features. However, it is not even WSS, since its mean is non-stationary. On the other hand, it is stationary for all central moments, because they subtract the mean value from $Z(t)$. It is not staionary for its pdf's, not even the first-order one, again due to the changing mean value.

Solution to question (5)

The problem with finding the power spectral density is that the usual formula

$G_Z(f) = F\{R_Z(\tau)\}$, which is valid for processes that are at least WSS, cannot be used since the autocorrelation of $Z(t)$ is non-stationary.

As a result, we need to try a different route. We recall the formula that is valid for non-stationary processes:

$$G_Z(f) = F\{< R_Z(t, t-\tau) >_{\mathbb{R}}\}$$

We then substitute the found value for $R_Z(t, t-\tau)$ and write:

$$\begin{aligned} & G_Y(f) = \\ & = F\{< R_Y(\tau) + x(t)x(t-\tau) >_{\mathbb{R}}\} \\ & = F\{< R_Y(\tau) >_{\mathbb{R}} + < x(t)x(t-\tau) >_{\mathbb{R}}\} \\ & = F\{< R_Y(\tau) >_{\mathbb{R}}\} + F\{< x(t)x(t-\tau) >_{\mathbb{R}}\} \end{aligned}$$

But $< R_Y(\tau) >_{\mathbb{R}} = R_Y(\tau)$ because $R_Y(\tau)$ does not depend on time t, which is the variable that the time-average operates on. Also, we have by definition (see Chapter 10):

$$< x(t)\, x(t-\tau) >_{\mathbb{R}} = \Phi_x(\tau)$$

where $\Phi_x(\tau)$ is the deterministic-signal autocorrelation for finite-average-power signals, like we assume $x(t)$ to be. As a result:

$$G_Z(f) = F\left\{R_Y(\tau)\right\} + F\left\{\Phi_x(\tau)\right\}$$

which can be re-written as:

$$G_Z(f) = G_Y(f) + G_x(f)$$

This is an interesting result, which shows that the average PSD of $Z(t)$ is in the end just the sum of the average PSD of $Z(t)$ with the deterministic PSD of the signal $x(t)$.

Solution to question (6)

We can concentrate on the two contributions to $G_Z(f)$ (see above) independently. We first look at:

$$G_Y(f) = G_N(f) \cdot \left|H(f)\right|^2 = \frac{N_0}{2}\left|\mathrm{F}\left\{h(t)\right\}\right|^2 =$$

$$= \frac{N_0}{2}\left|\mathrm{F}\left\{t \cdot e^{-t}u(t)\right\}\right|^2 = \frac{N_0}{2}\left|\frac{1}{\left(a + j2\pi f\right)^2}\right|^2 = \frac{N_0}{2}\frac{1}{\left(a + j2\pi f\right)^4}$$

As for $G_x(f)$, this is the deterministic PSD of a periodic signal. From the definition of $x(t)$ we see that we can write it as:

$$x(t) = \sum_{n=-\infty}^{\infty} x_T(t - nT)$$

where $x_T(t)$ is the truncated signal, that we can write as:

$$x_T(t) = \pi_{1/2}(t - n)$$

Clearly, the period of the periodic signal $x(t)$ is simply $T = 1$.

From Chapter 10, the expression of the PSD of a periodic signal is:

$$G_x(f) = f_0^2 \sum_{n=-\infty}^{\infty} \left| X_T(nf_0) \right|^2 \delta(f - f_0)$$

Here we have:

$$X_T(f) = \frac{T}{2} \mathrm{Sinc}\left(\frac{fT}{2} \right) e^{-j2\pi f \frac{T}{4}}$$

and so:

$$X_T(nf_0) = \frac{T}{2}\text{Sinc}\left(\frac{n}{2}\right)e^{-j\pi n\frac{n}{2}}$$

Substituting:

$$G_x(f) = f_0^2 \sum_{n=-\infty}^{\infty} \left|\frac{T}{2}\text{Sinc}\left(\frac{n}{2}\right)e^{-j\pi n\frac{n}{2}}\right|^2 \delta(f - f_0)$$

$$= \frac{1}{4}\sum_{n=-\infty}^{\infty} \text{Sinc}^2\left(\frac{n}{2}\right)\cdot\delta(f - f_0)$$

Putting everything together:

$$G_Z(f) = G_Y(f) + G_x(f) =$$

$$= \frac{N_0}{2}\frac{1}{(a + j2\pi f)^4} + \frac{1}{4}\sum_{n=-\infty}^{\infty} \text{Sinc}^2\left(\frac{n}{2}\right)\cdot\delta(f - f_0)$$

The average PSD of $Z(t)$ has a delta-like contribution from the periodic deterministic signal $x(t)$ and a continuous (in frequency) contribution due to the random process $Y(t)$.

12.8.9

Consider the process that was discussed in Section 12.2.3.5:

$$X(t) = \pi_T(t - \xi)$$

where ξ is uniformly distributed in $[0, T]$.

Its first-order pdf was found to be:

$$f_X(x; t_s) = \left[1 - \Lambda_T\left(t_s - T\right)\right]\delta\left(x\right) + \Lambda_T\left(t_s - T\right)\delta\left(x - 1\right)$$

- find the mean and variance of the process
- what would be a possible procedure for finding the autocorrelation and autocovariance of the process?
- [optional] actually find the autocorrelation and autocovariance of the process

- what would be a possible procedure for finding the second-order pdf of the process?
- [optional] actually calculate the second order pdf of the process

12.8.10

Consider the process that was discussed in Section 12.2.3.3

$$X(t) = \pi_{\xi}(t).$$

where ξ is uniformly distributed over the interval $[0,T]$, that is: $f_{\xi}(z) = \frac{1}{T}\pi_T(z)$.

Its first-order pdf was found to be:

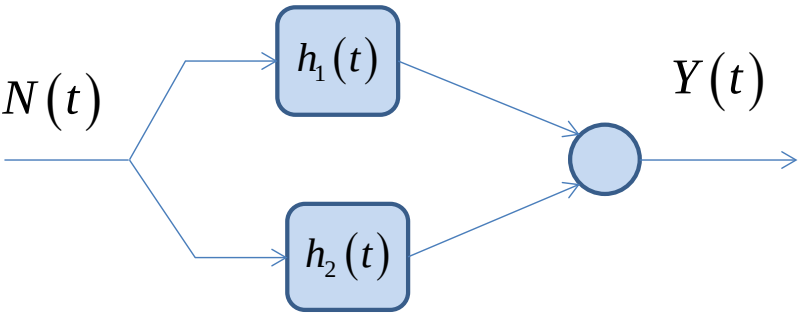
$$f_X(x;t)\begin{cases} = \delta(x) & t<0 \\ = \delta(x-1/2) & t=0 \\ = \frac{t}{T}\delta(x) + \left(1-\frac{t}{T}\right)\delta(x-1) & t \in]0,T[\\ = \delta(x) & t \geq T \end{cases}$$

In the following, you can approximate it as:

$$f_X(x;t)\begin{cases} = \delta(x) & t<0 \\ = \dfrac{t}{T}\delta(x)+\left(1-\dfrac{t}{T}\right)\delta(x-1) & t\in[0,T] \\ = \delta(x) & t>T \end{cases}$$

- find the mean and variance of the process
- what would be a possible procedure for finding the autocorrelation and autocovariance of the process?
- [optional] actually find the autocorrelation and autocovariance of the process
- what would be a possible procedure for finding the second-order pdf of the process?
- [optional] actually calculate the second order pdf of the process

12.8.11



The signal $N(t)$ is a WGN process, with average power spectral density $G_N(f) = \frac{N_0}{2}$. The signals $h_1(t)$ and $h_2(t)$ are the impulse responses of two LTI systems.

1. Find the autocorrelation, autocovariance and autocorrelation-coefficient

of the RPs $Z(t)$.

2. Re-calculate the results of the previous question, assuming $h_1(t) = \pi_{T_1}(t)$ and $h_2(t) = \pi_{T_2}(t)$.
3. what is the minimum value of $|\tau|$ such that the RVs $Z(t)$ and $Z(t-\tau)$ are statistically independent?

Solution to question (1)

The processes $Y_1(t)$ and $Y_2(t)$ both originate from the same WGN process $N(t)$, which goes through the LTI systems $h_1(t)$ and $h_2(t)$. As such, each one is (see Problem 12.8.3) Jointly Gaussian, SSS and zero-mean.

The process $Z(t) = Y_1(t) + Y_2(t)$ is the sum of two jointly Gaussian processes. The sum of two or more JG RPs is always a jointly Gaussian process (both in the case they are statistically dependent or independent). This property derives from the general statistical theory result that the sum of two Gaussian random variables

always produces a Gaussian RV, whether they are SI or dependent (see Chapter 11).

The sum operator is LTI, so stationarity is preserved too: since both $Y_1(t)$ and $Y_2(t)$ are SSS, so is $Z(t)$. This fact will also be verified after calculating the mean and autocorrelation of $Z(t)$, which will turn out to be stationary. Overall, $Z(t)$ is WSS and, being JG, also SSS.

The autocorrelation is:

$$
\begin{aligned}
&R_Z(t_1,t_2) = \mathrm{E}\{Z(t_1)Z(t_2)\} = \mathrm{E}\{[Y_1(t_1)+Y_2(t_1)][Y_1(t_2)+Y_2(t_2)]\} \\
&= \mathrm{E}\{Y_1(t_1)Y_1(t_2)\} + \mathrm{E}\{Y_2(t_1)Y_2(t_2)\} \\
&+ \mathrm{E}\{Y_2(t_1)Y_1(t_2)\} + \mathrm{E}\{Y_1(t_1)Y_2(t_2)\} \\
&= R_{Y_1}(t_1-t_2) + R_{Y_2}(t_1-t_2) + \mathrm{E}\{Y_2(t_1)Y_1(t_2)\} + \mathrm{E}\{Y_1(t_1)Y_2(t_2)\}
\end{aligned}
$$

Rgearding the two autocorrelations, from Problem 12.8.3 we know that:

$$R_{Y_1}(t_1,t_2)=R_{Y_1}(\tau)=\frac{N_0}{2}R_{h_1}(\tau)$$

$$R_{Y_2}(t_1,t_2)=R_{Y_2}(\tau)=\frac{N_0}{2}R_{h_2}(\tau)$$

We then need to find out about the terms $\mathrm{E}\{Y_2(t_1)Y_1(t_2)\}$ and $\mathrm{E}\{Y_1(t_1)Y_2(t_2)\}$. These terms are technically *cross-correlations*, of the two processes $Y_1(t)$ and $Y_2(t)$. So we name them accordingly (although this is not strictly necessary; we do it for convenience):

$$\mathrm{E}\{Y_1(t_1)Y_2(t_2)\}=R_{Y_1Y_2}(t_1,t_2)$$

$$\mathrm{E}\{Y_2(t_1)Y_1(t_2)\}=R_{Y_2Y_1}(t_1,t_2)$$

We start with the former:

$$R_{Y_1Y_2}(t_1,t_2) = \mathrm{E}\left\{Y_1(t_1)Y_2(t_2)\right\} =$$

$$= \mathrm{E}\left\{\int_{-\infty}^{\infty} h_1(\theta_1)N(t_1-\theta_1)d\theta_1 \int_{-\infty}^{\infty} h_2(\theta_2)N(t_2-\theta_2)d\theta_2\right\}$$

$$= \mathrm{E}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)N(t_1-\theta_1)N(t_2-\theta_2)d\theta_2 d\theta_1\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)\mathrm{E}\left\{N(t_1-\theta_1)N(t_2-\theta_2)\right\}d\theta_2 d\theta_1$$

Clearly:

$$\mathrm{E}\left\{N(t_1-\theta_1)N(t_2-\theta_2)\right\}$$

$$= R_N(t_1-\theta_1, t_2-\theta_2) = R_N\left(t_1-t_2-\left[\theta_1-\theta_2\right]\right)$$

$$= \frac{N_0}{2}\delta\left(\tau-\left[\theta_1-\theta_2\right]\right)$$

and therefore:

$$\mathrm{E}\left\{Y_1(t_1)Y_2(t_2)\right\}$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_2)\delta\left(\tau-\left[\theta_1-\theta_2\right]\right)d\theta_2 d\theta_1 =$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h_1(\theta_1)\int_{-\infty}^{\infty} h_2(\theta_2)\delta\left(\theta_2+\tau-\theta_1\right)d\theta_2 d\theta_1 =$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h_1(\theta_1)h_2(\theta_1-\tau)d\theta_1 = \frac{N_0}{2}R_{h_1h_2}(\tau)$$

In summary:

$$R_{Y_1Y_2}(\tau) = \frac{N_0}{2}R_{h_1h_2}(\tau)$$

where $R_{h_1h_2}(\tau)$ is the cross-correlation (for deterministic signals) of $h_1(t)$ and $h_1(t)$. Note that the cross-correlation $R_{Y_1Y_2}(\tau)$ turns out to be stationary. With a completely analogous derivation we can get:

$$R_{Y_2Y_1}(\tau)=\frac{N_0}{2}R_{h_2h_1}(\tau)$$

Therefore, in summary:

$$R_Z(t_1,t_2)=R_Z(\tau)=\frac{N_0}{2}\left[R_{h_1h_1}(\tau)+R_{h_2h_2}(\tau)+R_{h_1h_2}(\tau)+R_{h_2h_1}(\tau)\right]$$

which is clearly stationary.

The autocovariance is in general:

$$K_Z(t_1,t_2)=R_Z(t_1,t_2)-\mu_Z(t_1)\mu_Z(t_2)$$

Regarding the mean, we directly have:

$$\mu_Z(t)=\mathrm{E}\{Z(t)\}=\mathrm{E}\{Y_1(t)+Y_2(t)\}=0+0=0$$

so it is stationary and it is zero. As a result, the autocovariance coincides with the autocorrelation:

$$K_Z(\tau)=R_Z(\tau)=\frac{N_0}{2}\left[R_{h_1h_1}(\tau)+R_{h_2h_2}(\tau)+R_{h_1h_2}(\tau)+R_{h_2h_1}(\tau)\right]$$

The variance of the process is

$$\sigma_Z^2 = K_Z(0) =$$

$$= \frac{N_0}{2}\left[R_{h_1 h_1}(0) + R_{h_2 h_2}(0) + R_{h_1 h_2}(0) + R_{h_2 h_1}(0) \right]$$

$$= \frac{N_0}{2}\left[\|h_1\|^2 + \|h_1\|^2 + \left(h_1(t), h_2(t)\right) + \left(h_2(t), h_1(t)\right) \right]$$

where $\|h_1\|^2$ and $\|h_2\|^2$ are the norm squared (or energy) of the LTI systems impulse responses, and $\left(h_1(t), h_2(t)\right)$, $\left(h_1(t), h_2(t)\right)$ are the mutual inner products.

The correlation coefficient of the process $Z(t)$ then comes to:

$$\rho_Z(\tau) = \frac{K_Z(\tau)}{\sigma_Z^2}$$

$$= \frac{\frac{N_0}{2}\left[R_{h_1h_1}(\tau) + R_{h_2h_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)\right]}{\frac{N_0}{2}\left[\|h_1\|^2 + \|h_1\|^2 + \left(h_1(t), h_2(t)\right) + \left(h_2(t), h_1(t)\right)\right]}$$

$$= \frac{R_{h_1h_1}(\tau) + R_{h_2h_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)}{\|h_1\|^2 + \|h_1\|^2 + \left(h_1(t), h_2(t)\right) + \left(h_2(t), h_1(t)\right)}$$

Solution to question (2)

Some of the calculations have already been done in previous problems:

$$R_{h_1h_1}(\tau) = T_1\Lambda_{T_1}(\tau)$$

$$R_{h_2h_2}(\tau) = T_2\Lambda_{T_2}(\tau)$$

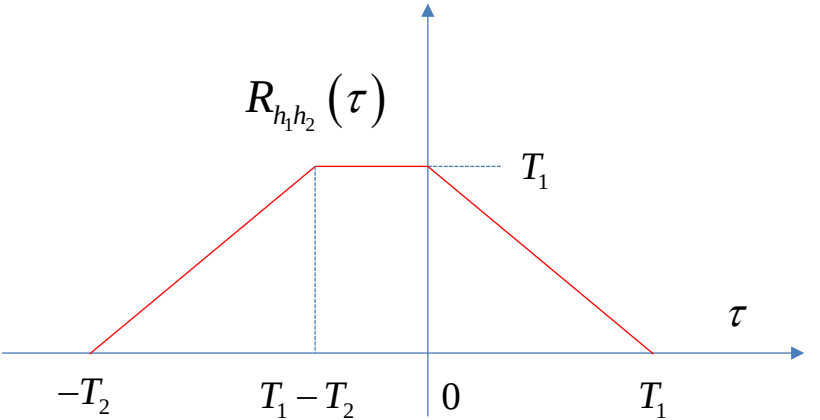
$$\|h_1\|^2 = T_1 \qquad , \qquad \|h_2\|^2 = T_2$$

Regarding $R_{h_1h_2}(\tau)$, the result depends on whether $T_1 < T_2$ or viceversa. We can avoid to lose generality by assuming that we are going to call T_1 the smaller of the two time durations, which is always possible. With this assumption, the result is univoquely:

$$R_{h_1h_2}(\tau) = \int_{-\infty}^{\infty} \pi_{T_1}(t)\,\pi_{T_2}(t-\tau)\,dt = \int_{0}^{T_1} \pi_{T_2}(t-\tau)\,dt =$$

$$= \begin{cases} \tau \le -T_2 & 0(\tau) \\ -T_2 < \tau \le T_1 - T_2 & \tau + T_2 \\ T_1 - T_2 < \tau \le 0 & T_1 \\ 0 < \tau \le T_1 & T_1 - \tau \\ \tau > T_1 & 0(\tau) \end{cases}$$

which can be plotted as:



Regarding $R_{h_2h_1}(\tau)$, it is easily shown that:

$$R_{h_2h_1}(\tau) = R_{h_1h_2}(-\tau)$$

Finally:

$$\left(h_1(t), h_2(t)\right) = \left(h_2(t), h_1(t)\right) = R_{h_2h_1}(0) = R_{h_1h_2}(0) = T_1$$

Substituting:

$$K_Z(\tau) = R_Z(\tau) = \frac{N_0}{2}\left[R_{h_1h_1}(\tau) + R_{h_2h_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)\right]$$

$$= \frac{N_0}{2}\left[T_1\Lambda_{T_1}(\tau) + T_2\Lambda_{T_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)\right]$$

$$\rho_Z(\tau) = \frac{R_{h_1h_1}(\tau) + R_{h_2h_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)}{\|h_1\|^2 + \|h_1\|^2 + (h_1(t), h_2(t)) + (h_2(t), h_1(t))}$$

$$= \frac{T_1\Lambda_{T_1}(\tau) + T_2\Lambda_{T_2}(\tau) + R_{h_1h_2}(\tau) + R_{h_2h_1}(\tau)}{3T_1 + T_2}$$

It is immediately seen that $\rho_Z(0) = 1$, as expected, since $Z(t)$ must be perfectly correlated with itself.

Solution to question (3)

Since $Z(t)$ is JG, statistical independence of any two samples of the process is equivalent to the correlation coefficient being 0. From the above formula, it is easy to find that $\rho_Z(\tau) = 0$ for $|\tau| > T_2$. So, the RVs $Z(t)$ and $Z(t-\tau)$ are statistically independent for $|\tau| > T_2$.

In other words, once T_2 seconds are gone by, the process decorrelates completely with itself and becomes statistically independent with itself.