Chapter 6.
The Fourier Transform

As shown in the previous chapter, given any signal $s(t) \in L^2_{[t_0,t_1]}$, we can represent it by means of the Fourier basis.

A problem with this approach is that the Fourier components change depending on the time interval chosen. For instance, considering $s(t) = \Pi_1(t)$, the s_n's are different when they are evaluated assuming $\left[t_0, t_1\right] = \left[-1, 1\right]$ or, say, $\left[t_0, t_1\right] = \left[-1.5, 1.5\right]$ (check for this on your own).

This happens with any signal and is somewhat inconvenient, especially when studying signals at a theoretical level.

One way of avoiding this, would be to select a "large enough" standard interval $[t_0, t_1]$ which could reasonably allow dealing with all signals. In fact, this idea is used to its extreme and what is done is to let $[t_0, t_1] \to [-\infty, \infty]$.

This turns the Fourier series into the Fourier integral and in this limit each signal has a unique set of "Fourier components". This chapter is devoted to the study of such alternative, time-interval independent, signal analysis and representation "tool".

It is important to notice, however, that the Fourier integral, or Fourier "transform", is a mathematical tool, useful to derive a large number of results of great interest, but it is almost never used *directly* for practical applications.

For practical applications it is customary to use Fourier series, though they are typically not called that way, but rather DFT (Discrete Fourier Transform), and just accept the need to use different time-intervals $\left[t_0, t_1\right]$, as needed. This need is obvious when one thinks that in physics there are "signals" that last a billionth of a second and "signals" that last years. Adapting the observation interval $\left[t_0, t_1\right]$ to the time-scale of the signal is clearly unavoidable.

However, the theoretical foundation of all of Signal Analysis hinges on the Fourier Transform, including its discrete-time counterpart as well as the DFT, either directly or indirectly. So it is essential to have a good understanding of this topic.

6.1 From the Fourier series to the Fourier Integral

By definition, the Fourier basis is the orthonormal set:

$$\Phi = \left\{ \hat{\varphi}_n \left(t \right) \right\}_{n=-\infty}^{+\infty} = \left\{ \frac{1}{\sqrt{T_0}} e^{j\frac{2\pi \cdot t}{T_0} n} \right\}_{n=-\infty}^{+\infty}$$

$$t \in [t_0,\ t_1], \qquad T_0 = t_1 - t_0$$

$$\left(\frac{1}{\sqrt{T_0}} e^{j\frac{2\pi \cdot t}{T_0} n}, \frac{1}{\sqrt{T_0}} e^{j\frac{2\pi \cdot t}{T_0} m} \right) = \delta_{mn}$$

Orthonormality ensures that all standard formulas of inner product spaces can be used (for instance Parseval's formula to compute energy, etc.).

However, it is also possible to use other versions of the same signal set, which are **just** **orthogonal**, but **normalized differently**.

For instance, if one chooses:

$$\Phi' = \left\{ e^{j\frac{2\pi \cdot t}{T_0}n} \right\}_{n=-\infty}^{+\infty} \qquad t \in [t_0,\ t_1], \qquad T_0 = t_1 - t_0$$

then:

$$\left(e^{j\frac{2\pi \cdot t}{T_0}n}, e^{j\frac{2\pi \cdot t}{T_0}m} \right) = T_0 \delta_{mn}$$

Also, one could choose:

$$\Phi'' = \left\{ \frac{1}{T_0} e^{j\frac{2\pi \cdot t}{T_0}n} \right\}_{n=-\infty}^{+\infty} \qquad t \in [t_0,\ t_1], \qquad T_0 = t_1 - t_0$$

$$\left(\frac{1}{T_0} e^{j\frac{2\pi \cdot t}{T_0}n}, \frac{1}{T_0} e^{j\frac{2\pi \cdot t}{T_0}m} \right) = \frac{1}{T_0}\delta_{mn}$$

It is still possible to use such bases to project and reconstruct signals, as shown here:

$$\Phi : s_n = \left(s(t), \frac{e^{j2\pi n f_0 t}}{\sqrt{T_0}} \right) = \int_{t_0}^{t_1} s(t) \frac{e^{-j2\pi n f_0 t}}{\sqrt{T_0}} dt \quad , \quad s(t) = \sum_{n=-\infty}^{+\infty} s_n \frac{e^{j2\pi n f_0 t}}{\sqrt{T_0}}$$

$$\Phi' : q_n = \left(s(t), e^{j2\pi n f_0 t} \right) = \int_{t_0}^{t_1} s(t) e^{-j2\pi n f_0 t} dt = s_n \sqrt{T_0} \quad , \quad s(t) = \sum_{n=-\infty}^{+\infty} q_n \frac{e^{j2\pi n f_0 t}}{T_0}$$

$$\Phi'' : \mu_n = \left(s(t), \frac{e^{j2\pi n f_0 t}}{T_0} \right) = \int_{t_0}^{t_1} s(t) \frac{e^{-j2\pi n f_0 t}}{T_0} dt = \frac{s_n}{\sqrt{T_0}} \quad , \quad s(t) = \sum_{n=-\infty}^{+\infty} \mu_n e^{j2\pi n f_0 t}$$

In essence, such bases are all equivalent. In fact, the non-normalized basis Φ'' is in wide use and can be found in many books.

However, when using non-normalized bases, all standard formulas of inner product spaces need *correction factors*. For instance, Parseval's formula for the energy becomes:

$$\mathcal{E}_{[t_0,t_1]}\{s(t)\} = \|s(t)\|^2 = \int_{t_0}^{t_1} |s(t)|^2$$

$$= \sum_{n=-\infty}^{+\infty} |s_n|^2 \quad = T_0 \sum_{n=-\infty}^{+\infty} |\mu_n|^2 \quad = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} |q_n|^2$$

To avoid correction factors, in this course we always use the orthonormal basis Φ when dealing with *Fourier series*.

As an exception, only here we will make use of the non-normalized basis Φ' because it simplifies the transition between Fourier series and Fourier transforms.

6.1.1 Finding the Fourier integral by "stretching time"

Just for convenience, and without any loss of generality, we will consider symmetric intervals. Therefore:

$$q_n = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) e^{-j2\pi n f_0 t} dt \qquad s(t) = \sum_{n=-\infty}^{+\infty} q_n \frac{e^{j2\pi n f_0 t}}{T_0}$$

$$t \in \mathbf{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2} \right]$$

We also assume that the parameter T_0 in $\mathbf{I}$ has been chosen so as to capture the whole signal $s(t)$, that is, the support $\mathbf{S}$ of $s(t)$ is: $\mathbf{S} \subset \mathbf{I}$, as shown in Fig. 6.1.

We then increase T_0 to a larger T_0'.

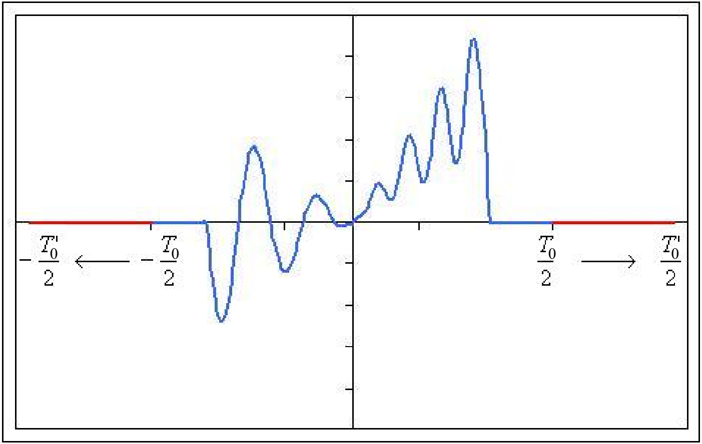


Fig. 6.1: The graph shows a generic finite energy signal with support

$$\mathbf{S} \subset \mathbf{I} = [-T_0/2, T_0/2]$$

When increasing the interval in which a signal is analyzed, then f_0 decreases because $f_0 = 1/T_0$.

Therefore, the spectrum of $s(t)$ becomes "denser" over the frequency axis.

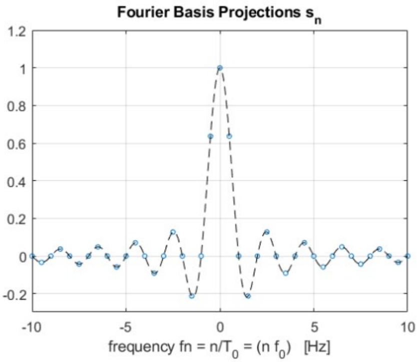
This is what happens for example in the case of the Fourier components of a rectangular signal $\Pi_T(t)$, whose generic components are given by (in the notation of this chapter):

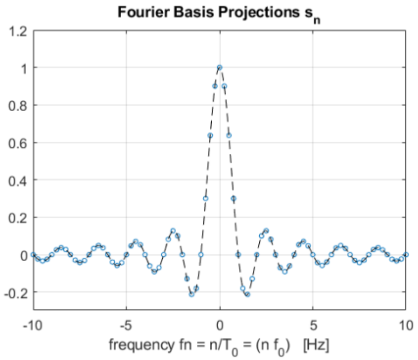
$$q_n = T \cdot \mathrm{Sinc}\left(\frac{T}{T_0} n\right)$$

and are shown in Fig. 6.2. With $T_0 = 2T$ the components are spaced $f_0 = \frac{1}{T_0} = \frac{1}{2T}$ and consist of just the blue dots.

With $T_0^{'} = 4T$ the components are spaced $f_0^{'} = \frac{1}{T_0^{'}} = \frac{1}{4T} = \frac{f_0}{2}$ and consist of both the blue and red dots.

Note that using the Φ' basis then all the possible components, independently of the value of, lie on the same curve: $T \cdot \text{Sinc}(Tf)$, which is shown as a thin line in Fig. 6.2.





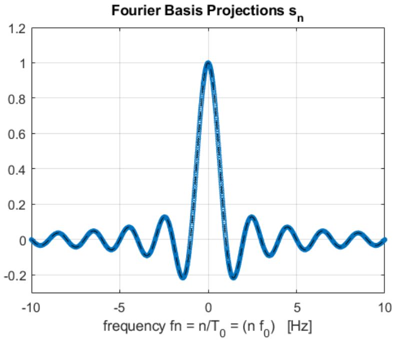


Fig. 6.2: Signal spectrum of a rectangular signal. Frequency components of $\Pi_T(t)$ according to the ONS Φ', when $T=1$ and T_0 is equal to 2, 4 and 100.

If we now continue increasing the interval size, then clearly the "density" of frequency components (the "dots") over the spectrum grows in kind. In fact, as $T_0 \to \infty$, then $f_0 \to 0$, i.e., the dots in the signal spectrum plot, occurring at $f = nf_0$, **become infinitesimally "close", or infinitely "dense"**.

In other words, their "distance" in frequency, which is f_0, tends to go to zero. As a result, $f = nf_0$ tends to become a "continuous" frequency variable f.

Before proceeding we make a simple change of notation. We can formally write, without any theoretical problems: $q_n \to q(nf_0)$.

If we now look at the projection formula below, and we let $T_0 \to \infty$ and then also substitute $nf_0 \to f$, we get:

$$
\begin{array}{ccc}
q(nf_0) & = & \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) \quad e^{-j2\pi nf_0 t} dt \\
\downarrow & & \quad\downarrow \qquad\qquad \downarrow \\
q(f) & = & \int_{-\infty}^{\infty} s(t) \quad e^{-j2\pi f t} dt
\end{array}
$$

Eq. 6-1

As for the reconstruction formula, we first make just formal notation changes:

$$
s(t) = \sum_{n=-\infty}^{+\infty} q_n e^{j2\pi nf_0 t} \frac{1}{T_0} \longrightarrow s(t) = \sum_{n=-\infty}^{+\infty} q(nf_0)\, e^{j2\pi nf_0 t} f_0
$$

Then, by again letting $T_0 \to \infty$, we claim that:

$$\begin{array}{ccccc} s(t) = & \displaystyle\sum_{n=-\infty}^{+\infty} & q(nf_0) & e^{j2\pi n f_0 t} & f_0 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ s(t) = & \displaystyle\int_{-\infty}^{\infty} & q(f) & e^{j2\pi f t} & df \end{array}$$

Eq. 6-2

In other words, when $T_0 \to \infty$ what happens is:

- the discrete sequence nf_0 becomes a continuous variable f ;
- the summation becomes an integral;
- $f_0 = \dfrac{1}{T_0}$ gets ‘infinitesimal’ and becomes the “differential factor” $\to df$

6.1.1.1 optional: comments

The above arguments appeal to intuition but are not entirely rigorous. However, the transition from Fourier series to Fourier transforms could be rigorously justified, though this will not be done here.

For the interested reader we point out that:

- given our assumption that the support $\mathbf{S}$ of $s(t)$ is $\mathbf{S} \subset \mathbf{I}$, then extending the integration range of the projection integral is not a problem, since: $q(nf_0) = \int_{-T/2}^{T/2} s(t)\, e^{-j2\pi n f_0 t} dt = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi n f_0 t} dt$;
- then, the change $nf_0 \to f$ is also not a problem, since nf_0 is just a parameter within the integral;

- the reconstruction formula $s(t)=\sum_{n=-\infty}^{+\infty} q(nf_0)\, e^{j2\pi n f_0 t} f_0$ is clearly a *Riemann sum*; if for $f_0 \to 0$ it converges, then it converges to precisely this value: $\int_{-\infty}^{\infty} q(f)\, e^{j2\pi f t} df$

What we omit here is the proof that such convergence is in fact ensured by the assumptions that we made on $s(t)$. Such proof can be found in Mathematics textbooks.

End of optional material.

6.1.2 The Fourier Transform

The result of Eq. 6-1, $q(f)$, is a function of the continuous variable f. Note that f always spans all of $\mathbb{R}$. Such function is called "Fourier transform" of the signal $s(t)$ and, in this course, it is written using the same letter as the signal, capitalized.

We therefore formally *define* the *direct Fourier transform* or simply *Fourier transform* of a signal $s(t)$ as:

$$S(f) \triangleq \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft} dt = \mathrm{F}\{s(t)\}$$

Eq. 6-3

where $\mathrm{F}\{s(t)\}$ is a shorthand in *operator notation* for the Fourier transform. Note that the Fourier transform is an *operator* because it takes in a *function* and returns a *function*.

From Eq. 6-2 we know that we should be able to reconstruct $s(t)$ from $S(f)$. The "reconstruction" integral is called the "*inverse Fourier transform*":

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df = \mathrm{F}^{-1}\left\{S(f)\right\}$$

Eq. 6-4

Again, note the operator notation, where the superscript "-1" simply means "inverse".

Note though that the "proof" given by Eq. 6-2 that the reconstruction is possible is far from rigorous. In any case, we don't know exactly what restrictions apply to the signals $s(t)$. Since Eq. 6-2 is based on Fourier series, at most we could claim that the direct/inverse Fourier transform can reproduce the signals that the Fourier series could handle, that is *finite-energy signals*.

In the following we will accumulate a few preliminary results. Then we will come back to the topic of what kind of signals the direct/inverse Fourier transform can handle.

6.1.3 The Fourier transform of $\delta(t)$

In this section, we will show that $\delta(t)$ can be Fourier-transformed and then reconstructed by means of the inverse Fourier transform. This result will be of the utmost importance and will be used to derive several other important results.

6.1.3.1 a preliminary result

We start out by discussing the following important mathematical identity:

$$\int_{-\infty}^{\infty} e^{j2\pi f t} df = \delta(t)$$

Eq. 6-5

To justify it, we first compute the integral over a limited frequency span:

$$\int_{-B/2}^{B/2} e^{j2\pi f t} df = \left. \frac{e^{j2\pi f t}}{j2\pi t} \right|_{-B/2}^{B/2} = \frac{e^{j\pi Bt} - e^{-j\pi Bt}}{j2\pi t}$$

$$= \frac{\sin(\pi t B)}{\pi t} = B \cdot \mathrm{Sinc}(B t)$$

We now try and formally extend the integration range to infinity by taking the limit for $B \to \infty$ of both sides:

$$\lim_{B \to \infty} \int_{-B/2}^{B/2} e^{j2\pi f t} df = \lim_{B \to \infty} B \cdot \mathrm{Sinc}(B t)$$

Eq. 6-6

We then recall from Chapter 2, Eq. 2.16, that one of the many ways of "generating" $\delta(t)$ is:

$$\lim_{T \to 0} \frac{1}{T} \mathrm{Sinc}\left(\frac{t}{T}\right) = \delta(t)$$

Eq. 6-7

The left-hand side of Eq. 6-7 is identical to the right-hand side of Eq. 6-6 if the parameter B is replaced with $T = 1/B$. Therefore, Eq. 6-6 and Eq. 6-7 coincide and we can simply write:

$$\lim_{B \to \infty} \int_{-B/2}^{B/2} e^{j2\pi f t} df = \delta(t)$$

Eq. 6-8

Remember that, strictly speaking, the limit operator in Eq. 6-8 does not converge in the sense of ordinary functions. It only converges if we think of it in the sense of "distributions". As already mentioned in Chapter 2, we should therefore use a different symbol for the limit:

$$\lim_{\substack{\text{dist} \\ B\to\infty}} \int_{-B/2}^{B/2} e^{j2\pi f t} df = \delta(t)$$

Eq. 6-9

The actual meaning of $\lim_{\substack{\text{dist} \\ B\to\infty}}$ is: "in the limit of $B \to \infty$ the left-hand side tends to assume the same properties as $\delta(t)$; specifically it tends to assume the properties that $\delta(t)$ has when it is placed within an integral".

Eq. 6-9 is often written without the limit, taking for granted that people know how to correctly interpret it:

$$\int_{-\infty}^{\infty} e^{j2\pi f t} df = \delta(t)$$

Eq. 6-10

Note again that this equation makes sense only if the equal sign is viewed in the sense of "distributions", that is, the left-hand-side object has the same properties as $\delta(t)$, when inserted into an integral.

Since $\delta(t)$ is an "even function", that is: $\delta(t) = \delta(-t)$, we can generalize Eq. 6-5 to:

$$\int_{-\infty}^{\infty} e^{\pm j2\pi f t} df = \delta(t) = \delta(-t)$$

Eq. 6-11

6.1.3.2 evaluating the transform of $\delta(t)$

We can now evaluate the Fourier transform of $\delta(t)$. This is easy, as it is enough to follow the integration rule of Chapter 2:

$$\mathrm{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{j2\pi f t} dt =$$

$$\int_{-\infty}^{\infty} \delta(t) \cdot \left[e^{j2\pi f t} \right]_{t=0} dt = \int_{-\infty}^{\infty} \delta\left(t\right) dt = 1(f)$$

Eq. 6-12

The result is 1 everywhere, independently of the value of f , so it is $1(f)$.

We would now like to try and see if by taking the inverse Fourier transform we can get back $\delta(t)$. In fact:

$$\mathrm{F}^{-1}\left\{1(f)\right\} = \int_{-\infty}^{\infty} 1(f)\cdot e^{j2\pi f t}df = \int_{-\infty}^{\infty} e^{j2\pi f t}df = \delta(t)$$

Eq. 6-13

where we have used Eq. 6-10 to perform the last passage.

So, in summary:

$$\mathrm{F}\left\{\delta(t)\right\} = 1(f)$$

$$\mathrm{F}^{-1}\left\{1(f)\right\} = \delta(t)$$

Eq. 6-14

These results are quite important. They prove that $\delta(t)$ can be Fourier-transformed and then reconstructed. In compact form, we can write:

$$\delta(t) = \mathrm{F}^{-1}\left\{\mathrm{F}\left\{\delta(t)\right\}\right\}$$

Using a very similar derivation to the one presented above, it is then easy to show the important results:

$$\mathrm{F}\left\{1(t)\right\} = \delta(f)$$

$$\mathrm{F}^{-1}\left\{\delta(f)\right\} = 1(t)$$

Eq. 6-15

$\boxed{\text{On your own}}$ Prove Eq. 6-15.

Hint: redo the derivation of Eq. 6-14, simply exchanging frequency with time.

6.1.4 The “Fourier Inversion Theorem”

The Fourier inversion theorem provides indications on what signals can be Fourier-transformed and then inverse-Fourier transformed back to themselves. In the following, there is an optional section that you can read if interested. A practically useful summary of the main results is provided instead in Sect. 6.1.4.2.

6.1.4.1 optional:

Whether the inverse Fourier transform can indeed “reconstruct” a signal from its Fourier transform or not, is a rather complex matter. We provide here a few results which identify some general conditions under which the “Fourier Inversion Theorem” holds.

The Fourier Inversion Theorem has one hypothesis:

hypothesis: given $s(t)$ *defined for all* $t \in \mathbb{R}$, $S(f) = F\{s(t)\}$ *exists for all* $f \in \mathbb{R}$;

and a <u>statement</u>:

then the Fourier transform is invertible, i.e.,

$$s(t) = F^{-1}\{S(f)\} = F^{-1}\{F\{s(t)\}\},$$

provided that the order of integration can be swapped in the left-hand side of the following formula, yielding the right-hand side (i.e., provided that the left-hand side and the right-hand side produce the same result):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\tau)e^{j2\pi f(t-\tau)}d\tau\, df = \int_{-\infty}^{\infty} s(\tau)\int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)}df\, d\tau$$

Proof:

We want to see if, given the hypothesis, we can “get $s(t)$ back” from its Fourier transform, that is:

$$s(t) \overset{?}{=} F^{-1}\left\{S(f)\right\}$$

Eq. 6-16

where the question mark indicates that this is what we want assess. To see if such equality holds, we use the theorem hypothesis $S(f) = F\left\{s(t)\right\}$ to replace $S(f)$ in the right-hand side of Eq. 6-16, obtaining:

$$s(t) \overset{?}{=} \mathrm{F}^{-1}\left\{S(f)\right\} = \mathrm{F}^{-1}\left\{\mathrm{F}\left\{s(t)\right\}\right\}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} s(\tau) \cdot e^{-j2\pi f \tau} d\tau\right] \cdot e^{j2\pi f t} df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\tau) e^{j2\pi f (t-\tau)} d\tau \, df$$

Eq. 6-17

So far, we have just used the hypothesis and the formal definition of Fourier transform and inverse Fourier transform. In order to get ahead, we need to be able to perform a swap in the order of integration of the last integral, that is:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\tau) e^{j2\pi f (t-\tau)} d\tau \, df = \int_{-\infty}^{\infty} s(\tau) \int_{-\infty}^{\infty} e^{j2\pi f (t-\tau)} df \, d\tau$$

Eq. 6-18

If so, then the following calculations are easy, based on the result of Eq. 6-10:

$$\begin{aligned}
s(t) &\overset{?}{=} \mathrm{F}^{-1}\left\{S(f)\right\} = \mathrm{F}^{-1}\left\{\mathrm{F}\left\{s(t)\right\}\right\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\tau) e^{j2\pi f(t-\tau)} d\tau\, df \\
&= \int_{-\infty}^{\infty} s(\tau) \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)} df\, d\tau \\
&= \int_{-\infty}^{\infty} s(\tau)\, \delta(t-\tau)\, d\tau \\
&= s(t)
\end{aligned}$$

Eq. 6-19

which shows that indeed, if the swap in the order of integration of Eq. 6-18 is possible, then we can remove the “question mark” above the first equal sign, because Eq. 6-19 shows that indeed we can get back to $s(t)$ from $S(f)$ through the inverse Fourier transform. This proves the theorem as stated.

The Fourier Inversion Theorem shows that the *critical point* of going from $s(t)$ to $S(f)$, and then back to $s(t)$ *or, in other words, of making the Fourier transform an invertible operator*, is indeed whether the exchange in the order of integration of Eq. 6-18 is possible or not.

It turns out that such exchange is possible under certain conditions. We list here a few conditions which are significant for our applications.

End of optional material.

6.1.4.2 conditions ensuring inversion

A signal $s(t)$ can be Fourier transformed, and inverse-Fourier transformed back to itself[1], that is:

$$s(t) = \mathrm{F}^{-1}\{S(f)\} = \mathrm{F}^{-1}\{F\{s(t)\}\}$$

<u>in any of these four cases:</u>

1. if $\int_{-\infty}^{\infty} |s(t)|\, dt < \infty$ and $\int_{-\infty}^{\infty} |S(f)|\, df < \infty$

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[1] Or, equivalently, with reference to the *optional* section 6.1.4.1, a signal $s(t)$ is such that the order of integration of Eq. 6-18 can be swapped.

2. if $s(t)$ is in $L^2_{\mathbb{R}}$
3. if $s(t)$ *is a "tempered distribution"*, such as for instance $\delta(t)$ (see Wikipedia).
4. when the signal is periodic and is finite-energy over its period (see Sect. 6.4.4)

Note that these are *sufficient conditions.*

6.1.4.3 The special case $L^2_{\mathbb{R}}$

Case (2) above is very important and we provide further details on it. The signal space $L^2_{\mathbb{R}}$ is the space of all *finite-energy signals over the whole of* $\mathbb{R}$. In other words, a signal $s(t)$ is in $L^2_{\mathbb{R}}$ if:

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt < \infty$$

$L^2_{\mathbb{R}}$ can be shown to be an inner product space. Its inner product has the customary definition of a signal inner product, with integration limits extended to the whole of $\mathbb{R}$:

$$\left(s(t), w(t)\right) = \int_{-\infty}^{\infty} s(t)\, w^*(t)\, dt$$

$L^2_{\mathbb{R}}$ can also be shown to be a Hilbert space, i.e., a *complete inner product space (see optional Sect. 4.7.1)*.

So, according to point (2) above, if a signal $s(t) \in L^2_{\mathbb{R}}$ has a Fourier transform $S(f)$, then $s(t) = F^{-1}\{S(f)\}$. In fact, for this space, we can say more:

given <u>any</u> signal $s(t) \in L^2_{\mathbb{R}}$ *then*

- $S(f)$ *always exists, for all* $f \in \mathbb{R}$

- $S(f) \in L^2_{\mathbb{R}}$ *as well*

So, the Fourier transform of a finite-energy signal always exists, is itself a finite-energy signal, and is always invertible back to the original signal.

6.1.5 Fourier transforms and inner-product spaces

We have shown at the start of this chapter that the Fourier transform and inverse-transform can be viewed as an extension from a finite-interval $\mathbf{I} = [t_0, t_1]$ to the whole of $\mathbb{R}$ of the Fourier-basis representation of a signal.

In particular, the summation formula can express all signals belonging to the (Hilbert) inner-product space $L^2_{[t_0,t_1]}$ as a linear combination of elements of the Fourier basis Φ, each one multiplied by the respective components, found through an inner product integral.

One might wonder whether this interpretation is still valid for the Fourier transform and inverse-transform. The interesting answer is that indeed this is still true, albeit with a few adjustments.

6.1.5.1 the continuous-frequency Fourier ONS

We look now at the Fourier transform and notice that it has the form of an inner product over the whole of $\mathbb{R}$, namely:

$$\mathrm{F}\left\{s(t)\right\} = S(f) = \int_{-\infty}^{\infty} s(t) \cdot e^{-j2\pi f t} dt = \left(s(t), e^{j2\pi f t}\right)$$

Eq. 6-20

In other words, the Fourier transform can be thought of as the "projection" of the signal $s(t)$ over a signal of the form $e^{j2\pi f t}$. The signal $e^{j2\pi f t}$ has a parameter f which identifies it and therefore $S(f) = \left(s(t), e^{j2\pi f t}\right)$ is the collection of the projections of $s(t)$ over all possible signals $e^{j2\pi f t}$, indexed by the continuous parameter f.

The role of Eq. 6-20 therefore seems to be the same as that of the "projection" formula for the Fourier series, which was:

$$s_n = \int_{t_0}^{t_1} s(t)\sqrt{f_0}\, e^{-j2\pi n f_0 t} = \left(s(t), \sqrt{f_0}\, e^{j2\pi n f_0 t}\right)$$

Eq. 6-21

Eq. 6-21 is indeed a "projection" formula because the functions $\left\{\sqrt{f_0}\ e^{j2\pi n f_0 t}\right\}_n$ are an orthonormal basis for $L^2_{[t_0, t_1]}$. So, to be able to claim that Eq. 6-20 is a projection formula too, **we need to prove that the set** $\Phi_{\mathbb{R}} = \left\{e^{j2\pi f t}\right\}_{f=-\infty}^{\infty}$ **is an orthonormal basis**.

First, we check orthonormality. Taken any two elements of the set $\Phi_{\mathbb{R}}$, i.e., choosing two "frequencies" f_1, f_2, we get, using Eq. 6-15:

$$\left(e^{j2\pi f_1 t}, e^{j2\pi f_2 t}\right) = \int_{-\infty}^{+\infty} e^{j2\pi f_1 t} e^{-j2\pi f_2 t} dt =$$

$$= \int_{-\infty}^{+\infty} e^{j2\pi(f_1 - f_2)t} dt = \delta\left(f_1 - f_2\right) = \delta\left(f_2 - f_1\right)$$

As a result, for any $f_1 \neq f_2$ the inner product $\left(e^{j2\pi f_1 t}, e^{j2\pi f_2 t}\right)$ is zero. This proves that $\Phi_{\mathbb{R}}$ is an *orthogonal* set. Note, though, that the result is different from what we found for the Fourier basis Φ of $L^2_{[t_0,t_1]}$, which had a Kronecker's delta:

$$\left(\sqrt{f_0}\, e^{j2\pi n f_0 t}, \sqrt{f_0}\, e^{j2\pi m f_0 t}\right) = \delta_{nm}\,.$$

Here, for the set $\Phi_{\mathbb{R}}$ we have Dirac's delta:

$$\left(e^{j2\pi f_1 t}, e^{j2\pi f_2 t}\right) = \delta\left(f_2 - f_1\right)$$

Eq. 6-22

However, it turns out that this new formula is the appropriate "orthonormality" relation for a set such as $\Phi_{\mathbb{R}}$. We now investigate whether $\Phi_{\mathbb{R}}$ is a basis for $L^2_{\mathbb{R}}$.

6.1.5.2 $\Phi_{\mathbb{R}}$ forming a basis for $L^2_{\mathbb{R}}$

In Appendix 6.6 we provide a proof that $\Phi_{\mathbb{R}}$ is indeed a basis for $L^2_{\mathbb{R}}$. We leave such proof as **optional**. What is important to remember, is that $\Phi_{\mathbb{R}}$ is an "orthonormal" set (in the new orthonormality sense of Eq. 6-22) and that it is also an ***orthonormal basis*** for $L^2_{\mathbb{R}}$.

As a consequence, Eq. 6-20 is a proper "projection" formula, because it projects a signal onto all the signals of an *orthonormal basis*. In addition, this suggests that the inverse Fourier transform may be a proper "reconstruction formula". In fact, it "sums" together all of the components of a signal $s(t)$ vs. the basis in a way that is analogous to the Fourier series reconstruction formula (except for the fact that here the continuous frequency index f is used rather than the discrete index nf_0).

6.1.5.3 Parseval's and Plancharel's formulas

When dealing with Fourier series, we saw that, given any two signals $s(t), w(t) \in L^2_{[t_0,t_1]}$, then their inner product could also be computed as:

$$\left(s(t), w(t)\right) = \sum_{n=-\infty}^{\infty} s_n w_n^*$$

From this, the famous Parseval's formula could be found, for any $s(t) \in L^2_{[t_0,t_1]}$:

$$\left\|s(t)\right\|^2 = \left(s(t), s(t)\right) = \sum_{n=-\infty}^{\infty} \left|s_n\right|^2$$

We may then wonder whether in $L^2_{\mathbb{R}}$ a similar property still exists. A direct way to find out is to start from the inner product in $L^2_{\mathbb{R}}$:

$$\left(s(t), w(t)\right) = \int_{-\infty}^{+\infty} s(t)\, w^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} S(f_1) e^{j2\pi f_1 t} df_1 \right] \left[\int_{-\infty}^{+\infty} W(f_2) e^{j2\pi f_2 t} df_2 \right]^* dt =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(f_1) W^*(f_2) e^{j2\pi(f_1 - f_2)t} df_1 df_2 dt =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(f_1) W^*(f_2) \int_{-\infty}^{+\infty} e^{j2\pi(f_1 - f_2)t} dt\, df_1 df_2 =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(f_1) W^*(f_2) \delta\left(f_2 - f_1\right) df_1 df_2 =$$

$$= \int_{-\infty}^{+\infty} S(f_1) W^*(f_1) df_1 = \int_{-\infty}^{+\infty} S(f) W^*(f) df$$

Eq. 6-23

Note that in the above derivation there is a delicate point that is the swapping of the order of integration. Provided that the signals $s(t), w(t) \in L^2_{[t_0,t_1]}$, then the swapping can certainly take place[2].

In summary:

$$\left(s(t), w(t)\right) = \int_{-\infty}^{+\infty} S(f) W^*(f) df$$

Eq. 6-24

This important formula, which is sometimes called Plancharel's Theorem, shows that it is still possible to compute an inner product based on the components of the signals with respect to the basis of the space. As it could be

---

[2] This is related to the Fourier Inversion Theorem, where a similar swap of integration order was the key point. If interested, read optional Sect. 6.1.4.1. Any pair of signals for which the Fourier Inversion Theorem holds, also allow the swapping of the integration order in Eq. 6-23

expected, being now the components characterized by a continuous index f rather than an integer parameter n, the summation is replaced by an integral.

Parseval's formula can also be extended as follows:

$$\left\|s(t)\right\|^2 = \left(s(t), s(t)\right) = \int_{-\infty}^{+\infty} S(f) S^*(f) df = \int_{-\infty}^{+\infty} \left|S(f)\right|^2 df$$

6.1.5.4 the time-frequency correspondence of $L^2_{\mathbb{R}}$

We remark that there is a striking correspondence between what are generally called "time-domain" and "frequency-domain" (that is, signals and their Fourier transforms).

We stress the following identities:

$$\int_{-\infty}^{+\infty} s(t)\, w^*(t)\, dt = \int_{-\infty}^{+\infty} S(f) W^*(f) df$$

$$\int_{-\infty}^{+\infty} \left| s(t) \right|^2 dt = \int_{-\infty}^{+\infty} \left| S(f) \right|^2 df$$

We can also be tempted to view the RHS of Plancharel's Theorem Eq. 6-24 as an inner product itself:

$$\int_{-\infty}^{+\infty} S(f) W^*(f) df = \left(S(f), W(f) \right)$$

Eq. 6-25

Being formally identical to the inner product for signals in time, Eq. 6-25 is a perfectly legitimate inner product, in the sense that it has all the properties required of an inner product. As a result, the set of all "finite energy signals" (according to the inner product in frequency) $S(f)$, with $f \in [-\infty, \infty]$, form themselves an inner-product space $L^2_{\mathbb{R}}$.

A following question could then be:
what is the actual relationship between the $L^2_{\mathbb{R}}$ space in t and the one in f ?

The interesting result is that there is a <u>one-to-one correspondence</u> between $s(t) \in L^2_{t\in[-\infty,\infty]}$ and $S(f) \in L^2_{f\in[-\infty,\infty]}$. Such correspondence is established by:

$$s(t) \xleftrightarrow{\mathrm{F}} S(f),$$

In the following paragraphs, we better explain what we mean by this.

We had already stated in Sect. 6.1.4.3 that the Fourier transform $S(f)$ of a signal $s(t) \in L^2_{t\in[-\infty,\infty]}$ always exists, it belongs to $L^2_{f\in[-\infty,\infty]}$ and its inverse Fourier transform always exists and coincides with the original signal $s(t)$. But now we can say more.

Note that, given $s(t) \in L^2_{t\in[-\infty,\infty]}$ and its Fourier transform $S(f) \in L^2_{f\in[-\infty,\infty]}$, these two signals *have the same energy*:

$$\mathcal{E}\{s(t)\} = \int_{-\infty}^{+\infty} \left|s(t)\right|^2 dt = \int_{-\infty}^{+\infty} \left|S(f)\right|^2 df = \mathcal{E}\{S(f)\}$$

Note also that, given any two signals $s(t), w(t) \in L^2_{t\in[-\infty,\infty]}$, and their "corresponding signals" in frequency domain $S(f), W(f) \in L^2_{f\in[-\infty,\infty]}$, then, using Plancharel's Theorem Eq. 6-24, their inner products have the same value:

$$\begin{aligned}\left(s(t), w(t)\right) &= \int_{-\infty}^{+\infty} s(t) w^*(t) dt \\ &= \int_{-\infty}^{+\infty} S(f) W^*(f) df = \left(S(f), W(f)\right)\end{aligned}$$

These results allow us to say that the correspondence between $L^2_{t\in[-\infty,\infty]}$ and $L^2_{f\in[-\infty,\infty]}$ is more than just a coincidence. In fact, we can conclude that **they are the same space** whose elements are simply represented through two different "bases", the time-basis and the "frequency-basis". Each signal, in this picture, is an abstract object that can be described using "time" or "frequency", while retaining its "norm" and its "relationships" with all the other elements of the space. Note for instance that two signals orthogonal in $L^2_{t\in[-\infty,\infty]}$ are also orthogonal in $L^2_{f\in[-\infty,\infty]}$.

So, what is the Fourier transform in this new picture?

The Fourier transform can be viewed as a **change-of-basis operator**, the equivalent of a $N\times N$ orthogonal matrix for $\mathbb{R}^N$ spaces. Like in $\mathbb{R}^N$ spaces, where we can represent a certain vector using different orthonormal bases, and inner products and norms are invariant with respect to the basis, we can represent a "signal" in the time-basis or frequency-basis, and norms and inner products are invariant.

On your own (advanced, **optional**): Show that $\left\{e^{j2\pi f_0 t}\right\}_{f_0}$ is the frequency basis expressed over the time-basis.

Can you find out what is the frequency-basis expressed over frequency?

Result: $\left\{\delta\left(f-f_0\right)\right\}_{f_0}$

Also find the time-basis expressed over frequency and over time.

Results: $\left\{e^{-j2\pi f t_0}\right\}_{t_0}$, $\left\{\delta\left(t-t_0\right)\right\}_{t_0}$

6.1.5.5 distance in time and frequency-domain

Since the Fourier transform can be viewed as a change of basis, but the set of elements of the (Hilbert) inner-product space and their relationships do not change, we should expect *distance* also not to change between time and frequency-domain.

It is in fact well-known that, given an inner product space, the distance between any two elements remains the same, independently of the orthonormal basis used to represent them (think for example of space vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$). In other words, given $s(t), w(t) \in L^2_{t\in[-\infty,\infty]}$, whose distance is:

$$\Delta\{s(t),w(t)\}=\|s(t)-w(t)\|=\sqrt{(s(t)-w(t),s(t)-w(t))}$$

Given the corresponding Fourier transforms and their distance:

$$\Delta\{S(f),W(f)\}=\|S(f)-W(f)\|=$$
$$=\sqrt{(S(f)-W(f),S(f)-W(f))}$$

then, it should be:

$$\Delta\{s(t),w(t)\}=\Delta\{S(f),W(f)\}$$

It is easy to prove that this is in fact true (do it on your own starting from Eq. 6-24).

Distances among signals are of great importance in the analysis of digital transmission systems, where it directly affects the *error probability* of such systems, that is, the probability of making a mistake about the bits that were transmitted.

6.1.6 A few examples of Fourier transforms of finite-energy signals

The following is a table of the Fourier transforms of some of the most common finite-energy signals that are encountered in this class. All of them are elementary and could be calculated on your own, except the transform of the Gaussian signal. Some will be explicitly calculated in the following.

$s(t)$	$S(f)$
$\Pi_T(t)$	$\dfrac{\sin(\pi f T)}{\pi f} = T\,\text{Sinc}(fT)$
$\Lambda_T(t)$	$\dfrac{1}{T}\left[\dfrac{\sin(\pi f T)}{\pi f}\right]^2 = T\,\text{Sinc}^2(fT)$
$e^{-t^2/2\sigma^2}$	$\sqrt{2\pi\sigma^2}\,e^{-2\pi^2 f^2 \sigma^2}$
$e^{-\alpha t}u(t) \quad \alpha > 0$	$\dfrac{1}{\alpha + j2\pi f}$

$te^{-\alpha t}u(t) \quad \alpha > 0$	$\dfrac{1}{(\alpha + j2\pi f)^2}$
$\dfrac{t^n}{n!}e^{-\alpha t}u(t) \quad \alpha > 0$	$\dfrac{1}{(\alpha + j2\pi f)^{n+1}}$
$e^{-\alpha\lvert t\rvert} \quad \alpha > 0$	$\dfrac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$
$\rho_{T,\alpha}(t)$	$T\,\mathrm{Sinc}(fT)\dfrac{\cos(\pi\alpha fT)}{1-(2\alpha fT)^2}$

Table 6-1: a few frequently-used Fourier transforms

You can check all of the above results numerically, by using Matlab. An example program that does it is reported at the end of this chapter.

6.1.6.1 calculations

Transform of the rectangular signal:

$$\mathrm{F}\{\Pi_T(t)\} = \int_{-\infty}^{\infty} \Pi_T(t) \cdot e^{-j2\pi f t} dt$$

$$= \int_{-T/2}^{T/2} e^{-j2\pi f t} dt = -\left. \frac{e^{-j2\pi f t}}{j2\pi f} \right|_{-T/2}^{T/2}$$

$$= \frac{e^{j\pi f T} - e^{-j\pi f T}}{j2\pi f} = \frac{\sin(\pi f T)}{\pi f} = T \cdot \mathrm{Sinc}(f T)$$

Transform of the unilateral decreasing exponential signal:

Let's first recall that for the exponential to be decreasing, it must be: $\alpha > 0$. Then:

$$\mathrm{F}\left\{e^{-\alpha t}u(t)\right\} = \int_{-\infty}^{\infty} e^{-\alpha t}u(t) \cdot e^{-j2\pi f t}dt = \int_{0}^{\infty} e^{-\alpha t}e^{-j2\pi f t}dt =$$

$$= \int_{0}^{\infty} e^{-(j2\pi f+\alpha)t}dt = -\left.\frac{e^{-(j2\pi f+\alpha)t}}{\alpha + j2\pi f}\right|_{0}^{\infty} = \frac{1}{\alpha + j2\pi f}$$

Transform of the bilateral decreasing exponential signal:

As above, for the exponential to be decreasing, it must be: $\alpha > 0$. Then this transform can actually be broken into two:

$$\mathrm{F}\left\{e^{-\alpha|t|}\right\} = \mathrm{F}\left\{e^{-\alpha t}u(t)\right\} + \mathrm{F}\left\{e^{\alpha t}u(-t)\right\}$$

Then the first was already found above. As for the second:

$$\mathrm{F}\left\{e^{\alpha t}u(-t)\right\} = \int_{-\infty}^{\infty} e^{\alpha t}u(-t)\cdot e^{-j2\pi f t}dt = \int_{-\infty}^{0} e^{\alpha t}e^{-j2\pi f t}dt =$$

$$= \int_{-\infty}^{0} e^{(\alpha - j2\pi f)t}dt = \left.\frac{e^{(\alpha - j2\pi f)t}}{\alpha - j2\pi f}\right|_{-\infty}^{0} = \frac{1}{\alpha - j2\pi f}$$

Putting the two together, one finds:

$$\mathrm{F}\left\{e^{-\alpha|t|}\right\} = \mathrm{F}\left\{e^{-\alpha t}u(t)\right\} + \mathrm{F}\left\{e^{\alpha t}u(-t)\right\} = \frac{1}{\alpha + j2\pi f} + \frac{1}{\alpha - j2\pi f} =$$
$$= \frac{\alpha - j2\pi f + \alpha + j2\pi f}{(\alpha + j2\pi f)(\alpha - j2\pi f)} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

It is interesting to remark that the resulting function:

$$S(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

is famous in Physics, where it goes by the name of *Lorentzian curve*. For instance, it is quite important in the discussion of laser emission spectra.

Optional:

Transform of the unilateral decreasing exponential signal times "t": $te^{-\alpha t}u(t) \quad \alpha > 0$

$$\mathrm{F}\left\{te^{-\alpha t}u(t)\right\} = \int_{-\infty}^{\infty} te^{-\alpha t}u(t) \cdot e^{-j2\pi f t}dt = \int_{0}^{\infty} te^{-\alpha t}e^{-j2\pi f t}dt$$

The rule of integration by parts should then be applied. As a reminder, the rule says that: $\int w'g = wg - \int wg'$. The justification is as follows:

$$
\begin{gathered}
w'g + wg' = (wg)' \\
\Downarrow \\
\int w'g + \int wg' = \int (wg)' = wg \\
\Downarrow \\
\int w'g = wg - \int wg'
\end{gathered}
$$

In the following, we identify:

$$
\begin{aligned}
&g = t \\
&w' = e^{-\alpha t} e^{-j2\pi f t}
\end{aligned}
$$

As a result, we have:

$$\int_0^{\infty} te^{-\alpha t} e^{-j2\pi f t} dt = -\frac{te^{-\alpha t} e^{-j2\pi f t}}{\alpha + j2\pi f}\bigg|_0^{\infty} - \int_0^{\infty} \frac{e^{-\alpha t} e^{-j2\pi f t}}{\alpha + j2\pi f} dt$$

However:

$$-\frac{te^{-\alpha t} e^{-j2\pi f t}}{\alpha + j2\pi f}\bigg|_0^{\infty} = 0$$

And:

$$-\int_0^{\infty} \frac{e^{-\alpha t} e^{-j2\pi f t}}{\alpha + j2\pi f} dt = \frac{-1}{\alpha + j2\pi f} \frac{-e^{-\alpha t} e^{-j2\pi f t}}{\alpha + j2\pi f}\bigg|_0^{\infty} = \frac{1}{\left(\alpha + j2\pi f\right)^2}$$

Transform of the triangular signal:

$$\mathrm{F}\left\{\Lambda_T\left(t\right)\right\} = \int_{-\infty}^{\infty} \Lambda_T\left(t\right) \cdot e^{-j2\pi f t} dt = \int_{-T}^{T} \Lambda_T\left(t\right) \cdot e^{-j2\pi f t} dt =$$

$$= \int_{-T}^{T} \Lambda_T\left(t\right) \cdot \cos(2\pi f t)\, dt$$

The last passage is justified because the integral of an odd integrand between symmetric limits is zero. Specifically:

$$-j\int_{-T}^{T} \Lambda_T\left(t\right) \cdot \sin(2\pi f t)\, dt = 0$$

We then have:

$$\mathrm{F}\left\{\Lambda_T\left(t\right)\right\} = \int_0^T [1 - t/T]\cos(2\pi f t)\,dt + \int_{-T}^0 [1 + t/T]\cos(2\pi f t)\,dt =$$

$$= \int_{-T}^T \cos(2\pi f t)\,dt - \int_0^T [t/T]\cos(2\pi f t)\,dt + \int_{-T}^0 [t/T]\cos(2\pi f t)\,dt =$$

$$= \int_{-T}^T \cos(2\pi f t)\,dt - 2\int_0^T [t/T]\cos(2\pi f t)\,dt$$

$$= \left.\frac{\sin(2\pi f t)}{2\pi t}\right|_{-T}^{T} - 2\int_0^T [t/T]\cos(2\pi f t)\,dt$$

After the above elementary steps, we are left with two terms. The first yields:

$$\left.\frac{\sin(2\pi f t)}{2\pi f}\right|_{-T}^{T} = \frac{\sin(2\pi f T)}{2\pi f} - \frac{\sin(-2\pi f T)}{2\pi f} = 2\frac{\sin(2\pi f T)}{2\pi f} = \frac{\sin(2\pi f T)}{\pi f}$$

The second term needs to be integrated by parts. We recall once again the general formula for integration by parts:

$$\int w' g = wg - \int w g'$$

By assigning:

$$\frac{t}{T} = g$$
$$\cos(2\pi f t) = w'$$

we then have:

$$\int_0^T \frac{t}{T}\cos(2\pi f t)\,dt = \frac{t}{T}\frac{\sin(2\pi f t)}{2\pi f}\Bigg|_0^T - \int_0^T \frac{1}{T}\frac{\sin(2\pi f t)}{2\pi f}\,dt =$$
$$= \frac{\sin(2\pi f T)}{2\pi f} + \frac{1}{T}\frac{\cos(2\pi f t)}{4\pi^2 f^2}\Bigg|_0^T =$$
$$= \frac{\sin(2\pi f T)}{2\pi f} + \frac{1}{T}\frac{\cos(2\pi f T) - 1}{4\pi^2 f^2}$$

We now put the two terms together again to obtain:

$$
\begin{aligned}
&\mathrm{F}\left\{\Lambda_T\left(t\right)\right\} = \\
&= \left.\frac{\sin(2\pi f t)}{2\pi t}\right|_{-T}^{T} - 2\int_0^T [t/T]\cos(2\pi f t)\,dt \\
&= 2\frac{\sin(2\pi f T)}{2\pi f} - 2\left[\frac{\sin(2\pi f T)}{2\pi f} + \frac{1}{T}\frac{\cos(2\pi f T)-1}{4\pi^2 f^2}\right] = \\
&= 2\frac{\sin(2\pi f T)}{2\pi f} - 2\frac{\sin(2\pi f T)}{2\pi f} - \frac{2}{T}\cdot\frac{\cos(2\pi f T)-1}{4\pi^2 f^2} = \\
&= \frac{2}{T}\cdot\frac{1-\cos(2\pi f T)}{4\pi^2 f^2} = \frac{1}{T}\cdot\frac{1-\cos(2\pi f T)}{2\pi^2 f^2}
\end{aligned}
$$

We can further simplify the result by recalling the trigonometric identity:

$$\frac{1-\cos(x)}{2} = \sin^2(x/2)$$

so that finally:

$$\mathrm{F}\left\{\Lambda_T(t)\right\} = \frac{1}{T}\frac{\sin^2(\pi fT)}{\pi^2 f^2} = T\frac{\sin^2(\pi fT/2)}{\pi^2 f^2 T^2} = T\cdot\mathrm{Sinc}^2(fT)$$

End of optional material.

6.2 Properties of the Fourier Transform

6.2.1 Linearity

The Fourier Transform is a linear operator:

$$F\left\{\alpha x(t)+\beta y(t)\right\} = \alpha X(f)+\beta Y(f)\,.$$

This property is obvious because the Fourier transform is an inner product whose second operand is fixed. So the distributive property of the inner product can be used:

$$\mathrm{F}\left\{\alpha x(t)+\beta y(t)\right\}=\left(\alpha x(t)+\beta y(t),e^{j2\pi ft}\right)=$$

$$\alpha\left(x(t),e^{j2\pi ft}\right)+\beta\left(y(t),e^{j2\pi ft}\right)=\alpha\mathrm{F}\left\{x(t)\right\}+\beta\,\mathrm{F}\left\{y(t)\right\}=$$

$$\alpha X(f)+\beta Y(f)$$

One can also simply reason that the Fourier transform is an integral operator and the integral is a linear operator itself.

6.2.2 Time-delay

Given $s(t)\xleftrightarrow{\mathrm{F}}S(f)$, then:

$$\mathrm{F}\left\{s(t-t_d)\right\}=S(f)e^{-j2\pi f t_d}\,.$$

In fact:

$$\mathrm{F}\left\{s(t-t_d)\right\} = \int_{-\infty}^{+\infty} s(t-t_d)\, e^{-j2\pi f t} dt$$

We now substitute $\tau = t - t_d$, that is:

$$t = \tau + t_d, \quad dt = d\tau$$

$$\tau_{\mathrm{btm}} = -\infty - t_d = -\infty = t_{\mathrm{btm}},$$

$$\tau_{\mathrm{top}} = \infty - t_d = -\infty = t_{\mathrm{top}},$$

As a result:

$$\mathrm{F}\left\{s(t-t_d)\right\} = \int_{-\infty}^{+\infty} s(\tau) e^{-j2\pi f(\tau+t_d)} d\tau =$$

$$= e^{-j2\pi f t_d} \int_{-\infty}^{+\infty} s(\tau) e^{-j2\pi f \tau} d\tau$$
$$= e^{-j2\pi f t_d} \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f t} dt = e^{-j2\pi f t_d} S(f).$$

where in the last passage we have simply substituted $\tau = t$ to go back to the customary time-variable used in Fourier transforms.

Note that, considering this property, the location where a signal "emerges" along the time-axis is not due to its frequency content, which remains essentially unchanged, but is due to the phase with which such frequencies add up. By orderly shifting such phases, the signal "is created" at a different time.

6.2.3 Fourier transform symmetries

Given $s(t) \in \mathbb{R}$ then $S(f)$ is such that:

$$S(f) = S^*(-f)$$

Eq. 6-26

The proof is immediate:

$$S^*(-f) = \left[\int_{-\infty}^{+\infty} s(t) e^{-j2\pi(-f)t} dt \right]^* = \int_{-\infty}^{+\infty} s^*(t) e^{-j2\pi ft} dt = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi ft} dt = S(f)$$

having used the fact that $s(t) = s^*(t)$.

From the relation $S(f) = S^*(-f)$, many derived results follow. Specifically:

- the absolute value of $S(f)$ is even
- the phase of $S(f)$ is odd
- the real part of $S(f)$ is even
- the imaginary part of $S(f)$ is odd

We prove them one by one.

The absolute value of $S(f)$ *is even:* $\left|S(f)\right| = \left|S(-f)\right|$.

First, we take the absolute value of both sides of Eq. 6-26:

$$\left|S(f)\right| = \left|S^*(-f)\right|$$

But, given a complex number z, it is also true in general that $\left|z^*\right| = \left|z\right|$. So, the RHS of the above equation $\left|S^*(-f)\right|$ is equal to $\left|S(-f)\right|$. Therefore, we can write:

$$\left|S(f)\right| = \left|S(-f)\right|$$

which proves the property.

The phase of $S(f)$ *is odd:* $\angle S(-f) = -\angle S(f)$

To prove this, we convert both $S(f)$ and $S(-f)$ into polar coordinates:

$$S(f) = |S(f)| e^{j\varphi(f)}$$
$$S(-f) = |S(-f)| e^{j\varphi(-f)}$$

But we have just shown that the two absolute values above coincide: $|S(f)| = |S(-f)|$. Therefore, we can re-write the above equalites as:

$$S(f) = |S(f)| e^{j\varphi(f)}$$
$$S(-f) = |S(f)| e^{j\varphi(-f)}$$

Using these polar forms, we can then rewrite Eq. 6-26 as:

$$|S(f)| e^{j\varphi(f)} = \left[|S(f)| e^{j\varphi(-f)} \right]^*$$

from which we immediately get:

$$|S(f)| e^{j\varphi(f)} = |S(f)| e^{-j\varphi(-f)}$$

and from here, dividing both sides by $|S(f)|$:

$$|S(f)|e^{j\varphi(f)} = |S(f)|e^{-j\varphi(-f)}$$
$$\downarrow$$
$$e^{j\varphi(f)} = e^{-j\varphi(-f)}$$
$$\downarrow$$
$$\varphi(f) = -\varphi(-f)$$

which can also be written as:

$$\angle S(-f) = -\angle S(f).$$

which proves the property.

Proving that the real part of $S(f)$ is even and the imaginary part of $S(f)$ is odd.

The Fourier transform of a signal is, in general, a complex function, that is: $S(f) \in \mathbb{C}$. Therefore, we can always write:

$$S(f) = \mathrm{Re}\{S(f)\} + j\,\mathrm{Im}\{S(f)\} = S_{\mathrm{Re}}(f) + jS_{\mathrm{Im}}(f)$$

Keeping in mind the assumption that $s(t) \in \mathbb{R}$, then we have:

$$S_{\mathrm{Re}}(f) = \mathrm{Re}\{S(f)\} = \mathrm{Re}\left\{\int_{-\infty}^{+\infty} s(t) e^{-j2\pi ft} dt\right\} =$$
$$\int_{-\infty}^{+\infty} s(t)\cos\left(-2\pi ft\right) dt = \int_{-\infty}^{+\infty} s(t)\cos\left(2\pi ft\right) dt$$

Eq. 6-27

So, the real part is the "*cosine transform*" of the signal and the last equality shows that $S_{\mathrm{Re}}(f)$ is *even* over f.

Similarly:

$$S_{\mathrm{Im}}(f) = \mathrm{Im}\{S(f)\} = \mathrm{Im}\left\{\int_{-\infty}^{+\infty} s(t)e^{-j2\pi ift}\,dt\right\} =$$
$$\int_{-\infty}^{+\infty} s(t)\sin\left(-2\pi ft\right)dt = -\int_{-\infty}^{+\infty} s(t)\sin\left(2\pi ft\right)dt$$

Eq. 6-28

So, the imaginary part is the "*sine transform*" of the signal and the last equality clearly shows that such transform is odd over f.

$\boxed{\text{On your own.}}$ The same proof that that the real part of $S(f)$ is even and the imaginary part of $S(f)$ is odd can also be given by using again directly the relation: $S(-f) = S^*(f)$. Try to do it yourself.

Two related **important properties** are the following:

if $s(t)$ ***is real and even, then*** $S(f)$ ***is real and even;***

if $s(t)$ ***is real and odd, then*** $S(f)$ ***is purely imaginary and odd.***

To prove these properties, we first introduce the following general result: *the integral of an odd function over an interval* $[-a,a]$ *(including* $[-\infty,\infty]$*) is zero.* You can easily prove it on your own.

Another general result that we need is the following:
the product of an even and an odd function is odd.
This too, you can easily prove it on your own.

Then, we first assume that $s(t)$ is real ad even.

If so, the integrand of Eq. 6-28 is the product of an even function of time, $s(t)$, and an odd function of time, $\sin(2\pi f t)$. Therefore, the integrand is an odd function of time. Its integral over $[-\infty,\infty]$ therefore is zero, that is:

$$S_{\mathrm{Im}}(f) = -\int_{-\infty}^{+\infty} s(t)\sin(2\pi ft)\,dt = 0(f)$$

Eq. 6-27 is instead, in general, non-zero, and therefore $S(f)$ <u>is purely real and even</u>.

Then, we assume that $s(t)$ <u>is real ad odd</u>.

If so, the integrand of Eq. 6-27 is the product of an odd function of time, $s(t)$, and an even function of time, $\cos(2\pi f t)$. Therefore, the integrand is an odd function of time and its integral over $[-\infty, \infty]$ is zero, that is:

$$S_{\mathrm{Re}}(f) = \int_{-\infty}^{+\infty} s(t)\cos(2\pi ft)\,dt = 0(f)$$

Eq. 6-28 is instead, in general, non-zero, and therefore $S(f)$ is purely imaginary and odd.

On your own: verify the relevant symmetry properties on the Fourier transform of the following signals:

$s(t) = \Pi_T(t)$,

$$s(t) = e^{-\alpha|t|} \quad \alpha > 0$$

$$s(t) = \Pi_T\left(t + \frac{T}{2}\right) - \Pi_T\left(t - \frac{T}{2}\right)$$

$$s(t) = e^{-\alpha t}u(t) - e^{\alpha t}u(-t) \qquad \alpha > 0$$

(hint: here use the time-delay property to compute the Fourier transform)

6.2.4 Optional: Fourier transform of the complex conjugate

Given $s(t) \xleftrightarrow{\mathrm{F}} S(f)$, then:

$$\mathrm{F}\left\{s^*(t)\right\} = S^*(-f)$$

In fact:

$$S^*(-f) = \left[\int_{-\infty}^{+\infty} s(t)e^{-j2\pi(-f)t}dt\right]^* = \left[\int_{-\infty}^{+\infty} s(t)e^{j2\pi(f)t}dt\right]^* =$$

$$= \int_{-\infty}^{+\infty} s^*(t)e^{-j2\pi ft}dt = \mathrm{F}\left\{s^*(t)\right\}$$

This property appears to be just an oddity, but in fact it is a fundamental one for digital receivers for data transmission.

End of optional material.

6.2.5 Time-scaling

Given $s(t) \xleftrightarrow{\mathrm{F}} S(f)$, then:

$$\mathrm{F}\left\{s(a \cdot t)\right\} = \frac{1}{|a|} S\left(\frac{f}{a}\right), \quad a \neq 0$$

There are two possible cases which we analyse separately: $a > 0$ and $a < 0$. Let us start with $a > 0$:

$$\mathrm{F}\left\{s(a \cdot t)\right\} = \mathrm{F}\left\{s(|a|t)\right\} = \int_{-\infty}^{+\infty} s(|a|\tau) e^{-j2\pi\tau} d\tau$$

Applying the following substitution in the integral:

$$t = |a|\tau, \quad \frac{dt}{|a|} = d\tau, \quad \tau_{\text{btm}} = -\infty = t_{\text{btm}}, \quad \tau_{\text{top}} = \infty = t_{\text{top}},$$

we have:

$$\mathrm{F}\left\{s(|a|t)\right\} = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f \frac{t}{|a|}} \frac{dt}{|a|} = \frac{1}{|a|} \int_{-\infty}^{+\infty} s(t) e^{-j2\pi t \frac{f}{|a|}} dt = \frac{1}{|a|} S\left(\frac{f}{|a|}\right)$$

Looking now at $a < 0$:

$$\mathrm{F}\left\{s(a\cdot t)\right\}=\mathrm{F}\left\{s(-|a|t)\right\}=\int_{-\infty}^{+\infty} s(-|a|\tau)e^{-j2\pi\tau}d\tau$$

Applying the following substitution in the integral:

$$t=-|a|\tau,\quad -\frac{dt}{|a|}=d\tau,\quad \tau_{\text{btm}}=-\infty=-t_{\text{btm}},\quad \tau_{\text{top}}=\infty=-t_{\text{top}},$$

we have:

$$\mathrm{F}\left\{s(-|a|t)\right\}=\int_{+\infty}^{-\infty} s(t)e^{-j2\pi f\left(-\frac{t}{|a|}\right)}\left(-\frac{dt}{|a|}\right)=$$

$$=\frac{1}{|a|}\int_{-\infty}^{+\infty} s(t)e^{-j2\pi t\left(-\frac{f}{|a|}\right)}dt=\frac{1}{|a|}S\left(-\frac{f}{|a|}\right)=\frac{1}{|a|}S\left(\frac{f}{a}\right)$$

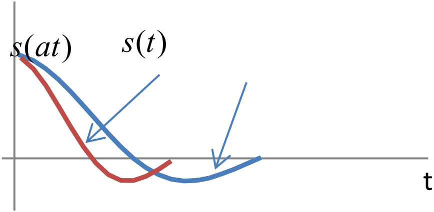
The last equality shows that the result for $a > 0$ and $a < 0$ are indeed identical and the general result is proved.

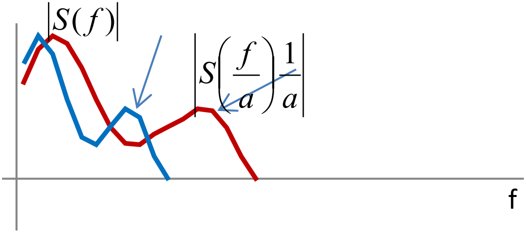
Notice that by setting $a = -1$ we get the following noteworthy result, called the *time-inversion property*:

given $s(t) \xleftrightarrow{\mathrm{F}} S(f)$, *then* $\mathrm{F}\{s(-t)\} = S(-f)$.

If $a > 1$

$$\left|S\left(\frac{f}{a}\right)\frac{1}{a}\right|$$





If $0 < a < 1$

$\left|S(f)\right|$

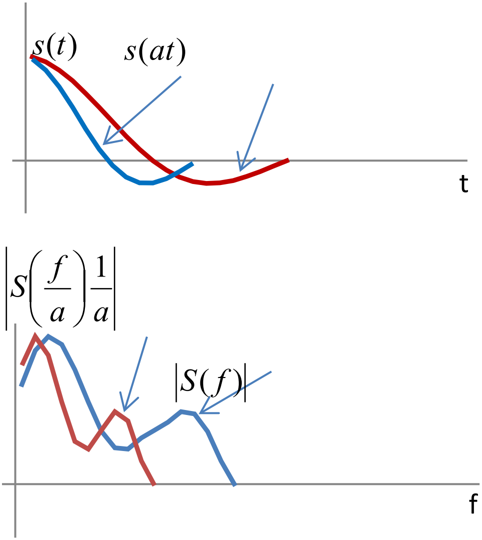


Fig. 6.3

It can be seen that faster-evolving signals (in time) are characterized by having spectra that extend further into higher frequencies. Conversely, slower-evolving signals (in time) tend to be characterized by spectra that extend less into high frequencies.

$\boxed{\text{On your own}}$

Find the relationship between $\mathcal{E}_{\mathbb{R}}\left\{s(t)\right\}$ and $\mathcal{E}_{\mathbb{R}}\left\{s(\alpha \cdot t)\right\}$

Answer: $\mathcal{E}_{\mathbb{R}}\left\{s(\alpha \cdot t)\right\} = \dfrac{1}{|\alpha|}\mathcal{E}_{\mathbb{R}}\left\{s(t)\right\}$

$\boxed{\text{On your own}}$ check that the energy identity between a signal and its Fourier transform is preserved after time-scaling.

In other words, given that:

$$\mathcal{E}_{\mathbb{R}}\{s(t)\} = \mathcal{E}_{\mathbb{R}}\{S(f)\}$$

check by direct calculation that:

$$\mathcal{E}_{\mathbb{R}}\{s(\alpha \cdot t)\} = \mathcal{E}_{\mathbb{R}}\left\{\frac{1}{|\alpha|} S\left(\frac{f}{\alpha}\right)\right\}$$

$\boxed{\text{On your own}}$ Given the signal:

$$s(t) = \Lambda_1(t-1)$$

consider $s(\alpha \cdot t)$ with $\alpha = 1$, 1/2, 2 . Draw the three signals.

Calculate their Fourier transforms using the rule presented above. Then also calculate their Fourier transform by first converting each signal to the form $\Lambda_T(t-t_d)$ and then using the table of transforms for signals of that type. Check that the results coincide.

Answers:

$$\mathrm{F}\left\{s(t)\right\} = \mathrm{F}\left\{\Lambda_1(t-1)\right\} = \mathrm{Sinc}^2(f)\cdot e^{-j2\pi f}$$

$$\mathrm{F}\left\{s(2t)\right\} = \mathrm{F}\left\{\Lambda_1(2t-1)\right\} = \mathrm{F}\left\{\Lambda_{1/2}(t-1/2)\right\} = \frac{1}{2}\mathrm{Sinc}^2\left(\frac{f}{2}\right)\cdot e^{-j\pi f}$$

$$\mathrm{F}\left\{s\left(\frac{1}{2}t\right)\right\} = \mathrm{F}\left\{\Lambda_1(t/2-1)\right\} = \mathrm{F}\left\{\Lambda_2(t-2)\right\} = 2\,\mathrm{Sinc}^2(2f)\cdot e^{-j4\pi f}$$

6.2.6 The frequency translation property

Given $s(t) \xleftrightarrow{\mathrm{F}} S(f)$, *then:*

$$\mathrm{F}\left\{s(t)e^{j2\pi f_0 t}\right\} = S(f - f_0)$$

Eq. 6-29

To show it, we directly write:

$$\mathrm{F}\left\{s(t)e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} s(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt =$$

$$= \int_{-\infty}^{\infty} s(t)e^{-j2\pi(f - f_0)t}dt = S(f - f_0)$$

The frequency translation property is extremely important for practical applications. It allows taking a signal from a portion of the spectrum and relocating it onto any other portion of the spectrum. Thanks to this property, for instance, radio transmission is possible, including FM radio, cell phones, TV transmission and so on.

This aspect was extensively commented on in class and the student is supposed to be able to expand on it based on the discussion in class.

In particular, in class the aspect of using the real signal $\cos\left(2\pi f_0 t\right)$ rather than the non-physical complex signal $e^{j2\pi f_0 t}$ was discussed. The relevant result is the following:

$$\mathrm{F}\left\{s(t)\cdot 2\cos\left(2\pi f_0 t\right)\right\} = S(f - f_0) + S(f + f_0)$$

Note also that this property is essentially the dual property of translation in time. The latter could also be written as:

$$\mathrm{F}^{-1}\left\{S(f)e^{-j2\pi f t_d}\right\} = s(t - t_d)$$

from which the complete duality (apart from the sign of the exponent of the exponential function) is evident.

Such time-frequency symmetry exists in all Fourier transform properties, essentially because the Fourier transform integral and inverse transform integral have an almost identical structure (except for a sign change). Time-frequency symmetry also exists for the Fourier transforms themselves, as shown in the next property.

6.2.7 Time-frequency symmetry

The time-frequency symmetry property states the following;

$$given\ \ a(t) \xleftrightarrow{\mathrm{F}} b(f),\ then\ \ b(t) \xleftrightarrow{\mathrm{F}} a(-f)$$

Note the non-standard notation for signals, which is necessary here not to get confused in the statement. Specifically, we have refrained from using the capitalized letter to indicate a Fourier transform.

Optional:

To prove it, we start off by re-writing the hypothesis as follows:

$$b(f) = \mathrm{F}\{a(t)\} = \int_{-\infty}^{+\infty} a(t) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} a(\theta) e^{-j2\pi f \theta} d\theta$$

So, in short, the hypothesis can be written as:

$$b(f) = \int_{-\infty}^{+\infty} a(\theta) e^{-j2\pi f \theta} d\theta$$

Eq. 6-30

Note that equation Eq. 6-30 is *our assumption, so it is true.*

Based on such assumption we want to prove that: $b(t) \xleftrightarrow{\mathrm{F}} a(-f)$. This formula, *which is what we want to prove and we do not know yet if it is true (which we indicate with a question mark)*, can be re-written as:

$$b(t) \overset{?}{=} \mathrm{F}^{-1}\{a(-f)\} = \int_{-\infty}^{+\infty} a(-f) e^{j2\pi ft} df = \int_{-\infty}^{+\infty} a(\theta) e^{-j2\pi\theta t} d\theta$$

Taking just the leftmost and rightmost terms above, we can write:

$$b(t) \overset{?}{=} \int_{-\infty}^{+\infty} a(\theta) e^{-j2\pi\theta t} d\theta$$

Eq. 6-31

which, again, is what needs to be proved, but *we do not know yet if it is true or not*.

So, in summary, *we want to check whether*

$$b(t) \overset{?}{=} \int_{-\infty}^{+\infty} a(\theta)\, e^{-j2\pi\theta t} d\theta$$

given that

$$b(f)=\int_{-\infty}^{+\infty} a(\theta)\, e^{-j2\pi f\theta} d\theta$$

To prove the above, it is enough to formally change $f \to t$ in the second equation: the result shows that the second equation, which we know it is true by assumption, is in fact identical to the first. So the two equations are in fact the same equation and therefore if the second is true, the first is true too. As a result:

$$b(t) \xleftrightarrow{\mathrm{F}} a(-f) \ \textit{given} \ \ a(t) \xleftrightarrow{\mathrm{F}} b(f), \textit{ is proved.}$$

End of optional material.

6.2.7.1 Examples

As an example, we know that:

$$e^{-\alpha t}u(t), \quad \alpha > 0 \quad \xleftrightarrow{\mathrm{F}} \quad \frac{1}{\alpha + j2\pi f}$$

A direct application of the symmetry property tells us that:

$$\frac{1}{\alpha - j2\pi t} \quad \xleftrightarrow{\mathrm{F}} \quad e^{-\alpha f}u(f), \quad \alpha > 0$$

Another example is:

$$\Pi_a(t) \quad \xleftrightarrow{\mathrm{F}} \quad a \cdot \mathrm{Sinc}\left(a f\right)$$

$$a \cdot \mathrm{Sinc}\left(a t\right) \quad \xleftrightarrow{\mathrm{F}} \quad \Pi_a(f)$$

The latter signal, $a \cdot \text{Sinc}(a\,t)$, is of exceptional importance in the theory of digital signal reconstruction, as we shall see.

$\boxed{\text{On your own:}}$ take

Table 6-1 and complete it with all the "dual" pairs. What happens regarding the Gaussian signal?

6.2.8 Time-derivative property

$$\textit{Given } s(t) \xleftrightarrow{\mathrm{F}} S(f), \textit{ then: } \mathrm{F}\left\{\frac{d}{dt}s(t)\right\} = j2\pi f \cdot S(f)$$

This result can be easily proved as follows. We know that:

$$s(t) = \int_{-\infty}^{+\infty} S(f)e^{j2\pi ft}df$$

We know take the time-derivative of both sides:

$$\frac{d}{dt}s(t) = \frac{d}{dt}\left[\int_{-\infty}^{+\infty} S(f)e^{j2\pi ft}df\right]$$

Both the integral and the derivative are linear operators. In addition, we remark that the integral operates on the integration variable f whereas the derivative operator relates to the variable t. For these reasons they can be swapped:

$$\frac{d}{dt}s(t)=\int_{-\infty}^{+\infty}\frac{d}{dt}\left[S(f)e^{j2\pi ft}\right]df=$$

$$\int_{-\infty}^{+\infty}S(f)\frac{d}{dt}\left[e^{j2\pi ft}\right]df=\int_{-\infty}^{+\infty}j2\pi f\,S(f)e^{j2\pi ft}df$$

$$=\mathrm{F}^{-1}\left\{j2\pi f\cdot S(f)\right\}$$

This property can be extended to any order of derivation, by simply applying it repeatedly. It can be easily shown that:

$$\textit{given } s(t) \xleftrightarrow{\mathrm{F}} S(f), \textit{ then: } \mathrm{F}\left\{ \frac{d^n}{dt^n} s(t) \right\} = \left(j2\pi f \right)^n \cdot S(f).$$

The time-derivative property is used extensively to deal with differential equations, where time-derivatives are present. By applying a Fourier transformation such equations can be reduced to algebraic equations. The Laplace transform, which is closely related to the Fourier transform, is also used for similar purposes.

This property always works if $s(t) \in L^2_{\mathbb{R}}$ (provided that the signal is not too pathological). When applied to signals that are not finite-energy, it may run into problems and each case must be carefully discussed. If interested, read on.

So, in general, caution should be used and results checked. However, for the signals and distributions used in this course, the property always works unless otherwise pointed out.

Optional: One case in point is when the signal does not go to zero for $|t| \to \infty$. This can be seen by integrating by part:

$$\mathrm{F}\left\{\frac{d}{dt}s(t)\right\} = \int_{-\infty}^{+\infty}\left[\frac{d}{dt}s(t)\right]e^{-j2\pi ft}dt$$

$$= s(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{+\infty}\left(-j2\pi f\right)e^{-j2\pi ft}s(t)dt$$

$$= j2\pi f\; S\left(f\right) + s(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty}$$

The last term $s(t)e^{-j2\pi ft}\Big|_{-\infty}^{\infty}$ is clearly problematic if $s(t)$ does not go to zero for $|t| \to \infty$.

End of optional material.

6.2.9 Problems

6.2.9.1 energy of a signal using Parseval's formula

Compute the energy of $\Pi_T(t)$.

We first directly calculate the energy of the signal in time-domain:

$$\mathcal{E}\left\{\Pi_T(t)\right\} = \int_{-\infty}^{+\infty} \Pi_T^2(t)\,dt = \int_{-T/2}^{T/2} 1(t)\,dt = T$$

We then try to see whether calculating it in frequency-domain yields the same result.

First:

$$\mathrm{F}\{\Pi_T(t)\} = T \cdot \mathrm{Sinc}(Tf)$$

We then know that the energy of the time-domain and the energy of the frequency-domain representation of the signal are the same:

$$\mathcal{E}\{\Pi_T(t)\} = \mathcal{E}\left\{T \cdot \mathrm{Sinc}(Tf)\right\}$$

and:

$$\mathcal{E}\left\{T\cdot \mathrm{Sinc}\left(Tf\right)\right\} = T^2 \int_{-\infty}^{+\infty} \mathrm{Sinc}^2\left(Tf\right)\, df$$

Eq. 6-32

Calculating the right-hand side integral directly is difficult, so we try to use some already-known result and adapt it to our case. In fact, by looking at the table of transforms (

Table 6-1), we see that:

$$\Lambda_T(t) = \mathrm{F}^{-1}\left\{T\cdot \mathrm{Sinc}^2\left(Tf\right)\right\} = T\int_{-\infty}^{+\infty} \mathrm{Sinc}^2\left(Tf\right)\, e^{j2\pi f t} df$$

There is a great similarity between the latter integral and the energy integral in Eq. 6-32, that we need to calculate.

In fact:

$$\Lambda_T(t)\Big|_{t=0} = T\int_{-\infty}^{+\infty} \mathrm{Sinc}^2\left(Tf\right) e^{j2\pi ft} df\Bigg|_{t=0} = T\int_{-\infty}^{+\infty} \mathrm{Sinc}^2\left(Tf\right) df$$

As a result:

$$\mathcal{E}\{\Pi_T(t)\} = T^2\int_{-\infty}^{+\infty} \mathrm{Sinc}^2\left(Tf\right) df = T\cdot\Lambda_T(t)\Big|_{t=0} = T\cdot\Lambda_T(0) = T$$

So, we find again that the energy of the signal is T, both evaluating it over its time-representation and its frequency-representation (Fourier transform).

6.2.9.2 inner product using the component formula

It is easy to see that the two signals $s(t), w(t)$ are orthogonal in time-domain: $(s(t), w(t)) = 0$ (do it $\boxed{\text{on your own}}$), hint: use symmetries).

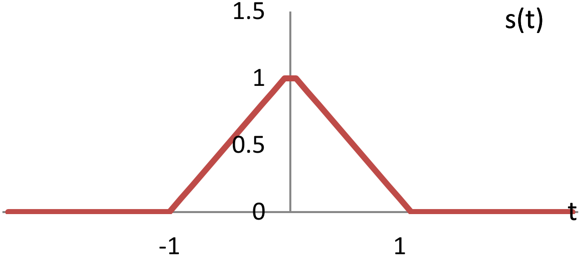


Fig. 6.4

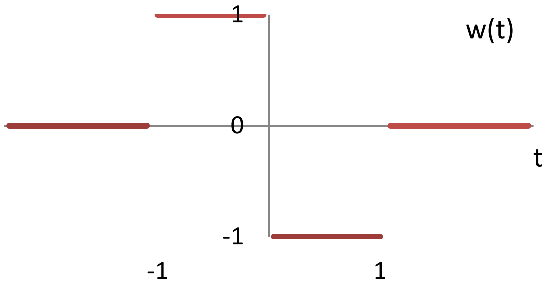


Fig. 6.5

They should stay orthogonal after the basis change to frequency-domain. We want to make sure that this is true.

First of all, we find their Fourier transforms:

$$S(f) = F\left\{\Lambda_1(t)\right\} = \text{Sinc}^2(f)$$

$$\begin{aligned}W(f) &= F\left\{\Pi_1\left(t-\tfrac{1}{2}\right)-\Pi_1\left(t+\tfrac{1}{2}\right)\right\} \\ &= \text{Sinc}(f)e^{j\pi f} - \text{Sinc}(f)e^{-j\pi f} \\ &= \text{Sinc}(f)\left[e^{j\pi f} - e^{-j\pi f}\right] = 2j\text{Sinc}(f)\sin(\pi f)\end{aligned}$$

Then we have:

$$\left(S(f),W(f)\right)=\int_{-\infty}^{+\infty}S(f)\,W^{*}(f)\,df$$

$$=\int_{-\infty}^{+\infty}\mathrm{Sinc}^{2}\left(f\right)\cdot\left[2j\,\mathrm{Sinc}\left(f\right)\sin\left(\pi f\right)\right]^{*}df$$

$$=-2j\int_{-\infty}^{+\infty}\mathrm{Sinc}^{3}\left(f\right)\sin\left(\pi f\right)df=0$$

The integral is 0 because $\mathrm{Sinc}^{3}\left(f\right)$ is even, while $\sin\left(\pi f\right)$ is odd. The overall integrand function is therefore *odd*. As is well-known, the integral of an odd function between symmetric integration limits (identical in absolute value and opposite in sign) is always 0. Hence, the signals are orthogonal in frequency-domain too, as expected.

$\boxed{\text{On your own:}}$ Consider the Fourier transforms of the signals $s(t), w(t)$ and check that their Fourier Transforms have the expected features and symmetries, given that $s(t), w(t)$ a real signals and also have symmetries in time.

6.2.9.3 inner product of a real signal and its derivative

We want to compute the inner product of a signal and its time-derivative:

$$\left(s(t), \frac{d}{dt} s(t) \right)$$

under the assumption that $s(t) \in \mathbb{R}$.

Recalling that the inner product yields the same result in frequency-domain, and using the derivative property of the Fourier transform, we get:

$$\left(s(t), \frac{d}{dt} s(t) \right) = (S(f), \, j2\pi f S(f)) =$$

$$-\int_{-\infty}^{+\infty} S(f) j2\pi f S^*(f) df = -j2\pi \int_{-\infty}^{+\infty} f \, |S(f)|^2 df$$

Since $s(t) \in \mathbb{R}$, then $|S(f)|^2$ is an even function, while f is an odd function. As a result, the overall integrand $f\,|S(f)|^2$ is odd. As well-known, the integral of an odd function between symmetric integration limits is always 0.

Therefore, we can conclude that **a real signal $s(t)$ is always orthogonal to its derivative $\frac{d}{dt}s(t)$**.

One example of this is given by the problem of Sect. 6.2.9.2. The two signals of that example are in the following relationship: $w(t) = \frac{ds(t)}{dt}$. Check for this $\boxed{\text{on your own.}}$ In fact, we indeed found that their inner product was zero.

$\boxed{\text{On your own.}}$ Consider the signal:

$s(t) = (t+2)\Pi_1(t+1) + \Pi_1(t) + (2-t)\Pi_1(t-1)$

Calculate its derivative (suggestion: do it graphically) and then show that $s(t)$ and its derivative are orthogonal.

Optional $\boxed{\text{On your own:}}$ take the signal:

$$s(t) = u(t)\, e^{-at}$$

and prove by direct inner product calculation in time-domain that it is orthogonal to its derivative signal. Note: be sure you know how to handle the case of a Dirac's delta placed at the time of a step-discontinuity in another signal. Specifically, given a signal $s(t)$ that has a step-discontinuity at time $t = 0$, then:

$$\int_{-\infty}^{\infty} \delta(t)\, s(t)\, dt = \frac{s(0^+) + s(0^-)}{2}$$

where $s(0^+)$ and $s(0^-)$ are the limits of $s(t)$ for $t \to 0$ from positive and negative times, respectively. In other words, at time $t = 0$ the signal $s(t)$ jumps from a value $s(0^-)$ to a value $s(0^+)$.

Note that the above result can be easily proved by replacing $\delta(t)$ with a sequence of rectangular signals, as shown in Chapter 2. In particular, it is easy to show that:

$$\lim_{T\to 0}\frac{1}{T}\int_{-\infty}^{\infty}\Pi_T(t)\,s(t)\,dt=\frac{s(0^+)+s(0^-)}{2}$$

The orthogonality property does not work in certain cases. For instance, it does not work for finite-average power signals such that:

$$\lim_{t\to+\infty}s(t)+\lim_{t\to-\infty}s(t)=0$$

One example is $s(t)=u(t)$, for which $\lim_{t\to+\infty} s(t)+\lim_{t\to-\infty} s(t)=1$. Instead, it works if $s(t)=\mathrm{sign}(t)$. You can check this on your own, after reading Section 6.3.

End of optional material

6.2.10 A few more problems

6.2.10.1

On your own:

Prove that the formula for combined scaling and delay is, in general:

$$\mathrm{F}\{s(a[t-t_d])\}=\frac{1}{|a|}S\left(\frac{f}{a}\right)e^{-j2\pi f t_d}$$

6.2.10.2

On your own: evaluate the Fourier transform of the following signal: $s(-t-t_d)$, knowing that $s(t) \xleftrightarrow{\mathrm{F}} S(f)$.

Note that one could use the combined property found above to try and find the result directly. Try that on your own and then *compare the result* with the full direct calculation:

$$\mathrm{F}\{s(-t-t_d)\} = \int_{-\infty}^{\infty} s(-\tau - t_d) e^{-j2\pi f \tau} d\tau$$

$$t = -\tau - t_d, \quad \tau = -(t_d + t) \quad -dt = d\tau,$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} s(t) e^{j2\pi f(t_d+t)} dt = \int_{-\infty}^{\infty} s(t) e^{-j2\pi(-f)(t_d+t)} dt$$

$$= e^{j2\pi f t_d} \int_{-\infty}^{\infty} s(t) e^{-j2\pi(-f)t} dt = S(-f) e^{j2\pi f t_d}$$

Did you get the same result? In what order should you apply the Fourier transform properties to get the correct result? Hint: notice that $s(-t-t_d) = s(-[t+t_d])$.

6.2.10.3

$\boxed{\text{On your own:}}$

compute the Fourier transform of:

$s(t) = \Pi_2(-t-3)$

$w(t) = 3\left[\Pi(2t-4) - \Pi(t/2-6)\right] e^{j2\pi t}$

6.2.10.4

$\boxed{\text{On your own:}}$

Prove that the Fourier transform of:

$$s(t) = \Lambda\left(t + \frac{1}{2}\right) + \Lambda\left(t - \frac{1}{2}\right) + \frac{1}{2}\Lambda(2t)$$

is:

$$S(f) = \frac{9}{4} \cdot \mathrm{Sinc}^2\left(\frac{3}{2}f\right)$$

Hint: draw the signal in time first! And see if you can show that it can be written in a much simpler way.

6.2.11 Optional: The Data Transmission Signal

The following signal:

$$s(t) = \sum_{n=0}^{N-1} b_n \, q(t - nT)$$

Eq. 6-33

is an example of a possible **data transmission signal**.

In practice, the transmitter emits a pulse $q(t)$ every T seconds. Each one of these pulses is multiplied by one of the b_n 's, which can be used to encode information, and delayed by a multiple of T .

For instance, if the n-th pulse is meant to carry a bit equal to zero, then $b_n = 0$. If it is meant to carry a bit at 1, then $b_n = 1$. This simple signal and encoding principle is in fact in very wide use. Other more complex encodings are possible.

In the following, we calculate the Fourier transform of $s(t)$, assuming that the b_n's are real numbers and $q(t)$ is a real signal with $\mathrm{F}\{q(t)\} = Q(f)$.

We also want to find the support of $S(f)$ knowing that $\operatorname{supp}\{Q(f)\} = [-B/2, B/2]$. Note that this quantity is related to the so-called "spectral occupation" or "bandwidth" taken by the signal.

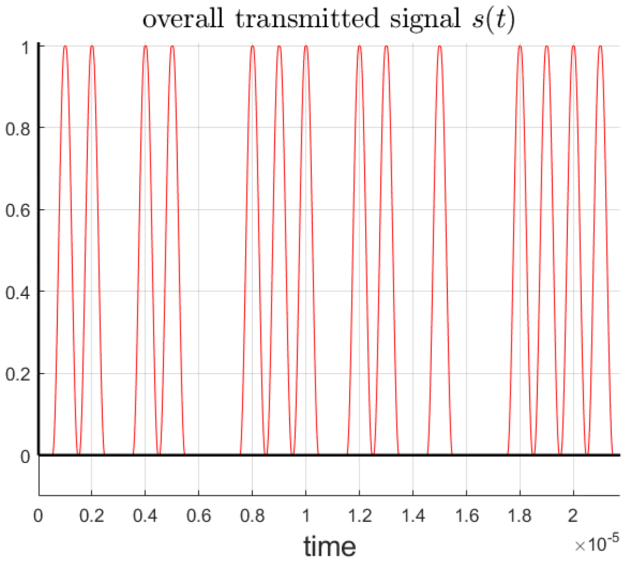


Fig. 6.6: One example of a data transmission signal such as Eq. 6-33, where the b_n 's are either zero or one.

Solution

We can directly calculate:

$$S(f) = F\{s(t)\} = F\left\{\sum_{n=0}^{N-1} b_n\, q(t-nT)\right\} = \sum_{n=0}^{N-1} b_n\, F\{q(t-nT)\}$$

where we exploited the linearity of the Fourier transform. We then remember the delay property:

$$F\{q(t-nT)\} = Q(f)\, e^{-j2\pi fnT}$$

so that finally:

$$S(f) = F\{s(t)\} = Q(f)\sum_{n=0}^{N-1} b_n\, e^{-j2\pi fnT}$$

Eq. 6-34

We remark that we can write:

$$S(f) = Q(f)D(f)$$

Eq. 6-35

where:

$$D(f) = \sum_{n=0}^{N-1} b_n \, e^{-j2\pi f n T}$$

is a factor that depends only on the transmitted data, whereas $Q(f)$ is due solely to the pulse used to carry the information.

Note that $D(f)$ has infinite support and therefore the support of the product $S(f)=Q(f)D(f)$ is determined by $Q(f)$:

$$\mathrm{supp}\{S(f)\}=\mathrm{supp}\{Q(f)\}=[-B/2,B/2]$$

$\boxed{\text{On your own}}$:

Redo the previous problem assuming:

$$s(t)=\sum_{n=0}^{N-1} b_n\, q(t-nT)\, e^{j2\pi f_0 t}$$

with $f_0>0$.

Also redo the previous problem assuming:

$$s(t)=\sum_{n=0}^{N-1} b_n\, q(t-nT)\cos(2\pi f_0 t)$$

with $f_0 > 0.$

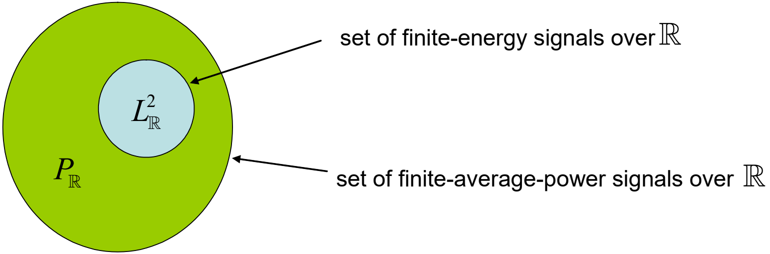
End of optional.

6.3 Fourier Transforms of Finite Average-Power Signals

We have seen that the Fourier transform can be viewed as a "change of basis" for signals $s(t) \in L^2_{\mathbb{R}}$. As such, among other things, it preserves the energy of the signals: $\|s(t)\|^2 = \|S(f)\|^2$. In other words, if $s(t)$ is finite-energy, then also $S(f)$ is finite-energy. The Fourier transform (and inverse-transform) have no convergence problem with finite-energy signals, so this is another way of saying that, for finite energy signals, there is no problem going back and forth from $s(t)$ and $S(f)$.

The Fourier basis $\{e^{j2\pi ft}\}_f$ is however capable of resolving many signals that are not finite-energy. In fact, it is in general capable of resolving finite time-averaged-power signals.

Optional: Remember from previous chapters that the set of finite-energy and finite time-averaged-power signals do not coincide:



We will not discuss here the properties of the set $P_{\mathbb{R}}$ of finite average-power signals over $\mathbb{R}$. We will just show that certain significant signals, or classes of signals, belonging to $P_{\mathbb{R}}$, can be represented by $\left\{e^{j2\pi ft}\right\}_f$, which means that their Fourier transform and inverse transform exist. In addition, all the properties of the Fourier transforms are still valid for these signals. Somewhat surprisingly, many of the Hilbert-space related properties involving the Fourier transform do still apply, such as the fundamental formula Eq. 6-24 establishing the equivalence of the inner product in time and frequency-domain.

Incidentally, the Fourier transform of finite time-averaged-power signals are typically tempered distributions. In Sect. 6.1.4.2, we stated that tempered distributions can be Fourier-transformed and the inversion theorem holds for these objects. This circumstance further ensures that finite time-averaged-power signals can be dealt with using Fourier transforms in a meaningful way, though their transforms will not be ordinary functions, but tempered distributions.

End of optional material

6.3.1 The signals $1(t)$, $\mathrm{sign}(t)$ and $\mathrm{u}(t)$

These three signals are non-finite energy signals, whose average power is however finite. Specifically, $1(t)$, $\mathrm{sign}(t)$ have average power equal to 1, whereas $\mathrm{u}(t)$ has average power $1/2$. In passing, we remark that $1(t)$ could be thought of as a periodic signal, but the other two are non-periodic.

Remember that average power is defined as follows:

$$\left\langle P_s(t)\right\rangle_{\mathbb{R}} \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} P_s(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \left|s(t)\right|^2 dt$$

The Fourier transform of $1(t)$ was already discussed:

$$1(t) \xleftrightarrow{\mathrm{F}} \delta(f)$$

6.3.1.1 the transform of $\operatorname{sign}(t)$

The transform of $\operatorname{sign}(t)$ is: $\mathrm{F}\{\operatorname{sign}(t)\} = \dfrac{1}{j\pi f}$, with the assumption that for $f = 0$ this function exists and its value is 0. This result will be discussed in the following.

The following discussion is **OPTIONAL**

$$\mathrm{F}\{\operatorname{sign}(t)\} = \int_{-\infty}^{\infty} \operatorname{sign}(t) e^{-j2\pi ft} dt$$

Note that $\text{sign}(t)$ is real and odd. As a result its Fourier transform is purely imaginary and is given by the "sine" transform alone:

$$\mathrm{F}\{\text{sign}(t)\} = j\int_{-\infty}^{\infty} \text{sign}(t)\sin(-2\pi ft)\,dt = -2j\int_{0}^{\infty} \sin(2\pi ft)\,dt$$

The latter integral cannot be calculated in an ordinary sense, because the result would be:

$$-2j\int_{0}^{\infty} \sin(2\pi ft)\,dt = -2j\left.\frac{-\cos(2\pi ft)}{2\pi f}\right|_0^{\infty}$$

$$= j\frac{\cos(\infty)}{\pi f} - j\frac{1}{\pi f} = \frac{1}{j\pi f}\left[1 - \frac{\cos(\infty)}{\pi f}\right]$$

but the expression $\cos(\infty)$ has no meaning. The only value of f for which we can find the result is $f = 0$, since in that case:

$$-2j\int_0^{\infty}\sin(2\pi ft)\,dt\Big|_{f=0} = -2j\int_0^{\infty}0(t)\,dt = 0$$

For all other values of f, one can try and see if, by gradually increasing the upper integration limit, the result converges to any known quantity:

$$\lim_{T\to\infty}(-2j)\int_0^{T}\sin(2\pi ft)\,dt = \lim_{T\to\infty}\frac{j\cos(2\pi ft)}{\pi f}\Big|_0^T =$$
$$= \lim_{T\to\infty}\frac{1}{j\pi f}\left[1-\cos(2\pi fT)\right]$$

The expression $\lim_{T\to\infty}\left[1-\frac{\cos(2\pi fT)}{\pi f}\right]$ still does not have a meaning in an ordinary sense, since the ordinary limit does not converge. However, it does have a meaning in the same context as that of the “delta” (technically, the context of “distributions”).

It turns out that:

$$\lim_{T\to\infty}{}_{\text{dist}} \frac{1}{j\pi f}\left[1-\cos\left(2\pi fT\right)\right]=\begin{cases}\dfrac{1}{j\pi f}, & f\neq 0\\[2ex] 0, & f=0\end{cases}$$

which means that if we enclose this function within an integral together with a suitable test function, it tends to have the same *integral properties* as the function on the right-hand-side.

End of optional material.

The result is:

$$\mathrm{F}\{\mathrm{sign}(t)\} = \begin{cases} \dfrac{1}{j\pi f}, & f \neq 0 \\ 0, & f = 0 \end{cases}$$

Note that for brevity the above cumbersome notation on the right-hand-side is typically replaced by simply $\frac{1}{j\pi f}$, with the implied assumption that the value for $f = 0$ does exist and it is 0. In this class we will always assume that $\frac{1}{j\pi f}$ is 0 for $f = 0$

6.3.1.2 Transform of $\mathrm{u}(t)$

Regarding $\mathrm{u}(t)$, we first remark that:

$$u(t) = \frac{1(t)}{2} + \frac{1}{2}\mathrm{sign}(t)$$

Therefore, its Fourier transform is easily found as follows:

$$\mathrm{F}\{\mathrm{u}(t)\} = \mathrm{F}\left\{\frac{1(t)}{2} + \frac{1}{2}\mathrm{sign}(t)\right\}$$

Exploiting the results on $1(t)$ and $\mathrm{sign}(t)$ one immediately finds:

$$\mathrm{F}\{\mathrm{u}(t)\} = \frac{\delta(f)}{2} + \frac{1}{2j\pi f}$$

6.3.2 The signals $e^{j2\pi f_0 t}, \cos(2\pi f_0 t), \sin(2\pi f_0 t)$

These are very important and crucial signals, so students are expected to know their transforms by heart and to be able to derive them.

They are infinite-energy, finite-average-power signals. They are also periodic, a class of signals that will be dealt with in the following section. However, being fundamental in many ways, and easy to deal with, we look at them separately from the other periodic signals.

The transform of $e^{j2\pi f_0 t}$ can be calculated directly very easily as follows:

$$\mathrm{F}\left\{e^{j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} e^{j2\pi f_0 t} e^{j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt = \delta\left(f - f_0\right)$$

In the last passage we have used the well-known result Eq. 6-5, swapping frequency with time as shown in Eq. 6-15.

Using this result, the transforms of cosine and sine are immediately found as:

$$\mathrm{F}\left\{\cos(2\pi f_0 t)\right\} = \mathrm{F}\left\{\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\} = \frac{1}{2}\delta\left(f - f_0\right) + \frac{1}{2}\delta\left(f + f_0\right)$$

$$\mathrm{F}\left\{\sin(2\pi f_0 t)\right\} = \mathrm{F}\left\{\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right\} = -\frac{j}{2}\delta\left(f - f_0\right) + \frac{j}{2}\delta\left(f + f_0\right)$$

Note that these signals have a spectrum that consists or one or two pure "spectral lines", or deltas. This clearly identifies their "frequency content" as being one or two specific frequencies, or "pure tones", as one could expect.

Note also that the $e^{j2\pi f_0 t}$ is a function of the frequency-continuous Fourier Basis Φ_f. The delta result is simply a rewriting of the orthonormality relation for this basis, introduced in Sect. 6.1.5.1.

$\boxed{\text{On your own:}}$ Find the following Fourier transforms:

$$\mathrm{F}\left\{e^{j(2\pi f_0 t+\varphi_0)}\right\}$$

$$\mathrm{F}\left\{\cos(2\pi f_0 t+\varphi_0)\right\}$$

$$\mathrm{F}\left\{\sin(2\pi f_0 t+\varphi_0)\right\}$$

Answers:

$$\mathrm{F}\left\{e^{j(2\pi f_0 t+\varphi_0)}\right\}=\delta\left(f-f_0\right)e^{j\varphi_0}$$

$$\mathrm{F}\left\{\cos(2\pi f_0 t+\varphi_0)\right\}=\frac{1}{2}\delta\left(f-f_0\right)e^{j\varphi_0}+\frac{1}{2}\delta\left(f+f_0\right)e^{-j\varphi_0}$$

$$\mathrm{F}\left\{\sin(2\pi f_0 t+\varphi_0)\right\}=\frac{1}{2j}\delta\left(f-f_0\right)e^{j\varphi_0}-\frac{1}{2j}\delta\left(f+f_0\right)e^{-j\varphi_0}$$

Note that the third result can be easily derived from the second by setting $\varphi_0 = -\frac{\pi}{2}$.

6.3.3 Signal Time-Average and Fourier Transform

Given a signal $s(t)$*, whose spectrum contains a Dirac's delta located at* $f = 0$ *whose coefficient is* d *, then the time average of* $s(t)$ *over all times* $t \in \mathbb{R}$ *is given by:*

$$\left\langle s(t) \right\rangle_{\mathbb{R}} = d$$

This result will be proved later on.

Note that this result is similar to the result that we obtained for the Fourier series representation in L^2_{I}, with $\mathrm{I}=\left[t_0,t_1\right]$, $T_0=t_1-t_0$.

There we had:

$$\left\langle s(t)\right\rangle_{\mathrm{I}}=\frac{s_0}{\sqrt{T_0}}$$

where s_0 is the frequency component of the signal vs. the zero-frequency unit element of the basis: $s_0=\left(s(t),\hat{\phi}_0(t)\right)_{\mathrm{I}}=\left(s(t),\frac{1(t)}{\sqrt{T_0}}\right)_{\mathrm{I}}$

Example

Let $s(t)$ have Fourier Transform:

$$S(f) = F\{s(t)\} = d \cdot \delta(f) + W(f)$$

where $W(f)$ does not have a delta centered at the origin. Then, the time average over $t \in \mathbb{R}$ of $s(t)$ is $\langle s(t) \rangle_{\mathbb{R}} = d$.

6.4 Periodic Signals

A signal is periodic if it satisfies the following relationship:

$$x(t - T_0) = x(t) \quad t \in [-\infty, \infty]$$

Eq. 6-36

It is easy to show that if this is true, then it is also true that:

$$x(t - kT_0) = x(t) \quad t \in [-\infty, \infty], \forall k \text{ integer}$$

Eq. 6-37

On your own: Prove Eq. 6-37 using Eq. 6-36.

$\boxed{\text{On your own}}$: Verify the periodicity of $y(t)$ using the *definition of periodic signal Eq. 6-36*.

$$y(t) = e^{j\frac{2\pi Nt}{T}}$$

Answer

When we introduced complex exponential, we learned that its "period" is everything that divides $\left(j2\pi t\right)$ in the exponent. In the case above, we can write the exponent as: $\dfrac{j2\pi t}{\left(T/N\right)}$, so the quantity which is the "period" appears to be T/N. Let us verify that is the case.

We guess that indeed the period of $y(t)$ is $T_0 = \dfrac{T}{N}$. If so, then $y(t)$ should satisfy Eq. 6-36 with period $T_0 = \dfrac{T}{N}$. We directly write:

$$y\left(t-\frac{T}{N}\right)=e^{j\frac{2\pi}{T}N\left(t-\frac{T}{N}\right)}=e^{j\frac{2\pi Nt}{T}}e^{-j\frac{2\pi N}{T}\frac{T}{N}}$$

The factor $e^{-j\frac{2\pi N}{T}\frac{T}{N}}=e^{-j2\pi}$ is always 1 so indeed:

$$y(t)=y\left(t-\frac{T}{N}\right)=y\left(t-T_0\right)$$

$\boxed{\text{On your own}}$: verify that $y(t)$ does not have any other period smaller than $T_0=\dfrac{T}{N}$

6.4.1 Energy and power of periodic signals

We will restrict our interest to **periodic signals that are finite average-power signals**. This will be implicitly assumed throughout the remainder of Section 6.4.

We will start the discussion on periodic signal by providing some preliminary results on their energy and power.

For a periodic signal, *the energy of the signal is the same over any interval whose length is exactly one period*, independently of where the interval starts. Specifically, given a periodic signal $s(t)$ of period T_0 then the quantity:

$$\mathcal{E}_{[t_0,t_0+T_0]}\{s(t)\} = \int_{t_0}^{t_0+T_0} P_s(t)\,dt = \int_{t_0}^{t_0+T_0} |s(t)|^2\,dt$$

Eq. 6-38

is the same, $\forall t_0$.

In other words:

$$\mathcal{E}_{[t_0, t_0+T_0]}\{s(t)\} = \mathcal{E}_{T_0}\{s(t)\} \qquad \forall t_0$$

where the symbol $\mathcal{E}_{T_0}\{s(t)\}$ is introduced to denote such constant value of the energy of the periodic signal, over any period.

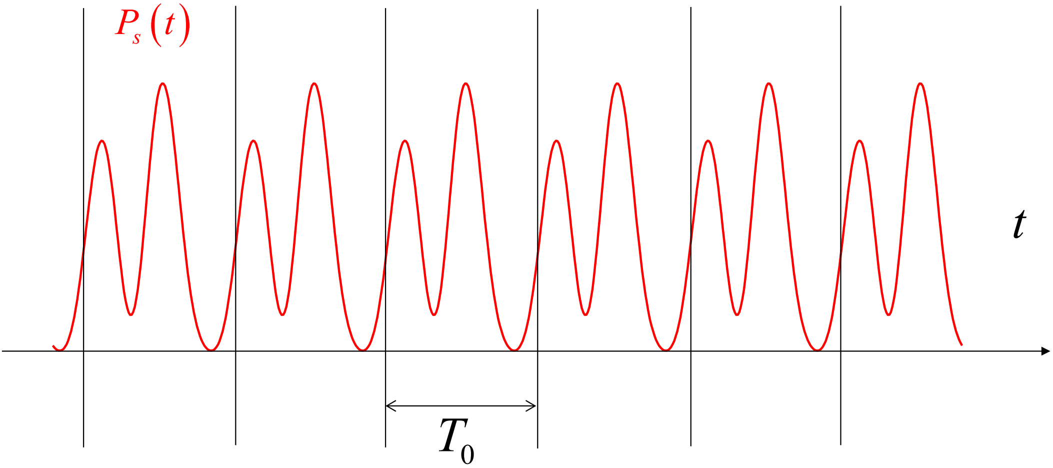
This result is easily proved graphically, after recognizing that *if a signal is periodic, then also its instantaneous power is periodic, with the same period*.
In fact, given that:

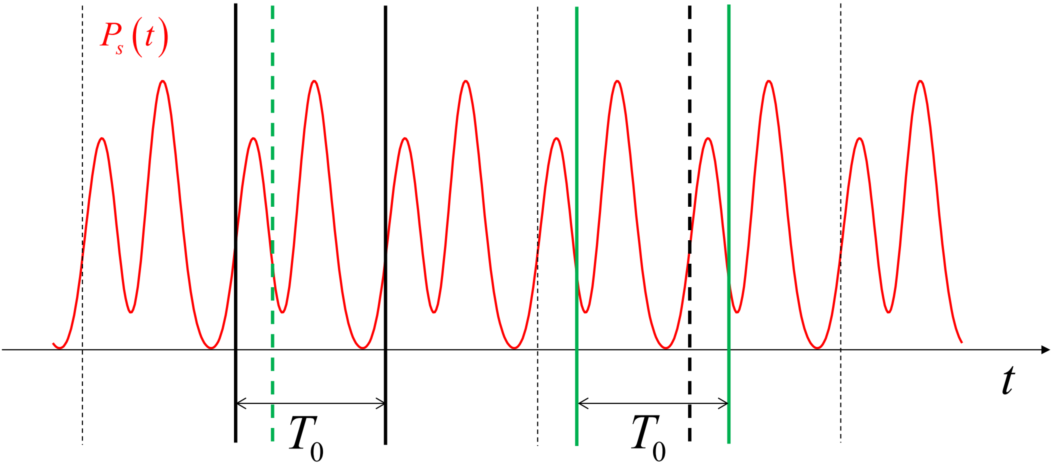
$$s(t) = s(t - T_0)$$

then:

$$P_s(t) = |s(t)|^2 = |s(t - T_0)|^2 = P_s(t - T_0)$$

Then graphically it is easy to see that integrating $P_s(t)$ over any arbitrary interval of time-length one period yields the same result.




$P_s(t)$
t
T_0
T_0

A formal proof is possible and is left as optional material here below.

Optional Proof:

First of all, given any value of t_0 we can always write it:

$$t_0 = t_1 + kT, \qquad 0 \le t_1 < T, \qquad k = \left\lfloor \frac{t_0}{T} \right\rfloor$$

Note that t_1 is positive and smaller than a full period T. The substitution is depicted in the figure below:

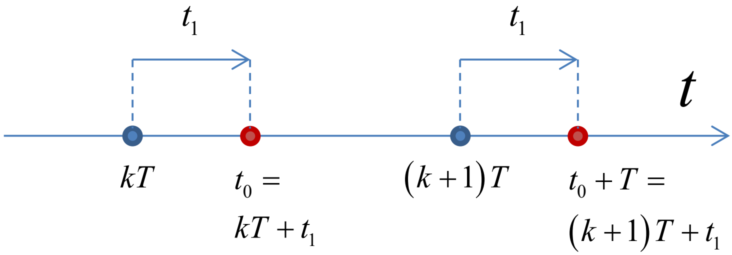


Fig. 6.7

We can then substitute:

$$\int_{t_0}^{t_0+T} P_s(t)\,dt = \int_{t_1+kT}^{t_1+kT+T} |s(t)|^2\,dt = \int_{t_1+kT}^{t_1+(k+1)T} |s(t)|^2\,dt$$

We can then always split the integral into the following two integrals:

$$\int_{t_1+kT}^{t_1+(k+1)T} |s(t)|^2 \, dt = \int_{t_1+kT}^{(k+1)T} |s(t)|^2 \, dt + \int_{(k+1)T}^{t_1+(k+1)T} |s(t)|^2 \, dt$$

Eq. 6-39

so nothing has really changed from the original integral of Eq. 6-38. In particular, the overall integration range is still *exactly* the same.

We then concentrate on the first integral in the right-hand side of Eq. 6-39 and change variable:

$$\tau = t - kT, \qquad d\tau = dt$$

$$\int_{t_1+kT}^{(k+1)T} |s(t)|^2 \, dt = \int_{t_1}^{T} |s(\tau + kT)|^2 \, d\tau$$

However, since the signal is periodic: $s(\tau + kT) = s(\tau)$ so we have:

$$\int_{t_1}^{T} \left|s(\tau + kT)\right|^2 d\tau = \int_{t_1}^{T} \left|s(\tau)\right|^2 d\tau$$

We then concentrate on the second integral in the right-hand side of Eq. 6-39 and change variable:

$$\tau = t - \left(k+1\right)T, \qquad d\tau = dt$$

$$\int_{(k+1)T}^{t_1+(k+1)T} \left|s(t)\right|^2 dt = \int_0^{t_1} \left|s(\tau + \left[k+1\right]T)\right|^2 d\tau$$

However, since the signal is periodic: $s(\tau + \left[k+1\right]T) = s(\tau)$ so we have:

$$\int_0^{t_1} \left|s(\tau + \left[k+1\right]T)\right|^2 d\tau = \int_0^{t_1} \left|s(\tau)\right|^2 d\tau$$

Putting together the two results obtained by changing variables, we can finally re-write Eq. 6-38 and Eq. 6-39 as:

$$\mathcal{E}_{[t_0,t_0+T]}\{s(t)\} = \int_{t_0}^{t_0+T} |s(t)|^2\, dt = \int_{t_1+kT}^{t_1+(k+1)T} |s(t)|^2\, dt =$$

$$= \int_0^{t_1} |s(\tau)|^2\, d\tau + \int_{t_1}^{T} |s(\tau)|^2\, d\tau = \int_0^{T} |s(\tau)|^2\, d\tau$$

$$= \mathcal{E}_{[0,T]}\{s(t)\} = \mathcal{E}_T\{s(t)\} \qquad \forall t_0 \in \mathbb{R}$$

End of optional material.

6.4.2 Average power of a periodic signal

Using the previous result, it is immediately found that the average power over any period is a constant as well, and its value is:

$$\mathcal{P}_{[t_0, t_0+T_0]}\{s(t)\} = \langle P_s(t) \rangle_{[t_0, t_0+T_0]} =$$

$$= \frac{\mathcal{E}_{[t_0, t_0+T_0]}\{s(t)\}}{T_0} = \frac{\mathcal{E}_{T_0}\{s(t)\}}{T_0} = \mathcal{P}_{T_0}\{s(t)\} \qquad \forall t_0$$

Eq. 6-40

where the last symbol $\mathcal{P}_{T_0}\{s(t)\}$ has the meaning of time-averaged power over any interval of extension T_0, since the starting time t_0 is irrelevant.

It is then obvious that the time-averaged power over any integer number of intervals is the same as $\mathcal{P}_{T_0}\left\{s(t)\right\}$:

$$\left\langle P_s(t)\right\rangle_{[t_0, t_0+kT_0]} = \frac{k\mathcal{E}_{T_0}\left\{s(t)\right\}}{kT_0} = \frac{\mathcal{E}_{T_0}\left\{s(t)\right\}}{T_0} = \mathcal{P}_{T_0}\left\{s(t)\right\} \qquad \forall t_0$$

We then remark that nothing changes if we write the interval as $\left[t_0 - \frac{kT_0}{2}, t_0 + \frac{kT_0}{2}\right]$:

$$\left\langle P_s(t)\right\rangle_{\left[t_0-\frac{kT_0}{2}, t_0+\frac{kT_0}{2}\right]} = \frac{k\mathcal{E}_{T_0}\left\{s(t)\right\}}{kT_0} = \frac{\mathcal{E}_{T_0}\left\{s(t)\right\}}{T_0} = \mathcal{P}_{T_0}\left\{s(t)\right\} \qquad \forall t_0$$

Then, by by *letting* k *go to infinity*, we can extend the interval to $\left[-\infty, +\infty\right] = \mathbb{R}$. Since the right-hand side is independent of k, we can surmise that *the average power of a periodic signal over the whole of* $\mathbb{R}$ *is identical to the average power over a single period*:

$$\mathcal{P}_{\mathbb{R}}\left\{s(t)\right\} = \mathcal{P}_{T_0}\left\{s(t)\right\} = \frac{\mathcal{E}_{T_0}\left\{s(t)\right\}}{T_0}$$

Eq. 6-41

This is an *interesting result* which clearly shows that *if a periodic signal is finite-energy over a period, then it is finite-average power over a period and finite-average power over the whole of* $\mathbb{R}$. The converse is also clearly true.

Optional: Note that the reasoning leading to Eq. 6-41 , letting $k \to \infty$, is *not a rigorous proof*. A rigorous proof is provided below:

$$\langle P_s(t) \rangle_{\mathbb{R}} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} P_s(t)dt =$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-\frac{kT-\varepsilon}{2}}^{\frac{kT+\varepsilon}{2}} P_s(t)dt + \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-\frac{kT-\varepsilon}{2}}^{-\frac{kT}{2}} P_s(t)dt + \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{\frac{kT}{2}}^{\frac{kT+\varepsilon}{2}} P_s(t)dt$$

$$= \lim_{T_0 \to \infty} \frac{1}{T_0} \left[\frac{k\mathcal{E}_T\{s(t)\}}{kT} + \frac{\mathcal{E}_{\varepsilon-}}{kT} + \frac{\mathcal{E}_{\varepsilon+}}{kT} \right] = \lim_{T_0 \to \infty} \frac{1}{T_0} \frac{\mathcal{E}_T\{s(t)\}}{T} = \mathcal{P}_T\{s(t)\}$$

where $\varepsilon = (T_0 \bmod T)$, $k = \lfloor T_0 / T \rfloor$ and:

$$\mathcal{E}_{\varepsilon-} = \int_{-\frac{kT-\varepsilon}{2}}^{-\frac{kT}{2}} P_s(t)dt \quad , \quad \mathcal{E}_{\varepsilon+} = \int_{\frac{kT}{2}}^{\frac{kT+\varepsilon}{2}} P_s(t)dt$$

Note that $\mathcal{E}_{\varepsilon-}, \mathcal{E}_{\varepsilon+} \leq \mathcal{E}_T$. This inequality is obvious since both $\mathcal{E}_{\varepsilon-}, \mathcal{E}_{\varepsilon+}$ are energy integrals over an interval of *less than a period* T, so the resulting energy must be lower than or at most equal to $\leq \mathcal{E}_T$. **End of optional material.**

6.4.3 Writing periodic signals

In the following we address three ways to mathematically write down periodic signals. This will be instrumental to finding the Fourier transform of a generic periodic signal.

6.4.3.1 representation 1

Let $s(t)$ be a periodic signal of period T_0. A first intuitive way of formally representing (or “writing”) it is to “cut out” a single period of it and then replicate it at intervals equal to the period, through a summation:

$$s(t) = \sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0)$$

where $s_{T_0}(t)$ is a single full period of the signal, taken anywhere along the real axis.

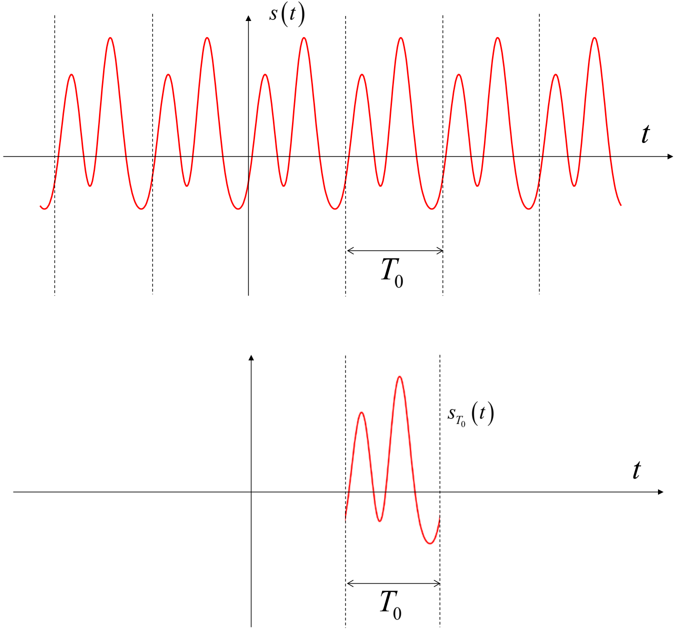


Fig. 6.8: a periodic signal $s(t)$ and a single period $s_{T_0}(t)$

6.4.3.2 representation 2

Let $s(t)$ be a periodic signal of period T_0. Let $s_{T_0}(t)$ be a single full period of it, taken anywhere along the real axis, for instance between $[t_0, t_1]$, with $t_1 - t_0 = T_0$.

Since $s_{T_0}(t)$ is a finite-energy signal, we can write it using its Fourier basis representation over $L^2_{[t_0,t_1]}$:

$$s_{T_0}(t) = \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}}}{\sqrt{T_0}} \quad t \in [t_0, t_1]$$

We now stop restricting time to $t \in [t_0, t_1]$ and instead let time span the whole of $\mathbb{R}$. The resulting signal $s(t)$ is periodic with period T_0 and $s_{T_0}(t)$ is indeed its "cut out" version between $t \in [t_0, t_1]$. Periodicity is easily proved:

$$s(t - T_0) = \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi n}{T_0}[t - T_0]}}{\sqrt{T_0}} = \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}} e^{-j\frac{2\pi n T_0}{T_0}}}{\sqrt{T_0}}$$

$$= \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}} e^{-j2n\pi}}{\sqrt{T_0}} = \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}}}{\sqrt{T_0}} = s(t)$$

In summary, the second way of representing periodic signals is through its Fourier basis representation computed over $s_{T_0}(t)$ and then extended to the whole of $\mathbb{R}$:

$$s(t) = \sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}}}{\sqrt{T_0}}$$

Eq. 6-42

Finally, note that the Fourier components can also be evaluated from Fourier transforms, as pointed out earlier:

$$s_{T_0,n} = \int_{t_0}^{t_1} s_{T_0}(t) \frac{e^{-j\frac{2\pi nt}{T_0}}}{\sqrt{T_0}} dt = \frac{1}{\sqrt{T_0}} \int_{-\infty}^{\infty} s_{T_0}(t) e^{-j\frac{2\pi nt}{T_0}} dt =$$

$$= F\left\{s_{T_0}(t)\right\}\Big|_{f = n/T_0} = \frac{S_{T_0}\left(n/T_0\right)}{\sqrt{T_0}} = \frac{S_{T_0}\left(nf_0\right)}{\sqrt{T_0}} = \sqrt{f_0} \cdot S_{T_0}\left(nf_0\right)$$

So, in fact, this second representation has a variant:

$$s(t) = \sum_{n=-\infty}^{\infty} \frac{S_{T_0}\left({}^{n}\!/\!{}_{T_0} \right)}{\sqrt{T_0}} \frac{e^{j\frac{2\pi n t}{T_0}}}{\sqrt{T_0}} =$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} S_{T_0}\left({}^{n}\!/\!{}_{T_0} \right) e^{j\frac{2\pi n t}{T_0}} = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}\left(n f_0 \right) e^{j 2\pi n f_0 t}$$

Eq. 6-43

6.4.3.3 representation 3

Let $q(t)$ be a generic finite-energy signal. Then, in general, we can write:

$$s(t) = \sum_{n=-\infty}^{\infty} q(t - nT_0)$$

Eq. 6-44

If the series converges for all $t \in \mathbb{R}$, then $w(t)$ is a periodic signal. In fact:

$$s(t-T_0) = \sum_{n=-\infty}^{\infty} q(t-T_0-nT_0) = \sum_{n=-\infty}^{\infty} q(t-[n+1]T_0)$$

Now formally substituting: $m = n+1$ we get:

$$s(t-T_0) = \sum_{m=-\infty}^{\infty} q(t-mT_0)$$

Eq. 6-45

But the right-hand side of Eq. 6-45 is identical to that of Eq. 6-44, so we proved that:

$$s(t) = s(t - T_0) \quad \forall t$$

Note that $q(t)$ can have either limited support or unlimited support. If its support is limited, then $\sum_{n=-\infty}^{\infty} q(t - nT_0)$ always converges. Otherwise, convergence should be checked. However, note that if:

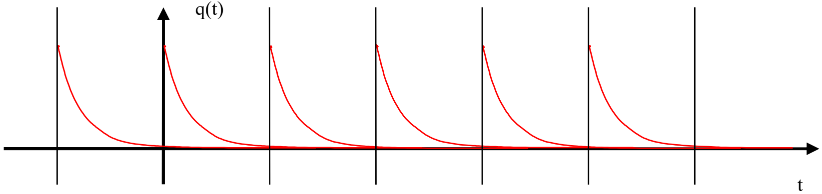
$$\begin{gathered} q(t) = o\left(\frac{1}{|t|}\right) \\ t \to \pm\infty \end{gathered}$$

then the series certainly converges.

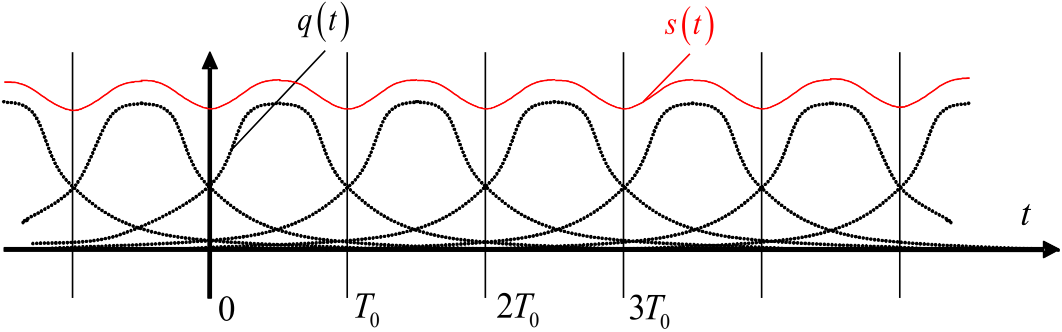
Note also that even though $\sum_{n=-\infty}^{\infty} q(t-nT_0)$ has a similar structure as $\sum_{n=-\infty}^{\infty} s_{T_0}(t-nT_0)$, the signal $q(t)$ is *not* the "cut out" single period of $s(t)$.

Two possible examples of unlimited support functions that can generate a periodic signal are the unilateral decreasing exponential and the Gaussian signals:

$$q(t) = u(t)e^{-at} \quad \rightarrow \quad s(t) = \sum_{n=-\infty}^{\infty} u(t-nT_0)e^{-a[t-nT_0]}$$



$$q(t) = e^{-t^2/2\sigma^2} \quad \rightarrow \quad s(t) = \sum_{n=-\infty}^{\infty} e^{-[t-nT_0]^2/2\sigma^2}$$



6.4.4 The Fourier transform of periodic signals

We will now evaluate the Fourier transform of $s(t)$ starting from the three above-described representations.

6.4.4.1 representation 1

Given the "cut-out" signal $s_{T_0}(t)$ in the full period $[t_0, t_1]$, with $T_0 = t_1 - t_0$,

$$s(t) = \sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0)$$

we can directly compute the Fourier transform:

$$\mathrm{F}\{s(t)\} = \mathrm{F}\left\{\sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0)\right\} = \sum_{n=-\infty}^{\infty} \mathrm{F}\{s_{T_0}(t - nT_0)\} =$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{F}\{s_{T_0}(t)\} e^{-j2\pi f n T_0} = S_{T_0}(f) \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_0} = S_{T_0}(f) \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi n}{f_0} f}$$

Eq. 6-46

We now recall the important result found when discussing the fact that the Fourier ONS is also an ONB (a basis):

$$\sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{-j\frac{2\pi n}{T_0}t} = \sum_{n=-\infty}^{\infty} \delta\left(t - nT_0\right)$$

By formally exchanging $f \rightarrow t$ and $f_0 \rightarrow T_0$ we get:

$$\sum_{n=-\infty}^{\infty} \frac{1}{f_0} e^{-j\frac{2\pi n}{f_0}f} = \sum_{n=-\infty}^{\infty} \delta\left(f - nf_0\right)$$

Eq. 6-47

Using this formula, we can directly find from Eq. 6-46:

$$\mathrm{F}\{s(t)\} = f_0 \; S_{T_0}(f) \sum_{n=-\infty}^{\infty} \delta\left(f - nf_0\right)$$

Note also the obvious result:

$$S_{T_0}(f) \cdot \delta\left(f - nf_0\right) = S_{T_0}(nf_0) \cdot \delta\left(f - nf_0\right)$$

from which we finally get:

$$\mathrm{F}\{s(t)\} = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}(nf_0) \delta\left(f - nf_0\right)$$

Eq. 6-48

which can also be written as:

$$S(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} S_{T_0}\left(\frac{n}{T_0}\right) \delta\left(f - \frac{n}{T_0}\right)$$

Eq. 6-49

6.4.4.2 representation 2

We directly take the Fourier transform:

$$\mathrm{F}\left\{s(t)\right\} = S(f) = \mathrm{F}\left\{\sum_{n=-\infty}^{\infty} s_{T_0,n} \frac{e^{j\frac{2\pi nt}{T_0}}}{\sqrt{T_0}}\right\} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{s_{T,n}}{\sqrt{T_0}} \mathrm{F}\left\{e^{j\frac{2\pi nt}{T_0}}\right\} = \sum_{n=-\infty}^{\infty} \frac{s_{T,n}}{\sqrt{T_0}} \delta\left(f - n/T_0\right)$$

We also remember the "alternative" representation 2, based on the relationship between Fourier series coefficients and Fourier transforms:

$$\mathrm{F}\{s(t)\} = S(f) = \mathrm{F}\left\{\frac{1}{T_0}\sum_{n=-\infty}^{\infty} S_{T_0}\left(nf_0\right)e^{j2\pi nf_0 t}\right\}$$

$$= f_0 \sum_{n=-\infty}^{\infty} S_{T_0}\left(nf_0\right)\mathrm{F}\left\{ e^{j2\pi nf_0 t}\right\} =$$

$$= f_0 \sum_{n=-\infty}^{\infty} S_{T_0}\left(nf_0\right)\delta\left(f - nf_0\right) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} S_{T_0}\left(\frac{n}{T_0}\right)\delta\left(f - \frac{n}{T_0}\right)$$

This result is identical to Eq. 6-49, as it should of course be.

6.4.4.3 representation 3

In this case the periodic signal is represented as:

$$s(t) = \sum_{n=-\infty}^{\infty} q(t - nT_0)$$

Its Fourier transform can again be directly obtained, following the same steps as for representation 1. We have:

$$\mathrm{F}\{s(t)\} = S(f) = \mathrm{F}\left\{\sum_{n=-\infty}^{\infty} q(t - nT_0)\right\} = \sum_{n=-\infty}^{\infty} \mathrm{F}\{q(t - nT_0)\} =$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{F}\{q(t)\} e^{-j2\pi f n T_0} = Q(f) \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_0} = Q(f) \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi n}{f_0} f}$$

Using again:

$$\sum_{n=-\infty}^{\infty} \frac{e^{-j\frac{2\pi n}{f_0}f}}{f_0} = \sum_{n=-\infty}^{\infty} \delta\left(f - nf_0\right)$$

we immediately arrive at:

$$\mathrm{F}\left\{s(t)\right\} = f_0 \sum_{n=-\infty}^{\infty} Q(nf_0)\delta\left(f - nf_0\right)$$

Eq. 6-50

which can also be written as:

$$S(f)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}Q\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right)$$

Eq. 6-51

6.4.4.4 comments on spectra of periodic signals

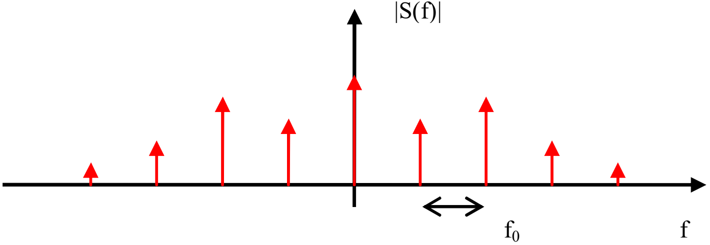
Pulling everything together, we now have various ways to spectrally represent the same periodic signal $s(t)$:

$$S(f)=\sqrt{f_0}\sum_{n=-\infty}^{\infty}s_{T_0,n}\,\delta\left(f-nf_0\right)$$

$$S(f)=f_0\sum_{n=-\infty}^{\infty}S_{T_0}\left(nf_0\right)\delta\left(f-nf_0\right)$$

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} Q(nf_0)\delta\left(f - nf_0\right)$$

First of all we remark that they are collections of deltas (or spectral "tones" or spectral "lines"), occurring at intervals of $f_0 = 1/T_0$. Note that the whole spectrum is made up of lines: there is no part of $S(f)$ which is a "continuous" function of f.



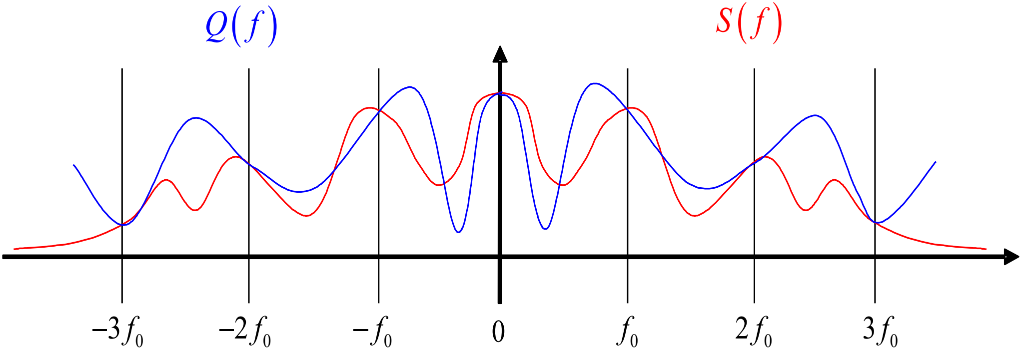
In addition, it is clear that the three different ways of expressing the spectrum must be identical. By equating them and eliminating the common factors, we get the fundamental relationship:

$$\frac{s_{T_0,n}}{\sqrt{f_0}} = S_{T_0}\left(nf_0\right) = Q\left(nf_0\right)$$

The first equality is obvious and we have already found it when dealing with the relationship between Fourier series and Fourier transforms.

The second equality is less obvious and essentially it states than not just one signal $q(t)$ can generate the periodic signal $s(t)$, but any signal $q(t)$ such that $Q(nf_0) = S_{T_0}(nf_0)$.

Note that **it is not required that** $Q(f) = S_{T_0}(f), \forall f$. In fact, for $f \neq nf_0$ the two spectra $Q(f), S_{T_0}(f)$ can be completely different as shown in the plot below.



Example

A very simple example is provided by these three signals. Let:

$$q_1(t) = \Lambda_{T_0}(t)\exp(j2\pi f_0 t) \qquad q_2(t) = \Pi_{T_0}(t)\exp(j2\pi f_0 t)$$
$$q_3(t) = \rho_{T_0,1}(t)\exp(j2\pi f_0 t)$$

Their respective transforms are:

$$Q_1(f) = T_0 \operatorname{Sinc}([f - f_0]T_0)$$
$$Q_2(f) = T_0 \operatorname{Sinc}^2([f - f_0]T_0)$$
$$Q_3(f) = T_0 \operatorname{Sinc}([f - f_0]T_0)\frac{\cos(\pi[f - f_0]T_0)}{1-(2[f - f_0]T_0)^2}$$

It can be easily seen that the same periodic signal is obtained as:

$$s(t) = \sum_{n=-\infty}^{\infty} q_1(t - nT_0) = \sum_{n=-\infty}^{\infty} q_2(t - nT_0) = \sum_{n=-\infty}^{\infty} q_3(t - nT_0)$$

$\boxed{\text{On your own,}}$ do this easy check and also draw $S(f)$.

$\boxed{\text{On your own,}}$ assume now that:

$$q_1(t) = \Lambda_{T_0}(t) \qquad q_2(t) = \Pi_{T_0}(t) \qquad q_3(t) = \rho_{T_0,1}(t)$$

and check again that:

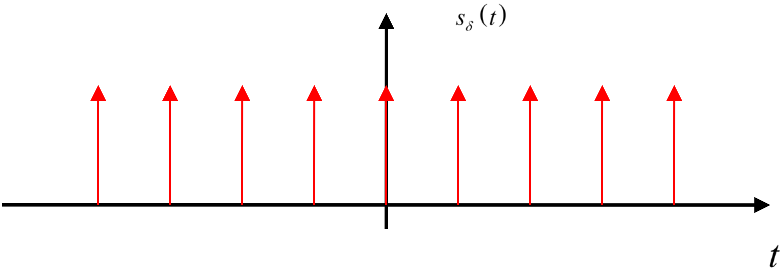
$$s(t) = \sum_{n=-\infty}^{\infty} q_1(t - nT_0) = \sum_{n=-\infty}^{\infty} q_2(t - nT_0) = \sum_{n=-\infty}^{\infty} q_3(t - nT_0)$$

is periodic. Do it also graphically in time-domain. What *special kind* of periodic signal is $s(t)$?

6.4.4.5 Fourier transform of the "train of deltas"

We now look at a very special periodic signal:

$$s_\delta(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - nT_0\right)$$



We already know that it can be written as:

$$s_\delta(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - nT_0\right) = \sum_{n=-\infty}^{\infty} \frac{e^{-j\frac{2\pi n}{T_0}t}}{T_0}$$

Its Fourier transform can then be calculated directly:

$$F\left\{s_\delta(t)\right\} = F\left\{\sum_{n=-\infty}^{\infty} \delta\left(t - nT_0\right)\right\} = F\left\{\sum_{n=-\infty}^{\infty} \frac{e^{-j\frac{2\pi n}{T_0}t}}{T_0}\right\} =$$

$$\sum_{n=-\infty}^{\infty} F\left\{\frac{e^{-j\frac{2\pi n}{T_0}t}}{T_0}\right\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) = f_0 \sum_{n=-\infty}^{\infty} \delta\left(f - nf_0\right)$$

So, in the end, we have the quite important result:

$$F\left\{\sum_{n=-\infty}^{\infty}\delta\left(t-nT_0\right)\right\}=f_0\sum_{n=-\infty}^{\infty}\delta\left(f-nf_0\right)$$

Eq. 6-52

That is, the Fourier transform of a "train of deltas" is again a "train of deltas". This result is of extreme importance in dealing with the theory of sampled and digital signals.

6.4.5 Optional: Periodicity of a Line-Spectrum Signal

We have seen that periodic signal necessarily have a spectrum that is exclusively made up of spectral “lines”, that is of Dirac’s deltas.

The question then arises whether the converse is true, that is whether a signal $w(t)$ whose spectrum is exclusively made up of Dirac’s deltas is necessarily periodic. Such a signal could be written as:

$$W(f) = \sum_{n=1}^{N} w_n \cdot \delta(f - f_n)$$

where the f_n’s are the center frequencies of the deltas and the w_n’s are the constants multiplying each delta.

The interesting result is that, in general, $w(t)$ is *non-periodic*. For $w(t)$ to be periodic, the set of the deltas’ center frequencies $\{f_n\}_{n=1}^{N}$ in $W(f)$ must be *commensurable* with one another. In other words, there must be a number $\alpha \in \mathbb{R}^+$ such that:

$$f_1 = k_1\alpha, \quad f_2 = k_2\alpha, ..., f_N = k_N\alpha$$

Eq. 6-53

where $k_1, k_2, ..., k_N$ are all integers. If so, then the period of the periodic signal is $T = 1/f_0$ where $f_0 = K\alpha$ and K is the Greatest Common Factor (or GCF) of the integers $k_1, k_2, ..., k_N$. Note that GCF has also various other names, including Greatest Common Divisor, or GCD, and others. Note also that α may be non-unique, but this does not create any problems. Any choice of α yields the same result regarding $T = 1/f_0$.

Examples

Let $W(f)$ be the following:

$$W(f) = w_{25} \cdot \delta(f-25) + w_{35} \cdot \delta(f-35) + w_{70} \cdot \delta(f-70)$$

which means that $\{f_n\} = \{25, 35, 70\}$. The signal is clearly periodic, because if we choose for instance $\alpha = 1$ then clearly each frequency is an integer multiple of α. To find out what the actual period is, it is then necessary to find the GCF of the integer coefficients $\{k_n\} = \{25, 35, 70\}$. We have:

$$\mathrm{GCF}\{25, 35, 70\} = 5$$

The GCF is $K = 5$ and therefore $T = 1/(K\alpha) = 1/5$. Note that we could have chosen a different α and the result would not change. For instance, we could have chosen $\alpha = 0.5$. Then $\{k_n\} = \{50, 70, 140\}$. Clearly $\text{GCF} = \{50, 70, 140\} = 10$ and $T = 1/(K\alpha) = 1/(10 \cdot 0.5) = 1/5$. In other words, any α such that Eq. 6-53 holds, is fine.

Let now:

$$p(t) = p_1 e^{j2\pi f_1 t} + p_2 e^{j2\pi f_2 t}$$

where $\{f_n\}_{n=1}^{2} = \{1, \pi\}$. $P(f)$ is of course the following:

$$P(f) = p_1 \cdot \delta(f - 1) + p_2 \cdot \delta(f - \pi)$$

Therefore, $P(f)$ is a delta-only spectrum and a possible candidate for being a periodic signal in time. However, as number theory states, the numbers 1 and π are *incommensurable*. In other words there is no real number $\alpha \in \mathbb{R}^{+}$ such $1 = f_1 = k_1\alpha, \quad \pi = f_2 = k_2\alpha$, with k_1, k_2 integers.

Therefore, $P(f)$ is non-periodic, even though it is made up of two exponentials that, by themselves, are periodic of period 1 and period $1/\pi$.

(End of optional part.)

6.5 *A few problems*

On your own, try to solve the following problems.

6.5.1.1

Given the signal $s(t) \in L^2_{\mathbb{R}}$, whose energy is $\mathcal{E}\{s(t)\} = \mathcal{E}_0$, find the energy of the signal $\mathcal{E}\{s(\alpha[t - t_d])\}$.

Answer:

$$\mathcal{E}\{s(\alpha[t - t_d])\} = \frac{\mathcal{E}_0}{|\alpha|}$$

6.5.1.2

Prove that, given two signals $s(t), w(t) \in L^2_{\mathbb{R}}$, then:

$$\left(s(t), w(t)\right) = \left(S(f), W(f)\right)$$

Hint: write down the definition of inner product in time and then try to express $s(t)$ and $w(t)$ through their Fourier transforms.

6.5.1.3

Consider the signal $s(t) \in \mathbb{R}$ and its Fourier transform $S(f)$ Then answer/do the following.

1. Prove that $S(f) = S^*(-f)$
2. What symmetry does the quantity $|S(f)|$ have? Prove your statement. Provide an example.
3. If $s(t)$ is even, what can be said about $S(f)$ being even or odd, real or complex? Prove your statement. Provide an example.
4. If $s(t)$ is odd, what can be said about $S(f)$ being even or odd, real or complex? Prove your statement. Provide an example.

6.5.1.4

Given the signals:

$$s(t) = \Lambda_{T_0}(t+T_0) - \Lambda_{T_0}(t-T_0)$$

$$w(t) = \Lambda_{T_0}(t+T_0) + \Lambda_{T_0}(t-T_0)$$

answer the following questions.

- Are they odd, even, or neither?
- find their Fourier transform $S(f)$ and $W(f)$
- discuss the symmetry properties of $S(f)$ and $W(f)$

Answer:

Regarding $s(t)$, the signal is odd versus time.

Its Fourier transform is $S(f) = 2jT_0 \mathrm{Sinc}^2(T_0 f)\sin(2\pi T_0 f)$.

$S(f)$ is purely imaginary and odd. This could be expected since $s(t)$ is real and odd, given the symmetry properties of the Fourier transform.

Regarding $w(t)$, the signal is even versus time.

Its Fourier transform is $W(f) = 2T_0 \text{Sinc}^2(T_0 f)\cos(2\pi T_0 f)$.

$W(f)$ is purely real and even. This could be expected since $w(t)$ is real and even, given the symmetry properties of the Fourier transform.

6.5.1.5

Given the signal:

$$q(t) = e^{-at}u(t) \qquad a > 0$$

find $\mathcal{E}\{q(t)\}$ and $\mathcal{E}\{Q(f)\}$. Carry out the calculations explicitly both in time and in frequency.

Answer:

The energy is the same for both the time and frequency versions of the signal:

$$\mathcal{E}\{q(t)\} = \mathcal{E}\{Q(f)\} = 1/(2a)$$

To find $\mathcal{E}\{Q(f)\}$ it is convenient to use the same procedure as used in Problem 6.2.9.1.

6.5.1.6

Given the signals:

$$s(t) = \exp(-at)\mathrm{u}(t) \qquad a > 0$$

and

$$w(t) = s(t_0 - \beta t) + s(\beta t - t_0) \qquad \beta \in \mathbb{R}, \beta \neq 0$$

find $W(f) = \mathrm{F}\{w(t)\}$.

Solution:

We first put the expression of the signal into the "canonical form" indicated in Section 6.2.10 :

$$w(t) = s\left(-\beta\left[t - t_0/\beta\right]\right) + s\left(\beta\left[t - t_0/\beta\right]\right)$$

We can then use the general formula indicated in 6.2.10 :

$$\mathrm{F}\left\{s\left(a\left[t - t_d\right]\right)\right\} = \frac{1}{|a|} S\left(\frac{f}{a}\right) e^{-j2\pi f t_d}$$

Simply by comparison we can the write:

$$\mathrm{F}\left\{w(t)\right\} = \mathrm{F}\left\{s\left(-\beta\left[t - t_0/\beta\right]\right)\right\} + \mathrm{F}\left\{s\left(\beta\left[t - t_0/\beta\right]\right)\right\}$$

$$= \frac{1}{|\beta|} S\left(-\frac{f}{\beta}\right) e^{-j2\pi f t_0/\beta} + \frac{1}{|\beta|} S\left(\frac{f}{\beta}\right) e^{-j2\pi f t_0/\beta}$$

We know that $S(f) = \dfrac{1}{a + j2\pi f}$ so in the end:

$$W(f) = \mathrm{F}\{w(t)\} = \mathrm{F}\{s(-\beta[t - t_0/\beta])\} + \mathrm{F}\{s(\beta[t - t_0/\beta])\}$$

$$= \frac{1}{|\beta|} S\left(-\frac{f}{\beta}\right) e^{-j2\pi f t_0/\beta} + \frac{1}{|\beta|} S\left(\frac{f}{\beta}\right) e^{-j2\pi f t_0/\beta}$$

$$= \frac{1}{|\beta|} e^{-j2\pi f t_0/\beta} \left(\frac{1}{a + j2\pi f/\beta} + \frac{1}{a - j2\pi f/\beta} \right)$$

$$= \frac{1}{|\beta|} e^{-j2\pi f t_0/\beta} \frac{2a}{a^2 + 4\pi^2 f^2 / \beta^2}$$

So, in conclusion:

$$W(f) = \frac{1}{|\beta|} e^{-j2\pi f t_0/\beta} \frac{2a}{a^2 + 4\pi^2 f^2 / \beta^2}$$

6.5.1.7

Given the signal:

$$s(t) = t\,e^{-at}u(t) \qquad a > 0$$

and the signal:

$$w(t) = \frac{d}{dt}\left[t\,e^{-at}u(t)\right] \qquad a > 0$$

verify by explicit calculation that $s(t) \perp w(t)$.

Hints:

- remember that $w(t) = \frac{d}{dt}u(t) = \delta(t)$ and that $t \cdot \delta(t) = 0(t)$
- carry out the inner product in frequency domain

6.5.1.8

Consider the signal: $q(t) = e^{-t^2/\left(2\sigma^2\right)}$.

Consider then the periodic signals:

$$s(t) = \sum_{n=-\infty}^{\infty} q(t - nT_0)$$

$$w(t) = \sum_{n=-\infty}^{\infty} (-1)^n q(t - nT_0)$$

1. What is the period of $s(t)$? What is period of $w(t)$?
2. Draw (qualitatively) $s(t)$ and $w(t)$ vs. time over the interval $[-3T_0, 3T_0]$, assuming that $T_0 \gg \sigma$.
3. Calculate the Fourier transform of $s(t)$ and $w(t)$.
4. Draw the two Fourier transforms (qualitatively) vs. frequency, over the interval $[-3.5/T_0\,, 3.5/T_0]$
5. What are the main differences between the two transforms?

6. Verify the required symmetries of the Fourier transform, based on the features of $s(t)$ and $w(t)$.

(Note: ask yourself: is $s(t)$ real? Does it have any symmetry?)

Answer:

$$S(f) = f_0\sqrt{2\pi\sigma^2}\sum_{n=-\infty}^{\infty} e^{-2\pi^2\sigma^2 n^2 f_0^2} \cdot \delta(f - nf_0)$$

$$f_0 = 1/T_0$$

The signal is real and even, so its Fourier transform must be real and even. The result complies with this requirement.

6.5.1.9

Given the signal $q(t) = e^{-at}u(t) \quad a > 0$, build the corresponding periodic signal $s(t)$ of period $T_0 = 1/f_0$ using the representation (3). Draw it (approximately). Then calculate and draw its Fourier transform. How many plots do you need to draw the transform? Verify the required symmetries of the Fourier transform, based on the features of $s(t)$.

(Note: ask yourself: is $s(t)$ real? Does it have any symmetry?)

Answer:

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{1}{a + j2\pi n f_0} \cdot \delta(f - nf_0)$$

To draw the above Fourier transform, two plots are needed: one for the real part:

$$S_{\mathrm{Re}}(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{a}{a^2 + 4\pi^2 n^2 f_0^2} \cdot \delta(f - nf_0)$$

and one for the imaginary part:

$$S_{\mathrm{Im}}(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{-2\pi n f_0}{a^2 + 4\pi^2 n^2 f_0^2} \cdot \delta(f - nf_0)$$

The signal is real, so the real part of its Fourier transform must be even, while the imaginary part must be odd. The result complies with this requirement.

6.5.1.10

Given the signal $q(t) = e^{-at} u(t) \cdot e^{j 2\pi f_1 t} \qquad a > 0$, build the corresponding periodic signal $s(t)$ of period $T_0 = 1/f_0$ using the representation (3). Assume $f_1 = 20 \cdot f_0$.

Draw it approximately. How many plots are necessary?

Then calculate and draw its Fourier transform. How many plots do you need to draw the transform? Are there symmetries in the Fourier transform, based on the features of $s(t)$?

Answer:

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{1}{a + j2\pi(n-20) f_0} \cdot \delta(f - nf_0)$$

To draw the above Fourier transform, two plots are needed: one for the real part:

$$S_{\mathrm{Re}}(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{a}{a^2 + 4\pi^2 (n-20)^2 f_0^2} \cdot \delta(f - nf_0)$$

and one for the imaginary part:

$$S_{\mathrm{Im}}(f) = f_0 \sum_{n=-\infty}^{\infty} \frac{-2\pi(n-20) f_0}{a^2 + 4\pi^2 (n-20)^2 f_0^2} \cdot \delta(f - nf_0)$$

The signal is complex, so the Fourier transform has no symmetry.

6.5.1.11

Given the signal $s_{T_0}(t) = -\Pi_T\left(t+\frac{T}{2}\right)+\Pi_T\left(t-\frac{T}{2}\right)$, build the corresponding periodic signal $s(t)$ of period $T_0 = 2T$ using the representation (3). Draw it. Then calculate and draw its Fourier transform. How many plots do you need to draw the transform? Verify the required symmetries of the Fourier transform, based on the features of $s(t)$.

(Note: ask yourself: is $s(t)$ real? Does it have any symmetry?).

Do you know what type of signal is this in electrical engineering? Are all the deltas of the spectrum present?

Answer:

$$S(f) = -j\sum_{n=-\infty}^{\infty} \mathrm{Sinc}\left(\frac{n}{2}\right)\sin\left(n\frac{\pi}{2}\right)\delta(f - nf_0)$$

$$f_0 = 1/T_0$$

The result can be further manipulated to yield:

$$S(f) = -\frac{j}{\pi}\sum_{k=-\infty}^{\infty} \frac{1}{k+1/2}\delta\left(f - [2k+1]f_0\right)$$

6.5.1.12

Given the signal $s_{T_0}(t) = \Lambda_{T_0/2}(t)$, build the corresponding periodic signal $s(t)$ of period $T_0 = 1/f_0$ using the representation (2), that is using a Fourier series, and the representation (1), that is the truncated signal. Draw the signal. Then calculate and draw its Fourier transform. How many plots do you need to draw the transform? Verify the required symmetries of the Fourier transform, based on the features of $s(t)$.

(Note: ask yourself: is $s(t)$ real? Does it have any symmetry?).

Answer:

The representation in terms of Fourier series is:

$$s(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Sinc}^2\left(\frac{n}{2}\right) e^{j2\pi n f_0 t}$$

The representation using the truncated signals is:

$$s(t) = \sum_{n=-\infty}^{\infty} \Lambda_{T_0/2}\left(t - nT_0\right)$$

The Fourier transform is:

$$S(f) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Sinc}^2\left(\frac{n}{2}\right) \cdot \delta\left(f - nf_0\right)$$

Only one plot is needed to draw $S(f)$ because its imaginary part is zero. This is because the periodic signal is real and even.

6.5.1.13

Given the signal $q(t)=\Lambda_{T_0}(t)$, build the corresponding periodic signal $s(t)$ of period $T_0=1/f_0$ using the representation (3).

Then calculate and draw its Fourier transform. Do you notice anything strange? Do you know why the result looks like it does?

(Take the inverse Fourier transform of $S(f)$ and discuss the result).

Answer:

Since $q(t)=\Lambda_{T_0}(t)$, the signal can be written as:

$$s(t)=\sum_{n=-\infty}^{\infty} q(t-nT_0)=\sum_{n=-\infty}^{\infty} \Lambda_{T_0}(t-nT_0)$$

The Fourier transform, calculated according to the rules of periodic signals, comes out as:

$$S(f) = \sum_{n=-\infty}^{\infty} \mathrm{Sinc}^2(n) \cdot \delta(f - nf_0)$$

However, since $\mathrm{Sinc}(n) = 0, \forall n \neq 0,$ in fact the Fourier transform reduces to:

$$S(f) = \delta(f).$$

The result is like this because $s(t)$ is in fact just $1(t)$. To prove it, draw the signal in time domain.

6.5.1.14

Consider the signal:

$$x(t) = |\cos(2\pi f_0 t)| \qquad t \in \mathbb{R}$$

Do the following:

1. plot $x(t)$ over the interval $t \in \left[-\frac{2}{f_0}, \frac{2}{f_0} \right]$
2. is $x(t)$ a periodic signal ?
3. evaluate the Fourier transform of $x(t)$

6.5.1.15

Given the signal $w(t) = \Pi_T\left(t - \frac{T}{2}\right)$, consider the signal:

$$s(t)=\sum_{n=-\infty}^{\infty} w(t-nT)\cdot(-1)^n$$

Is $s(t)$ periodic? If so, what is its period ? What is a possible suitable truncated signal $s_{T_0}(t)$? Calculate the Fourier Transform of $s(t)$ and draw it.

6.5.1.16 Optional

A signal has the following spectrum:

$$\begin{aligned} W(f) = {} & 3\delta(f-210)+j\delta(f-21)+\delta(f-35)+(1+j)\delta(f-28)+ \\ & 3\delta(f+210)-j\delta(f+21)+\delta(f+35)+(1-j)\delta(f+28) \end{aligned}$$

Is the corresponding signal $w(t)$ periodic? What is its period? Is the signal $w(t)$ real?

Answer:

Yes, it is periodic. The period is $T_0 = 1/7$, that is $f_0 = 7$. This is because 7 is the MCD of $\{210, 21, 35, 28\}$.

The resulting spectral lines are located at $3f_0$, $4f_0$, $5f_0$, and $30f_0$.

The signal is real because the $W(f)$ has even real part and odd imaginary part.

6.6 *Optional: appendix*

We assume we have a generic orthonormal set $\{u(\rho,t)\}_\rho$ where $\rho \in \mathbb{R}$ is a parameter indexing the signals of the set. We assume that $(u(\rho_1,t),u(\rho_2,t)) = \delta(\rho_1 - \rho_2)$, that is, $\{u(\rho,t)\}_\rho$ is an orthonormal set.

We want to first find out under what specific conditions an orthonormal set is a basis for $L^2_{[-\infty,\infty]}$.

To do so, we assume as *hypothesis* that any $s(t) \in L^2_{[-\infty,\infty]}$ can be fully reconstructed using a "linear combination" of the elements of $\{u(\rho,t)\}_\rho$ as follows:

$$s(t) = \int_{-\infty}^{\infty} S(\rho) u^*(\rho,t) d\rho$$

Eq. 6-54

where $S(\rho)$ is the projection of $s(t)$ over each signal of $\{u(\rho,t)\}_\rho$, that is:

$$S(\rho) = \left(s(t), u(\rho,t)\right)$$

Eq. 6-55

Note that we also have to assume that $S(\rho)=\left(s(t),u(\rho,t)\right)$ converges for all values of ρ, otherwise, Eq. 6-54 would lose meaning. Then, we find out what constraints the hypothesis that Eq. 6-54 is verified, poses on $\left\{u(\rho,t)\right\}_{\rho}$.

Note in passing that if we set: $u(\rho,t)=e^{j2\pi f t}\Big|_{f=\rho}$ we would get from Eq. 6-54 the inverse Fourier transform and from Eq. 6-55 the Fourier transform. However, we want to keep this proof general so we will keep referring to a generic orthonormal set $\left\{u(\rho,t)\right\}_{\rho}$. We now combine Eq. 6-54 and Eq. 6-55:

$$
\begin{aligned}
s(t) &= \int_{-\infty}^{\infty} S(\rho)u(\rho,t)\,d\rho \\
&= \int_{-\infty}^{\infty} \left(s(\tau),u(\rho,\tau)\right)u(\rho,t)\,d\rho \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\tau)u^{*}(\rho,\tau)\,d\tau\, u(\rho,t)\,d\rho
\end{aligned}
$$

Since integrals are *linear operators*, it is possible to swap the integration order (with certain cautions that will not be addressed here) and get:

$$s(t) = \int_{-\infty}^{\infty} s(\tau) \int_{-\infty}^{\infty} u^*(\rho,\tau) u(\rho,t) d\rho \, d\tau$$

Comparing this result with this other well-known result:

$$s(t) = \int_{-\infty}^{\infty} s(\tau) \delta(t-\tau) d\tau$$

we then immediately get the *relationship that an orthonormal set must comply with to be a basis*:

$$\int_{-\infty}^{\infty} u^*(\rho,\tau) u(\rho,t) d\rho = \delta(t-\tau)$$

This general formula can now be used to check whether the orthogonal set $\left\{e^{j2\pi f t}\right\}_f$ is a basis for $L^2_{[-\infty,\infty]}$. In fact:

$$\int_{-\infty}^{\infty} e^{-j2\pi f \tau}\, e^{j2\pi f t}\, df = \int_{-\infty}^{\infty} e^{j2\pi f (t-\tau)}\, df = \delta(t-\tau)$$

This shows that the continuous-frequency Fourier set is a basis for $L^2_{[-\infty,\infty]}$. It can also be viewed as a proof of the *Fourier inversion theorem* of Sect. 6.1.4 for signals $L^2_{[-\infty,\infty]}$. In fact, the Fourier inversion theorem and the fact that the Fourier orthogonal set $\left\{e^{j2\pi f t}\right\}_f$ is a basis, are essentially equivalent in $L^2_{[-\infty,\infty]}$. The theorem is however slightly more general because it does also apply to certain non-finite-energy signals, which are not in $L^2_{[-\infty,\infty]}$. One noteworthy example is given by periodic signals, whose support is the whole of $\mathbb{R}$ and that are *not* finite-energy signals, so they are outside of $L^2_{[-\infty,\infty]}$. Nonetheless, the inversion theorem works for them as well.

6.7 Matlab program

This short Matlab program calculates the Fourier transforms of some of the signals used in this chapter.

An extended version is available in the "material" section of the course web-page.

```
%% Fourier Transforms
clear all;
close all;
% setting the time interval for displaying the time-domain signal
t=[-10:0.02:10];
% setting the frequency points where the Fourier transform is calculated
f=[-10:0.02:10];
% counting how many there are
```

```
Nt=numel(t);
Nf=numel(f);
% string initialization for print out of calculation progress
strPerc = 0; strMsg = sprintf('\n percentage executed %d/%d',strPerc);
%
% signal definition
% T0=1; s= @(t) HPi(T0,t);
% T0=1; s= @(t) HLambda(T0,t);
% a=1; s= @(t) u(t).*exp(-a*t);
% a=1; s= @(t) u(-t).*exp(a*t);
% a=1; s= @(t) u(t).*t.*exp(-a*t);
% a=1; n=2; s= @(t) u(t).*(t.^n).*exp(-a*t);
% alpha=10; a=1; n=2; s= @(t) u(alpha*t).*(alpha*t.^n).*exp(-a*alpha*t);
% a=1; s= @(t) exp(-a*abs(t));
% T0=1;alpha=0.2; s= @(t) rho(t,alpha,T0,0,0);
% alpha=-1;a=1; s= @(t) u(alpha*t).*(alpha*t).*exp(-a*alpha*t);
% T0=1;td=0; s= @(t) HPi(T0,t-td);
% T0=1;td=0; s= @(t) t.*HPi(T0,t-td);
% T0=1; s= @(t) t.*HLambda(T0,t);
% a=1;s= @(t) 1./(a-1i*2*pi*t);

%
% calculation loop
for n=1:Nf;
    %
    % actual calculation takes place here
    S(n)=integral(@(t) s(t) .* exp(-1i*2*pi*f(n)*t) , -10, 10);
    %
    % displaying progress
    strErase = sprintf(repmat('\b',1,numel(strMsg)));
    strMsg = sprintf('\n percentage executed %i',round(n/Nf*100));
    fprintf([strErase strMsg]);
    %
```

```
end
%
disp(' ');
%
figure;plot(t,real(s(t)),t,imag(s(t)),'linewidth',2);
grid on;xlabel('time [s]');legend('real part', 'imaginary part');
pause;
%
figure;plot(f,real(S),f,imag(S),'linewidth',2);
grid on;xlabel('frequency [Hz]');legend('real part', 'imaginary part');
pause;
%
figure;plot(f,abs(S),'k','linewidth',2);
grid on;xlabel('frequency [Hz]');legend('absolute value');
```

Chapter 7. Asymptotic Behavior of Fourier Transforms

7.1 Differentiability Classes

A signal $s(t)$ belongs to the "differentiability class" C^n if its n-th derivative is everywhere continuous *and* its $(n+1)$-th derivative is not everywhere continuous.

7.1.1.1 example

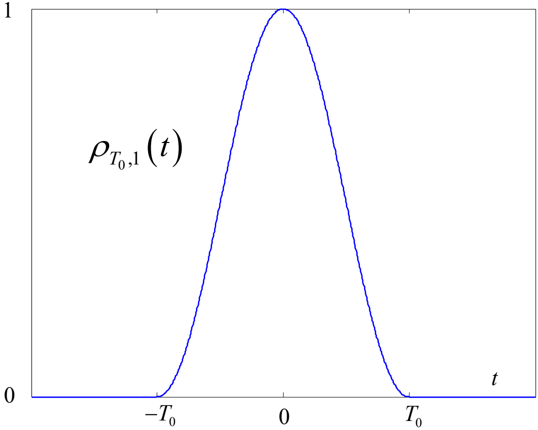


Fig. 7.1 – Raised cosine

The graph of Fig. 7.1 represents a raised-cosine signal with roll-off parameter $\alpha = 1$. It is described by the following formula:

$$\rho_{T_0,1}(t) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi t}{T_0}\right)\right]\Pi_{2T_0}(t)$$

Eq. 7-1

with:

$$t \in [-\infty, \infty]$$

It is immediately visible that $\rho_{T_0,1}(t)$ is **continuous.** It can also be easily checked that it is **differentiable once**.

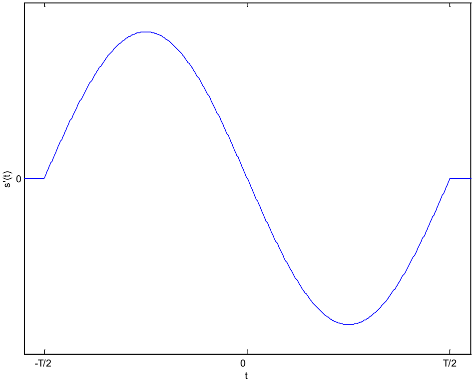


Fig. 7.2 – $\rho_{T_0,1}(t)$ first derivative

In fact, the graph of Fig. 7.2 represents $\frac{d}{dt}\rho_{T_0,1}(t)$, which is still **continuous**.

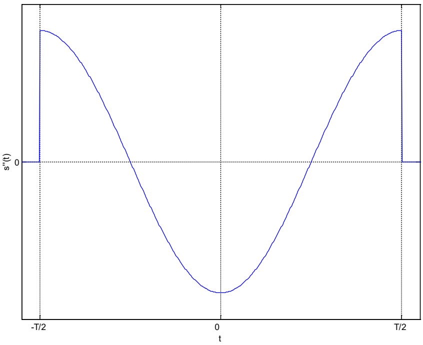


Fig. 7.3 – $\rho_{T_0,1}(t)$ **second derivative**

The graph of Fig. 7.3 shows that $\frac{d^2}{dt^2}\rho_{T_0,1}(t)$ is **discontinuous** at $-T_0$ and T_0. Therefore $\rho_{T_0,1}(t)$ is *not* differentiable twice. In conclusion:

$$\rho_{T_0,1}(t) \in C^1$$

$\boxed{\text{On your own}}$:

1) Compute analytically the first derivative of $\rho_{T_0,1}(t)$ and check that it is still continuous. Remember that:

$$\frac{d}{dt}\Pi_{T_0}(t) = \delta(t + T_0/2) - \delta(t - T_0/2)$$
$$t \cdot \delta(t) = 0$$

2) Show that the differentiability class of all raised cosine signals $\rho_{T_0,\alpha}(t)$ is C^1, for any value of $\alpha \in [0,1]$.

7.1.1.2 example

We now consider the signal:

$$s(t) = te^{-t}u(t)$$

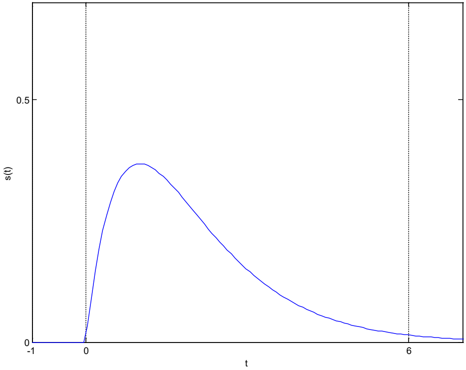


Fig. 7.4 - $s(t) = te^{-t}u(t)$

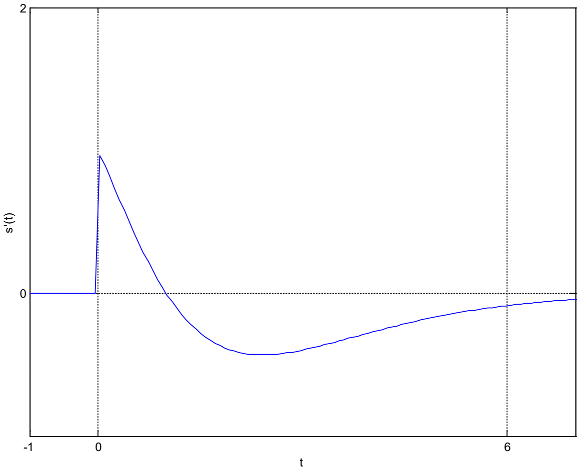


Fig. 7.5 – $s(t)$ **first derivative**

As shown in Fig. 7.5 , $\frac{d}{dt}s(t)$ is **discontinuous**, so:

$$s(t) \in C^0$$

$\boxed{\text{On your own}}$:

Compute analytically the first derivative of $s(t)$ and check that it is discontinuous. Remember that:

$$\frac{d}{dt}u(t) = \delta(t) \qquad t \cdot \delta(t) = 0 \;.$$

7.1.1.3 example

The next signal we look at is:

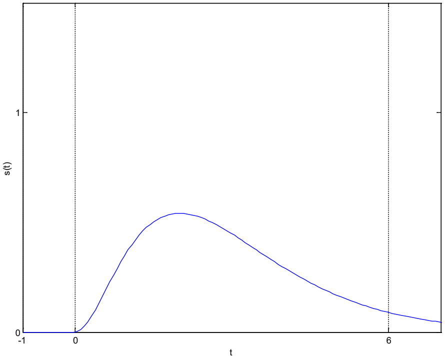


Fig. 7.6 - $s(t) = t^2 e^{-t} u(t)$

It can be easily shown that this signal belongs to C^1.
Moreover, generalizing:

$$t^n e^{-t} u(t) \in C^{n-1}$$

Eq. 7-2

Optional, prove it $\boxed{\text{on your own}}$.

7.1.1.4 example

We now consider:

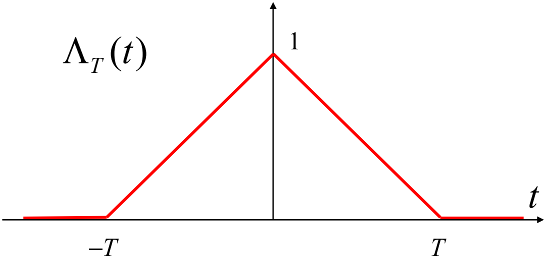


Fig. 7.7 - $s(t) = \Lambda_T(t)$

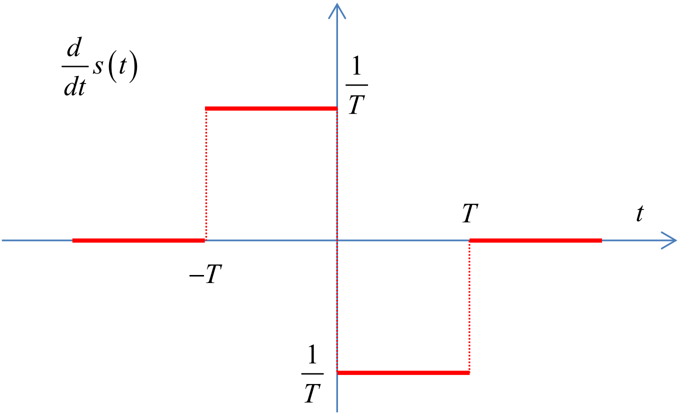


Fig. 7.8 – $s(t)$ first derivative

As shown in Fig. 7.8 $\frac{d}{dt}s(t)$ is **discontinuous**, so: $s(t) \in C^0$

7.1.2 Negative differentiability classes

The differentiability class definition based on ordinary derivatives $\frac{d^n}{dt^n}, \quad n \geq 0,$ manages to classify all signals that are at least continuous. It cannot classify signals that are *discontinuous*.

However, derivatives can be easily generalized to include negative indices, that is $n < 0$. For instance, considering $n = -1$, we could define:

$$\frac{d^{-1}}{dt^{-1}} s(t) \triangleq \int_{-\infty}^{t} s(\theta)\, d\theta$$

Eq. 7-3

Can we get back the original signal if we differentiate the above result? Indeed:

$$\frac{d}{dt}\int_{-\infty}^{t} s(\theta)\, d\theta = \frac{d}{dt}\left[s_p(\theta)\right]_{-\infty}^{t} = \frac{d}{dt}\left[s_p(t) - s_p(-\infty)\right] = \frac{d}{dt}s_p(t) - \frac{d}{dt}s_p(-\infty) = s(t)$$

where $s_p(t)$ is any primitive of $s(t)$.

7.1.2.1 example

We now concentrate on the signal:

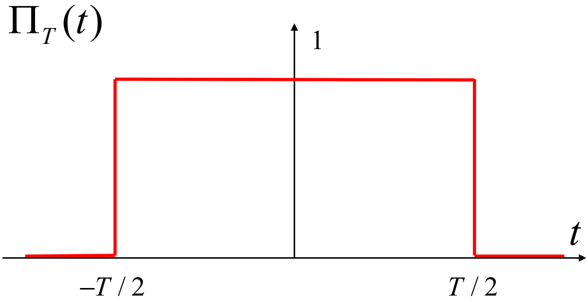


Fig. 7.9: $s(t) = \Pi_T(t)$

function

This signal is **discontinuous,** it is not even of class C^0.

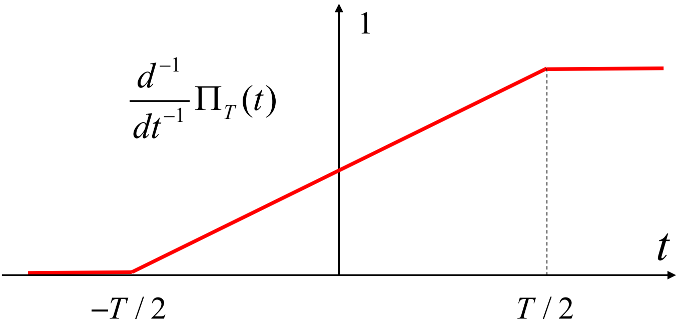


Fig. 7.10: $\frac{d^{-1}}{dt^{-1}}s(t)$

We notice from the graph of Fig. 7.10 that $\frac{d^{-1}}{dt^{-1}}s(t)$ turns out to be **continuous**.

So, we can conclude that: $\Pi_T(t) \in C^{-1}$

7.1.2.2 example 6

Let's take another function that is not only discontinuous, but actually has a singularity at $t = 0$:

$$s(t) = \delta(t)$$

Eq. 7-6

We take one step at a time and begin with the first "negative" derivative. We get:

$$\frac{d^{-1}}{dt^{-1}} s(t) = \int_{-\infty}^{t} \delta(\theta) d\theta = \mathrm{u}(t)$$

The unilateral step function is still discontinuous, so $\delta(t) \notin C^{-1}$. We try then by taking another negative derivative:

$$\frac{d^{-2}}{dt^{-2}} s(t) = \int_{-\infty}^{t} \mathrm{u}(\theta) d\theta = \int_{0}^{t} 1(\theta) d\theta = t \cdot \mathrm{u}(t)$$

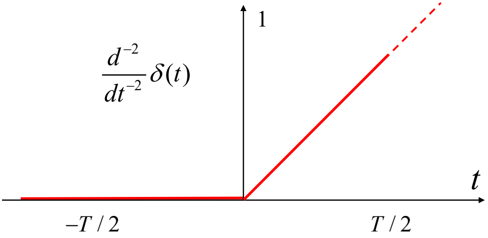


Fig. 7.11 - $\frac{d^{-2}}{dt^{-2}}s(t)$

From the graph of Fig. 7.11, $\frac{d^{-2}}{dt^{-2}}s(t)$ turns out to be **continuous**. As a result, we can say that: $\delta(t) \in C^{-2}$

7.2 Asymptotic behaviour of Fourier transforms

Given a signal $s(t) \in C^n$, its Fourier transform (if it exists) has an asymptotic behaviour for $f \to \pm\infty$ of the kind:

$$S(f) = O\left(\frac{1}{|f|^{n+2}}\right)$$

Eq. 7-7

Conversely, if a signal $s(t)$ has a Fourier transform that is

$$S(f) = O\left(\frac{1}{|f|^{n+2}}\right)$$

then $s(t)$ *belongs to the differentiability class* C^n .

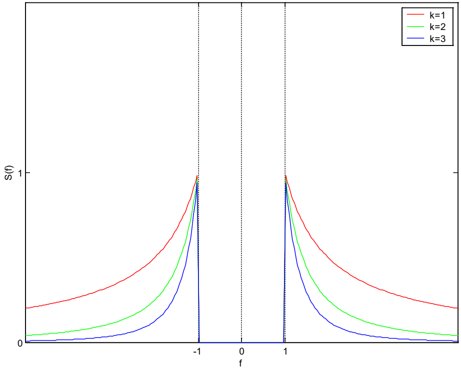


Fig. 7.12 – Plot of $\dfrac{1}{|f|^m}$ **for** $m = 1, 2, 3$**, shown for** $|f| > 1$

So, we can say that from a certain frequency on, the Fourier transform will remain comprised within $\pm\frac{A}{|f|^{n+2}}$.

The table below shows the trend assumed by specific functions, according to the Theorem.

$s(t)$	$S(f)$	**Trend**
$\Pi_T(t) \in C^{-1}$	$T \cdot \mathrm{Sinc}(fT) = T\frac{\sin(\pi fT)}{\pi fT}$	$O\left(\frac{1}{\lvert f \rvert}\right)$
$\delta(t) \in C^{-2}$	$1(f)$	$O(1)$

$\Lambda_T\left(t\right) \in C^0$	$T \cdot \mathrm{Sinc}^2\left(fT\right) = T\dfrac{\sin^2\left(\pi fT\right)}{\left(\pi fT\right)^2}$	$O\left(\dfrac{1}{\lvert f\rvert^2}\right)$
$e^{-at}u(t) \in C^{-1}$	$\dfrac{1}{a + j2\pi f}$	$O\left(\dfrac{1}{\lvert f\rvert}\right)$
$te^{-at}u(t) \in C^0$	$\dfrac{1}{\left(a + j2\pi f\right)^2}$	$O\left(\dfrac{1}{\lvert f\rvert^2}\right)$
$\dfrac{t^2}{2}e^{-at}u(t) \in C^1$	$\dfrac{1}{\left(a + j2\pi f\right)^3}$	$O\left(\dfrac{1}{\lvert f\rvert^3}\right)$
$\dfrac{t^n}{n!}e^{-at}u(t) \in C^{n-1}$	$\dfrac{1}{\left(a + j2\pi f\right)^{n+1}}$	$O\left(\dfrac{1}{\lvert f\rvert^{n+1}}\right)$

$\rho_{T,1}(t) =$ $\frac{1}{2}\left[\cos\left(\frac{\pi t}{T}\right)+1\right]\cdot\Pi_{2T}(t)\in C^1$	$T\frac{\sin(\pi fT)}{\pi fT}\frac{\cos(\pi fT)}{1-(2fT)^2}$	$O\left(\frac{1}{\lvert f\rvert^3}\right)$
$\rho_{T,\alpha}(t)\in C^1$ (see Chapter 2 for definition)	$T\frac{\sin(\pi fT)}{\pi fT}\frac{\cos(\pi\alpha fT)}{1-(2\alpha fT)^2}$	$O\left(\frac{1}{\lvert f\rvert^3}\right)$

Tab. 1 – Trends according to the Theorem

We remark that the higher the differentiability class, the "smoother" (or "more gradual") the signal is. In other words, "smoother" signals have larger n. According to theorem, their spectrum goes to 0 faster, as $f\to\pm\infty$. So, we can conclude that the "smoother" a signal is, the less high frequency content it has.

Smooth signals are favoured in practical applications, such as in digital radio transmission systems, because their faster spectrum decay makes them occupy less “bandwidth”. In other words, they occupy fewer “frequencies” and so, given a certain frequency interval, it is possible to fit more “channels” using smooth signals than using “harsh” or “irregular” signals.

Specifically, compare the frequency occupation of a pulse of FWHM T, when it is implemented as $\Pi_T(t)$ or $\rho_{T,\alpha}(t)$.

7.2.1 C^∞ signals

Certain signals are “infinitely smooth”, that is, they can be differentiated an infinite number of times. A straightforward application of the theorem would suggest that such signals are:

$$O\left(\frac{1}{|f|^\infty}\right)$$

Unfortunately, it is not entirely clear what this formula means. Specifically, while one can plot $1/|f|^m$ and compare it to a specific Fourier transform, doing so with $1/|f|^\infty$ is not possible.

However, one can still try and give the expression $O\left(1/|f|^\infty\right)$ meaning for $n=\infty$ by saying that the spectrum tails decay faster than any finite power of f, that is, the spectrum is $o\left(1/|f|^k\right)$ for any k big at will.

We now show that this interpretation of the theorem for C^∞ still appears to work. In fact, if we take the Gaussian signal:

$$e^{\frac{-t^2}{2\sigma^2}} \in C^\infty$$

Eq. 7-8

Its Fourier transform is:

$$F\left\{e^{-\frac{t^2}{2\sigma^2}}\right\} = S(f) = \sqrt{2\pi\sigma^2}\, e^{-2\pi^2\sigma^2 f^2}$$

It is then well-known that the Gaussian function decays more rapidly than any finite power of its independent variable (f in this case). As a result, it appears that the theorem can be extended, as shown, to C^∞ signals too.

The signals of the Gaussian type may occupy a very small bandwidth as compared to other kinds of signals. They are therefore frequently used, for instance in GSM telephony.

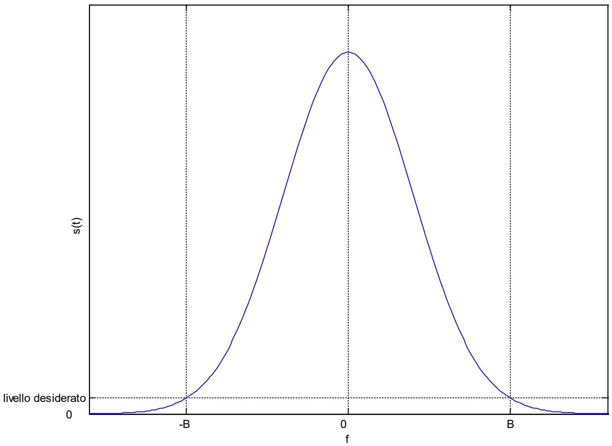


Fig. 7.13 – A Gaussian Spectrum

The graph of Fig. 7.13 shows a Gaussian spectrum e^{-f^2}.

7.2.2 Time-Frequency Duality

As it is the case for many other properties, the relationship between differentiability class in time domain vs. trend for $|f| \to \infty$ in frequency domain, has its dual. Specifically:

$$given\ \ S(f) \in C^n\ \ then\ \ s(t) = O\left(\frac{1}{|t|^{n+2}}\right)$$

where of course the differentiability class is referred to derivatives vs. frequency.

So, for example, the function $\Pi_B\left(f\right) \in C^{-1}$ has inverse Fourier transform, $B \cdot \mathrm{Sinc}\left(Bt\right) = B\dfrac{\sin\left(\pi Bt\right)}{\pi Bt} = O\left(\dfrac{1}{|t|}\right)$, as expected.

The function $\rho_{B,\alpha}\left(f\right) \in C^2$ and therefore its inverse Fourier transform falls off as $O\left(1/\left|t^3\right|\right)$