Chapter 4. Signal Spaces

In this chapter we will show that signals can be "structured" in a way quite similar to vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$.

By this we mean that similar operators to those that can be applied to "vectors" in $\mathbb{R}^n$ or $\mathbb{C}^n$ can also be applied to "signals". Among such operators: the sum, the product times a real or a complex number and the so-called inner-product, or "scalar" product.

Thanks to this, it will be possible to treat suitable sets of signals as linear (or vector) spaces and as **inner product spaces**. We will find out that under certain conditions, certain sets of signals may even form a Hilbert space.

We will then see that it is possible to "project" a signal onto a set of "orthonormal signals", similarly to what happens in $\mathbb{R}^n$ or $\mathbb{C}^n$, where a vector can be projected onto a set of orthonormal vectors (called unit vectors).

Also, like in $\mathbb{R}^n$ or $\mathbb{C}^n$, a "basis" for the set of signals can be found.

These specific aspects pave the way for a large number of applications, especially in the realm of *digital transmission and digital signal processing*, including modern audio and video compression techniques.

4.1 Linear Spaces

A linear space $\mathcal{L}$ is a set of elements such that a "sum" operator and a "multiplication by a number" operator can be defined in such a way that they satisfy certain properties.

The theory of linear spaces is well described in countless textbooks and is also introduced in other courses, so it will not be dealt with here in detail[1]. It is however important to recall at least a few fundamental properties regarding such spaces.

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[1] If you are interested, a clear axiomatic definition of Linear Spaces can be found in this Wikipedia article.

Given any two elements $\overline{x}, \overline{y} \in \mathcal{L}$*, then the result of their sum:*

$$\overline{z} = \overline{y} + \overline{x}$$

must still be an element of $\mathcal{L}$*, i.e.,* $\overline{z} \in \mathcal{L}$.

The "multiplication by a number" operator requires that, *given any* $\overline{x} \in \mathcal{L}$ *and* $\alpha \in \mathbb{R}$ *then:*

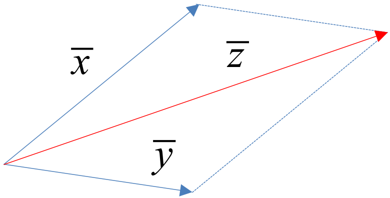
$$\overline{z} = \alpha \overline{x}$$

must still be an element of $\mathcal{L}$*, i.e.,* $\overline{z} \in \mathcal{L}$.

Note that these may seem obvious properties because we are accustomed to such linear spaces as $\mathbb{R}^2$ or $\mathbb{R}^3$, which we know well and are simple to deal with. However, for generic spaces $\mathcal{L}$ whose elements may be quite unusual objects, these fundamental properties may not be so straightforward.

One simple example of a linear space is $\mathbb{R}^2$, the Euclidean space of two-dimensional vectors lying on a plane.

In this space, it is possible to define the sum and product operator in such a way that their results is still an element of $\mathbb{R}^2$. In particular, the sum of two vectors can be geometrically defined through the "**parallelogram**" rule, which clearly *always yields another vector lying on the same plane*.



As for the "multiplication times a number" operator, it can be defined as a new vector whose direction is the same as the old one and whose length (or "magnitude") is the old one multiplied by α. If α is negative, the direction is reversed. In any case, the result is clearly still a vector on the same plane.

Note that if numbers are allowed to be complex, i.e., $\alpha \in \mathbb{C}$, then the simple multiplication rule and the geometric "parallelogram rule" for the sum of two (now possibly complex) vectors cannot be used anymore.

Extensions to handle complex α 's and vectors can be devised, but this shows that even "obvious" properties are in fact not so obvious as soon as one departs from simple objects such as vectors in $\mathbb{R}^2$.

The sum and multiplication operators are not "independent" in the sense that they must be able to interact according to certain properties, the most important of which is **the distributive property of the multiplication over the sum.**

It states that *given* $\alpha \in \mathbb{R}$, $\overline{x}, \overline{y} \in \mathcal{L}$, *then the following must be true:*

$$\alpha\left(\overline{y}+\overline{x}\right)=\alpha\overline{y}+\alpha\overline{x}$$

Using the parallelogram rule and the rule of multiplication, it is quite easy to see that this property is satisfied if $\mathcal{L} = \mathbb{R}^2$. Once again, just by allowing $\alpha \in \mathbb{C}$ things are not quite so simple and extended rules of sum and multiplication must be used.

There are countless examples of linear spaces.

For instance:

- the set of 2x2 or 3x3 or N x N matrices are linear spaces
- the set of polynomials of degree N is a linear space
- the set of zero-mean random variables is a linear space

and many more exotic ones…

In the following, we will concentrate on linear spaces formed by *signals*.

4.2 Signals and Linear Spaces

Let us now look at signals to see whether a properly defined set of signals can be viewed as a linear space.

First of all, let us define $\mathcal{S}_{\mathbf{I}}$ as **the set of all signals**, i.e., of all functions of time, whose domain is $\mathbf{I} = \left[t_0, t_1\right]$. We assume that $\mathbf{I} \subseteq \mathbb{R}$ can be any finite time-interval, which can also coincide with $\mathbb{R}$.

$\mathcal{S}_{\mathbf{I}}$ can be easily shown to be a linear space because a sum and multiplication operator satisfying the necessary properties can be defined in a very simple way.

Given the signals $x(t), y(t) \in \mathcal{S}_{\mathbf{I}}$, then their "sum" is written as:

$$z(t) = x(t) + y(t),$$

where the '+' operator operates as the sum of real or complex numbers. It is then obvious that $z(t) \in \mathcal{S}_{\mathbf{I}}$ as well.

Note however that, though "intuitive", this definition of the sum operator is not so trivial. The formula means that for each time t, we must assign to the signal $z(t)$ the sum of the values of the signals $x(t)$ and $y(t)$ at the same time t.

So, this is more than a simple "sum", it is in fact a rule for executing an infinite number of sums, one for each $t \in [t_0, t_1]$.

Also, the rule $z(t) = \alpha\, x(t)$ clearly yields $z(t) \in \mathcal{S}_{\mathbf{I}}$. Finally, there is no doubt that:

$$\alpha\,[x(t) + y(t)] = \alpha\, x(t) + \alpha\, y(t).$$

Since we have discovered that the signal set $\mathcal{S}_{\mathbf{I}}$ is a linear space, **we can use the "vector" notation in $\mathcal{S}_{\mathbf{I}}$ as well**. Given a signal $x(t) \in \mathcal{S}_{\mathbf{I}}$ we can also write it as $\overline{x}$, keeping in mind the "equivalence" $\overline{x} \equiv x(t)$ (the symbol $\equiv$ meaning "is equivalent to").

We can then write:

$$\overline{z} = \overline{x} + \overline{y}$$
$$\overline{z} = \alpha\,\overline{x}\,,$$

where the first line means $z(t) = x(t) + y(t)$ and the second line means $z(t) = \alpha\, x(t)$.

Optional

These definitions turn $\mathcal{S}_{\mathbf{I}}$ into a linear space, but some caution is in order. For instance, when a signal has a singularity for some $t = t_s \in \mathbf{I}$, such singularity must be discussed and eliminated.

One example is the signal $1/t$ at $t=0$. Note that $1/t$ is defined for any t other than $t=0$. If the linear space signals domain $\mathbf{I}$ comprises $t=0$, then we must remove such singularity.

In this example, we only need to assign a conventional value to $1/t$ for $t=0$ to solve the problem. For instance, the value 0 could be selected. The new signal:

$$s(t)=\begin{cases}1/t & t\neq 0\\ 0 & t=0\end{cases}$$

is then defined for $t=0$ too, and in fact it is defined for all $t\in\mathbb{R}$. It can be summed with any other signal and multiplied by any number. End of optional.

4.3 Inner Product Spaces

4.3.1 Preface to this section

The contents of this section are dealt more rigorously and in more detail in other courses, which the student is assumed to have already taken. Nonetheless, the main features of inner-product spaces are recalled here. This is done both to clearly outline the minimum set of basic knowledge that is required in this course, and to establish notation and terminology.

The student is therefore expected to at least read through Sect. 4.3 to make sure he/she is familiar with all the concepts recalled here, and that notation is understood and acquired.

4.3.2 From linear space to inner-product space

If a <u>linear space</u> $\mathcal{L}$ is such that for all pairs $\overline{x}, \overline{y} \in \mathcal{L}$ it is possible to compute a so-called "*inner product*" then $\mathcal{L}$ is an "**inner product space**", which we will call $\mathcal{P}$.

The inner product (or "scalar product" or "dot product") has to have certain specific properties, which we will recall later on. As for <u>notation</u>, many different writings are used to mean "inner product", such as:

$$\overline{x} \cdot \overline{y} \quad , \quad \langle \overline{x} \,|\, \overline{y} \rangle \quad , \quad (\overline{x}, \overline{y}) \quad , \quad \text{etc.}$$

We will use the last of the above:

$$(\overline{x}, \overline{y})$$

Typical examples of inner product spaces are the customary Euclidean spaces $\mathbb{R}^n$ or the complex spaces $\mathbb{C}^n$. These spaces are of course introduced in other courses.

There are countless more examples of inner-product spaces than $\mathbb{R}^n$ or $\mathbb{C}^n$. For instance the set of zero-mean random variables, or the set of 2x2 real or complex matrices, or sets of polynomials.

Also properly defined sets of ***signals*** can be inner product spaces.

We will first recall here certain general key properties of the inner product and of inner product spaces, and then we apply them to *inner product spaces made up of signals*.

4.3.3 Properties of the inner product

The inner product takes two elements of an inner product space and *returns a number*:

$$\left(\overline{x}, \overline{y}\right) = \alpha$$

where $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$ depending on the space being real or complex.

Note that such product may be defined in different ways, depending on the nature of the elements of the space.

For instance, if the elements are vectors in $\mathbb{R}^2$, we know that the "native rule" for the inner product between two vectors $\overline{x}$ and $\overline{y}$ is to find the "length" of both vectors, called $|\overline{x}|$ and $|\overline{y}|$, find the angle subtended between the two, call it θ, and then $(\overline{x},\overline{y}) = |\overline{x}| \cdot |\overline{y}| \cdot \cos(\theta)$.

Different spaces may have of course completely differente rules. In all cases, though. the inner product must have the following key properties.

Property 1 – Conjugate commutativity

Given $(\overline{x},\overline{y}) = \alpha$, then $(\overline{y},\overline{x}) = \alpha^*$, for any $\overline{x},\overline{y} \in \mathcal{P}$.

Property 2 – Product of an element by itself

$$\left(\overline{x},\overline{x}\right)=\mu\geq 0,\quad \mu\in\mathbb{R},$$

for all $\overline{x}\in\mathcal{P}$.

In fact, there is only one element of the space for which:

$$\left(\overline{x},\overline{x}\right)=0$$

Such element is called the "zero element of the space", or $\overline{0}$. Note that, for all $\overline{x}\in\mathcal{P}$, it must be:

$$\left(\overline{x},\overline{0}\right)=\left(\overline{0},\overline{x}\right)=0$$

Property 3 – Distributivity

The inner product is distributive:

$$\left(\alpha\overline{x}+\beta\overline{y},\overline{z}\right)=\alpha\left(\overline{x},\overline{z}\right)+\beta\left(\overline{y},\overline{z}\right)$$

Note however that due to Property (1) there is a difference between the left and right argument distribution property:

$$\left(\overline{z},\alpha\overline{x}+\beta\overline{y}\right)=\alpha^{*}\left(\overline{z},\overline{x}\right)+\beta^{*}\left(\overline{z},\overline{y}\right)$$

<u>Based on the inner product, we can define certain important quantities.</u>

These quantities can of course be computed in all inner product spaces. Two of them are *the norm and the energy*. Directly derived from the norm, we then have *the distance of two elements in* $\mathcal{P}$.

4.3.3.1 norm of an element in $\mathcal{P}$

The norm of an element of an inner product space is defined as:

$$\|\overline{x}\| = \sqrt{(\overline{x}, \overline{x})}$$

Due to Property 2 of the inner product, then $(\overline{x}, \overline{x}) \in \mathbb{R}, \geq 0$ and therefore the norm $\|\overline{x}\| = \sqrt{(\overline{x}, \overline{x})} \in \mathbb{R}, \geq 0$ too.

Elements in $\mathcal{P}$ whose norm is 1 are called unit elements, or *normal* elements. Normal elements are often represented with a "hat" on top, rather than the usual "bar". In other words, writing $\hat{x}$ implies that $\|\hat{x}\| = \sqrt{(\hat{x}, \hat{x})} = 1$

4.3.3.2 energy of an element in $\mathcal{P}$

The definition of energy of an element in $\mathcal{P}$ is

$$\mathcal{E}\{\overline{x}\} = (\overline{x}, \overline{x})$$

Eq. 4-1

Clearly, norm and energy are strictly related:

$$\mathcal{E}\{\overline{x}\} = \|\overline{x}\|^2$$

Due again to Property 1 and 2 of the inner product, energy is always real and positive, as one would expect it to be in accordance with physical constraints.

4.3.3.3 distance between two elements in $\mathcal{P}$

A very important quantity derived from the norm is the *distance* of two elements of an inner product space. *Given* $\overline{x}, \overline{y} \in \mathcal{P}$*, then the distance between these two elements is defined as:*

$$\Delta\{\bar{x},\bar{y}\} = \|\bar{x}-\bar{y}\| = \sqrt{(\bar{x}-\bar{y}\,,\bar{x}-\bar{y})}$$

Thanks to the properties and the quantities defined above, inner product spaces have an interesting underlying structure which is going to be explored in the following.

4.3.4 Orthonormal sets in $\mathcal{P}$

We first define the concept of orthogonality.

4.3.4.1 orthogonality between two elements in $\mathcal{P}$

Given $\bar{x},\bar{y}\in\mathcal{P}$, $\bar{x},\bar{y}\neq\bar{0}$, they are said to be *orthogonal*, if:

$$\left(\overline{x},\overline{y}\right)=0$$

which can be written in short as: $\overline{x} \perp \overline{y}$

A set of normal elements in $\mathcal{P}$ that are all mutually orthogonal is said to form an ***orthonormal set (ONS).*** In other words, an ONS $\mathcal{U}$ made up of N orthonormal elements of $\mathcal{P}$ is such that:

$$\mathcal{U}\equiv\left\{\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_N\right\} \qquad , \qquad \hat{u}_n\in\mathcal{P}$$

$$\left(\hat{u}_n,\hat{u}_k\right)=\delta_{nk}$$

where we have used the symbol δ_{nk}, which is called Kronecker's delta, to mean:

$$\delta_{nk} = \begin{cases} 0 \text{ if } n \neq k \\ 1 \text{ if } n = k \end{cases}$$

It is then easy to see that the condition $(\hat{u}_n, \hat{u}_k) = \delta_{nk}$ ensures that all elements in $\mathcal{U}$ are orthonormal.

4.3.4.2 projecting an element of $\mathcal{P}$ over an orthonormal set

Given any $\overline{x} \in \mathcal{P}$ and an ONS $\mathcal{U} \subset \mathcal{P}$ we can "project" $\overline{x}$ over $\mathcal{U}$. By this, we mean computing N inner products, one for each element in $\mathcal{U}$, as follows:

$$x_n = (\overline{x}, \hat{u}_n), \quad n = 1, 2, \ldots N$$

The number x_n is called the "projection" or **component** of $\overline{x}$ with respect to the unit element $\hat{u}_n$.

The N components of $\overline{x}$ with respect to the N unit elements in $\mathcal{U}$ are often written as an N-tuple, as follows:

$$\left[x_1, x_2, \ldots, x_N\right]$$

4.3.5 approximating an element using its components

Using the components x_n of $\overline{x}$ with respect to $\mathcal{U}$, we can construct an element in $\mathcal{P}$ as follows:

$$\overline{x}_{\text{app}} = \sum_{n=1}^{N} x_n \hat{u}_n = x_1 \hat{u}_1 + x_2 \hat{u}_2 + \ldots + x_N \hat{u}_N$$

Eq. 4-2

The subscript in $\overline{x}_{\text{app}}$ stands for "approximation". It has been chosen because we will show that there is a "resemblance" or "likeness" between $\overline{x}$ and $\overline{x}_{\text{app}}$, which we will precisely analyze.

However, the element $\overline{x}_{\text{app}}$ is, in general, *different from* $\overline{x}$. Specifically, in general, they differ by a non-zero 'error':

$$\overline{x} - \overline{x}_{\text{app}} = \overline{x}_{\text{err}} \quad , \quad \overline{x}_{\text{err}} \neq \overline{0}$$

We call the element $\overline{x}_{\text{app}}$ built as Eq. 4-2 the ***canonical approximation of*** $\overline{x}$ ***using the ONS*** $\mathcal{U}$.

The name *canonical approximation* is to distinguish it from other possible approximations, such as for instance:

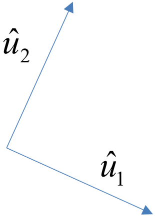
$$\overline{x}'_{\text{app}} = \sum_{n=1}^{N} x'_n \, \hat{u}_n$$

where the multiplying coefficients x'_n *are not* the projections $x_n = \left(\overline{x}, \hat{u}_n\right)$, that is, where $x'_n \neq x_n$.

Canonical approximations have special properties vs. all other possible approximations, as we shall see later.

4.3.5.1 Example

We consider the inner product space $\mathbb{R}^3$. We then choose **two orthonormal elements** in the space. For instance, the ones shown below, that lie on the plane of the slide:



Clearly, $\mathcal{U} \equiv \{\hat{u}_1, \hat{u}_2\}$ is an ONS in $\mathbb{R}^3$.

Given a generic vector $\overline{x} \in \mathbb{R}^3$, *its canonical approximation* $\overline{x}_{\text{app}}$ based on the ONS $\mathcal{U}$ would then be:

$$\overline{x}_{\text{app}} = x_1\hat{u}_1 + x_2\hat{u}_2$$

$$x_1 = (\overline{x}, \hat{u}_1) \quad , \quad x_2 = (\overline{x}, \hat{u}_2)$$

It is obvious that in general $\overline{x}_{\text{app}}$ *will not coincide with* $\overline{x}$*,* that is:

$$\overline{x}_{\text{app}} \neq \overline{x}$$

In fact, a vector pointing out of the plane of the page could not be exactly reproduced as a linear combination of $\left\{ \hat{u}_1, \hat{u}_2 \right\}$, which lie on the plane of the page.

In that case there would be an **error**: $\overline{x}_{\text{err}} = \overline{x} - \overline{x}_{\text{app}} \neq \overline{0}$

4.3.5.2 orthonormal bases

Only in *special cases* it happens that, given an ONS $\mathcal{U} \subset \mathcal{P}$, then:

$$\forall \overline{x} \in \mathcal{P} \quad \rightarrow \quad \overline{x} = \overline{x}_{\text{app}}$$

or equivalently:

$$\forall \overline{x} \in \mathcal{P} \quad \rightarrow \quad \overline{x}_{\text{err}} = \overline{x} - \overline{x}_{\text{app}} = \overline{0}$$

In other words, *for certain specific ONS* $\mathcal{U}$ in $\mathcal{P}$, given ***any*** element $\overline{x}$ in $\mathcal{P}$, its canonical approximation $\overline{x}_{\text{app}}$ always coincides with $\overline{x}$.

If this happens, **the orthonormal set $\mathcal{U}$ is said to be an orthonormal basis for** $\mathcal{P}$.

If $\mathcal{U}$ is an orthonormal basis for $\mathcal{P}$, then one can unambiguously describe any $\overline{x} \in \mathcal{P}$ by supplying the orthonormal basis $\mathcal{U}$ and just the components of $\overline{x}$, i.e., $\left[x_1, x_2, \dots, x_N\right]$, with respect to $\mathcal{U}$.

This is because, given $\mathcal{U} \equiv \{\hat{u}_1, \hat{u}_2, ..., \hat{u}_N\}$ and $[x_1, x_2, ..., x_N]$ then

$$\bar{x} = \sum_{n=1}^{N} x_n \hat{u}_n \qquad , \qquad \forall \bar{x} \in \mathcal{P}$$

where the x_n are the components collected in $[x_1, x_2, ..., x_N]$ and the unit elements are the one specified by $\mathcal{U} \equiv \{\hat{u}_1, \hat{u}_2, ..., \hat{u}_N\}$.

4.3.6 Dimensions of an inner-product space

Given a generic inner product space $\mathcal{P}$, such space is said to be N **-dimensional** if there is an orthonormal set $\mathcal{U} = \{\hat{u}\}_{n=1}^{N}$ which is also an *orthonormal basis* for $\mathcal{P}$, i.e., it is such that it can generate all $\bar{x} \in \mathcal{P}$.

Notice that it is precisely the number N of elements needed to obtain a basis that establishes the "dimension" of the space.

Notice that it can be shown that if an orthonormal basis is found for $\mathcal{P}$ with N elements, then any other orthonormal basis for $\mathcal{P}$ has precisely N elements (no more, no less). As a corollary, this means that it is *impossible* to find $N+K$ orthonormal elements in $\mathcal{P}$, for any $K>0$.

4.3.6.1 calculating inner products when an orthonormal basis is available

The inner product of any two elements $\overline{x}, \overline{y} \in \mathcal{P}$ can be computed according to the "native" rule of the space $\mathcal{P}$. These rules are typically different depending on the nature of the space.

However, if an ONS $\mathcal{U}$ is an orthonormal basis for $\mathcal{P}$, we know that all elements in $\mathcal{P}$ can be represented as:

$$\overline{x} = \sum_{n=1}^{N} x_n \hat{u}_n$$

As a result, the inner product $(\overline{x}, \overline{y})$ can be written as:

$$(\overline{x}, \overline{y}) = \left(\sum_{n=1}^{N} x_n \hat{u}_n , \sum_{m=1}^{N} y_m \hat{u}_m \right)$$

Now using the distributive property of the inner product, we have:

$$\left(\overline{x},\overline{y}\right)=\left(\sum_{n=1}^{N}x_n\hat{u}_n,\sum_{m=1}^{N}y_m\hat{u}_m\right)$$

$$=\sum_{n=1}^{N}\sum_{m=1}^{N}\left(x_n\hat{u}_n,y_m\hat{u}_m\right)=\sum_{n=1}^{N}\sum_{m=1}^{N}x_n y_m^*\left(\hat{u}_n,\hat{u}_m\right)$$

$$=\sum_{n=1}^{N}\sum_{m=1}^{N}x_n y_m^*\,\delta_{mn}=\sum_{n=1}^{N}x_n y_m^*$$

This result is one of the most important results of the theory of inner product spaces. It says that:

given an inner product space $\mathcal{P}$ and given an orthonormal basis $\mathcal{U}$ for $\mathcal{P}$, then one can carry out the inner product using a standardized rule, based on the sum of the products of same-index components, independently of the nature of the objects in $\mathcal{P}$:

$$\left(\overline{x}, \overline{y}\right) = \sum_{n=1}^{N} x_n y_n^*$$

As a corollary to this, in such spaces with an orthonormal basis, then the **norm** becomes "standardized" too:

$$\left\|\overline{x}\right\| = \sqrt{\left(\overline{x}, \overline{x}\right)} = \sqrt{\sum_{n=1}^{N} x_n x_n^*} = \sqrt{\sum_{n=1}^{N} \left|x_n\right|^2}$$

This rule is extremely famous and goes under the name of **Parseval's rule**.

Note that for N=2 or 3 such rule has been known for millennia because in $\mathbb{R}^2$ or $\mathbb{R}^3$ it is equivalent to the celebrated Pythagorean theorem!

(Can you show it for $\mathbb{R}^2$? Prove it on your own!)

Finally, the distance between two elements of $\mathcal{P}$ enjoys a standardized form too:

$$\Delta\{\overline{x},\overline{y}\} = \|\overline{x}-\overline{y}\| = \left\|\sum_{n=1}^{N} x_n \hat{u}_n - \sum_{m=1}^{N} y_n \hat{u}_n\right\| =$$

$$\left\|\sum_{n=1}^{N} (x_n - y_n)\hat{u}_n\right\| = \sqrt{\sum_{n=1}^{N} |x_n - y_n|^2}$$

$\boxed{\text{On your own}}$: try and verify as many of the above properties as possible based on what you know about $\mathbb{R}^2$ and $\mathbb{C}^2$.

4.3.6.2 <u>important</u>: generating an inner product space from an orthonormal set $\mathcal{U}$

Let $\mathcal{U}$ be an orthonormal set (ONS) in $\mathcal{P}$.

Let $\mathcal{P}'$ be the set of all elements $\overline{x}$ created by all possible linear combinations of the unit elements of $\mathcal{U}$:

$\mathcal{P}'$ is the set of all $\overline{x}$ such that: $$\overline{x} = x_1\hat{u}_1 + x_2\hat{u}_2 + \ldots + x_N\hat{u}_N = \sum_{n=1}^{N} x_n\hat{u}_n$$

Eq. 4-3

Using a compact *equivalent* notation, we can write:

$$\mathcal{P}' = \text{span}\{\mathcal{U}\}$$

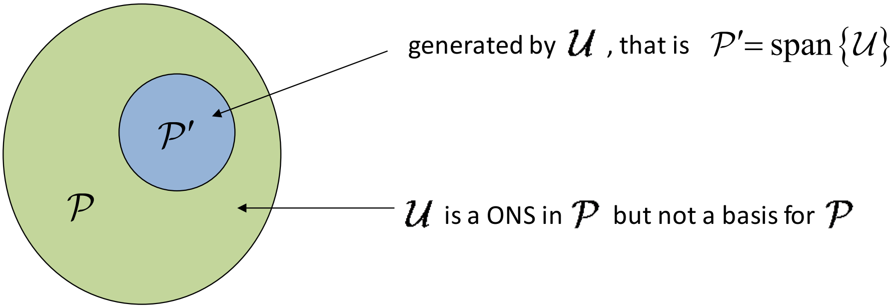
where $\text{span}\{\mathcal{U}\}$ means precisely the set of all elements generated as linear combinations of the elements of $\mathcal{U}$, i.e., all elements generated as Eq. 4-3.

Then $\mathcal{P}'$ **is an inner product space**.

In addition $\mathcal{U}$ is clearly an **orthonormal basis** for $\mathcal{P}'$, because Eq. 4-3 by definition can generate *all elements* of $\mathcal{P}'$.

However, note that in general, $\mathcal{P}' \subseteq \mathcal{P}$.

Specifically, only if $\mathcal{U}$ is an *orthonormal basis* for $\mathcal{P}$ as well, then clearly $\mathcal{P}' \equiv \mathcal{P}$. Otherwise, in the more general case in which $\mathcal{U}$ is not an orthonormal basis for $\mathcal{P}$, then $\mathcal{P}' \subset \mathcal{P}$. Note also that the case $\mathcal{P}' \supset \mathcal{P}$ is *impossible*.



On your own: verify that $\mathcal{P}'$ is an inner product space. Also, why is it impossible that $\mathcal{P}' \supset \mathcal{P}$?

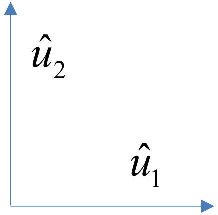
Solution hints: The proof is very simple. We first remark that $\overline{x}, \overline{y} \in \mathcal{P}'$ are elements of $\mathcal{P}$ too, that is $\overline{x}, \overline{y} \in \mathcal{P}$ as well. So, it is always possible to take their inner product

$(\overline{x},\overline{y})$ because $\mathcal{P}$ is an inner product space. All properties of the inner product are also guaranteed to be satisfied, for the same reason.

Regarding the sum, it is always possible to sum $\overline{x},\overline{y} \in \mathcal{P}'$, because $\overline{x},\overline{y} \in \mathcal{P}$ too, and $\mathcal{P}$ is a linear space. So we will always obtain a result $\overline{z} = \overline{x} + \overline{y}$.

Note that for sure $\overline{z} \in \mathcal{P}$ but we do not know whether $\overline{z} \in \mathcal{P}'$ or perhaps $\overline{z} \notin \mathcal{P}'$ However, For $\mathcal{P}'$ to be an inner product space, it must be that the result of the sum of any two of its elements, $\overline{x},\overline{y}$, is still an element of $\mathcal{P}'$, so it must be that $\overline{z} \in \mathcal{P}'$. Check on your own that indeed the sum of any two elements of $\mathcal{P}'$ is still an element of $\mathcal{P}'$ and therefore $\mathcal{P}'$ is an inner product space.

As an ***example***, considering $\mathbb{R}^3$ and extracting from it an orthonormal set $\mathcal{U} \equiv \{\hat{u}_1, \hat{u}_2\}$ then $\text{span}\{\mathcal{U}\}$ is just a single plane in the three-dimensional space $\mathbb{R}^3$.



$\mathcal{U}$ is an ONS in $\mathbb{R}^3$ but *it is not an orthonormal basis of* $\mathbb{R}^3$, since not all elements of $\mathbb{R}^3$ can be generated as linear combinations of $\mathcal{U}$.

However, $\mathcal{U}$ *is an orthonormal basis for the* $\mathbb{R}^2$ *plane where* $\hat{u}_1, \hat{u}_2$ *lie*. The plane $\mathbb{R}^2$ is a linear space and an inner product space as well.

4.4 Signals Spaces as Inner product Spaces

4.4.1 The inner product for signals

We already know that the set of all signals over an interval $\mathbf{I}$, which we called $\mathcal{S}_{\mathbf{I}}$, is a linear (or vector) space. We shall now see that an inner product can be defined over it.

The inner product between two signals $x(t)$ e $y(t)$ is defined as:

$$\big(x(t), y(t)\big) = \int_{\mathbf{I}} x(t) y^*(t) dt$$

Eq. 4-4

where $\mathbf{I}$ is as usual the time-interval t belongs to.

Its result is a number, which can be either real or complex depending on whether the signals in $\mathcal{S}_{\mathbf{I}}$ are real or complex.

Important: for certain pairs of signal in $\mathcal{S}_{\mathbf{I}}$, the inner-product integral Eq. 4-4 might not converge.

It can be shown that this problem can be removed by looking at a *subset* of $\mathcal{S}_{\mathbf{I}}$. Such subset is *the set of all finite-energy signals over a finite interval*, called $L^2_{[t_0,t_1]}$ or equivalently $L^2_{\mathbf{I}}$.

For a generic $s(t)$ to be in $L^2_{\mathbf{I}}$ it must be:

$$\mathcal{E}_{\mathbf{I}}\{s(t)\} = \int_{\mathbf{I}} |s(t)|^2 dt < \infty$$

Note: see that the concept of energy of a signal, introduced in Chapter 3, is preserved here and notice also that in this new context we find an equivalent definition in terms of the inner product:

$$\mathcal{E}_{\mathbf{I}}\{s(t)\} = \int_{\mathbf{I}} |s(t)|^2 dt = \int_{\mathbf{I}} s(t)s^*(t)dt = \left(s(t), s(t)\right)$$

Theorems assure that if any two signals $x(t)$ and $y(t)$ are both finite-energy, that is, if they both belong to $L^2_{\mathbf{I}}$, then their inner product Eq. 4-4 converges to a finite number. This ensures that $L^2_{\mathbf{I}}$ is indeed a proper *inner product space.*

Note that $L^2_{\mathbf{I}} \subset \mathcal{S}_{\mathbf{I}}$, that is $L^2_{\mathbf{I}}$ has fewer elements than $\mathcal{S}_{\mathbf{I}}$. In particular, it does not contain all those elements of $\mathcal{S}_{\mathbf{I}}$ whose energy is *infinite*.

In the following, when not otherwise specified, when dealing with signals we will assume to be operating in $L^2_{\mathbf{I}}$.

4.4.1.1 "element" or "native" notation

In Eq. 4-4 we could also have written the left-hand side as: $(\overline{x}, \overline{y})$, rather than $(x(t), y(t))$, using the element notation, which is "standard" for all linear and inner product spaces, independently of the actual nature of the element and of the inner product space.

In the following, for signals we will interchangeably use either the "element" notation as in $\overline{x}$, $\overline{y}$ or the "native" notation $x(t), y(t)$.

4.4.1.2 properties of the signal inner product

We should now show that the required properties of the inner product are satisfied by the definition Eq. 4-4. This is easily proved, so we will omit it.

We will only show that the distributive property holds:

$$\left(\alpha\overline{x}+\beta\overline{y}\,,\overline{z}\right)=\int_I\left[\alpha x(t)+\beta y(t)\right]\,z^*(t)dt=$$

$$\alpha\int_I x(t)\,z^*(t)\,dt+\beta\int_I y(t)\,z^*(t)\,dt=\alpha\left(\overline{x},\overline{z}\right)+\beta\left(\overline{y},\overline{z}\right)$$

Show $\boxed{\text{on your own}}$ that $\left(\overline{z},\alpha\overline{x}+\beta\overline{y}\right)=\alpha^*\left(\overline{z},\overline{x}\right)+\beta^*\left(\overline{z},\overline{y}\right)$

Based on the inner product, we can for instance compute the energy of an element $\overline{x}$. By following the definition Eq. 4-1 the energy of $\overline{x}$ is:

$$\mathcal{E}\{\overline{x}\} = \left(\overline{x}, \overline{x}\right) = \int_{\mathbf{I}} \left|x(t)\right|^2 dt$$

Eq. 4-5

Note that this value of the energy is completely consistent with that of Eq. 3-2 which was given as an independent and unrelated definition.

In other words: $\mathcal{E}\{\overline{x}\} = \mathcal{E}_{\mathbf{I}}\left\{x(t)\right\}$. Since they coincide, like $\mathcal{E}_{\mathbf{I}}\left\{x(t)\right\}$, $\mathcal{E}\{\overline{x}\}$ too returns results that are consistent with physical arguments regarding energy and power.

The **norm** of an element $\overline{x}$ is the square-root of its energy, so:

$$\|\overline{x}\| = \sqrt{\left(\overline{x}, \overline{x}\right)} = \sqrt{\mathcal{E}\{\overline{x}\}} = \sqrt{\int_{\mathbf{I}} \left|x(t)\right|^2 dt}$$

The **distance** between two signals is then:

$$\Delta\{\overline{x},\overline{y}\} = \|\overline{x}-\overline{y}\| = \sqrt{\int_{\mathbf{I}} \left|x(t)-y(t)\right|^2 dt}$$

4.5 Simple Signal Spaces

4.5.1 generating signal inner product spaces by means of simple orthonormal sets

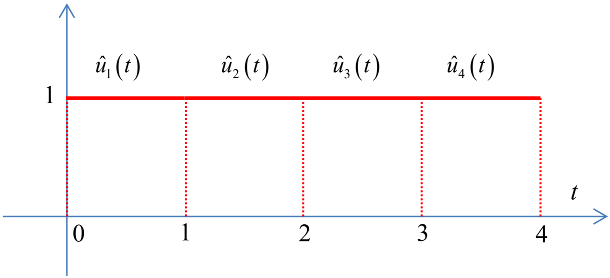
In Sect. 4.3.5.4 we showed that an orthonormal set (ONS) $\mathcal{U}$ can be used to generate an inner product space $\mathcal{P}'$ of which $\mathcal{U}$ is, by construction, an *orthonormal basis*.

In the following, we concentrate on ***signal*** spaces. We assume that $\mathcal{U}$ is a set of orthonormal elements taken from $L^2_{\mathbf{I}}$. Then, all possible linear combinations of the elements of $\mathcal{U}$ generate an inner product space Σ, such that $\Sigma \subseteq L^2_{\mathbf{I}}$.

Since $\mathcal{U}$ is an orthonormal basis for Σ, we can use both the *'native rules'* of signal spaces to compute norms, distances, inner products, etc., or we can use the *standardized* general rules, operating on the components of the signals in Σ with respect to $\mathcal{U}$, as shown in Section 4.3.5.3.

4.5.1.1 example of a simple signal orthonormal set

A simple example of an orthonormal signal set is the following:



$$\mathcal{U} = \{\hat{u}_n\}_{n=1}^{N=4}, \quad \hat{u}_n \equiv \Pi(t - n + 1/2) = \pi(t - [n-1])$$

with t limited to the interval $\mathbf{I} = [0,4]$.

Let us check the orthogonality on the unit elements $\hat{u}_1, \hat{u}_2$:

$$\left(\hat{u}_1, \hat{u}_2\right) = \int_0^4 \pi\left(t\right)\left[\pi\left(t-1\right)\right]^* dt = 0$$

They are clearly orthogonal because the integrand function is zero everywhere. $\boxed{\text{On your own}}$ check the orthogonality of all other pairs.

Unit norm is also straightforward. For element $\hat{u}_1$:

$$\left\|\hat{u}_1\right\| = \sqrt{\left(\hat{u}_1, \hat{u}_1\right)} = \sqrt{\int_0^4 \Pi\left(t-\frac{1}{2}\right)\cdot \Pi^*\left(t-\frac{1}{2}\right) dt} = \sqrt{\int_0^1 1(\mathrm{t})\, dt} = 1$$

and likewise for the other elements.

$\boxed{\text{On your own:}}$ take the orthogonal (but <u>not</u> orthonormal) set:

$$\tilde{\mathcal{U}} = \left\{ \overline{u}_n \right\}_{n=1}^{N=4} \quad , \quad \overline{u}_n \equiv \Pi_{\frac{1}{4}}\left(t - \frac{n}{4} + \frac{1}{8} \right)$$

where the tilde above $\tilde{\mathcal{U}}$ is used to point out that it is not an ONS.

Plot the elements of the set. Check their mutual orthogonality. Then, *normalize* them appropriately, i.e., find multiplying constants for each one of them, so that their norm becomes 1.

***Answer**:* the normalizing constant is 2, that is, the signals must be multiplied by 2.

4.5.1.2 example of generating signals using a simple orthonormal set

We now consider the very simple orthonormal set made up of the signals:

$$\mathcal{U} = \left\{ \hat{u}_n \right\}_{n=0}^{3}, \quad \hat{u}_n \equiv \Pi\left(t - n + 1/2\right) \quad \text{with} \quad \mathbf{I} = [-1, 3]$$

which is depicted in Fig. 4.1.

This orthonormal set generates a signal inner product space Σ, which clearly does not coincide with $L^2_{\mathbf{I}}$. In other words, $\Sigma \subset L^2_{\mathbf{I}}$. Of course, $\mathcal{U}$ is an orthonormal basis for Σ.

$\boxed{\text{On your own}}$: write down and draw a few possible signals in Σ.

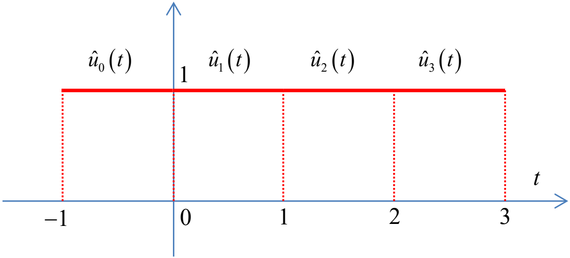


Fig. 4-1: the orthonormal set $\mathcal{U}$

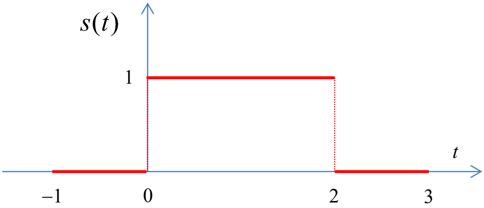
We use $\mathcal{U}$ to generate two simple signals. Specifically, we generate $s(t)$ as:

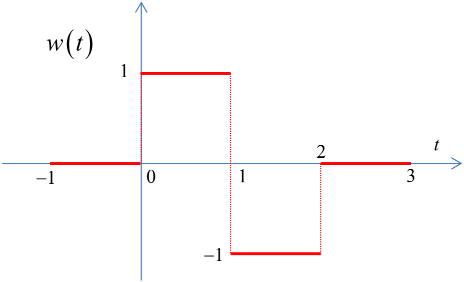
$$s(t) = 1 \cdot \hat{u}_1(t) + 1 \cdot \hat{u}_2(t)$$

and $w(t)$ as:

$$w(t) = 1 \cdot \hat{u}_1(t) - 1 \cdot \hat{u}_2(t)$$

These signals are shown below:





Using the n-tuple (or array) format, we can also write them as:

$$\overline{s} = [0,1,1,0], \ \overline{w} = [0,1,-1,0]$$

4.5.1.3 taking the inner product of two signals using the standardized rule

Both $s(t)$ and $w(t)$ belong to the *inner product space* Σ of which we have a orthonormal basis (it is $\mathcal{U}$ shown in Fig. 4.1). We can therefore compute the inner product of these two signals either by using the "native rule" for signals, or by using the standardized rule which is valid for all inner product spaces.

Using the native rule, we get:

$$\left(\overline{s},\overline{w}\right)=\int_{-1}^{3} s(t)\,w^{*}(t)\,dt=$$

$$=\int_{-1}^{3}\Pi_2\left(t-1\right)\left[\Pi\left(t-\frac{1}{2}\right)-\Pi\left(t-\frac{3}{2}\right)\right]dt$$

$$=\int_{-1}^{3}\Pi_2\left(t-1\right)\Pi\left(t-\frac{1}{2}\right)dt-\int_{-1}^{3}\Pi_2\left(t-1\right)\Pi\left(t-\frac{1}{2}\right)dt$$

$$=\int_{0}^{1}1(t)\;dt-\int_{1}^{2}1(t)\;dt=1-1=0$$

that is, the two signals are orthogonal.

We now verify that the same result is obtained when using the "standardized rule" based on the n-tuple representation of the two signals vs. the orthonormal basis $\mathcal{U}$:

$$\left(\overline{s},\overline{w}\right)=\sum_{n=0}^{3} s_n w_n^* = s_0\cdot w_0^* + s_1\cdot w_1^* + s_2\cdot w_2^* + s_3\cdot w_3^*$$

$$=0\cdot 0+1\cdot 1+1\cdot(-1)+0\cdot 0=0$$

So, in fact, both the "native" rule and the standardized rule produce the same result.

4.5.1.4 remarks on orthogonality of signals

To discuss signal orthogonality, we first introduce the definition of *support of a signal*.

In mathematics, the support of a function is the set of points where the function is not zero. In signal theory, *the support of a signal is the set of all time-instants where the signal is not zero*.

A straightforward result about supports is found when two signals are multiplied:

Given the signal: $z(t) = x(t) \cdot y(t)$*, and given the support of* $x(t)$*, called* $\mathbf{X}$ *and the support of* $y(t)$*, called* $\mathbf{Y}$ *, then the support of* $z(t)$ *is the intersection of the supports of* $x(t)$ *and* $y(t)$*:* $\mathbf{Z} = \mathbf{X} \bigcap \mathbf{Y}$.

As a corollary, *if two signals* $x(t)$ *and* $y(t)$ *have completely disjoint supports, that is, if* $x(t) \neq 0$ *then* $y(t) = 0$ *and vice-versa, then* $\mathbf{Z} = \mathbf{X} \bigcap \mathbf{Y} = \varnothing$.

That is, the support of $z(t)$ is an empty set; in other words, $z(t) = 0$ at all times or, equivalently, $z(t) = 0(t)$.

This allows us to find a sufficient condition for two signals to be orthogonal. Given $x(t)$ and $y(t)$, they are orthogonal if their supports are disjoint, i.e., if $\mathbf{X} \bigcap \mathbf{Y} = \varnothing$.

$\boxed{\text{On your own}}$: why? It should be obvious, based on the definition of inner product.

For instance, based on this result, it is immediately seen that the four signals:

$$\mathcal{U} = \left\{ \hat{u}_n \right\}_{n=1}^{4}, \quad \hat{u}_n \equiv \Pi\left(t - n + 1/2\right)$$

are orthogonal to one another, <u>because their supports are all mutually disjoint</u>.

This condition for orthogonality is *sufficient* but *not necessary*. In other words, there are signals whose supports are not disjoint, that are orthogonal nonetheless. We saw an example in Sect. 4.5.1.2. The supports of the signals $s(t)$ and $w(t)$ were not disjoint, but such two signals were orthogonal nonetheless.

On your own Consider the two signals $x(t) = \cos\left(2\pi f_0 t\right)$ and $y(t) = \sin\left(2\pi f_0 t\right)$ over the interval $\left[0, T_0\right]$ with $T_0 = 1/f_0$. Carry out the calculations for their inner product.

Answer

These two signals have supports that are fully superimposed. Nonetheless, they are orthogonal: $\left(x(t), y(t)\right) = 0$

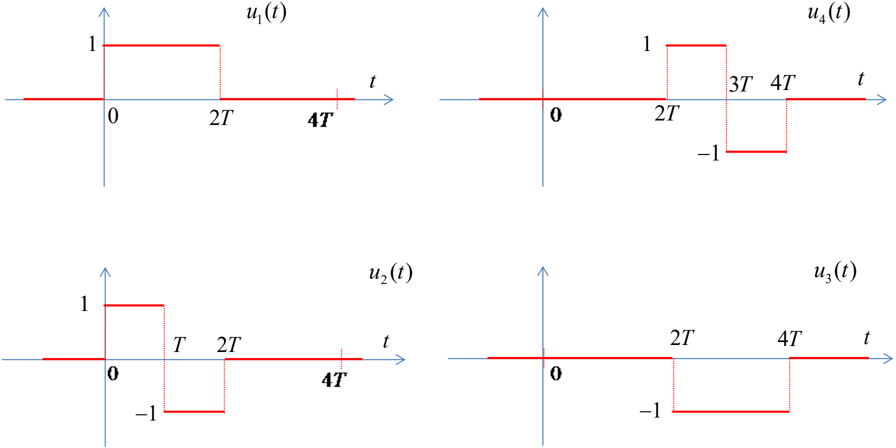


Fig. 4-2

$\boxed{\text{On your own}}$

Consider the four signals from Fig. 4-2.

- Are they unit norm? If not, make them all unit norm by dividing them by their norm.
- Are they orthogonal to one another? Check for this condition.

Answers

The four signals can be made unit norm by dividing them by their norm, which is $\sqrt{2T}$. They are all mutually orthogonal.

$\boxed{\text{On your own}}$

Consider the ONS $\mathcal{U} = \left\{\hat{u}_n\left(t\right)\right\}_{n=1}^{4}$ based on the four signals from Fig. 4-2, normalized as found in the previous exercise (i.e., dividing them by their norm, $\sqrt{2T}$).

Consider the signals

$s(t) = \pi_T(t)$

$w(t) = \Pi_{2T}(t-2T)$.

- Find their components vs. $\mathcal{U}$.
- Write down their approximation based on $\mathcal{U}$ and find out if there is any error. (In other words, check whether $s(t), w(t) \in \Sigma = \text{span}\{\mathcal{U}\}$.)
- compute the inner product $\left(s(t), w(t)\right)$

Answers

The projections s_n (or components) of $s(t)$ on $\mathcal{U}$ are $\left[\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}, 0, 0\right]$ and the projections w_n of $w(t)$ on $\mathcal{U}$ are $\left[\sqrt{\frac{T}{2}}, -\sqrt{\frac{T}{2}}, -\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}\right]$. Both signals are reconstructed with no error and therefore they both belong to $\Sigma = \text{span}\{\mathcal{U}\}$. The two signal are orthogonal, that is $\left(s(t), w(t)\right) = 0$. This can be easily seen by performing the inner product using the general rule:

$$\left(s(t), w(t)\right) = \sum_{n=1}^{4} s_n w_n^*$$

$\boxed{\text{On your own}}$ Consider the following three signals:

$$s_1(t) = \Pi_2(t)$$

$$s_2(t) = -\Pi_1(t - 1/2) + \Pi_1(t + 1/2)$$

$$s_3(t) = \Pi_{1/2}(t + 3/4) - \Pi_{1/2}(t + 1/4) + \Pi_{1/2}(t - 1/4) - \Pi_{1/2}(t - 3/4)$$

over the interval $t \in [-1,1]$.

- Draw the signals
- Calculate the inner product of each one of them vs. any other.

Answer

These three signals have supports that are fully superimposed. Nonetheless all the inner products are zero, that is every signal is orthogonal to every other.

4.6 Orthonormal Sets, Approximations and Errors

Reminder: an orthonormal set:

$$\mathcal{U} = \left\{ \hat{u}_n(t) \right\}_{n=1}^{N}$$

whose elements are taken from a generic signal inner product space Σ, that is:

$$\hat{u}_n(t) \in \Sigma$$

constitutes an **orthonormal basis** for the signal inner product space Σ if it can generate all the elements of that space.

In other words, $\mathcal{U}$ is an orthonormal basis, if:

$$v(t)=\sum_{n=1}^{N} v_n \hat{u}_n(t) \quad \text{for any } v(t)\in\Sigma$$

or, equivalently, using a compact notation, if: $\Sigma = \text{span}\{\mathcal{U}\}$.

Otherwise, if

$$\text{span}\{\mathcal{U}\} = \Sigma' \subset \Sigma$$

then $\mathcal{U}$ is just an ONS in Σ *but not a basis for* Σ, because it cannot generate all the elements of Σ. ($\mathcal{U}$ is however an orthonormal basis for the smaller inner product space $\Sigma' \subset \Sigma$ that it can fully generate.)

4.6.1 ONSs and definition of approximation "error"

We assume that $\mathcal{U}$ is an ONS in the inner product space Σ, *but not a basis for* Σ, that is $\text{span}\{\mathcal{U}\} = \Sigma' \subset \Sigma$.

If we take a signal $w(t) \in \Sigma$, we know that we can write the *canonical approximation of the signal* $w(t)$ *by means of the ONS* $\mathcal{U}$, as the signal $w_{app}(t)$:

$$w_{app}(t) = \sum_{n=1}^{N} w_n \hat{u}_n(t)$$

Eq. 4-6

We can also say that the signal $w_{app}(t)$ is identified by the N-tuple:

$$w_{app}(t) \equiv [w_1, w_2, w_3, \ldots, w_N]$$

Eq. 4-7

In general, $w(t)$ and its canonical approximation $w_{app}(t)$ **will not coincide**. An *error signal* can then be defined as follows:

$$w_{err}(t) = w(t) - w_{app}(t)$$

Eq. 4-8

Note that the signal $w(t)$ can always be written *exactly* as:

$$w(t) = w_{app}(t) + w_{err}(t)$$

Eq. 4-9

4.6.2 Properties of the error signal

The error signal presents **important features** which are summarized in the three theorems that follow.

Theorem 1

Given:

$$w(t) = w_{app}(t) + w_{err}(t)$$

Eq. 4-10

the error signal $w_{err}(t)$ is orthogonal to each unit element (or unit signal) of the ONS used to generate $w_{app}(t)$.

Proof: if we consider the representation:

$$w_{app}(t) = \sum_{n=1}^{N} w_n \hat{u}_n(t)$$

Eq. 4-11

and remember from Eq. 4-8 that $w_{err}(t) = w(t) - w_{app}(t)$, we have:

$$\begin{aligned}
\left(w_{err}(t), \hat{u}_n(t)\right) &= \left(w(t), \hat{u}_n(t)\right) - \left(w_{app}(t), \hat{u}_n(t)\right) \\
&= w_n - \left(\sum_{m=1}^{N} w_m \hat{u}_m(t), \hat{u}_n(t)\right) \\
&= w_n - \sum_{m=1}^{N} w_m \left(\hat{u}_m(t), \hat{u}_n(t)\right) \\
&= w_n - \sum_{m=1}^{N} w_n \delta_{n,m} = w_n - w_n = 0
\end{aligned}$$

Eq. 4-12

In short:

$$\left(\overline{w}_{err}, \hat{u}_n\right) = 0, \quad \forall n$$

Theorem 2

The canonical approximation $w_{app}(t)$ *and the error signal* $w_{err}(t)$ *are orthogonal to each other.*

Proof: we directly calculate the inner product between $w_{app}(t)$ and $w_{err}(t)$:

$$\left(w_{app}(t), w_{err}(t)\right) =$$

$$= \left(\sum_{m=1}^{N} w_m \hat{u}_m(t), w_{err}(t) \right) = \sum_{m=1}^{N} w_m \left(\hat{u}_m(t), w_{err}(t) \right)$$

Eq. 4-13.

But from Theorem 1 we know that:

$$\left(\hat{u}_m(t), w_{err}(t)\right) = 0 \quad \forall m$$

So all the terms in the last summation are zero and therefore:

$$\left(w_{app}(t), w_{err}(t)\right) = 0$$

Theorem 3

Given $w(t) = w_{app}(t) + w_{err}(t)$*, the energy of the signal* $w(t)$ *is equal to the sum of the energy of its canonical approximation with the energy of the error signal:*

$$\mathcal{E}\left\{w(t)\right\} = \mathcal{E}\left\{w_{app}(t)\right\} + \mathcal{E}\left\{w_{err}(t)\right\}$$

Eq. 4-14

Proof: again by direct calculation:

$$\begin{aligned}\mathcal{E}\left\{w(t)\right\} &= \mathcal{E}\left\{w_{app}(t) + w_{err}(t)\right\} \\ &= (w_{app}(t) + w_{err}(t)\,,\, w_{app}(t) + w_{err}(t)) \\ &= (w_{app}(t), w_{app}(t)) + (w_{err}(t), w_{err}(t)) + \\ &\qquad + (w_{app}(t), w_{err}(t)) + (w_{err}(t), w_{app}(t))\end{aligned}$$

Eq. 4-15

But from Theorem 2 we know that

$$(w_{app}(t), w_{err}(t)) = (w_{err}(t), w_{app}(t)) = 0$$

So:

$$\begin{aligned} \mathcal{E}\left\{w(t)\right\} &= (w_{app}(t), w_{app}(t)) + (w_{err}(t), w_{err}(t)) + \\ &\quad + (w_{app}(t), w_{err}(t)) + (w_{err}(t), w_{app}(t)) \\ &= (w_{app}(t), w_{app}(t)) + (w_{err}(t), w_{err}(t)) + 0 + 0 \\ &= (w_{app}(t), w_{app}(t)) + (w_{err}(t), w_{err}(t)) \\ &= \mathcal{E}\left\{w_{app}(t)\right\} + \mathcal{E}\left\{w_{err}(t)\right\} \end{aligned}$$

Eq. 4-16

We remark that, given a certain signal, according to the above result, *the higher is the energy of the canonical approximation, the lower is the one of the error signal*.

As a corollary of Theorem 3, note the following obvious results:

$$\mathcal{E}\left\{\overline{w}_{\text{app}}\right\} \leq \mathcal{E}\left\{\overline{w}\right\}$$

$$\mathcal{E}\left\{\overline{w}_{\text{err}}\right\} \leq \mathcal{E}\left\{\overline{w}\right\}$$

On your own: show that the energy of the error signal coincides with the *distance squared* between the signal and its canonical approximation, and that the distance coincides with the norm of the error. That is:

$$\mathcal{E}\left\{\overline{w}_{err}\right\} = \left(\Delta\left\{\overline{w}, \overline{w}_{app}\right\}\right)^2$$

$$\left\|\overline{w}_{err}\right\| = \Delta\left\{\overline{w}, \overline{w}_{app}\right\} = \sqrt{\mathcal{E}\left\{\overline{w}_{err}\right\}}$$

$\boxed{\text{On your own}}$: prove that if you add one more unit element to the ONS, the energy of the error can only decrease or at most remain the same. This is an important results because it shows that by adding more unit elements to the ONS the quality of the approximation can only improve.

4.6.2.2 optimality of the canonical approximation

Given $w(t) \in \Sigma$ with $t \in [t_0, t_1]$, we have introduced its canonical approximation with respect to an ONS $\mathcal{U}$ as:

$$w_{app}(t) = \sum_{n=1}^{N} w_n \cdot \hat{u}_n(t)$$

Eq. 4-17

The **key aspect** is that the w_n's are the "canonical components" of $w(t)$ with respect to the ONS $\mathcal{U}$, i.e., they are *uniquely and unambiguously found* through the inner product: $w_n = \left(w(t), \hat{u}_n(t) \right)$.

One may however wonder: ***is there a better approximation*** of the signal $w(t)$ than the canonical one? (Of course still based on the same ONS).

The critical aspect here is how to define "better". One obvious and very reasonable criterion may be that of the **minimization of the energy of the error signal**.

The question would then be: is it possible to find an alternative set of numbers $w'_n \neq w_n$ such that a new approximation, still built using the unit elements of the same ONS:

$$w'_{app}(t) = \sum_{n=1}^{N} w'_n \cdot \hat{u}_n(t)$$

generates an error signal whose energy is lower than the energy of the error related to the canonical approximation? That is, such that:

$$\mathcal{E}\left\{w'_{err}\left(t\right)\right\} < \mathcal{E}\left\{w_{err}\left(t\right)\right\}$$

The interesting and remarkable answer is that **the canonical approximation is optimum** from the viewpoint of the energy of the related error signal, that is, ***it minimizes the energy of the error***. No alternative set of numbers $w'_n \neq w_n$ can generate an error signal with lower energy.

Note also that the norm of the error coincides with the distance between the signal and its approximant and with the square root of the error energy:

$$\Delta\left\{\overline{w}, \overline{w}_{app}\right\} = \left\|\overline{w} - \overline{w}_{app}\right\| = \left\|\overline{w}_{err}\right\| = \sqrt{\mathcal{E}\left\{\overline{w}_{err}\right\}}$$

So, saying that the canonical approximation minimizes the energy of the error **is the same** as saying that *the canonical approximation minimizes the distance between the original signal (or element) and its approximant*.

In fact, we can state that:

the canonical approximant is the closest signal (or element) in $\Sigma' = \mathrm{span}\{\mathcal{U}\}$ ***to the original element*** $\overline{w} \in \Sigma$

The proof is optional and is reported below:

Proof: ***optional***

We assume $w(t) \in \Sigma$ with $\mathrm{I} = [t_0, t_1]$ and we consider its canonical approximation with respect to an ONS $\mathcal{U}$:

$$w_{app}(t) = \sum_{n=1}^{N} w_n \cdot \hat{u}_n(t)$$

We can write *any other element* in the inner product space generated by $\mathcal{U}$, which could potentiall be a better approximant $w'_{app}(t)$, as:

$$w'_{app}(t) = \sum_{n=1}^{N} w'_n \cdot \hat{u}_n(t)$$

The distance between $w'_{app}(t)$ and $w(t)$ is given by:

$$\Delta\left\{w(t), w'_{app}(t)\right\}$$

Note that:

$$\mathcal{E}\left\{w'_{err}(t)\right\} = \mathcal{E}\left\{w(t) - w'_{app}(t)\right\} = \left\|w(t) - w'_{app}(t)\right\|^2 = \Delta^2\left\{w(t), w'_{app}(t)\right\}$$

So, minimizing $\Delta^2\left\{w(t), w'_{app}(t)\right\}$ is the same as minimizing the energy of the error $\mathcal{E}\left\{w'_{err}(t)\right\}$. Also, minimizing $\Delta^2\left\{w(t), w'_{app}(t)\right\}$ is the same as minimizing $\Delta\left\{w(t), w'_{app}(t)\right\}$, so for convenience we will try to minimize $\Delta^2\left\{w(t), w'_{app}(t)\right\}$.

In this formula, we can substitute $w(t)$ with:

$$w(t) = w_{app}(t) + w_{err}(t)$$

and write:

$$\Delta^2\left\{w(t), w'_{app}(t)\right\} = \Delta^2\left\{w_{app}(t) + w_{err}(t), w'_{app}(t)\right\}$$

$$= \left(w_{app}(t) + w_{err}(t) - w'_{app}(t), w_{app}(t) + w_{err}(t) - w'_{app}(t)\right) =$$

$$\left(\sum_{n=1}^{N} w_n \cdot \hat{u}_n(t) - \sum_{n=1}^{N} w'_n \cdot \hat{u}_n(t) + w_{err}(t), \sum_{n=1}^{N} w_n \cdot \hat{u}_n(t) - \sum_{n=1}^{N} w'_n \cdot \hat{u}_n(t) + w_{err}(t)\right.$$

$$= \left(\sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t), \sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t)\right) +$$

$$+ \left(\sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t), w_{err}(t)\right) +$$

$$\left(w_{err}(t), \sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t)\right) + \left(w_{err}(t), w_{err}(t)\right)$$

It is easy to see that:

$$\left(\sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t), w_{err}(t)\right) = 0$$

$$\left(w_{err}(t), \sum_{n=1}^{N}\left(w_n - w'_n\right) \cdot \hat{u}_n(t)\right) = 0$$

because of Theorem 1.

Then of course: $\left(w_{err}(t), w_{err}(t)\right) = \mathcal{E}\left\{w_{err}(t)\right\}$

Finally:

$$\left(\sum_{n=1}^{N}\left(w_n - w_n'\right)\cdot \hat{u}_n(t), \sum_{m=1}^{N}\left(w_m - w_m'\right)\cdot \hat{u}_m(t)\right)$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \left(w_n - w_n'\right)\left(w_m - w_m'\right)^* \left(\hat{u}_n(t), \hat{u}_m(t)\right)$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \left(w_n - w_n'\right)\left(w_m - w_m'\right)^* \delta_{nm}$$

$$= \sum_{n=1}^{N}\left|w_n - w_n'\right|^2$$

So, putting everything together:

$$\Delta^2\left\{w(t), w'_{app}(t)\right\} = \sum_{n=1}^{N}\left|w_n - w_n'\right|^2 + \mathcal{E}\left\{w_{err}(t)\right\}$$

We now try to minimize this distance squared, to find the closest element of $\Sigma' = \text{span}\{\mathcal{U}\}$ to the original signal $w(t)$. We cannot do anything about $\mathcal{E}\{w_{err}(t)\}$ which, given $w(t)$ and given $\mathcal{U}$ is fixed.

We can therefore try to minimize: $\sum_{n=1}^{N} |w_n - w'_n|^2$.

However, it is quite obvious that the minimum is found for $w_n = w'_n$, that is for $w'_{app}(t) = w_{app}(t)$. In other words, indeed:

among all possible elements $w'_{app}(t) \in \text{span}\{\mathcal{U}\}$ ***the one that is closest to*** $w(t)$ ***is the canonical approximant*** $w_{app}(t)$***;***

equivalently, we can say that the canonical approximant $w_{app}(t)$ ***is the element in*** $\text{span}\{\mathcal{U}\}$ ***that minimizes the energy of the error.***

End of optional material.

Notice that in practice other criteria of "optimality" may be used, leading to different approximations. One example is audio signals, where certain compression algorithms may generate approximation signals whose error energy or error norm can be *larger* than the one of a canonical approximation.

The key there is that the larger generated error signal consists of sound that the human ear is relatively unable to perceive, so that although such error is larger in energetic terms, it may be less noticeable than the one generated by a canonical approximation.

This matter is of course very complex, but MP3 is an example of an approximation algorithm that relies heavily on the inability of the human ear to perceive certain kinds of 'errors' in the reproduced signal.

4.6.2.4 A visual example in $\mathbb{R}^2$

We look at a vector $\overline{v} \in \mathbb{R}^2$, that is a vector in a conventional Euclidean two-dimensional real space (a plane).

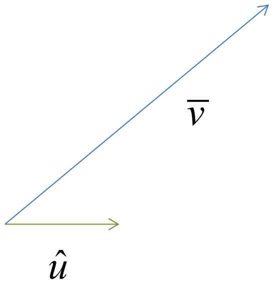


Fig. 4-3

We decide to use an ONS in $\mathbb{R}^2$ which consists of only one unit vector instead of two. That is: $\mathcal{U} = \{\hat{u}\}$. We try to obtain the best approximation of $\overline{v}$:

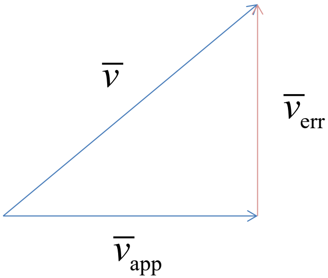


Fig. 4-4

$$\overline{v} \cong \overline{v}_{app} = (\overline{v} \cdot \hat{u}) \cdot \hat{u} = v_u \cdot \hat{u}$$

$$\overline{v} = \overline{v}_{app} + \overline{v}_{err}$$

Now by carrying out the same process as previously done, we try to find a v_u different from the one that we found with the canonical projection and we try to see if the approximation improves or it becomes worse:

$$\overline{v}'_{app} = v'_u \cdot \hat{u}$$

where v' is a number that changes the length of the approximation vector.

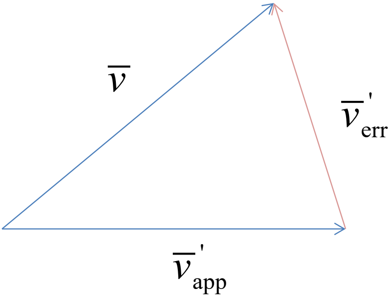


Fig. 4-5

By choosing a different approximation vector we can visually see that the error increases: $\left|\overline{v}'_{err}\right| > \left|\overline{v}_{err}\right|$

We now try with an approximation vector shorter than the projection we have used before:

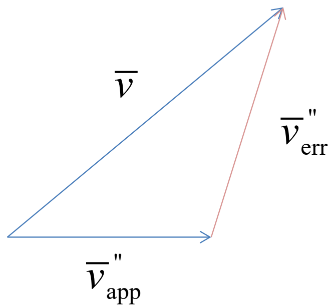
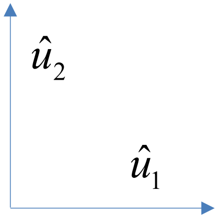


Fig. 4-6

$$\overline{v}''_{app} = v''_u \cdot \hat{u}$$

Also in this case we can visually see that the error is increased: $\left|\overline{v}''_{err}\right| > \left|\overline{v}_{err}\right|$

$\boxed{\text{On your own}}$: redo this visual example by looking at vectors in $\mathbb{R}^3$ and choosing as ONS the one given below:



that is, take $\mathcal{U} = \{\hat{u}_n\}_{n=1}^{2}$. Then visually show that given any vector coming out of the plane of the slide is approximated with an error and that such error is minimum if the canonical approximation is used.

4.6.3 Example problems

4.6.3.1 $\boxed{\text{on your own}}$ problem 1

Given the set of signals:

$$\mathcal{U} = \left\{ \hat{u}_n(t) \right\}_{n=1}^{10}$$

Eq. 4-27

with $t \in \mathbf{I} = \left[0, T_0 \right]$, where:

$$\hat{u}_n(t) = A_n \cos\left(\frac{2\pi}{T_0} nt + \phi \right)$$

Eq. 4-28

we want to calculate the values of φ and A_n so that $\mathcal{U}$ is an orthonormal set, that is:

$$(\hat{u}_n(t), \hat{u}_m(t)) = \delta_{nm}$$

Eq. 4-29

We first calculate the inner product between any two $\hat{u}_n(t)$ and $\hat{u}_m(t)$:

$$(\hat{u}_n(t), \hat{u}_m(t)) = \int_0^{T_0} A_n \cos\left(\frac{2\pi}{T_0} nt + \phi\right) A_m^* \cos\left(\frac{2\pi}{T_0} mt + \phi\right) dt$$

Eq. 4-30

We then use the following trigonometric identity:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$

and therefore get:

$$(\hat{u}_n(t), \hat{u}_m(t)) =$$

$$\frac{1}{2} A_n A_m^* \int_0^{T_0} \left[\cos\left(\frac{2\pi}{T_0} n \cdot t + \phi + \frac{2\pi}{T_0} m \cdot t + \phi \right) + \cos\left(\frac{2\pi}{T_0} n \cdot t + \phi - \frac{2\pi}{T_0} m \cdot t - \phi \right) \right] dt$$

$$\frac{1}{2} A_n A_m^* \int_0^{T_0} \left[\cos\left(\frac{2\pi}{T_0} [n+m] t + 2\phi \right) + \cos\left(\frac{2\pi}{T_0} [n-m] t \right) \right] dt$$

$$= \frac{1}{2} A_n A_m^* \int_0^{T_0} \cos\left(\frac{2\pi}{T_0} [n+m] t + 2\phi \right) dt + \frac{1}{2} A_n A_m^* \int_0^{T_0} \cos\left(\frac{2\pi}{T_0} [n-m] t \right) dt$$

The first integral in the right-hand side can be solved as follows:

$$\int_0^{T_0} \cos\left(\frac{2\pi}{T_0}[n+m]t + 2\varphi\right) dt =$$

$$= \frac{1}{\frac{2\pi}{T_0}[n+m]} \sin\left(\frac{2\pi}{T_0}[n+m]t + 2\varphi\right)\Bigg|_0^{T_0}$$

$$= \frac{1}{\frac{2\pi}{T_0}[n+m]} \left[\sin\left(2\pi[n+m] + 2\varphi\right) - \sin\left(2\varphi\right)\right]$$

$$= \frac{1}{\frac{2\pi}{T}[n+m]} \left[\sin\left(2\varphi\right) - \sin\left(2\varphi\right)\right] = 0$$

The result could be *foreseen* because the integral of a circular function (sine, cosine, complex exponential) calculated over a period, or over an exact multiple of the period, is always null.

The same result can be obtained for the second term of the sum, that is:

$$\frac{1}{2} A_n A_m^* \int_0^{T_0} \cos\left(\frac{2\pi}{T_0} [n-m] t \right) dt = 0 \qquad n \neq m$$

So, in summary, if $n \neq m$ we get:

$$\left(\hat{u}_n(t), \hat{u}_m(t) \right) = \frac{1}{2} A_n A_m^* (0+0) = 0$$

Vice versa if $n = m$ we get :

$$\left(\hat{u}_n(t),\hat{u}_n(t)\right)=\int_0^{T_0}\left|A_n\right|^2\cos^2\left(\frac{2\pi}{T_0}nt+\phi\right)dt=$$

$$=\frac{1}{2}\left|A_n\right|^2\int_0^{T_0}dt+\frac{1}{2}\left|A_n\right|^2\int_0^{T_0}\cos\left(\frac{4\pi}{T_0}nt+2\phi\right)dt=$$

$$=\frac{\left|A_n\right|^2 T_0}{2}+0=\frac{\left|A_n\right|^2 T_0}{2}$$

where we have used: $\cos^2(\alpha)=1/2+1/2\cdot\cos(2\alpha)$.

Therefore, to ensure normalization, i.e., unit norm, it must be:

$$\frac{\left|A_n\right|^2 T_0}{2}=1$$

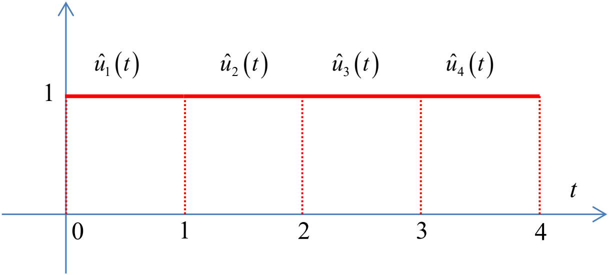
that is: $\left|A_n\right| = \sqrt{\dfrac{2}{T_0}}$.

To conclude, $\mathcal{U}$ is an ONS for *any* value of ϕ and for $\left|A_n\right| = \sqrt{\dfrac{2}{T_0}}$.

4.6.3.2 problem 2

We use the ONS over the interval $t \in \mathbf{I} = [0,4]$:

$$\mathcal{U} = \{\hat{u}_n\}_{n=1}^{N=4}, \quad \hat{u}_n \equiv \Pi\left(t-n+1/2\right) = \pi\left(t-[n-1]\right)$$



Given the signal:

$$v(t) = 1 - \frac{t}{4}$$

with $t \in \mathbf{I} = [0, 4]$, shown in Fig. 4.6, $v(t)$ is approximated through the signal $v_{app}(t)$ defined as:

$$v_{app}(t) = \sum_{n=1}^{4} v_n \hat{u}_n(t)$$

<u>We want to find the n-tuple representation of</u> $v_{app}(t)$:

$$v_{app}(t) \equiv [v_1, v_2, v_3, v_4]$$

that is, we want to calculate the coefficients v_n which represent it.

We also want to find the error signal and we want to compare the energy of the error with the energy of the approximant and the energy of the original signal.

Solution

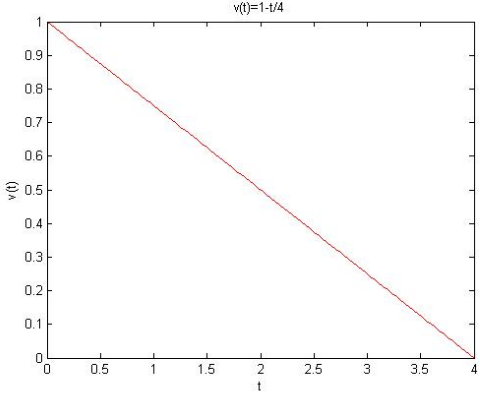


Fig. 4-7: the signal to be approximated

Each v_n is defined as the inner product between $v(t)$ and $\hat{u}_n(t)$, so the various v_n can be calculated as:

$$v_1 = \left(v(t), \hat{u}_1(t)\right) = \int_0^4 \left(1 - \frac{t}{4}\right) \hat{u}_1(t)\, dt$$

$$= \int_0^1 \left(1 - \frac{t}{4}\right) dt = \left[t - \frac{t^2}{8}\right]_0^1 = 1 - \frac{1}{8} = \frac{7}{8}$$

$$v_2 = \left(v(t), \hat{u}_2(t)\right) = \int_0^4 \left(1 - \frac{t}{4}\right) \hat{u}_2(t)\, dt$$

$$= \int_1^2 \left(1 - \frac{t}{4}\right) dt = \left[t - \frac{t^2}{8}\right]_1^2 = 2 - \frac{4}{8} - 1 + \frac{1}{8} = \frac{5}{8}$$

$$v_3 = \left(v(t), \hat{u}_3(t)\right) = \int_0^4 \left(1 - \frac{t}{4}\right) \hat{u}_3(t)\, dt$$

$$= \int_2^3 \left(1 - \frac{t}{4}\right) dt = \left[t - \frac{t^2}{8}\right]_2^3 = 3 - \frac{9}{8} - 2 + \frac{4}{8} = \frac{3}{8}$$

$$v_4 = \left(v(t), \hat{u}_4(t)\right) = \int_0^4 \left(1 - \frac{t}{4}\right) \hat{u}_4(t)\, dt$$

$$= \int_3^4 \left(1 - \frac{t}{4}\right) dt = \left[t - \frac{t^2}{8}\right]_3^4 = 4 - \frac{16}{8} - 3 + \frac{9}{8} = \frac{1}{8}$$

It can also be calculated in general, for any n, and it is easy to see that the same results as above are obtained (check it on your own, solution below):

$$v_n = \left(v(t), \hat{u}_n(t)\right) = \int_0^4 \left(1 - \frac{t}{4}\right) \hat{u}_n^*(t)\, dt =$$

$$\int_0^4 \left(1 - \frac{t}{4}\right) \pi\left(t - [n-1]\right) dt = \int_{n-1}^{n} \left(1 - \frac{t}{4}\right) dt = \left[t - \frac{t^2}{8}\right]_{n-1}^{n} =$$

$$n - (n-1) - \frac{1}{8}\left[n^2 - (n-1)^2\right] = 1 + \frac{1}{8} - \frac{n}{4} = \frac{9-2n}{8}$$

As a result $v_{app}(t)$ in n-tuple form will be:

$$v_{app}(t) \equiv \overline{v}_{app} \equiv \left[\frac{7}{8}, \frac{5}{8}, \frac{3}{8}, \frac{1}{8}\right]$$

If we want to calculate the energy of the signal $v(t)$ we proceed as follows:

$$\mathcal{E}\left\{1-\frac{t}{4}\right\}=\left(1-\frac{t}{4},1-\frac{t}{4}\right)=\int_0^4\left(1-\frac{t}{4}\right)^2 dt=\int_0^4\left(1+\frac{t^2}{16}-\frac{t}{2}\right)dt=$$

$$=\left(t+\frac{t^3}{48}-\frac{t^2}{4}\right)\Bigg|_0^4=4+\frac{64}{48}-\frac{16}{4}=\frac{64}{48}=\frac{4}{3}$$

From Theorem 3 we learn that this energy is equal to the sum of the energy of the approximated signal $v_{app}(t)$ and the error $v_{err}(t)$. If we calculate the energy of the approximated signal $v_{app}(t)$ we can therefore obtain the energy of the error.

We use Parseval's rule:

$$\mathcal{E}\left\{v_{app}(t)\right\}=\sum_{n=1}^{4}v_n^2=\frac{49}{64}+\frac{25}{64}+\frac{9}{64}+\frac{1}{64}=\frac{84}{64}=\frac{21}{16}$$

We then calculate the difference between the energy of the signal and the one of its approximation, to find the energy of the error:

$$\mathcal{E}\{v_{err}(t)\} = \frac{4}{3} - \frac{21}{16} = \frac{64-63}{48} = \frac{1}{48}$$

Note that the ratio between the energy of the error and the energy of the signal is quite small:

$$\frac{\mathcal{E}\{v_{err}(t)\}}{\mathcal{E}\{v(t)\}} = \frac{1/48}{64/48} = \frac{1}{64} \cong 0{,}016 = 1.6\%$$

We plot the signals $v_{app}(t)$ and $v_{err}(t)$:

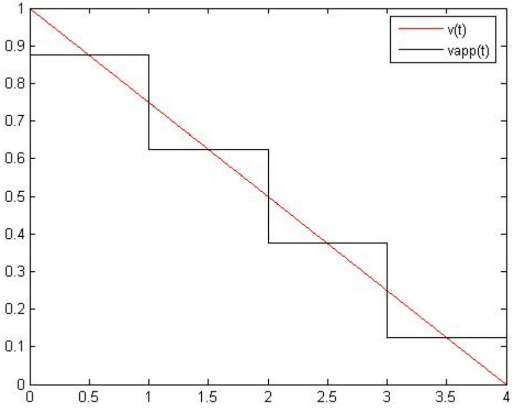


Fig. 4-8: the signal and its approximation

As the chart shows, despite the fact that the ONS used for the calculation is rather elementary, and clearly insufficient to reproduce the signal without error, the energy of the error is low compared to the energy of the signal. In fact the energy of the error corresponds to only 1.6% of the energy of the signal.

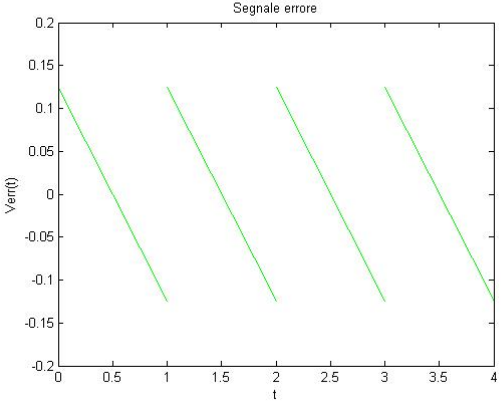


Fig. 4-9: the error signal

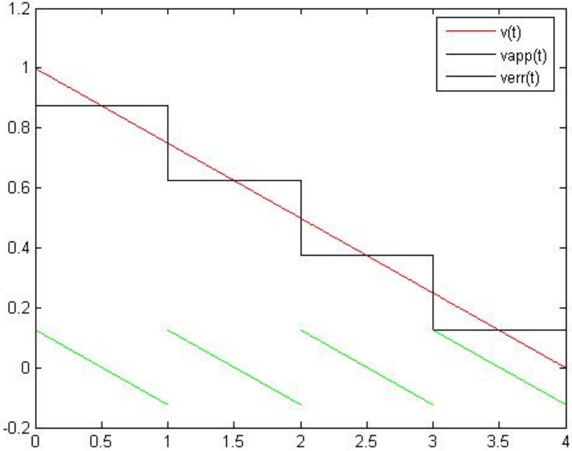


Fig. 4-10: all the signals together

$\boxed{\text{On your own}}$: check that the error signal in the previous exercise is orthogonal to the approximant.

$\boxed{\text{On your own}}$: redo the same problem as above, assuming as orthogonal set:

$$\mathcal{U} = \{\overline{u}_n\}_{n=1}^{8}, \quad \overline{u}_n \equiv \Pi_{1/2}(t - n/2 + 1/4)$$

Note that the orthogonal functions need to be *normalized* first because their norm is no longer 1.

What is the new error energy? Can you guess what would happen to the error energy if you used the orthonormal set derived from the orthogonal set:

$$\mathcal{U} = \{\overline{u}_n\}_{n=1}^{16}, \quad \overline{u}_n \equiv \Pi_{1/4}(t - n/4 + 1/8)$$

Can you guess what happens in general when doubling the number of rectangular elements in the ONS?

Answers: The energy of the error goes down to about 0.4% when using an 8-signal ONS and to 0.1% when using a 16-signal ONS.

In general, *when doubling the number of rectangular unit-norm signals in the ONS, the energy of the error signal decreases by a factor of 4*.

$\boxed{\text{On your own}}$: redo the same problem, assuming as ONS the following:

$$\mathcal{U} = \{\hat{u}_n\}_{n=1}^{4}, \quad \hat{u}_n \equiv \Pi(t - n + 1/2)$$

and as signals to approximate, the following:

$$s(t) = e^{-t/2} \cdot u(t)$$

$$s(t) = \sin\left(\frac{\pi}{4} t\right)$$ (optional, because of lengthier calculations)

Suggestion: start out by drawing $s(t)$. Also, do the elements of $\mathcal{U}$ need to be normalized? Highly recommended: write a piece of Matlab code that does the calculations.

4.6.4 Extracting an orthonormal set from a generic set of elements

Given a generic set of elements belonging to an inner product space $\mathcal{P}$:

$$W = \left\{ \overline{w}_n \right\}_{n=1}^{N} \qquad W \subset \mathcal{P}$$

we want to extract an orthonormal set $\mathcal{U} = \left\{ \hat{u}_m \right\}_{m=1}^{M}$ out of W, that must have a ***special feature***: *given any element* $\overline{w}_n \in W$ *then it must be possible to express it as*:

$$\bar{w}_n = \sum_{n=1}^{N} w_{nm}\,\hat{u}_m \quad , \quad w_{nm} = \left(\bar{w}_n, \hat{u}_m\right)$$

with no error.

The procedure to follow is called **the Gram-Schmidt algorithm**. In the following we introduce it.

We initially assume that all the elements in W are *linearly independent* of one another, that is none of them can be represented exactly as a linear combination of the others. Then we generalize.

Step1

Pick one of the elements in W, for instance $\bar{w}_1$ (but it could be any other). Then use it as the first element of $\mathcal{U}$, after proper normalization:

$$\bar{u}_1 = \bar{w}_1 \qquad \rightarrow \qquad \hat{u}_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|} = \frac{\bar{u}_1}{\|\bar{w}_1\|}$$

Step 2

Pick another element of W, for instance $\bar{w}_2$. Then use it as the second element of $\mathcal{U}$, after subtracting any component it may have vs. $\hat{u}_1$ and after proper normalization:

$$\bar{u}_2 = \bar{w}_2 - \left(\bar{w}_2, \hat{u}_1\right)\hat{u}_1 \qquad \rightarrow \qquad \hat{u}_2 = \frac{\bar{u}_2}{\|\bar{u}_2\|}$$

Regarding the normalization factor $\|\bar{u}_2\|$, by direct calculation it can be found:

$$\left\|\overline{u}_2\right\|^2 = \left(\overline{u}_2, \overline{u}_2\right)$$

$$= \left(\overline{w}_2 - \left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1 \;,\; \overline{w}_2 - \left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1\right) =$$

$$= \left(\overline{w}_2, \overline{w}_2\right) - \left(\overline{w}_2 \;,\left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1\right) - \left(\left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1 \;,\; \overline{w}_2\right) + \left(\left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1 \;,\left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1\right) =$$

$$= \left\|\overline{w}_2\right\|^2 - \left(\overline{w}_2, \hat{u}_1\right)\left(\overline{w}_2, \hat{u}_1\right)^* - \left(\overline{w}_2, \hat{u}_1\right)\left(\hat{u}_1, \overline{w}_2\right) + \left(\overline{w}_2, \hat{u}_1\right)\left(\overline{w}_2, \hat{u}_1\right)^*\left(\hat{u}_1, \hat{u}_1\right)$$

$$= \left\|\overline{w}_2\right\|^2 + \left|\left(\overline{w}_2, \hat{u}_1\right)\right|^2 - \left|\left(\overline{w}_2, \hat{u}_1\right)\right|^2 - \left|\left(\overline{w}_2, \hat{u}_1\right)\right|^2 = \left\|\overline{w}_2\right\|^2 - \left|\left(\overline{w}_2, \hat{u}_1\right)\right|^2$$

The key question to ask at this point is whether this new element is *orthogonal* to the previous one, that is whether $\hat{u}_2 \perp \hat{u}_1$.

We can directly check:

$$
\begin{aligned}
\left(\hat{u}_2,\hat{u}_1\right) &= \left(\left\|\overline{u}_2\right\|\hat{u}_2,\hat{u}_1\right)\cdot\left\|\overline{u}_2\right\|^{-1} = \left(\overline{u}_2,\hat{u}_1\right)\cdot\left\|\overline{u}_2\right\|^{-1} \\
&= \left(\overline{w}_2 - \left(\overline{w}_2,\hat{u}_1\right)\hat{u}_1,\hat{u}_1\right)\cdot\left\|\overline{u}_2\right\|^{-1} \\
&= \left[\left(\overline{w}_2,\hat{u}_1\right) - \left(\overline{w}_2,\hat{u}_1\right)\left(\hat{u}_1,\hat{u}_1\right)\right]\cdot\left\|\overline{u}_2\right\|^{-1} \\
&= \left[\left(\overline{w}_2,\hat{u}_1\right) - \left(\overline{w}_2,\hat{u}_1\right)\right]\cdot\left\|\overline{u}_2\right\|^{-1} = 0
\end{aligned}
$$

In other words, the subtraction from $\overline{w}_2$ of its component vs. $\hat{u}_1$ ensures that $\overline{u}_2 \perp \hat{u}_1$ and therefore $\hat{u}_2 \perp \hat{u}_1$.

Each of the further steps generates one more element of $\mathcal{U}$, so that after K steps there are K unit elements in $\mathcal{U}$.

Step K

The generic step procedure can be expressed in a compact way as follows. Assuming we have completed the $(K-1)$-th step, then pick element $\overline{w}_K$ from W and do:

$$\overline{u}_K = \overline{w}_K - \sum_{k=1}^{K-1} \left(\overline{w}_K, \hat{u}_k\right)\hat{u}_k \qquad \rightarrow \qquad \hat{u}_K = \frac{\overline{u}_K}{\left\|\overline{u}_K\right\|}$$

with:

$$\left\|\overline{u}_K\right\|^2 = \left\|\overline{w}_K\right\|^2 - \sum_{k=1}^{K-1} \left|\left(\overline{w}_K, \hat{u}_k\right)\right|^2$$

This generates the K-th orthogonal and unit-norm element $\overline{u}_K$.

Note that, again, *to find* $\overline{u}_K$ *it is necessary to subtract from* $\overline{w}_K$ *the components that* $\overline{w}_K$ *has vs. all the previously found* $K-1$ *unit elements in* $\mathcal{U}$. This ensures that $\hat{u}_K$ be orthogonal to all previously found unit elements $\hat{u}_{K-1}, \hat{u}_{K-2}, ..., \hat{u}_1$.

$\boxed{\text{On your own}}$: prove the orthogonality of $\hat{u}_K$ vs. all previous K-1 orthonormal elements, by generalizing the proof given in step 2 about $\hat{u}_2 \perp \hat{u}_1$.

After N steps, all the N elements of W have been used and N orthonormal unit elements are found in $\mathcal{U}$.

Optional ***Mini-Challenge:***

How can it be immediately shown that $\overline{w}_K$ can be expressed exactly as a linear combination of the unit elements generated thus far ? That is, of the unit elements: $\hat{u}_1, \hat{u}_2, \hat{u}_3 ..., \hat{u}_{K-1}, \hat{u}_K$

Or, in other words, can it be immediately and easily shown that:

$$\overline{w}_K = \sum_{k=1}^{K} \left(\overline{w}_K , \hat{u}_k \right) \hat{u}_k \qquad ?$$

(with no error)

Case of W containing linearly dependent elements

We now remove the assumption that the elements of W are linearly independent. In other words, it may happen that an element $\overline{w}_k$ *is a linear combination of the other elements of* W :

$$\overline{w}_k = \sum_{\substack{n=1 \\ n \neq k}}^{N} a_n \, \overline{w}_n$$

If so, then it is easy to show that one or more steps of the previous procedure, except of course the first, end up producing a zero element. That is, for some K, the result is zero, or more precisely, the zero element:

$$\overline{u}_K = \overline{w}_K - \sum_{k=1}^{K-1} \left(\overline{w}_K, \hat{u}_k\right)\hat{u}_k = \overline{0}$$

This simply means that $\overline{w}_K$ is a linear combination of the first $K-1$ unit elements $\left\{\hat{u}_k\right\}_{k=1}^{K-1}$. However, this is not a problem. When a linearly dependent element is found, it is sufficient to remove it from W and continue the procedure by picking the next element in W.

The only consequence of the presence of linearly dependent elements in W is that the total number of orthonormal elements that can be extracted out of it is going to be less than the total number of elements in W. That is:

$$W = \left\{\overline{w}_n\right\}_{n=1}^{N} \quad \Rightarrow \quad \mathcal{U} = \left\{\hat{u}_m\right\}_{m=1}^{M}$$

with $M < N$.

4.6.4.2 Example

Let's consider the time interval $t \in [0,3]$ and the set of signals:

$$W = \left\{ \overline{w}_n \right\}_{n=1}^{4}$$

with:

$$\begin{aligned}
w_1(t) &= \Pi_1\left(t - \tfrac{1}{2}\right) \\
w_2(t) &= \Pi_2\left(t - 1\right) \\
w_3(t) &= \Pi_3\left(t - \tfrac{3}{2}\right) \\
w_4(t) &= \Pi_1\left(t - \tfrac{1}{2}\right) - \Pi_1\left(t - \tfrac{3}{2}\right) + \Pi_1\left(t - \tfrac{5}{2}\right)
\end{aligned}$$

We want to find an orthonormal set derived from W.

Solution:

The first step is:

$$\overline{u}_1 = \overline{w}_1 \qquad \rightarrow \qquad \hat{u}_1 = \frac{\overline{u}_1}{\left\|\overline{u}_1\right\|} = \frac{\overline{u}_1}{\left\|\overline{w}_1\right\|}$$

So we need to find:

$$\left\|w_1(t)\right\|^2 = \int_0^3 \Pi_1\left(t - \tfrac{1}{2}\right)\Pi_1^*\left(t - \tfrac{1}{2}\right)dt = \int_0^1 \left|\Pi_1\left(t - \tfrac{1}{2}\right)\right|^2 dt = \int_0^1 1(t)\,dt = 1$$

Therefore:

$$\hat{u}_1 = \overline{w}_1 = \Pi_1\left(t - \tfrac{1}{2}\right)$$

The second step is:

$$\overline{u}_2 = \overline{w}_2 - \left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1 \qquad \rightarrow \qquad \hat{u}_2 = \frac{\overline{u}_2}{\left\|\overline{u}_2\right\|}$$

So we need to calculate two unknown quantities: $\left(\overline{w}_2,\hat{u}_1\right)$ and $\left\|\overline{u}_2\right\|$.

$$\left(\overline{w}_2,\hat{u}_1\right)=\int_0^3 \Pi_2\left(t-1\right)\Pi_1^*\left(t-\tfrac{1}{2}\right)dt=\int_0^1 \Pi_2\left(t-1\right)\Pi_1^*\left(t-\tfrac{1}{2}\right)dt=\int_0^1 1\left(t\right)dt=1$$

As for $\left\|\overline{u}_2\right\|$, we can find it as: $\left\|\overline{u}_2\right\|^2=\left\|\overline{w}_2\right\|^2-\left|\left(\overline{w}_2,\hat{u}_1\right)\right|^2$.

But we already know $\left(\overline{w}_2,\hat{u}_1\right)$. So what we actually need is:

$$\left\|\overline{w}_2\right\|^2=\left(\overline{w}_2,\overline{w}_2\right)=\int_0^3 \Pi_2\left(t-1\right)\Pi_2^*\left(t-1\right)dt=\int_0^2 \left\|\Pi_2\left(t-1\right)\right\|^2 dt=\int_0^2 1\left(t\right)dt=2$$

So we have:

$$\left\|\overline{u}_2\right\|^2=\left\|\overline{w}_2\right\|^2-\left|\left(\overline{w}_2,\hat{u}_1\right)\right|^2=2-1=1$$

We can now put everything together:

$$\overline{u}_2 = \overline{w}_2 - \left(\overline{w}_2, \hat{u}_1\right)\hat{u}_1 = \Pi_2\left(t-1\right) - 1 \cdot \Pi_1\left(t - \tfrac{1}{2}\right)$$

$$\hat{u}_2 = \frac{\overline{u}_2}{\left\|\overline{u}_2\right\|} = \frac{\Pi_2\left(t-1\right) - 1 \cdot \Pi_1\left(t - \tfrac{1}{2}\right)}{1} = \Pi_2\left(t-1\right) - 1 \cdot \Pi_1\left(t - \tfrac{1}{2}\right) = \Pi_1\left(t - \tfrac{3}{2}\right)$$

The third step is:

$$\overline{u}_3 = \overline{w}_3 - \left(\overline{w}_3, \hat{u}_1\right)\hat{u}_1 - \left(\overline{w}_3, \hat{u}_2\right)\hat{u}_2 \qquad \to \qquad \hat{u}_3 = \frac{\overline{u}_3}{\left\|\overline{u}_3\right\|}$$

So we need to calculate three unknown quantities: $\left(\overline{w}_3, \hat{u}_1\right)$, $\left(\overline{w}_3, \hat{u}_2\right)$ and $\left\|\overline{u}_3\right\|$.

$$\left(\overline{w}_3,\hat{u}_1\right)=\int_0^3 \Pi_3\left(t-\tfrac{3}{2}\right)\Pi_1^*\left(t-\tfrac{1}{2}\right)dt=\int_0^1 \Pi_3\left(t-\tfrac{3}{2}\right)\Pi_1^*\left(t-\tfrac{1}{2}\right)dt=\int_0^1 1\left(t\right)dt=1$$

$$\left(\overline{w}_3,\hat{u}_2\right)=\int_0^3 \Pi_3\left(t-\tfrac{3}{2}\right)\Pi_1^*\left(t-\tfrac{3}{2}\right)dt=\int_1^2 \Pi_3\left(t-\tfrac{3}{2}\right)\Pi_1^*\left(t-\tfrac{3}{2}\right)dt=\int_1^2 1\left(t\right)dt=1$$

As for $\left\|\overline{u}_2\right\|$, we can find it as: $\left\|\overline{u}_3\right\|^2=\left\|\overline{w}_3\right\|^2-\left|\left(\overline{w}_3,\hat{u}_1\right)\right|^2-\left|\left(\overline{w}_3,\hat{u}_2\right)\right|^2$.

But we already know $\left(\overline{w}_3,\hat{u}_1\right)$ and $\left(\overline{w}_3,\hat{u}_2\right)$. So what we actually need is:

$$\left\|\overline{w}_3\right\|^2=\left(\overline{w}_3,\overline{w}_3\right)=\int_0^3 \Pi_3\left(t-\tfrac{3}{2}\right)\Pi_3^*\left(t-\tfrac{3}{2}\right)dt=\int_0^3 \left\|\Pi_3\left(t-\tfrac{3}{2}\right)\right\|^2 dt=\int_0^3 1\left(t\right)dt=3$$

So we have:

$$\left\|\overline{u}_3\right\|^2 = \left\|\overline{w}_3\right\|^2 - \left|\left(\overline{w}_3, \hat{u}_1\right)\right|^2 - \left|\left(\overline{w}_3, \hat{u}_2\right)\right|^2 = 3 - 1 - 1 = 1$$

We can now put everything together:

$$\overline{u}_3 = \overline{w}_3 - \left(\overline{w}_3, \hat{u}_1\right)\hat{u}_1 - \left(\overline{w}_3, \hat{u}_2\right)\hat{u}_2 = \Pi_1\left(t - \tfrac{3}{2}\right) - 1 \cdot \Pi_1\left(t - \tfrac{1}{2}\right) - 1 \cdot \Pi_1\left(t - \tfrac{3}{2}\right) = \Pi_1\left(t - \tfrac{5}{2}\right)$$

$$\hat{u}_3 = \frac{\overline{u}_3}{\left\|\overline{u}_3\right\|} = \frac{\Pi_1\left(t - \tfrac{5}{2}\right)}{1} = \Pi_1\left(t - \tfrac{5}{2}\right)$$

The fourth step does not contribute any new element because it is easy to see that $w_4\left(t\right)$ is a linear combination of $\left\{\hat{u}_n\right\}_{n=1}^{3}$, that is, when we calculate:

$$\overline{u}_4 = \overline{w}_4 - \sum_{k=1}^{3} \left(\overline{w}_4, \hat{u}_k\right)\hat{u}_k = \overline{0}$$

Verify this simple result $\boxed{\text{on your own}}$.

<u>*In conclusion*</u>, considering the time interval $t \in [0,3]$ and the set of signals:

$$W = \left\{\overline{w}_n\right\}_{n=1}^{4}$$

the found ONS is:

$$U = \left\{\hat{u}_n\right\}_{n=1}^{3}$$

with:

$$\begin{aligned}
\hat{u}_1(t) &= \Pi_1\left(t - \tfrac{1}{2}\right) \\
\hat{u}_2(t) &= \Pi_1\left(t - \tfrac{3}{2}\right) \\
\hat{u}_3(t) &= \Pi_1\left(t - \tfrac{5}{2}\right)
\end{aligned}$$

Note again that, although the set $W = \left\{ \overline{w}_n \right\}_{n=1}^{4}$ contains four elements, only three of them are linearly independent. As a result, the ONS U consists of only three elements.

We can now check whether indeed we can express all elements of the original set $W = \left\{ \overline{w}_n \right\}_{n=1}^{4}$ in terms of the found ONS $U = \left\{ \hat{u}_n \right\}_{n=1}^{3}$.

$\boxed{\text{On your own:}}$ It is easy to see that by taking the "canonical approximation" of each one of the $\overline{w}_n$ using U, no error is incurred and the corresponding arrays of components are given by:

$$
\begin{aligned}
w_1(t) &= \Pi_1\left(t - \tfrac{1}{2}\right) \equiv [1,0,0] \\
w_2(t) &= \Pi_2\left(t - 1\right) \equiv [1,1,0] \\
w_3(t) &= \Pi_3\left(t - \tfrac{3}{2}\right) \equiv [1,1,1] \\
w_4(t) &= \Pi_1\left(t - \tfrac{1}{2}\right) - \Pi_1\left(t - \tfrac{3}{2}\right) + \Pi_1\left(t - \tfrac{5}{2}\right) \equiv [1,-1,1]
\end{aligned}
$$

Prove this on your own.

4.6.4.3 On your own

Given the following set of signals over $t \in [0,2]$:

$$W = \{\overline{w}_n\}_{n=1}^{N=3}$$

$$= \{\Pi_2(t-1)\,,\,\Pi_1(t-3/2)\,,\,-2\Pi_1(t-1/2)+3\Pi_1(t-3/2)\}$$

extract an orthonormal set from it, which can represent all of the elements in W with no error.

Result:

Applying Gram-Schmidt in order on the elements of W, we get from the first two steps:

$$U = \{\hat{u}_n\}_{n=1}^{2} = \left\{ \frac{1}{\sqrt{2}} \Pi_2(t-1), \frac{1}{\sqrt{2}} \left[\Pi_1(t-3/2) - \Pi_1(t-1/2) \right] \right\}$$

The third step of Gram-Schmidt produces zero, meaning that the third element of W is actually a linear combination of the previous two.

4.6.4.4 $\boxed{\textbf{On your own}}$

Given the following set of signals over $t \in [-1,1]$:

$$W = \{\overline{w}_n\}_{n=1}^{3} = \{1(t)\ ,\ t\ ,\ t^2\}$$

extract an orthonormal set from it, which can represent all of the elements in W with no error.

Answer

$$U = \left\{ \hat{u}_n \right\}_{n=1}^{3} = \left\{ \frac{1(t)}{\sqrt{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{5}{8}}\left(3t^2 - 1\right) \right\}$$

4.7 Infinite-Dimensional Inner Product Spaces

We remind the reader that a generic inner product space $\mathcal{P}$ is said to be N-**dimensional** if there is an orthonormal set $\mathcal{U} = \left\{ \hat{u} \right\}_{n=1}^{N}$ which is *orthonormal basis* for $\mathcal{P}$, i.e., it is such that it can generate all $\overline{x} \in \mathcal{P}$.

Signal inner-product spaces contain quite complicated elements (the signals) and thus it is reasonable to think that very large bases may be needed to represent them.

In fact, for certain signal spaces, a basis can only be found if N is allowed to become *infinite*, that is, the basis is made up of an infinite number of orthonormal elements: $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$. In this case, the inner product space is said to be of *infinite dimension*.

One example of such infinite-dimension inner product space is *the set of all finite-energy signals over a finite interval* $L^2_{[t_0,t_1]}$.

As a reminder, for $s(t)$ to be in $L^2_{[t_0,t_1]}$ it must be:

$$\mathcal{E}_{[t_0,t_1]}\{s(t)\} = \int_{t_0}^{t_1} |s(t)|^2 < \infty$$

Theorems assure that $L^2_{[t_0,t_1]}$ does have a *basis*.

In fact, this space has an *infinite number of possible bases*. All of them have an infinite number of elements. In the next chapter we will specifically deal with one such basis, called the Fourier Basis.

Remarkably, the fact that an inner product space is infinite-dimensional does not invalidate any of the key properties that we have found in finite-dimensional spaces. In particular, given a basis $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$, any element $\overline{x}$ is fully represented by its projections onto the basis:

$$\overline{x} \equiv [x_1, x_2, x_3, x_4 ...]$$

in the sense that we can write it with no error:

$$\overline{x} = \sum_{n=1}^{\infty} x_n \hat{u}_n$$

Also, the inner-product between two elements $\overline{x}, \overline{y}$ can be calculated using the generalized rule:

$$\left(\overline{x}, \overline{y}\right) = \sum_{n=1}^{\infty} x_n y_m^*$$

Parseval's rule continues to hold:

$$\|\overline{x}\| = \sqrt{\left(\overline{x}, \overline{x}\right)} = \sqrt{\sum_{n=1}^{\infty} x_n x_n^*} = \sqrt{\sum_{n=1}^{\infty} \left|x_n\right|^2}$$

as well as the distance generalized formula:

$$\Delta\left\{\overline{x}, \overline{y}\right\} = \|\overline{x} - \overline{y}\| = \sqrt{\sum_{n=1}^{\infty} \left|x_n - y_n\right|^2}$$

The only caution to be used is that an element in $\mathcal{P}$ cannot have infinite norm (or equivalently infinite energy). Using Parseval's rule, this means that we accept only elements $\overline{x}$ such that:

$$\|\overline{x}\|^2 = \sum_{n=1}^{\infty} |x_n|^2 < \infty$$

One example of a infinite-dimension space and related basis is the inner product space $L^2_{\mathbf{I}}$ of the signals defined over the time-interval $\mathrm{I} = [-1,1]$ and the set of *Legendre polynomials*.

On your own:

Consider the set:

$$\mathcal{U} = \{u_0(t), u_1(t), u_2(t)\}$$

with:

$$u_0(t) = 1(t) \quad , \quad u_1(t) = t \quad , \quad u_2(t) = \frac{1}{2}\left(3t^2 - 1\right)$$

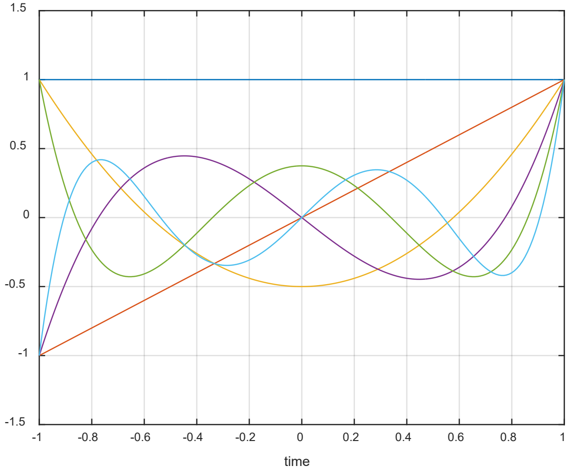
Verify that these three signals are all mutually orthogonal. Then turn $\mathcal{U}$ into an orthonormal set by finding the suitable normalization that makes them unit-norm elements.

Note: these signals are the three lowest-order ***Legendre polynomials***.

You can plot the Legendre polynomials of any order using these statements in *Matlab*:

```
% Plots the first six Legendre polynomials, from 0 to 5
N=5;
figure;
t=[-1:0.01:1];
for n=0:N
plot(t,legendreP(n,t)); hold on;
end;
   hold off; grid on; axis([-1 1 -1.5 1.5]);
xlabel('time');
```

This is what you get:



Answer: Legendre polynomials are all orthogonal to one another, for any order. As for normalization, the norm of the n-th Legendre polynomial has the general formula:

$$\left\| u_n(t) \right\| = \sqrt{2/(2n+1)}$$

4.7.1 Optional: Hilbert spaces

Hilbert spaces and inner product spaces are almost synonyms. All Hilbert spaces are inner product spaces. However, by saying that an inner product space is a Hilbert space too, we are stressing a property that Hilbert spaces must have by definition: *completeness*.

To properly deal with this concept we should introduce the so-called Cauchy sequences and other concepts, which we'll omit. However, in essence, a "complete" inner product space is such that it contains *all* those elements whose mutual distance is "infinitesimally" small, having defined distance as usual as:

$$\Delta\{\bar{x}, \bar{y}\} = \|\bar{x} - \bar{y}\|.$$

We state without proof the following very important result.

Optional Result:

$L^2_{[t_0, t_1]}$ is a complete inner product space, i.e., a Hilbert space.

An example of an incomplete inner product space is the set of all finite-energy *continuous* signals over an interval $[t_0, t_1]$, which we can call C. This is a legitimate inner product space. However, it can be shown that given a step-discontinuous signal $q(t)$ over $[t_0, t_1]$, such as for instance a rectangular signal, which is therefore not in C, the distance of such signal with respect to some of the signals in C can be made "infinitesimally" small. But $q(t)$ is not in C, so C does not contain all those elements whose mutual distance is infinitesimally small. Hence, C is not a Hilbert space. *End of optional material.*

Optional: verifying that an orthonormal set in $L^2_{\mathbf{I}}$ is a basis

We assume to be dealing with a Hilbert space of signals $L^2_{\mathbf{I}}$, with $\mathbf{I} = [t_1, t_2]$, $t_1, t_2 < \infty$

We also assume to have found an orthonormal set $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$ in $L^2_{\mathbf{I}}$. We want to find out whether such orthonormal set is a basis for $L^2_{\mathbf{I}}$.

To do so, we will try to find some conditions that, if verified, tell us that for sure $\mathcal{U}$ is basis.

We first recall that if $\mathcal{U}$ was a basis, then it must be, for any $s(t) \in L^2_{\mathbf{I}}$:

$$s(t) = \sum_{n=1}^{\infty} s_n \cdot \hat{u}_n(t) = \sum_{n=1}^{\infty} \left(s(t), \hat{u}_n(t) \right) \cdot \hat{u}_n(t)$$

$$= \sum_{n=1}^{\infty} \hat{u}_n(t) \int_{t_1}^{t_2} s(\tau)\, \hat{u}_n^*(\tau)\, d\tau = \int_{t_1}^{t_2} s(\tau) \left[\sum_{n=1}^{\infty} \hat{u}_n(t) \hat{u}_n^*(\tau) \right] d\tau$$

We then recall the well-known result:

$$\int_{-\infty}^{\infty} s(\tau)\,\delta\left(t-\tau\right) d\tau = s(t)$$

This result is also valid if the integration range is just $\left[t_1, t_2\right]$, provided that $t \in \left[t_1, t_2\right]$[2].
We can then rewrite the above formula as:

$$\int_{t_1}^{t_2} s(\tau)\ \delta\left(t-\tau\right) d\tau = s(t), \qquad t \in \left[t_1, t_2\right]$$

We then compare the previous two results:

[2] The range should actually be $t \in \left]t_1, t_2\right[$. It can be extended to the extremes using certain caution and assumptions. We skip these details which are unimportant in this context and simply refer to the overall interval, assuming that proper caution has been exercised.

$$s(t) = \int_{t_1}^{t_2} s(\tau) \left(\sum_{n=1}^{\infty} \hat{u}_n(t) \hat{u}_n^*(\tau) \right) d\tau$$

Eq. 4-31

$$s(t) = \int_{t_1}^{t_2} s(\tau) \ \delta(t-\tau) d\tau$$

Eq. 4-32

The comparison tells us that, for Eq. 4-31 to be verified, it must be:

$$\sum_{n=1}^{\infty} \hat{u}_n(t) \hat{u}_n^*(\tau) = \delta(t-\tau), \quad t, \tau \in [t_1, t_2]$$

If so, Eq. 4-31 is correct and therefore $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$ is a basis for $L_{\mathbf{I}}^2$. We can therefore state the very important result:

given a Hilbert space $L^2_{\mathbf{I}}$ *with* $\mathbf{I} = [t_1, t_2],\ t_1, t_2 < \infty$, *and given an orthonormal set* $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$ *in* $L^2_{\mathbf{I}}$, *such orthonormal set is a basis for* $L^2_{\mathbf{I}}$ *if and only if:*

$$\sum_{n=1}^{\infty} \hat{u}_n(t)\hat{u}_n^*(\tau) = \delta(t-\tau), \quad t, \tau \in [t_1, t_2]$$

Eq. 4-33

This result is sometimes re-phrased as follows: *an orthonormal set* $\mathcal{U} = \{\hat{u}\}_{n=1}^{\infty}$ *in* $L^2_{\mathbf{I}}$ *is a basis for* $L^2_{\mathbf{I}}$ *if it can "represent" Dirac's delta* (or it can "resolve" Dirac's delta) over the whole interval. *End of optional material.*

4.8 Questions

4.8.1

Let $t \in \mathbf{I} = [0,6]$.

Let $\mathcal{U}$ be an orthonormal set of signals over $\mathbf{I}$, defined as:

$$\mathcal{U} = \left\{ \hat{u}_n(t) \right\}_{n=0}^{5}$$

where:

$$\hat{u}_n(t) = \pi(t-n)$$

Consider the signals $s(t)$, $w(t)$:

$$s(t) = \sum_{n=0}^{5} s_n \, \hat{u}_n(t) \qquad\qquad w(t) = \sum_{n=0}^{5} w_n \, \hat{u}_n(t)$$

where:

$$s_n \equiv \left[1, -1, -1, 1, -1, -1 \right]$$

$$w_n \equiv \left[\sqrt{2}, 0, 1, 1, 0, \sqrt{2} \right]$$

Draw the signals and answer the following:

1) is $s(t)$ orthogonal to $w(t)$?

2) what is the energy of $s(t)$ and of $w(t)$?

3) what is the distance between the two signals, $\Delta\{s(t), w(t)\}$?

4) defining: $p(t) = s(t) - w(t)$; $q(t) = s(t) + w(t)$; what is the value of $\left(p(t), q(t)\right)$?

Answers

The two signals are orthogonal.

$\mathcal{E}\{s(t)\} = \mathcal{E}\{w(t)\} = 6$

$\Delta\{s(t), w(t)\} = 2\sqrt{3}$

$\left(p(t), q(t)\right) = 0$; note that this can be found without any calculation. How?

4.8.2

Consider the orthonormal set:

$$\mathcal{U} = \left\{\hat{u}_n(t)\right\}_{n=-3}^{2}, \quad \hat{u}_n(t) \equiv \pi(t-n)$$

Consider the signal:

$$v(t) = \Lambda_3(t)$$

with $t \in [-3, 3]$. Do the following:

1. Draw the signal $v(t)$ and the signals $\mathcal{U} = \{\hat{u}_n(t)\}_{n=-3}^{2}$ vs. time, over the interval $t \in [-3,3]$
2. Find the canonical approximation $v_{\text{app}}(t)$ of $v(t)$, using the orthonormal set $\mathcal{U}$, then draw $v_{\text{app}}(t)$ over $t \in [-3,3]$
3. Find the energy of $v(t)$, of $v_{\text{app}}(t)$ and of the error signal $v_{\text{err}}(t)$
4. Find the distance between $v(t)$ and $v_{\text{app}}(t)$. How does it relate to the energy of the error signal?

Answers

According to theory the canonical approximant is:

$$v_{app}(t) = \sum_{n=-3}^{2} v_n \hat{u}_n(t)$$

The components defining the canonical approximant are:

$$v_n = \left[1/6\,,1/2\,,5/6\,,5/6\,,1/2\,,1/6\right]$$

The energy of the approximant is:

$$\mathcal{E}\left\{v_{app}\left(t\right)\right\} = \sum_{n=-3}^{2}\left|v_n\right|^2 = 35/18$$

The energy of the signal is:

$$\mathcal{E}\left\{v\left(t\right)\right\} = 2$$

The energy of the error is:

$$\mathcal{E}\left\{v_{err}\left(t\right)\right\} = 1/18$$

The relative error energy is therefore:

$$\mathcal{E}\left\{v_{err}\left(t\right)\right\} / \mathcal{E}\left\{v\left(t\right)\right\} = 1/36 = 2.\overline{7}\ \%$$

The distance is:

$$\Delta\left\{v\left(t\right), v_{app}\left(t\right)\right\} = \sqrt{\mathcal{E}\left\{v_{err}\left(t\right)\right\}} = \frac{1}{3\sqrt{2}}$$

The problem is solved numerically by the following Matlab code:

```
% defining the ONS
% notice: the index cannot be negative
U.u{1}= @(t) Hspi(1,t+3)
U.u{2}= @(t) Hspi(1,t+2)
U.u{3}= @(t) Hspi(1,t+1)
U.u{4}= @(t) Hspi(1,t)
U.u{5}= @(t) Hspi(1,t-1)
U.u{6}= @(t) Hspi(1,t-2)
%
% index offset between Matlab and problem
offset=4
%
%% loop for components calculation
for n=-3:2
    % computing the n-th component
    comp(n+offset)=integral(@(t)...
    U.u{n+offset}(t).*HLambda(3,t),-3,3 );
end;
%
display(['components:  ', num2str(comp)])
```

```
%
energy_app=sum(comp.*comp);
display(['energy of the approximant:  ', num2str(energy_app)])
energy_sig=integral(@(t) HLambda(3,t).^2,-3,3);
display(['energy of the signal:  ', num2str(energy_sig)])
energy_err=energy_sig-energy_app;
display(['energy of the error:  ', num2str(energy_err)])
distance=sqrt(energy_err);
display(['distance between signal and approximant:  ',
num2str(distance)])
```

In the code, `Hspi(T,t)` implements $\pi_T(t)$ and `HLambda(T,t)` implements $\Lambda_T(t)$

4.8.3

Consider the orthonormal set made up of the signals:

$$\mathcal{U} = \{\hat{u}_n(t)\}_{n=0}^{3}, \quad \hat{u}_n(t) \equiv \pi_1(t-n)$$

$$\text{with } t \in \mathbf{I} = [0,4]$$

Consider the signals $s(t),\ w(t)$:

$$s(t) = \sum_{n=0}^{3} s_n\, \hat{u}_n(t) \qquad\qquad w(t) = \sum_{n=0}^{3} w_n\, \hat{u}_n(t)$$

where:

$$s_n \equiv [-1, 1, -1, 1]$$

$$w_n \equiv [1, 1, 1, 1]$$

Do the following:

1. Draw the signals $s(t),\ w(t)$ over the interval $t \in \mathbf{I} = [0, 4]$
2. Find the energy of the signals $s(t),\ w(t)$.
3. Calculate the inner product $(s(t), w(t))$.
4. Find the distance between $s(t)$ and $w(t)$.

5. Given the signal $z(t)=s(t)-w(t)$, find the components of $z(t)$ versus the elements of the orthonormal set $\mathcal{U}$

Is it possible to exactly represent $z(t)$ as a linear combination of the elements of $\mathcal{U}$?

4.8.4

Prove the relationship:

$$\mathcal{E}\left\{w(t)\right\}=\mathcal{E}\left\{w_{app}(t)\right\}+\mathcal{E}\left\{w_{err}(t)\right\}$$

4.8.5

Consider the following set of signals belonging to $L^2_{\mathbf{I}}$, with $\mathbf{I}=\left[0,3\right]$:

$$W=\left\{\overline{w}_n(t)\right\}_{n=1}^{4}$$

$$\begin{aligned}\bar{w}_1(t) &= \Pi_1(t-5/2)\\ \bar{w}_2(t) &= \Pi_2(t-1)\\ \bar{w}_3(t) &= \Pi_1(t-1/2)\\ \bar{w}_4(t) &= \Pi_3(t-3/2)\end{aligned}$$

Do the following.

1. Draw all four signals given above.
2. Apply the Gram-Schmidt procedure and find an orthonormal set $\mathcal{U} = \{\hat{u}_m(t)\}_{m=1}^{M}$
3. Draw the signals $\hat{u}_m(t)$
4. Write the component arrays of the elements $\bar{w}_n$ vs. $\mathcal{U}$ and verify that the "canonical approximation" of the $\bar{w}_n$ based on such components is exact, that is, the error signal is $0(t)$.

Answers:

The signals that are found by applying the Gram-Schmidt procedure are:

$$\hat{u}_1(t) = \Pi_1\left(t - \frac{5}{2}\right)$$

$$\hat{u}_2(t) = \frac{1}{\sqrt{2}}\Pi_2(t-1)$$

$$\hat{u}_3(t) = \sqrt{2}\,\Pi_1\left(t - \frac{1}{2}\right) - \frac{\sqrt{2}}{2}\Pi_2(t-1)$$

The signal $\overline{w}_4(t)$ turns out to be a linear combination of the others and therefore does not generate a fourth unit element.

4.8.6

Given the orthonormal set over the interval $\mathbf{I} = [-3,3]$:

$$\mathcal{U} = \{\hat{u}_m(t)\}_{m=1}^{6}$$

$$\hat{u}_m(t) = \Pi(t-[m-4]-1/2) = \pi(t-[m-4])$$

Find the canonical approximations $s_{\text{app}}(t)$ and $w_{\text{app}}(t)$ of the signals:

$$s(t) = \sin\left(\frac{2\pi}{T}t\right) \quad , \quad w(t) = \cos\left(\frac{2\pi}{T}t\right)$$

where $T = 12$.

Find the result of the inner product: $\left(s_{\text{app}}(t), w_{\text{app}}(t)\right)$.

Draw $s(t)$, $w(t)$, $s_{\text{app}}(t)$, $w_{\text{app}}(t)$.

Find the energy of $s(t)$, $w(t)$, $s_{\text{app}}(t)$, $w_{\text{app}}(t)$ and of the error signals $s_{\text{err}}(t)$, $w_{\text{err}}(t)$.

Answers

$$s_{\text{app}} \equiv \frac{3}{\pi}\left[-1, 1-\sqrt{3}, \sqrt{3}-2, 2-\sqrt{3}, \sqrt{3}-1, 1\right]$$

$$w_{\text{app}} \equiv \frac{3}{\pi}\left[2-\sqrt{3}, \sqrt{3}-1, 1, 1, \sqrt{3}-1, 2-\sqrt{3}\right]$$

$$\left(s_{\text{app}}(t), w_{\text{app}}(t)\right) = 0$$

$$\mathcal{E}\left\{s(t)\right\} = \mathcal{E}\left\{w(t)\right\} = 3$$

$$\mathcal{E}\left\{s_{\text{app}}(t)\right\} = \mathcal{E}\left\{w_{\text{app}}(t)\right\} = \frac{108}{\pi^2}\left(2-\sqrt{3}\right) \approx 2.932$$

$$\mathcal{E}\left\{s_{\text{err}}(t)\right\} = \mathcal{E}\left\{w_{\text{err}}(t)\right\} = 3-\frac{108}{\pi^2}\left(2-\sqrt{3}\right) \approx 0.06791$$

In both cases, the energy of the error is just 2.26%. of the energy of the signal.

4.8.7

Let $t \in \mathbf{I} = [-1,1]$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two sets of signals over $\mathbf{I}$:

$$\mathcal{U}_1 = \left\{\pi(t), \pi(-t)\right\}$$

$$\mathcal{U}_2 = \left\{\pi(t), \pi(-t), 2\sqrt{3}\left(t-1/2\right)\pi(t), 2\sqrt{3}\left(t+1/2\right)\pi(-t)\right\}$$

Let $s(t) = t^2$. Do the following:

- draw $s(t)$ and draw all the signals belonging to $\mathcal{U}_1$ and $\mathcal{U}_2$
- verify that $\mathcal{U}_1$ and $\mathcal{U}_2$ are orthonormal sets
- find the best (canonical) approximation of $s(t)$, first using $\mathcal{U}_1$ and the using $\mathcal{U}_2$
- draw the two approximants $s_{\text{app}_1}(t)$ and $s_{\text{app}_2}(t)$
- find the energy of the error in the two cases
- find the energy of $s(t)$
- find the relative error energy, as percentage vs. the signal energy, in the two cases.

Answers

Using $\mathcal{U}_2$, the components of $s(t)$ turn out to be: $\left[\frac{1}{3},\frac{1}{3},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\right]$.

The energy of $s(t)$ is $\frac{2}{5}$. The energy of $s_{\mathrm{app}_1}(t)$ is $\frac{2}{9}$, the energy of $s_{\mathrm{app}_2}(t)$ is $\frac{7}{18}$.

The relative error energy, in the case of the approximant $s_{\mathrm{app}_2}(t)$, is 2.8%.

Chapter 5. The Fourier Basis

The ***Fourier basis*** is by far the most famous basis for the signal space $L^2_{[t_0,t_1]}$.

Other bases are also well-known, such as the *Legendre polynomial* basis, but the Fourier basis has unique properties which make it very successful in many practical applications. It should be mentioned that modern applications make use of other bases as well, such as for instance different types of *wavelet* bases, but this topic is outside of the scope of this course.

In the following, the Fourier basis is introduced and its main properties are described. Examples are provided. At the end of the chapter, a few hints will be given as to the proof of the Fourier orthonormal set actually being a basis.

5.1 Definitions

Given the inner-product (Hilbert) space $L^2_{[t_0,t_1]}$ of finite-energy signals over the finite interval $\mathbf{I} \in [t_0,\ t_1]$, then a ***Fourier basis*** for such signal space is the following:

$$\Phi = \left\{ \hat{\varphi}_n(t) \right\}_{n=-\infty}^{+\infty} = \left\{ \frac{1}{\sqrt{T_0}} e^{j\frac{2\pi \cdot t}{T_0} n} \right\}_{n=-\infty}^{+\infty}$$

$$t \in [t_0,\ t_1],\ \ T_0 \triangleq t_1 - t_0$$

Eq. 5-1

Let's first check the orthonormality of this set. We compute the inner product of two elements with different index $n \neq m$:

$$\left(\hat{\varphi}_n(t),\hat{\varphi}_m(t)\right)=$$

$$=\left(\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}n},\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}m}\right)=\frac{1}{T_0}\int_{t_0}^{t_1}e^{j\frac{2\pi\cdot t}{T_0}n}\cdot e^{-j\frac{2\pi\cdot t}{T_0}m}dt$$

$$=\frac{1}{T_0}\int_{t_0}^{t_1}e^{j\frac{2\pi\cdot t}{T_0}[n-m]}dt=\frac{1}{j\dfrac{2\pi}{T_0}[n-m]}\frac{1}{T_0}\left[e^{j\frac{2\pi\cdot t}{T_0}[n-m]}\right]_{t_0}^{t_1}$$

$$=\frac{1}{j2\pi[n-m]}\left[e^{j\frac{2\pi}{T_0}[n-m]t_1}-e^{j\frac{2\pi}{T_0}[n-m]t_0}\right]\qquad n\neq m$$

We can then simplify the result as follows:

$$\left(\hat{\varphi}_n(t),\hat{\varphi}_m(t)\right)=\frac{1}{j2\pi[n-m]}\left[e^{j\frac{2\pi}{T_0}[n-m]t_1}-e^{j\frac{2\pi}{T_0}[n-m]t_0}\right]$$

$$=\frac{1}{j2\pi[n-m]}e^{j\frac{2\pi}{T_0}[n-m]t_0}\left[e^{-j\frac{2\pi}{T_0}[n-m]t_0}e^{j\frac{2\pi}{T_0}[n-m]t_1}-e^{-j\frac{2\pi}{T_0}[n-m]t_0}e^{j\frac{2\pi}{T_0}[n-m]t_0}\right]$$

$$=\frac{1}{j2\pi[n-m]}e^{j\frac{2\pi}{T_0}[n-m]t_0}\left[e^{j\frac{2\pi}{T_0}[n-m](t_1-t_0)}-1\right]$$

$$=\frac{1}{j2\pi[n-m]}e^{j\frac{2\pi}{T_0}[n-m]t_0}\left[e^{j2\pi[n-m]}-1\right]=0,\quad n\neq m$$

As a result, this is certainly an ***orthogonal*** set. Regarding normalization to unit norm, we look at the case $n=m$:

$$\left(\hat{\varphi}_n(t),\hat{\varphi}_n(t)\right)=\left\|\hat{\varphi}_n(t)\right\|^2$$

$$=\left(\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}n},\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}n}\right)=\frac{1}{T}\int_{t_0}^{t_1}e^{j\frac{2\pi\cdot t}{T_0}n}\cdot e^{-j\frac{2\pi\cdot t}{T_0}n}dt$$

$$=\frac{1}{T_0}\int_{t_0}^{t_1}e^{j\frac{2\pi\cdot t}{T_0}[n-n]}dt=\frac{1}{T_0}\int_{t_0}^{t_1}1(t)\cdot dt=\frac{t_1-t_0}{T_0}=\frac{T_0}{T_0}=1$$

From this we have:

$$\left(\hat{\varphi}_n(t),\hat{\varphi}_m(t)\right)=\left(\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}n},\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi\cdot t}{T_0}m}\right)=\delta_{mn}$$

which means that Φ is indeed an orthonormal set.

In fact the result can even be made more general. One can add arbitrary phases to the functions and orthonormality still holds:

$$\Phi = \left\{ \hat{\varphi}_n(t) \right\}_{n=-\infty}^{+\infty} = \left\{ \frac{1}{\sqrt{T_0}} e^{j\left[\frac{2\pi \cdot t}{T_0} n + \theta_n\right]} \right\}_{n=-\infty}^{+\infty}$$

$$t \in [t_0,\ t_1],\ \ T_0 \triangleq t_1 - t_0$$

$$\left(\hat{\varphi}_n(t), \hat{\varphi}_m(t) \right) = \left(\frac{1}{\sqrt{T_0}} e^{j\left[\frac{2\pi \cdot t}{T_0} n + \theta_n\right]}, \frac{1}{\sqrt{T_0}} e^{j\left[\frac{2\pi \cdot t}{T_0} m + \theta_m\right]} \right) = \delta_{mn}$$

A symbolic Matlab calculation can support this result. By changing the value of n and m it is easily seen that the result is δ_{nm}. In the program, random phases are added, showing that even with extra random phases, orthonormality still holds.

Warning: while adding an arbitrary phase to each one of the Fourier basis unit elements is ok, when this is done some properties of the signal components calculated using the "zero-phase" Fourier basis are no longer true. In particular, property 5-1955.5.1 does not hold anymore.

```
% symbolic computation
syms t T0
n=2,m=5, Phi_n=rand*2*pi, Phi_m=rand*2*pi
int( (1/sqrt(T0) * exp(j*n*2*pi/T0*t+Phi_n)) *...
    (1/sqrt(T0) * exp(-j*m*2*pi/T0*t-Phi_m)),t,0,T0 )
```

5.2 *Fourier Signal Reconstruction*

Given any signal $s(t) \in L^2_{[t_0,t_1]}$, then *it is possible to exactly express it is as a linear combination of the elements of the Fourier basis*, in the **canonical way**:

$$s(t) = \sum_n s_n \cdot \hat{\varphi}_n(t) = \sum_{n=-\infty}^{+\infty} s_n \cdot \frac{e^{j\frac{2\pi \cdot t}{T_0}n}}{\sqrt{T_0}}$$

Eq. 5-2

where the coefficients s_n are the "projections" of the signal $s(t)$ over each orthonormal element $\hat{u}_n(t)$ of the basis Φ. That is, the s_n's are given by:

$$s_n = \left(s(t), \hat{\varphi}_n(t)\right) = \left(s(t), \frac{e^{j\frac{2\pi \cdot t}{T_0}n}}{\sqrt{T_0}} \right) = \frac{1}{\sqrt{T_0}} \int_{t_0}^{t_1} s(t) \cdot e^{-j\frac{2\pi \cdot t}{T_0}n} dt$$

Formulas similar to Eq. 5-2 are often called *Fourier series,* but we will not use this terminology here.

As mentioned, the Fourier orthonormal set has the remarkable property of being a ***basis*** for $L^2_{[t_0,t_1]}$. This means that, given any signal $s(t) \in L^2_{[t_0,t_1]}$, such signal is fully represented through the infinite array of the projections (inner products) of the signal vs. the basis elements:

$$s(t) = \overline{s} \equiv [s_0, s_1, s_{-1}, s_2, s_{-2}, \ldots, s_n, s_{-n}, \ldots]$$

As it is the case for any element of an inner product space with an orthonormal basis, the energy and the norm of a signal $s(t)$ can be computed based on the components vs. the basis elements, using Parseval's rule. In this case, the summations are over an infinity of terms:

$$\mathcal{E}\{s(t)\} = \sum_{n=-\infty}^{+\infty} |s_n|^2$$

$$\|s(t)\| = \sqrt{\sum_{n=-\infty}^{+\infty} |s_n|^2}$$

Also, the inner product of any two signals $s(t), w(t)$, with Fourier components s_n, w_n can be computed as:

$$\left(s(t), w(t)\right) = \sum_{n=-\infty}^{+\infty} s_n \cdot w_n^*$$

Finally, the distance between any two such signals can be calculated as:

$$\Delta\left\{s(t), w(t)\right\} = \|s(t) - w(t)\| = \sqrt{\sum_{n=-\infty}^{+\infty} |s_n - w_n|^2}$$

5.2.1 Example of Fourier components calculation: the Heaviside Pi signal

We look at $s(t) = \Pi_T(t)$, over the interval $\mathbf{I} \in \left[-\frac{T_0}{2}, \frac{T_0}{2} \right]$, with $T < T_0$.

We have already shown that: $s(t) \in L^2_{[-T_0/2\,,\,T_0/2]}$.

As a result, it is certainly possible to compute its Fourier components and it is also possible to write the signal exactly based on a Fourier basis representation.

We first evaluate its Fourier components:

$$s_n = \left(\Pi_T(t), \hat{\varphi}_n(t)\right) = \left(\Pi_T(t), \frac{e^{j\frac{2\pi \cdot t}{T_0}\cdot n}}{\sqrt{T_0}}\right) = \frac{1}{\sqrt{T_0}} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \Pi_T(t) \cdot e^{-j\frac{2\pi \cdot t}{T_0}\cdot n} dt$$

$$= \frac{1}{\sqrt{T_0}} \int_{-\frac{T}{2}}^{+\frac{T}{2}} e^{-j\frac{2\pi \cdot t}{T_0}\cdot n} dt = \frac{1}{\sqrt{T_0}} \frac{1}{-j\frac{2\pi}{T_0}n} \left[e^{-j\frac{2\pi \cdot t}{T_0}\cdot n} \right]_{-\frac{T}{2}}^{+\frac{T}{2}}$$

$$= \frac{1}{\sqrt{T_0}} \frac{1}{-j\frac{2\pi}{T_0}n} \cdot \left[e^{-j\frac{2\pi \cdot \frac{T}{2}}{T_0}\cdot n} - e^{j\frac{2\pi \cdot \frac{T}{2}}{T_0}\cdot n} \right]$$

The result can be further manipulated as follows:

$$s_n = \frac{1}{\sqrt{T_0}} \frac{1}{-j\dfrac{2\pi}{T_0}n} \cdot \left[e^{-j\frac{2\pi \cdot \frac{T}{2}}{T_0}\cdot n} - e^{j\frac{2\pi \cdot \frac{T}{2}}{T_0}\cdot n} \right]$$

$$= \frac{1}{\sqrt{T_0}} \frac{1}{-j\dfrac{2\pi}{T_0}n} \cdot 2j\sin\left(-\frac{2\pi}{T_0}n\frac{T}{2}\right)$$

$$= \frac{1}{\sqrt{T_0}} \cdot \frac{T_0}{\pi \cdot n} \cdot \sin\left(\frac{\pi \cdot n}{T_0} \cdot T\right) = \frac{\sqrt{T_0}}{\pi \cdot n} \cdot \sin\left(n\frac{T}{T_0}\pi\right) \quad n \neq 0$$

where we have used the well-known formula: $e^{j\alpha} - e^{-j\alpha} = 2j\sin(\alpha)$

Note that the final result of the above calculation is fine for all n except $n = 0$, because then we end up with a form $\dfrac{\sin(0)}{0}$ which we do not know how to handle.

Instead, for $n = 0$ we restart the calculation from scratch and easily find:

$$s_0 = \left(\Pi_T(t), \hat{\varphi}_0(t)\right)$$

$$= \left(\Pi_T(t), \frac{\left. e^{j\frac{2\pi \cdot t}{T_0}\cdot n} \right|_{n=0}}{\sqrt{T_0}} \right) = \left(\Pi_T(t), \frac{1(t)}{\sqrt{T_0}} \right)$$

$$= \frac{1}{\sqrt{T_0}} \int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} \Pi_T(t) \cdot 1(t)\, dt = \frac{1}{\sqrt{T_0}} \int_{-\frac{T}{2}}^{+\frac{T}{2}} 1(t)\, dt = \frac{T}{\sqrt{T_0}}$$

Note that s_n can then be written in a compact form valid for all n as:

$$s_n = \frac{\sqrt{T_0}}{\pi \cdot n} \cdot \sin\left(n \frac{T}{T_0} \pi \right) = \sqrt{T_0} \frac{T}{T_0} \cdot \frac{\sin\left(n \frac{T}{T_0} \pi \right)}{\pi \cdot n \frac{T}{T_0}} = \frac{T}{\sqrt{T_0}} \cdot \mathrm{Sinc}\left(n \frac{T}{T_0} \right)$$

Eq. 5-3

The case $n = 0$ is not a problem, because by definition $\mathrm{Sinc}(0) = 1$.

According to theory, we can then "reconstruct" $\Pi_T(t)$ as a canonical linear combination of the Fourier basis orthonormal signals:

$$\Pi_T(t) = \sum_{n=-\infty}^{+\infty} s_n \cdot \hat{\varphi}_n(t) = \sum_{n=-\infty}^{+\infty} \frac{T}{\sqrt{T_0}} \mathrm{Sinc}\left(n\frac{T}{T_0}\right) \cdot \frac{e^{j\frac{2\pi \cdot t}{T_0}n}}{\sqrt{T_0}}$$

$$= \frac{T}{T_0} \sum_{n=-\infty}^{+\infty} \mathrm{Sinc}\left(n\frac{T}{T_0}\right) \cdot e^{j\frac{2\pi \cdot t}{T_0}n}$$

The energy of $\Pi_T(t)$ is:

$$\mathcal{E}\left\{\Pi_T(t)\right\} = \sum_{n=-\infty}^{+\infty} \left|s_n\right|^2 = \sum_{n=-\infty}^{+\infty} \frac{T^2}{T_0} \mathrm{Sinc}^2\left(n\frac{T}{T_0}\right) =$$

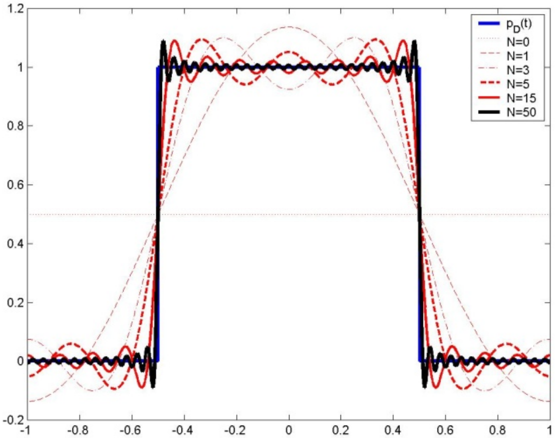
$$\sum_{n=-\infty}^{+\infty} \frac{T^2}{T_0} \frac{\sin^2\left(n\frac{T}{T_0}\pi\right)}{n^2\pi^2\frac{T^2}{T_0^2}} = = \sum_{n=-\infty}^{+\infty} \frac{T_0}{\pi^2 \cdot n^2} \cdot \sin^2\left(n\frac{T}{T_0}\pi\right)$$

$\boxed{\text{On your own}}$: write a computer program that verifies that the energy value resulting from numerically adding up a sufficient number of series coefficients coincides with the energy value found by direct calculation using the "native" definition of the inner product, whose result is $\mathcal{E}\left\{\Pi_T(t)\right\} = T$.

$\boxed{\text{On your own}}$: write a computer program that sums an increasing number of contributions in the formula:

$$\Pi_{T,\text{app}}(t) = \frac{T}{\sqrt{T_0}} \sum_{n=-N}^{+N} \text{Sinc}\left(n \frac{T}{T_0} \right) \cdot \frac{e^{j\frac{2\pi \cdot t}{T_0} n}}{\sqrt{T_0}},$$

that is, N is increased gradually. Plot the result and compare $\Pi_{T,\text{app}}(t)$ with $\Pi_T(t)$ as N goes up. Verify you get a figure similar to the following:



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
$p_D(t)$
N=0
N=1
N=3
N=5
N=15
N=50

5.3 The signal spectrum

The *spectrum* of a signal $s(t)$ is a representation of the Fourier components s_n of the signal itself. For instance, for the case of the Heaviside Pi signal, the components are, once again:

$$s_n = \frac{T}{\sqrt{T_0}} \cdot \mathrm{Sinc}\left(n \frac{T}{T_0} \right)$$

If we assume $T = 1/3$ and $T_0 = 1$, over an interval symmetric with respect to the origin $\mathbf{I} \in \left[-T_0/2, T_0/2 \right]$, we get the following:

$$s_n = \frac{1}{3} \cdot \mathrm{Sinc}\left(\frac{n}{3} \right)$$

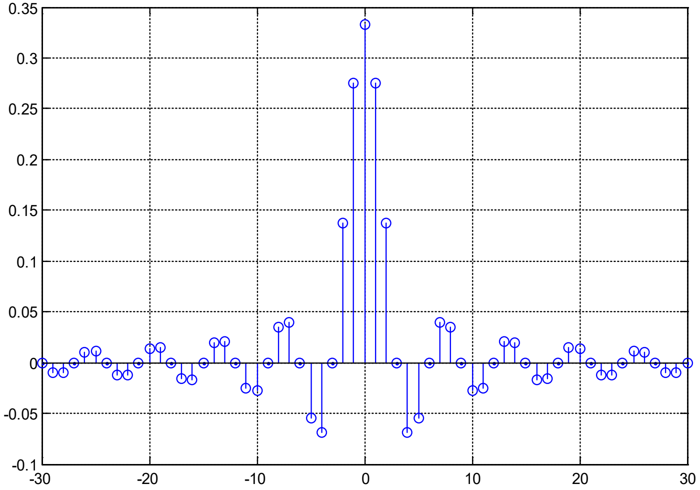


Fig. 5-1 – Spectrum (or Fourier components) of the signal $\Pi_T(t)$ for $T = 1/3$ and $T_0 = 1$ (s). The abscissa is frequency $(n \cdot f_0)$ in Hz, where in this case $f_0 = 1/T_0 = 1$.

If instead of s_n we plot $|s_n|^2$, then we have the *energy spectrum* of the signal. The name derives from the fact that each $|s_n|^2$ contributes to the total signal energy according to the well-known formula:

$$\mathcal{E}\{s(t)\} = \sum_{n=-\infty}^{+\infty} |s_n|^2$$

For the same example as above, the energy spectrum is given by:

$$|s_n|^2 = \frac{T^2}{T_0}\mathrm{Sinc}^2\left(n\frac{T}{T_0}\right) = \frac{1}{9}\cdot\mathrm{Sinc}^2\left(\frac{n}{3}\right)$$

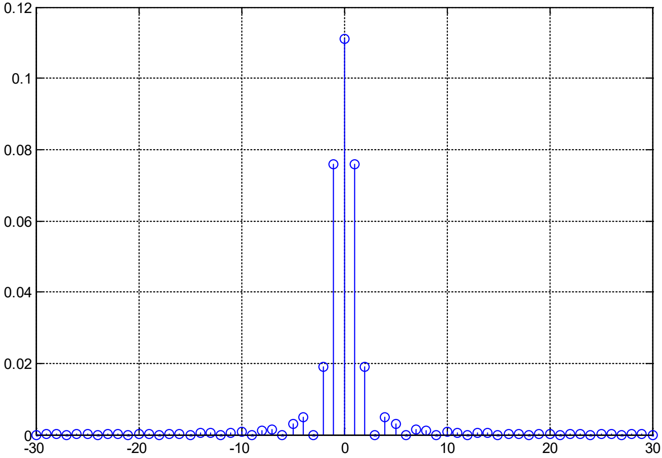


Fig. 5-2 – Energy spectrum of the signal $\Pi_T(t)$ for $T=1/3$ and $T_0=1$. The abscissa is n, corresponding to frequency $(n \cdot f_0)$.

5.4 *More examples of Fourier Series*

5.4.1 The unilateral exponential signal

The signal:

$$s(t) = e^{-at}u(t) \qquad a > 0$$

has a support that is infinitely extended towards $+\infty$. Nonetheless, its Fourier series can be calculated on a truncated version of it.

Assuming that the signal space is defined over $\mathbf{I} \in \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ and that we use the Fourier basis Φ, then the result is easily found to be:

$$s_n = \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} \cdot (-1)^n}{a + j2\pi \frac{n}{T_0}}$$

These are the calculations:

$$s_n = \left(s(t), \hat{\varphi}_n(t)\right) = \left(e^{-at}u(t), \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}} \right) = \int_{-T_0/2}^{T_0/2} e^{-at}u(t)\frac{e^{-j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}}dt =$$

$$\frac{1}{\sqrt{T_0}}\int_0^{T_0/2} e^{-at}e^{-j\frac{2\pi}{T_0}nt}dt = \frac{1}{\sqrt{T_0}}\int_0^{T_0/2} e^{-\left(j\frac{2\pi}{T_0}n+a\right)t}dt = \frac{1}{\sqrt{T_0}}\frac{\left[e^{-\left(j\frac{2\pi}{T_0}n+a\right)t}\right]_0^{\frac{T_0}{2}}}{-\left(j\frac{2\pi}{T_0}n+a\right)}$$

$$= \frac{1}{\sqrt{T_0}}\frac{1}{-\left(j\frac{2\pi}{T_0}n+a\right)}\left[e^{-\left(j\frac{2\pi}{T_0}n+a\right)\frac{T_0}{2}} - 1\right] = \frac{1}{\sqrt{T}}\frac{e^{-j\pi n}e^{-aT_0/2} - 1}{-\left(j\frac{2\pi}{T_0}n+a\right)}$$

$$= \frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}e^{-j\pi n}}{a+j\frac{2\pi}{T_0}n} = \frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}\cdot(-1)^n}{a+j\frac{2\pi}{T_0}n}$$

Important: Note that the components of $s(t) = e^{-at}u(t)$, which is a ***real*** signal, are nonetheless ***complex***. Remember that, in general, the Fourier components of a real signal are in fact complex and that only in special cases the Fourier components of a real signal are all real themselves.

In this example, we have:

$$\mathrm{Re}\{s_n\} = \mathrm{Re}\left\{ \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} \cdot (-1)^n}{a + j\dfrac{2\pi}{T_0}n} \right\} = \mathrm{Re}\left\{ \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} \cdot (-1)^n}{a + j\dfrac{2\pi}{T_0}n} \left(\frac{a - j\dfrac{2\pi}{T_0}n}{a - j\dfrac{2\pi}{T_0}n} \right) \right\}$$

$$= \mathrm{Re}\left\{ \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} \cdot (-1)^n}{a^2 + \dfrac{4\pi^2}{T_0^2}n^2} \left(a - j\frac{2\pi}{T_0}n \right) \right\} = \frac{a}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} \cdot (-1)^n}{a^2 + \dfrac{4\pi^2}{T_0^2}n^2}$$

$$\mathrm{Im}\left\{s_n\right\} = \mathrm{Im}\left\{\frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}\cdot\left(-1\right)^n}{a+j\dfrac{2\pi}{T_0}n}\right\} = \mathrm{Im}\left\{\frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}\cdot\left(-1\right)^n}{a+j\dfrac{2\pi}{T_0}n}\left(\frac{a-j\dfrac{2\pi}{T_0}n}{a-j\dfrac{2\pi}{T_0}n}\right)\right\}$$

$$= \mathrm{Im}\left\{\frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}\cdot\left(-1\right)^n}{a^2+\dfrac{4\pi^2}{T_0^2}n^2}\left(a-j\frac{2\pi}{T_0}n\right)\right\} = -\frac{2\pi}{T_0}n\frac{1}{\sqrt{T_0}}\frac{1-e^{-aT_0/2}\cdot\left(-1\right)^n}{a^2+\dfrac{4\pi^2}{T_0^2}n^2}$$

$\boxed{\text{On your own (recommended)}}$ Find $\left|s_n\right|$ and $\angle s_n$

$\boxed{\text{On your own (recommended)}}$ prove that if, instead of the interval $\mathbf{I} = \left[-T_0/2, T_0/2\right]$, we chose the interval: $\mathbf{I} = \left[0, T_0\right]$, the result would be:

$$s_n = \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0}}{a + j2\pi \frac{n}{T_0}}$$

$\boxed{\text{On your own}}$ find out what the result is when using a *generic* interval $\mathbf{I} = [t_0, t_1]$ with as usual $T_0 = |t_1 - t_0|$. Note that you need to deal with three distinct cases separately:

1) $t_0, t_1 < 0$
2) $t_0 < 0, t_1 > 0$
3) $t_0, t_1 > 0$

Answers

Case (1) is trivial: $s_n = 0, \forall n$

Case (2) is: $$s_n = \frac{1}{\sqrt{T_0}} \frac{1 - e^{-a \cdot t_1} e^{-j\frac{2\pi}{T_0} n \cdot t_1}}{j\frac{2\pi}{T_0} n + a}$$

Case (3) is: $$s_n = \frac{1}{\sqrt{T_0}} \frac{\left(1 - e^{-aT_0}\right)}{j\frac{2\pi}{T_0} n + a} e^{-\left(j\frac{2\pi}{T_0} n + a\right) \cdot t_0}$$

5.4.2 The triangular signal

The Fourier series coefficients for the triangular signal with support $\left[-T, T\right]$, i.e., FWHM equal to T, that is $\Lambda_T\left(t\right)$, are as follows:

$$s_n = \frac{T}{\sqrt{T_0}} \mathrm{Sinc}^2\left(\frac{n}{T_0}T\right)$$

Notice the similarity with the components of the rectangular signal.

5.4.3 Sine and Cosine

We assume now that the signal is $x(t) = \cos(2\pi K f_0 t)$ where K is an integer. We also assume the interval $\mathbf{I} = [-T_0/2, T_0/2]$.

$\boxed{\text{On your own}}$ go through the derivation below.

We have:

$$x_n = \left(\cos\left(2\pi K f_0 t\right), \hat{\varphi}_n\left(t\right)\right) = \left(\cos\left(2\pi K f_0 t\right), \sqrt{f_0} e^{j2\pi n f_0 t}\right) =$$

$$= \sqrt{f_0} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi K f_0 t\right) e^{-j2\pi n f_0 t} dt =$$

$$\sqrt{f_0} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi K f_0 t\right) \cos\left(2\pi n f_0 t\right) dt - j\sqrt{f_0} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi K f_0 t\right) \sin\left(2\pi n f_0 t\right) dt$$

Using the standard trigonometric formulas:

$$\cos\left(\alpha\right)\cos\left(\beta\right) = \frac{1}{2}\cos\left(\alpha+\beta\right) + \frac{1}{2}\cos\left(\alpha-\beta\right)$$

$$\sin\left(\alpha\right)\cos\left(\beta\right) = \frac{1}{2}\sin\left(\alpha+\beta\right) + \frac{1}{2}\sin\left(\alpha-\beta\right)$$

we can write:

$$x_n = \left(\cos\left(2\pi K f_0 t\right), \hat{\varphi}_n\left(t\right)\right) =$$

$$\sqrt{f_0} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi K f_0 t\right) \cos\left(2\pi n f_0 t\right) dt - j\sqrt{f_0} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi K f_0 t\right) \sin\left(2\pi n f_0 t\right) dt$$

$$= \frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi\left[n+K\right] f_0 t\right) dt + \frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \cos\left(2\pi\left[n-K\right] f_0 t\right) dt +$$

$$-j\frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \sin\left(2\pi\left[n+K\right] f_0 t\right) dt - j\frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \sin\left(2\pi\left[n-K\right] f_0 t\right) dt +$$

It is is now easy to see that all integrals integrate either a sine or a cosine function over an exact integer number (either $\left[n+K\right]$ or $\left[n-K\right]$) of periods. The result is of course zero in all these cases. Therefore, we can so far write:

$$x_n = \left(\cos\left(2\pi K f_0 t\right), \hat{\varphi}_n\left(t\right)\right) = 0 \qquad \text{for} \quad \left[n+K\right] \neq 0 \quad \text{or} \quad \left[n-K\right] \neq 0$$

We then need to discuss the two residual cases:

$$\left[n-K\right] = 0 \quad \rightarrow \quad n = K$$

$$\left[n+K\right] = 0 \quad \rightarrow \quad n = -K$$

In the first case we get:

$$x_K = \left(\cos\left(2\pi K f_0 t\right), \hat{\varphi}_K\left(t\right)\right) =$$

$$= \frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \cos\left(4\pi K f_0 t\right) dt + \frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} 1\left(t\right) dt$$

$$- j\frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} \sin\left(4\pi K f_0 t\right) dt - j\frac{\sqrt{f_0}}{2} \int_{-T_0/2}^{T_0/2} 0\left(t\right) dt$$

$$= \frac{1}{2\sqrt{f_0}} = \frac{\sqrt{T_0}}{2}$$

where again the integrals with sine and cosine are zero because they are performed over an exact integer number of periods.

A very similar calculation for $n = -K$ leads to the same result:

$$x_{-K} = \left(\cos\left(2\pi K f_0 t\right), \hat{\varphi}_{-K}(t)\right) = \frac{1}{2\sqrt{f_0}} = \frac{\sqrt{T_0}}{2}$$

$\boxed{\text{On your own:}}$ Find the result for $x(t) = \sin\left(2\pi K f_0 t\right)$. You may need:

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$$

Answer:

$$x_K = \frac{-j}{2\sqrt{f_0}} = \frac{-j\sqrt{T_0}}{2} \quad , \quad n = K \qquad\qquad x_{-K} = \frac{j}{2\sqrt{f_0}} = \frac{j\sqrt{T_0}}{2} \quad , \quad n = -K$$

$\boxed{\text{End of “on your own”}}$

5.4.3.1 summarizing the results for sine and cosine

A cosine that has a period that is an exact divisor of T_0, and hence frequency exactly Kf_0, with K an integer, has only two Fourier components that are non-zero:

$$x_K = x_{-K} = \frac{\sqrt{T_0}}{2}$$

A sine that has a period that is an exact divisor of T_0, and hence frequency exactly Kf_0, with K an integer, has only two Fourier components that are non-zero:

$$x_K = \frac{-j}{2\sqrt{f_0}} = \frac{-j\sqrt{T_0}}{2}$$

$$x_{-K} = \frac{j}{2\sqrt{f_0}} = \frac{j\sqrt{T_0}}{2}$$

$\boxed{\text{On your own:}}$ What is the significance of this result?

$\boxed{\text{On your own:}}$

Prove that the results of this section do not change if the chosen interval is a generic

assume the interval $\mathbf{I} = [t_0, t_1]$, provided that $t_1 - t_0 = T_0$.

5.4.4 *optional exercise*

Given the signal $s(t) = \cos(2\pi f_1 t)$ find its components vs. the Fourier basis, over the interval $\mathrm{I} = [0, T_0]$, for the general case $f_1 \neq \dfrac{1}{T_0}$.

Then, assume that $f_1 = \dfrac{K}{T_0}$ with K an integer greater than 1. What happens to the result?

Answer

$$s_n = \frac{\sqrt{T_0}}{2}(-1)^n \left[\mathrm{Sinc}(f_1 T_0 + n) e^{-j\pi f_1 T_0} + \mathrm{Sinc}(f_1 T_0 - n) e^{j\pi f_1 T_0} \right]$$

If $f_1 = \frac{K}{T_0}$ then one gets:

$$s_K = s_{-K} = \frac{\sqrt{T_0}}{2}$$

$$s_n = 0 \quad n \neq K$$

End of optional material.

5.4.5 exercise

$\boxed{\text{On your own}}$: given the signal $s(t) = e^{-2|t|}$:

1. find its components vs. the Fourier basis, over the interval $\mathrm{I} = [-1,1]$.

2. calculate the energy of the signal $s(t)$ and write the formula for the energy of the approximant $s_{\text{app}}(t) = \sum_{n=-N}^{N} s_n \, \hat{\varphi}_n(t)$

3. write a Matlab program that does the following:

   a. it plots the spectrum of $s(t)$, i.e., plots the s_n vs. frequency $f = nf_0$

   b. it plots the *energy* spectrum of $s(t)$

   c. it plots the approximating signal $s_{\text{app}}(t) = \sum_{n=-N}^{N} s_n \, \hat{\varphi}_n(t)$ for a given value of N

Answer (partial)

$$s_n = \frac{1}{\sqrt{2}} \frac{1 - e^{-2} \cdot (-1)^n}{1 + \left(n \frac{\pi}{2}\right)^2}$$

5.5 *Some properties of the Fourier components*

5.5.1 property 1: conjugate symmetry

Given $s(t) \in L^2_{\mathbf{I}}$ with $s(t) \in \mathbb{R}$, then:

$$s_n = s^*_{-n}$$

Note that from this property, the following can be directly derived:

$$|s_n| = |s_{-n}| \quad , \quad \angle s_n = -\angle s_{-n}$$

Proof:

$$s_n = \left(s(t), \hat{\varphi}_n\left(t\right)\right) = \int_{\mathbf{I}} s(t) \frac{1}{\sqrt{T_0}} e^{-j\frac{2\pi}{T_0}\cdot n\cdot t} dt \ ,$$

$$s_{-n} = \int_{\mathbf{I}} s(t)\cdot \frac{1}{\sqrt{T_0}} e^{-j\frac{2\pi}{T_0}\cdot(-n)\cdot t} dt = \int_{\mathbf{I}} s(t)\cdot \frac{1}{\sqrt{T_0}} e^{j\frac{2\pi}{T_0}\cdot n\cdot t} dt$$

and therefore:

$$s_{-n}^{*} = \int_{\mathbf{I}} s^{*}(t)\cdot \frac{1}{\sqrt{T_0}} e^{-j\frac{2\pi}{T_0}\cdot n\cdot t} dt = \int_{\mathbf{I}} s(t)\cdot \frac{1}{\sqrt{T_0}} e^{-j\frac{2\pi}{T_0}\cdot n\cdot t} dt = s_n$$

because $s(t) = s^{*}(t)$ since $s(t) \in \mathbb{R}$.

Note also that obviously $s_n = s_{-n}^{*} \rightarrow s_{-n} = s_n^{*}$ too.

Finally, it is easy to see that $s_0 \in \mathbb{R}$.

$\boxed{\text{On your own}}$: consider the real signal $s(t) = e^{-at}u(t)$ over an interval of your choice (for example $\mathbf{I} = [-T_0/2, T_0/2]$ or $\mathbf{I} = [0, T_0]$). Check that indeed $s_n = s_{-n}^*$ using the result already calculated. Also check the expected symmetries on $|s_n|$, $\angle s_n$. Finally, can you see any symmetries in $\mathrm{Re}\{s_n\}$, $\mathrm{Im}\{s_n\}$ as well?

5.5.2 property 2: delay

Given a signal . $s(t) \in L_{\mathbf{I}}^2$., then the following property holds:

$$s(t) \leftrightarrow s_n$$

$$s(t - t_d) \leftrightarrow s_n \cdot e^{-j\frac{2\pi}{T_0}\cdot n \cdot t_d}$$

However, the *assumption that the support of* $s(t-t_d)$ *remains comprised within the interval* $\mathbf{I}=[t_0, t_1]$ *is necessary*. Otherwise, the signal would get truncated and the property would not be valid.

Optional proof:

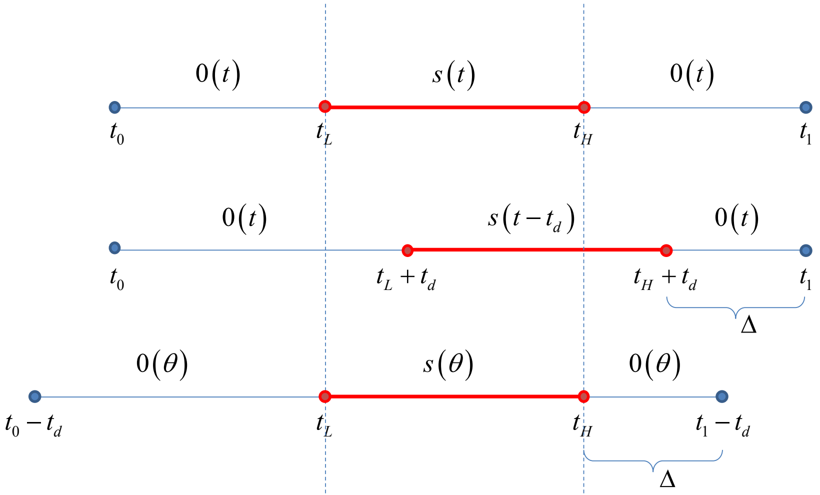


Fig. 5-3

We assume we have a situation as represented in the first line of Fig. 5.3, where $s(t)$ is identically zero to the left and to the right of the interval $[t_L, t_H]$. Then we have formally:

$$s_n = \left(s(t), \phi_n(t)\right) = \frac{1}{\sqrt{T_0}} \int_{t_0}^{t_1} s(t) e^{-j\frac{2\pi}{T_0}nt} dt = \frac{1}{\sqrt{T_0}} \int_{t_L}^{t_H} s(t) e^{-j\frac{2\pi}{T_0}nt} dt$$

Eq. 5-4

We then look at the time-shifted signal $s(t - t_d)$. This signal is identically zero to the left and to the right of the interval $\left[t_L + t_d, t_H + t_d\right]$. We then add the ***essential assumption*** that $\Delta = t_1 - \left(t_H + t_d\right) > 0$, which in words means that shifting $s(t)$ to the right by t_d does not bring the non-zero part of the signal (the red interval) outside of the interval $\mathbf{I} = \left[t_0, t_1\right]$. We then have:

$$\left(s\left(t - t_d\right), \phi_n(t)\right) = \frac{1}{\sqrt{T_0}} \int_{t_0}^{t_1} s\left(t - t_d\right) e^{-j\frac{2\pi}{T_0}nt} dt$$

which is the situation depicted in the second line of Fig. 5.3.

We then change variable in the integral as: $\theta = t - t_d$, which yields:

$$\left(s\left(t-t_d\right),\phi_n\left(t\right)\right) = \frac{1}{\sqrt{T_0}} \int_{t_0-t_d}^{t_1-t_d} s\left(\theta\right) e^{-j\frac{2\pi}{T_0}n\left(\theta+t_d\right)} d\theta$$

$$= \frac{1}{\sqrt{T_0}} \int_{t_L}^{t_H} s\left(\theta\right) e^{-j\frac{2\pi}{T_0}n\left(\theta+t_d\right)} d\theta = e^{-j\frac{2\pi}{T}nt_d} \frac{1}{\sqrt{T_0}} \int_{t_L}^{t_H} s\left(\theta\right) e^{-j\frac{2\pi}{T_0}n\theta} d\theta$$

This is the situation shown in the third line of Fig. 5.3. We then remark that the last integral is the same as that shown in Eq. 5-4, so that in the end we can write, by comparison with Eq. 5-4:

$$\left(s\left(t-t_d\right),\phi_n\left(t\right)\right) = s_n \cdot e^{-j\frac{2\pi}{T_0}nt_d}$$

which is what we wanted to prove.

In this proof we have assumed to shift the signal to the right (that is, $t_d > 0$), but a completely analogous proof can be written assuming to shift the signal to the left (that is, $t_d < 0$), which the student can figure out on his own. ***End of optional material.***

5.5.3 (optional) property 3: time inversion

Given a signal $s(t) \in L_{\mathrm{I}}^2$, then the following property holds:

$$\begin{aligned} s(t) &\leftrightarrow s_n \\ s(-t) &\leftrightarrow s_{-n} \end{aligned}$$

However, the additional assumption that the support of $s(-t)$ remains comprised within the interval I is necessary. Otherwise, the signal would get truncated and the property would not be valid.

End of optional material.

5.5.4 property 4: inner product formula for real signals

Given two signals, $v(t), w(t) \in L^2_{\mathbf{I}}$, $v(t), w(t) \in \mathbb{R}$, then their inner product can be written as:

$$\left(v(t), w(t)\right) = 2\sum_{n=1}^{\infty} \mathrm{Re}\left\{v_n w_n^*\right\} + v_0 w_0$$

Eq. 5-5

Proof:

$$\left(v(t), w(t)\right) = \sum_{n=-\infty}^{\infty} v_n w_n^*$$

$$= \sum_{n=1}^{\infty} v_n w_n^* + \sum_{n=-\infty}^{-1} v_n w_n^* + v_0 w_0^* = \sum_{n=1}^{\infty} v_n w_n^* + \sum_{n=1}^{\infty} v_{-n} w_{-n}^* + v_0 w_0^*$$

$$= \sum_{n=1}^{\infty} v_n w_n^* + \sum_{n=1}^{\infty} v_n^* w_n + v_0 w_0^* = \sum_{n=1}^{\infty} \left(v_n w_n^* + v_n^* w_n \right) + v_0 w_0^*$$

$$= 2\sum_{n=1}^{\infty} \mathrm{Re}\left\{ v_n w_n^* \right\} + v_0 w_0^*$$

Note that the term for $n = 0$ apparently is different than in Eq. 5-5: $v_0 w_0^*$ rather than $v_0 w_0$. However, we know that:

$$w_0 = \left(w(t), \hat{\varphi}_0(t)\right) = \left(w(t), \left. \frac{e^{j\frac{2\pi n t}{T_0}}}{\sqrt{T_0}} \right|_{n=0} \right)$$

$$= \left(w(t), \frac{1}{\sqrt{T_0}} \right) = \frac{1}{\sqrt{T_0}} \int_I w(t)\, dt$$

Since by assumption $w(t) \in \mathbb{R}$ and since the integral of a real function is real, then $w_0 \in \mathbb{R}$ and therefore $w_0^* = w_0$. As a result, Eq. 5-5 is verified.

5.5.5 Problem

$\boxed{\text{On your own}}$

Given the signals: $x(t)$ and $y(t)$, both in $L^2_{[-2;2]}$:

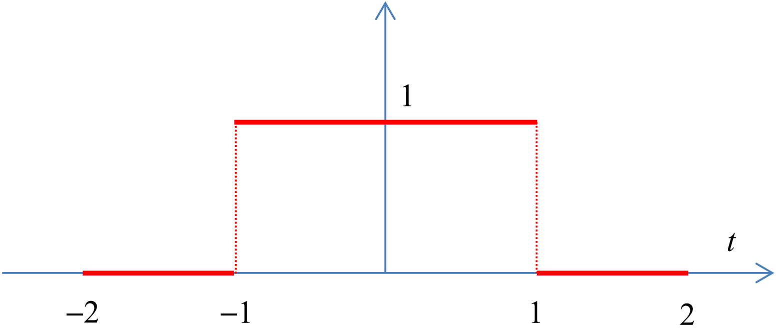


Fig. 5-4: signal $x(t)$

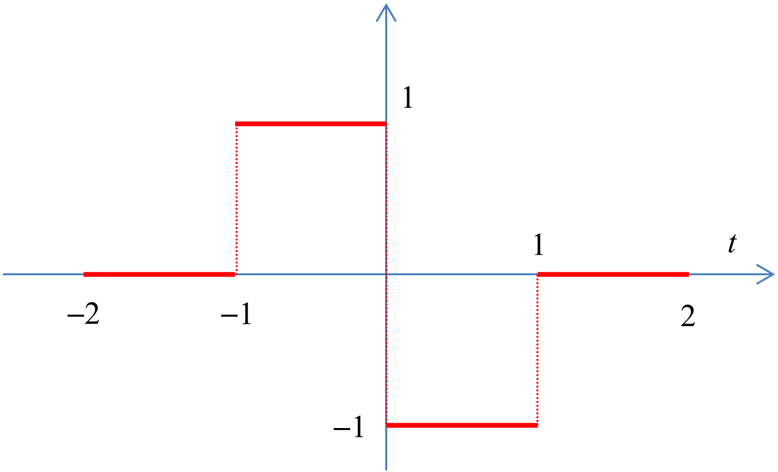


Fig. 5-5: signal $y(t)$

we want to compute their inner product: $\left(x(t), y(t)\right)$.

We want to do it both using the native rule of $L_{\mathbf{I}}^2$ and the standardized rule of the inner product spaces.

Solution

We first look at the "native" rule for calculating the inner product:

$$\left(x(t), y(t)\right) = \int_{-2}^{2} x(t), y^*(t)\, dt$$

The integrand function is: $z(t) = x(t) y^*(t)$. If we plot it over $[-2, 2]$ we get:

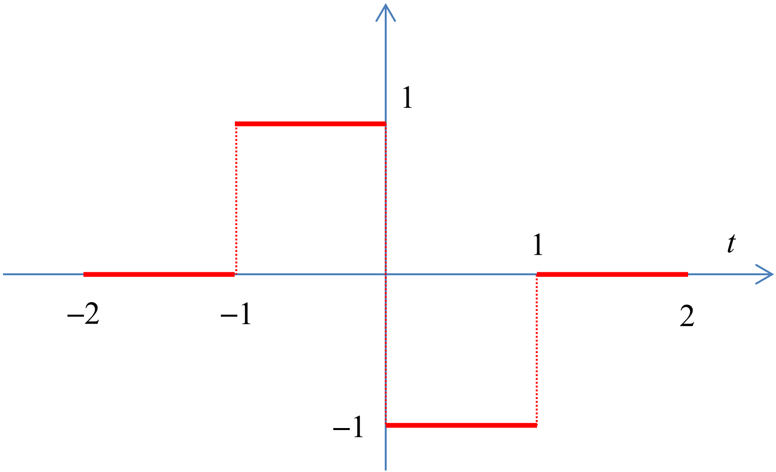


Fig. 5-6: plot of the integrand signal $z(t)$

The integral result is zero because the area subtended by $z(t)$ when $t < 0$ is equal to the area subtended by $z(t)$ when $t > 0$, but the latter must be counted with a minus sign since the signal is negative.

The same result can also be found by simply remarking that $z(t)$ is an odd function, that is, $z(t) = -z(-t)$. ***Any odd function integrated between symmetric integration limits has zero integral.***

Therefore, we can conclude that

$$\left(x(t), y(t)\right) = \int_{-2}^{2} x(t)\, y^*(t)\, dt = 0$$

So $x(t)$ and $y(t)$ are *orthogonal*.

We now redo the same calculation with the generalized inner product rule based on the signal components with respect to a basis:

$$\left(x(t), y(t)\right) = \sum_{n=-\infty}^{+\infty} x_n y_n^*$$

As basis, we choose the Fourier basis Φ. The signal $x(t)$ is in fact a rectangular signal $\Pi_2(t)$. From Sect. 5.2.1 we immediately find:

$$x_n = \frac{T}{\sqrt{T_0}} \cdot \text{Sinc}\left(\frac{T}{T_0} \cdot n\right) = \text{Sinc}\left(\frac{n}{2}\right)$$

Eq. 5-6

As for $y(t)$, we have:

$$y_n = \left(y(t), \hat{\varphi}_n(t)\right)$$

However, we notice that $y(t) = \Pi_1\left(t + \frac{1}{2}\right) - \Pi_1\left(t - \frac{1}{2}\right)$. The inner product is "distributive" so:

$$y_n = \left(y(t), \hat{\varphi}_n(t)\right) = \left(\Pi_1\left(t + \frac{1}{2}\right) - \Pi_1\left(t - \frac{1}{2}\right), \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}}\right)$$

$$= \left(\Pi_1\left(t + \frac{1}{2}\right), \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}}\right) - \left(\Pi_1\left(t - \frac{1}{2}\right), \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}}\right)$$

Then :

$$\left(\Pi_1\left(t+\frac{1}{2} \right), \hat{\varphi}_n(t) \right) = \int_{-2}^{2} \Pi_1\left(t+\frac{1}{2} \right) \frac{1}{\sqrt{T_0}} e^{-j\frac{2\pi}{T_0}nt} dt$$

The integral can be easily solved directly. It can also be solved by recalling property 5.5.2 , which allows us to directly write:

$$\left(\Pi_1\left(t+\frac{1}{2} \right), \hat{\varphi}_n(t) \right) = \left(\Pi_1(t), \hat{\varphi}_n(t) \right) \cdot e^{-j\frac{2\pi}{T_0} \cdot n \cdot t_d}$$

The Fourier components $\left(\Pi_1(t), \hat{\varphi}_n(t) \right)$ were calculated in 5.2.1 , Eq. 5-3, and were:

$$\left(\Pi_1(t), \hat{\varphi}_n(t) \right) = \frac{1}{\sqrt{T_0}} \cdot \mathrm{Sinc}\left(\frac{n}{T_0} \right)$$

Therefore:

$$\left(\Pi_1\left(t+\frac{1}{2}\right),\hat{\varphi}_n(t)\right)=e^{j\frac{\pi}{4}n}\frac{1}{2}\cdot\mathrm{Sinc}\left(\frac{n}{4}\right)$$

Similarly for the other contribution:

$$\left(\Pi_1\left(t-\frac{1}{2}\right),\hat{\varphi}_n(t)\right)=e^{-j\frac{\pi}{4}n}\frac{1}{2}\cdot\mathrm{Sinc}\left(\frac{n}{4}\right)$$

As a result:

$$y_n=\frac{1}{2}\cdot\mathrm{Sinc}\left(\frac{n}{4}\right)\left[e^{j\frac{\pi}{4}n}-e^{-j\frac{\pi}{4}n}\right]=j\ \mathrm{Sinc}\left(\frac{n}{4}\right)\sin\left(\frac{\pi}{4}n\right)$$

Eq. 5-9

We can now compute the inner product using Parseval's formula:

$$\left(x(t),y(t)\right)=\sum_{n=-\infty}^{+\infty}x_n y_n^*=-j\sum_{n=-\infty}^{+\infty}\mathrm{Sinc}\left(\frac{n}{2}\right)\mathrm{Sinc}\left(\frac{n}{4}\right)\sin\left(\frac{\pi}{4}n\right)$$

For $n = 0$, it is $x_0 y_0^* = 0$ because $\sin(0) = 0$. For all non-zero values of n, it easy to see that: $x_n y_n^* + x_{-n} y_{-n}^* = 0$, because the sine function is odd. So each term of the summation for a positive index n cancels out with the term for the corresponding negative value of the index. Therefore:

$$\left(x(t), y(t)\right) = \sum_{n=-\infty}^{+\infty} x_n y_n^* = 0$$

The same result can be obtained using the equivalent formula for the inner product which is valid for real signals (see Sect. 5.5.4):

$$\left(x(t), y(t)\right) = 2\sum_{n=1}^{+\infty} \mathrm{Re}\left\{x_n y_n^*\right\} + x_0 y_0$$

Eq. 5-10

The last term is zero because $y_0 = 0$. All the other terms are of the form:

$$\mathrm{Re}\left\{x_n y_n^*\right\} = \mathrm{Re}\left\{-j\,\mathrm{Sinc}\left(\frac{n}{2}\right)\mathrm{Sinc}\left(\frac{n}{4}\right)\sin\left(\frac{\pi}{4}n\right)\right\}$$

Eq. 5-11

However, each one of the $x_n y_n^*$ is purely imaginary, so $\mathrm{Re}\left\{x_n y_n^*\right\} = 0, \forall n$. As a result, every term of the inner product (Eq. 5-10) is zero and therefore:

$$\left(x(t), y(t)\right) = 0$$

Eq. 5-12

5.5.6 Problem

Find the energy of the following signal.

$$x(t) = \cos(2\pi f_0 t) + 2\sin(4\pi f_0 t) + \sqrt{2}\cos(8\pi f_0 t)$$

$$I = [0, T_0] \qquad f_0 = \frac{1}{T_0}$$

Solution:

This problem can be solved either directly, using the "native rule":

$$\mathcal{E}_{[0,T_0]}\{x(t)\} = (x(t), x(t)) = \int_0^{T_0} |x(t)|^2$$

or it can be solved using Parseval's formula:

$$\mathcal{E}\{x(t)\} = \sum_{n=-\infty}^{+\infty} |x_n|^2$$

We choose to use Parseval's formula but, to be able to use it, we need to project $x(t)$ over a suitable basis. We use the Fourier basis. As a result, we would have to evaluate the Fourier components:

$$x_n = \left(x(t), \hat{\varphi}_n(t)\right) = \frac{1}{\sqrt{T_0}} \int_0^{T_0} x(t) e^{-j\frac{2\pi}{T_0}nt} dt$$

However, from 5.4.3.1 we know that:

$\cos\left(2\pi K f_0 t\right)$ has only two non-zero Fourier components: $x_K = x_{-K} = \dfrac{\sqrt{T_0}}{2}$

$\sin\left(2\pi K f_0 t\right)$ has only two non-zero Fourier components:

$$x_K = \frac{-j}{2\sqrt{f_0}} = \frac{-j\sqrt{T_0}}{2}, \qquad x_{-K} = \frac{j}{2\sqrt{f_0}} = \frac{j\sqrt{T_0}}{2}$$

By inspecting $x(t)$ and using the above formulas, it is easy to immediately find:

$$
\begin{array}{ll}
x_0 = 0 & \\
x_1 = \tfrac{1}{2}\sqrt{T_0} & x_{-1} = \tfrac{1}{2}\sqrt{T_0} \\
x_2 = -j\sqrt{T_0} & x_{-2} = j\sqrt{T_0} \\
x_3 = 0 & x_{-3} = 0 \\
x_4 = \tfrac{\sqrt{2}}{2}\sqrt{T_0} & x_{-4} = \tfrac{\sqrt{2}}{2}\sqrt{T_0} \\
x_n = 0, \quad n > 4 & x_{-n} = 0, \quad n > 4
\end{array}
$$

Note that $x(t) \in \mathbb{R}$ so it must be: $x_n = x_{-n}^*$. In fact, a quick check of the above results shows that such condition is satisfied.

Finally, we apply Parseval's formula and get:

$$\mathcal{E}\{x(t)\} = \sum_{n=-\infty}^{+\infty} |x_n|^2 = \left(\tfrac{1}{4}+\tfrac{1}{4}+1+1+\tfrac{1}{2}+\tfrac{1}{2}\right)T_0 = \tfrac{7}{2}T_0$$

5.5.7 Problem On your own

On your own find the inner product $\left(x(t), y(t)\right)$ of the following signals:

$$x(t) = \frac{1}{2}\cos(2\pi f_0 t) + 2\sin(2\pi f_0 t) + \cos(4\pi f_0 t) + 2\sin(8\pi f_0 t)$$

$$y(t) = -\frac{1}{2}\cos(2\pi f_0 t) - \sin(2\pi f_0 t) - \frac{1}{3}\cos(4\pi f_0 t)$$

$$I = [0, T_0] \qquad f_0 = \frac{1}{T_0}$$

Solution:

This is found in a similar way to the previous exercise.

By direct inspection of $x(t)$ and $y(t)$ using the formulas 5.4.3.1 we directly find:

$$
\begin{array}{ll}
x_0 = 0 & \\
x_1 = \sqrt{T_0}\left(\tfrac{1}{4} - j\right) & x_{-1} = \sqrt{T_0}\left(\tfrac{1}{4} + j\right) \\
x_2 = \sqrt{T_0}\,\tfrac{1}{2} & x_{-2} = \sqrt{T_0}\,\tfrac{1}{2} \\
x_3 = 0 & x_3 = 0 \\
x_4 = -j\sqrt{T_0} & x_{-4} = j\sqrt{T_0}
\end{array}
$$

$$
\begin{array}{ll}
y_0 = 0 & \\
y_1 = \left(-\tfrac{1}{4} + \tfrac{1}{2}j\right)\sqrt{T_0} & y_{-1} = \left(-\tfrac{1}{4} - \tfrac{1}{2}j\right)\sqrt{T_0} \\
y_2 = -\tfrac{1}{6}\sqrt{T_0} & y_{-2} = -\tfrac{1}{6}\sqrt{T_0}
\end{array}
$$

We must now carry out the inner product. We remark that since $x(t), y(t) \in \mathbb{R}$, then:

$$\left(x(t), y(t)\right) = \sum_{n=-\infty}^{+\infty} x_n y_n^* = 2\sum_{n=1}^{+\infty} \mathrm{Re}\left\{x_n y_n^*\right\} + x_0 y_0$$

The easy calculations are as follows:

$$\left(x(t), y(t)\right) = x_0 y_0 + 2\,\mathrm{Re}\left\{x_1 y_1^*\right\} + 2\,\mathrm{Re}\left\{x_2 y_2^*\right\} + 0 + 0 + 0 \ldots$$

$$2\,\mathrm{Re}\left\{\left(\tfrac{1}{4} - j\right)\left(-\tfrac{1}{4} - \tfrac{1}{2} j\right)\right\} T_0 = 2T_0\left(-\tfrac{1}{16} - \tfrac{1}{2}\right) = -\tfrac{18}{16} T_0$$

$$2\,\mathrm{Re}\left\{\tfrac{1}{2}\left(-\tfrac{1}{6}\right)\right\} T_0 = -\tfrac{1}{6} T_0$$

$$\left(x(t), y(t)\right) = \left(-\tfrac{18}{16} - \tfrac{1}{6}\right) T_0 = \frac{-54-8}{48} T_0 = -\frac{31}{24} T_0$$

5.5.8 Optional: The raised-cosine signals

We have discussed at length the case of the Fourier series concerning an ideal rectangular signal. Here we address the case of a whole class of signals that have $\Pi(t)$ as a "limiting" case.

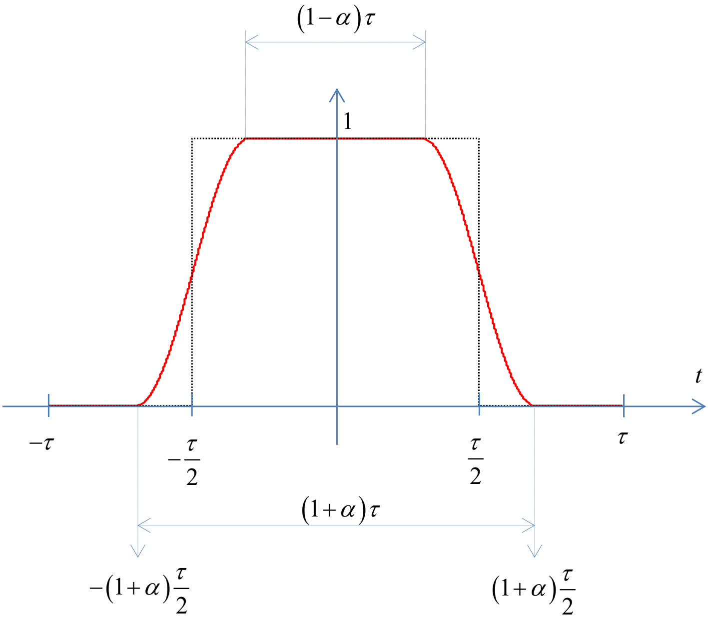
They are, however, "smoother" signals than $\Pi(t)$, because they are all continuous, and differentiable once. (The second derivative is instead discontinuous, check this on your own).

They are called "raised cosine signals" or "raised cosine pulses". The general form of such signals is:

$$\rho_{\tau,\alpha}(t) = \begin{cases} 1, & |t| \le \dfrac{1-\alpha}{2}\tau \\ \dfrac{1}{2}\left[1+\cos\left(\dfrac{\pi}{\alpha\tau}\left[|t| - \dfrac{1-\alpha}{2}\tau\right]\right)\right] & \dfrac{1-\alpha}{2}\tau < |t| < \dfrac{1+\alpha}{2}\tau \\ 0 & \text{elsewhere} \end{cases}$$

Eq. 5-13

The detailed plot of one of these signals is as follows:


$(1-\alpha)\tau$
1
t
$-\tau$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
τ
$(1+\alpha)\tau$
$-(1+\alpha)\frac{\tau}{2}$
$(1+\alpha)\frac{\tau}{2}$

$\boxed{\text{On your own}}$: given the plot above, noting that the curved sections are half of a cosine-waveform (properly raised and translated), derive the analytical definition given in formula Eq. 5-13.

The corresponding Fourier series coefficients, calculated assuming an interval $\mathbf{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2} \right]$, with $T_0 > \tau$ so that the whole support of the signal is comprised within $\mathbf{I}$, are:

$$\rho_n = \frac{\tau}{\sqrt{T_0}} \operatorname{Sinc}\left(n \frac{\tau}{T_0} \right) \frac{\cos\left(\pi \alpha n \frac{\tau}{T_0} \right)}{1 - 4\alpha^2 n^2 \frac{\tau^2}{T_0^2}}$$

$$= \tau \sqrt{f_0} \operatorname{Sinc}\left(n \tau f_0 \right) \frac{\cos\left(\pi \alpha n \tau f_0 \right)}{1 - 4\alpha^2 n^2 \tau^2 f_0^2}$$

Note that the key parameter is the so-called roll-off parameter, $0 \le \alpha \le 1$. Note also that, when $\alpha = 0$, the signal $\rho_{\tau,\alpha}(t)$ becomes a rectangle, specifically:

$$\rho_{\tau,0}(t) = \Pi_{\tau}(t)$$

In fact, the Fourier coefficients for the case $\alpha = 0$ coincide with those found for the rectangular signal $\Pi_{\tau}(t)$ in Eq. 5-3.

Instead, when $\alpha = 1$, the signal is just simply a single period of a cosine function, "raised up" by a constant $1/2$ so that it is all positive:

$$\rho_{\tau,1}(t) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}\cos\left(\pi \dfrac{t}{\tau}\right) & -\tau < t < \tau \\ 0 & \text{elsewhere} \end{cases}$$

The plots for the signal $\rho_{\tau,\alpha}(t)$ and the Fourier series coefficients are shown below:

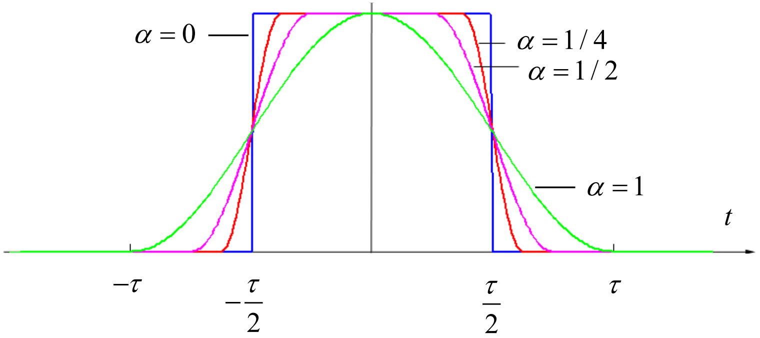


Fig. 5-7 Plot of the signal $\rho_{\tau,\alpha}(t)$ for several values of the roll-off factor α .

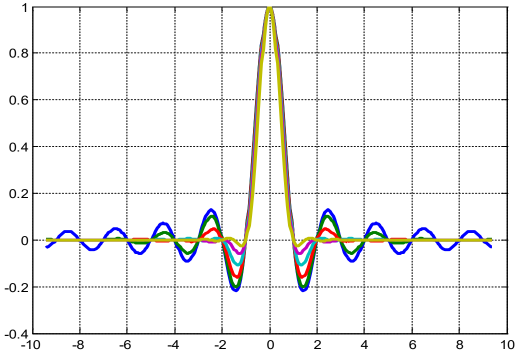


Fig. 5-8 Loci of points where the Fourier series coefficients ρ_n lie for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$. The dark blue curve is for $\alpha = 0$ and the yellow-green curve is for $\alpha = 1$. The abscissa is $n\tau / T_0$.

End of optional

5.6 Recording a Signal

Given an audio track of duration 200 seconds,

$$s(t) = \sum_{n=-\infty}^{+\infty} s_n \, \hat{\varphi}_n(t) = \sum_{n=-\infty}^{+\infty} s_n \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}} = \sum_{n=-\infty}^{+\infty} s_n \sqrt{f_0} e^{j2\pi n f_0 t} \qquad t \in [0, 200]$$

$$s(t) \in L^2[0, 200]$$

Eq. 5-14

we would like to 'record' the signal and store it digitally on a computer. We are going to discuss how to do it and how much information is generated, in MBytes.

First off, we remark that the representation basis is known beforehand. It can be considered as part of the 'recording standard'. Therefore, to record the signal, we need only store the Fourier components s_n of the signal.

However, looking at Eq. 5-14, we see an obvious problem: the number of components s_n to store is infinite.

$$s(t) = \sum_{n=-\infty}^{\infty} s_n \sqrt{f_0} e^{j2\pi n f_0 t}$$

We therefore need to resort to some approximation. To this goal, a discipline called 'psycho-acoustics' comes to our aid. It tells us that the human ear is capable of hearing frequencies only up to 20 kHz.

Therefore, we can neglect the part of the signal $s(t)$ that is made up of frequencies exceeding 20 kHz. In other words, we are going to store an approximated signal which does not keep all frequencies but only enough of them to ensure that the listening experience is not degraded.

In this respect, the Fourier basis representation is ideal, since it allows us to easily identify the signal components exceeding 20 kHz. The approximant will then be:

$$s_{\text{app}}(t) = \sum_{n=-N}^{N} s_n \sqrt{f_0} e^{j2\pi n f_0 t}$$

where N must be selected so that only frequencies exceeding 20 kHz are discarded. The simple calculation is:

$$Nf_0 = 20{,}000 \text{ Hz} \qquad f_0 = \frac{1}{T_0} = \frac{1}{200} \Rightarrow N = \frac{20.000}{f_0} = \frac{20.000}{f_0} \cdot 200 = 4{,}000{,}000$$

Theoretically we would have to store $2N$ components, N for the positive frequencies and N for the negative ones. However, the signal is an acoustic pressure wave (or an electric signal proportional to it coming out of a microphone) and therefore it is certainly a *real* signal. So $s_{-n} = s_n^*$. As a result, we only need to store components for $n > 0$, that is, we need to store only N components (4 million) and not $2N$.

However, each s_n is a complex number and so we need to store *two real numbers* for each one of them (8 million real numbers). Moreover, the signal is typically stereo (in fact, two different signals) and so we have to store twice the information (16 million real numbers).

We then assume that each real number is represented using 2 Bytes (16 bits). This value is enough to ensure a good fidelity. Then:

$$\left(16,000,000 \quad \text{real numbers}\right)\cdot\left(2 \;\; \text{bytes}\right) = 32 \;\; \text{Mbytes (MB)}$$

This result is consistent with the typical size of any audio track of the same time-length that can be found on an audio CD (approximately 30-40 MBytes).

Note however that the format in which the signal is stored in an audio CD is *different* than the one we discussed here. A comparison with the actual audio CD format will be proposed later on, after the sampling theorem has been introduced.

Note also that typical MP3 tracks are only about 2-3 Mbytes, that is about 1/10 the value found above. This is because MP3 *compresses the signal further* using various techniques. MP3 compression is however *lossy*, that is, there is a difference between the original signal and the one recorded and reproduced after MP3. Some of the fidelity is therefore lost.

5.7 Periodicity of the Fourier Reconstruction Formula

The Fourier sum, if viewed as the formula for the canonical reconstruction of a signal in $L^2_{\mathbf{I}}$ using the Fourier basis, is by definition limited to the independent variable *t* being within the interval where the signal space $L^2_{\mathbf{I}}$ is defined:

$$t \in \mathbf{I} = \left[t_1, t_2\right]$$

However, if a Fourier sum is taken out of this context, it can be viewed as a generic function of time and it is formally possible to evaluate it over larger time intervals and even up to $t \in \mathbb{R}$. The interesting result is that the obtained signal, which we call:

$$s_p\left(t\right) = \sum_{n=-N}^{N} s_n \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}} \qquad t \in \mathbb{R}$$

turns out to be periodic, with period exactly equal to T_0. This peculiar circumstance will be used in later chapters to describe and analyze *periodic signals*.

Proof: given the Fourier sum:

$$s_p(t) = \sum_{n=-N}^{N} s_n \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}} \qquad t \in \mathbb{R}$$

we write:

$$s_p(t - T_0) = \sum_{n=-N}^{N} s_n \frac{e^{j\frac{2\pi}{T_0}n(t-T_0)}}{\sqrt{T_0}}$$

$$= e^{j\frac{2\pi}{T_0}n(-T_0)} \sum_{n=-N}^{N} s_n \frac{e^{j\frac{2\pi}{T_0}nt}}{\sqrt{T_0}} = e^{j2n\pi} s_p(t) = s_p(t)$$

This conclusively proves that ***a Fourier sum is periodic of period*** T_0.

This property is used in various contexts, both to carry out calculations, but also for practical applications in digital signal processing.

5.8 Generating Dirac's delta using a 'Fourier series'

A very interesting result is found by *assuming that a signal has all identical Fourier components equal to* $1/\sqrt{T_0}$.

It should be pointed out that, by assuming that all components have the same value, *we clearly set ourselves outside of* $L^2_{\mathbf{I}}$. In fact, using Parserval's rule, we immediately find that the energy of the signal is:

$$\mathcal{E}\{s(t)\} = \sum_{n=-\infty}^{+\infty} |s_n|^2 = \sum_{n=-\infty}^{\infty} \left| \frac{1}{\sqrt{T_0}} \right|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} = \infty$$

so, the resulting signal, if any, is certainly not in $L^2_{\mathbf{I}}$. Indeed, it turns out that:

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T_0}} \hat{\varphi}_n(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{j\frac{2\pi}{T_0}nt} = \delta(t), \quad \mathbf{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2} \right]$$

Eq. 5-15

This should more correctly be written in the sense of distributions, as:

$$\lim_{\substack{\text{dist} \\ N\to\infty}} \frac{1}{T_0}\sum_{n=-N}^{N} e^{j\frac{2\pi}{T_0}nt} = \delta(t), \quad \mathbf{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$$

Eq. 5-16

It is a very important result which we will use repeatedly in the remainder of the course.

$\boxed{\text{On your own:}}$ show that:

$$\int_{-T_0/2}^{T_0/2} \left(\frac{1}{T_0}\sum_{n=-N}^{N} e^{j\frac{2\pi}{T_0}nt} \right) = 1 \quad , \quad \forall n$$

So all the "pulses" that tend to "approximate" $\delta(t)$ as N goes up, all integrate to 1. This is similar to what is the case with other pulses that can be used to obtain delta as a limit in the sense of distributions. Remember that delta itself integrates to 1.

$\boxed{\text{On your own:}}$ write a computer program that sums increasing numbers of series terms of Eq. 5-16 and plot the result. Does it appear to "converge" to $\delta(t)$?

$\boxed{\text{On your own:}}$ We just saw that assuming $s_n = 1/\sqrt{T_0}$ then the $\delta(t)$ is generated by the "reconstruction" formula. Could this value of the Fourier components s_n of $\delta(t)$ be also derived in a canonical way? Try to calculate the projections $s_n = \left(\delta(t), \hat{\phi}_n(t)\right)$ over the interval $\mathbf{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ and see what the results is. Does it coincide with $s_n = 1/\sqrt{T_0}$?

5.8.1 (Recommended) optional material

We provide in the following a computational indication that in fact the generated "signal" behaves as $\delta(t)$.

We know that the key defining property of $\delta(t)$ is the following:

$$\int_{-\infty}^{\infty} s(t)\,\delta(t)\,dt = s(0)$$

We then point out that the integral does not necessarily have to extend over the whole of $\mathbb{R}$. In fact, it is enough that it occurs over an interval which fully contains the origin. We can pick for instance the interval $\mathbf{I} = \left[-T_0/2, T_0/2\right]$ which clearly includes the origin. It is certainly still true that:

$$\int_{-T_0/2}^{T_0/2} s(t)\,\delta(t)\,dt = s(0)$$

We would then like to check that:

$$a=\int_{-T_0/2}^{T_0/2} s(t)\left(\lim_{N\to\infty}\frac{1}{T_0}\sum_{n=-N}^{N} e^{j\frac{2\pi}{T_0}nt}\right)dt \overset{?}{=} s(0)$$

where the question mark indicates that this is the equality that we are investigating.

We start out by pulling out the limit and the summation, and we also split the constant $1/T_0$:

$$a=\lim_{N\to\infty}\frac{1}{\sqrt{T_0}}\sum_{n=-N}^{N}\int_{-T_0/2}^{T_0/2} s(t)\frac{1}{\sqrt{T_0}}e^{j\frac{2\pi}{T_0}nt}dt$$

We then remark that the integral corresponds to the Fourier component s_{-n} of the signal $s(t)$. We can therefore write:

$$a = \lim_{N \to \infty} \frac{1}{\sqrt{T_0}} \sum_{n=-N}^{N} s_{-n}$$

Then, by simply swapping the sign of index in the summation:

$$a = \lim_{N \to \infty} \sum_{n=-N}^{N} s_n \frac{1}{\sqrt{T_0}}$$

Eq. 5-17

But from Fourier series theory we know that:

$$s(0)=\sum_{n=-\infty}^{\infty} s_n \frac{1}{\sqrt{T_0}} e^{j\frac{2\pi}{T_0}nt}\bigg|_{t=o} = \sum_{n=-\infty}^{\infty} s_n \frac{1}{\sqrt{T_0}}$$

Eq. 5-18

Comparing Eq. 5-17 and Eq. 5-18 it is then obvious that $a=s(0)$ and therefore:

$$a=\int_{-T_0/2}^{T_0/2} s(t)\left(\lim_{N\to\infty}\frac{1}{T_0}\sum_{n=-N}^{N} e^{j\frac{2\pi}{T_0}nt}\right)dt = s(0)$$

which in turn means that indeed

$$\lim_{N\to\infty}{}_{\text{dist}} \frac{1}{T_0}\sum_{n=-N}^{N} e^{j\frac{2\pi}{T_0}nt} = \delta(t), \quad \mathbf{I}=\left[-\frac{T_0}{2},\frac{T_0}{2}\right]$$

End of optional material.

5.8.2 Generalizing the delta formula

We should point out that Eq. 5-16 can be viewed as a stand-alone mathematical relationship, not necessarily related to Fourier series. Eq. 5-16 can be recast in a more general form using a generic parameter a and a generic independent variable x:

$$\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{a}nx} = \delta(x), \quad x \in \left[-\frac{a}{2},\frac{a}{2}\right]$$

Eq. 5-19

Also, it is easy to show that:

$$\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{a}nx} = \sum_{n=-\infty}^{\infty} e^{-j\frac{2\pi}{a}nx}$$

so, in fact, Eq. 5-19 can be further generalized as:

$$\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{\pm j\frac{2\pi}{a}nx} = \delta(x), \quad x \in \left[-\frac{a}{2},\frac{a}{2}\right]$$

Eq. 5-20

Depending on the context, *a* and *x* can be times, frequencies or other appropriate quantities, with or without physical dimensions. In particular, in a later chapter, the following version of the same equality will be used, where *a* and *x* are frequencies:

$$\frac{1}{f_0}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{f_0}nf} = \delta(f), \quad f \in \left[-\frac{f_0}{2},\frac{f_0}{2}\right]$$

5.9 The ‘train of deltas’

Using the result on the periodicity of the Fourier series, it can be easily shown that the general formula Eq. 5-20 generates a so-called *train of deltas*, that is a periodic signal made up of an infinite series of deltas whose support is spaced exactly *a*. In math:

$$\frac{1}{a}\sum_{n=-\infty}^{\infty} e^{\pm j\frac{2\pi}{a}nx} = \sum_{k=-\infty}^{\infty} \delta\left(x - n \cdot a\right) \qquad x \in \mathbb{R}$$

Eq. 5-21

This result too will be of great importance in later chapters.

$\boxed{\text{On your own:}}$ show that Eq. 5-16, due to the periodicity of the Fourier series, actually generates a train of deltas, with period T_0.

5.10 *Optional: Is the Fourier orthonormal set really a basis for L^2_I ?*

We have shown that given a Hilbert space $L^2_{\mathbf{I}}$ with $\mathbf{I} = [t_1, t_2]$ $t_1, t_2 < \infty$, and given an orthonormal set $\mathcal{U} = \{\hat{u}_n(t)\}_{n=1}^{\infty}$ in $L^2_{\mathbf{I}}$, such orthonormal set is a basis for $L^2_{\mathbf{I}}$ if:

$$\sum_{n=1}^{\infty} \hat{u}_n(t)\hat{u}_n^*(\tau) = \delta(t-\tau), \quad t, \tau \in [t_1, t_2]$$

Eq. 5-22

We would like to check whether this is the case for the Fourier basis Φ. By substituting the Fourier basis into Eq. 5-22, we find:

$$\sum_{n=1}^{\infty} \hat{\varphi}_n(t)\, \hat{\varphi}_n^*(\tau) = \sum_{n=-\infty}^{\infty} \frac{e^{j\frac{2\pi}{T}nt}}{\sqrt{T}} \frac{e^{-j\frac{2\pi}{T}n\tau}}{\sqrt{T}} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{T}n(t-\tau)}$$

So, we have to check whether:

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{T}n(t-\tau)} = \delta(t-\tau), \quad t,\tau \in [t_1, t_2]$$

However, it is easy to see that the above equation is the same as Eq. 5-21. We can then conclude that the Fourier orthonormal set Φ is indeed a basis for $L^2_{\mathbf{I}}$.

5.11 Appendix: frequently used formulas

Here are a few formulas that are frequently used.

$$\sin(-\alpha) = -\sin(\alpha)$$
$$\cos(-\alpha) = \cos(\alpha)$$

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos(\alpha)$$
$$\cos\left(\alpha - \frac{\pi}{2}\right) = \sin(\alpha)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\beta)$$

$$\cos(\alpha)\cos(\beta)=\frac{1}{2}\cos(\alpha+\beta)+\frac{1}{2}\cos(\alpha-\beta)$$

$$\sin(\alpha)\cos(\beta)=\frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta)$$

$$\sin(\alpha)\sin(\beta)=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$$

$$\cos^2(\alpha)=\frac{1}{2}+\frac{1}{2}\cos(2\alpha)$$

$$\sin^2(\alpha)=\frac{1}{2}-\frac{1}{2}\cos(2\alpha)$$

$$\int\cos(2\pi f_0 t+\varphi)\,dt=\frac{\sin(2\pi f_0 t+\varphi)}{2\pi f_0}+C$$

$$\int\sin(2\pi f_0 t+\varphi)\,dt=-\frac{\cos(2\pi f_0 t+\varphi)}{2\pi f_0}+C$$

$$e^{j\alpha} = \cos(\alpha) + j\sin(a)$$

$$\cos(\alpha) = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$\sin(\alpha) = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

5.12 *Questions*

5.12.1

Prove that the set of signals Φ defined as:

$$\Phi = \left\{\hat{\varphi}_n(t)\right\}_{n=-\infty}^{+\infty} = \left\{\frac{1}{\sqrt{T_0}} e^{j\frac{2\pi(t-\tau_n)}{T_0}n} e^{j\varphi_n}\right\}_{n=-\infty}^{+\infty}$$

$$t \in \mathbf{I} = [t_0,\ t_1],\ \ |t_1 - t_0| = T_0$$

is an orthonormal set over the interval $\mathbf{I}$. Note that the numbers τ_n and φ_n depend on the index n, that is they may be different for different values of n. In other words, for instance, $\varphi_3 \neq \varphi_7$, $\tau_{-5} \neq \tau_4$, etc.

5.12.2

A voice signal $v(t)$ should be digitally recorded. The maximum frequency that needs to be reproduced for good voice quality is 8 KHz. The duration of the signal is 1 hour. The signal is "mono" (not "stereo"). Each recorded number must be represented using 12 bits, again sufficient for good voice quality.

Find the size of the file in MBytes, assuming to use a canonical approximation based on the Fourier basis.

Answer: 86.4 MBytes

5.12.3

Prove that, if $s(t) \in \mathbb{R}$, then its Fourier components are such that $s_n = s_{-n}^*$. Try first on your own, then look in the chapter for the proof.

5.12.6

Prove that, given a signal $s(t) \in \mathbb{R}$, $t \in \mathbf{I} = [t_0,\ t_1]$, then the following relationship holds:

$$< s(t) >_{\mathbf{I}} = \frac{s_0}{\sqrt{T_0}}$$

where $< s(t) >_{\mathbf{I}}$ means time-average of the signal over the interval $\mathbf{I}$ and $T_0 = t_1 - t_0$

5.12.7

Consider the signals:

$$s(t) = \Pi_1(t - 1/2)$$
$$w(t) = \Pi_1(t + 1/2)$$

Find the Fourier series coefficients for the two signals, over the interval $t \in \mathbf{I} = [-2,\ 2]$ Calculate the inner product between the two signals using the generalized rule of the inner product, $\left(s(t), w(t)\right) = \sum_{n=-\infty}^{\infty} s_n w_n^*$. Do it also using the "native" rule. Check that the two results are identical.

Note: <u>you will need to use the following formula</u>:

$$\sum_{n=-\infty}^{\infty} \mathrm{Sinc}^2(n/4) \cdot (-j)^n = 0$$

Eq. 5-23

Answer

The two signals, once drawn, clearly appear to have disjoint supports. So, their inner product carried out using the native formula is obviously zero.

The signal components are:

$$s_n = \frac{1}{2}\mathrm{Sinc}\left(\frac{n}{4}\right)e^{-j\frac{n\pi}{4}}$$

$$w_n = \frac{1}{2}\mathrm{Sinc}\left(\frac{n}{4}\right)e^{j\frac{n\pi}{4}}$$

Using Eq. 5-23, it is easy to see that $\sum_{n=-\infty}^{\infty} s_n w_n^* = 0$, which confirms the orthogonality of the two signals.

5.12.8

Prove that the Fourier series is periodic in time. Try first on your own, then look in the chapter for the proof.

5.12.9

Find the energy and the inner product of the two signals over $L^2_{\mathbf{I}}$:

$$x(t) = \tfrac{1}{2}\sin(4\pi f_0 t) + 2\cos(8\pi f_0 t)$$

$$y(t) = -\tfrac{1}{2}\cdot 1(t) - \tfrac{1}{3}\sin(4\pi f_0 t)$$

$$\mathrm{I}=[0, T_0] \qquad f_0 = \frac{1}{T_0}$$

Answers

$$x_2 = \frac{\sqrt{T_0}}{4j}, \quad x_{-2} = -\frac{\sqrt{T_0}}{4j}, \quad x_4 = \sqrt{T_0}, \quad x_{-4} = \sqrt{T_0}$$

$$y_0 = -\frac{\sqrt{T_0}}{2}, \quad y_2 = -\frac{\sqrt{T_0}}{6j}, \quad y_{-2} = +\frac{\sqrt{T_0}}{6j}$$

$$\left(x(t),y(t)\right)=-\frac{T_0}{12}$$

$$\mathcal{E}\left\{x(t)\right\}=\frac{17}{8}T_0$$

$$\mathcal{E}\left\{y(t)\right\}=\frac{11}{36}T_0$$

5.12.10

Let the signals: $s(t) = \Pi_T(t)$ and $w(t) = e^{-at}u(t)$, with $a \in \mathbb{R}$, and $a > 0$. Both signals are considered in L^2_{I}, with $\mathrm{I} = \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ and $T < T_0$.

- Calculate the inner product of the two signals, both using the native rule and the generalized rule.
- Assuming $a = 1$, $T = 4$ and $T_0 = 6$, check that the two results computed above coincide (requires numerical computation).

Solution

The direct calculation of the inner product using the native rule is:

$$\left(s(t), w(t)\right) = \int_{-T_0/2}^{T_0/2} s(t) w^*(t)\, dt = \int_{-T_0/2}^{T_0/2} e^{-at} u(t) \Pi_T(t)\, dt$$

$$= \int_0^{T/2} e^{-at} dt = \frac{1 - e^{-aT/2}}{a}$$

Substituting the actual values into the parameters, we get:

$$\left(s(t), w(t)\right) = 1 - e^{-2} \approx \ 0.864664716763387$$

The calculation using the generalized inner product rule is:

$$\left(s(t), w(t)\right) = \sum_{n=-\infty}^{\infty} s_n w_n^* = \sum_{n=-\infty}^{\infty} \frac{T}{\sqrt{T_0}} \operatorname{Sinc}\left(n \frac{T}{T_0}\right) \cdot \frac{1}{\sqrt{T_0}} \frac{1 - e^{-aT_0/2} (-1)^n}{a - j 2\pi \frac{n}{T_0}}$$

The two results should match.

The following very simple few Matlab statements calculate the sum:

$$\sum_{n=-N}^{N} \frac{T}{\sqrt{T_0}} \mathrm{Sinc}\left(n\frac{T}{T_0}\right) \cdot \frac{1}{\sqrt{T_0}} \frac{1-e^{-aT_0/2}(-1)^n}{a-j2\pi\frac{n}{T_0}}$$

where the index ranges between $-N$ and N, with N set to 1000:

```
T0=6;T=4;a=1;N=1000;n=-N:N;
T/T0*sum(sinc(n*T/T0).*(1-exp(-a*T0/2)*(-1).^n)./(a-j*2*pi*n/T0))
```

The result is:

0.864664717098127 + 0.000000000000000i

This clearly shows the agreement between the two procedures.