Contents

1 Affine Varieties

Definition 1.1: Affine variety.
We define an **affine variety** to be the solution set of systems of polynomial equations.

Remark 1   We will work over a fixed algebraically closed field. For example we will work over $\mathbb{C}$ or over the algebraic closure of the finite fields $\mathbb{F}_q$, denoted with $\overline{\mathbb{F}}_q$.

Definition 1.2: Affine space.
We denote with $\mathbb{A}^n(\mathbb{K}) = \mathbb{A}^n$ the **affine** n-space:

$$\mathbb{A}^n(\mathbb{K}) := \{(a_1, a_2, \ldots, a_n) \in \mathbb{K}^n\}. \tag{1.1}$$

Definition 1.3: Polynomial in n indeterminates.
A **polynomial** with coefficients in $\mathbb{K}$ in indeterminates $x_1, x_2, \ldots, x_n$ is an expression of the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{I = (i_1, \ldots, i_n) \in \mathbb{N}^n} c_I \, x_i^{i_1} \cdot \ldots \cdot x_n^{i_n}, \tag{1.2}$$

where $c_I \in \mathbb{K}$ for all I and it is zero for all but finitely many choices of I.

Definition 1.4: Polynomial ring.
We define the set

$$\mathbb{K}[x_1, \ldots, x_n] := \{\text{polynomials in } x_1, \ldots, x_n \text{ with coefficients in } \mathbb{K}\}. \tag{1.3}$$

Definition 1.5: (affine) Algebraic set.
Let $S \subset \mathbb{K}[x_1, \ldots, x_n]$ a set of polynomials. We define the **zero set** (or **vanishing locus**) of S as

$$\mathbb{V}(S) = Z(S) := \{p \in \mathbb{A}^n \mid f(p) = 0 \, \forall \, f \in S\} \subset \mathbb{A}^n. \tag{1.4}$$

Subsets of $\mathbb{A}^n$ of the form of $\mathbb{V}(S)$ are called **(affine) algebraic sets**.
For finitely many polinomials $S = \{f_1, \ldots, f_n\}$ we denote

$$\mathbb{V}(S) =: \mathbb{V}(f_1, \ldots, f_n). \tag{1.5}$$

Remark 2   The set S defining the algebraic susbset $X = \mathbb{V}(S)$ is not unique. For instance:

- if f and g both vanish on X, then $f + g$ does so,

- if f vanishes on X, then for all $h \in \mathbb{K}[x_1, \ldots, x_n]$, also hf vanishes on X.

In particular we have, denoting by (S) the ideal generated by S, the following equality

$$\mathbb{V}(S) = \mathbb{V}((S)). \tag{1.6}$$

Remark 3   In this course a ring R will always be a commutative ring with unity.

1.1   Set-theoretical properties of algebraic sets

Lemma 1.6. *If S_1 and S_2 are sets of polynomials, then*

$$\mathbb{V}(S_1) \cup \mathbb{V}(S_2) = \mathbb{V}(S_1 \cdot S_2), \tag{1.7}$$

where $S_1 \cdot S_2 := \{fg \mid f \in S_1, g \in S_2\}$.

Moreover, for a family $\{S_i\}_{i \in I}$, with $S_i \in \mathbb{K}[x_1, \ldots, x_n]$, we have

$$\bigcap_{i \in I} \mathbb{V}(S_i) = \mathbb{V}(\bigcup_{i \in I} S_i). \tag{1.8}$$

Corollary 1.7. *Finite unions of algebraic sets and arbitrary intersections of algebraic sets are, again, algebraic sets.*

Definition 1.8: Zariski topology.
The **algebraic** subsets of $\mathbb{A}^n$ satisfy the properties for **closed** subsets in a topology:

- $\emptyset, \mathbb{A}^n$ are algebraic,

- finite unions and arbitrary intersections of algebraic subsets are algebraic.

This means they induce a topology, which is called the **Zarisky** topology.

From the above definition, in general, we will not refer to algebraic subsets of $\mathbb{A}^n$ as such, but as **closed** subsets in the **Zariski** topology.

Remark 4 The **Zariski** topology is not **Hausdorff**. For example in $\mathbb{A}^1$ the **Zariski** topology is exactly the cofinite topology.

1.2 Hilbert's Nullstellensatz

Definition 1.9: Associated ideal.
Let $X \subset \mathbb{A}^n$, the ideal associated to X is

$$\mathbb{I}(X) := \{f \in \mathbb{K}[x_1, \ldots, x_n] \mid f(p) = 0 \ \forall p \in X\}. \tag{1.9}$$

Lemma 1.10. *Let $S, T \subset \mathbb{K}[x_1, \ldots, x_n]$ and $X, Y \subset \mathbb{A}^n$, then*

1. *$\mathbb{V}$ and $\mathbb{I}$ reverse inclusions, i.e.*

$$S \subset T \implies \mathbb{V}(T) \subset \mathbb{V}(S) \tag{1.10}$$
$$X \subset Y \implies \mathbb{I}(Y) \subset \mathbb{I}(X), \tag{1.11}$$

2. *$X \subset \mathbb{V}(\mathbb{I}(X))$ and $S \subset \mathbb{I}(\mathbb{V}(S))$,*

3. *if X is algebraic, then $X = \mathbb{V}(\mathbb{I}(X))$.*

Remark 5 In general, given an ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$, the identity

$$\mathcal{I} = \mathbb{I}(\mathbb{V}(\mathcal{I})) \tag{1.12}$$

is false.

Lemma 1.11. *Let $(a_1, \ldots, a_n) \in \mathbb{A}^n$, then*

$$\mathbb{I}(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n). \tag{1.13}$$

Theorem 1.12 (Hilbert's Nullstellensatz). *Let $\mathbb{K}$ be an algebraically closed field and $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$. If $(f_1, \ldots, f_n) \neq (1)$ (i.e. it is a proper ideal), then $\mathbb{V}(f_1, \ldots, f_n) \neq \emptyset$ in $\mathbb{A}^n(\mathbb{K})$.*

Remark 6 This theorem is the reason why we work with algebraically closed fields, moreover, if $\mathbb{K}$ is not algebraically closed, then a common zero of $f_1, \ldots, f_n$ will exist over a finite extension of $\mathbb{K}$.

Corollary 1.13. *If $\mathbb{K}$ is algebraically closed, then $\mathbb{I}$ and $\mathbb{V}$ are bijections between the set of points in $\mathbb{A}^n$ and the set of maximal ideals in $\mathbb{K}[x_1, \ldots, x_n]$.*

Definition 1.14: Radical of an ideal.
Let R be a ring and $\mathcal{I} \subset R$ be an ideal of R. We define the radical of $\mathcal{I}$, denoted with $\sqrt{\mathcal{I}}$, to be

$$\sqrt{\mathcal{I}} := \{x \in R \mid \exists\, n \in \mathbb{N}_+, \text{ s.t. } x^n \in \mathcal{I}\}. \tag{1.14}$$

If $\mathcal{I} = \sqrt{\mathcal{I}}$ we say that the ideal $\mathcal{I}$ is radical.

Proposition 1.15 (Nullstellensatz)**.** *Let $\mathbb{K}$ be algebraically closed and $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal, then*

$$\mathbb{I}(\mathbb{V}(\mathcal{I})) = \sqrt{\mathcal{I}}. \tag{1.15}$$

Corollary 1.16. *The maps $\mathbb{I}$ and $\mathbb{V}$, from algebraic sets in $\mathbb{A}^n(\mathbb{K})$ to radical ideals in $\mathbb{K}[x_1, \ldots, x_n]$ are inclusion-reversing correspondences that are inverse to one another.*

Let's now state some basic operations on algebraic sets:

Lemma 1.17. *Let $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbb{K}[x_1, \ldots, x_n]$ ideals, then*

1. *$\mathbb{V}(\mathcal{I}_1) \cap \mathbb{V}(\mathcal{I}_2) = \mathbb{V}(\mathcal{I}_1 + \mathcal{I}_2)$,*

2. *$\mathbb{V}(\mathcal{I}_1) \cup \mathbb{V}(\mathcal{I}_2) = \mathbb{V}(\mathcal{I}_1 \mathcal{I}_2) = \mathbb{V}(\mathcal{I}_1 \cap \mathcal{I}_2)$.*

Lemma 1.18. *Let $X, Y \subset \mathbb{A}^n$ algebraic subsets, then*

1. *$\mathbb{I}(X \cup Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$,*

2. *$\mathbb{I}(X \cap Y) = \sqrt{\mathbb{I}(X) + \mathbb{I}(Y)}$.*

1.3 Reducibility

Definition 1.19: Reducibile subsets.
A topological space X is **reducible** iff there exist proper closed subsets $X_1, X_2 \subsetneq X$ s.t.

$$X = X_1 \cup X_2. \tag{1.16}$$

If X is not reducible, it is said to be **irreducible**.

Definition 1.20: Connected space.
If X is reducible and it is a disjoint union of closed subsets

$$X = X_1 \cup X_2, \tag{1.17}$$

then we say that X is **disconnected**. If X is not disconnected it is said to be **connected**.

Remark 7 In order to be **disconnected**, a space X has to be **reducible**, i.e. there exist no irreducible disconnected spaces.

Remark 8 If a space is **Hausdorff**, then only the singletons are **irreducible**.

Definition 1.21: Affine variety.
An irreducible algebraic set in $\mathbb{A}^n$, with the Zariski topology, is called an **affine variety**.

Lemma 1.22 (Algebraic characterization of irreducibility). *An algebraic set $X \subset \mathbb{A}^n$ is an affine variety iff $\mathbb{I}(X) \subset \mathbb{K}[x_1, \ldots, x_n]$ is a prime ideal.*

Remark 9   Recall: $\mathcal{I} \subset R$ is a prime ideal iff $R/\mathcal{I}$ is an integral domain.

Definition 1.23: Noetherian topological space.
A topological space X is called **Noetherian** iff every descending chain of closed subset

$$X \supset X_1 \supset X_2 \supset X_3 \supset \ldots \supset X_n \supset X_{n+1} \supset \ldots \tag{1.18}$$

is stationary.

Remark 10   Each algebraic set, in the Zariski topology, is **Noetherian**. This follows from the fact that $\mathbb{K}[x_1, \ldots, x_n]$ (hence every quotient of this ring) is a Noetherian ring, therefore, by Nullstellensatz,

$$\mathbb{I}(X) \subset \mathbb{I}(X_1) \subset \mathbb{I}(X_2) \subset \ldots \subset \mathbb{I}(X_n) \subset \mathbb{I}(X_{n+1}). \tag{1.19}$$

Since the maps $\mathbb{V}$ and $\mathbb{I}$ reverse inclusions we have our claim.

Proposition 1.24. *Every **Noetherian** topological space C can be written as*

$$X = X_1 \cup X_2 \cup \ldots \cup X_r, \tag{1.20}$$

a finite union of disjoint irreducible closed subsets, with $X_1 \not\subset X_j$ for $j \neq i$. Moreover this decomposition is unique up to order of $X_1, \ldots, X_r$.

Definition 1.25: Irreducible components.
The irreducible subsets $X_1, \ldots, X_r$ are called the **irreducible components** of X.

Corollary 1.26. *Every affine algebraic set X is the finite union of affine varieties $X_1, \ldots, X_r$, i.e.*

$$X = X_1 \cup \ldots \cup X_r. \tag{1.21}$$

From the algebraic point of view this can be viewed using Hilbert Nullstellensatz: let $\mathbb{I}(X)$ be the radical ideal associated to X and $\mathbb{I}(X_i)$ the prime ideals associated to X_i, then

$$\mathbb{I}(X) = \mathbb{I}(X_1 \cup \ldots \cup X_r) = \mathbb{I}(X_1) \cap \ldots \cap \mathbb{I}(X_r), \tag{1.22}$$

i.e. every radical ideal in $\mathbb{K}[x_1, \ldots, x_n]$ is the finite intersection of prime ideals.

Remark 11   If one reads (and adapts) the proof of the above proposition, one can prove that each algebraic set is the disjoit union of finitely many connected components.

Definition 1.27: Dimension of a Noetherian topological space.
Let X be a nonempty irreducible topological space. We define the dimension of X to be the largest index n s.t. there exist a chain of length $n + 1$

$$\emptyset \neq X_0 \subsetneq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X \tag{1.23}$$

of irreducible closed subsets of X.

If X is an arbitrary Noetherian topological space (it might be reducible), the dimension of X is defined to be the supremum (hence the maximum) of the dimensions of its irreducible components.

Remark 12 If all components of X have the same dimension we say that X is **pure-dimensional**. One gives special names to small pure-dimensional topological spaces:

- if X is of pure dimension 1, we call X a **curve**,

- if X is of pure dimension 2, we call X a **surface.**

Remark 13: Characterization of irreducibility.
The following properties are equivalent:

- X is irreducible,

- any two nonempty open subsets of X have a nonempty intersection,

- every nonempty open subset U is dense in X, i.e. $\overline{U} = X$,

- every nonempty open subset of X is connected.

Moreover, the image of an irreducible subset under a continuous map is again irreducible.

Definition 1.28: Hypersurface.
$X \subset \mathbb{A}^n$ is an **hypersurface** iff $X = \mathbb{V}(f)$ for a single polynomial f of positive degree.

Example: Irreducible varieties.

- Every linear subspace of $\mathbb{A}^n$ is irreducible (i.e. it is an affine variety). As a consequence we obtain that $\dim \mathbb{A}^n \geq n$.

- An hypersurface $X = \mathbb{V}(f)$ is irreducible iff f is a power of an irreducible polynomial.

1.4 Functions and Morphisms

Definition 1.29: Regular function.
Given $X \subset \mathbb{A}^n$ a **Zariski** closed subset, a function

$$f : X \to \mathbb{K} \tag{1.24}$$

is said to be **regular** iff there exist $F \in \mathbb{K}[x_1, \ldots, x_n]$ s.t.

$$f(a) = F(a) \quad \forall\, a \in X. \tag{1.25}$$

Remark 14 Notice that, in the above definition, F is unique up to an element in $\mathbb{I}(X)$.

Definition 1.30: Coordinate ring.
The coordinate ring of X is

$$\underbrace{\mathbb{K}[X] := \mathbb{A}(X)}_{\text{alternative notations}} := \mathbb{K}[x_1, \ldots, x_n] / \mathbb{I}(X). \tag{1.26}$$

It is the set of all regular functions on X

Construction 1.31: Localization.
Let R be a ring. A set $S \subset R$ is **multiplicatively closed** iff $1 \in S$ and $\forall f, g \in S$ then also $fg \in S$.

We construct on $R \times S$ the equivalence relation

$$(f, g) \sim (f', g') \iff \exists h \in S \text{ s.t. } h(fg' - f'g) = 0. \tag{1.27}$$

We then define the localization of R at S as

$$S^{-1}R := (R \times S)/\sim . \tag{1.28}$$

We will denote the elements of the localization as fractions:

$$[(f, g)] =: \frac{f}{g}. \tag{1.29}$$

Remark 15

- If R is a **domain**, then we can always take $h = 1$ in the definition of the equivalence relation. In this case, if $S \subset S'$ are multiplicatively closed subsets, then

$$S^{-1}R \hookrightarrow (S')^{-1}R \tag{1.30}$$

$$\frac{f}{g} \mapsto \frac{f}{g} \tag{1.31}$$

  is injective, provided $0 \notin S'$.

- $S^{-1}R$ is a ring, with addition and multiplication defined as usual with fractions:

$$\frac{f}{g} + \frac{f'}{g'} := \frac{fg' + f'g}{gg'}, \tag{1.32}$$

$$\frac{f}{g} \cdot \frac{f'}{g'} := \frac{ff'}{gg'}. \tag{1.33}$$

Definition 1.32: Local ring.
Let R be a ring. We say that R is **local** iff it has a unique maximal ideal.

Example

1. Let $a \in R$ be an arbitrary element. Let $S := \{a^n \mid n \in \mathbb{N}\}$, then we define

$$R_a := S^{-1}R, \tag{1.34}$$

   and we call it the localization of R at a.

2. Let $\mathcal{P} \subset R$ a **prime** ideal of R. let $S := R \setminus \mathcal{P}$, then we denote with

$$R_{\mathcal{P}} := S^{-1}R \tag{1.35}$$

   the localization of R at $\mathcal{P}$. Here we can describe the ideals of $R_{\mathcal{P}}$, they are of the form

$$\left\{ \frac{f}{g} \;\middle|\; f \in \mathcal{I}, g \notin \mathcal{P} \right\}, \tag{1.36}$$

   for some $\mathcal{I} \subset \mathcal{P}$ ideal of R.

In particular $R_{\mathcal{P}}$ has a unique maximal ideal

$$M = \left\{ \frac{f}{g} \;\middle|\; f \in \mathcal{P}, g \notin \mathcal{P} \right\} \tag{1.37}$$

hence it is a **local** ring.

3. Let $S := R \setminus \{\text{zero divisors}\}$. This clearly is a multiplicatively closed subset, since $\{\text{zero divisors}\}$ is a prime ideal of R.

   If R is an integral domain, then $S = R \setminus \{0\}$ and we denote the localization as

$$Q(R) := S^{-1}R, \tag{1.38}$$

the field of fractions of R.

Remark 16 When X is an affine **variety**, then $\mathbb{I}(X)$ is prime, hence $\mathbb{K}[X]$ is an integral domain.

Definition 1.33: Field of rational functions.
Let $X \subset \mathbb{A}^n$ be an affine variety. We define $\mathbb{K}(X) := Q\left(\mathbb{K}[X]\right)$ to be the field of **rational functions** on X.

Remark 17 We call the elements of $\mathbb{K}(X)$ rational functions even if their values may not be defined on all points of X.

Definition 1.34: Regular functions.

1. We say that $\varphi \in \mathbb{K}(X)$ is **regular at** $p \in X$ iff

$$\varphi = \frac{f}{g}, \text{ with } f, g \in \mathbb{K}[X] \text{ and } g(p) \neq 0, \tag{1.39}$$

   i.e. φ is well defined at p,

2. We define the **local ring of** X at $p \in X$ to be

$$\mathcal{O}_{X,p} := \{\varphi \in \mathbb{K}(X) \mid \varphi \text{ is regular at } p\}, \tag{1.40}$$

3. For any nonempty open $U \subset X$, we define

$$\mathcal{O}_X(U) := \bigcap_{p \in U} \mathcal{O}_{X,p}, \tag{1.41}$$

the ring of regular functions on U.

Remark 18 In general

$$\mathcal{O}_X(U) \neq \left\{ \frac{f}{g} \;\middle|\; f, g \in \mathbb{K}[X], \; g(p) \neq 0 \; \forall \, p \in U \right\}. \tag{1.42}$$

Definition 1.35: $\mathbb{K}$-algebra.
A $\mathbb{K}$-algebra is a ring R which is also a $\mathbb{K}$-Vector Space, s.t. the ring multiplication is $\mathbb{K}$-bilinear, i.e.

$$\lambda(fg) = (\lambda f)g = f(\lambda g), \tag{1.43}$$

for $f, g \in R$ and $\lambda \in \mathbb{K}$.

Moreover we define morphisms of $\mathbb{K}$-algebras as those maps between $\mathbb{K}$-algebras that are both $\mathbb{K}$-linear and ring homomorphisms.

Remark 19 Rings of functions are always $\mathbb{K}$-algebras

Lemma 1.36 (Identity principle). *Let $U \subset V \subset X$ be nonempty open subsets of an affine variety X. If $\phi_1, \phi_2 \in \mathcal{O}_X(V)$ coincide when restricted to U, then they are the same function also on V, i.e. for $\phi_1, \phi_2 \in \mathcal{O}_X(V)$*

$$\phi_1|_U = \phi_2|_U \in \mathcal{O}_X(U) \implies \phi_1 = \phi_2 \in \mathcal{O}_X(V). \tag{1.44}$$

It is the same property that holds for holomorphic functions.

Remark 20 Recall that an affine variety X is irreducible, this means that any open subset in X is dense. With this in mind the above result should not sound too strange. It actually seems less strong than what can be proved for holomorphic functions.

Definition 1.37: Distinguished open subsets.
Let $X \subset \mathbb{A}^n$ be an affine variety and $f \in \mathbb{K}[X]$ a regular function on X, then

$$X_f := \{p \in X \mid f(p) \neq 0\} = X \setminus \mathbb{V}(F), \tag{1.45}$$

for some $F \in \mathbb{K}[x_1, \ldots, x_n]$ s.t. $[F] = f \in \mathbb{K}[X]$, is an open subset of X (in the Zariski topology). It is in fact the complement of a hypersurface, which is closed in X. Such open subsets are called **distinguished open** subsets of X.

Remark 21 The distinguished open subsets form a base for the Zariski topology on X. It means that any open subset can be written as a union of distinguished open subsets.

By Hilbert's basis theorem we can say even more: every closed subset is the intersection of finitely many hypersurfaces, hence (taking the complements) every open subset is a finite union of distinguished open subsets.

Proposition 1.38. *For any $f \in \mathbb{K}[X]$ we have*

$$\mathcal{O}_X(X_f) = \left\{ \phi \in \mathbb{K}(X) \ \middle| \ \phi = \frac{g}{f^r} \text{ for some } r \geq 0, \ g \in \mathbb{K}[X] \right\}. \tag{1.46}$$

In particular, $\mathcal{O}_X(X_f) \cong K[X]_f$.

Corollary 1.39. *If $f = 1$, then $X_f = X$, hence the proposition grants*

$$\mathcal{O}_X(X) \cong \mathbb{K}[X], \tag{1.47}$$

hence the rational functions which are regular at each point of X are exactly the regular functions on X.

Remark 22 Also the local ring $\mathcal{O}_{X,p}$ is a localization of $\mathbb{K}[X]$:

$$\mathfrak{m}_{X,p} := \{g \in \mathbb{K}[X] \mid g(p) = 0\} = \mathbb{I}(\{p\})/\mathbb{I}(X) \subset \mathbb{K}[X] \tag{1.48}$$

is a maximal, hence prime, ideal. This is true, since the map

$$\mathbb{K}[X]/\mathfrak{m}_{X,p} \to \mathbb{K} \tag{1.49}$$

$$[g] \mapsto g(p) \tag{1.50}$$

is an isomorphism. Moreover one can prove that the natural map

$$\mathcal{O}_{X,p} \to \mathbb{K}[X]_{\mathfrak{m}_{X,p}} \tag{1.51}$$

$$\frac{f}{g} \mapsto \frac{f}{g}, \tag{1.52}$$

with $g(p) \neq 0$, is an isomorphism.

Remark 23 In general, for $U \subset X$ an arbitrary open subset, in order to compute $\mathcal{O}_X(U)$ we proceed as follows:

- we write U as the union of finitely many distinguished open subsets (coming from the generators of $\mathbb{I}(X \setminus U)$),

- we use the decomposition $U = X_{f_1} \cup X_{f_2} \cup \ldots \cup X_{f_l}$ and $\mathcal{O}_X(X_{f_j}) = \mathbb{K}[X]_{f_j}$ to figure out what $\mathcal{O}_X(U)$ is.

Example Let $X = \mathbb{A}^2$ and $X \supset U := \mathbb{A}^2 \setminus \{(0,0)\}$. We can write $U = \mathbb{A}^2_x \cup \mathbb{A}^2_y$. Carrying out some computations on these distinguished open subsets we can arrive at the conclusion that

$$\mathcal{O}_{\mathbb{A}^2}\left(\mathbb{A}^2 \setminus \{(0,0)\}\right) = \mathbb{K}[x,y] = \mathbb{K}[\mathbb{A}^2]. \tag{1.53}$$

This result can be interpeted as: "the point $(0,0)$ is a removable singularity".

Remark 24 Since we can check regularity at a neighborhood of single points, then regularity is a **local** property.

2 Sheaves

Let X be a topological space.

Definition 2.1: Presheaf.
A **presheaf** of rings $\mathcal{F}$ on X is the assignment of a ring $\mathcal{F}(U)$ to each open subsets $U \subset X$, and to each pair of comparable open subset U, V (i.e. $U \subset V$), of a ring homomorphism

$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U), \tag{2.1}$$

satisfying the following requirements:

    S1 $\mathcal{F}(\emptyset) = 0$,

    S2 $\rho_{U,U} = id_{\mathcal{F}(U)}$,

    S3 for all $U \subset V \subset W$, then

$$\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}. \tag{2.2}$$

We call the rings $\mathcal{F}(U)$ sections of $\mathcal{F}$ over U and the morphism $\rho_{V,U}$ the restriction morphisms, which we denote as

$$\rho_{V,U}(f) =: \left. f \right|_V. \tag{2.3}$$

Definition 2.2: Sheaf.
A presheaf $\mathcal{F}$ of rings is called a **sheaf** if it fulfills the gluing condition:

    S4 for each open subset $U \subset X$, each open covering $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of U and each family of sections $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in \mathcal{A}}$ s.t. s_α and s_β have the same restriction to $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in \mathcal{A}$, there is a **unique** section $s \in \mathcal{F}(U)$ s.t. $\rho_{U,U_\alpha}(s) = s_\alpha$ for all $\alpha \in \mathcal{A}$.

Remark 25

    S2,S3 ensure that $\mathcal{F}$ is a contravariant functor

$$\mathcal{F} : \{U \subset X \mid U \text{ is open in } U\} =: \mathsf{Op}(X) \to \mathsf{Rings}. \tag{2.4}$$

    S1 is customary in algebraic geometry, but it is a trivial requirement: given a functor $\mathcal{F}$ as above, then

$$\widetilde{\mathcal{F}}(U) = \begin{cases} 0 & \text{if } U = \emptyset, \\ \mathcal{F}(U) & \text{if } U \neq \emptyset \end{cases} \tag{2.5}$$

    is a presheaf.

    S4 $\implies$ **S1** it follows from the case when $\mathcal{A} = \emptyset$, hence any **sheaf** is also a **presheaf**.

Remark 26 We can actually define (pre-)sheaves of abelian groups, by requiring $\mathcal{F}(U)$ to be abelian groups and the maps $\rho_{U,V}$ to be group homomorphisms.

Remark 27 We can also define (pre-)sheaves of $\mathbb{K}$-algebras, as before: $\mathcal{F}(U)$ are $\mathbb{K}$-algebras, and $\rho_{U,V}$ are moprhisms of $\mathbb{K}$-algebras.

Example Let $X \subset \mathbb{A}^n$ be an affine variety, then the rings $\mathcal{O}_X(U)$, with the natural restriction maps

$$\mathcal{O}_X(V) \to \mathcal{O}_X(U) \quad \text{for } U \subset V \tag{2.6}$$

form a sheaf of rings ($\mathbb{K}$-algebras). We denote this sheaf by $\mathcal{O}_X$ and it is called the **sheaf of regular functions**, or the **structure sheaf**, on X.

Definition 2.3: Ringed space.
A pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X is called a **ringed space**. Moreover $\mathcal{O}_X$ is called the **structure sheaf** of the ringed space.

Remark 28 In general, spaces of functions easily form a presheaf of rings. However, they only form a sheaf if the conditions imposed on the functions are local. In fact the condition **S4** guarantees that we can check whether a function belongs to the sections locally.

Definition 2.4: Restrictions of (pre-)sheaves.
Let $U \subset X$ be an open subset. Given a (pre-)sheaf $\mathcal{F}$ on X, we define its restriction $\mathcal{F}|_U$ as the sheaf defined by

$$\mathcal{F}|_U (V) := \mathcal{F}(V), \tag{2.7}$$

for each open subset $V \subset U$ and we take the same restriction morphisms as $\mathcal{F}$. Notice that, given a sheaf $\mathcal{F}$ also $\mathcal{F}|_U$ is a sheaf.

Generalization of the **local rings** $\mathcal{O}_{X,p}$ to sheaves:

Definition 2.5: Stalk of a (pre-)sheaf.
Consider $\mathcal{F}$ a presheaf on X and a point $p \in X$. Consider the quotient space

$$\mathcal{F}_p := \left\{ (U, \phi) \,\middle|\, p \in U \overset{\text{open}}{\subset} X, \phi \in \mathcal{F}(U) \right\} / \sim, \tag{2.8}$$

where the equivalence relation is defined as follows: $(U_1, \phi_1,) \sim (U_2, \phi_2)$ iff $\exists V \overset{\text{open}}{\subset} X$ s.t. $p \in V \subset U_1 \cap U_2$, and $\phi_1|_V = \phi_2|_V$.

This quotient space is called the **stalk** of $\mathcal{F}$ at p, whereas its elements $[(U, \phi)] \in \mathcal{F}_p$ are called the **germs** of $\mathcal{F}$ at p.

Remark 29 $\mathcal{F}_p$ inherits the algebraic structure from $\mathcal{F}(U)$. (We just need to check that the operations defined on $\mathcal{F}(U)$ are preserved by the equivalence relation). In other words if $\mathcal{F}$ is a sheaf of rings, then $\mathcal{F}_p$ is a ring for any $p \in X$.

Lemma 2.6. *Let X be an affine variety and $p \in X$, then the stalk of $\mathcal{F} = \mathcal{O}_X$ at p is (canonically isomorphic to) the local ring $\mathcal{O}_{X,p}$.*

3   Morphisms between affine varieties

Definition 3.1: Morphism of ringed spaces.
Given two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a **morphism of ringed spaces** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ consists of:

- a continuous map $f : X \to Y$,

- for each open subset $U \subset Y$, a ring homomorphism

$$f_U : \mathcal{O}_Y(U) \to \mathcal{O}_X\left(f^{-1}(U)\right), \tag{3.1}$$

  s.t. the following diagram commutes for all $U \overset{\text{open}}{\subset} V \overset{\text{open}}{\subset} Y$

$$
\begin{array}{ccc}
\mathcal{O}_Y(V) & \xrightarrow{\ \ \rho_{V,U}\ \ } & \mathcal{O}_Y(U) \\
{\scriptstyle f_V}\big\downarrow & & \big\downarrow{\scriptstyle f_U} \\
\mathcal{O}_X\left(f^{-1}(V)\right) & \xrightarrow[\ \rho_{f^{-1}U, f^{-1}U}\ ]{} & \mathcal{O}_X\left(f^{-1}(U)\right)
\end{array}
\quad . \tag{3.2}
$$

Definition 3.2: Isomorphism of ringed spaces.
An **isomorphism of ringed spaces** is a morphism of ringed spaces

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \tag{3.3}$$

s.t. there exists another morphism of ringed spaces

$$g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X) \tag{3.4}$$

with $g \circ f = id_{(X, \mathcal{O}_X)}$ and $f \circ g = id_{(Y, \mathcal{O}_Y)}$.

Remark 30   Notice that we only work with ringed spaces $(X, \mathcal{O}_X)$ s.t.

- the elements of $\mathcal{O}_X(U)$, for any $U \overset{\text{open}}{\subset} X$ are functions $U \to \mathbb{K}$,

- the restriction maps are given by

$$\mathcal{O}_X(V) \to \mathcal{O}_X(U) \tag{3.5}$$
$$\phi \mapsto \phi|_U . \tag{3.6}$$

This implies that a continuous map $f : X \to Y$ induces, for any $U \overset{\text{open}}{\subset} Y$, a ring homomorphism f^*:

$$\mathcal{O}_Y(U) \xrightarrow{f^*} \mathcal{O}_X\left(f^{-1}(U)\right) \tag{3.7}$$
$$\phi \mapsto f^*\phi := \phi \circ f. \tag{3.8}$$

We will almost always take these ring homomorphisms as our f_U.

Remark 31   In our case, then, a morphism of ringed space $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a continuous map

$$f : X \to Y \tag{3.9}$$

such that $f^*\phi \in \mathcal{O}_X\left(f^{-1}(U)\right)$ for any $U \overset{\text{open}}{\subset} Y$ and $\phi \in \mathcal{O}_Y(U)$.

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Definition 3.3: Morphism of affine varieties.
A **morphism of affine varieties**

$$f : X \to Y \tag{3.10}$$

is a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ (with $f_U := f^*$ for all $U \overset{\text{open}}{\subset} Y$).

Lemma 3.4. *Let X, Y be **affine varieties** and $f : X \to Y$ a continuous map. TFAE:*

1. *f is a morphism,*

2. *for every $\phi \in \mathcal{O}_Y(Y)$ we have $f^*\phi \in \mathcal{O}_X(X)$, i.e. the pull-back of any regular function on Y is regular everywhere on X,*

3. *for every $p \in X$ and $\phi \in \mathcal{O}_{Y,f(p)}$ we have $f^*\phi \in \mathcal{O}_{X,p}$, i.e. the pull-back of a function regular at $f(p)$ is regular at p.*

Example

1. The following is a morphism of varieties

$$f : \mathbb{A}^1 \to \mathbb{A}^1 \tag{3.11}$$

$$x \mapsto x^2, \tag{3.12}$$

2. Let X, Y be affine varieties, $f_1, \ldots, f_n \in \mathbb{K}[X]$. Then a map of the following form is always a morphism

$$f : X \to Y \tag{3.13}$$

$$a \mapsto (f_1(a), \ldots, f_n(a)) . \tag{3.14}$$

Lemma 3.5. *Let $X \subset \mathbb{A}^m, Y \subset \mathbb{A}^n$ be affine varieties. We have a one to one correspondance*

$$\left\{ \begin{array}{c} \text{morphisms of varieties} \\ f : X \to Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{K} - \text{algebra homomorphisms} \\ \phi : \mathbb{K}[Y] \to \mathbb{K}[X] \end{array} \right\} . \tag{3.15}$$

$$f \longmapsto f^*$$

Remark 32 Applying the lemma we can view the ring of regular functions on a variety as

$$\mathbb{K}[X] = \left\{ f : X \to \mathbb{A}^1 \mid f \text{ is a morphism of varieties} \right\} . \tag{3.16}$$

Lemma 3.6. *Let $f : X \to Y$ be a morphism of affine varieties and let $f^* : \mathbb{K}[Y] \to \mathbb{K}[X]$ be the associated map of $\mathbb{K}$-algebras.*

- *If f is surjective, then f^* is injective.*

- *f^* is injective iff $f(X)$ is dense in Y.*

- *If f^* is surjective, then f is injective.*

- *f^* is surjective iff f restricts to an isomorphism between X and $f(X)$. In such case we call f an embedding.*

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4 Product of varieties

Lemma 4.1. *If $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are **affine varieties**, then $X \times Y \subset \mathbb{A}^{m+n}$ is an **affine variety**.*

Remark 33 While it is true that, for generic irreducible topological spaces X and Y, then $X \times Y$ is irreducible with the product topology. It is also true that the Zariski topology on $\mathbb{A}^{m+n}$ is not the product topology, it is actually finer (there are more closed subsets than in the product topology, hence it is harder to prove irreducibility).

Remark 34 The natural projections

$$
\begin{array}{ccc}
 & X \times Y & \\
\pi_X \swarrow & & \searrow \pi_Y \\
X & & Y
\end{array}
\tag{4.1}
$$

are morphisms of varieties.

Proposition 4.2 (Universal property of products of affine varieties)**.** *Let X and Y be **affine varieties**, then for every Z affine variety with morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$, then $\exists! f : Z \to X \times Y$ s.t. the following diagram commutes*

$$
\tag{4.2}
$$

i.e. $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$.

In particular, given Z, X, Y affine varieties, there is a one to one correspondance

$$
\left\{ \begin{array}{c} \textit{morphisms of varieties} \\ f : Z \to X \times Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{pairs of morphisms} \\ \left(Z \xrightarrow{f_X} X,\ Z \xrightarrow{f_Y} Y \right) \end{array} \right\} .
\tag{4.3}
$$

Remark 35
$$
\mathbb{K}[X \times Y] \cong \mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y].
\tag{4.4}
$$
This means that regular functions on $X \times Y$ are finite linear combinations of products $f \cdot g$, with $f \in \mathbb{K}[X]$ and $g \in \mathbb{K}[Y]$.

Definition 4.3: Affine variety as ringed spaces.
A ringed space $(X, \mathcal{O}_X)$ is an **affine variety** over $\mathbb{K}$ iff $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-algebras and $(X, \mathcal{O}_X)$ is isomorphic to an affine variety $Y \subset \mathbb{A}^n(\mathbb{K})$.

Theorem 4.4. *An algebra A over the field $\mathbb{K}$ is isomorphic to the coordinate ring $\mathbb{K}[X]$ of an affine variety X iff A is an integral domain, finitely generated, as a $\mathbb{K}$-algebra.*

Remark 36 We have an equivalence of categories:

$$\left\{ \begin{array}{c} \text{Affine} \\ \text{varieties} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finitely generated } \mathbb{K}\text{-algebras} \\ \text{that are integral domains} \end{array} \right\} , \qquad (4.5)$$

$$(X, \mathcal{O}_X) \longleftarrow\!\!\!\!\shortmid A$$

with $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X(U) = \{\phi \in \mathcal{Q}(A) \mid \phi \text{ is regular at } p \text{ for any } p \in U\}$ for any $U \overset{\text{open}}{\subset} X$.

Remark 37 One can prove that an algebra A is $A \cong \mathbb{K}[X]$ for some algebraic set $X \subset \mathbb{A}^n$ iff A is a finitely generated $\mathbb{K}$-algebra with no nilpotent elements (i.e. it is a **reduced** algebra).

Recall that f is nilpotent iff $\exists\, r \in \mathbb{N}$ s.t. $f^r = 0$.

Example: An affine variety from the new definition. Let X be an affine variety and $f \neq 0$ a fixed regular function on X. Clearly $(X_f, \mathcal{O}_{X_f})$, with $\mathcal{O}_{X_f} := \mathcal{O}_X|_{X_f}$ is a ringed space. $A := \mathcal{O}_X(X_f)$ is a finitely generated algebra. Then X_f is an affine variety with coordinate ring $\mathbb{K}[X]_f$.

Remark 38 Not all open subsets of an affine variety are affine varieties. For example $U := \mathbb{A}^2 \setminus \{(0,0)\}$ is not an affine variety.

Though we can construct U by patching together two affine varieties: $\mathbb{A}^2_x$ and $\mathbb{A}^2_y$. In fact any open subset of an affine variety can be covered by a finite family of distinguished open subsets (which are affine varieties indeed).

5 Prevarieties

Definition 5.1: Prevariety.
A **prevariety** over $\mathbb{K}$ is a ringed space $(X, \mathcal{O}_X)$ satisfying

1. X is irreducible,

2. $\mathcal{O}_X$ is a **sheaf** of $\mathbb{K}$-algebras s.t. the elements of $\mathcal{O}_X(U)$ are $\mathbb{K}$-valued functions on U, hence the restriction morphisms are restrictions of functions,

3. there is a **finite** open conver $\{U_i\}_{i \in I}$ of X s.t., for all i, $\left(U_i, \mathcal{O}_X|_{U_i}\right)$ is an affine variety.

Open subsets $U \subset X$ which are isomorphic to affine varieties are called affine open sets. Morphisms of prevarieties are just morphisms of ringed spaces.

Remark 39 An open cover $\{U_i\}_{i \in I}$ of X, in which each $\left(U_i, \mathcal{O}_X|_{U_i}\right)$ is an affine variety, is called **affine open** cover of X.

Example

1. Affine varieties,

2. open subsets of affine varieties (sometimes called *quasi-affine varieties*), which are finite unions of distinguished open subsets.

Definition 5.2: Gluing of prevarieties.
Consider two prevarieties X_1 and X_2, and $U_1 \overset{\text{open}}{\subset} X_1$, $U_2 \overset{\text{open}}{\subset} X_2$ with an isomorphism $f : U_1 \to U_2$. We define

$$X = X_1 \cup_f X_2 := (X_1 \sqcup X_2) / \mathcal{R}_f, \tag{5.1}$$

where $\mathcal{R}_f$ is the equivalence relation defined by

$$\mathcal{R}_f := \{(x, x) \mid x \in X_1 \sqcup X_2\} \cup \{(x, f(x)), (f(x), x) \mid x \in U_1\}. \tag{5.2}$$

Finally we endow X with the quotient topology.
  We now define a **structure sheaf** on X to make it a prevariety

$$\mathcal{O}_X(U) := \left\{(\phi_1, \phi_2) \in \mathcal{O}_{X_1}(X_1 \cap U) \times \mathcal{O}_{X_2}(X_2 \cap U) \mid \phi_1|_{U_1 \cap U} = f^* \phi_2|_{U_2 \cap U}\right\}. \tag{5.3}$$

(Explicitly it just states that, on the intersection of the union over f, the two functions have to coincide).

Example: $\mathbb{P}^1(\mathbb{K})$ the projective line. Let $X_1 = X_2 := \mathbb{A}^1 \supset \mathbb{A}^1 \setminus \{0\} =: U_1 = U_2$ and

$$f : U_1 \to U_2 \tag{5.4}$$

$$x \mapsto \frac{1}{x}. \tag{5.5}$$

We then define the projective space

$$\mathbb{P}^1(\mathbb{K}) := \mathbb{P}^1 := \mathbb{A}^1 \cup_f \mathbb{A}^1. \tag{5.6}$$

Set theoretically $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.

Remark 40 The set of fixed points, by a morphism of prevarieties, can be an open subset of a variety, a different behaviour from what we'd expect from morphisms of algebraic sets.

Lemma 5.3 (gluing of prevarieties). *Let $X_1, \ldots, X_r$ be **prevarieties**, $\{(U_{ij}, f_{ij})\}_{1 \leq i,j \leq r}$ with $U_{ij} \overset{open}{\subset} X_i$, $f_{ij} : U_{ij} \overset{\cong}{\to} U_{ji}$ s.t.*

1. *$U_{ii} = X_i$ and $f_{ii} = id_{X_i}$ for any i,*

2. *$f_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$ and $f_{jk} \circ f_{ij} = f_{ik}$ for all $i, j, k \in \{1, \ldots, r\}$.*

As a special case for the second condition we obtain that $f_{ij}^{-1} = f_{ji}$.

Then there exists a unique prevariety X up to isomorphism, obtained by gluing $X_1, \ldots, X_r$ along the U_{ij} via the isomorphisms f_{ij}.

5.1 Regular functions on prevarieties

Proposition 5.4. $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \{\phi : \mathbb{P}^1 \to \mathbb{K} \mid \phi \text{ is constant}\} \cong \mathbb{K}$.

Corollary 5.5. $\mathbb{P}^1$ *is not an **affine variety**. In fact, for X affine we have $\mathbb{K}[X] \cong \mathbb{K} \iff X = \{pt\}$. (By gluing prevarieties together we can obtain spaces which are not affine varieties).*

Remark 41 This is analogous to the theory of holomorphic functions (again). (The only holomorphic functions on the Riemann sphere are the constant functions).

Lemma 5.6. *Let $f : X \to Y$ a set theoretic map between two prevarieties X and Y. If there exists an open cover $\{U_1, \ldots, U_r\}$ of X and an affine open cover $\{V_1, \ldots, V_r\}$ of Y s.t.*

$$\begin{cases} f(U_i) \subset V_i \\ (f|_{U_i})^* \phi \in \mathcal{O}_X(U_i) \end{cases} \tag{5.7}$$

for all i and all $\phi \in \mathcal{O}_Y(V_i)$, then f is a morphism of prevarieties.

5.2 Geometrical remarks

Lemma 5.7. *Given a prevariety $(X, \mathcal{O}_X)$, then X is a **noetherian** topological space.*

Definition 5.8: Dimension of a prevariety.
Given a **prevariety** $(X, \mathcal{O}_X)$, we proved that X is a Noetherian topological space, hence we can consider $\dim X$, as usual, as the largest integer d s.t. we can construct a chain of irreducible closed subsets of X

$$\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_d = X. \tag{5.8}$$

Lemma 5.9. *Any open subset U of a prevariety $(X, \mathcal{O}_X)$ is itself a prevariety, with structure sheaf $\mathcal{O}_X|_U$.*

Definition 5.10: Subprevarieties.
Given a prevariety $(X, \mathcal{O}_X)$ we define its subprevarieties corresponding to:

- $U \overset{open}{\subset} X$, then $(U, \mathcal{O}_X|_U)$ is a prevariety and it is called **open subprevariety** of X,

- $Y \subset X$ is a closed subset, we define a a sheaf $\mathcal{O}_Y$ on Y by setting, on $U \overset{\text{open}}{\subset} Y$,

$$\mathcal{O}_Y(U) := \left\{ \phi : U \to \mathbb{K} \ \middle| \ \forall\, x \in U, \exists\, x \in V_x \overset{\text{open}}{\subset} X, \psi_x \in \mathcal{O}_X(V_x) \right. \tag{5.9}$$

$$\text{s.t. } \left. \phi|_{U \cap V_x} = \psi_x|_{U \cap V_x} \right\}. \tag{5.10}$$

$(Y, \mathcal{O}_Y)$ is called **closed subprevariety** of X,

- Given $Y \overset{\text{open}}{\subset} X \overset{\text{closed}}{\supset} Y$, then, by a combination of the previous constructions, $U \cap Y$ is a prevariety. It is called **locally closed subprevariety** of X.

Definition 5.11: Product of prevarieties.
Given $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ prevarieties, we define their product, as prevarieties, as the prevariety P, with the two morphisms

$$
\begin{array}{ccc}
 & P & \\
\pi_X \swarrow & & \searrow \pi_Y \\
X & & Y
\end{array}
\tag{5.11}
$$

satisfying the universal property for pruducts, i.e. for every prevariety Z with morphisms

$$
\begin{array}{ccc}
 & Z & \\
f_X \swarrow & & \searrow f_Y \\
X & & Y
\end{array}
\qquad .
\tag{5.12}
$$

then there exists a unique morphism $f : Z \to P$ s.t. the following diagram commutes

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow^{f_X} \ \nearrow^{\pi_X} & \\
Z \xrightarrow{\exists! f} P & & \\
 & \searrow_{\pi_Y} \ \searrow_{f_Y} & \\
 & & Y
\end{array}
\tag{5.13}
$$

We can construct the product prevariety $(P, \mathcal{O}_P)$ as the topological space $P := X \times Y$. The structure of prevariety, then, is obtained from X and Y considering their open covers $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$, where U_i and V_j are affine varieties, respectively for each i and j. In fact, then $\{U_i \times V_j\}_{(i,j) \in I \times J}$ is an open cover for P, in which every element is an affine variety (the product variety). We can now obtain P by gluing all of these affine varieties (viewed as prevarieties) together.

5.3 Abstract varieties

Definition 5.12: Variety.
Let X be a prevariety. We say that X is **separated** iff, for every prevariety Y and morphisms $f_1, f_2 : Y \to X$ we have that

$$\{ p \in Y \mid f_1(p) = f_2(p) \} \overset{\text{closed}}{\subset} Y. \tag{5.14}$$

Separated prevarieties are called **varieties**.

Remark 42   Given a variety X and two morphisms $Y \to X$ that coincide on a dense subset of Y, then they coincide on all of Y.

Definition 5.13: Diagonal morphism.
Let X be a **prevariety**, then we have the diagonal morphism

$$\Delta : X \to X \times X \qquad\qquad (5.15)$$
$$p \mapsto (p, p) \,. \qquad\qquad (5.16)$$

Its image, $\Delta(X)$, is the diagonal in $X \times X$. (Notice: Δ, in universal property's terms is given by the morphisms (id_X, id_X)).

Lemma 5.14. *A prevariety X is **separated** iff $\Delta(X)$ is closed in $X \times X$. Often this is taken as the definition of separatedness.*

Remark 43   There is a similar result for **Hausdorff** spaces: a topological space X is Hausdorff iff its diagonal is closed in $X \times X$ (taken with the product topology). Notice that, in our situation, the topology on $X \times X$ is not the product topology.

Lemma 5.15.

- *Every **affine** variety is a **variety**.*

- *If X ia a variety, then every locally closed subprevariety of X is a variety.*

6 Projective spaces

Definition 6.1: Projective space.
Let $\mathbb{K}$ be a field. The projective space of dimension n over the field $\mathbb{K}$, denoted by $\mathbb{P}^n(\mathbb{K}) = \mathbb{P}^n$, is the set of all 1-dimensional linear subspaces in $\mathbb{K}^{n+1}$.

Remark 44 Each 1-dimensional linear subspace of $\mathbb{K}^{n+1}$ is spanned by a nonzero vector $\mathbf{v} \in \mathbb{K}^{n+1}$, and two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{K}^{n+1} \setminus \{\mathbf{0}\}$ have the same span iff there exists $\lambda \in \mathbb{K}$ s.t. $\mathbf{v} = \lambda \mathbf{w}$. Hence, set theoretically,

$$\mathbb{P}^n = \left(\mathbb{K}^{n+1} \setminus \{\mathbf{0}\} \right) / \sim, \tag{6.1}$$

where $(v_0, \ldots, v_n) \sim (w_0, \ldots, w_n)$ if there is $\lambda \in \mathbb{K} \setminus \{\mathbf{0}\}$ s.t. $\lambda w_i = v_i$ for every $i = 0, \ldots, n$.

The equivalence classe of $\mathbf{v} = (v_0, \ldots, v_n)$ is denoted as

$$[\mathbf{v}] = (v_0 : v_1 : \ldots : v_n) = [v_0, \ldots, v_n]. \tag{6.2}$$

Example For $\mathbb{P}^1$ we can see its points with

$$U_0 := \left\{ [p_0, p_1] \in \mathbb{P}^1 \mid p_0 \neq 0 \right\} \to \mathbb{A}^1 \tag{6.3}$$

$$[p_0, p_1] \mapsto \frac{p_1}{p_0} \tag{6.4}$$

$$[1, t] \leftarrow\!\shortmid t, \tag{6.5}$$

which is a bijection, and with the points $[0, p_1]$ with $p_1 \neq 0$, which is the unique point $[0, 1]$ at infinity. In particular

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{[0, 1]\}. \tag{6.6}$$

Remark 45 Analogously $\mathbb{P}^n$ can be covered by $n + 1$ copies of $\mathbb{A}^n$. In particular we have the $n + 1$ bijections

$$U_0 := \{[p_0, \ldots, p_n] \in \mathbb{P}^n \mid p_0 \neq 0\} \to \mathbb{A}^n \tag{6.7}$$

$$[p_0, p_1, \ldots p_n] \mapsto \left(\frac{p_1}{p_0}, \ldots, \frac{p_n}{p_0} \right) \tag{6.8}$$

$$[1, t_1 \ldots, t_n] \leftarrow\!\shortmid (t_1, \ldots, t_n). \tag{6.9}$$

These give rise to the decomposition

$$\mathbb{P}^n = U_0 \cup \{[0, p_1, \ldots, p_n]\} \cong \mathbb{A}^n \cup \mathbb{P}^{n-1}. \tag{6.10}$$

As expected $\mathbb{P}^n$ can be covered by the open subsets U_i, with $i + 1$-th coordinate equal to one. Study what happens on the overlap $U_i \cap U_j$, for $i \neq j$.

Remark 46 $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are **compact**, when endowed with the quotient topology of the standard topology on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ and $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ respectively.

6.1 Projective algebraic sets

Definition 6.2: Homogeneous polynomials of degree d.
We define the set

$$\mathbb{K}[x_1, \ldots, x_n]_d := \text{Span} \left\{ x_0^{i_0} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \mid i_0 + i_1 + \ldots + i_n = d \right\}. \tag{6.11}$$

The elements of this set are called **homogeneous polynomials** of degree d.

Remark 47 $f \in \mathbb{K}[x_1, \ldots, x_n]_d$ implies, for any $\lambda \in \mathbb{K}$,

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n). \tag{6.12}$$

In particular the vanishing of f gives a well-defined locus in $\mathbb{P}^n$.

6.2   Graded rings

Definition 6.3: Graded ring.
A **graded ring** R is a ring with a decomposition

$$R = \bigoplus_{d \in \mathbb{N}} R_d, \tag{6.13}$$

with R_d an abelian subgroup of R for any $d \in \mathbb{N}$, satisfying the compatibility condition with the product: $f_d \cdot f_e \in R_{d+e}$ for all $f_d \in R_d$ and $f_e \in R_e$.

Definition 6.4: Graded algebra.
A graded $\mathbb{K}$-algebra R is a graded ring which is also a $\mathbb{K}$-algebra and s.t. the R_d are also linear subspaces.

Remark 48 Notice that

$$\mathbb{K}[x_1, \ldots, x_n] = \bigoplus_{d \in \mathbb{N}} \mathbb{K}[x_1, \ldots, x_n]_d \tag{6.14}$$

is a graded $\mathbb{K}$-algebra.

Definition 6.5: Homogeneous element.
Given a graded ring R, an element $f \in R$ is called **homogeneous** iff $\exists d \in \mathbb{N}$ s.t. $f \in R_d$. Since $R = \oplus_{d \in \mathbb{N}} R_d$, then any $f \in R$ has a *unique* decomposition

$$f = \sum_{m \in \mathbb{N}} f_m, \tag{6.15}$$

with $f_m \in R_m$. The various f_m are called the **homogeneous components** of f.

Definition 6.6: degree of an element.
Given $0 \neq f \in R$ a graded ring, with the decomposition $\sum_{m \in \mathbb{N}} f_m$ as above, the **degree** of f is the maximal m s.t. $f_m \neq 0$. Equivalently, if $d = \deg f$, then

$$f = f_0 + \ldots + f_d, \tag{6.16}$$

for $f_j \in R_j$.

Definition 6.7: Homogeneous ideal.
An ideal $\mathcal{I}$ in a graded ring R is called **homogeneous** iff it is generated by homogeneous elements.

Lemma 6.8. *Let $\mathcal{I}, \mathcal{J}$ be ideals in a graded ring R.*

1. *$\mathcal{I}$ is **homogeneous** iff for all $f \in \mathcal{I}$, with decomposition in homogeneous components*

$$f = \sum_{d \in \mathbb{N}} f_d \tag{6.17}$$

   then $f_d \in R_d \cap \mathcal{I}$ for all d.

2. *If $\mathcal{I}$ and $\mathcal{J}$ are both homogeneous ideals, then also $\mathcal{I} + \mathcal{J}, \mathcal{I} \cdot \mathcal{J}$ and $\mathcal{I} \cap \mathcal{J}$ are homogeneous. Furthermore $\sqrt{\mathcal{J}}$ is homogeneous if $\mathcal{J}$ is.*

3. *If $\mathcal{I}$ is homogeneous, then*

$$R/\mathcal{I} = \bigoplus_{d \in \mathbb{N}} R_d / (R_d \cap \mathcal{I}) \tag{6.18}$$

   gives $R/\mathcal{I}$ the structure of graded ring.

6.3 Projective algebraic sets

Definition 6.9: Projective algebraic set.
For any set $S \subset \mathbb{K}[x_0, \dots, x_n]$ of *homogeneous* polynomials we define the **projective zero locus** of S by

$$\mathbb{V}_p(S) := \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in S\} \subset \mathbb{P}^n. \tag{6.19}$$

Analogously, for any homogeneous ideal $\mathcal{I}$ we set

$$\mathbb{V}_p(\mathcal{I}) := \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in \mathcal{I}\} \subset \mathbb{P}^n. \tag{6.20}$$

Subsets of $\mathbb{P}^n$ of one of the above form are called **projective algebraic sets**.
   Furthermore, for a subset $X \subset \mathbb{P}^n$ we define the ideal of X by

$$\mathbb{I}_p(X) := (f \in \mathbb{K}[x_0, \dots, x_n] \mid f \text{ is homogeneous}, f(x) = 0 \text{ for all } x \in X) \subset \mathbb{K}[x_0, \dots, x_n]. \tag{6.21}$$

Example

- $\emptyset = \mathbb{V}_p(1)$ and $\mathbb{P}^n = \mathbb{V}_p(0)$ are projective algebraic sets.

- Given $L_0, \dots, L_m$ homogeneous linear forms, then $\mathbb{V}_p(L_0, \dots, L_m)$ is a projective algebraic set. In particular points in $\mathbb{P}^n$ are algebraic sets.

Proposition 6.10.

1. *If $\{\mathcal{I}_i\}_{i \in I}$ is a family of homogeneous ideals in $\mathbb{K}[x_0, \dots, x_n]$, then*

$$\bigcap_{i \in I} \mathbb{V}_p(\mathcal{I}_i) = \mathbb{V}_p\left(\bigcup_{i \in I} \mathcal{I}_i\right) \subset \mathbb{P}^n. \tag{6.22}$$

   In other words, arbitrary intersections of projective algebraic sets are algebraic.

2. *If $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbb{K}[x_1, \dots, x_n]$ are homogeneous ideals, then*

$$\mathbb{V}_p(\mathcal{I}_1) \cup \mathbb{V}_p(\mathcal{I}_2) = \mathbb{V}_p(\mathcal{I}_1 \cdot \mathcal{I}_2) = \mathbb{V}_p(\mathcal{I}_1 \cap \mathcal{I}_2) \subset \mathbb{P}^n. \tag{6.23}$$

   As a consequence, finite union of algebraic sets are algebraic.

Definition 6.11: Zariski topology on $\mathbb{P}^n$.
The **Zariski topology** on $\mathbb{P}^n$ is the topology whose closed subsets are precisely the projective algebraic sets. If $X \subset \mathbb{P}^n$ we define the Zariski topology on X as the subset topology, induced by the Zariski topology on $\mathbb{P}^n$.

Remark 49 $\mathbb{P}^n$ is a **prevariety**, hence it is a **noetherian** topological space. In particular we can consider $\dim X$ for all closed subsets $X \subset \mathbb{P}^n$.

6.4 Projective varieties

Definition 6.12: Projective variety.
A projective variety is a **Zariski closed** subset $X \subset \mathbb{P}^n$.

Definition 6.13: Cone.
An affine algebraic set $C \subset \mathbb{A}^{n+1}$ is called a **cone** iff

- C contains the origin,

- for any $x \in C$, then $\lambda x \in C$ for any $\lambda \in \mathbb{K}$.

Definition 6.14: Cone over an algebraic set.
Given any $X \subset \mathbb{P}^n$ a projective algebraic set, then we can define the cone over X as

$$C(X) := \{\mathbf{0}\} \cup \left\{ x \in \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \,\middle|\, [x] \in X \right\}. \tag{6.24}$$

Notice that the second part of the union is the preimage of X under $\mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$.

Lemma 6.15 (Relation between X and $C(X)$).

- *Let $X = \mathbb{V}_p\left(\mathcal{I}\right) \subset \mathbb{P}^n$ be defined by a homogeneous ideal $\mathcal{I} \subsetneq (1)$. Then*

$$C(X) = \mathbb{V}\left(\mathcal{I}\right) \subset \mathbb{A}^{n+1}. \tag{6.25}$$

- *If $X \subset \mathbb{P}^n$ is a projective algebraic set, then*

$$\mathbb{I}\left(C(X)\right) = \mathbb{I}_p(X). \tag{6.26}$$

6.5 Projective Nullstellensatz

Lemma 6.16. *Let $\mathcal{I} \subset \mathbb{K}\left[x_0, \ldots, x_n\right]$ be an homogeneous ideal. The following properties are equivalent:*

1. $\mathbb{V}_p\left(\mathcal{I}\right) = \emptyset \subset \mathbb{P}^n$,

2. *either $\sqrt{\mathcal{I}} = \mathbb{K}\left[x_0, \ldots, x_n\right]$ or $\sqrt{\mathcal{I}} = (x_0, \ldots, x_n)$,*

3. *there is $d \geq 1$ a degree, s.t. $\mathbb{K}\left[x_0, \ldots, x_n\right]_d \subset \mathcal{I}$.*

Theorem 6.17 (Projective Nullstellenstaz).
There is a one to one inclusion reversing correspondence

$$\left\{ \begin{array}{c} \textit{algebraic sets} \\ \textit{in } \mathbb{P}^n(\mathbb{K}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homogeneous} \\ \textit{radical ideals} \\ \mathcal{I} \neq (x_0, \ldots, x_n)_d \end{array} \right\}.$$

$$X \longmapsto \mathbb{I}_p(X)$$

$$\mathbb{V}_p\left(\mathcal{I}\right) \longleftarrow\!\shortmid \mathcal{I} \tag{6.27}$$

Furthermore, for all subsets $X \subset \mathbb{P}^n$ and all homogeneous ideals $\mathcal{I}$ with $\mathbb{V}_p\left(\mathcal{I}\right) \neq \emptyset$, we have

$$\mathbb{I}_p\left(\mathbb{V}_p\left(\mathcal{I}\right)\right) = \sqrt{\mathcal{I}} \tag{6.28}$$

$$\mathbb{V}_p\left(\mathbb{I}_p(X)\right) = \overline{X}. \tag{6.29}$$

6.6 Regular functions on projective varieties

Definition 6.18: Homogeneous coordinate ring.
Given $X \subset \mathbb{P}^n$ a projective algebraic set, then the $\mathbb{K}$-algebra

$$S(X) := \frac{\mathbb{K}[x_0, \ldots, x_n]}{\mathbb{I}_p(X)} \tag{6.30}$$

is called the **homogeneous coordinate ring** of X.

Remark 50 Elements of $S(X)$ are functions on the cone $C(X)$, but not on X (they are not constant up to multiplication by a scalar). However quotients $\frac{f}{g}$ with $\deg f = \deg g$ give well-defined functions outside $\mathbb{V}(g)$. In fact

$$\frac{f(\lambda x_0, \ldots, \lambda x_n)}{g(\lambda x_0, \ldots, \lambda x_n)} = \frac{\lambda^d f(x_0, \ldots, x_n)}{\lambda^d g(x_0, \ldots, x_n)} = \frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} \tag{6.31}$$

hold for all $\lambda \neq 0$ and $[x_0, \ldots, x_n] \in X$.

Let's now introduce the notation $S(X)_d = \mathbb{K}[x_0, \ldots, x_n]_d / \mathcal{I}_d$.

Definition 6.19: Field of rational functions.
Let $X \subset \mathbb{P}^n$ be a projective variety. The **field of rational functions** on X is

$$\mathbb{K}(X) := \left\{ \frac{f}{g} \;\middle|\; f, g \in S(X)_d, g \neq 0 \right\}. \tag{6.32}$$

Definition 6.20: Regular functions.

1. We say that $\varphi \in \mathbb{K}(X)$ is **regular at** $p \in X$ iff

$$\varphi = \frac{f}{g} \in \mathbb{K}(X), \text{ with } g(p) \neq 0, \tag{6.33}$$

   i.e. φ is well defined at p,

2. We define the **local ring of** X at p to be

$$\mathcal{O}_{X,p} := \{\varphi \in \mathbb{K}(X) \mid \varphi \text{ is regular at } p\}, \tag{6.34}$$

3. For any $\emptyset \neq U \overset{\text{open}}{\subset} X$, we define

$$\mathcal{O}_X(U) := \bigcap_{p \in U} \mathcal{O}_{X,p}, \tag{6.35}$$

   the ring of regular functions on U.

Remark 51 We can show that only the constant functions on $\mathbb{P}^n$ are regular on the whole $\mathbb{P}^n$, hence that

$$\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = \mathbb{K}. \tag{6.36}$$

Proposition 6.21. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $(X, \mathcal{O}_X)$ is a prevariety.*

Remark 52 From the proof of the proposition we obtain a one-to-onte correspondance

$$\left\{ \begin{array}{c} \text{affive viarieties} \\ \text{in } \mathbb{A}^n(\mathbb{K}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{projective varieties in } \mathbb{P}^n \text{ not} \\ \text{contained in the hyperplane at } \infty \end{array} \right\} . \quad (6.37)$$

Remark 53 All known constructions for prevarieties apply also to projective varieties. In particular we know how to define

- morphisms,

- products,

- the function field of X as a prevariety, which asctually is the same thing as the definition $\mathbb{K}(X)$.

Let's now consider, for simplicity, $X_j := X \cap U_j \cong Y_j \subset \mathbb{A}^n$, with $j = 1$.

Definition 6.22: Homogenization of a polynomial.
Let $f \in \mathbb{K}[x_1, \ldots, x_n], g \in \mathbb{K}[x_0, \ldots, x_n]_d$. The **homogenization** of f, with respect to x_0, is

$${}^h f(x_0, \ldots, x_n) := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{K}[x_0, \ldots, x_n]_{\deg f}. \quad (6.38)$$

Conversely, the dehomogenization of g, with respect to x_0, is

$${}^a g(x_1, \ldots, x_n) := g(1, x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]. \quad (6.39)$$

Definition 6.23: Homogenization of an ideal.
Let $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. We define its **homogenization** by

$${}^h \mathcal{I} := \left({}^h f \mid f \in \mathcal{I}\right). \quad (6.40)$$

Remark 54 It is easy to check that ${}^h f$ is a homogeneous polynomial of degree $\deg {}^h f = \deg f$. With regards to the dehomogenization, instead, the degree might decrease, hence $\deg {}^a g \leq \deg g$, with $<$ if x_0 divides g.

Moreover it can be easily checked that

$${}^a\left({}^h f\right) = f \quad \text{and} \quad {}^h\left({}^a g\right) = g \text{ iff } x_0 \nmid g. \quad (6.41)$$

Lemma 6.24.

- *For $X = \mathbb{V}(\mathcal{I}) \subset \mathbb{A}^n$ we have $\mathbb{V}_p\left({}^h \mathcal{I}\right) = \overline{X}$, where $\overline{X}$ is the projective closure of X, defined as the Zariski closure of $X \subset \mathbb{A}^n \cong U_0$ in $\mathbb{P}^n$.*

- *If $X = \mathbb{V}(f)$, then $\overline{X} = \mathbb{V}_p\left({}^h f\right)$.*

- *The map $\mathcal{I} \to {}^h \mathcal{I}$ induces a one-to-one correspondance*

$$\left\{ \begin{array}{c} \text{Zariski closed} \\ \text{subsets of } \mathbb{A}^n(\mathbb{K}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Zariski closed subsets of } \mathbb{P}^n \text{ s.t. no} \\ \text{irr. comp. is contained in } H_\infty = \mathbb{V}(x_0) \end{array} \right\} .$$

$$\mathbb{V}(\mathcal{I}) \longmapsto \mathbb{V}\left({}^h \mathcal{I}\right)$$

$$(6.42)$$

Remark 55 For hypersurfaces we just proved that

$$X = \mathbb{V}(f) \implies \overline{X} = \mathbb{V}_p\left({}^h f\right). \tag{6.43}$$

For an arbitrary number of polynomials, instead, in general

$$\overline{\mathbb{V}(F_1, \ldots, F_r)} \neq \mathbb{V}_p\left({}^h F_1, \ldots, {}^h F_r\right). \tag{6.44}$$

Only for sufficiently good sets of generators the ideal generated by their homogeneization corresponds with the homogeneization of the ideal.

7 Morphisms and products of projective varieties

Definition 7.1: Quasi-projective variety.
Let $X \subset \mathbb{P}^n$. X is a **quasi-projective** variety iff it is a dense open subset of a projective variety.

Remark 56 All quasi-projective varieties are prevarieties.

Lemma 7.2. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. Let $F_0, \ldots, F_m \in \mathbb{K}\,[x_0, \ldots, x_n]_d$ s.t., for every $x \in X$, there is at least one F_i with $F_i(x) \neq 0$. Then the F_is define a map*

$$F : X \to \mathbb{P}^m \tag{7.1}$$
$$x \mapsto [F_0(x), \ldots, F_m(x)] \tag{7.2}$$

which is always a morphism of quasi-projective varieties.

Remark 57 Not all morphisms $X \to \mathbb{P}^m$ can be obtained in this way.

Example The map

$$f : \mathbb{P}^1 \to \mathbb{P}^2 \tag{7.3}$$
$$[s, t] \mapsto \left[s^2, st, t^2\right] \tag{7.4}$$

is a morphism with image $X = \mathbb{V}_p\left(xz - y^2\right)$. In fact it is an isomorphism on its image. This means that its inverse is also a morphism

$$f^{-1} : X \to \mathbb{P}^1 \tag{7.5}$$
$$[x, y, z] \mapsto \begin{cases} [x, y] & \text{for } (x, y) \neq 0 \\ [y, z] & \text{for } (y, z) \neq 0 \end{cases}. \tag{7.6}$$

However f^{-1} cannot be defined globally by a pair of homogeneous polynomials in x, y and z.

Remark 58 As a corollary, the example shows that all irreducible conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$.

In the following we'll denote $N_{n,m} := (n + 1)\,(m + 1) - 1$.

Definition 7.3: Segre embedding.
The **Segre embedding** is the morphism

$$s_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{N_{n,m}} \tag{7.7}$$
$$([x_0, \ldots, x_n], [y_0, \ldots, y_m]) \mapsto [z_{ij} \mid 0 \leq i \leq n,\ 0 \leq j \leq m], \tag{7.8}$$

with $z_{ij} := x_i y_j$.

Lemma 7.4.

1. *The image of $s_{n,m}$ is the projective variety $X \subset \mathbb{P}^n$ whose ideal is generated by*

$$z_{ij} z_{kl} - z_{il} z_{kj} \text{ for all } 0 \leq i, k \leq n \text{ and } 0 \leq j, l \leq m. \tag{7.9}$$

   *X is called the **Segre** variety.*

2. $s_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to X$ is an isomorphism of prevarieties. In particular $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety.

3. The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are the zero loci of polynomials in $\mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ that are bihomogeneous in the x and y coordinates.

Remark 59 The ring $R := \mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is a bigraded ring, i.e. a graded ring over the monoid $\mathbb{N} \times \mathbb{N}$.

$$R = \bigoplus_{(d,e) \in \mathbb{N} \times \mathbb{N}} R_{d,e} \tag{7.10}$$

$$R_{d,e} := \mathrm{span}_{\mathbb{K}} \left\{ x_0^{i_0} \ldots x_n^{i_n} y_0^{j_0} \ldots y_m^{j_m} \,\middle|\, i_0 + \ldots + i_n = d \text{ and } j_0 + \ldots + j_m = e \right\}. \tag{7.11}$$

The elements of $R_{d,e}$ are called bihomogeneous polynomials of bidegree (d, e). Moreover, as expected, when multiplying polynomials, the bidegrees are additive.

Example Let's compute the **Segre embedding** of $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{V}_p (u_0 u_3 = u_1 u_2) \subset \mathbb{P}^3. \tag{7.12}$$

Explicitly the embedding is given by

$$([x_0, x_1], [y_0, y_1]) \mapsto [x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1]. \tag{7.13}$$

Any rank 4 quadric in $\mathbb{P}^3$ is isomorphic to the Segre product X, after a linear change of coordinates. Hence it is covered by two families of (pairwise disjoint) lines. This is very different from $\mathbb{P}^2$, in which any two curves (and in particular any two lines) intersect. (In $\mathbb{P}^1 \times \mathbb{P}^1$ lines from the same family either coincide or do not intersect at all).

Proposition 7.5 (Corollary). *Projective varieties are varieties (i.e. they are **separated** prevarieties).*

Remark 60 Since projective varieties are separated, also quasi-projective varieties (i.e. dense open subsets of projective varieties) are varieties.

8 Main theorem on projective varieties

Remark 61 Let's recall a few notions:

- $\mathbb{P}^n(\mathbb{C})$ is compact in the standard topology. This means that every projective variety is compact in the analytic topology.

- The image of a compact subset under a continuous map is compact.

Moreover, coming back to concepts concerning algebraic geometry: the image of an affine variety under a morphism may not be closed. An example is $\mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1$, then

$$\pi\left(\mathbb{V}\left(xy - 1\right)\right) = \mathbb{A}^1 \setminus \{\mathbf{0}\}. \tag{8.1}$$

The first clearly is closed, whereas the latter is open.

For the Zariski topology on projective varieties we will, instead, prove that the image of a projective variety under a morphism is always closed. In other words the image of a projective variety under a morphism is always a projective variety. (That's the reason why we needed to extend the definition of affine varieties to abstract varieties. In the case of projective ones we do not need to do so.)

Theorem 8.1 (Projection map is closed). *Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a closed subset. Let $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ be the canonical projection. Then $\pi(X) \subset \mathbb{P}^n$ is closed. In other words the projection map is closed.*

Remark 62 This theorem has application in computational algebra. It is sometimes called the *main theorem of elimination theory*. In elimination theory, starting from a variety (i.e. the zero locus of a finite collection of bihomogeneous polynomials)

$$X : F_1(x, y) = F_2(x, y) = \ldots = F_r(x, y) = 0. \tag{8.2}$$

Then the projection sens X to

$$\pi(X) = \{x \in \mathbb{P}^n \mid \exists\, y \in \mathbb{P}^m \text{ s.t. } F_i(x, y) = 0 \,\forall\, i\}. \tag{8.3}$$

We are trying to remove the y variables from the first system. Since $\pi(X)$ is closed, then it is the solution set of some polynomial equations.

Corollary 8.2. *The projection morphism $\pi : \mathbb{P}^n \times Y \to Y$ is closed for all varieties Y.*

Definition 8.3: Complete variety.
A variety X is **complete** iff the projection $\pi : X \times Y \to Y$ is closed for any variety Y.

Example

- $\mathbb{P}^n$ is complete, by the above corollary.

- Any projective variety $X \subset \mathbb{P}^n$ is complete: Any closed subset of $X \times Y$ is also closed in $\mathbb{P}^n \times Y$. Since the image by the projection is the same, then also its projection to Y is closed.

- There also exist complete varieties that are not projective, but it is hard to prove this last statement.

Corollary 8.4. *Let $f : X \to Y$ be a morphism of varieties. Let X be a complete variety, then $f(X)$ is closed in Y.*

Proof. The idea is to study an arbitrary map as a projection. ∎

Let's now see a few corollaries to the above theorem.

Corollary 8.5. *Let $X \subset \mathbb{P}^n$ be a projective variety that contains more than one point (equivalently $\dim X \geq 1$). Then, for any non constant homogeneous polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$, $X \cap \mathbb{V}_p(f) \neq \emptyset$.*

Corollary 8.6. *Any two curves in $\mathbb{P}^2$ intersect. In particular $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (recall that $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by a family of lines that do not intersect each other).*

Corollary 8.7. *Every regular fucntion on a complete variety is constant, i.e. for any X complete*

$$\mathcal{O}_X(X) = \mathbb{K}. \tag{8.4}$$

8.1 Veronese variety

Let $n, d \geq 1$. Consider $\mathbb{K}[x_0, \ldots, x_n]_d$, a $\mathbb{K}$-Vector Space of dimension $\binom{n+d}{n}$. Let $F_0, \ldots, F_\nu$ be a basis for this space, hence

$$\nu = \nu_{n,d} := \binom{n+d}{n} - 1. \tag{8.5}$$

One could choose the monomials of degree d (ordered in some way, for example lexicographically), as a basis. For instance

$$F_0 = x_0^d, \qquad F_1 = x_0^{d-1}x_1, \qquad \ldots, \qquad F_\nu = x_n^d. \tag{8.6}$$

The basis induces a morphism of varieties

$$v_d : \mathbb{P}^n \to \mathbb{P}^\nu \tag{8.7}$$

$$[x_0, \ldots, x_n] \mapsto [F_0(x), \ldots, F_\nu(x)]. \tag{8.8}$$

By the main theorem on projective varieties, the image of this morphism, $V_{n,d} := v_d(\mathbb{P}^n)$, is a projective variety. We call it the Veronese variety of degree d.

Proposition 8.8. *The morphism $v_d : \mathbb{P}^n \to V_{n,d}$ is an isomorphism.*

Remark 63 When we view $\mathbb{P}^n$ as $V_{n,d} \subset \mathbb{P}^\nu$, then the degree d homogeneous polynomials in $x_0, \ldots, x_n$ give **linear** forms in the coordinates of $\mathbb{P}^\nu$.

Example: $n = 1$**: the rational normal curve of degree** d**.** $V_{1,d} \subset \mathbb{P}^d$ is called a *rational normal curve of degree d* in $\mathbb{P}^d$. It is given by

$$v_d : \mathbb{P}^1 \to \mathbb{P}^d \tag{8.9}$$

$$[x_0, x_1] \mapsto [x_0^d, x_0^{d-1}x_1, \ldots, x_1^d] = [u_0, u_1, \ldots, u_d]. \tag{8.10}$$

As a subvariety of $\mathbb{P}^d$, $V_{1,d}$ is defined by the condition

$$\mathrm{rk}\begin{pmatrix} u_0 & u_1 & \ldots & u_{d-1} \\ u_1 & u_2 & \ldots & u_d \end{pmatrix} = 1. \tag{8.11}$$

Corollary 8.9. *Let $X \subset \mathbb{P}^n$ be a projective variety and $f \in \mathbb{K}[x_0, \ldots, x_n]$ any non constant homogeneous polynomial. Then $X \setminus \mathbb{V}_p(f)$ is an affine variety.*

9 Dimension

Recall that the dimension of a *Noetherian* topological space X is the largest index d for which there is exists a chain

$$\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_d = X \tag{9.1}$$

of irreducible closed subsets $X_i \subset X$. We will call such chain a *longest chain* in X.

Remark 64   What do we know about dimension as of now?

- $\dim \{pt\} = 0$.

- $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$, since the only irreducible closed subsets are the points.

- *Bezout's theorem*: two curves $\mathbb{V}(f), \mathbb{V}(g) \subset \mathbb{A}^2$ with no common irreducible component, intersect in $(\deg f)(\deg g)$ points (counted with multiplicity). This means that the irreducible closed subsets of $\mathbb{A}^2$ are:

    - $\mathbb{A}^2$,
    - irreducible curves $X = \mathbb{V}(f)$, for $f \in \mathbb{K}[x,y]$ irreducible,
    - points.

  Hence longest chains are of the form

$$\emptyset \subsetneq \{pt\} \subsetneq \mathbb{V}(f) \subsetneq \mathbb{A}^2, \tag{9.2}$$

  with $f(p) = 0$ and f is irreducible. This implies that $\dim \mathbb{A}^2 = 2$. Since, moreover, $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{V}_p(x_0)$, we obtain that $\dim \mathbb{P}^2 = 2$.

- We also noticed that $\dim \mathbb{A}^n \geq n$, in fact we have the *longest chain*

$$\emptyset \subsetneq \mathbb{V}(x_1, \ldots, x_n) \subsetneq \mathbb{V}(x_2, \ldots, x_n) \subsetneq \ldots \subsetneq \mathbb{V}(x_n) \subsetneq \mathbb{A}^n. \tag{9.3}$$

  Analogously we can show that $\dim \mathbb{P}^n \geq n$.

Lemma 9.1.

1. *If X is a Noehterian topological space and we consider a longest chain*

$$\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X, \tag{9.4}$$

   then $\dim X_j = j$ for all $j = 0, \ldots, n$.

2. *If X is irreducible and $Y \subsetneq X$ is closed, then $\dim Y < \dim X$.*

   A comment on what's to come:

- We'll prove that dimension is a local property, i.e. that given a variety X and an affine open subset $U \subset X$, then $\dim U = \dim X$. Moreover, for any $U \subset \mathbb{A}^n$, then $\dim Y = \dim U$, for $Y := \overline{U} \subset \mathbb{P}^n$. This means that we can concentrate our study only on projective varieties.

- Given a surjective morphism $f : X \to Y$ of projective varieties, then $\dim Y \leq \dim X$. From this idea we get the expectation that, given X a projective variety, then $\dim X$ is the largest integer d s.t. there is a surjective morphism $f : X \to \mathbb{P}^d$.

Lemma 9.2. *Let $f : X \to Y$ be a surjective morphism of projective varieties. Then every longest chain*

$$\emptyset \subsetneqq Y_0 \subsetneqq Y_1 \subsetneqq \ldots \subsetneqq Y_n = Y \tag{9.5}$$

can be lifted to a chain

$$\emptyset \subsetneqq X_0 \subsetneqq X_1 \subsetneqq \ldots \subsetneqq X_n = X \tag{9.6}$$

of irreducible closed subsets $X_j \subset X$, s.t. $f(X_j) = Y_j$ for all $j = 0, \ldots, n$. In particular $\dim X \geq \dim Y$.

Remark 65: Next step.

Our next aim is to show that every projective variety $X \subset \mathbb{P}^n$ admits a surjective morphism to $\mathbb{P}^m$, for some $m \leq n$. It will be constructed by taking a sequence of projections.

Construction. Let $X \subsetneqq \mathbb{P}^n$ a projective variety and $p \in \mathbb{P}^n$ s.t. $p \notin X$. Let $H \subset \mathbb{P}^n$ be a linear subspace of codim 1 no passing through p, i.e. H is the zero locus of a degree 1 homogeneous polynomial. For instance $H = \mathbb{V}_p(a_0 x_0 + \ldots + a_n x_n)$, with $a_i \neq 0$ for some i. WLOG $a_n \neq 0$. Then H is isomorphic to $\mathbb{P}^{n-1}$, via

$$\sigma : H = \mathbb{V}_p(a_0 x_0 + \ldots + a_n x_n) \xrightarrow{\cong} \mathbb{P}^{n-1} \tag{9.7}$$

$$[x_0, \ldots, x_n] \mapsto [x_0, \ldots, x_{n-1}], \tag{9.8}$$

where $x_n = -\frac{a_0}{a_n} x_0 - \ldots - \frac{a_{n-1}}{a_n}$. We then define the projection $\pi : X \to \mathbb{P}^{n-1}$ as the map sending $q \in X$ to the point of $H \cong \mathbb{P}^{n-1}$ corresponding to the intersection of H with the line pq. Up to a linear change of coordinates we may assume $p = [0, \ldots, 0, 1]$, and $H = \mathbb{V}_p(x_n)$. Then

$$\pi : X \to \mathbb{P}^{n-1} \tag{9.9}$$

$$[x_0, \ldots, x_n] \mapsto [x_0, \ldots, x_{n-1}]. \tag{9.10}$$

Let's now construct this change of coordinates. Recall that a linear change of coordinates $A \in \mathrm{PGL}(n+1)$ is determined by the image of $n+2$ points in general position (which means that no $n+1$ of them are contained in a hyperplane). In particular we choose n points, $q_0, \ldots, q_{n-1}$ in general position s.t. $\mathrm{span}(q_0, \ldots, q_n) = H$, $p \notin H$ and r any point on pq_{n-1} different both from p and q_{n-1}, then A is defined as follows

$$
\begin{aligned}
q_0 & \mapsto & [1, 0, \ldots, 0] \\
& \vdots & \\
q_{n-1} & \mapsto & [0, \ldots, 1, 0] \\
p & \mapsto & [0, \ldots, 0, 1] \\
r & \mapsto & [1, 1, \ldots, 1, 1]
\end{aligned} \quad \cdot \tag{9.11}
$$

$\blacksquare$

Remark 66

- The map $\pi : X \to \mathbb{P}^{n-1}$ is a morphism, since $x_0 = \ldots = x_{n-1} = 0$ has no common solution on X ($p \notin X$).

- All fibers of π are finite sets: for all $a \in H$, $\pi^{-1}(\{a\}) = pa \cap X$ is a closed subset of pa (a line), different from the whole pa, since $p \notin X$. The closed subsets of a line are finite sets by the classification of closed sybsets of $\mathbb{P}^1$.

- We can iterate

$$X \xrightarrow{\pi_{p_1}} \mathbb{P}^{n-1} \xrightarrow{\pi_{p_2}} \ldots \xrightarrow{\pi_{p_{n-m}}} \mathbb{P}^m, \tag{9.12}$$

so long as $\exists\, p_i \notin \mathbb{P}^{n-i}$, until $f = \pi_{p_{n-m}} \circ \ldots \circ \pi_{p_2} \circ \pi_{p_1}$ is surjective. We can interpret geometrically f as a projection of X onto a linear subspace $H' \cong \mathbb{P}^m$, with center a linear subspace of dimension $n - m - 1$, spanned by $p_1, \ldots, p_{n-m}$

Lemma 9.3. *Let $X \subsetneq \mathbb{P}^n$ be a projective variety not containing $p = [0, \ldots, 0, 1]$. Then, for every $f \in S(X) = \mathbb{K}[x_0, \ldots, x_n] / \mathbb{I}_p(X)$ there exists a degree $D \geq 1$, and homogeneous polynomials $a_0, \ldots, a_{D-1} \in \mathbb{K}[x_0, \ldots, x_{n-1}]$ s.t.*

$$f^D + a_{D-1} f^{D-1} + \ldots + a_1 f + a_0 = 0 \in S(X). \tag{9.13}$$

Remark 67 If we consider $f = \bar{x}_n$, we obtain again that the fibers of π are finite. However it is a stronger condition: let's restrict to an affine open subset $p \notin U \cong \mathbb{A}^n$ of $\mathbb{P}^n$. (Let, for simplicity, $U = U_0$), then we have the projection

$$\pi|_Y : X \cap U_0 \subset \mathbb{A}^n \to \mathbb{A}^{n-1} \tag{9.14}$$

$$(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{n-1}). \tag{9.15}$$

Moreover the formula (9.13) in lemma 9.3 says that $(\pi|_Y)^*$ realizes $\mathbb{K}[Y]$ as an integral ring extension of $\mathbb{K}[t_1, \ldots, t_{n-1}] = \mathbb{K}[\mathbb{A}^{n-1}]$. Morphisms satisfying this property (on each affine open set) are called **finite morphisms**.

Lemma 9.4. *Let $\pi : X \to \mathbb{P}^{n-1}$ be the projection of $X \subset \mathbb{P}^n$ with center $p \notin X$. If $Y \subset X$ is a closed subset and $\pi(X) = \pi(Y)$, then $Y = X$.*

Remark 68 Equivalently, for a closed $Y \subsetneq X$, then $\pi(Y) \subsetneq \pi(X)$.

Corollary 9.5. *Let $\pi : X \to \mathbb{P}^{n-1}$ be the projections from $p \notin X$. Then $\dim X = \dim \pi(X)$.*

Corollary 9.6. $\dim \mathbb{P}^n = n$.

9.1 Dimension of the intersection with a hypersurface

9.1.1 Projective case

Proposition 9.7. *Let $X \subset \mathbb{P}^n$ be a projective variety, and $f \in \mathbb{K}[x_0, \ldots, x_n]$ be a non constant homogeneous polynomial, s.t. $f \notin \mathbb{I}_p(X)$. Then*

$$\dim (X \cap \mathbb{V}_p(f)) = \dim X - 1. \tag{9.16}$$

Remark 69 If the intersection $X \cap \mathbb{V}_p(f)$ is reducible, we don't know if all the components have maximal dimension ($\dim X - 1$). However we will later prove that all components have the same dimension, i.e. $X \cap \mathbb{V}_p(f)$ is of pure dimension $\dim X - 1$.

Proposition 9.8. *Let X be a variety and $\emptyset \neq U \subset X$ be an open subset. Then $\dim X = \dim U$.*

Remark 70: Some consequences.

1. $\mathbb{A}^n \subset \mathbb{P}^n$ is open, hence $\dim \mathbb{A}^n = \dim \mathbb{P}^n$.

2. $\mathbb{A}^{n+m} = \mathbb{A}^n \times \mathbb{A}^m \overset{\text{open}}{\subset} \mathbb{P}^n \times \mathbb{P}^m$, hence $\dim (\mathbb{P}^n \times \mathbb{P}^m) = n + m$.

3. Given $f \in \mathbb{K}[x_1, \ldots, x_n]$ a non-constant polynomial, then $\mathbb{V}(f) \subset \mathbb{A}^n$ has dimension $n - 1$. This follows from $X \subset \overline{X} = \mathbb{V}_p \left({}^h f \right) \subset \mathbb{P}^n$.

4. Moreover every irreducible component of $\mathbb{V}(f)$, with f the above polynomial, has dimension $n - 1$. This follows from the factorization in prime components (possible, since $\mathbb{K}[x_1, \ldots, x_n]$ is a UFD)

$$f = f_1 \cdot \ldots \cdot f_r, \tag{9.17}$$

with $f_1, \ldots, f_r$ prime. Then the irreducible components of $\mathbb{V}(f)$ are exactly $\mathbb{V}(f_i)$, and, by the above remark, they are exactly of dimension $n - 1$.

5. If $f \in \mathbb{K}[x_0, \ldots, x_n]$ is a non-constant polynomial and $X := \mathbb{V}_p(f) \subset \mathbb{P}^n$, then each irreducible component of X has dimension $n - 1$.

   This is true, since we can always find a hyperplane $H \not\subset X (\subsetneq \mathbb{P}^n)$. Then $\mathbb{A}^n \supset X \cap (\mathbb{P}^n \setminus H) \simeq \mathbb{V}(g)$, for some $g \in \mathbb{K}[x_1, \ldots, x_n]$ non-constant. Then the irreducible components of X are the projective closures of the irreducible components of $\mathbb{V}(g)$, which have dimension $n - 1$ by the above remark.

   In particular, given a factorization

$$f = f_1 \cdot \ldots \cdot f_r \tag{9.18}$$

   for $f \in \mathbb{K}[x_0, \ldots, x_n]$, in prime components (which can be obtained homogenizing the prime components of g). Then

$$X = \mathbb{V}_p(f) = \underbrace{\mathbb{V}_p(f_1) \cup \ldots \cup \mathbb{V}_p(f_r)}_{\text{irreducible components of } X}. \tag{9.19}$$

Definition 9.9: Hypersurface.
A **hypersurface** in $\mathbb{A}^n$, resp. $\mathbb{P}^n$, is a closed subset of dimension $n - 1$. The minimal degree of a generator of its ideal is called the *degree* of the hypersurface X.

Hypersurfaces of degree 1 are called *hyperplanes*.

Remark 71 Up to scaling, we have the following correspondances

$$\left\{ \begin{matrix} \text{irreducible degree } d \\ \text{hypersurfaces in } \mathbb{A}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducible degree } d \\ \text{polynomials} \end{matrix} \right\}_{/\mathbb{K}^*} \subset \mathbb{P}\left(\mathbb{K}[x_1, \ldots, x_n]_{\leq d} \right) . \tag{9.20}$$

$$\left\{ \begin{matrix} \text{irreducible degree } d \\ \text{hypersurfaces in } \mathbb{P}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irred. homogoneneous} \\ \text{polynomials of deg. } d \end{matrix} \right\}_{/\mathbb{K}^*} \subset \mathbb{P}\left(\mathbb{K}[x_0, \ldots, x_n]_d \right) . \tag{9.21}$$

One can actually prove that the locus of irreducible polynomials is a dense open subset in the projective space of all homogenous polynomials of a fixed degree. This is indeed a consequence of Main theorem on projective varieties.

Definition 9.10: Codimension.
Let $Y \subset X$ be a closed subset. We define the **codimension** of Y in X as

$$\text{codim}_X Y := \dim X - \dim Y. \tag{9.22}$$

9.1.2 Affine case

Remark 72 The projective case does not imply the affine one. In fact, let $X := \mathbb{V}\left(x_2 - x_1^2\right) \subset \mathbb{A}^2$ and $f = x_1$. Let $\overline{X} = \mathbb{V}_p\left(x_0 x_2 - x_1^2\right) \subset \mathbb{P}^2$. Then

$$\dim\left(\overline{X} \cap \mathbb{V}_p(x_1)\right) = \dim \overline{X} - 1 = 0. \tag{9.23}$$

More explicitly

$$\overline{X} \cap \mathbb{V}_p(x_1) = \left\{ [1, 0, 0],\ \underbrace{[0, 0, 1]}_{\substack{\text{contained in} \\ H = \mathbb{V}_p(x_0)}} \right\}. \tag{9.24}$$

Moreover factorizing f does not help to construct all irreducible components.

Proposition 9.11. *Let $X \subset \mathbb{A}^n$ be an affine variety, and $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial not vanishing identically on X. Then, if $X \cap \mathbb{V}(f) \neq \emptyset$, we have $\dim(X \cap \mathbb{V}(f)) = \dim X - 1$.*

Corollary 9.12. *If $X \subset \mathbb{A}^n$ is an affine variety, and $f \notin \mathbb{I}(X)$ is a non constant polynomial, then all irreducible components of $X \cap \mathbb{V}(f)$ have dimension equal to $\dim X - 1$.*

Theorem 9.13 (Dimension of a fiber (weak version))**.** *Let $f : X \to Y$ be a morphism of varieties and assume there is a nonempty open subset $U \subset Y$ s.t. $f^{-1}(p) \neq \emptyset$ for all $p \in U$ and $\dim f^{-1}(p) = n$ for all $p \in U$, then*

$$\dim X = \dim Y + n. \tag{9.25}$$

Theorem 9.14 (Dimension of a fiber (complete version))**.** *Let $f : X \to Y$ be a surjective morphism of varieties, then $\dim X \geq \dim Y$ and*

1. *for every $y \in Y$ and every component F of $f^{-1}(y)$, we have $\dim F \geq \dim X - \dim Y$,*

2. *there exists a non-empty open subset $U \subset Y$ s.t. $\dim f^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.*

Corollary 9.15. *Let $X \subset \mathbb{P}^n$ be Zariski closed and $f : X \to Y$ be a morphism to a variety Y (in the sense that f is the restriction of a morphism defined on some larger subvariety of $\mathbb{P}^n$). If all fibers $f^{-1}(y)$, for $y \in Y$, are irreducible and of the same dimension, then X is irreducible.*

Corollary 9.16. *For any two varieties X and Y, we have*

$$\dim X \times Y = \dim X + \dim Y. \tag{9.26}$$

[Consider $\pi_Y : X \times Y \to Y$, with fibers isomorphic to X].

Corollary 9.17. *Let $X, Y \subset \mathbb{A}^n$ be affine varieties s.t. $X \cap Y \neq \emptyset$, then every irreducible component Z of $X \cap Y$ has dimension*

$$\dim Z \geq \dim X + \dim Y - n. \tag{9.27}$$

Corollary 9.18. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties. Then every irreducible component of $X \cap Y$ has dimension $\geq \dim X + \dim Y - n$ and, if $\dim X + \dim Y \geq n$, then $X \cap Y \neq 0$.*

9.2 Dimension - again

Remark 73 Let X be a variety. Denote by $\mathbb{K}(X)$ the function field of X, i.e. the field of rational functions on X.

If $X \subset \mathbb{A}^n$ is affine, then $\mathbb{K}(X) = Q(\mathbb{K}[X])$, e.g.

$$\mathbb{K}(\mathbb{A}^n) = \mathbb{K}(x_1, \ldots, x_n) \qquad \text{the field of rational functions.} \tag{9.28}$$

If $X \subset \mathbb{P}^n$ is projective, then

$$\mathbb{K}(\mathbb{P}^n) = \left\{ \frac{p(x_0, \ldots, x_n)}{q(x_0, \ldots, x_n)} \; \middle| \; p, q \in S(X)_d \text{ and } q \neq 0 \right\}. \tag{9.29}$$

If X is an arbitrary variety, by ex. 6, ex-sh 3, we have

$$\mathbb{K}(X) = \{(U, \phi) \mid \phi \in \mathcal{O}_X(U)\} / \sim, \tag{9.30}$$

where $(U, \phi) \sim (U', \phi')$ iff $\phi|_V = \phi'|_V$ for some $\emptyset \subsetneq V \subset U \cap U'$ open. The fundamental property is that $\mathbb{K}(X) = \mathbb{K}(U)$ for any (affine) open set $U \neq \emptyset$.

Definition 9.19: Transcendence degree.
Given $\mathbb{K} \subset E$ a field extension, we define the **transcendence degree**, denoted by $\text{tr.deg.}(E/\mathbb{K})$, is the largest integer l s.t. there exists l algebraically independent elements $\zeta_1, \ldots, \zeta_l \in E$. In the above algebraically independent means that the valuation morphism

$$\phi : \mathbb{K}[x_1, \ldots, x_l] \to E \tag{9.31}$$
$$F(x_1, \ldots, x_l) \mapsto F(\zeta_1, \ldots, \zeta_l) \tag{9.32}$$

has trivial kernel.

Theorem 9.20. *If X is a variety, then $\dim X$ equals the transendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$.*

Remark 74 If $X \subset \mathbb{A}^m$ is an affine variety, then there is a finite map $X \to \mathbb{A}^n$, with $n = \dim X$, given by a sequence of projections. Hence $\mathbb{K}(X)$ is an integral extension of $\mathbb{K}[x_1, \ldots, x_n]$, then $\text{tr.deg.} \mathbb{K}(X) = n$.

10 Blow-ups

Definition 10.1: Rational map.
Let X, Y be varieties. A **rational map** $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$, for some $\emptyset \neq U \subset X$ open. (Notice that we denote rational maps via a dashed arrow, since it might not be defined on the whole variety X). If $f : U \to Y$ and $f' : V \to Y$ are morphisms with $\emptyset \neq U \subset X \supset V \neq \emptyset$ both open, then f and g define the same rational map iff

$$f|_{U \cap V} = g|_{U \cap V} \,. \tag{10.1}$$

(Recall that, since X is irreducible, then $U \cap V \neq \emptyset$).

Definition 10.2

- A rational map $f : X \dashrightarrow Y$, given by a morphism $f : U \to Y$, is dominant iff $f(U)$ contains a non-empty open subset $V \subset Y$ (in particular its image is dense in Y).

- If $f : X \dashrightarrow Y$ is dominant, for every $g : Y \dashrightarrow Z$, the composition $g \circ f : X \dashrightarrow Z$ is a rational map.

- A birational map is a rational map $f : X \dashrightarrow Y$ with a rational inverse, i.e. f is a dominant rational map s.t. there exists a dominant $g : Y \dashrightarrow X$ satisfying

$$g \circ f = id_X \qquad \text{and} \qquad f \circ g = id_Y \qquad \text{(as rational maps).} \tag{10.2}$$

- Two varieties X and Y are said to be birational, or birationally equivalent, iff there is a birational map $f : X \dashrightarrow Y$.

Theorem 10.3. *X and Y are birational iff there exist non-empty open subsets $U \subset X$ and $V \subset Y$ that are isomorphic to each other.*

Let's now construct the most important class of birational maps: the blow-ups. Since this is a local construction, we'll start from affine varieties.

Definition 10.4: Construction of blow-up.
Let $X \subset \mathbb{A}^n$ be an affine variety, and $f_0, \ldots, f_r \notin \mathbb{I}(X)$ be polynomials in $x_1, \ldots, x_n$. Then $U := X \setminus \mathbb{V}(f_0, \ldots, f_r)$ is a non-empty open subset of X. We can consider the morphism

$$f : U \to \mathbb{P}^r \tag{10.3}$$
$$x \mapsto [f_0(x), \ldots, f_r(x)] \,. \tag{10.4}$$

The graph $\Gamma := \{(p, f(p)) \mid p \in U\} \subset X \times \mathbb{P}^r$ is isomorphic to U via the projection $(p, q) \mapsto p$. We define the **blow-up** of X at $(f_0, \ldots, f_r)$ to be the Zariski closure

$$\widetilde{X} := \overline{\Gamma} \subset X \times \mathbb{P}^r. \tag{10.5}$$

Remark 75

- By definition $\widetilde{X}$ is closed in $X \times \mathbb{P}^r$, hence it is a quasi-projective variety.

- $\widetilde{X}$ is irreducible, since $U \cong \Gamma$ is irreducible.

- $\dim \widetilde{X} = \dim X$.

- The projections $p_X : X \times \mathbb{P}^r \to X$ and $p_{\mathbb{P}^r} : X \times \mathbb{P}^r \to \mathbb{P}^r$ induce the projection morphisms

$$\pi : \widetilde{X} \to X \qquad \text{and} \qquad p : \widetilde{X} \to \mathbb{P}^r. \tag{10.6}$$

Example Let $r = 0$, then the blow-up of X at $f_0 \notin \mathbb{I}(X)$ is isomorphic to X. In fact $\widetilde{X} \subset X \times \mathbb{P}^0 \cong X$, since $\mathbb{P}^0$ is just a point. Then an irreducible subvariety of $X \times \mathbb{P}^0$, with the same dimension as X, is isomorphic to X.

Example Let $X = \mathbb{A}^2$, with coordinates x_0, x_1. Set $f_0 := x_0$ and $f_1 := x_1$. Then $\widetilde{X} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Then

$$f : U = \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1 \tag{10.7}$$
$$(x_0, x_1) \mapsto [x_0, x_1]. \tag{10.8}$$

The graph of f is

$$\Gamma = \{((x_0, x_1), [y_0, y_1]) \mid x_0 y_1 = x_1 y_0\} \subset U \times \mathbb{P}^1. \tag{10.9}$$

Then, if we compute its closure, we obtain

$$\widetilde{X} = \{((x_0, x_1), [y_0, y_1]) \mid x_0 y_1 = x_1 y_0\} \subset \mathbb{A}^2 \times \mathbb{P}^1, \tag{10.10}$$

in which we allow $(x_0, x_1) = (0,0)$. Denoting $\pi : \widetilde{X} \to X$ the projection, we obtain

$$\pi^{-1}((x_0, x_1)) = \{((x_0, x_1), [x_0, x_1])\} \qquad \text{if } (x_0, x_1) \neq (0,0) \tag{10.11}$$
$$\pi^{-1}((x_0, x_1)) = ((0,0)) \times \mathbb{P}^1 \cong \mathbb{P}^1. \tag{10.12}$$

Definition 10.5: Strict transform.
Let $Y \subset X$ be a closed subset with $Y \cap U \neq \emptyset$, then the Zariski closure of $Y \cap U$ in $\widetilde{X}$ is called the **strict transform** $\widetilde{Y}$ of Y. This is the same as the blow-up of Y at $(f_0, \ldots, f_r)$.

Example In the above example the strict transform is interesting for a curve C in $\mathbb{A}^2$ passing through $(0,0)$. Since C is a curve, then $C = \mathbb{V}(g)$, for

$$g(x_0, x_1) = \sum_{i,j \geq 0} a_{ij} x_0^i x_1^j = a_{00} + a_{10} x_0 + a_{01} x_1 + h.o.t. \tag{10.13}$$

Let's assume that $a_{00} = 0$, i.e. $(0.0) \in C$, and $(a_{01}, a_{10}) \neq (0,0)$, so that

$$L := \mathbb{V}(a_{10} x_0 + a_{01} x_1) \tag{10.14}$$

is a line, which we call the tangent line to C at $(0,0)$. If $((x_0, x_1), [y_0, y_1]) \in \widetilde{D}$, then it satisfies

$$\begin{cases} a_{10} x_0 + a_{01} x_1 + a_{20} x_0^2 + \ldots = 0 \\ x_0 y_1 - x_1 y_0 = 0 \end{cases}. \tag{10.15}$$

But the solution set of (10.15) is larger than $\widetilde{C}$, since it contains the whole $\pi^{-1}((0,0)) \cong \mathbb{P}^1$. But $\widetilde{C}$ is irreducible, hence it cannot contain such additional component.

To get the equations for $\widetilde{C}$, we work first on $\mathbb{A}^2 \setminus \{x_0 x_1 = 0\}$. There we can multiply the equation of C by $\frac{y_0}{x_0}$, which is a regular function on $\mathbb{A}^2 \setminus \{x_0 x_1 = 0\} \times \mathbb{P}^1$, and obtain

$$a_{10}y_0 + a_{01}\frac{y_0}{x_0}x_1 + a_{20}y_0 x_0 + a_{11}y_0 x_1 + \dots . \tag{10.16}$$

Using $\frac{y_0}{x_0} = \frac{y_1}{x_1}$ this is equivalent to

$$\begin{cases} a_{10}y_0 + a_{01}y_1 + a_{20}y_0 x_0 + a_{11}y_0 x_1 + a_{02}y_1 x_1 + \dots = 0 \\ x_0 y_1 - x_1 y_0 = 0 \end{cases} . \tag{10.17}$$

This equation should hold on $\widetilde{C}$. For every $C \ni (x_0, x_1) \neq (0,0)$, the above equations give the single point $((x_0, x_1), [x_0, x_1])$. Instead over $(0,0)$ we obtain the point $((0,0), [y_0, y_1])$ with $a_{10}y_0 + a_{01}y_1 = 0$, which is exactly the equation of the tangent line at $(0,0)$.

Example: Blow-up of $\mathbb{A}^2$ at (x_0, x_1). It is a subset

$$\widetilde{\mathbb{A}^2} \subset \mathbb{A}^2 \times \mathbb{P}^1, \tag{10.18}$$

whose points are

$$((x_0, x_1), [y_0, y_1]) \quad \text{s.t.} \quad \mathrm{rk} \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} = 1. \tag{10.19}$$

Moreover it comes endowed with two morphisms (since they are given by the restriction of the projection morphisms)

- $p : \widetilde{\mathbb{A}^2} \to \mathbb{P}^1$ that extends the map used to define the projective space $\mathbb{P}^1$

$$\mathbb{K}^2 \setminus \{\mathbf{0}\} \to \mathbb{P}^1 \tag{10.20}$$
$$(y_0, y_1) \mapsto [y_0, y_1]. \tag{10.21}$$

  Then the fibers of p are isomorphic to the lines of $\mathbb{A}^2$ through the origin via π.

- $\pi : \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$, which is an isomorphism outside $(0,0) = \mathbf{0}$. In fact $\pi^{-1}(\{\mathbf{0}\}) = \{\mathbf{0}\} \times \mathbb{P}^1$.

In particular we can say that the points of $\pi^{-1}(\{\mathbf{0}\})$ correspond to the directions of the lines through the origin $\mathbf{0}$.

Looking at the strict transform, then we can identify $\pi^{-1}(\{\mathbf{0}\})$ with the set of tangent directions to $\mathbb{A}^2$ at the origin $\mathbf{0}$.

Lemma 10.6. *The blow-up of an affine variety X at $(f_0, \dots, f_r)$ only depends on the ideal generated by $(f_0, \dots, f_r)$ in $\mathbb{K}[X]$.*

Definition 10.7: Blow-up at an ideal.
Given an affine variety X and $\mathcal{I} \subset \mathbb{K}[X]$ an ideal, we define the blow-up of X at $\mathcal{I}$ as the blow-up of X at $(f_0, \dots, f_r)$, for any $f_0, \dots, f_r$ generators of $\mathcal{I}$.

In particular, for $Y \subset X$ a closed subset, we can define the blow-up of X along Y as the blow-up of X at $\mathbb{I}(Y)$.

Remark 76 The construction of the blow-up is local. In fact let X be an arbitrary variety and $Y \subset X$ a closed subset. If $\{U_i\}_{i \in I}$ is an affine open cover of X, let us denote by $\widetilde{U}_i$ the blow-up of the affine variety U_i along $Y \cap U_i$.

It is easy to check that we can glue together the $\widetilde{U}_i$ in a natural way, to construct a variety $\widetilde{X}$, which we will call the blow-up of X along Y.

Actually we can blow-up a sheaf of ideals on X. In fact the construction of the blow-up of an ideal can be generalized to arbitrary varieties. In order to do so we need

- $I(U) \subset \mathcal{O}_X(U)$, for each open subset $U \subset X$,

- gluing conditions.

Let's specialize the above remark to the case of projective varieties.

Definition 10.8: Blow-up for projective varieties.
Let X be a projective variety, and let $f_0, \ldots, f_r$ homogeneous polynomials of degree d, not in $\mathbb{I}_p(X)$. Let $U := X \setminus \mathbb{V}_p(f_0, \ldots, f_r)$ and

$$\Gamma := \{(x, [f_0(x), \ldots, f_r(x)]) \mid x \in U\} \subset X \times \mathbb{P}^r. \tag{10.22}$$

Then we define $\widetilde{X} := \Gamma \times \mathbb{P}^r$ to be the **blow-up** of X at $(f_0, \ldots, f_r)$.

Remark 77 In particular, the blow-up of a projective variety is a projective variety.

Lemma 10.9.

1. $\widetilde{X} \subset \{(x, [y_0, \ldots, y_r]) \mid y_i f_j(x) - y_j f_i(x) = 0, \text{ for all } i, j = 0, \ldots, r\} \subset X \times \mathbb{P}^r$.

2. *The inverse image* $E := \pi^{-1}(\mathbb{V}_p(f_0, \ldots, f_r))$ *is of pure dimension* $\dim X - 1$. E *is called the* exceptional hypersurface *of the blow-up.*

Remark 78 The equations $y_i f_j(x) = y_j f_i(x)$, in general, are not the only equations defining $\widetilde{X}$ inside $X \times \mathbb{P}^r$.

In case they actually are the only ones, then

$$E = (X \cap \mathbb{V}(f_0, \ldots, f_r)) \times \mathbb{P}^r. \tag{10.23}$$

The dimensions are: $\dim E = \dim X - 1$, whereas $X \cap \mathbb{V}(f_0, \ldots, f_r)$ might have dimension $> \dim X - r - 1$, then the r.h.s. might have dimension $> \dim X - 1$.

Definition 10.10: Initial polynomial/ideal.
Consider $0 \neq f \in \mathbb{K}[x_1, \ldots, x_n]$.

- We can decompose in homogeneous components $f = \sum_{i \in \mathbb{N}} f_i$, for $f_i \in \mathbb{K}[x_1, \ldots, x_n]_i$ homogeneous of degree i. The minimal $i \in \mathbb{N}$ s.t. $f_i \neq 0$ is called the **minimal degree**, denoted with $\operatorname{mindeg} f$, of f. For $i = \operatorname{mindeg} f$, we call $f^{in} := f_i$ the **initial polynomial** of f.

- For an ideal $(0) \neq \mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$, we define the **initial ideal** of $\mathcal{I}$ to be

$$\mathcal{I}^{in} := \left(f^{in} \mid f \in \mathcal{I}\right), \tag{10.24}$$

the homogeneous ideal generated by the initial polynomials of the elements of $\mathcal{I}$.

Remark 79 Notice that, given $\mathcal{I} = (g_1, \ldots, g_l)$, then it is not always true that $\mathcal{I}^{in} = (g_1^{in}, \ldots, g_l^{in})$.

Remark 80 Let's now study the blow-up of an affine variety $X \subset \mathbb{A}^n$ at a point, wlog $(0, \ldots, 0)$. Let us denote by $\widetilde{\mathbb{A}^n}$ the blow up of $\mathbb{A}^n$ at $(x_1, \ldots, x_n)$. We want to cover it with affine varieties: let's start covering $\mathbb{P}^{n-1}$ with the affine varieties $U_i := \{[y_1, \ldots, y_n] \in \mathbb{P}^{n-1} \mid y_i \neq 0\}$. In particular, for $i = 1$, we have $y_1 \neq 0$. Then, by the condition on the rank we obtain $x_j = x_1 y_j$ for the elements in $\widetilde{\mathbb{A}^n}$, hence we can define a map φ, with which we can cover the blow-up

$$\varphi : \mathbb{A}^n \to \widetilde{\mathbb{A}^n} \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \supset \mathbb{A}^n \times U_1 \qquad (10.25)$$

$$\left(x_1, \underbrace{x_1 y_2, \ldots, x_1 y_n}_{\text{inhom. coord. on } U_1}\right) \mapsto \left((x_1, x_1 y_2, \ldots, x_1 y_n), [1, y_2, \ldots, y_n]\right), \qquad (10.26)$$

Then, by choosing the various affine charts of $\mathbb{P}^{n-1}$, we can cover the blow-up $\widetilde{\mathbb{A}^n}$ by affine spaces.

Then the blow-up of any $X := \mathbb{V}(\mathcal{I}) \subset \mathbb{A}^n$ containing $\mathbf{0}$ is defined, on this coordinate patch, by the vanishing of

$$\frac{f(x_1, x_1 y_2, \ldots, x_1 y_n)}{x_1^{\text{mindeg } f}} \qquad \text{for all } f \in \mathcal{I}, f \neq 0. \qquad (10.27)$$

The exceptional hypersurface $E \subset \widetilde{X}$ is

$$\{\mathbf{0}\} \times \mathbb{V}_p\left(\mathcal{I}^{in}\right) \subset \{\mathbf{0}\} \times \mathbb{P}^{n-1}. \qquad (10.28)$$

The geometric interpretation of E is similar to taking the lower terms of the Taylor expansion of a function. In fact $\mathbb{I}(X)^{in}$ approximates $\mathbb{I}(X)$ by keeping only the terms of smallest degree. Since $\mathbb{I}^{in}$ is a homogeneous ideal, then $\hat{Y} := \mathbb{V}\left(\mathbb{I}(X)^{in}\right) \subset \mathbb{A}^n$ is a cone with vertex in the origin. By construction, $\hat{Y}$ is the cone (of the same dimension as X) that approximates X best around $\mathbf{0}$. In fact

$$\dim \hat{Y} = \dim \underbrace{\mathbb{V}_p\left(\mathbb{I}(X)^{in}\right)}_{E} + 1 = \dim X - 1 + 1 = \dim X. \qquad (10.29)$$

Definition 10.11: Tangent cone.
Let $p \in X \subset \mathbb{A}^n$ a point of the Zariski closed subset X (wlog $p = (0, \ldots, 0) \in \mathbb{A}^n$). Then the tangent cone to X at the point p is

$$C_{X,p} := \mathbb{V}\left(\mathbb{I}(X)^{in}\right) \subset \mathbb{A}^n. \qquad (10.30)$$

Remark 81 Then the exceptional hypersurface $E \subset \widetilde{X}$ is the projectivization of this cone and corresponds to the limit tangent directions to X at p.

Example

1. Let $X := \mathbb{V}\left(y - x^2 + x\right) \subset \mathbb{A}^2$. Let $f(x, y) = y - x^x + x$, then $\text{mindeg } f = 1$. Then $C_{X,0} = \mathbb{V}(y + x) \subset \mathbb{A}^2$ is the tangent line to X at the origin $\mathbf{0}$.

2. Let $X := \mathbb{V}\left(x^2 + x^3 - y^2\right) \subset \mathbb{A}^2$. Let $f(x, y) = x^2 + x^3 - y^2$, then $\text{mindeg } f = 2$. Then $C_{X,0} = \mathbb{V}\left(x^2 - y^2\right) = \mathbb{V}(x - y) \cup \mathbb{V}(x + y) \subset \mathbb{A}^2$ is the tangent line to X at the origin $\mathbf{0}$. If we blow up X at $\mathbf{0}$, then $\widetilde{X}$ separates the two branches of X at the origin $\mathbf{0}$.

3. Let $X := \mathbb{V}\left(x^3 - y^2\right) \subset \mathbb{A}^2$. Let $f(x, y) = x^3 - y^2$, then $\mathrm{mindeg}\, f = 2$. Then $C_{X,0} = \mathbb{V}\left(y^2\right) = \mathbb{V}\left(y\right) \subset \mathbb{A}^2$ is the tangent line to X at the origin $\mathbf{0}$. But this line comes with some nontrivial multiplicity. In fact, if we blow-up the origin $\widetilde{X}$ is tangent to $\{\mathbf{0}\} \times \mathbb{P}^1 = E \subset \widetilde{\mathbb{A}^2}$ at its point corresponding to the line $y = 0$.

Remark 82 Given a rational map $f : X \dashrightarrow \mathbb{P}^r$, given by $x \mapsto [f_0(x), \ldots, f_f(x)]$ for $r + 1$ polynomials of the same degree (not all of them in the homogeneous ideal of X), then f is a well defined morphism only on an open subset $U := X \setminus \mathbb{V}(f_0, \ldots, f_r)$. Then we have a closed subset on which f is not determined: $\mathbb{V}(f_0, \ldots, f_r)$. We will call it the *indeterminacy locus* of f.

However, we can always extend f to a morphism $\tilde{f} : \widetilde{X} \to \mathbb{P}^r$, where $\widetilde{X}$ is a the blow-up of X at $(f_0, \ldots, f_r)$, and f is defined by the composition $\widetilde{X} \hookrightarrow X \times \mathbb{P}^r \xrightarrow{\pi} \mathbb{P}^r$. If we blow-up the indeterminacy locus, we can always extend $f : X \dashrightarrow \mathbb{P}^r$ to a projective variety birational to X.

If, moreover, we start from a birational map $f : X \dashrightarrow Y$ this gives rise to interesting situations. Let's look at how to extend such a map to isomorphisms $\widetilde{X} \to \widetilde{Y}$.

Example: Cremona transformation. It is a birational map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \tag{10.31}$$

$$[x_0, x_1, x_2] \mapsto [x_1 x_2, x_0 x_2, x_0 x_1]. \tag{10.32}$$

Its indeterminacy locus is $\mathbb{V}_p\left(x_1 x_2, x_0 x_2, x_0 x_1\right) = \{p_0 := [1, 0, 0], p_1 := [0, 1, 0], p_3 := [0, 0, 1]\}$. Since there are no relations in $S(\mathbb{P}^2) = \mathbb{K}[x_0, x_1, x_2]$, we cannot extend f to a morphism outside $U := \mathbb{P}^2 \setminus \{p_0, p_1, p_2\}$.

Let's show that f is a birational map: in such case there are $U \subset \mathbb{P}^2 \supset V$ open subsets s.t. $f|_U : U \to V$ is an isomorphism. In fact, on $V := \mathbb{V}_p\left(x_0 x_1 x_2\right)$, we can divide by $x_0 x_1 x_2$, and obtain that

$$f([x_0, x_1, x_2]) = [x_1 x_2, x_0 x_2, x_0 x_1] = [\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}]. \tag{10.33}$$

Clearly, when restricted to V, $f \circ f = id_V$, hence $f|_U : V \to V$ is an isomorphism (f is an involution of V). Moreover, denoting by $L_i := \mathbb{V}_p\left(x_i\right)$, we can notice that $f(L_i) = p_i$, where p_i are the points defined above. In fact $f(L_0) = f([0, x_1, x_2]) = [x_1 x_2, 0, 0] = p_0$.

Finally we can extend f to a morphism by using blow-ups. Let's take $\widetilde{\mathbb{P}^2}$ the blow-up of $\mathbb{P}^2$ at $(x_1 x_2, x_0 x_2, x_0 x_1)$, i.e. the blow-up at p_0, p_1, p_2. Then we can extend f to $\tilde{f} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$. In fact $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$, and

$$([x], [u]) \in \widetilde{\mathbb{P}^2} \iff \mathrm{rk}\begin{pmatrix} x_1 x_2 & x_0 x_2 & x_0 x_1 \\ u_0 & u_1 & u_2 \end{pmatrix} = 1. \tag{10.34}$$

Notice that this is equivalent (in general), to asking that u_0, u_1, u_2 is proportional to $1/x_0, 1/x_1, 1/x_2$, hence the condition is symmetric in $[x]$ and $[u]$. The we can define

$$\tilde{f}\left([x_0, x_1, x_2], [u_0, u_1, u_2]\right) = \left([u_0, u_1, u_2], [x_0, x_1, x_2]\right), \tag{10.35}$$

and $[u_0, u_1, u_2] = f([x_0, x_1, x_2])$, where defined. Then this is clearly an extension of f and also clearly an isomorphism. We only need to check that its image is $\widetilde{\mathbb{P}^2}$. In

order to do so we only need to check that the image of the three exceptional divisors is contained in $\widetilde{\mathbb{P}^2}$. This is true, since f is a birational inverse to itself.

More explicitly, in $\widetilde{\mathbb{P}^2}$ the points p_i get mapped to the exceptional lines E_i, and L_i get mapped to their strict transform $\widetilde{L_i}$. Then $\tilde{f}$ sends $\widetilde{L_i}$ to E_i and vice-versa.

Example Another example of a couple of birational projective varieties is given by

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \tag{10.36}$$

since they contain the isomorphic open subsets $\mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1$. In particular we can explicitly write f as, in the affine chart $U_0 \subset \mathbb{P}^2$

$$[1, x, y] \xrightarrow{f} s_{1,1}([1, x], [1, y]) = [1, x, y, xy]. \tag{10.37}$$

In homogeneous coordinates (homogenizing everything with respect to x_0) it becomes

$$f([x_0, x_1, x_2]) = [x_0^2, x_0 x_1, x_0 x_2, x_1 x_2]. \tag{10.38}$$

Its indeterminacy locus is exactly

$$\mathbb{V}_p\left(x_0^2, x_0 x_1, x_0 x_2, x_1 x_2\right) = \mathbb{V}_p\left(x_0, x_1 x_2\right) = \{q_1 := [0, 1, 0], q_2 := [0, 0, 1]\}. \tag{10.39}$$

To extend f to a morphism (to resolve f) we need to blow-up $\mathbb{P}^2$ at $\mathcal{I} = \left(x_0^2, x_0 x_1, x_0 x_2, x_1 x_2\right) \subsetneq \mathbb{I}_p(\{q_1, q_2\})$ (notice that they have the same zero locus: they are related by saturations).

Let $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^3$ be the blow-up at $\mathcal{I}$. Then $\widetilde{\mathbb{P}^2}$ is isomorphic to the blow-up of $\mathbb{P}^2$ at q_1 and q_2. In $U_1 = \{x_1 \neq 0\} \ni q_1$, we can take $x_1 = 1$ and then $\mathcal{I}$ restricts to

$$\left(x_0^2, x_0, x_0 x_2, x_2\right) = (x_0, x_2) = \mathbb{I}(q_2) \subset \mathbb{K}[x_0, x_2] = \mathbb{K}[U_1]. \tag{10.40}$$

andalogously, on U_2, $\mathcal{I}$ restricts to

$$\left(x_0^2, x_0 x_1, x_0, x_1\right) = (x_0, x_1) = \mathbb{I}(q_1) \subset \mathbb{K}[x_0, x_1] = \mathbb{K}[U_2]. \tag{10.41}$$

The birational inverse of f is

$$g : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \tag{10.42}$$

$$[u_0, u_1, u_2, u_3] \mapsto [u_0, u_1, u_2]. \tag{10.43}$$

This is the projection from $p = [0, 0, 0, 1] = s_{1,1}([0, 1], [0, 1])$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Hence p is the indeterminacy locus and we need to blow it up, to obtain

$$\widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^3 \times \mathbb{P}^2. \tag{10.44}$$

Now we want to define an isomorphism between $\widetilde{\mathbb{P}^2}$ and $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$.

$$\tilde{f} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \tag{10.45}$$

$$([x_0, x_1, x_2], [y_0, y_1, y_2, y_3]) \mapsto ([y_0, y_1, y_2, y_3], [x_0, x_1, x_2]), \tag{10.46}$$

where the points $([x_0, x_1, x_2], [y_0, y_1, y_2, y_3]) \in \widetilde{\mathbb{P}^2}$ have to satisfy

$$1 = \mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ x_0^2 & x_0 x_1 & x_0 x_2 & x_1 x_2 \end{pmatrix} \Rightarrow \mathrm{rk}\begin{pmatrix} y_0 & y_1 & y_2 \\ x_0^2 & x_0 x_1 & x_0 x_2 \end{pmatrix} = 1 \ \text{ for } x_0 \neq 0. \tag{10.47}$$

This is the restriction to $\widetilde{\mathbb{P}^2}$ of an isomorphism between $\mathbb{P}^2 \times \mathbb{P}^3$ and $\mathbb{P}^3 \times \mathbb{P}^2$. Hence we only have to check that

$$\tilde{f}\left(\widetilde{\mathbb{P}^2}\right) \subset \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \qquad \text{and} \qquad \tilde{f}^{-1}\left(\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}\right) \subset \widetilde{\mathbb{P}^2}. \tag{10.48}$$

Since $\widetilde{\mathbb{P}^2}$ is an irreducible projective variety, it is sufficient to show the above inclusions for dense open subsets of the required varieties. Take $\mathbb{A}^2 \cong \mathbb{P}^2 \setminus \mathbb{V}_p(x_0) \subset \widetilde{\mathbb{P}^2}$, where $\mathbb{V}_p(x_0)$ is the line passing trhough q_1 and q_2. We know that

$$\tilde{f}\left(\mathbb{A}^2\right) = f\left(\mathbb{A}^2\right) = \mathbb{A}^1 \times \mathbb{A}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{V}_p(x_0 y_0) \hookrightarrow \widetilde{\mathbb{P}^1 \times \mathbb{P}^1}. \tag{10.49}$$

The result for $\tilde{f}^{-1}$ follows analogously.

Lemma 10.12. $\mathbb{P}^1 \times \mathbb{P}^1$ *blown-up at one point is isomorphic to the blow-up of* $\mathbb{P}^2$ *at two points.*

Proof. The proof is a generalzation of the above example. ∎

10.1 Smooth varieties

Definition 10.13: Tangent space at a point.
Let $X \subset \mathbb{A}^n$ be an affine variety and $p \in X$. After a linear change of coordinates

$$X_1 := x_1 - p_1, \qquad X_2 := x_2 - p_2, \qquad \dots, \qquad X_n := x_n - p_n \qquad (10.50)$$

we can assume that $p = (0, \dots, 0)$. We define the **tangent space** to X at p to be

$$T_{X,p} := \mathbb{V}\left(\{f_1 \in \mathbb{K}[x_1, \dots, x_n] \mid f \in \mathbb{I}(X)\}\right), \qquad (10.51)$$

in which f_1 represents the linear part of $f = \sum_{j \in \mathbb{N}} f_j$, with $f_j \in \mathbb{K}[x_1, \dots, x_n]_j$ are the homogeneous components of f.

Remark 83 Consider $p = (p_1, \dots, p_n)$. We can consider the Taylor expansion of $f \in \mathbb{K}[x_1, \dots, x_n]$ at p (after all it is a polynomial):

$$f(x_1, \dots, x_n) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + h.o.t.. \qquad (10.52)$$

Then $T_{X,p}$ is defined by the linear equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) = 0, \qquad (10.53)$$

for all $f \in \mathbb{I}(X)$ (for which, hence, $f(p) = 0$).

Remark 84 The degree 1 part of an ideal is always contained in the initial ideal, i.e. $\mathcal{I}_1 \subset \mathcal{I}^{in}$. It follows that $C_{X,p} \subset T_{X,p}$, i.e. the tangent cone is contained in the tangent space.

Let's give a more intrinsic definition to the tangent space:

Lemma 10.14. *Let $p \in X \subset \mathbb{A}^n$ a point of an affine variety. Then the Vector Space $T_{X,p} \subset \mathbb{K}^n$ is isomorphic to the space of linear forms on $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$, where*

$$\mathfrak{m}_{X,p} = \{\varphi \in \mathcal{O}_{X,p} \mid \varphi(p) = 0\} \subset \mathcal{O}_{X,p} \subset \mathbb{K}(X), \qquad (10.54)$$

which can be identified with the germs of functions vanishing at p. Analogously, for $\mathfrak{m}_{X,p}^2$, we have

$$\mathfrak{m}_{X,p}^2 = \langle \varphi \cdot \psi \mid \varphi, \psi \in \mathfrak{m}_{X,p} \rangle \subset \mathcal{O}_{X,p} \subset \mathbb{K}(X), \qquad (10.55)$$

i.e. it contains the germs of functions vanishing at order ≥ 2 at p. Then

$$T_{X,p} \cong \left(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2\right)^{\vee}. \qquad (10.56)$$

Proof. We can assume, wlog, that $p = (0, \dots, 0)$. Then, the coordinates $x_1, \dots, x_n$ of $\mathbb{A}^n \cong \mathbb{K}^n$ define linear forms in $V^{\vee}$. In particular $x_1, \dots, x_n$ is a basis of $V^{\vee} = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$. Let, now, $\mathcal{I} := \mathbb{I}(X)$ and $\mathcal{I}_1 \subset \mathbb{K}[x_1, \dots, x_n]_1 = V^{\vee}$ be a linear subspace.

> **Step 1:** $V/\mathcal{I}_1$ is canonically isomorphic to $T_{X,p}^{\vee} = \text{Hom}_{\mathbb{K}}(T_{X,p}, \mathbb{K})$.
>
> We define the restriction morphism $\varphi : V^{\vee} \to T_{X,p}^{\vee}$ that sends any linear form $l \in V^{\vee}$ to itself, as a linear form in $T_{X,p}^{\vee}$. This is clearly a linear map. Moreover $T_{X,p}^{\vee} = \text{span}(\varphi(x_1), \dots, \varphi(x_n))$, hence φ is surjective. Finally $T_{X,p}^{\vee} = \mathbb{V}(\mathcal{I}_1)$, hence $\ker \varphi = \mathcal{I}_1$.

Step 2: *There is a natural isomorphism*

$$\psi : V^\vee / \mathcal{I}_1 \to \mathfrak{m}_{X,p} / \mathfrak{m}_{X,p}^2 \tag{10.57}$$

$$x_j \mapsto [x_j]. \tag{10.58}$$

Recall that $p = 0$, hence any x_j identifies an equivalence class in $\mathfrak{m}_{X,p} / \mathfrak{m}_{X,p}^2$.
Injectivity: Let $0 \neq \alpha \in \mathbb{K}[x_1, \ldots, x_n]_1$ be s.t. $[\alpha] = 0 \in \mathfrak{m}_{X,p} / \mathfrak{m}_{X,p}^2$. Then $\alpha \in \mathcal{O}_{X,p}$ belongs to $\mathfrak{m}_{X,p}^2$, i.e.

$$\alpha = \frac{F_1}{G_1} \cdot \frac{F_2}{G_2} \in \mathbb{K}(x_1, \ldots, x_n), \tag{10.59}$$

for $F_i, G_i \in \mathbb{K}[x_1, \ldots, x_n]$, with $F_i(p) = 0$ and $G_i(p) \neq 0$ for $i = 1, 2$. Then

$$G_1 G_2 \alpha = F_1 F_2 \in \mathbb{K}[x_1, \ldots, x_n]. \tag{10.60}$$

Then, since α is of degree 1, it is irreducible (hence prime). Since G_1 cannot divide F_i (they do not vanish at p, then α divides the product $F_1 F_2$, then α divides one of the two, let's say $F_1 = \alpha \widetilde{F_1}$. Then $G_1 G_2 = \widetilde{F_1} F_2$, but

$$0 \neq (G_1 G_2)(p) = \left(\widetilde{F_1} F_2 \right)(p) = \widetilde{F_1}(p) F_2(p) = 0. \tag{10.61}$$

This is a contradiction, hence $\alpha = 0$.
Surjectivity: Let $\phi = \frac{f}{g} \in \mathfrak{m}_{X,p}$, wlog $g(p) = 1$. We want to prove that

$$\phi' := \sum_i \frac{\partial \phi}{\partial x_i}(p) x_i \qquad \text{where} \qquad \frac{\partial \phi}{\partial x_i} = \frac{\frac{\partial f}{\partial x_i} \cdot g - \frac{\partial g}{\partial x_i} \cdot f}{g^2} \tag{10.62}$$

is the inverse image of ϕ, i.e. $\psi(\phi') - \phi \in \mathfrak{m}_{X,p}^2$. Let's multiply by g, to obtain

$$g(\phi' - \phi) = f - \frac{g}{g^2} \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(p) g(p) - \frac{\partial g}{\partial x_i}(p) f(p) \right) x_i \tag{10.63}$$

$$\equiv f - \frac{g(p)}{g^2(p)} \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) x_i \mod \mathfrak{m}_{X,p}^2 \tag{10.64}$$

$$\equiv f - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) x_i \mod \mathfrak{m}_{X,p}^2. \tag{10.65}$$

$\blacksquare$

Remark 85 The same proof yields

$$T_{X,p}^\vee \cong M/M^2 \qquad \text{for} \qquad M = \mathbb{I}(p)/\mathbb{I}(X) \subset \mathbb{K}[X]. \tag{10.66}$$

Remark 86 The identification $T_{X,p}^\vee \cong \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ is intrinsic, i.e. independent of the embedding in $\mathbb{A}^n$. In particular we can use it to define $T_{X,p}$ for X any variety and $p \in X$ any point. In this case $T_{X,p}$ is an abstract Vector Space (not embedded in $\mathbb{A}^n$ anymore).

Remark 87: Comparison with tangent cones.
Let $X \subset \mathbb{A}^n$ be an affine variety, and $p \in X$. WLOG we assume that $p = (0, \ldots, 0)$. Then, for $f_0 + f_1 + \ldots + f_d = f \in \mathbb{K}[x_1, \ldots, x_n]$, with $f(p) = 0$ (i.e. $f_0 = 0$), we have $f_1 \neq 0 \implies \mathrm{mindeg}\, f = 1$ and $f_1 = f^{in}$. In particular $\mathbb{I}(X)_{in} = \{f^{in} \mid f \in \mathbb{I}(X)\} \supset \mathbb{I}(X)_1$. Then $C_{X,p} = \mathbb{V}(\mathbb{I}(X)_{in}) \subset \mathbb{V}(\mathbb{I}(X)_1) = T_{X,p}$ in $\mathbb{A}^n$.

In particular we obtain that $\dim X = \dim C_{X,p} \leq \dim T_{X,p}$. The dimension of the tangent (linear space) might be bigger than the dimension of the cone.

Definition 10.15: Regular and singular points/varieties.
Given a variety X we say that a point $p \in X$ is a **regular point** of X iff $\dim T_{X,p} = \dim X$. (Alternatively it can be called **simple**, **smooth**, **non-singular** point). If, instead, $\dim T_{X,p} > \dim X$, then p is called a **singular point**.

A variety X is called **singular** iff it has at least one singular point. Non-singular varieties are allso called **regular** or **smooth**.

Finally, we define the regular locus of X to be

$$X_{\mathrm{reg}} := \{p \in X \mid X \text{ is regular at } p\} \tag{10.67}$$

and the singular locus of X as

$$X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}. \tag{10.68}$$

Example: Curves in $\mathbb{A}^2$. Consider the parabola $X_1 := \mathbb{V}(x(x-1) - y)$, the nodal cubic $X_2 := \mathbb{V}(x^2 + x^3 - y^2)$ and the cuspidal cubic $X_3 := \mathbb{V}(x^3 - y^2)$. Then $T_{X_1,\mathbf{0}} = C_{X_1,\mathbf{0}}$, hence the parabola is regular at $\mathbf{0}$. Instead $T_{X_2,\mathbf{0}} = T_{X_3,\mathbf{0}} = \mathbb{A}^2$, hence X_2 and X_3 are both singular at $\mathbf{0}$.

In fact the blow-ups at $\mathbf{0}$ are as follows. For $\widetilde{X}_1 \to X_1$ is an isomorphism, in fact it coincides with the strict transform. For $\widetilde{X}_2 \to X_2$ it is not even injective: the two tangent directions corresponding to the two components of the tangent cone $C_{X_2,\mathbf{0}}$ get both mapped to $\mathbf{0}$. $\widetilde{X}_3 \to X_3$, instead, is injective. Indeed both $\widetilde{X}_2$ and $\widetilde{X}_3$ are smooth, whereas X_2 and X_3 are singular (and this gives the fact that the above maps cannot be iso). Moreover we call the above maps from the smooth varieties, obtained by blowing up the singular locus, a resolution of singularities. (Those are birational map inducing isomorphisms $p^{-1}(X_{i,\mathrm{reg}}) \to X_{i,\mathrm{reg}}$ s.t. $\widetilde{X}_i$ is smooth). In particular the maps $\widetilde{X}_i \xrightarrow{p} X_i$ (for $i = 2, 3$) are resolutions of the singularities at $\mathbf{0}$.

Let's check, for instance, that $\widetilde{X}_3$ is non-singular. We only need to check what happens at the preimage of $\mathbf{0}$.

$$\widetilde{\mathbb{A}^2} = \{((x,y), [\zeta, \eta]) \mid x\eta = y\zeta\} \subset \mathbb{A}^2 \times \mathbb{P}^1. \tag{10.69}$$

The local coordinates at $(p, [1, 0])$ are $(u, v) \in \mathbb{A}^2$. Recall that the chart we are using is induced by

$$\mathbb{A}^2 \to \widetilde{\mathbb{A}^2} = \mathbb{A}^2 \times \mathbb{P}^1 \tag{10.70}$$
$$(u, v) \mapsto ((v, uv), [1, u]) \tag{10.71}$$

for $u = \eta/\zeta$ (for $\zeta \neq 0$) and $v = x$. The equation for X_3 is $y^2 - x^3 = 0$. We substitute $x = v$ and $y = uv$ and obtain $(uv)^2 - v^3 = v^2(u^2 - v) = 0$. This has two components: $v = 0$, which defines the exceptional line of $\widetilde{\mathbb{A}^2} \to \mathbb{A}^2$, which is not contained in the

stric transform $\widetilde{X}_3$, by definition. And $u^2 - v = 0$, which gives the local equation for the strict transform $\widetilde{X}_3$. It has a nontrivial linear part, hence $\widetilde{X}_3$ is smooth at $(p, [1, 0])$.

With similar computations one can check that $\widetilde{X}_2$ is smooth at $(u, v) = (\pm 1, 0)$, since the local equations are $v = 1 - u^2 = (1 - u)(1 + u)$.

In general, one needs to be careful, since one needs more than one blow-up to obtain a resolution of singularities.

Definition 10.16: Jacobi matrix.
Given $F := (f_1, \ldots, f_r)$ an r-tuple of polynomials in $x_1, \ldots, x_n$, we define the **Jacobi matrix** of F at p as

$$J_{F,p} := \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j} \qquad \text{for } i = 1, \ldots, r \text{ and } j = 1, \ldots, n. \tag{10.72}$$

Proposition 10.17 (Jacobi criterion).

> **Affine case:** *Let $X \subset \mathbb{A}^n$ be an affine variety, and $f_1, \ldots, f_r$ be generators of $\mathbb{I}(C)$. Then $p \in X$ is regular iff*
>
> $$\operatorname{rk} J_{(f_1,\ldots,f_r),p} \geq n - \dim X. \tag{10.73}$$
>
> **Projective case:** *Let $X \subset \mathbb{P}^n$ be a projective variety and $f_1, \ldots, f_r$ homogeneous generators of $\mathbb{I}_p(X)$. Then $p \in X$ is regular iff*
>
> $$\operatorname{rk} J_{(f_1,\ldots,f_r),p} \geq n - \dim X. \tag{10.74}$$
>
> *Notice that, even though the Jacobi matrix itself depends on the choice of homogeneous coordinates for p, its rank does not change (each row gets multiplied by a nonzero constant).*

Moreover, in both cases, if $\operatorname{rk} J_{F,p} = r$, then X is smooth of dimension $n - r$ at p.

Remark 88

- Set $d := \dim X$, then in the above notation

$$X_{\text{sing}} = \left\{ p \in X \mid \operatorname{rk} J_{(f_1,\ldots,f_r),p} \leq n - d - 1 \right\} \tag{10.75}$$

$$= \mathbb{V}\left(\left\{ (n - d) \times (n - d)\text{-minors of } J_{(f_1,\ldots,f_r),p} \right\} \right), \tag{10.76}$$

  hence $X_{\text{sing}} \subset X$ is a closed subset.

- $X_{\text{reg}} \subset X$ is always a nonempty open subset. In particular it is always dense. Let's give a sketch of the argument: Since $d := \dim X$, there is a surjective projection on $\mathbb{A}^d$. Though let's stop a step earlier and consider $\pi : X \to X' \subset \mathbb{A}^{d+1}$. Clearly X' is a hypersurface in $\mathbb{A}^{d+1}$. Making a good choice of the projection there is a non empty open subset of X, on which π is an isomorphism. Hence π will be a birational map (it is a consequence of the primitive element theorem). It is easy to check that $X'_{\text{reg}} \neq \emptyset$, for hypersurfaces. Then (one should check that the inclusion holds) $X_{\text{reg}} \supset \pi^{-1}\left(X'_{\text{reg}} \cap V \right)$ (for $V = \pi(U)$) is non empty.

Remark 89 If X is a hypersurface, and

- $X = \mathbb{V}(f) \subset \mathbb{A}^n$, for a square-free polynomial f (not necessairily irreducible). A point $p \in \mathbb{A}^n$ is a singular point of X iff

$$f(p) = \frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0. \tag{10.77}$$

- $X = \mathbb{V}_p(f) \subset \mathbb{P}^n$, for a suqare-free and homogeneous polynomial f. Then

$$p \in X_{\text{sing}} \iff \frac{\partial f}{\partial x_0}(p) = \frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = f(p) = 0. \tag{10.78}$$

Moreover one can easily check that, set $d := \deg f$, multiplying each partial derivative by the corresponding linear form

$$d \cdot f(x) = x_0 \frac{\partial f}{\partial x_0}(x) + \ldots + x_n \frac{\partial f}{\partial x_n}. \tag{10.79}$$

(Clearly this only works for homogeneous polynomials). Moreover, if d is invertible in the ground field (e.g. if the field is of char 0), then one can omit the condition $f(p) = 0$. Then we have

$$X_{\text{sing}} = \mathbb{V}_p\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right). \tag{10.80}$$

Example: Fermat hypersurfaces. Consider $n, d \geq 1$. The Fermat hypersurfaces of degree d in $\mathbb{P}^n$ are defined by

$$X_d := \mathbb{V}_p\left(x_0^d + \ldots + x_n^d\right) \subset \mathbb{P}^n. \tag{10.81}$$

This is a degree d hypersurface if $\operatorname{char} \mathbb{K} = 0, p$ for $p \nmid d$ (otherwise we'd have $x_0^d + \ldots x_n^d = (x_0 + \ldots + x_n)^d$). Then the Jacobi matrix is

$$J_f = \left(dx_0^{d-1}, dx_1^{d-1}, \ldots, dx_n^{d-1}\right). \tag{10.82}$$

Hence X_d is a smooth hypersurface by the Jacobi criterion.

Example: Twisted cubic curve.

$$X = \left\{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\right\}. \tag{10.83}$$

Since $X \cong \mathbb{P}^1$, we already know that X is regular. Let's check it with the Jacobi criterions:

$$X = \mathbb{V}_p\left(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2\right) \tag{10.84}$$

(the above is determined by the determinant of the minors of the matrix $\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$).
The Jacobi matrix is

$$J = \begin{pmatrix} x_2 & -2x_1 & x_0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 \\ 0 & x_3 & -2x_2 & x_1 \end{pmatrix}. \tag{10.85}$$

We need to check that, at every point $\operatorname{rk} J \geq \dim \mathbb{P}^3 - \dim X = 3 - 1 = 2$. We need to find 2×2 minors that do not vanish simultaneosly:

$$\begin{vmatrix} x_0 & 0 \\ -x_1 & x_0 \end{vmatrix} = x_0^2 \qquad \text{and} \qquad \begin{vmatrix} x_3 & -x_2 \\ 0 & x_3 \end{vmatrix} = x_3. \tag{10.86}$$

The first vanishes only at $[0, 0, 0, 1] \in X$, whereas the second at $[1, 0, 0, 0] \in X$, i.e. they have no common zeros on X, hence $\operatorname{rk} J \geq 2$ at all $p \in X$. In particular X is regular.

11 Affine schemes

The idea is to extend the category of prevarieties in order to

- include also reducible objects,

- expand the (Nullstellensatz) one to one correspondance also to non-radical ideals:

$$\left\{ \begin{array}{c} \text{affive viarieties} \\ \text{in } \mathbb{A}^n(\mathbb{K}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } \mathbb{K}[x_1, \ldots, x_n] \end{array} \right\} . \tag{11.1}$$

In fact, for $X_1, X_2 \subset \mathbb{A}^n$, the ideal $\mathbb{I}(X_1) + \mathbb{I}(X_2)$ contains more refined information then the associated ideal to the intersection $X_1 \cap X_2$ and, in general, it is not radical.

$$\mathbb{I}(X_1 \cap X_2) = \sqrt{\mathbb{I}(X_1) + \mathbb{I}(X_2)}. \tag{11.2}$$

For example, given $X_1' := \mathbb{V}(x^3 - y)$ and $X_2 := \mathbb{V}(x)$, then $\mathbb{I}(X_1') + \mathbb{I}(X_2) = (x, y)$. If, instead, we consider the curve $X_1 := \mathbb{V}(y^2 - x)$, then $\mathbb{I}(X_1) + \mathbb{I}(X_2) = (x, y^2)$.

We have two different geometrical situations (transverse intersection, against non-transverse intersection), signified by different sums of associated ideals, but clearly with the same radical ideal.

For instance, in the second case, if we blow up $\mathbb{A}^2$ at $\mathbb{I}(X_1) + \mathbb{I}(X_2)$, then the strict transforms $\widetilde{X}_1$ and $\widetilde{X}_2$ are disjoint. If, instead, we blow up $\mathbb{I}(X_1 \cap X_2) = \sqrt{\mathbb{I}(X_1) + \mathbb{I}(X_2)}$, then $\widetilde{X}_1 \cap \widetilde{X}_2 \neq 0$. Another reason why the radical ideal is not always the right choice.

Moreover, from

$$\left\{ \begin{array}{c} \text{Affine} \\ \text{varieties} \end{array} \right\}_{/\text{isom}} \longleftrightarrow \left\{ \begin{array}{c} \text{finitely generated } \mathbb{K}\text{-algebras} \\ \text{that are integral domains} \end{array} \right\}_{/\text{isom}} , \tag{11.3}$$

we will drop all assumptions on the right hand side and associate a ringed space (i.e. a space on which we can do geometry) to any commutative ring with unity. (The idea is that, when defining the Zariski topology, we didn't use the $\mathbb{K}$-algebra structure, but only ideals, hence the ring structure).

Definition 11.1: Spectrum of a ring.
Let R be a ring. We define the **spectrum** of R to be

$$\operatorname{Spec} R := \{ \mathfrak{p} \triangleleft R \mid \mathfrak{p} \text{ is a prime ideal} \} . \tag{11.4}$$

$\operatorname{Spec} R$ is also called the **affine scheme** associated with R. Moreover, for every $\mathfrak{p} \in \operatorname{Spec} R$, we define the residue field of $\operatorname{Spec} R$ at $\mathfrak{p}$ to be

$$k(\mathfrak{p}) := Q(R/\mathfrak{p}). \tag{11.5}$$

(Notice that we will use the convention that $(1) = R$ is not a prime ideal).

Moreover we can define $\operatorname{MaxSpec} R := \{ \mathfrak{m} \triangleleft R \mid \mathfrak{m} \text{ is a maximal ideal} \}$ the maximal specturm of R.

Remark 90 R is the analogue of the coordinate ring: we can view any element $f \in R$ as a function on $\operatorname{Spec} R$ by

$$
\begin{array}{ccc}
R & \twoheadrightarrow R/\mathfrak{p} & \hookrightarrow k(\mathfrak{p}) \\
f & \longmapsto & f(\mathfrak{p})
\end{array}
\tag{11.6}
$$

Hence f can be regarded as a function, although $f(\mathfrak{p})$ might lie in different fields, depending on $\mathfrak{p}$.

If, in particular, $R = \mathbb{K}[X]$, for some affine variety X, we can use point $x \in X$ to construct prime ideals, namely $\mathfrak{p} = \mathfrak{m}_x := \mathbb{I}(x) \in \operatorname{Spec} R$. In particular $\mathbb{K}[X]/\mathfrak{m}_x \cong \mathbb{K}$. Then, for all $f \in \mathbb{K}[X]$, we have $f(\mathfrak{p}) = f(x) \in \mathbb{K}$ in which the left hand side intereprets f as a function on $\operatorname{Spec} R$, whereas the right hand side interepresentationets it as a function on X.

If, instead, $\mathfrak{p} \subset \mathbb{K}[X]$ is prime, but not maximal, then $\mathfrak{p} = \mathbb{I}(Y) \subset \mathbb{K}[X]$, for some irreducible closed subset $Y \subset X$. Then $f(\mathfrak{p}) \in k(\mathfrak{p}) = \mathbb{K}(Y)$ is the restriction of $f : X \to \mathbb{K}$ to Y.

Example

1. If $\mathbb{K}$ is a field (not necessairily algebraically closed), then $\operatorname{Spec} \mathbb{K} = \{(0)\}$ is a single point.

2. $\operatorname{Spec} \mathbb{C}[x] = \{(x - a) \mid a \in \mathbb{C}\} \cup \{(0)\}$. The same description holds for any algebraically closed field $\mathbb{K}$ instead of $\mathbb{C}$.

3. Let X be an affine variety over the algebraically closed field $\mathbb{K}$. Consider $R := \mathbb{K}[X]$. Then we have the bijection

$$
\begin{array}{ccc}
\operatorname{Spec} R & \longleftrightarrow & \{\text{subvarieties of } X\} \\
\mathbb{I}(Y) & \longleftarrow & Y
\end{array}
\tag{11.7}
$$

in which subvarieties are exactly the irreducible closed subsets of X. Then, restricting to the maximal spectrum of R, we obtain the correspondance

$$
\begin{array}{ccc}
\operatorname{MaxSpec} R & \longleftrightarrow & X \\
\mathfrak{m}_x & \longleftarrow & x
\end{array}
\tag{11.8}
$$

The element $\mathbb{I}(Y) \in \operatorname{Spec} R$ is called **generic point** of Y. If we don't want to consider the additional points of $\operatorname{Spec} R$ coming from $\mathbb{I}(Y)$, for $\dim Y \geq 1$, we may want to restrict to the maximal spectrum of R.

4. Let's consider the special case in which the field is not algebraically closed, i.e. $\mathbb{R}$:

$$
\operatorname{Spec} \mathbb{R} = \{(x - a) \mid a \in \mathbb{R}\} \cup \{((x - a)^2 + b^2) \mid a, b \in \mathbb{R} \text{ and } b > 0\} \cup \{(0)\}.
\tag{11.9}
$$

Then the residue fields are:

(a) $k((x - a)) \cong \mathbb{R}$,

(b) $k(((x - a)^2 + b^2)) \cong k((x^2 + 1)) \cong \mathbb{C}$ (by scaling things),

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(c) $k\left((0)\right) \cong \mathbb{R}(x)$.

5. $\operatorname{Spec} \mathbb{Z} = \{(2), (3), \dots, (p), \dots\} \cup \{(0)\}$ for p prime. (0) is the generic point of $\mathbb{Z}$. Then $k\left((p)\right) \cong Q\left(\mathbb{Z}/p\mathbb{Z}\right) = \mathbb{F}_p$ and, clearly, $k\left((0)\right) = \mathbb{Q}$.

   FInally, by definition, given $f \in \mathbb{Z}$, $f\left((p)\right) = f \mod p \in \mathbb{F}_p$.

We want to defien a ringed space for $\operatorname{Spec} R$, hence we need to define a topology on it. Ideally we should mimic the Zariski topology on affine varieties.

Definition 11.2: Vanishing locus.
Let R be a ring and $S \subset R$. We define the **vanishing locus** of S as

$$\mathbb{V}\left(S\right) := \{\mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) = 0 \text{ for all } f \in S\} \tag{11.10}$$
$$= \{\mathfrak{p} \in \operatorname{Spec} R \mid S \subset \mathfrak{p}\} \subset \operatorname{Spec} R. \tag{11.11}$$

Remark 91 We clearly have $\mathbb{V}\left(S\right) = \mathbb{V}\left((S)\right)$, where we denote the ideal generated by S by (S).

Lemma 11.3.

1. $\mathbb{V}\left(0\right) = \operatorname{Spec} R$ and $\mathbb{V}\left(1\right) = \emptyset$.

2. *If $\{\mathcal{I}_j\}_{j \in J}$ is an arbitrary family of ideals, then*

$$\bigcap_{j \in J} \mathbb{V}\left(\mathcal{I}_j\right) = \mathbb{V}\left(\sum_{j \in J} \mathcal{I}_j\right) \subset \operatorname{Spec} R. \tag{11.12}$$

3. *Given $\mathcal{I}_1, \mathcal{I}_2 \lhd R$, then*

$$\mathbb{V}\left(\mathcal{I}_1\right) \cup \mathbb{V}\left(\mathcal{I}_2\right) = \mathbb{V}\left(\mathcal{I}_1 \cdot \mathcal{I}_2\right) \subset \operatorname{Spec} R. \tag{11.13}$$

4. *Given $\mathcal{I}_1, \mathcal{I}_2 \lhd R$, we have*

$$\mathbb{V}\left(\mathcal{I}_1\right) \subset \mathbb{V}\left(\mathcal{I}_2\right) \iff \sqrt{\mathcal{I}_2} \subset \sqrt{\mathcal{I}_1}. \tag{11.14}$$

Definition 11.4: Zariski topology.
The topology on $\operatorname{Spec} R$, whose closed subsets are of the form $\mathbb{V}\left(S\right)$, for $S \subset R$, is called the **Zariski topology**.

Remark 92 We can use the Zariski topology to study the irreducibility of $\operatorname{Spec} R$ and to define its dimension.

Remark 93 Not all point of $\operatorname{Spec} R$ are closed. In fact we have

$$\overline{\{\mathfrak{p}\}} = \mathbb{V}\left(\mathfrak{p}\right) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supset \mathfrak{p}\}. \tag{11.15}$$

Then $\{\mathfrak{p}\}$ is closed iff $\mathfrak{p}$ is a maximal ideal. In particular, in $\operatorname{Spec} R$, the closed points are maximal ideals (i.e. the elements that correspond to points in the ring of functions on an affine variety). In other words $\{\text{closed points}\} = \operatorname{MaxSpec} R$. Whereas non-closed points are generic points of $\mathbb{V}\left(\mathfrak{p}\right)$.

Example: Generic points. Let $\mathbb{K}$ be an algebraically closed field. Consider $X :=$ $\mathrm{Spec}\,\mathbb{K}[x_1, x_2]$ the spectrum of the coordinate ring of the affine plane. Consider $L :=$ $\mathbb{V}(x_2) \subset X$ the x_1-axis. What is $X \setminus L$? It contains

- all closed points not lying in L,

- generic points (e.g. $\mathfrak{p} = x_1$, the x_2-axis).

The x_1-axis and the x_2-axis have common points, but the generic points of the x_2-axis does not lie on the x_1-axis, so that $(x_1) \in X \setminus L$.

Definition 11.5: Distinguished open subsets.
Let R be a ring, and $X := \mathrm{Spec}\,R$. The open subsets of the form $X_f := X \setminus \mathbb{V}(f)$, for some $f \in R$ are called the **distinguished open subsets** of X.

Remark 94 As for varieties, the distinguished open subsets will form a basis for the Zariski topology. Though, since we don't know whether this Zariski topology makes the spectrum of the ring R a Noetherian space, we cannot say that every open subset can be obtained by finite union of distinguished open subsets.

Definition 11.6: Regular functions on an open subset.
Let R be a ring, $X = \mathrm{Spec}\,R$ and $U \subset X$ an open subset. We define the ring of regular functions on U, denoted by $\mathcal{O}_X(U)$, as

$$\mathcal{O}_X(U) := \left\{ \varphi = (\varphi_\mathfrak{p})_{\mathfrak{p} \in U} \in \prod_{\mathfrak{p} \in U} R_\mathfrak{p} \,\middle|\, \text{for every } \mathfrak{p} \in U \text{ there is } V \subset U \right. \tag{11.16}$$

$$\text{an open neighbourhood of } \mathfrak{p} \text{ and } f, g \in R, \text{ with } g \notin Q, \tag{11.17}$$

$$\left. \varphi_Q = \frac{f}{g} \in R_Q \text{ for all } Q \in V \right\}. \tag{11.18}$$

The idea is thet a regular function on U should locally be a quotient f/g, with $f, g \in R$. In particular g should be "ivertible" at some $Q \in U$, i.e. $Q \notin \mathbb{V}(g)$. Equivalently $g \notin Q$, for some $Q \in U$.

Remark 95: Structure sheaf.
It is easy to see that $\mathcal{O}_X$ is a sheaf on X (the defining condition on φ is local). Moreover we call it the *structure sheaf* on X.

Proposition 11.7. *Let R be a ring and $X = \mathrm{Spec}\,R$.*

1. *For all $\mathfrak{p} \in X$ we have $\mathcal{O}_{X,\mathfrak{p}} \simeq R_\mathfrak{p}$ (the stalk of $\mathcal{O}_X$ at $\mathfrak{p}$ is isomorphic to the localization of R at $\mathfrak{p}$),*

2. *For any $f \in R$, $\mathcal{O}_X(X_f) \simeq R_f$.*

In particular, for $f = 1$, we obtain $\mathcal{O}_X(X) \simeq R$.

Remark 96 A regular function is not determined by its value at each point of $X :=$ $\mathrm{Spec}\,R$.

Example Let $R := \mathbb{K}[x]/(x^2)$ and $X := \mathrm{Spec}\,R$. Clearly R is not a domain and X contains only the point (x), with residue field $k((x)) \cong \mathbb{K}[x]/(x) \cong \mathbb{K}$. Since $R \cong$ $\mathcal{O}_X(X)$ we have that 0 and $x \in R$ define different functions on X. However both functions have value 0 at the only point of X.

11.1 Locally ringed spaces

Remark 97 To define morphisms between affine schemes one needs to keep in mind that the structure sheaf is not necessairily a sheaf of $\mathbb{K}$-valued functions. This means that, if we want to define a morphism $f : X \to Y$ between affine schemes, we need to also define the pull-back maps

$$f^* : \mathcal{O}_Y\left(U\right) \to \mathcal{O}_X\left(f^{-1}(U)\right) \tag{11.19}$$

in the data required to define f.

Recall, moreover, that the pull-back should:

- be compatible with restriction morphisms, i.e. for all $V \overset{\text{open}}{\subset} U$ in Y, the following diagram should commute

$$
\begin{array}{ccc}
\mathcal{O}_Y\left(U\right) & \xrightarrow{\ \ \rho_{U,V}\ \ } & \mathcal{O}_Y\left(V\right) \\
{\scriptstyle f_U^*}\downarrow & & \downarrow{\scriptstyle f_V^*} \\
\mathcal{O}_X\left(f^{-1}(U)\right) & \xrightarrow[\rho_{f^{-1}(U),f^{-1}(V)}]{} & \mathcal{O}_X\left(f^{-1}(V)\right)
\end{array} \quad ; \tag{11.20}
$$

- preserve the structure of the stalks as local rings: $\mathcal{O}_{X,p}$ has a unique macimal ideal

$$\mathfrak{m}_{X,p} := \{\phi \in \mathcal{O}_{X,p} \mid \phi(p) = 0\} \tag{11.21}$$

and $\mathcal{O}_{Y,f(p)}$ has maximal ideal $\mathfrak{m}_{Y,f(p)}$. Then we require that the induced map

$$f_p^* : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p} \tag{11.22}$$
$$(U,\phi) \mapsto \left(f^{-1}(U), f^*\phi\right) \tag{11.23}$$

satisfies $\left(f_p^*\right)^{-1}\left(\mathfrak{m}_{X,p}\right) = \mathfrak{m}_{Y,f(p)}$.

Definition 11.8: Locally ringed space.
A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ s.t. at all $p \in X$ the stalk $\mathfrak{O}_{X,p}$ is a local ring, wth maximal ideal denoted by $\mathfrak{m}_{X,p}$ and residue field $k(p) := \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$.

Definition 11.9: Morphism of locally ringed spaces.
Consider two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. A morphism of locally ringed spaces is given by a continuous map $f : X \to Y$ and by a family, for all $U \subset Y$ open, of ring homomorphisms

$$f_U^* : \mathcal{O}_Y\left(U\right) \to \mathcal{O}_X\left(f^{-1}(U)\right) \tag{11.24}$$

satisfying

1. compatibility with restrictions, i.e. for all $U \subset V \subset Y$ open, the diagram commutes

$$
\begin{array}{ccc}
\mathcal{O}_Y\left(V\right) & \xrightarrow{\ f_V^*\ } & \mathcal{O}_X\left(f^{-1}(V)\right) \\
{\scriptstyle \rho_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho_{f^{-1}(V),f^{-1}(U)}} \\
\mathcal{O}_Y\left(U\right) & \xrightarrow[f_U^*]{} & \mathcal{O}_X\left(f^{-1}(U)\right)
\end{array} \quad , \tag{11.25}
$$

2. the induced maps $f_p^* : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ (whose existance is granted by the previous property) satisfy $\left(f_p^*\right)^{-1}(\mathfrak{m}_{X,p}) = \mathfrak{m}_{Y,f(p)}$ for all $p \in X$.

Notice that, in general, the subset $U \subset Y$ will be clear from the context, hence one will write just f^*, instead of f_U^*.

Definition 11.10: Morphism of affine schemes.
A morphism of affine schemes is a morphism of locally ringed spaces, between affine schemes.

Proposition 11.11. *Let R, S be rings, $X := \operatorname{Spec} R$ and $Y := \operatorname{Spec} S$. Then there is a one-to-one correspondance*

$$\{morphisms\ X \to Y\} \longleftrightarrow \{ring\ homomorphisms\ S \to R\}$$

$$f \longmapsto f^* \qquad , \qquad (11.26)$$

$$g^\# \longleftarrow\!\!\dashv g$$

in which, for a ring homomorphism $g : S \to R$ we define

$$g^\# : X = \operatorname{Spec} R \to Y = \operatorname{Spec} S \qquad (11.27)$$

$$\mathfrak{p} \mapsto g^{-1}(\mathfrak{p}). \qquad (11.28)$$

Example Let $X := \operatorname{Spec} R$ and $\mathcal{I} \lhd R$ an ideal. Define $Y := \operatorname{Spec} R/\mathcal{I}$. The ring homomorphism $q : R \to R/\mathcal{I}$ gives rise to a morphism

$$i : Y \to X \qquad (11.29)$$

$$\mathfrak{p} \mapsto \mathfrak{p} + \mathcal{I} = q^{-1}(\mathfrak{p}). \qquad (11.30)$$

Let us observe that $i = q^\#$ gives a one-to-one correspondance between prime ideals of $R/\mathcal{I}$ and prime ideals of R containing $\mathcal{I}$. Thus i is injective and its image is $\mathbb{V}(\mathcal{I}) \subset X$. In other words: ideals of R give rise to affine closed subschemes of X.

Definition 11.12: Intersection scheme.
Let Y_1 and Y_2 be affine subschemes of X, then the intersection scheme of Y_1 and Y_2 in X is

$$Y_1 \cap Y_2 := \operatorname{Spec} \frac{R}{\mathcal{I}_1 + \mathcal{I}_2}. \qquad (11.31)$$

Definition 11.13: Union scheme.
Let Y_1 and Y_2 be affine subschemes of X, then the union scheme of Y_1 and Y_2 in X is

$$Y_1 \cup Y_2 := \operatorname{Spec} \frac{R}{\mathcal{I}_1 \cap \mathcal{I}_2}. \qquad (11.32)$$

Example: Affine plane. Consider the affine plane $X := \operatorname{Spec} \mathbb{C}[x_1, x_2]$ and $Y_1 := \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2)$ (the x axis), $Y_2 := \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2 - x_1^2 + a^2)$ for a fixed $a \in \mathbb{C}$ (a parabola). Then

$$Y_1 \cap Y_2 = \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2, x_1^2 - a^2) = \operatorname{Spec} \mathbb{C}[x_1]/((x_1 - a)(x_1 + a)). \qquad (11.33)$$

If, in particular $a = 0$, $Y_1 \cap Y_2 \cong \operatorname{Spec} \mathbb{C}[x]/(x^2)$: it is a double point, it encodes the point $(0,0)$ together with a tangent direction (along the x_1-axis).

Lemma 11.14. *Let R be a ring, $X := \operatorname{Spec} R$, and $f \in R$. Then*

$$X_f \cong \operatorname{Spec} R_f. \tag{11.34}$$

12 Schemes

When gluing together affine varieties one obtains prevarieties. Let's define, in an analogous way, schemes, as locally ringed spaces which can be covered by affine schemes.

Definition 12.1: Scheme.
A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that can be covered by open subsets $U_i \subset X$ s.t.
$$\left(U_i, \mathcal{O}_X|_{U_i}\right) \cong \left(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}\right), \tag{12.1}$$
for some ring R_i and for all i.

Moreover a morphism of schemes is just a morphism of locally ringed spaces between schemes.

Remark 98 Each affine variety X over a field $\mathbb{K}$ has an associated affine scheme
$$X_{\text{sch}} := \operatorname{Spec} \mathbb{K}[X]. \tag{12.2}$$

Definition 12.2: Scheme associated to a prevariety.
Let $\mathbb{K}$ be a algebraically closed field and X a prevariety over $\mathbb{K}$. Then the scheme associated with X is
$$X_{\text{sch}} := \{Z \subset X \mid \emptyset \neq Z \text{ is irreducible}\} \tag{12.3}$$
with the topology defined by $\left\{ U_{\text{sch}} \ \middle| \ U \overset{\text{open}}{\subset} X \right\}$ with

$$U_{\text{sch}} := \{V \subset U \mid V \text{ is an irreducible closed subset of } U\} \hookrightarrow X_{\text{sch}} \tag{12.4}$$
$$U \supset Z \mapsto \overline{Z} \subset X, \tag{12.5}$$

and structure sheaf $\mathcal{O}_{X_{\text{sch}}}(U_{\text{sch}}) := \mathcal{O}_X(U)$, with natural restrictions.

Proposition 12.3. X_{sch} *is a scheme for every prevariety X. Moreover any morphism of prevarieties $f : X \to Y$ naturally extends to a morphism of schemes*

$$f : X_{sch} \to Y_{sch} \tag{12.6}$$
$$Z \mapsto \overline{f(Z)} \tag{12.7}$$

for $Z \subset X$ an irreducible closed subset.

Moreover a scheme X comes from a prevariety over $\mathbb{K}$ if locally it is given by $\operatorname{Spec} R_i$, for $R_i = \mathbb{K}[Y_i]$ for Y_i an affine variety over $\mathbb{K}$. (More explicitly R_i are all finitely generated $\mathbb{K}$-algebras which are also domains).

Definition 12.4 Let Y be a scheme. A scheme over Y is a scheme X, together with a morphism $f : X \to Y$. A morphism of schemes $X_1 \to Y$, $X_2 \to Y$ over Y is a morphism of schemes s.t. the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{\ f\ } & X_2 \\ \downarrow & & \downarrow \\ Y & = \!\!=\!\!= & Y \end{array} \quad . \tag{12.8}$$

If $Y = \operatorname{Spec} R$, then we call schemes over Y, schemes over R.

Definition 12.5: Scheme over Y of finite type.
A scheme X over Y is of finite type iff there is a covering of Y by affine open subsets $V_i := \operatorname{Spec} B_i$ and a covering of $f^{-1}(V_i)$ by affine open subsets

$$U_{ij} := \operatorname{Spec} A_{ij} \qquad \text{for all indeces } i \tag{12.9}$$

with the property that each A_{ij} is a finitely generated B_i-algebra (induced by $U_{ij} \to V_i$).

Remark 99 Recall that, given a ring B, then a B-algebra is a ring A together with an injective ring homomorphism $B \hookrightarrow A$. The B-algebra A is finitely generated iff there are finitely many $\alpha_1, \ldots, \alpha_k \in A$ s.t.

$$B[t_1, \ldots, t_k] \to A \tag{12.10}$$
$$f \mapsto f(\alpha_1, \ldots, \alpha_k) \tag{12.11}$$

is surjective.

Moreover recall that, if $X \subset \mathbb{A}^n$ is a Zariski closed subset, then $\mathbb{I}(X) \subset \mathbb{K}[x_1, \ldots, x_n]$ is a radical ideal and

$$\mathbb{K}[X] = \frac{\mathbb{K}[x_1, \ldots, x_n]}{\mathbb{I}(X)} \tag{12.12}$$

does not contain any nilpotent element.

Definition 12.6: Reduced ring.
A ring R is reduced iff, for every $f \in R$,

$$f^r = 0 \implies f = 0. \tag{12.13}$$

Definition 12.7: Reduced scheme.
A scheme X is called reduced iff $\mathcal{O}_X(U)$ is a reduced ring for all $U \overset{\text{open}}{\subset} X$.

Proposition 12.8. *Let $\mathbb{K}$ be an algebraically closed field, then*

- *For any affine variety X over $\mathbb{K}$, the associated scheme $X_{sch} = \operatorname{Spec} \mathbb{K}[X]$ is a reduced and irreducible affine scheme of finite type over $\mathbb{K}$. Moreover any reduced and irreducible affine scheme of finite type over $\mathbb{K}$ is of this form.*

- *Let X, Y be affine schemes over $\mathbb{K}$, then there are one to one correspondences*

$$\begin{Bmatrix} morphisms \\ X \to Y \\ as\ varieties \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \mathbb{K}\text{-}algebra \\ homomorphisms \\ \mathbb{K}[X] \to \mathbb{K}[Y] \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} morphisms\ as \\ schemes\ over\ \mathbb{K} \\ X_{sch} \to Y_{sch} \end{Bmatrix} .$$

  In particular the category of affine $\mathbb{K}$-varieties is equivalent to the category of reduced and irreducible affine schemes of finite type over $\mathbb{K}$. Moreover both are equivalent to the opposite of the category of finitely generated reduced $\mathbb{K}$-algebras.

Proposition 12.9. *Let $\mathbb{K}$ be an algebraically closed field. Then there is an equivalence of categories between the category of* prevarieties *over $\mathbb{K}$ and the category of* reduced and irreducible schemes of finite type over $\mathbb{K}$.

In particualr all such schemes arise from prevarieties over $\mathbb{K}$. Moreover, for any two prevarieties X and Y, we have a one to one correspondance

$$\left\{\begin{array}{c} \textit{morphisms} \\ X \to Y \\ \textit{as prevarieties} \end{array}\right\} \xleftrightarrow{\ 1:1\ } \left\{\begin{array}{c} \textit{morphisms as} \\ \textit{schemes over } \mathbb{K} \\ X_{sch} \to Y_{sch} \end{array}\right\} .$$

Remark 100 Schemes can be constructed by gluing other schemes. In fact we need

- a collection $\{X_i\}_{i \in I}$ of shcemes (I may even be infinite)

- a collection $\{U_{ij}, f_{ij}\}_{i,j \in I}$ of open subsets $U_{ij} \subset X$ and isomorphisms

$$f_{ij} : U_{ij} \to U_{ji}. \tag{12.14}$$

All of these satifying the conditions:

1. $U_{ii} = X_i$ and $f_{ii} = id_{X_i}$ for all $i \in I$,

2. $f_{ij}(U_{ij} \cap U_{ik}) \subset U_{jk}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in I$.

The case where U_{ij} is empty gives rise to a reducible scheme. Then a morphism from the scheme X, obtained by gluing the X_i, to a scheme Y is a collection of morphisms of scheme

$$\{X_i \to Y\}_{i \in I} \tag{12.15}$$

compatible with the overla maps $f_{ij} : U_{ij} \to U_{ji} \subset X_j$.

Proposition 12.10. *Let X be a scheme and $Y := \operatorname{Spec} R$ an affine scheme. Thene there is a one to one correspondence*

$$\left\{\begin{array}{c} \textit{morphisms} \\ X \to Y \\ \textit{as schemes} \end{array}\right\} \xleftrightarrow{\ 1:1\ } \left\{\begin{array}{c} \textit{ring homomorphisms} \\ \mathcal{O}_Y(Y) = R \to \mathcal{O}_X(X) \end{array}\right\} .$$

Remark 101 Every ring R has a natural ring homomorphism

$$\mathbb{Z} : \to R \tag{12.16}$$

$$0 \le m \mapsto \underbrace{1 + 1 + \ldots + 1}_{m \text{ times}} \tag{12.17}$$

$$0 > m \mapsto \underbrace{-1 - 1 - \ldots - 1}_{-m \text{ times}} . \tag{12.18}$$

In view of the proposition, this yields that every scheme X is a scheme over $\mathbb{Z}$, by using

$$\mathbb{Z} \to \mathcal{O}_X(X) \tag{12.19}$$

$$X \mapsto \operatorname{Spec} \mathbb{Z}. \tag{12.20}$$

Is X is a scheme over $\mathbb{C}$, then the image of this morphism is the point $(0) \in \operatorname{Spec} \mathbb{Z}$.

12.1 Fiber products

Definition 12.11: Fiber product of schemes.

Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be two schemes over a fixed scheme S. The fiber product $(X \times_S Y, \pi_X, \pi_Y)$ is a scheme endowed with two morphisms s.t. the following diagram commutes

$$
\begin{array}{ccc}
X \times_S Y & \xrightarrow{\pi_X} & X \\
{\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{g} & S
\end{array}
\tag{12.21}
$$

and satisfies the following universal property. For every couple of morphism ϕ_X and ϕ_Y s.t. $f \circ \phi_X = g \circ \phi_Y$, i.e. the following diagram commutes

$$
\tag{12.22}
$$

there exists a unique morphism $\phi : Z \to X \times_S Y$ s.t. the whole diagram commutes, i.e. $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$.

Remark 102   As with the usaul argument with universal properties, if $X \times_S Y$ exists, then it is unique up to isomorphism.

Moreover one can construct $X \times_S Y$ by a gluing construction. The key point is that, called $X := \operatorname{Spec} M$, $Y := \operatorname{Spec} N$ and $S := \operatorname{Spec} R$ (in which M and N are R-algebras, since both X and Y are schemes over S) then

$$
\operatorname{Spec}(M \otimes_R N) = X \times_S Y.
\tag{12.23}
$$

13 Projective schemes

Definition 13.1: Irrelevant ideal of a graded ring.

Let $R := \bigoplus_{d \geq 0} R_d$ be a graded ring. The irrelevant ideal of R is

$$
R_+ := \bigoplus_{d>0} R_d.
\tag{13.1}
$$

Definition 13.2: Projective scheme.

Let R be a graded ring. The projective scheme associated with R is

$$
\operatorname{Proj} R := \{\mathfrak{p} \lhd R \mid \mathfrak{p} \text{ is a homogeneous prime ideal, and } R_+ \not\subset \mathfrak{p}\},
\tag{13.2}
$$

with the topology defined by the closed subsets

$$
\mathbb{V}(\mathcal{I}) = \{\mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{p} \supset \mathcal{I}\}
\tag{13.3}
$$

for all homogeneous ideals $\mathcal{I} \lhd R$.

Remark 103 The Zariski topology on $\operatorname{Proj} R$ is well defines, since

$$\bigcap_{j \in \mathcal{J}} \mathbb{V}\left(\mathcal{I}_j\right) = \mathbb{V}\left(\sum_{j \in \mathcal{J}} \mathcal{I}_j\right) \subset \operatorname{Proj} R \qquad (13.4)$$

$$\mathbb{V}\left(\mathcal{I}_1\right) \cup \mathbb{V}\left(\mathcal{I}_2\right) = \mathbb{V}\left(\mathcal{I}_1 \cdot \mathcal{I}_2\right) \subset \operatorname{Proj} R \qquad (13.5)$$

also hold in this new setting.

Definition 13.3: Structure sheaf.
Let R be a graded ring and $X := \operatorname{Proj} R$.

1. For all $\mathfrak{p} \in \operatorname{Proj} R$ one sets

$$R_{(\mathfrak{p})} := \left\{ \frac{f}{g} \in R_{\mathfrak{p}} \; \middle| \; g \notin \mathfrak{p}, \, f, g \in R_d \text{ for some } d \in \mathbb{N} \right\}. \qquad (13.6)$$

2. For all open subsets $U \subset X$ we define

$$\mathcal{O}_X\left(U\right) := \left\{ \phi := (\phi_{\mathfrak{p}})_{\mathfrak{p} \in U} \in \prod_{\mathfrak{p} \in U} R_{(\mathfrak{p})} \; \middle| \; \text{for every } \mathfrak{p} \in U \text{ there is an open} \right.$$

$$\text{neighbourhood } V \text{ of } \mathfrak{p} \text{ and } f, g \in R_d \text{ for some } d \in \mathbb{N}$$

$$\left. \text{s.t. } g \notin Q \text{ and } \phi_Q = \frac{f}{g} \, \forall Q \in V \right\}.$$

$\mathcal{O}_X$ is called the *structure sheaf* of $X = \operatorname{Proj} R$.

Proposition 13.4. *Let R be a graded ring and $X := \operatorname{Proj} R$. Then $(O, \mathcal{O}_O)$ is a scheme with*

1. *$\mathcal{O}_{X,\mathfrak{p}} \cong R_{(\mathfrak{p})}$ for all $\mathfrak{p} \in X$,*

2. *for each homogeneous element $f \in R_d$, for $d > 0$, we define the distinguished open subset*
$$X_f := X \setminus \mathbb{V}\left(f\right) = \left\{ \mathfrak{p} \in \operatorname{Proj} R \mid f \notin \mathfrak{p} \right\}. \qquad (13.7)$$

The distinguishe open subsets X_f cover X and satisfy

$$\left(X_f, \mathcal{O}_X|_{X_f}\right) \cong \operatorname{Spec} R_{(f)}, \qquad (13.8)$$

(i.e. distinguished open subsets are affine schemes), with

$$R_{(f)} := \left\{ \frac{g}{f^r} \in R_f \; \middle| \; g \in R_{rd}, \, d = \deg f \right\}. \qquad (13.9)$$

Proof. The proof is analogous to the affine case, apart from the fact that X_f cover X. ∎

Example Consider $(\mathbb{P}^n_{\mathbb{K}})_{\operatorname{sch}} := \operatorname{Proj} \mathbb{K}\left[x_0, \dots, x_n\right]$. If $X \subset \mathbb{P}^n_{\mathbb{K}}$ is a projective variety, then

$$X_{\operatorname{sch}} = \operatorname{Proj}\left(\mathbb{K}\left[x_0, \dots, x_n\right] / \mathbb{I}\left(X\right)\right), \qquad (13.10)$$

where $S(X) := \mathbb{K}\left[x_0, \dots, x_n\right] / \mathbb{I}\left(X\right)$ is the homogeneous coordinate ring of X.

Definition 13.5: Projective subscheme.
Let $\mathbb{K}$ be an algebraically closed field. A projective subscheme of $\mathbb{P}^n_{\mathbb{K}}$ is a scheme of the form

$$\mathrm{Proj}\left(\mathbb{K}\left[x_0,\ldots,x_n\right]/\mathcal{I}\right) \tag{13.11}$$

for $\mathcal{I} \triangleleft \mathbb{K}\left[x_0,\ldots,x_n\right]$ a homogeneous ideal.

Remark 104 Projective subschemes can be reducible, e.g.

$$\mathrm{Proj}\,\mathbb{K}\left[x_0,x_1\right]/(x_1^2) \tag{13.12}$$

or even non-reduced, e.g.

$$\mathrm{Proj}\,\mathbb{K}\left[x_0,x_1,x_2\right]/(x_1 x_2). \tag{13.13}$$

Remark 105 Different homogeneous ideals may define the same projective subscheme in $\mathbb{P}^n$. For example

$$\mathrm{Proj}\,\mathbb{K}\left[x_0,\ldots,x_n\right]/(f) = \mathrm{Proj}\,\mathbb{K}\left[x_0,\ldots,x_n\right]/(fx_0, fx_1,\ldots, fx_n). \tag{13.14}$$

Definition 13.6: Saturation of an ideal.
Let $\mathcal{I} \triangleleft R := \mathbb{K}\left[x_0,\ldots,x_n\right]$ be a homogeneous ideal. The saturation of $\mathcal{I}$ is the homogeneous ideal

$$\overline{\mathcal{I}} := \{f \in R \mid x_0^m f,\ldots, x_n^m f \in \mathcal{I} \text{ for some } m \in \mathcal{I}\}. \tag{13.15}$$

A homogeneous ideal $\mathcal{I}$ is called saturated iff $\mathcal{I} = \overline{\mathcal{I}}$.

Lemma 13.7.
$$\mathrm{Proj}\,R/I = \mathrm{Proj}\,R/J \iff \overline{I} = \overline{J}. \tag{13.16}$$

Theorem 13.8. *There is a one to one correspondence*

$$\left\{\begin{array}{c} \textit{projective} \\ \textit{subschemes} \\ \textit{of } \mathbb{P}^n_{\mathbb{K}} \end{array}\right\} \xleftrightarrow{\ 1:1\ } \left\{\begin{array}{c} \textit{saturated} \\ \textit{homogeneous ideals} \\ \mathcal{I} \triangleleft \mathbb{K}\left[x_0,\ldots,x_n\right] \end{array}\right\}, \tag{13.17}$$

$$X \longmapsto \mathbb{I}(X) = \overline{\mathcal{I}}$$

where $\mathcal{I}$ is any ideal defining X.

Definition 13.9: Homogeneous coordinate ring of a projective subscheme.
Let X be a projective subscheme in $\mathbb{P}^n_{\mathbb{K}}$. The *homogeneous coordinate ring* of X is

$$S(X) := \mathbb{K}\left[x_0,\ldots,x_n\right]/\mathbb{I}(X). \tag{13.18}$$