Contents

1 Sheaves

1.1 Basic definitions

Let's begin with a couple of preliminary definitions.

Definition 1.1: Presheaf.
Let X be a topological space. A *presheaf* P of *abelian groups* on X consists of the data of:

- for any $\mathcal{U} \subset X$ open, an abelian group $P(\mathcal{U})$,
- for any inclusion $\mathcal{V} \subset \mathcal{U}$ of open subsets of X, of a morphism of abelian groups

  $$\rho_{\mathcal{U}\mathcal{V}} \colon P(\mathcal{U}) \longrightarrow P(\mathcal{V}) \tag{1.1}$$

  such that

  1. $\rho_{\mathcal{U}\mathcal{U}} = \mathrm{id}_{P(\mathcal{U})}$,
  2. For any $\mathcal{W} \subset \mathcal{V} \subset \mathcal{U}$, then the following diagram commutes

  $$\begin{tikzcd} P(\mathcal{U}) \arrow[rr, "\rho_{\mathcal{U}\mathcal{W}}"] \arrow[dr, "\rho_{\mathcal{U}\mathcal{V}}"'] & & P(\mathcal{W}) \\ & P(\mathcal{V}) \arrow[ur, "\rho_{\mathcal{V}\mathcal{W}}"'] & \end{tikzcd} . \tag{1.2}$$

Notation 1.2: Restriction.
Let $\mathcal{U} \subset X$ be an open subset, and $s \in P(\mathcal{U})$. Given $\mathcal{V} \subset \mathcal{U}$ open, we define the following:

$$s|_{\mathcal{V}} := \rho_{\mathcal{U}\mathcal{V}}(s) \in P(\mathcal{V}). \tag{1.3}$$

The element $s|_{\mathcal{V}}$ is called the *restriction* of s to $\mathcal{V}$ and $\rho_{\mathcal{U}\mathcal{V}}$ is called the *restriction morphism* from $\mathcal{U}$ to $\mathcal{V}$.

Remark 1.3: Category of open subset.
Given a topological space X, we can define the category $\mathsf{Op}(X)$, characterized by:

- Objects: $\mathcal{U} \in \mathrm{Ob}\,(\mathsf{Op}(X))$ are given by open subsets $\mathcal{U} \subset X$,
- Morphisms: The set of morphisms are given by

  $$\mathrm{Hom}_{\mathsf{Op}(X)}\,(\mathcal{U}, \mathcal{V}) := \begin{cases} \{*\} & \text{if } \mathcal{U} \subset \mathcal{V} \\ \emptyset & \text{otherwise} \end{cases}, \tag{1.4}$$

  i.e. there is an arrow $\mathcal{U} \hookrightarrow \mathcal{V}$ iff $\mathcal{U} \subset \mathcal{V}$.

Definition 1.4: Presheaf (again).
A *presheaf* of abelian groups on a topological space X is a (contravariant) functor

$$\begin{aligned} P : \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{U} &\longmapsto P(\mathcal{U}) \\ (\mathcal{U} \hookrightarrow \mathcal{V}) &\longmapsto \rho_{\mathcal{U}\mathcal{V}} \colon P(\mathcal{U}) \to P(\mathcal{V}). \end{aligned} \tag{1.5}$$

Recall that, in $\mathsf{Op}(X)^{op}$, the arrow $\mathcal{U} \to \mathcal{V}$ corresponds to the inclusion $\mathcal{V} \subset \mathcal{U}$.

Definition 1.5: Presheaf of rings, groups, sets.
A presheaf of *rings, groups* or *sets* is simply a functor on the appropriate category.

Definition 1.6: Sheaf.
A presheaf $\mathcal{F}$ of abelian groups on X is called a *sheaf* iff it satisfies the following conditions:

- $\mathcal{F}(\emptyset) = 0$,
- **Uniqueness:** let $\mathcal{U} \subset X$ an open subset, and $\{\mathcal{U}_i\}_{i\in I}$ an open cover of $\mathcal{U}$. Given a section $s \in \mathcal{F}(\mathcal{U})$ s.t. $s|_{\mathcal{U}_i} = 0$ for all $i \in I$, then $s = 0 \in \mathcal{F}(\mathcal{U})$.
- **Gluing:** let $\mathcal{U} \subset X$ an open subset, and $\{\mathcal{U}_i\}_{i\in I}$ an open cover of $\mathcal{U}$. Given a family of sections $s_i \in \mathcal{F}(\mathcal{U}_i)$ for all $i \in I$, s.t. for all $(i, j) \in I^2$

  $$s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j|_{\mathcal{U}_i \cap \mathcal{U}_j}, \tag{1.6}$$

  then there exists $s \in \mathcal{F}(\mathcal{U})$ s.t. $s|_{\mathcal{U}_i} = s_i$ for all $i \in I$.

Remark 1.7 Uniqueness above grants that, given a family of compatible sections on an open cover of an open set, their gluing is unique.

Remark 1.8 In the same way, just changing the target category, one defines sheaves of rings, groups and sets.

Remark 1.9: Condition 1 is redundant tk: fix it, please.
(The fact that the product over an empty set is a final object is explained in section 3, then I'd also add the exact sequence satisfied by sheaves). In fact the covering $\bigcup_{i\in\emptyset} U_i = \emptyset$, then $\prod_{i\in\emptyset} F(\mathcal{U}_i)$ is a final object in our category Then, in the exact sequence for the definition of sheaf we have

$$0 \longrightarrow \mathcal{F}(\emptyset) \longrightarrow 0 \longrightarrow 0\ . \tag{1.7}$$

By exactness $\mathcal{F}(\emptyset) = 0$.

Definition 1.10: Section.
An element $s \in \mathcal{F}(\mathcal{U})$ is called a *section* of $\mathcal{F}$ over $\mathcal{U}$.

Example: Continuous functions. Let X be a topological space and $\mathcal{U} \subset X$ an open subset of X. One defines the ring

$$\mathcal{C}(\mathcal{U}) := \mathcal{C}^0(\mathcal{U}, \mathbb{R}) := \{f : \mathcal{U} \to \mathbb{R} \mid f \text{ is continuous on } \mathcal{U}\} \in \mathrm{Ob}(\mathsf{Rings})\ . \tag{1.8}$$

Notice that above we consider $\mathcal{C}(\mathcal{U})$ with its natural ring structure. Then, given $\mathcal{V} \subset \mathcal{U}$ an open subset of X, to the inclusion we associate the restriction map

$$\begin{aligned} \rho_{\mathcal{U}\mathcal{V}} : \mathcal{C}(\mathcal{U}) &\longrightarrow \mathcal{C}(\mathcal{V}) \\ (f : \mathcal{U} \to \mathbb{R}) &\longmapsto (f|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R})\ . \end{aligned} \tag{1.9}$$

Clearly $\mathcal{C}$ is a sheaf of rings on X, in fact:

- $\mathcal{C}(\emptyset) = 0$,
- By the gluing lemma, one can glue together continuous functions, defined on arbitrarily many (possibly overlapping) open subsets of X, in a unique way in order to obtain a continuous function defined on the union.

Example Let $A \neq 0$ be a nontrivial abelian group. Let $X := X_1 \sqcup X_2$ be a topological space with two connected components, i.e. $X_i \neq \emptyset$ and it is connected for $i = 1, 2$. Let's define the presheaf A_X on X, on objects, by

$$A_X(\emptyset) := 0 \qquad \text{and} \qquad A_X(\mathcal{U}) := A \quad \forall\, \emptyset \neq \mathcal{U} \subset X, \tag{1.10}$$

and on morphisms (i.e. inclusions), for all $\emptyset \neq \mathcal{V} \subset \mathcal{U} \subset X$, by setting $\rho_{\mathcal{U}\mathcal{V}} : A_X(\mathcal{U}) \to A_X(\mathcal{V})$ to be $id_A : A \to A$ and $\rho_{\mathcal{U}\emptyset} : A_X(\mathcal{U}) \to A_X(\emptyset) = 0$ to be, of course, the zero morphism.

Clearly A_X is not a sheaf. In fact, given $s_1 \neq s_2 \in A$, with $s_i \in A_X(X_i) = A$, Then we have $X_1 \cap X_2 = \emptyset$, hence $A_X(X_1 \cap X_2) = A_X(\emptyset) = 0$ and

$$s_1|_{X_1 \cap X_2} = 0 = s_2|_{X_1 \cap X_2}\ . \tag{1.11}$$

Then, if A_X were a sheaf, we'd be able to glue the two sections together and obtain a global section $s \in A_X(X) = A$ s.t.

$$s = id_A(s) = s|_{X_1} = s_1 \neq s_2 = s|_{X_2} = id_A(s) = s, \tag{1.12}$$

which is, clearly, impossible.

Definition 1.11: Restriction sheaf.
Let $\mathcal{U} \subset X$ be an open subset of a topological space X, and $\mathcal{F}$ a sheaf of abelian groups on X. Then one can define the *restriction sheaf*

$$\begin{aligned} \mathcal{F}|_{\mathcal{U}} : \mathsf{Op}(\mathcal{U})^{op} &\longrightarrow \mathsf{Ab} \\ (\mathcal{V} \subset \mathcal{U}) &\longmapsto \mathcal{F}(\mathcal{V}). \end{aligned} \tag{1.13}$$

This functor acts the same way as $\mathcal{F}$, but only on open subsets of $\mathcal{U}$. (Notice that, since $\mathcal{U} \subset X$ is open, then open subsets of $\mathcal{U}$ are already open in X).

Definition 1.12: Basis for a topology.
Let's recall the definition for the *basis* of a topology. Let X be a topological space. A family of open subsets β forms a basis for the topology on X iff

- every open subset $\mathcal{U} \subset X$ is the union of elements of β, i.e.

$$\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i \qquad \text{s.t.} \qquad \mathcal{U}_i \in \beta, \ \forall\, i \in I; \tag{1.14}$$

- β is closed under finite intersections, i.e. given $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \beta$, then

$$\bigcap_{i=1}^{n} \mathcal{U}_i \in \beta. \tag{1.15}$$

Remark 1.13: Category β.
We can consider β as a standalone category. In fact it is a full subcategory of $\mathsf{Op}(X)$.

Definition 1.14: β-sheaf.
Let $X \in \mathsf{Top}$ and β a basis of open subsets for the topology on X. A β-preasheaf of abelian groups is, as one would expect, a functor

$$P \colon \beta^{op} \to \mathsf{Ab}. \tag{1.16}$$

A β-*sheaf* is a β-presheaf satisfying the sheaf conditions (only for subsets in β).

Remark 1.15 Every β-sheaf extends uniquely to a sheaf on X. Let, in fact, $\mathcal{F}_0$ be a β-sheaf of abelian groups and $\mathcal{U} \subset X$ be an open subset. Then there exists $\{\mathcal{U}_i\}_{i \in I}$ a family in β s.t. $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$. We can now extend $\mathcal{F}_0$ to $\mathcal{U}$ by setting

$$\mathcal{F}(\mathcal{U}) := \ker \begin{pmatrix} \prod_{i \in I} \mathcal{F}_0(\mathcal{U}_i) & \longrightarrow & \prod_{(i,j) \in I^2} \mathcal{F}_0\left(\mathcal{U}_i \cap \mathcal{U}_j\right) \\ (s_i)_{i \in I} & \longmapsto & \left(s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} - s_j|_{\mathcal{U}_i \cap \mathcal{U}_j} \right)_{(i,j) \in I^2} \end{pmatrix}. \tag{1.17}$$

If, instead, $\mathcal{F}_0$ is a β-sheaf of sets, then

$$\mathcal{F}(\mathcal{U}) = \varprojlim \left(\prod_{i \in I} \mathcal{F}_0(\mathcal{U}_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}_0\left(\mathcal{U}_i \cap \mathcal{U}_j\right) \right). \tag{1.18}$$

Definition 1.16: Complex of groups.
A *complex of groups* is a sequence, indexed in $\mathbb{Z}$, of morphisms of abelian groups

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \ldots, \tag{1.19}$$

s.t. $d^{i+1} \circ d^i = 0$.

Example Let $\mathcal{U} \subset X$ be an open subset, and $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ be an open cover. Consider any presheaf of abelian groups P on X; Then one can define the complex $\mathcal{C}\left((\mathcal{U}_i)_{i \in I}, P \right)$ by

$$\begin{array}{ccccccc} 0 \xrightarrow{d^{-1}} & P(\mathcal{U}) & \xrightarrow{d^0} & \prod_{i \in I} P(\mathcal{U}_i) & \xrightarrow{d^1} & \prod_{(i,j) \in I^2} P(\mathcal{U}_{ij}) \\ & s & \longmapsto & \left(s|_{\mathcal{U}_i} \right)_{i \in I} & & \\ & & & (s_i)_{i \in I} & \longmapsto & \left(s_i|_{\mathcal{U}_{ij}} - s_j|_{\mathcal{U}_{ij}} \right)_{(i,j) \in I^2} \end{array}, \tag{1.20}$$

where $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$. This is a complex, since

$$
\begin{aligned}
d^1 \circ d^0(s) &= d^1\left(\left(s|_{\mathcal{U}_i}\right)_{i\in I}\right) = && (1.21)\\
&= \left(s|_{\mathcal{U}_i}\right)\big|_{\mathcal{U}_{ij}} - \left(s|_{\mathcal{U}_j}\right)\Big|_{\mathcal{U}_{ij}}\\
&= s|_{\mathcal{U}_{ij}} - s|_{\mathcal{U}_{ij}} = 0. && (1.22)
\end{aligned}
$$

Definition 1.17: Exact complex.
A complex is said to be *exact* iff, for all $i \in \mathbb{Z}$,

$$\operatorname{im}\left(d^{i-1}\right) = \ker\left(d^i\right). \tag{1.23}$$

Lemma 1.18. *Let P be a* presheaf *on X. Then P is a sheaf on X iff, given any $\mathcal{U} \subset X$ and any open cover $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i$ of $\mathcal{U}$, the complex $\mathcal{C}\left((\mathcal{U}_i)_{i\in I}, P\right)$ is exact. In other words iff d^0 is injective (uniqueness condition) and $\ker d^1 = \operatorname{im} d^0$ (gluing condition).*

Definition 1.19: Stalk of a presheaf.
Let P be a presheaf on X and $x \in X$. The stalk of P at x is defined to be the abelian group (resp. ring, set, ...)

$$P_x := \varinjlim_{\mathcal{U}\ni x} P(\mathcal{U}). \tag{1.24}$$

Remark 1.20: Recall.
For sets

$$P_x = \varinjlim_{\mathcal{U}\ni x} P(\mathcal{U}) = \coprod_{\mathcal{U}\ni x} P(\mathcal{U})/\sim, \tag{1.25}$$

where $\coprod$ denotes the disjoint union of sets, and the equivalence relation is defined by $(\mathcal{U}, s) \sim (\mathcal{V}, t)$, for $\mathcal{U}, \mathcal{V} \subset X$ open subsets and $s \in P(\mathcal{U})$, $t \in P(\mathcal{V})$, iff there exists an open neighbourhood $x \in \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$ s.t.

$$s|_{\mathcal{W}} = t|_{\mathcal{W}}. \tag{1.26}$$

We denote by $[\mathcal{U}, s]$, or by s_x the equivalence class of $(\mathcal{U}, s)$. For any $x \in X$ and any $x \in \mathcal{U}$ there is a canonical map

$$
\begin{aligned}
\pi : P(\mathcal{U}) &\longrightarrow P_x && (1.27)\\
s &\longmapsto s_x. && (1.28)
\end{aligned}
$$

Finally s_x is called the *germ* of s at x.

Lemma 1.21. *Let $\mathcal{F}$ be a sheaf on X, and $s, t \in \mathcal{F}(X)$ two global sections. If, for every $x \in X$, $s_x = t_x$, then $s = t$.*

Proof. We know that $s_x = t_x$ iff there exists $x \in \mathcal{U}_x \neq \emptyset$ on which $s|_{\mathcal{U}_x} = t|_{\mathcal{U}_x}$. Clearly $X = \bigcup_{x\in X} \mathcal{U}_x$ is an open cover. Then

$$(s-t)|_{\mathcal{U}_x} = s|_{\mathcal{U}_x} - t|_{\mathcal{U}_x} = 0. \tag{1.29}$$

This holds for any $x \in X$, i.e. on an open cover of X. By uniqueness we obtain that $s - t = 0$, hence the desired global equality. ■

Definition 1.22: Morphism of presheaves.
Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves of abelian groups (resp. rings, sets, ...) on X. A morphism of

presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ is simply a morphism of functors, sometimes called natural transformation. More explicitly it is the data, for any $\mathcal{U} \in \mathrm{Ob}\,(\mathsf{Op}(X))$, of a morphism in Ab

$$\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \ . \tag{1.30}$$

Moreover this family of morphisms has to satisfy the following naturality (or functoriality) condition: for any $\mathcal{V} \subset \mathcal{U}$ the diagram commutes

$$\begin{array}{ccc} \mathcal{F}(\mathcal{U}) & \xrightarrow{\alpha(\mathcal{U})} & \mathcal{G}(\mathcal{U}) \\ {\scriptstyle \rho^{\mathcal{F}}_{\mathcal{U}\mathcal{V}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho^{\mathcal{G}}_{\mathcal{U}\mathcal{V}}} \\ \mathcal{F}(\mathcal{V}) & \xrightarrow[\alpha(\mathcal{V})]{} & \mathcal{G}(\mathcal{V}) \end{array} \ . \tag{1.31}$$

Definition 1.23: Injective, surjective, iso morphisms of presheaves.
A morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of presheaves is

- *injective* iff $\alpha(\mathcal{U})$ is a monomorphism for every $\mathcal{U} \subset X$ open,
- *surjective* iff $\alpha(\mathcal{U})$ is an epimorphism for every $\mathcal{U} \subset X$ open,
- an *isomorphism* iff $\alpha(\mathcal{U})$ is an isomorphism for every $\mathcal{U} \subset X$ open.

Notice that these coincide with the definitions of mono/epi/iso morphism in the category of presheaves, viewed as the category of functors from $\mathsf{Op}(X)$.

Remark 1.24: Morphism at the level of stalks.
A morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ induces, for every $x \in X$ a morphism of abelian groups (resp. rings, sets, ...) at the level of stalks

$$\begin{aligned} \alpha_x : \mathcal{F}_x &\longrightarrow \mathcal{G}_x \\ s_x &\longmapsto [\alpha(s)]_x \ . \end{aligned} \tag{1.32}$$

The above definition is independent of the chosen representative of α_x.

Definition 1.25: Morphism of sheaves.
A morphism of sheaves is simply a morphism of the underlying presheaves.

Remark 1.26: Forgetful functor.
The forgetful functor

$$\begin{aligned} \iota : \mathsf{Sh}(X) &\longrightarrow \mathsf{PSh}(X) \\ \mathcal{F} &\longmapsto \mathcal{F}, \end{aligned} \tag{1.33}$$

assigning to every sheaf $\mathcal{F}$ itself, viewed as a presheaf, is clearly fully faithful (in fact morphisms of sheaves are just morphisms of presheaves among sheaves).

Definition 1.27: Injective, surjective, iso morphisms of sheaves.
A morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of sheaves is

- *injective* iff α_x is a mono for every $x \in X$,
- *surjective* iff α_x is epi for every $x \in X$,
- an *isomorphism* iff α_x is an iso for every $x \in X$.

One can prove that this definition coincides with the notion of mono/epi/iso in the category of sheaves.

Proposition 1.28. *Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups (or modules), then*

- *α is an isomorphism iff*
$$\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \tag{1.34}$$
is an isomorphism for every $\mathcal{U} \subset X$ open,

- *α is a monomorphism iff*
$$\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \tag{1.35}$$
is a monomorphism for every $\mathcal{U} \subset X$ open.

Proof. Let's prove it for isomorphisms:

$\Leftarrow$ Assume that $\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ is an iso for all $\mathcal{U} \subset X$ open. Then it is easy to check that the induced maps
$$\alpha_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x \tag{1.36}$$
are isomorphisms for every $x \in X$.

$\Rightarrow$ Suppose that α_x is an isomorphism for all $x \in X$. Let's recall that, in abelian groups (and modules more in general), a morphism is an isomorphism iff it is both mono and epi.

mono Consider $\mathcal{U} \subset X$ and a section $s \in \mathcal{F}(\mathcal{U})$ s.t. $\alpha(\mathcal{U})(s) = 0 \in \mathcal{G}(\mathcal{U})$. Consider $x \in \mathcal{U}$, then $\alpha_x(s_x) = [\alpha(\mathcal{U})(s)]_x = 0$. Since α_x is an iso, it is in particular mono, hence $s_x = 0$ for all $x \in X$. By uniqueness, for sheaves, we then have that $s = 0 \in \mathcal{F}(\mathcal{U})$. Then $\alpha(\mathcal{U})$ is mono for each $\mathcal{U}$.

epi Let $\mathcal{U} \subset X$ open and $t \in \mathcal{G}(\mathcal{U})$. Consider $x \in \mathcal{U}$, then $\alpha_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an iso. It follows that, for every $x \in \mathcal{U}$, there exists $s_x \in \mathcal{F}_x$ s.t. $\alpha_x(s_x) = t_x$. This means, in particular, that there exists $\mathcal{U}' \subset \mathcal{U}$ open, and $s \in \mathcal{F}(\mathcal{U}')$ s.t. $\alpha(\mathcal{U}')(s) = t|_{\mathcal{U}'}$.
Let's extract an open cover $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{U}$ from the above construction. In particular, for every $i \in I$, there exists $s_i \in \mathcal{F}(\mathcal{U}_i)$ s.t. $\alpha(\mathcal{U}_i)(s_i) = t|_{\mathcal{U}_i}$. Let's now concentrate on $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then
$$\alpha(\mathcal{U}_{ij})(s_i|_{\mathcal{U}_{ij}}) = \alpha(\mathcal{U}_i)(s_i)|_{\mathcal{U}_{ij}} = (t|_{\mathcal{U}_i})\big|_{\mathcal{U}_{ij}} = t|_{\mathcal{U}_{ij}} = \tag{1.37}$$
$$= (t|_{\mathcal{U}_j})\Big|_{\mathcal{U}_{ij}} = \alpha(\mathcal{U}_{ij})(s_j|_{\mathcal{U}_{ij}}) = \alpha(\mathcal{U}_{ij})(s_j|_{\mathcal{U}_{ij}}). \tag{1.38}$$
We have already proved injectivity of $\alpha(\mathcal{U})$ at any open subset, hence $s_i|_{\mathcal{U}_{ij}} = s_j|_{\mathcal{U}_{ij}}$. By gluing and uniqueness ($\mathcal{F}$ is a sheaf) we obtain that there exists a unique $s \in \mathcal{F}(\mathcal{U})$ s.t. $s|_{\mathcal{U}_i} = s_i$.
Again by uniqueness we obtain that $\alpha(\mathcal{U})(s) = t$, in fact for any $i \in I$
$$\alpha(\mathcal{U})(s)|_{\mathcal{U}_i} = \alpha(\mathcal{U}_i)(s|_{\mathcal{U}_i}) = \alpha(\mathcal{U}_i)(s_i) = t|_{\mathcal{U}_i}\,. \tag{1.39}$$
∎

Remark 1.29 Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a surjective morphism of sheaves. In general the induced morphism $\alpha\,(\mathcal{U}) : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ is not epi.

1.2 Étalé space associated to a presheaf

Given a presheaf of abelian groups P on X, topological space, we want to associate it an étalé space. Let's consider $\widetilde{P} := \coprod_{x\in X} P_x$, the disjoint union of the stalks of the presheaf P, viewed as a set.

There is a canonical projection which can be easily defined

$$\begin{aligned} \pi : \coprod_{x\in X} P_x &\longrightarrow X \\ (x, s_x) &\longmapsto x . \end{aligned} \tag{1.40}$$

Moreover, for any section of the presheaf $s \in P(\mathcal{U})$, we can define a map:

$$\begin{aligned} \tilde{s} : \mathcal{U} &\longrightarrow \coprod_{x\in X} P_x \\ x &\longmapsto (x, s_x) . \end{aligned} \tag{1.41}$$

This map $\tilde{s}$ is, in fact, called a *section* of π over $\mathcal{U}$. More explicitly we have

$$\begin{tikzcd} \coprod_{x\in\mathcal{U}} P_x \arrow[r, hook] \arrow[drr, "\pi_{\mathcal{U}}"] & \coprod_{x\in X} P_x \arrow[r, "\pi"] & X \\ & & \mathcal{U} \arrow[u, hook] \arrow[ull, bend left, "\tilde{s}"] \end{tikzcd} . \tag{1.42}$$

This diagram commutes, i.e. $\pi_{\mathcal{U}} \circ \tilde{s} = id_{\mathcal{U}}$.

Moreover one defines on $\widetilde{P}$ the weakest topology making $\tilde{s}$ continuous for any $s \in P(\mathcal{U})$, with $\mathcal{U} \subset X$ open. More explicitly a subset $G \subset \widetilde{P}$ is open iff, for any open subset $\mathcal{U} \subset X$ and any local section $s \in P(\mathcal{U})$, the subset $\tilde{s}^{-1}(G) \subset X$ is open.

Exercise 1 Check that the above actually defines a topology.

Proposition 1.30. *The canonical projection $\pi\colon \widetilde{P} \to X$ is a local homeomorphism, i.e. for any point $x \in X$ there exists a neighbourhood $\mathcal{U}_x$ of x on which π restricts to a homeomorphism.*

Proof. Just some hints: Fix a point $s_{x_0} \in P_{x_0} \subset \widetilde{P}$. We need to find a subset $G \subset \widetilde{P}$, with $s_{x_0} \in G$ and s.t. $\pi|_G : G \to \mathcal{U} \subset X$, for $s_{x_0} = [\mathcal{U}, s]$, is a homemorphism. Consider

$$G := \tilde{s}(\mathcal{U}) = \{s_x \mid x \in \mathcal{U}\} . \tag{1.43}$$

■

Definition 1.31: Sheaf associated to a presheaf.
Let $\mathcal{F}$ be a presheaf on X. The sheaf associated to $\mathcal{F}$ is a sheaf $\mathcal{F}^{\#}$, endowed with a morphism of presheaves

$$\theta\colon \mathcal{F} \longrightarrow \mathcal{F}^{\#} \tag{1.44}$$

s.t. given any morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism $\tilde{\alpha}\colon \mathcal{F}^{\#} \to \mathcal{G}$ making the following diagram commutative

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\alpha"] \arrow[dr, "\theta"'] & & \mathcal{G} \\ & \mathcal{F}^{\#} \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd} . \tag{1.45}$$

Theorem 1.32. *Given a presheaf $\mathcal{F}$, its associated sheaf $\mathcal{F}^{\#}$ exists and is unique up to a unique isomorphism. Moreover, for every $x \in X$, the associated morphism at the level of stalks is an isomorphism*

$$\theta_x \colon \mathcal{F}_x \longrightarrow \left(\mathcal{F}^{\#}\right)_x \, . \tag{1.46}$$

Proof. Let's construct $F^{\#}$ and show that it actually is a sheaf. Let $\mathcal{U} \subset X$ be an open subset of X. We define, then

$$\mathcal{F}^{\#}(\mathcal{U}) := \left\{ f \colon \mathcal{U} \to \widetilde{\mathcal{F}} = \coprod_{x \in X} \mathcal{F}_x \;\middle|\; \forall\, x \in \mathcal{U}, \exists\, x \in \mathcal{V} \subset \mathcal{U} \text{ and } \exists\, s \in \mathcal{F}(\mathcal{V}) \text{ s.t. } f|_{\mathcal{V}} = \tilde{s} \right\}. \tag{1.47}$$

Clearly this set belongs to the same category as the target of the original presheaf (define sum and multiplication pointwise on stalks). In fact

$$\mathcal{F}^{\#}(\mathcal{U}) = \left\{ f \colon \mathcal{U} \to \widetilde{\mathcal{F}} \;\middle|\; f \text{ is continuous and } \pi \circ f = id_{\mathcal{U}} \right\} \tag{1.48}$$

is the set of continuous sections of $\pi \colon \widetilde{\mathcal{F}} \to X$ over $\mathcal{U}$, i.e. maps that make the following diagram commute:

$$\begin{tikzcd} & \widetilde{\mathcal{F}} \arrow[d, "\pi"] \\ \mathcal{U} \arrow[ur, "f"] \arrow[r, hook] & X \end{tikzcd} . \tag{1.49}$$

There is a canonical map

$$\begin{aligned} \theta(\mathcal{U}) : \mathcal{F}(\mathcal{U}) &\longrightarrow \mathcal{F}^{\#}(\mathcal{U}) \\ s &\longmapsto \tilde{s}, \end{aligned} \tag{1.50}$$

where, we recall, $\tilde{s}$ is defined as follows:

$$\begin{aligned} \tilde{s} : \mathcal{U} &\longrightarrow \coprod_{x \in X} \mathcal{F}_x \\ x &\longmapsto s_x . \end{aligned} \tag{1.51}$$

Clearly the family of maps $\theta(\mathcal{U})$ defines a morphism of sheaves. In particular it satisfies the following:

Lemma 1.33. *If $\mathcal{F}$ is a sheaf, then the above map*

$$\theta \colon \mathcal{F} \longrightarrow \mathcal{F}^{\#} \tag{1.52}$$

is an isomorphism of sheaves.

Proof. Consider $f \in \mathcal{F}^{\#}(\mathcal{U})$. Then there exists an open cover $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ s.t., for all $i \in I$, there is $s_i \in \mathcal{F}(\mathcal{U}_i)$ with $f|_{\mathcal{U}_i} = \tilde{s}_i$ (above construction of $\mathcal{F}^{\#}$). Let's prove that $s_i|_{\mathcal{U}_{ij}} = s_j|_{\mathcal{U}_{ij}}$ in order to glue the local sections together. By definition we have

$$\widetilde{s_i|_{\mathcal{U}_{ij}}} = \tilde{s}_i|_{\mathcal{U}_{ij}} = f|_{\mathcal{U}_{ij}} = \tilde{s}_j|_{\mathcal{U}_{ij}} = \widetilde{s_j|_{\mathcal{U}_{ij}}} . \tag{1.53}$$

Since $\mathcal{F}$ is a sheaf, then there exists a unique $s \in \mathcal{F}(\mathcal{U})$ s.t. $s|_{\mathcal{U}_i} = s_i$. Then, in particular, s is the unique section s.t. $\tilde{s} = f$. In particular $\theta(\mathcal{U})$ is an iso of abelian groups for all $\mathcal{U}$, hence (as proved above) θ is an iso of sheaves. ∎

Let's now check that the above construction satisfies the required the universal property. We have to consider the following morphisms of presheaves

$$\alpha\colon \mathcal{F} \longrightarrow \mathcal{G} \ , \tag{1.54}$$

for some $\mathcal{G} \in \mathsf{Sh}\,(X)$. Then α induces a family of maps

$$\begin{tikzcd} \widetilde{\mathcal{F}} = \coprod_{x\in X} \mathcal{F}_x \arrow[rr, "(\alpha_x)_{x\in X}"] \arrow[dr, "\pi_{\mathcal{F}}"'] & & \coprod_{x\in X} \mathcal{G}_x = \widetilde{\mathcal{G}} \arrow[dl, "\pi_{\mathcal{G}}"] \\ & X & \end{tikzcd} . \tag{1.55}$$

In turn, this induces a map

$$\alpha^{\#}(\mathcal{U}) : \mathcal{F}^{\#}(\mathcal{U}) \longrightarrow \mathcal{G}^{\#}(\mathcal{U}) \ . \tag{1.56}$$

Recall the above construction of the associated sheaf:

$$\mathcal{F}^{\#}(\mathcal{U}) = \left\{ f\colon \mathcal{U} \to \widetilde{\mathcal{F}} \,\middle|\, \exists \mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i, \exists\, s_i \in \mathcal{F}(\mathcal{U}_i) \text{ s.t. } f|_{\mathcal{U}_i} = \tilde{s}_i \right\}. \tag{1.57}$$

Analogously for $\mathcal{G}^{\#}$. Then the map induced by α maps $f \in \mathcal{F}^{\#}(\mathcal{U})$ to the map $g \in \mathcal{G}^{\#}(\mathcal{U})$, i.e. $g\colon \mathcal{U} \to \coprod_{x\in X} \mathcal{G}_x$, s.t.

$$g|_{\mathcal{U}_i} = \widetilde{\alpha(\mathcal{U}_i)(s_i)}. \tag{1.58}$$

Clearly, if we call $\tilde{\alpha} := (\alpha_x)_{x\in X}$, then we have just constructed $g = \tilde{\alpha} \circ f$.

Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\theta_{\mathcal{F}}"'] & \mathcal{G} \arrow[d, "\theta_{\mathcal{G}}"] \\ \mathcal{F}^{\#} \arrow[r, "\alpha^{\#}"'] \arrow[ur, "\hat{\alpha}"] & \mathcal{G}^{\#} \end{tikzcd} . \tag{1.59}$$

Notice that, since $\theta_{\mathcal{G}}$ is an iso (hence admits inverse), we can define

$$\hat{\alpha} = \theta_{\mathcal{G}}^{-1} \circ \alpha^{\#}, \tag{1.60}$$

which makes the diagram commute. Now, for exercise, check uniqueness, and we are done. ■

Definition 1.34: Constant presheaf.
Consider X a topological space, and A an abelian group (resp. ring, set, etc.). One defines the constant presheaf

$$\begin{aligned} P_A : \mathsf{Op}(X) &\longrightarrow \mathsf{Ab} \\ \mathcal{U} \subset X &\longmapsto A \\ \mathcal{V} \subset \mathcal{U} &\longmapsto (id_A\colon A \to A). \end{aligned} \tag{1.61}$$

Remark 1.35 Clearly the above is not a sheaf unless $A = 0$.

Definition 1.36: Constant sheaf.
The constant sheaf, with values in A, is the sheaf A_X, associated to the constant presheaf P_A.

Remark 1.37 Given any $\mathcal{U} \subset X$ open, then we have

$$A_X(\mathcal{U}) = A^{\pi_0(\mathcal{U})}, \tag{1.62}$$

where $\pi_0(\mathcal{U})$ is the set of connected components of $\mathcal{U}$, and

$$A^{\pi_0(\mathcal{U})} := \mathrm{Maps}\left(\pi_0(\mathcal{U}), A\right). \tag{1.63}$$

Proof. Let P be the constant presheaf s.t., for all $\mathcal{U} \subset X$ open, $P(\mathcal{U}) = A$. Its stalks are:

$$\begin{aligned} P_x &= \varinjlim_{\mathcal{U} \ni x} P(\mathcal{U}) && (1.64)\\ &= \varinjlim_{\mathcal{U} \ni x} A \simeq A. && (1.65) \end{aligned}$$

Moreover, the étalé space associated to P, as a set, is given by

$$\widetilde{P} = \coprod_{x \in X} P_x = \coprod_{x \in X} A = A \times X. \tag{1.66}$$

Then, in particular, the projection is given by

$$\begin{aligned} \pi_P : \widetilde{P} = A \times X &\longrightarrow X \\ (a, x) &\longmapsto x. \end{aligned} \tag{1.67}$$

Moreover the basic open subsets of $\widetilde{P}$ are given by

$$\left\{ s_x \in \coprod_{x \in X} P_x \;\middle|\; x \in \mathcal{U}, \text{ for some } \mathcal{U} \subset X \text{ open and } s \in P(\mathcal{U}) \right\}. \tag{1.68}$$

Notice that the above is just $\mathrm{im}(\tilde{s})$, for

$$\begin{aligned} \tilde{s} : \mathcal{U} &\longrightarrow \widetilde{P} = \coprod_{x \in X} P_x \\ x &\longmapsto s_x. \end{aligned} \tag{1.69}$$

Since $s \in P(\mathcal{U})$ is just an element $a \in A$, then for all $x \in \mathcal{U}$ we have $s_x = a$, hence

$$\mathrm{im}(\tilde{s}) = \{a\} \times \mathcal{U}. \tag{1.70}$$

Then the topology on $\widetilde{P} = A \times X$ is just the product topology of the discrete topology on A and the given topology on X. From this we obtain that

$$\begin{aligned} P^{\#}(\mathcal{U}) &= \{ f \colon \mathcal{U} \to A \times X \mid f \text{ is continuous and } \pi_P \circ f = id_{\mathcal{U}} \} && (1.71)\\ &= \{ f \colon \mathcal{U} \to A \mid f \text{ is continuous } \} = A^{\pi_0(\mathcal{U})}, && (1.72) \end{aligned}$$

where the last equality holds, since A has the discrete topology. Let's now consider $\mathcal{V} \subset \mathcal{U}$ and construct the restriction map

$$\begin{tikzcd} A_X(\mathcal{U}) \arrow[r, "\rho_{\mathcal{U}\mathcal{V}}"] \arrow[d, equal] & A_X(\mathcal{V}) \arrow[d, equal] \\ A^{\pi_0(\mathcal{U})} \arrow[r] & A^{\pi_0(\mathcal{V})} \end{tikzcd}. \tag{1.73}$$

This is induced by the inclusion $\pi_0(\mathcal{V}) \hookrightarrow \pi_0(\mathcal{U})$, which itself is induced by

$$\mathrm{Map}\left(\pi_0(\mathcal{V}), A\right) \longleftrightarrow \mathrm{Map}(\pi_0(\mathcal{U}), A).$$

■

Remark 1.38 Let $X = \{*\}$ a singleton. Let $\mathcal{F}$ be a sheaf on X, then

$$\begin{cases} \mathcal{F}(\emptyset) = 0 \\ \mathcal{F}(X) = A \end{cases} . \tag{1.74}$$

$\mathcal{F}$ is the constant sheaf associated to $A = \mathcal{F}(X)$. There is a one to one correspondance between sheaves on singletons and the objects of their target category.

Remark 1.39: Adjunction.
The functors $\iota\colon \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ (forgetful) and $(-)^{\#} : \mathsf{PSh}(X) \to \mathsf{Sh}(X)$ (associated sheaf) form the adjoint pair $((-)^{\#}, \iota)$. More explicitly, for any $\mathcal{F} \in \mathsf{PSh}(X)$ and $\mathcal{G} \in \mathsf{Sh}(X)$, we have an isomorphism

$$\mathrm{Hom}_{\mathsf{Sh}(X)}\left(\mathcal{F}^{\#}, \mathcal{G}\right) \simeq \mathrm{Hom}_{\mathsf{PSh}(X)}\left(\mathcal{F}, \iota(\mathcal{G})\right), \tag{1.75}$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$.

1.3 Exact sequences of sheaves

Let's give the necessary definitions, required to speak about exact sequences of sheaves. Let's start with kernel and image for morphisms of sheaves:

Definition 1.40: kernel and image.
Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. We define the *kernel* of α

$$\begin{aligned} \ker\alpha : \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{U} \subset X &\longmapsto \ker\left(\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})\right). \end{aligned} \tag{1.76}$$

This clearly is a presheaf, due to the universal property of kernels. Then we obtain the restriction morphisms

$$\begin{aligned} \rho_{\mathcal{U}\mathcal{V}}^{\ker\alpha}\colon \ \ker\left(\alpha(\mathcal{U})\right) &\longrightarrow \ker\left(\alpha(\mathcal{V})\right) \\ s &\longmapsto s|_{\mathcal{V}} . \end{aligned} \tag{1.77}$$

In fact this is a sheaf, since $\mathcal{F}$ is, and we denote this sheaf by $\ker(\mathcal{F})$.

Analogously we define the presheaf

$$\begin{aligned} \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{U} &\longmapsto \mathrm{im}\left(\alpha(\mathcal{U})\colon \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})\right). \end{aligned} \tag{1.78}$$

In general this is not a sheaf. Then we consider its associated sheaf, and call that the *image* of α, denoted by $\mathrm{im}(\alpha)$.

Exercise 2 Check that there is a natural, injective morphism of sheaves

$$\mathrm{im}(\alpha) \longrightarrow \mathcal{G} . \tag{1.79}$$

Definition 1.41: Quotient of sheaves.
Let now $\alpha\colon \mathcal{F} \to \mathcal{G}$ be an injective morphism of sheaves. Then the following is functorial in $\mathcal{U}$, hence a presheaf,

$$\begin{aligned} \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{U} &\longmapsto \mathcal{G}(\mathcal{U})/\mathcal{F}(\mathcal{U}). \end{aligned} \tag{1.80}$$

In general this is not a sheaf, but only a presheaf. Then we define the quotient $\mathcal{G}/\mathcal{F}$ to be its associated sheaf.

Remark 1.42 Recall that $\mathcal{F}_x \simeq \left(\mathcal{F}^{\#}\right)_x$ for all $x \in X$. It follows that

$$(\mathrm{im}(\alpha))_x \simeq \mathrm{im}(\alpha_x) \tag{1.81}$$
$$(\mathcal{G}/\mathcal{F})_x \simeq \mathcal{G}_x/\mathcal{F}_x \tag{1.82}$$

for all $x \in X$. Pleasecheckit.

Definition 1.43: Short exact sequence of sheaves.
Consider a sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0 \ . \tag{1.83}$$

We say that it is exact iff

- α is injective,
- $\ker \beta = \mathrm{im}\, \alpha$,
- β is surjective.

Proposition 1.44. *The sequence*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0 \tag{1.84}$$

is an exact sequence of sheaves of abelian groups on X iff

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{\alpha_x} \mathcal{F}_x \xrightarrow{\beta_x} \mathcal{F}''_x \longrightarrow 0 \tag{1.85}$$

is a short exact sequence of abelian groups for all $x \in X$.

Remark 1.45 If the following is an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0 \ , \tag{1.86}$$

one can show that, for any $\mathcal{U} \subset X$ open,

$$0 \longrightarrow \mathcal{F}'(\mathcal{U}) \xrightarrow{\alpha(\mathcal{U})} \mathcal{F}(\mathcal{U}) \xrightarrow{\beta(\mathcal{U})} \mathcal{F}''(\mathcal{U}) \tag{1.87}$$

is still exact, but, in general, $\beta(\mathcal{U})$ is not surjective.

Example tk: check the correctness of this sequence.

1. Let X be a Riemann surface, for example $X = \mathbb{C}^{\times}$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions. Let, now, $\mathcal{O}_X^*$ be the sheaf of invertible holomorphic functions. Then $\mathcal{O}_X(\mathcal{U})$ and $\mathcal{O}_X^*(\mathcal{U})$ are both abelian groups (with respect to the sum of functions) for any $\mathcal{U} \subset X$ open. Finally consider the constant sheaf $(2i\pi\mathbb{Z})_X$. We have an exact sequence of sheaves of abelian groups:

   $$0 \longrightarrow (2i\pi\mathbb{Z})_X \xrightarrow{\alpha} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0 \ , \tag{1.88}$$

   where the maps are defined as follows:

   $$\begin{aligned} \alpha : (2i\pi\mathbb{Z})_X &\longrightarrow \mathcal{O}_X \\ 2i\pi n &\longmapsto f, \end{aligned} \tag{1.89}$$

   s.t. $f(z) = 2i\pi n z$. Moreover the $\exp$ function acts as $f \mapsto \exp(f)$.

   Then the above sequence is indeed exact at the level of sheaves, but the last map is not exact at the level of sections, in fact $\exp\colon \mathcal{O}_{\mathbb{C}^*} \to \mathcal{O}^*_{\mathbb{C}^*}$ is not surjective. In fact $id|_{\mathbb{C}^*}$ does not admit an inverse image.

1.4 Inverse and direct image

Definition 1.46: Inverse and direct image of sheaves.
Let $f\colon X \to Y$ be a continuous map of topological spaces, $\mathcal{F} \in \mathsf{Sh}(X)$ and $\mathcal{G} \in \mathsf{Sh}(Y)$

- The *direct image* of $\mathcal{F}$ is the sheaf $f_*\mathcal{F} \in \mathsf{Sh}(Y)$ defined as follows:

$$\begin{aligned} f_*\mathcal{F} : \mathsf{Op}(Y)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{V} \subset Y &\longmapsto \mathcal{F}\left(f^{-1}(\mathcal{V})\right). \end{aligned} \tag{1.90}$$

  Notice that $f^{-1}(\mathcal{V}) \subset X$ is open by continuity of f. Pleasecheck that it indeed is a sheaf.

- The *inverse image* of $\mathcal{G}$ is the sheaf $f^*(\mathcal{G})$, defined as the associated sheaf to the following presheaf:

$$\begin{aligned} f^\dagger(\mathcal{G}) : \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \mathcal{U} \subset X &\longmapsto \varinjlim_{f(\mathcal{U})\subset\mathcal{V}} \mathcal{G}(\mathcal{V}), \end{aligned} \tag{1.91}$$

  where the limit is taken over $\mathcal{V} \subset Y$ open.

Proposition 1.47. *For any $x \in X$ we know that*

$$(f^*\mathcal{G})_x \simeq \mathcal{G}_{f(x)}. \tag{1.92}$$

Proof. By definition we have that

$$\begin{aligned} (f^*\mathcal{G})_x \simeq \left(f^\dagger\mathcal{G}\right)_x &\simeq \varinjlim_{x\in\mathcal{U}\subset Y} \left[\left(f^\dagger\mathcal{G}\right)(\mathcal{U})\right] &(1.93)\\ &\simeq \varinjlim_{x\in\mathcal{U}} \varinjlim_{f(\mathcal{U})\subset\mathcal{V}\subset Y} \mathcal{G}(\mathcal{V}) \simeq \varinjlim_{f(x)\in\mathcal{V}} \mathcal{G}(\mathcal{V}). &(1.94) \end{aligned}$$

tk: complete the proof :sweat: (look at d'Agnolo lecture notes :thinking:) ■

Remark 1.48: Étalé space language.
In the language of étalé spaces, we can define the fibered product, given $\mathcal{G} \in \mathsf{Sh}(Y)$ and its associated étalé space $\widetilde{\mathcal{G}}$, and a continuous map $f\colon X \to Y$,

$$\begin{tikzcd} X \times_Y \widetilde{\mathcal{G}} \arrow[r] \arrow[d] & \widetilde{\mathcal{G}} \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}. \tag{1.95}$$

This actually is a limit, in our categories, it can explicitly be described as:

$$X \times_Y \widetilde{\mathcal{G}} = \{(x, y) \mid f(x) = \pi_Y(y)\} \subset X \times \widetilde{\mathcal{G}}. \tag{1.96}$$

Then we have (pleasecheckit)

$$\widetilde{f^*\mathcal{G}} = \left(\pi\colon \widetilde{\mathcal{G}} \times_Y X \to X\right). \tag{1.97}$$

Proposition 1.49. *Let $X \in \mathsf{Top}$, $\mathcal{F} \in \mathsf{Sh}(X)$ and $x \in X$. Let $\iota_x\colon \{x\} \to X$ be the inclusion of $\{x\}$ in X. Then $i_X^*(\mathcal{F})$ is the constant sheaf $\mathcal{F}_x$ on $\{x\}$.*

Proof. It is clear, since $i_X^*(\mathcal{F})$ is the sheaf associated the presheaf

$$\begin{aligned} f^\dagger(\mathcal{F}) : \mathsf{Op}(X)^{op} &\longrightarrow \mathsf{Ab} \\ \{x\} &\longmapsto \varinjlim_{X \supset \mathcal{U} \supset \{x\}} \mathcal{F}(\mathcal{U}) = \mathcal{F}_x. \end{aligned} \tag{1.98}$$

■

Exercise 3 The following are functors

$$\begin{aligned} f_* \colon \mathsf{Sh}(X) &\longrightarrow \mathsf{Sh}(Y) \\ f^* \colon \mathsf{Sh}(Y) &\longrightarrow \mathsf{Sh}(X). \end{aligned} \tag{1.99}$$

Corollary 1.50. *f^* is an exact functor.*

Proof. Consider an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \longrightarrow 0 . \tag{1.100}$$

Then it is exact at the level of stalks, i.e. for all $y \in Y$

$$0 \longrightarrow \mathcal{F}'_y \xrightarrow{\alpha_y} \mathcal{F}_y \xrightarrow{\beta_y} \mathcal{F}''_y \longrightarrow 0 . \tag{1.101}$$

Now take any $x \in X$, we have the exact and commutative

$$\begin{tikzcd} 0 \arrow[r] & (f^*\mathcal{F}')_x \arrow[r, "\alpha_x"] \arrow[d, equal] & (f^*\mathcal{F})_x \arrow[r, "\beta_x"] \arrow[d, equal] & (f^*\mathcal{F})_x \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}'_{f(x)} \arrow[r, "\alpha_{f(x)}"] & \mathcal{F}_{f(x)} \arrow[r, "\beta_{f(x)}"] & \mathcal{F}''_{f(x)} \arrow[r] & 0. \end{tikzcd} \tag{1.102}$$

■

Proposition 1.51. *Consider two continuous maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z . \tag{1.103}$$

Then $(-)_*$ is covariant and $(-)^*$ is contravariant, i.e. (tk)

$$(g \circ f)_* \simeq g_* \circ f_* \tag{1.104}$$

$$(g \circ f)^* \simeq f^* \circ g^*. \tag{1.105}$$

Proof. Hint: use étalé spaces. ■

Proposition 1.52 (Adjunction of inverse and direct image)**.** *(f^*, f_*) is an adjoint pair, i.e. we have a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathsf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}). \tag{1.106}$$

Exercise 4 Let $\alpha \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on $X \in \mathsf{Top}$. Then α is surjective iff, for any $\mathcal{U} \subset X$ open and any $t \in \mathcal{G}(\mathcal{U})$, there exists an open cover $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ and local sections $s_i \in \mathcal{F}(\mathcal{U}_i)$, s.t.

$$\alpha(\mathcal{U}_i)(s_i) = t|_{\mathcal{U}_i} . \tag{1.107}$$

Definition 1.53: Support of a section.
Let P be a presheaf on X and $\mathcal{U} \subset X$ be an open subset of X. Consider a section $s \in \mathcal{F}(\mathcal{U})$, we define its support to be

$$\mathrm{Supp}(s) := \{x \in \mathcal{U} \mid s_x \neq 0\} . \tag{1.108}$$

Exercise 5 Show that, for any $s \in \mathcal{F}(\mathcal{U})$, $\mathrm{Supp}(s)$ is closed in $\mathcal{U}$.

Exercise 6 Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be an injective morphism of presheaves over $X \in \mathsf{Top}$.

1. Show that, for every $x \in X$, the induced morphism at the level of stalks is injective

$$\alpha_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x. \tag{1.109}$$

2. Show that the associated morphism of associated sheaves

$$\alpha^{\#}\colon \mathcal{F}^{\#} \longrightarrow \mathcal{G}^{\#} \tag{1.110}$$

is injective.

3. Let $\mathcal{U} \subset X$ be an open subset of X. What can one say about the morphism

$$\alpha^{\#}(\mathcal{U})\colon \mathcal{F}^{\#}(\mathcal{U}) \longrightarrow \mathcal{G}^{\#}(\mathcal{U})? \tag{1.111}$$

4. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Define an injective morphism of sheaves

$$\mathrm{im}(\alpha) \longrightarrow \mathcal{G}. \tag{1.112}$$

Exercise 7: Skyscraper sheaves. Let $X \in \mathsf{Top}$ and $x \in X$. Let $A \in \mathsf{Ab}$. We define the presheaf A_x on X as follows:

$$A_x(\mathcal{U}) := \begin{cases} A & \text{if } x \in \mathcal{U} \\ 0 & \text{if } x \notin \mathcal{U} \end{cases} \tag{1.113}$$

and the obvious maps (i.e. identity and zero map).

1. Prove that A_x is actually a sheaf.

2. Show that the stalks are given by

$$(A_x)_y = \begin{cases} A & \text{if } x \in \overline{\{x\}} \\ 0 & \text{if } x \notin \overline{\{x\}} \end{cases} . \tag{1.114}$$

3. Consider the inclusion $\iota\colon \overline{\{x\}} \to X$. Then show that

$$i_*(A) \simeq A_x, \tag{1.115}$$

for A the constant presheaf on $\overline{\{x\}}$.

2 Abelian categories

2.1 Preadditive categories

Definition 2.1: Preadditive category.
A category C is called **preadditive** iff it is a $\mathbb{Z}$ category, i.e. iff given any pair $X, Y \in \mathrm{Ob}(\mathsf{C})$ the set $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is a $\mathbb{Z}$-module (an abelian group) and the composition of morphisms is a bilinear map.

Lemma 2.2. *Let* C *be a* preadditive *category, then* C^{op} *is a preadditive category.*

Proof. By definition of opposite category we have

$$\mathrm{Hom}_{\mathsf{C}}(A, B) := \mathrm{Hom}_{\mathsf{C}^{op}}(B, A) \tag{2.1}$$

which is an abelian group. Moreover we have bilinearity of the composition, since it just reverses the order of composition. ■

Remark 2.3 We say that C has finite products iff given $\{X_i\}_{i=1}^n \subset \mathrm{Ob}(\mathsf{C})$, any finite family, we can define $\prod_{i=1}^n X_i \in \mathrm{Ob}(\mathsf{C})$, the universal representative of the projection maps.

Remark 2.4 C has finite products iff it has a final object $T := \prod_{\emptyset} X$ and, for all $A, B \in \mathrm{Ob}(\mathsf{C})$, their product $A \times B$ exists in C.

This is clearly enough by associativity of product.

Definition 2.5: Final/initial/zero object.

- $T \in \mathrm{Ob}(\mathsf{C})$ is a *final* or *terminal* object of C iff, given any $X \in \mathrm{Ob}(\mathsf{C})$, then
$$\mathrm{Hom}_{\mathsf{C}}(X, T) = \{*\} \tag{2.2}$$
is a singleton. Clearly T is unique up to a unique isomorphism.
- $I \in \mathrm{Ob}(\mathsf{C})$ is an *initial* object of C iff, given any $X \in \mathrm{Ob}(\mathsf{C})$, then
$$\mathrm{Hom}_{\mathsf{C}}(I, X) = \{*\} \tag{2.3}$$
is a singleton. Clearly I is unique up to a unique isomorphism.
- $0 \in \mathrm{Ob}(\mathsf{C})$ is a *zero* object iff it is both initial and terminal.

Example

- It is clear that $\{*\} \in \mathrm{Ob}(\mathsf{Sets})$ is a *final* object.
- Analogously $\emptyset \in \mathrm{Ob}(\mathsf{Sets})$ is an *initial* object.
- In Rings we know that $\mathbb{Z}$ is *initial* and $\{0\}$ is *final*.
- In Ab we know that $\{0\}$ is also *initial*.

Remark 2.6: Finite coproducts.
A category C admits finite coproducts iff it admits an initial object $I =: \coprod_{\emptyset} X$ and, for any $X, Y \in \mathsf{C}$, their coproduct $X \coprod Y$ is an object of C.

Lemma 2.7. *Let* C *be a* preadditive *category, then* C *has finite products iff it has finite coproducts. In such case the product of a finite family of objects coincide with its coproduct.*

Proof. Finite products in C correspond to finite coproducts in C^{op}. Then we will prove only one direction, the other one follows by applying this proof to the opposite category, which is also preadditive by lemma 2.2.

Assume that C has finite products. By remark 2.4 it has $T \in \mathsf{C}$ a final object, and for any pair of objects $A, B \in \mathsf{C}$ it has their product, $A \times B$.

1. Let's show that T is an initial object: for any $A \in \mathrm{Ob}(\mathsf{C})$ we know that

$$\emptyset \neq \mathrm{Hom}_{\mathsf{C}}(T, A) \in \mathsf{Ab}. \tag{2.4}$$

This means it contains $0_{T,A}$ the zero morphism. Moreover T is terminal, then

$$\{0_{T,T}\} \in \mathrm{Hom}_{\mathsf{C}}(T, T) = \{*\}, \tag{2.5}$$

and also $id_T \in \mathrm{Hom}_{\mathsf{C}}(T, T)$. Then $0_{T,T} = id_T$. It follows that, given any $a\colon T \to A$,

$$a = a \circ id_T = a \circ 0_{T,T} = 0, \tag{2.6}$$

where the last equality follows from bilinearity of composition.

2. Let's now show that, for any pair of elements $A, B \in \mathrm{Ob}(\mathsf{C})$, we have the isomorphism

$$A \times B \simeq A \coprod B. \tag{2.7}$$

Let's denote $0_{A,B} \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $0_{B,A} \in \mathrm{Hom}_{\mathsf{C}}(B, A)$. Let's define the embedding morphisms: $i_A\colon A \to A \times B$ as the unique map induced by the pair $(id_A, 0_{A,B})$ from A. Analogously $i_B\colon B \to A \times B$ is induced by $(0_{B,A}, id_B)$. Then the two maps correspond with the following commutative diagrams:

$$\begin{tikzcd} & A \\ A \arrow[ur, "id_A"] \arrow[r, "i_A"] \arrow[dr, "0_{A,B}"'] & A \times B \arrow[u, "p_A"'] \arrow[d, "p_B"] \\ & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \\ B \arrow[ur, "0_{B,A}"] \arrow[r, "i_A"] \arrow[dr, "id_B"'] & A \times B \arrow[u, "p_A"'] \arrow[d, "p_B"] \\ & B \end{tikzcd} \cdot \tag{2.8}$$

Let's now check the universal property of the coproduct on $(A \times B, i_A, i_B)$: let $C \in \mathrm{Ob}(\mathsf{C})$ and a pair of maps $f\colon A \to C$ and $g\colon B \to C$. One can define

$$h := (f \circ p_A) + (g \circ p_B) \in \mathrm{Hom}_{\mathsf{C}}(A \times B, C). \tag{2.9}$$

This gives rise to a diagram and we want to show it is commutative, i.e.

$$\begin{tikzcd} A \arrow[d, "i_A"'] \arrow[dr, "f"] \\ A \times B \arrow[r, "h"] & C \\ B \arrow[u, "i_B"] \arrow[ur, "g"'] \end{tikzcd} \tag{2.10}$$

$$h \circ i_A = f \qquad \text{and} \qquad h \circ i_B = g. \tag{2.11}$$

In fact, from the definition and bilinearity, we obtain

$$h \circ i_A = (f \circ p_A + g \circ p_B) \circ i_A = f \circ \underbrace{p_A \circ i_A}_{id_A} + g \circ \underbrace{p_B \circ i_A}_{0_{A,B}} = f. \tag{2.12}$$

With almost a carbon copy of this argument one proves that $h \circ i_B = g$.

Let's now show uniqueness of h. Let's, at first, show that

$$i_A \circ p_A + i_B \circ p_B = id_{A \times B}. \tag{2.13}$$

By universal property of the product it enough to show that

$$p_A \circ (i_A \circ p_A + i_B \circ p_B) = p_A, \tag{2.14}$$

and analogously for B. In fact we have

$$p_A \circ (i_A \circ p_A + i_B \circ p_B) = \underbrace{p_A \circ i_A}_{id_A} \circ p_A + \underbrace{p_A \circ i_B}_{0_{B,A}} \circ p_B = p_A. \tag{2.15}$$

Choose now $h' \colon A \times B \to C$ making the diagram in (2.10) commute. Then

$$\begin{aligned} h' &= h' \circ id_{A \times B} = h' \circ (i_A \circ p_A + i_B \circ p_B) \qquad (2.16) \\ &= \underbrace{h' \circ i_A}_{f} \circ p_A + \underbrace{h' \circ i_B}_{g} \circ p_B = f \circ p_a + g \circ p_B = h. \end{aligned}$$

■

Remark 2.8 The requirements in preadditive categories are crucial to have equality between products and coproducts. In Sets, for example, products are cartesian products, whereas coproducts are disjoint unions.

Remark 2.9 Let C be a *preadditive* category and $B \in \mathrm{Ob}(\mathsf{C})$. Suppose that $B \times B$ exists, then there is a map

$$\delta_B \colon B \times B \simeq B \coprod B \longrightarrow B \ , \tag{2.17}$$

called *codiagonal*, defined to be the one making the following diagram commute

$$\begin{tikzcd} B \arrow[d, "\epsilon_B"'] \arrow[dr, "id_B"] & \\ B \coprod B \arrow[r, "\delta_B"] & B \\ B \arrow[u, "\epsilon_B"] \arrow[ur, "id_B"'] & \end{tikzcd} . \tag{2.18}$$

As one could guess the name *codiagonal* comes from the fact that it satisfies the dual of the diagram for the *diagonal* morphism in a product.

2.2 Additive categories

Definition 2.10: Additive category.
An *additive category* is a *preadditive* category with finite products.

Remark 2.11 In an additive category, as seen in lemma 2.7, one has a zero element, since initial and terminal elements coincide, being (co)products indexed by the empty set.

In reality the structure of abelian group, for the hom sets in a additive category, does not depend on any external structure. It can be defined directly from the morphisms in the category itself.

Lemma 2.12. *Let* C *be an* additive *category and* $B \in \mathrm{Ob}(\mathsf{C})$*. Then, given any couple of morphisms* $f, g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$*, for any* $A \in \mathrm{Ob}(\mathsf{C})$*, one has*

$$f + g = \delta_B \circ (f, g), \tag{2.19}$$

where $(f, g) : A \to B \times B \simeq B \coprod B$ *is the map induced by the pair* f *and* g*.*

Proof. We have already shown that

$$id_{B \times B} = i_1 \circ p_1 + i_2 \circ p_2. \tag{2.20}$$

Then we have the commutative diagram

$$\begin{tikzcd}
 & B \arrow[d, "i_1"'] \arrow[dr, "id_B"] & \\
B \times B \arrow[ur, "p_1"] \arrow[r, "\sim"] \arrow[dr, "p_2"'] & B \coprod B \arrow[r, "\delta_B"] & B \, . \\
 & B \arrow[u, "i_2"] \arrow[ur, "id_B"'] &
\end{tikzcd} \tag{2.21}$$

From this we conclude, since

$$\begin{aligned}
\delta_B \circ (f, g) = \delta_B \circ (i_1 \circ p_1 + i_2 \circ p_2) \circ (f, g) = (\underbrace{\delta_B \circ i_1}_{id_B} \circ p_1 + \underbrace{\delta_B \circ i_2}_{id_B} \circ p_2) \circ (f, g) \\
= p_1 \circ (f, g) + p_2 \circ (f, g) = f + g.
\end{aligned}$$

■

Remark 2.13: Notation.
An additive category with finite products has also coproducts and viceversa (reason in C^{op}). Then we denote them, since they are isomorphic, by

$$A \oplus B := A \times B \simeq A \coprod B. \tag{2.22}$$

Remark 2.14: Zero morphism.
Given any $A, B \in \mathrm{Ob}(\mathsf{C})$ an additive category, one always has

$$0_{A,B} \in \mathrm{Hom}_{\mathsf{C}}(A, B). \tag{2.23}$$

In particular this corresponds to the composition

$$0_{A,B} : A \xrightarrow{\ 0\ } 0_{\mathsf{C}} \xrightarrow{\ 0\ } B \ . \tag{2.24}$$

In fact one needs no group structure on $\mathrm{Hom}_{\mathsf{C}}(A, B)$ to define a zero morphism between the two objects.

Definition 2.15: Zero morphism.
Let C be a category with zero object. Then, for all $A, B \in \mathrm{Ob}(\mathsf{C})$, one can define the zero morphism $0_{A,B} \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ as the composition

$$A \longrightarrow 0 \longrightarrow B \ . \tag{2.25}$$

Lemma 2.16. *Let* C *be a category with zero object, finite products and coproducts s.t. the map induced by the following diagram is an isomorphism*

$$\begin{tikzcd} B \arrow[d, "i_B"'] \arrow[dr, "(0,id_B)"] & \\ A\coprod B \arrow[r, "\sim"] & A\times B \\ A \arrow[u, "i_A"] \arrow[ur, "(id_A,0)"'] & \end{tikzcd} \cdot \tag{2.26}$$

Then $\mathrm{Hom}_{\mathsf{C}}(A,B)$ *is a monoid:*

Proof. Given $f, g \in \mathrm{Hom}_{\mathsf{C}}(A,B)$, then one can set

$$f+g : A \xrightarrow{(f,g)} B\oplus B \xrightarrow{\delta_B} B \tag{2.27}$$

and

$$0_{A,B} : A \longrightarrow 0 \longrightarrow B\ . \tag{2.28}$$

One only need to check that this effectively gives rise to a monoid. ∎

Definition 2.17: Additive category, take 2.
An additive category C is a category with a zero object, finite products and finite coproducts s.t.

- The map defined above is an iso

$$A\coprod B \xrightarrow{\sim} A\times B\ , \tag{2.29}$$

- The abelian monoid $\mathrm{Hom}_{\mathsf{C}}(A,B)$ is an abelian group.

Remark 2.18 The above definition makes clear that an additive category is just a category satisfying certain properties, with no additional structure. In fact the group structure on $\mathrm{Hom}_{\mathsf{C}}(A,B)$ is determined by the underlying category C.

Definition 2.19: (Co)equalizer.
Let f, g be two parallel morphisms $A \rightrightarrows B$ in a category C.

- An **equalizer** of f and g is a pair (C,e), with $e\colon C\to A$, satisfying

  eq1 $f\circ e = g\circ e$,

  eq2 for (C',e') with $C' \xrightarrow{e'} A$ s.t. $f\circ e' = g\circ e'$, then $\exists\,!\alpha : C'\to C$ s.t. $e\circ\alpha = e'$, i.e. the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ & C' \arrow[ul, dashed, "\exists\,!\alpha"] \arrow[u, "e'"'] & \end{tikzcd} . \tag{2.30}$$

- A **coequalizer** of f and g is an equalizer of f and g in C^{op}. In other words it is a pair (C,p), with $p\colon B\to C$ s.t.

  coeq1 $p\circ f = p\circ g$,

coeq2 for (C', p') with $B \xrightarrow{p'} C'$ s.t. $p' \circ f = p' \circ g$, then $\exists\, !\gamma : C \to C'$, with $\gamma \circ p = p'$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "P"] \arrow[d, "p'"] & C \arrow[dl, dashed, "\exists\, !\gamma"] \\ & C' & \end{tikzcd} . \tag{2.31}$$

Definition 2.20: (Co)kernel.
Let C be a category with zero object. Let $f\colon A \to B$ a morphism in C. One can define the kernel of f as the equalizer of f and 0, which corresponds with

$$\ker f = \varprojlim \begin{pmatrix} & 0 \\ & \downarrow \\ A \xrightarrow{f} & B \end{pmatrix} = A \times_B 0. \tag{2.32}$$

Analogously one defines the cokernel as the dual of the kernel, i.e.

$$\operatorname{coker} f = \varinjlim \begin{pmatrix} A \xrightarrow{f} B \\ \downarrow \\ 0 \end{pmatrix} = B \coprod_A 0. \tag{2.33}$$

Remark 2.21 The object $\ker f$ is given with a map

$$\ker f \longrightarrow A \tag{2.34}$$

and $\operatorname{coker} f$ with a map

$$B \longrightarrow \operatorname{coker} f\ . \tag{2.35}$$

Both of them satisfy a universal property. In particular the kernel satisfies the following: given any $g \in \operatorname{Hom}_{\mathsf{C}}(C, A)$ for any object $C \in \operatorname{Ob}(\mathsf{C})$ s.t. $f \circ g = 0$, then g factorizes uniquely through $\ker f$. In pictures the diagram commutes:

$$\begin{tikzcd} \ker f \arrow[r, "k"] & A \arrow[r, "f"] & B \\ & C \arrow[ul, dashed, "\exists!\, \alpha"] \arrow[u, "g"] \arrow[ur, "0"'] & \end{tikzcd} . \tag{2.36}$$

The cokernel, instead, satisfies the following: given any $g \in \operatorname{Hom}_{\mathsf{C}}(B, C)$ for any object $C \in \operatorname{Ob}(\mathsf{C})$ s.t. $g \circ f = 0$, then g factorizes uniquely through $\operatorname{coker} f$. In pictures the diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "c"] \arrow[d, "g"] & \operatorname{coker} f \arrow[dl, dashed, "\exists!\, \gamma"] \\ & C & \end{tikzcd} . \tag{2.37}$$

Remark 2.22 It follows from the definition that

$$\ker f \rightarrowtail A \tag{2.38}$$

is a monomorphism, whereas the morphism

$$B \twoheadrightarrow \operatorname{coker} f \tag{2.39}$$

is an epimorphism.

Proof. Let's give $\alpha, \beta\colon A \to \ker f$, and denote by $i\colon \ker f \to A$ the natural map. Assume that $i \circ \alpha = i \circ \beta$. Then $f \circ i \circ \alpha = f \circ i \circ \beta = 0$. Then by universal property of the kernel there is a unique map $\gamma\colon X \to \ker f$ s.t. $i \circ \alpha = i \circ \gamma$. By uniqueness of γ we have $\gamma = \alpha = \beta$. ■

Definition 2.23: (Co)image.
Let C be a category with zero object and $f\colon A \to B$ be a morphism in C. One can define the *image* of f, if it exists, as

$$\operatorname{im} f := \ker (B \longrightarrow \operatorname{coker} f) . \tag{2.40}$$

Analogously, if it exists, one defines the *coimage* of f as

$$\operatorname{coim} f := \operatorname{coker} \ (\ker f \longrightarrow A) . \tag{2.41}$$

Remark 2.24 From this definition one obtains natural maps

$$\begin{aligned} A &\rightarrowtail \operatorname{Coim} f \\ \operatorname{im} f &\twoheadrightarrow B. \end{aligned} \tag{2.42}$$

Exercise 8 Let $\mathsf{C} := \mathsf{Ab}$ and $f\colon A \to B$ a morphism. Show that

$$\operatorname{im} f \simeq \{b \in B \mid \exists\, a \in A \text{ s.t. } f(a) = b\} . \tag{2.43}$$

Analogously show that

$$\operatorname{coim} f \simeq \frac{A}{\ker f}, \tag{2.44}$$

where the quotient is the usual quotient of abelian groups. tk: what did I want to say? Moreover check the commutativity of the required diagrams (i.e. the arrows from/to A, B).

Notice that in Ab we have

$$\operatorname{coim} f \simeq \operatorname{im} f. \tag{2.45}$$

Exercise 9 Let C be a category with a zero object, kernels and cokernels.

1. Show that there exists a canonical map

$$\bar{f}\colon \operatorname{coim} f \longrightarrow \operatorname{im} f \tag{2.46}$$

s.t. the following diagram commutes

$$\begin{tikzcd} \ker f \arrow[r, tail] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[r, "\bar{f}"] & \operatorname{im} f \arrow[u, tail] & \end{tikzcd} . \tag{2.47}$$

2. Prove that $f = 0$ iff $\ker f = id_A$ iff $\operatorname{coker} f = id_B$.

3. Prove that $f = id_A$ implies that

$$\ker f = (0 \longrightarrow A) \tag{2.48}$$
$$\operatorname{coker} f = (A \longrightarrow 0) . \tag{2.49}$$

Proof. Since $\mathrm{coim}\, f = \mathrm{coker}\, (\ker f)$, by the universal property of cokernels, there is a unique map $\alpha\colon \mathrm{coim}\, f \to B$ s.t. $f = \alpha \circ \pi$, for π the map $\pi\colon A \to \mathrm{coim}\, f$. Then, since π is epi, denoted $\theta\colon B \to \mathrm{coker}\, f$, we have that $0 = \theta \circ f = \theta \circ \alpha \circ \pi = \theta \circ \alpha$. This means that α factorizes through $\mathrm{im}\, f$, i.e. we have a map $\bar{f}\colon \mathrm{coim}\, f \to \mathrm{im}\, f$ making the diagram commute. Here is a diagram with all of the maps used in the proof:

$$\begin{tikzcd}
\ker f \arrow[r, tail] & A \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & B \arrow[r, two heads, "\theta"] & \mathrm{coker}\, f \\
& \mathrm{coim}\, f \arrow[ur, dashed, "\alpha"] \arrow[r, "\bar{f}"'] & \mathrm{im}\, f \arrow[u, tail] &
\end{tikzcd} . \tag{2.50}$$

■

Remark 2.25 Clearly ker and coker in C correspond respectively to coker and ker in C^{op}.

Lemma 2.26. *Let C be a category with zero object, kernels and cokernels. Let $f\colon A \to B$ be a morphism, then*

$$\ker f \simeq \ker\left(A \longrightarrow \mathrm{coim}\, f\right) \tag{2.51}$$

$$\mathrm{coker}\, f \simeq \mathrm{coker}\left(\mathrm{im}\, f \longrightarrow B\right) . \tag{2.52}$$

Proof. It is enough to show the first one, then the second follows from the same argument in the opposite category. By definition

$$\mathrm{coim}\, f = \mathrm{coker}\, \left(\ker f \to A\right) . \tag{2.53}$$

In particular it follows that

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[rr, bend right, "0"'] & A \arrow[r] & \mathrm{coim}\, f
\end{tikzcd} . \tag{2.54}$$

Let's show that $\ker f$ is universal with this property. Let $C \in \mathsf{C}$ and consider a morphism $C \to A$ s.t.

$$\begin{tikzcd}
C \arrow[r] \arrow[rr, bend right, "0"'] & A \arrow[r] & \mathrm{coim}\, f
\end{tikzcd} . \tag{2.55}$$

Composing after this the map from the coimage to the image and the monomorphism from the image to B, we have a composition which gives zero:

$$\begin{tikzcd}
C \arrow[r] \arrow[rrrr, bend right, "0"'] & A \arrow[r] \arrow[rrr, bend left, "f"] & \mathrm{coim}\, f \arrow[r, "\bar{f}"] & \mathrm{im}\, f \arrow[r, tail] & B
\end{tikzcd} . \tag{2.56}$$

Then it factorizes through $\ker f$ and from the universal property of ker we have our result. ■

2.3 Abelian categories and complexes

Definition 2.27: Abelian category.
An *abelian* category is an additive category s.t.

1. any morphism has a kernel and a cokernel,

2. given any morphism $f\colon A \to B$, the canonical map

$$\bar{f}\colon \operatorname{coim} f \xrightarrow{\sim} \operatorname{im} f \tag{2.57}$$

is an isomorphism.

A *preabelian* category is an additive category satisfying only 1.

Remark 2.28 An abelian category behaves, in many ways, like the category of abelian groups (in other words, of $\mathbb{Z}$-modules). Actually there is a powerful theorem which states a similar result for small abelian category, i.e. they can be embedded into a category of modules, over a ring R, not necessairily $\mathbb{Z}$.

Remark 2.29 Consider a pair of composable morphism f, g in an abelian category. Assume $g \circ f = 0$. Then there exists a canonical map

$$\operatorname{im} f \longrightarrow \ker g \ . \tag{2.58}$$

Proof. Since C is an abelian category, any morphism $f\colon A \to B$ factors through $\operatorname{im} f$ as $f = \gamma \circ \delta$, for an epimorphism $\delta\colon A \to \operatorname{im} f$ (it is the composition of an epimorphism and an isomorphism). Then

$$0 = g \circ f = g \circ \gamma \circ \delta = g \circ \gamma, \tag{2.59}$$

since δ is epi. This means that γ factors through $\ker g$, giving the desired morphism $\tilde{\gamma}$. In pictures we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"'] & B \arrow[r, "g"] & C \\ \operatorname{im} f \arrow[ur, "\gamma"] \arrow[r, dashed, "\tilde{\gamma}"'] & \ker g \arrow[u, "\epsilon"'] & \end{tikzcd} \quad . \tag{2.60}$$

■

Definition 2.30 A sequence in a (pre)abelian category is a sequence of morphism

$$A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} \quad . \tag{2.61}$$

A sequence is called complex iff $d^{n+1} \circ d^n = 0$ for all n (in the set of indices of the complex).

A complex is called *acyclic*, also called *exact sequence*, iff the canonical map induced by the above condition

$$\operatorname{im} d^n \xrightarrow{\sim} \ker d^{n+1} \tag{2.62}$$

is an isomorphism for all n.

Example The following is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \ . \tag{2.63}$$

Exercise 10 Let C be a (pre)abelian category and $f\colon A \to B$ a morphism in C.

1. The following are equivalent

   (a) f is a monomorphism,

   (b) $\ker f = 0_{\mathsf{C}}$,

(c) the following is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \ . \tag{2.64}$$

2. The following are equivalent

   (a) f is an epimorphism,

   (b) $\operatorname{coker} f = 0_{\mathsf{C}}$,

   (c) the following is an exact sequence

$$A \xrightarrow{f} B \longrightarrow 0 \ . \tag{2.65}$$

Proof. Let's only prove the first point. The second follows applying the first in the opposite category.

Let f be a monomorphism, $C \in \mathsf{C}$ and $g\colon C \to A$ s.t. $f \circ g = 0$. Then

$$f \circ g = 0_{C,B} = f \circ 0_{C,A}. \tag{2.66}$$

Since f is a mono we obtain that $g = 0_{C,A}$, i.e. g factors through $0 = \ker f$ (uniqueness of the factorization, then, is obvious).

Let's prove the converse, $\ker f = 0$. Let $C \in \mathsf{C}$ and consider two maps $g_0, g_1\colon C \to A$ equalized by $f \circ g_0 = f \circ g_1$. Let's take the difference

$$0 = f \circ g_0 - f \circ g_1 = f \circ (g_0 - g_1). \tag{2.67}$$

Then $g_0 - g_1$ factors through $\ker f = 0$, hence $g_0 - g_1 = 0$, i.e. $g_0 = g_1$. ■

Exercise 11 Let C be an abelian category. Let $f\colon A \to B$ be a morphism. Then

1. f is a monomorphism iff $f = \operatorname{im} f$,
2. f is an epimorphism iff $f = \operatorname{coim} f$,
3. f is an isomorphism iff f is both a mono and an epimorphism.

Lemma 2.31. *Let* C *be an* abelian *category and*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.68}$$

a sequence in C. *TFAE:*

1. *The sequence is exact.*
2. *The following is exact in* C^{op}*:*

$$C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A \ . \tag{2.69}$$

3. $\operatorname{coker} f = \operatorname{coim} g$.

Proof.

1 $\implies$ 3: Notice that exactness is equivalent to $\operatorname{im} f = \ker g$. Then

$$\begin{tikzcd} \operatorname{coker}(\operatorname{im} f) \arrow[r, equal] \arrow[d, equal] & \operatorname{coker}(\ker g) \arrow[d, equal] \\ \operatorname{coker} f \arrow[r, equal] & \operatorname{coim} g \end{tikzcd} . \tag{2.70}$$

3 $\implies$ 1: By hypothesis

$$\operatorname{coker}(\operatorname{im} f) \simeq \operatorname{coker}(\ker g) \tag{2.71}$$
$$\ker(\operatorname{coker}(\operatorname{im} f)) \simeq \ker(\operatorname{coker}(\ker g)) \tag{2.72}$$
$$\operatorname{im}(\operatorname{im} f) \simeq \operatorname{im}(\ker g) . \tag{2.73}$$

By exercise 3, if h is a monomorphism, $\operatorname{im} h = h$. Then

$$\operatorname{im} f = \ker g. \tag{2.74}$$

3 $\iff$ 2: We know that images and kernels, in the opposite category, correspond (respectively) to coimages and cokernels, hence the following two lines are equivalent:

$$\operatorname{im}(g^{op}) \simeq \ker(g^{op}) \tag{2.75}$$
$$\operatorname{coim} g \simeq \operatorname{coker} f.$$ ■

Lemma 2.32. *Let* C *be an* abelian *category. TFAE:*

1. $f \simeq \ker g$ *iff the following is exact*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C . \tag{2.76}$$

2. $g \simeq \operatorname{coker} f$ *iff the following is exact*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 . \tag{2.77}$$

Proof. The second point follows from the first, applied in C^{op}. Let's show the first one, then. Let's recall the definition of exactness.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \tag{2.78}$$

is exact iff f is a monomorphism and $\operatorname{im} f \simeq \ker g$, but then, since f is a monomorphism we know that $\operatorname{im} f = f$, and we have proved one direction. The other is also clear, since $\ker g$ is a monomorphism, hence $f \simeq \operatorname{im} f$ and we have recovered the definition of exactness. ■

Remark 2.33: Hom functor.
Fix a category C, then the hom functor

$$\operatorname{Hom}_{\mathsf{C}} \colon \mathsf{C}^{op} \times \mathsf{C} \longrightarrow \mathsf{Sets} \tag{2.79}$$
$$(X, Y) \longmapsto \operatorname{Hom}_{\mathsf{C}}(X, Y) \tag{2.80}$$

is contravariant in the first entry, and covariant in the second one. In particolar, on morphisms, it acts as follows: let $g \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$, then

$$\begin{aligned} g_* := \mathrm{Hom}_{\mathsf{C}}(X, g) : \mathrm{Hom}_{\mathsf{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathsf{C}}(X, Z) \\ h &\longmapsto g \circ h. \end{aligned} \tag{2.81}$$

For $f \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$, instead

$$\begin{aligned} f^* := \mathrm{Hom}_{\mathsf{C}}(f, Y) : \mathrm{Hom}_{\mathsf{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathsf{C}}(Z, Y) \\ h &\longmapsto h \circ f. \end{aligned} \tag{2.82}$$

Lemma 2.34. *Let* C *be an* abelian *category.*

1. *A sequence in* C
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \tag{2.83}$$
is exact iff, for any $M \in \mathrm{Ob}(\mathsf{C})$*, the image sequence*
$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(M, A) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(M, B) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(M, C) \tag{2.84}$$
is exact in Ab.

2. *A sequence in* C
$$A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2.85}$$
is exact iff, for any $M \in \mathrm{Ob}(\mathsf{C})$*, the image sequence*
$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(C, M) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(B, M) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, M) \tag{2.86}$$
is exact in Ab.

Proof. 2 follows from 1. In fact the sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{2.87}$$

is exact in C iff

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \tag{2.88}$$

is exact in C^{op}. Then one can apply $\mathrm{Hom}_{\mathsf{C}^{op}}(M, -)$, obtaining

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(M, C) \longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(M, B) \longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(M, A) \ , \tag{2.89}$$

which is exact iff it is exact in the opposite category, i.e. iff

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(C, M) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(B, M) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, M) \tag{2.90}$$

is exact in C. In other words 1 is equivalent to 2, thanks to the previous lemma.

Let's now prove 1. Assume

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{f} C \tag{2.91}$$

is exact, i.e. $i = \ker f$, by the above proposition. But the universal property of $\ker f$ is equivalent to the exactness of the desired sequence. In fact, fixed $M \in \mathrm{Ob}(\mathsf{C})$, one obtains the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(M, A) \xrightarrow{i_*} \mathrm{Hom}_{\mathsf{C}}(M, B) \xrightarrow{f_*} \mathrm{Hom}_{\mathsf{C}}(M, C) \ . \tag{2.92}$$

Then, given any $h \in \mathrm{Hom}_{\mathsf{C}}(M, A)$, one has $f_* \circ i_*(h) = f \circ i \circ h = 0$, since $f \circ i = 0$. In other words $f_* \circ i_* = 0$. Then, by universal property of ker, on obtains a morphism

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(M, A) &\longrightarrow \ker f_* \\ \beta &\longmapsto i \circ \beta \in \mathrm{Hom}_{\mathsf{C}}(M, B) \end{aligned} . \tag{2.93}$$

Moreover, let $\alpha \in \ker f_*$, i.e. $f \circ \alpha = 0$ then, since $i = \ker f$, α factorizes through A, i.e. there exists a unique $\tilde{\alpha} \colon M \to A$ s.t. $\alpha = i \circ \tilde{\alpha}$.

$$\begin{tikzcd} M \arrow[r, "\exists ! \tilde{\alpha}"] \arrow[dr, "\alpha"'] & A \arrow[d, "i"] \\ & B \end{tikzcd} . \tag{2.94}$$

In other words the map in (2.93) is an isomorphism of abelian groups.

Let's prove the converse. Consider, at first, $M = A$. Then the following is exact

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, A) \xrightarrow{i_*} \mathrm{Hom}_{\mathsf{C}}(A, B) \xrightarrow{f_*} \mathrm{Hom}_{\mathsf{C}}(A, C) . \tag{2.95}$$

By exactness $f_* \circ i_* = 0$. In particular, taken $id_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$, we have $f \circ i \circ id_A = f \circ i = 0$.

Let's now consider a generic $M \in \mathsf{C}$. Given $\alpha \colon M \to B$ s.t. $f \circ \alpha = 0$, then there exists a unique $\tilde{a} \colon M \to A$ s.t. $\alpha = i \circ \tilde{\alpha}$. In fact this is true by injectivity of i_* and the fact that $\mathrm{im}\, i_* = \ker f_*$ (notice that we are working in Ab). Then i satisfies the universal property of the kernel of f, and we have exactness of the sequence in C. ■

Corollary 2.35. *Consider a commutative diagram in an* abelian *category* C*:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f_A"] & B \arrow[r, "p"] \arrow[d, "f_B"] & C \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd} \tag{2.96}$$

s.t. the rows are exact and f_A and f_C are isomorphisms. Then also f_B is an isomorphism.

Proof. Let's start by showing that f_B is a monomorphism. For any $M \in \mathrm{Ob}(\mathsf{C})$ we have a commuative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathsf{C}}(M, A) \arrow[r, "i_*"] \arrow[d, "(f_A)_*"] & \mathrm{Hom}_{\mathsf{C}}(M, B) \arrow[r, "p_*"] \arrow[d, "(f_B)_*"] & \mathrm{Hom}_{\mathsf{C}}(M, C) \arrow[d, "(f_C)_*"] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathsf{C}}(M, A') \arrow[r, "i'_*"] & \mathrm{Hom}_{\mathsf{C}}(M, B') \arrow[r, "p'_*"] & \mathrm{Hom}_{\mathsf{C}}(M, C') \end{tikzcd} . \tag{2.97}$$

Exactness of the rows follows from the previous lemma, whereas commutativity from the fact that $\mathrm{Hom}_{\mathsf{C}}(M, -)$ is a functor. Notice, moreover, that any functor sends isomorphisms to isomorphisms, hence both $(f_A)_*$ and $(f_C)_*$ are isomorphisms.

With a simple diagram chase we can prove that $(f_B)_*$ is injective for all $M \in \mathrm{Ob}(\mathsf{C})$. In fact, consider $x \in \mathrm{Hom}_{\mathsf{C}}(M, B)$ s.t. $f_B \circ x = 0$. Then

$$(f_C)_* \circ p_*(x) = p'_* \circ (f_B)_*(x) = 0. \tag{2.98}$$

Since $(f_C)_*$ is an isomorphism, it means that $p_*(x) = 0$, i.e. $x \in \ker p_* = \mathrm{im}\, i_*$, hence $x = i_*(y)$ for $y \in \mathrm{Hom}_{\mathsf{C}}(M, A)$. Moreover $(f_B)_*(x) = 0$ implies that $p'_* \circ (f_B)_*(x) = 0$, i.e. $(f_B)_*(x) \in \ker p'_* = \operatorname{im} i'_*$. Moreover i'_* is injective, its only inverse image is 0. Then $(f_A)_*(y) = 0 \in \operatorname{Hom}_{\mathsf{C}}(M, A')$. Since $(f_A)_*$ is an isomorphism by hypothesis, we have $y = 0$, hence $x = 0$.

But $(f_B)_*$ injective means, exactly, that f_B is a monomorphism. In fact, given $\alpha, \beta \in \operatorname{Hom}_{\mathsf{C}}(M, B)$ on has that $f_B \circ \alpha = f_B \circ \beta$ iff $(f_B)_*(\alpha) = (f_B)_*(\beta)$. Being $(f_B)_*$ injective this implies $\alpha = \beta$, i.e. f_B is a mono.

Reasoning with a similar diagram chase one proves that $(f_B)_*$ is also an epimorphism. (tk: copy the proof of epimorphism)

Finally we can conclude, since $(f_B)_*$ is both a monomorphism and an epimorphism which, in an abelian category, is equivalent to being an isomorphism. ∎

Definition 2.36: Split sequence.
An exact sequence in an abelian category

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd} \tag{2.99}$$

is said to *split* iff there exists a morphism $s\colon C \to B$ s.t. $p \circ s = id_C$. Such morphism is called a *splitting* of p.

Lemma 2.37. *Let C be an abelian category, and consider an exact sequence in C.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] & 0 \end{tikzcd} . \tag{2.100}$$

TFAE:

1. *The sequence splits.*

2. *We have an isomorphism $\varphi\colon A \oplus C \to B$ making the following diagram commute:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i_A"] & A \oplus C \arrow[r, "p_C"] \arrow[u, "\varphi"] & C \arrow[r] & 0 \end{tikzcd} . \tag{2.101}$$

3. *There exists a map $a\colon B \to A$ s.t. $s \circ i = id_A$.*

Proof.

$1 \implies 2$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, equal] \arrow[dr, "i"] & A \oplus C \arrow[r, "p_C"] \arrow[d, "f=(is)"] & C \arrow[r] \arrow[d, equal] \arrow[dl, "s"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd} . \tag{2.102}$$

We want to show that f is an isomorphism and the above diagram commutes.

Let's start by tackling commutativity of the diagram. By definition of f and direct sum we clearly have that $i = f \circ i_A$. Then we only need to check that $p_C = p \circ f$. To check the above equality we need to compute the composition with i_A and i_C. They clearly are given by:

$$p \circ f \circ i_C = p \circ (f \circ i_C) = p \circ s = id_C \tag{2.103}$$

$$p \circ f \circ i_A = p \circ i = 0. \tag{2.104}$$

Then, by universal property of coproduct, we have $p \circ f = p_C$ as desired.

Exactness is also clear: the second row is exact by assumption, whereas the first is exact by definition of direct sum: $i_A = \ker p_C$, one can explicitly check that it satisfies the universal property.

Then we can conclude applying the five lemma.

2 $\implies$ 1: It is obvious from the identity $p_C \circ i_C = id_C$ and commutativity of the diagram, i.e. $p_C = p \circ f$. In fact

$$p \circ f \circ i_C = p_C \circ i_C = id_C. \tag{2.105}$$

2 $\iff$ 3: This is proved almost in the exact same way as the above two implications. ■

Remark 2.38 The above lemma says that an exact sequence splits iff it is isomorphic, as exact sequence, to

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus C \xrightarrow{p_C} C \longrightarrow 0 \ . \tag{2.106}$$

Definition 2.39 Let R be a ring. We define the category $R\text{-}\mathsf{Mod}$ to be the category of right R-modules.

Remark 2.40 $R\text{-}\mathsf{Mod}$ is an *abelian* category.

Lemma 2.41 (Snake lemma)**.** *Consider the commutative diagram in $R\text{-}\mathsf{Mod}$ with exact rows*

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[d, "u"'] & B \arrow[r, "\beta"] \arrow[d, "v"'] & C \arrow[r] \arrow[d, "w"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' & \end{tikzcd} . \tag{2.107}$$

Then there is a canonical exact sequence

$$\ker u \xrightarrow{\underline{\alpha}} \ker v \xrightarrow{\underline{\beta}} \ker w \xrightarrow{\delta} \operatorname{coker} u \xrightarrow{\overline{\alpha'}} \operatorname{coker} v \xrightarrow{\overline{\beta'}} \operatorname{coker} w \ , \tag{2.108}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are the restrictions to the kernels, whereas $\overline{\alpha'}$ and $\overline{\beta'}$ the induced maps on the quotients.

Proof.

1. $\underline{\alpha}\colon \ker u \to \ker v$ is well defined, since $\alpha(x) \in \ker v$ for all $x \in \ker u$. In fact, since the diagram commutes, and $x \in \ker u$, we have

$$v\left(\alpha(x)\right) = \alpha'(u(x)) = 0. \tag{2.109}$$

   Analogously for $\underline{\beta}$. In fact, for any commutative diagram, the kernel of a vertical arrow gets mapped to the kernel of the following one, by the horizontal one.

2. $\overline{\alpha'}\colon \operatorname{coker} u \to \operatorname{coker} v$ is well define. In fact consider $x' \in A'/\operatorname{im} u$, then $\overline{\alpha'}(x') \in B'/\operatorname{im} v$. Clearly $x' \in \operatorname{im} u$ means that there exists $a \in A$ s.t. $u(a) = x'$. Then

$$\alpha'(x') = \alpha'(u(a)) = v(\alpha(a)) \in \operatorname{im} v. \tag{2.110}$$

   By the universal property of the quotient we obtain that $\overline{\alpha'}$ is well defined. Analogously one argues that $\overline{\beta'}$ is well defined.

3. Let's define $\delta\colon \ker w \to \operatorname{coker} u$. Consider $x \in \ker w$, then $w(x) = 0$. By exactness β is surjective, hence there exists $b \in B$ s.t. $\beta(b) = x$. Then we obtain $v(b) \in B'$ s.t.

$$\beta'(v(b)) = w(\beta(b)) = w(x) = 0, \tag{2.111}$$

i.e. $v(b) \in \ker \beta'$. Then, by exactness of the second row, there exists $a \in A'$ s.t. $\alpha'(a) = b$. We define $\delta(x) := \overline{a} \in \operatorname{coker} u$. Notice that α' is injective, hence we can identify $A' \simeq \operatorname{im} \alpha'$.

Let's show that this is a good definition, i.e. a is defined up to $\operatorname{im} u$: Consider $b_1, b_2 \in B$ s.t. $\beta(b_1) = \beta(b_2) = x \in \ker w$. Then, by commutativity and exacntess $v(b_1), v(b_2) \in \operatorname{im} \alpha'$. In particular $\beta(b_1 - b_2) = x - x = 0$. By exactness there exists $a \in A$ s.t. $\alpha(a) = b_1 - b_2$. Then by commutativity of the diagram

$$v(b_1 - b_2) = v(\alpha(a)) = \alpha'(u(a)). \tag{2.112}$$

In particular, being α' injective, $v(b_1)$ and $v(b_2)$ differ by an element in $\operatorname{im} u$, i.e. δ is well defined.

4. The sequence is exact:

Let's start by checking exactness at $\ker v$. Clearly $\underline{\beta} \circ \underline{\alpha} = 0$, they are just the restrictions. Let $x \in \ker \underline{\beta}$, then there exists $a \in A$ s.t. $\alpha(a) = x$. Since $x \in \ker v$, $0 = v(x) = v(\alpha(a)) = \alpha'(u(a))$, By injectivity of α' we obtain that $u(a) = 0$, hence $a \in \ker u$. In other words $x \in \operatorname{im} \underline{\alpha}$.

Let's now check exactness at $\operatorname{coker} v$. Again the composition $\overline{\beta'} \circ \overline{\alpha'}$ is trivial, since it is induced by the trivial composition $\beta' \circ \alpha' = 0$. Consider now $\overline{b'} \in \ker \overline{\beta'}$, i.e.

$$\overline{\beta'}\left(\overline{b'}\right) = \overline{\beta'(b')} = 0 \in \operatorname{coker} w. \tag{2.113}$$

In other words $\beta'(b') \in \operatorname{im} w$, i.e. there exists $c \in C$ s.t. $w(c) = \beta'(b')$. By surjectivity of β there is $b \in B$ s.t. $\beta(b) = c$. Then

$$\beta'(v(b)) = w(c) = \beta'(b'). \tag{2.114}$$

Then $b' - v(b) \in \ker \beta' = \operatorname{im} \alpha'$, i.e. there exists $a' \in A'$ s.t. $\alpha'(a') = b' - v(b)$. Then

$$\overline{\alpha'}(\overline{a'} = \overline{\alpha'(a')} = \overline{b' - v(b)} = \overline{b'} \in \operatorname{coker} v. \tag{2.115}$$

Then $\overline{b'} \in \operatorname{im} \overline{\alpha'}$.

Exactness at $\ker w$: Let $c \in \ker \delta$, i.e. $\delta(c) = 0$. By definition $\delta(c) = \alpha'^{-1}(v(b)) =: a'$ in the quotient. In particular it is zero iff $a' \in \operatorname{im}(u)$, i.e. $a' = u(a)$ for $a \in A$. Then $v(\alpha(a)) = v(b)$, i.e. $\alpha(a) - b \in \ker v$. Then $\underline{\beta}(b - \alpha(a)) = \beta(b) - \beta(\alpha(a)) = \beta(b) = c$.

$\delta \circ \underline{\beta} = 0$: in fact $x \in \ker v$ clearly implies $v(x) = 0$, and $\delta(\underline{\beta}(x)) = 0$ by definition of δ.

Exactness at $\operatorname{coker} u$: Let $\overline{a'} \in \ker \overline{\alpha'}$, then $\alpha'(a') \in \operatorname{im} v$, i.e. there is $b \in B$ s.t. $v(b) = \alpha'(a')$. By commutativity of the diagram (and exactness) we obtain that $\beta(b) \in \ker w$, since

$$w(\beta(b)) = \beta'(v(b)) = \beta'(\alpha'(a')) = 0. \tag{2.116}$$

And $\beta(b)$ gets mapped, by δ, to $\overline{a'}$. In fact $\delta(\beta(b)) = \overline{v(b)} = \overline{a'}$. ■

Remark 2.42 If α is injective and β' is surjective, and we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array} , \tag{2.117}$$

then we obtain the exact sequence:

$$0 \longrightarrow \ker u \xrightarrow{\alpha} \ker v \xrightarrow{\beta} \ker w \xrightarrow{\delta} \operatorname{coker} u \xrightarrow{\overline{\alpha'}} \operatorname{coker} v \xrightarrow{\overline{\beta'}} \operatorname{coker} w \longrightarrow 0 , \tag{2.118}$$

Theorem 2.43 (Freyd-Mitchell)**.** *Let* A *be a small abelian category. Then there exists a ring* R *and a fully faithful exact functor*

$$\iota \colon \mathsf{A} \longrightarrow R\text{-}\mathsf{Mod}. \tag{2.119}$$

Remark 2.44 Notice that any fully faithful and exact functor reflects exactness. This allows us to check exactness of sequences in any abelian category by using diagram chasing in an appropriate category of modules.

Corollary 2.45. *Snake lemma holds also for an arbitrary abelian category* A*.*

Proof. One needs to apply Freyd-Mitchell to the full subcategory of A given by the only objects of A appearing in the snake lemma diagram.

By exactness of ι the image of the diagram satisfies the hypothesis for snake lemma in R-Mod, hence we can apply it, obtain our result, and come back to A. Notice that we need to recall the last remark in order to come back to A. ∎

2.4 Functors between additive categories

Definition 2.46: Additive functor.
Let C, D be (pre)additive categories. A functor $F \colon \mathsf{C} \to \mathsf{D}$ is said to be *additive* iff for any $A, B \in \mathrm{Ob}\,(\mathsf{C})$, the map

$$F_{A,B} \colon \mathrm{Hom}_{\mathsf{C}}\,(A, B) \to \mathrm{Hom}_{\mathsf{D}}\,(FA, FB) \tag{2.120}$$

is a morphism of abelian group.

Remark 2.47 It would be natural to define an additive functor

$$F \colon \mathsf{C} \longrightarrow \mathsf{D} \tag{2.121}$$

as simply a functor which preserves finite products, (since finite products and coproducts coincide and are the only required structure in an additive category). In fact such two definitions coincide in the case of an additive category. Not in a preadditive category though.

Remark 2.48 In the next proof we will use the following fact, true in any preadditive category C: if $T \in \mathrm{Ob}\,(\mathsf{C})$ is a final object, then for any other $X \in \mathrm{Ob}\,(\mathsf{C})$, we have

$$X \times T \simeq X. \tag{2.122}$$

Lemma 2.49. *Let* $F \colon \mathsf{C} \to \mathsf{D}$ *be a functor between additive categories. TFAE:*

1. F *is additive (in the sense of the definition).*

2. F *preserves finite products (in particular the final object).*

3. F *preserves the product of any pair of objects.*

4. F *preserves finite coproducts (in particular the initial object).*

5. F *preserves the coproduct of any pair of objects.*

Proof.

2 $\iff$ 3: one direction is obvious, let's prove the inverse. We want to procede by induction on the number of objects in the product, in case $n = 3$ we use the associativity in the following way and generalize to arbitrary n:

$$\begin{aligned} F(A_1 \times A_2 \times A_3) &= F((A_1 \times A_2) \times A_3) = F(A_1 \times A_2) \times F(A_3) && (2.123)\\ &= F(A_1) \times F(A_2) \times F(A_3). && (2.124) \end{aligned}$$

Moreover we need to prove that F sends the empty product into the empty product, i.e. $F(0) = 0$: By definition of product we have

$$\mathrm{Hom}_{\mathsf{C}}\left(D, F(0) \times F(A)\right) \simeq \mathrm{Hom}_{\mathsf{C}}\left(D, F(0)\right) \times \mathrm{Hom}_{\mathsf{C}}\left(D, F(A)\right). \tag{2.125}$$

Then, from the above remark we obtain that

$$F(0 \times A) = F(A). \tag{2.126}$$

Then in the above we must have

$$\mathrm{Hom}_{\mathsf{C}}\left(D, F(0)\right) = \{*\} \tag{2.127}$$

for all $D \in \mathrm{Ob}\left(\mathsf{D}\right)$, i.e. $F(0)$ is a final (hence zero in an additive category) object.

3 $\iff$ 5: By additivity of C and D we obtain

$$F(A \coprod B) \simeq F(A \times B) \simeq F(A) \times F(B) \simeq F(A) \coprod F(B). \tag{2.128}$$

We also need to show that $F(i_A) = i_{F(A)}$. In fact we have $i_A := (id_A, 0)$, hence $F(i_A) = (id_{F(A)}, F(0)) = i_{F(A)}$, since F preserves terminal objects. Then $(F(A \coprod B), F(i_A), F(i_B))$ is a coproduct in D. The converse is the same, inverting product and coproduct.

2 $\implies$ 1: Let $A, B \in \mathrm{Ob}\left(\mathsf{C}\right)$. Consider the zero map

$$A \longrightarrow 0_{\mathsf{C}} \longrightarrow B\ . \tag{2.129}$$

Then F sends it into

$$F(A) \longrightarrow 0_{\mathsf{D}} \longrightarrow F(B)\ , \tag{2.130}$$

where we used the property we proved above, i.e. that a functor F preserving finite products sends zero objects into zero objects. In other words F preserves the zero map.

Let's now show that it is compatible with addition. In fact we have seen, in lemma 2.12, that in additive categories we have

$$f + g = \delta_B \circ (f, g), \tag{2.131}$$

where the above represents the diagram

$$f+g : A \xrightarrow{(fg)} B \times B =\!=\!=\!= B \coprod B \xrightarrow{\delta_B} B \ . \tag{2.132}$$

As shown before F preserves both finite products and coproducts, then

$$F(f+g) : F(A) \xrightarrow{(F(f)F(g))} F(B) \times F(B) =\!=\!=\!= F(B) \coprod F(B) \xrightarrow{F(\delta_B)} F(B) \ . \tag{2.133}$$

Here we need to notice that $F(\delta_B) = \delta_{F(B)}$, which is also a consequence of the fact that F preserves finite coproducts. Then, by lemma 2.12 again, we obtain $F(f+g) = F(f) + F(g)$.

1 $\implies$ 3: Consider $A, B \in \mathrm{Ob}\,(\mathsf{C})$, then we know

$$id_{A\times B} = i_A \circ p_A + i_B \circ p_B. \tag{2.134}$$

Being F an additive functor (preserves identity, composition and group operation, i.e. sum) we obtain that

$$id_{F(A\times B)} = F(id_{A\times B}) = F(i_A) \circ F(p_A) + F(i_B) \circ F(p_B). \tag{2.135}$$

Given any $D \in \mathrm{Ob}\,(\mathsf{D})$ and a pair of maps $f\colon D \to F(A)$ and $g\colon D \to F(B)$, we want to prove that there is a unique $h\colon D \to F(A \times B)$ making the diagram commute

$$\begin{tikzcd} & F(A) \arrow[d, shift left, "F(i_A)"] \\ D \arrow[ur, "f"] \arrow[r, dashed, "\exists ! h"] \arrow[dr, "g"'] & F(A\times B) \arrow[u, shift left, "F(p_A)"] \arrow[d, shift left, "F(p_B)"] \\ & F(B) \arrow[u, shift left, "F(i_B)"] \end{tikzcd} \ . \tag{2.136}$$

Let's start by showing uniqueness, assuming it exists. Clearly

$$\begin{aligned} h = id_{F(A\times B)} \circ h &= (F(i_A) \circ F(p_A) + F(i_B) \circ F(p_B)) \circ h \\ &= (F(i_A) \circ F(p_A) \circ h) + (F(i_B) \circ F(p_B) \circ h) \\ &= F(i_A) \circ f + F(i_B) \circ g, \end{aligned} \tag{2.137}$$

where the last equality is due to the commutativity of the diagram. Then h is uniquely determined by f and g, hence it is clearly unique.

Let's now check that the above formula satisfies the conditions. If

$$h := F(i_A) \circ f + F(i_B) \circ g. \tag{2.138}$$

We need to show that the triangles commute, i.e.

$$F(p_A) \circ h = F(\underbrace{p_A \circ i_A}_{id_A}) \circ f + F(\underbrace{p_A \circ i_B}_{0}) \circ g = f + 0 = f. \tag{2.139}$$

Analogously that

$$F(p_B) \circ h = F(\underbrace{p_B \circ i_A}_{0}) \circ f + F(\underbrace{p_B \circ i_B}_{id_B}) \circ g = g + 0 = g. \tag{2.140}$$

Then $(F(A \times B), F(p_A), F(p_B))$ is a product of $F(A)$ and $F(B)$. ■

Definition 2.50 Let C and D be *abelian* categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor.

1. We say that F is *left exact* iff, for any exact sequence
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \ . \tag{2.141}$$
in C, the image sequence is exact in D
$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \ . \tag{2.142}$$

2. We say that F is *right exact* iff, for any exact sequence
$$A \longrightarrow B \longrightarrow C \longrightarrow 0 \ . \tag{2.143}$$
in C, the image sequence is exact in D
$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \ . \tag{2.144}$$

3. We say that F is *exact* iff it is both left and right exact. In other words iff given any exact sequence
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \ . \tag{2.145}$$
in C, the image sequence is exact in D
$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \ . \tag{2.146}$$

Lemma 2.51. *Let $F\colon \mathsf{C} \to \mathsf{D}$ be a left exact functor between abelian categories, then F is additive.*

Proof. It's enough to show that F preserves the coproduct of two objects, by lemma 2.49. Let's consider the split exact sequence in C:
$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \longrightarrow 0 \ . \tag{2.147}$$
Then, since F is left exact, we obtain the exact sequence in D:
$$0 \longrightarrow F(A) \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(p_B)} F(B) \ . \tag{2.148}$$
Since the first sequence is split, we know that there is the section $i_B : B \to A \oplus B$, s.t. $p_B \circ i_B = id_B$. This implies that $F(i_B)$ is a section of $F(p_B)$, i.e.
$$F(p_B) \circ F(i_B) = id_B. \tag{2.149}$$
One can easily show that this condition implies that $F(p_B)$ is an epimorphism. Then the following is actually a split exact sequence:
$$0 \longrightarrow F(A) \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(p_B)} F(B) \longrightarrow 0 \ . \tag{2.150}$$
By the characterization of split exact sequences we know we have a commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r, "F(i_A)"] \arrow[d, equal] & F(A \oplus B) \arrow[r, "F(p_B)"] \arrow[d, "\varphi"] & F(B) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F(A) \arrow[r, "i_{F(A)}"] & F(A) \oplus F(A) \arrow[r, "p_{F(B)}"] & F(B) \arrow[r] & 0
\end{tikzcd} \ . \tag{2.151}$$
Since the diagram commutes, φ is an isomorphism, and by definition $\big(F(A) \oplus F(B), i_{F(A)}, i_{F(B)}\big)$ is a coproduct, it easily follows that $(F(A \oplus B), F(i_A), F(i_B))$ is a coproduct of $F(A)$ and $F(B)$. ■

Remark 2.52 For the sake of completeness let's show that

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \longrightarrow 0 \tag{2.152}$$

is exact in C. In fact it is exact on the left iff, for any $M \in \mathrm{Ob}\left(\mathsf{C}\right)$, the following

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}\left(M, A\right) \xrightarrow{i_{A*}} \mathrm{Hom}_{\mathsf{C}}\left(M, A \oplus B\right) \xrightarrow{p_{B*}} \mathrm{Hom}_{\mathsf{C}}\left(M, B\right) \ . \tag{2.153}$$

This is clearly exact in Ab, since by universal property of product we have the following isomorphism:

$$\mathrm{Hom}_{\mathsf{C}}\left(M, A \oplus B\right) \simeq \mathrm{Hom}_{\mathsf{C}}\left(M, A\right) \times \mathrm{Hom}_{\mathsf{C}}\left(M, B\right). \tag{2.154}$$

Finally the whole sequence is exact, since p_B is an epimorphism (it has a section, but can be proven in other ways too).

Remark 2.53 A morphism which admits a right inverse is an epimorphism: Recall that f is an epimorphism iff it is right erasable, i.e. for any pair of composable maps s.t.

$$\alpha \circ f = \beta \circ f \tag{2.155}$$

implies $\alpha = \beta$. But this is obvious, since

$$\alpha = \alpha \circ f \circ s = \beta \circ f \circ s = \beta, \tag{2.156}$$

in which we denoted by s the section (right inverse) of f.

3 Resolutions

3.1 Injective and projective objects

Remark 3.1 Let C be an abelian category, and $M \in \mathrm{Ob}\left(\mathsf{C}\right)$. Then the covariant functors

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}\left(M, -\right) : \mathsf{C} &\longrightarrow \mathsf{Ab} \\ A &\longmapsto \mathrm{Hom}_{\mathsf{C}}\left(M, A\right) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}\left(-, M\right) : \mathsf{C}^{op} &\longrightarrow \mathsf{Ab} \\ A &\longmapsto \mathrm{Hom}_{\mathsf{C}}\left(A, M\right) \end{aligned} \tag{3.2}$$

are both left exact. Both of these are consequences of lemma 2.34.

Though general it is not true, some objects make one of the two above functors (or both) also right exact. These are special objects, and we are going to study them in the following pages:

Definition 3.2: Injective and projective objects.
Let C be an abelian category.

1. An object $I \in \mathrm{Ob}\left(\mathsf{C}\right)$ is *injective* iff

$$\mathrm{Hom}_{\mathsf{C}}\left(-, I\right) : \mathsf{C}^{op} \longrightarrow \mathsf{Ab} \tag{3.3}$$

   is exact.

2. An object $P \in \mathrm{Ob}(\mathsf{C})$ is *projective* iff

$$\mathrm{Hom}_{\mathsf{C}}(P, -) : \mathsf{C} \longrightarrow \mathsf{Ab} \tag{3.4}$$

is exact.

Remark 3.3

- An object $I \in \mathrm{Ob}(\mathsf{C})$ is injective iff, given any monomorphism $i : A \hookrightarrow B$, then the induced map

$$\mathrm{Hom}_{\mathsf{C}}(B, I) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}_{\mathsf{C}}(A, I) \tag{3.5}$$

is surjective. In other words the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\exists\,\tilde{\alpha}"] \\ & I & \end{tikzcd} . \tag{3.6}$$

More explicitly, given any morphismh $\alpha\colon A \to I$, one can lift it to $\tilde{\alpha}\colon B \to I$ s.t. $\alpha = \tilde{\alpha} \circ i$. (Notice that the lift might not be unique).

- An object $P \in \mathrm{Ob}(\mathsf{C})$ is projective iff, given any epimorphism $p : B \twoheadrightarrow C$, then the induced map

$$\mathrm{Hom}_{\mathsf{C}}(P, B) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}_{\mathsf{C}}(P, C) \tag{3.7}$$

is surjective. In other words the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "p"] & C \arrow[r] & 0 \\ & P \arrow[u, "\beta"'] \arrow[ul, dashed, "\exists\,\tilde{\beta}"] & \end{tikzcd} . \tag{3.8}$$

More explicitly, given any morphismh $\beta\colon P \to C$, one can lift it to $\tilde{\beta}\colon P \to B$ s.t. $\beta = p \circ \tilde{\beta}$. (Notice that the lift might not be unique).

3.1.1 Injectives in R-Mod

In the following we will assume R is a commutative ring. Everything works also for non commutative rings, but in this case there are fewer notation problems.

We will denote by R-Mod the category of left R-modules. Clearly this is an *abelian* category.

Lemma 3.4 (Baer). *A module $M \in R$-Mod is injective iff, given an ideal $I \lhd R$, any R-linear map*

$$\alpha\colon I \longrightarrow M \tag{3.9}$$

extends to a morphism $\tilde{\alpha}\colon R \to M$. In other words we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\alpha"] & R \arrow[dl, dashed, "\tilde{\alpha}"] \\ & M & \end{tikzcd} . \tag{3.10}$$

Proof. tk: read again the proof, there will obviously be quite a few things to fix... The direct implication is obvious. Let's prove the converse. Let $A \subset B$ be a submodule of B, then the inclusion is a monomorphism. Consider a map $\alpha\colon A \to M$, we want to extend it to B. Let's consider the set $\mathcal{S}$ of pairs (A', α') s.t. $A' \subset B$ and $\alpha'\colon A' \to M$ coincides with α on A, i.e. $\alpha'|_A = \alpha$. This set is clearly endowed with a partial order: $(A', \alpha') \leq (A'', \alpha'')$ iff $A' \subset A''$ and $\alpha''|_{A'} = \alpha'$.

Consider now T a totally ordered subset of $\mathcal{S}$.

$$T = \{(A_a, \alpha_a)\}_{a \in \mathcal{A}}\,. \tag{3.11}$$

We want to define $A_\infty := \bigcup_{a \in \mathcal{A}} A_a \subset B$ an R-submodule, and

$$\alpha_\infty\colon A_\infty \longrightarrow M \tag{3.12}$$

s.t. $\alpha_\infty|_{A_a} = \alpha_a$ for all a. Then $(A_\infty, \alpha_\infty) \in \mathcal{S}$ is an upper bound for T, and by Zorn's lemma we obtain the existance of a maximal element $(C, \gamma) \in S$. Suppose that $C \neq B$, then we can take $x \in B \setminus C \neq \emptyset$. Let's define now

$$I := \{r \in R \,|\, r \cdot x \in C\} \triangleleft R. \tag{3.13}$$

Let's denote with r_* the multiplication by $r \in R$. Then we can define $\gamma \circ r_*\colon I \to M$. By assumption we can extend this to a map $\psi\colon R \to M$, making the diagram (tk) commute. In particular we can extend these two maps to $C \subsetneq C + R \cdot x \subset B$, by

$$\begin{aligned} \tilde{\gamma}\colon C + R \cdot x &\longrightarrow M \\ c + r \cdot x &\longmapsto \gamma(c) + \psi(r). \end{aligned} \tag{3.14}$$

This map is R-linear, well defined (on the intersection $C \cap R \cdot x$ ψ acts exactly as $\gamma \circ r_*$), and extends γ. Since $C \subsetneq C + R \cdot x$, we have a contradiction with maximality of (C, γ). Then $C = B$. ■

Remark 3.5 An element $r \in R$ is a non-zero-divisor iff the multiplication map

$$\begin{aligned} r_*\colon R &\longrightarrow R \\ x &\longmapsto r \cdot x \end{aligned} \tag{3.15}$$

is injective.

Definition 3.6: Divisible R-Module.
Consider $M \in R\text{-}\mathsf{Mod}$. We say that M is *divisible* iff, for any $r \in R$ non-zero-divisor, the multiplication map

$$\begin{aligned} r_*\colon M &\longrightarrow M \\ x &\longmapsto r \cdot x \end{aligned} \tag{3.16}$$

is surjective.

Proposition 3.7. *Consider $M \in R\text{-}\mathsf{Mod}$.*

1. *If M is an* injective *module, then it is also* divisible*.*
2. *If, moreover, R is a PID, then we have also the converse, i.e. M is* injective *iff it is* divisible*.*

Proof.

1. Consider a non-zero-divisor $r \in R$, let $x \in M$. Then, by injectivity, we can extend the map

$$\begin{aligned} \alpha \colon R &\longrightarrow M \\ r &\longmapsto r \cdot x \end{aligned} \tag{3.17}$$

to a map $\tilde{\alpha}$ making the diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "r_*"] \arrow[d, "\alpha"] & R \arrow[dl, "\tilde{\alpha}"] \\ & M & \end{tikzcd} . \tag{3.18}$$

In other words we know that $\alpha = \tilde{\alpha} \circ r_*$. In particular $x = \alpha(1) = \tilde{\alpha} \circ r_*(1) = r \cdot \tilde{\alpha}(1)$, i.e. M is divisible.

2. Assume now that R is a PID. Let $J \lhd R$ be an ideal, then $J = (r)$ for some $r \in R$. Let's use Baer's criterion: consider a map $\alpha \colon (r) \to M$ any ideal, let's extend this to the whole ring R. Let $x := \alpha(r) \in M$. Since M is divisible, then there exists $y \in M$ s.t. $r \cdot y = x$. We can now define $\tilde{\alpha}(1) := y$, then $\tilde{\alpha}$ is uniquely defined. Moreover it is clear that $\tilde{\alpha}$ extends α, since
$$\tilde{\alpha}(r \cdot 1) = r \cdot \tilde{\alpha}(1) = r \cdot y = x = \alpha(r).$$ ■

Corollary 3.8. *Let $R = \mathbb{Z}$, then in $\mathbb{Z}$-$\mathsf{Mod} \cong \mathsf{Ab}$, we have*

1. *$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective.*
2. *$\mathbb{Z}$ is not injective.*

Remark 3.9 For all $n \in \mathbb{Z}$, the multiplication by n is an iso between $\mathbb{Q}$ and $\mathbb{Q}$, but not between $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$. In fact we have the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{n\cdot} \mathbb{Q}/\mathbb{Z} \longrightarrow 0 . \tag{3.19}$$

In other words $\mathbb{Q}$ is uniquely divisible, i.e. the preimage of x from the multiplication by n is unique (for any n and x). On the other hand $\mathbb{Q}/\mathbb{Z}$ is not uniquely divisible. Instead the preimage of the multiplication by n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Definition 3.10 Let C be an abelian category.

1. We say that C has *enough injectives* iff for every $A \in \mathrm{Ob}(\mathsf{C})$ there is an injective $I \in \mathrm{Ob}(\mathsf{C})$ and a monomorphism
$$0 \longrightarrow A \longrightarrow I . \tag{3.20}$$

2. We say that C has *enough projectives* iff for every $A \in \mathrm{Ob}(\mathsf{C})$ there is a projective $P \in \mathrm{Ob}(\mathsf{C})$ and an epimorphism
$$P \longrightarrow A \longrightarrow 0 . \tag{3.21}$$

Proposition 3.11. Ab *has enough injectives.*

Proof. Let $A \in \mathsf{Ab}$ and $S \subset A$ be a set of generators (not necessarily finite). Then one gets a surjective map

$$\bigoplus_{s\in S} \mathbb{Z}s \xrightarrow{\ p\ } A \longrightarrow 0 \ . \tag{3.22}$$

This is clearly defined as the unique map induced by the inclusions $\mathbb{Z}s \hookrightarrow A$. In general this map is not injective, let's denote by

$$K := \ker p. \tag{3.23}$$

Then, by the first isomorphism theorem

$$\bigoplus_{s\in S} \mathbb{Z}s/K \simeq A. \tag{3.24}$$

Moreover we clearly have an injection (tk: define it explicitly)

$$0 \longrightarrow \bigoplus_{s\in S} \mathbb{Z}s/K \longrightarrow \bigoplus_{s\in S} \mathbb{Q}s/K \tag{3.25}$$

where $\bigoplus_{s\in S} \mathbb{Q}s/K$ is divisible, since it is the direct sum of divisible modules. ∎

Proposition 3.12. *Let R be a* commutative *ring. Then R-*Mod *has enough injectives.*

Proof. We have an adjucntion

$$\mathrm{res} : R\text{-}\mathsf{Mod} \rightleftarrows \mathbb{Z}\text{-}\mathsf{Mod} : \mathrm{Hom}_{\mathbb{Z}}\left(R, -\right) \ . \tag{3.26}$$

Where res is the restricition of scalars. fix $M \in R$-Mod and $A \in \mathbb{Z}$-Mod. Then we can let R act on $\mathrm{Hom}_{\mathbb{Z}}\left(R, A\right)$ giving it a structure of R-module.

In particular the adjunction is given by the following isomorphism

$$\mathrm{Hom}_R\left(M, \mathrm{Hom}_{\mathbb{Z}}\left(R, A\right)\right) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{res}(M), A\right). \tag{3.27}$$

functorial in both variables. Let's now fix $M \in R$-Mod. Clearly $\mathrm{res}(M) \in \mathsf{Ab}$. By adjunction any morphism

$$f \colon \mathrm{res}(M) \longrightarrow I, \tag{3.28}$$

for I an injective abelian group, gets mapped to

$$\begin{aligned} \tilde{f} \colon M &\longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(R, I\right) \\ m &\longmapsto \begin{pmatrix} R \to I \\ r \mapsto f(r \cdot m) \end{pmatrix}. \end{aligned} \tag{3.29}$$

Then $\tilde{f}$ is both R-linear and injective. We just need to show that $\mathrm{Hom}_{\mathbb{Z}}\left(R, I\right)$ is injective, i.e. that

$$\mathrm{Hom}_{\mathsf{C}}\left(-, \mathrm{Hom}_{\mathbb{Z}}\left(R, I\right)\right) \tag{3.30}$$

is exact. By adjunction this is isomorphic (tk: check this part) to the functor

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{res}(-), I\right) : R\text{-}\mathsf{Mod}^{op} \to \mathsf{Ab}. \tag{3.31}$$

In particular

$$\mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{res}(-), I\right) = \mathrm{Hom}_{\mathbb{Z}}\left(-, I\right) \circ \mathrm{res} \tag{3.32}$$

is the composition of two exact functors, hence it is exact. ∎

Proposition 3.13. *Let* C *and* D *be abelian categories. Consider an adjunction* (F, G)*. Then*

1. F *is right exact.*
2. G *is left exact.*
3. *If* F *is exact, then* G *preserves injectives.*

Proof.

1. See the following.
2. Take an exact sequence in D:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \ . \tag{3.33}$$

Consider the image sequence in C:

$$0 \longrightarrow G(A) \longrightarrow G(B) \longrightarrow G(C) \ . \tag{3.34}$$

In order to show that the above is exact it's enough to show that, for any $M \in \mathrm{Ob}\,(\mathsf{C})$, the following

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}\,(M, G(A)) \longrightarrow \mathrm{Hom}_{\mathsf{C}}\,(M, G(B)) \longrightarrow \mathrm{Hom}_{\mathsf{C}}\,(M, G(C)) \tag{3.35}$$

is exact. By adjunction (and functoriality of the isomorphism) the above is isomorphic (as a sequence) to

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{D}}\,(F(M), A) \longrightarrow \mathrm{Hom}_{\mathsf{D}}\,(F(M), B) \longrightarrow \mathrm{Hom}_{\mathsf{D}}\,(F(M), C) \tag{3.36}$$

which is exact, by exacntess of the original sequence.

3. Let $I \in \mathrm{Ob}\,(\mathsf{D})$ injective. We have a functorial isomorphism

$$\mathrm{Hom}_{\mathsf{C}}\,(-, G(I)) \simeq \mathrm{Hom}_{\mathsf{D}}\,(F(-), I)\,. \tag{3.37}$$

The last is exact since both I is injective and F is exact by hypothesis. Then $G(I)$ is injective, i.e. G preserves injectivity. ■

Corollary 3.14. *Let* $f\colon X \to Y$ *be a continuous map of topological spaces. Then* $f_*\colon \mathsf{Sh}(X) \to \mathsf{Sh}\,(Y)$ *preserves injectives.*

Proof. We have the adjunction (f^*, f_*). Since f^* behaves well with stalks it is exact, hence f_* preserves injectives. ■

4 Complexes

In the following let C be an *abelian* category.

Definition 4.1: (Cochain) complex in an abelian category.
A complex in C is a sequence

$$\ldots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \ldots \tag{4.1}$$

s.t. $A^n \in \mathrm{Ob}\,(\mathsf{C})$ and $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$. We will denote the complex by $(A^\bullet, d_A)$.

Definition 4.2: Morphism of complexes.
Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be two complexes in C. A morphism between the complexes,

$$f^\bullet\colon A^\bullet \longrightarrow B^\bullet \tag{4.2}$$

is a family of morphisms

$$f^n : A^n \longrightarrow B^n \tag{4.3}$$

s.t. for any $n \in \mathbb{Z}$ the square commutes

$$\begin{tikzcd} \dots \arrow[r] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\ \dots \arrow[r] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r] & \dots \end{tikzcd} . \tag{4.4}$$

Definition 4.3: Category of complexes in C.
We denote the category whose objects are complexes and morphism are morphism of complexes, as just defined, by $\mathrm{Ch}(\mathsf{C})$ and call it the category of complexes in C.

Exercise 12 Show that the category $\mathrm{Ch}(\mathsf{C})$ is abelian.

Exercise 13

1. In particular, given $f^\bullet\colon A^\bullet \to B^\bullet$ any morphism of complexes, then $\ker f^\bullet$ is the complex

$$\dots \longrightarrow \ker f^{n-1} \xrightarrow{d^{n-1}} \ker f^n \xrightarrow{d^n} \ker f^{n+1} \xrightarrow{d^{n+1}} \dots\ , \tag{4.5}$$

where d^n are the maps induced by d_A^n.

2. Analogously show that the cokernel of f is the complex

$$\dots \longrightarrow \operatorname{coker} f^{n-1} \xrightarrow{d^{n-1}} \operatorname{coker} f^n \xrightarrow{d^n} \operatorname{coker} f^{n+1} \xrightarrow{d^{n+1}} \dots\ , \tag{4.6}$$

where d^n is induced by d_B^n.

Remark 4.4 Recall that condition $d^{n+1} \circ d^n = 0$ induces a map

$$\operatorname{im} d^n \rightarrowtail \ker d^{n+1}\ . \tag{4.7}$$

Definition 4.5: Cohomology of the complex.
Fixed a complex $(A^\bullet, d_A)$, from the above map, for every $n \in \mathbb{Z}$, one denotes by

$$H^n(A^\bullet) := \operatorname{coker}\left(\operatorname{im} d^{n-1} \hookrightarrow \ker d^n\right) =: \ker d^n / \operatorname{im} d^{n-1}. \tag{4.8}$$

(The last notation can be viewed as purely formal, but it recalls the intuition developed in Ab). We call $H^n(A^\bullet)$ the n-th cohomology of $(A^\bullet, d_A)$.

Exercise 14: Functoriality of cohomology. Any morphism of complexes $f^\bullet\colon A^\bullet \to B^\bullet$ induces, for every index n, a morphism at the level of cohomologies:

$$H^n(f^\bullet)\colon H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \tag{4.9}$$

as a morphism in C. Then, for any $n \in \mathbb{Z}$, we have defined a functor

$$\begin{aligned} H^n \colon \mathrm{Ch}(\mathsf{C}) &\longrightarrow \mathsf{C} \\ A^\bullet &\longmapsto H^n(A^\bullet) \\ f^\bullet &\longmapsto H^n(f^\bullet). \end{aligned} \tag{4.10}$$

Moreover this functor is additive.

Lemma 4.6 (ker-coker sequence)**.** *Consider the following in* C*:*

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \ . \tag{4.11}$$

We have an exact sequence in C*:*

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r] & \ker g \circ f \arrow[r] & \ker g \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\ & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \circ f \arrow[r] & \operatorname{coker} g \arrow[r] & 0 \end{tikzcd} \ . \tag{4.12}$$

Proof. From the composition one obtains the diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "g \circ f"] & Y \arrow[r] \arrow[d, "g"] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z \arrow[r, equal] & Z \arrow[r] & 0 & \end{tikzcd} \ . \tag{4.13}$$

Clearly the diagram commutes and has exact rows. Then one can apply the snake lemma. We are only left to prove that

$$\ker\left(\ker f \circ g \to \ker g\right) \simeq \ker f \tag{4.14}$$

in order to conclude the proof. This is left as an exercise to the reader. ∎

Lemma 4.7. *Consider a short exact sequence in* $\mathrm{Ch}(\mathsf{C})$

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0 \ . \tag{4.15}$$

Then there exists, for all $n \in \mathbb{Z}$*, a canonical map*

$$\delta^n \colon H^n\left(C^\bullet\right) \longrightarrow H^{n+1}\left(A^\bullet\right) \tag{4.16}$$

s.t. the following is a long exact sequence in C*:*

$$\begin{tikzcd} \ldots \arrow[r] & H^n\left(A^\bullet\right) \arrow[r] & H^n\left(B^\bullet\right) \arrow[r] & H^n\left(C^\bullet\right) \arrow[dll, "\delta^n", rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\ & H^{n+1}\left(A^\bullet\right) \arrow[r] & H^{n+1}\left(B^\bullet\right) \arrow[r] & H^{n+1}\left(C^\bullet\right) \arrow[r] & \ldots \end{tikzcd} \ . \tag{4.17}$$

Moreover morphisms of short exact sequences of complexes induce morphisms of long exact sequences in C*. More explicitly, given any commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & ('A)^\bullet \arrow[r] & ('B)^\bullet \arrow[r] & ('C)^\bullet \arrow[r] & 0 \end{tikzcd} \tag{4.18}$$

one obtains a commutative diagram, whose arrows are induced by the above:

$$
\begin{tikzcd}
\dots \arrow[r] & H^n\left(A^\bullet\right) \arrow[r] \arrow[d] & H^n\left(B^\bullet\right) \arrow[r] \arrow[d] & H^n\left(C^\bullet\right) \arrow[r, "\delta^n"] \arrow[d] & H^{n+1}\left(A^\bullet\right) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^n\left({}'A^\bullet\right) \arrow[r] & H^n\left({}'B^\bullet\right) \arrow[r] & H^n\left({}'C^\bullet\right) \arrow[r, "{}'\delta^n"'] & H^{n+1}\left({}'A^\bullet\right) \arrow[r] & \dots
\end{tikzcd}
.
$$

Proof. For any $(A^\bullet, d_A) \in \mathrm{Ch}(\mathsf{X})$ we denote by $Z^n(A^\bullet) := \ker d_A^n$, a subobject of A^n, i.e. we have a canonical inclusion

$$Z^n(A^\bullet) \hookrightarrow A^n \ . \tag{4.19}$$

Analogously we define $I^n(A^\bullet) := \operatorname{im} d^{n-1}$. Since we have a complex we also have an inclusion

$$I^n(A^\bullet) \hookrightarrow Z^n(A^\bullet) \hookrightarrow A^n \ . \tag{4.20}$$

Then, by definition, we have the equality

$$H^n\left(A^\bullet\right) = Z^n(A^\bullet)/I^n(A^\bullet). \tag{4.21}$$

In the following, when there will be no possibility of confusion, we will denote the above simply by Z^n and I^n, omitting to specify the complex $A^\bullet$. Notice that we will only prove the first part of the lemma, since the last follows from the functoriality of snake lemma, the main ingredient of this proof.

Since we have a morphism of short exact sequences of complexes, we obtain a morphism of short exact sequences in C for all $n \in \mathbb{Z}$. That induces the following commutative diagram, with exact rows in the middle.

$$
\begin{tikzcd}
0 \arrow[r] & Z^n(A^\bullet) \arrow[r] \arrow[d] & Z^n(B^\bullet) \arrow[r] \arrow[d] & Z^n(C^\bullet) \arrow[d] \arrow[dddll, dashed, rounded corners] & \\
0 \arrow[r] & A^n \arrow[r] \arrow[d, "d_A^n"] & B^n \arrow[r] \arrow[d, "d_B^n"] & C^n \arrow[r] \arrow[d, "d_C^n"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r] \arrow[d] & B^{n+1} \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & 0 \\
 & A^{n+1}/I^{n+1}(A^\bullet) \arrow[r] & B^{n+1}/I^{n+1}(B^\bullet) \arrow[r] & C^{n+1}/I^{n+1}(C^\bullet) \arrow[r] & 0
\end{tikzcd}
. \tag{4.22}
$$

Then we apply snake lemma and obtain the following long exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & Z^n(A^\bullet) \arrow[r] & Z^n(B^\bullet) \arrow[r] & Z^n(C^\bullet) \arrow[dll, rounded corners] \\
A^{n+1}/I^{n+1} \arrow[r] & B^{n+1}/I^{n+1} \arrow[r] & C^{n+1}/I^{n+1} \arrow[r] & 0.
\end{tikzcd}
\tag{4.23}
$$

Since $d_A^{n+1} \circ d_A^n = 0$, we obtain that d_A^n factors through $i^{n+1} \colon Z^{n+1} \hookrightarrow A^{n+1}$. Then we obtain the commutative

$$
\begin{tikzcd}
I^n \arrow[r, hook, "i_A^n"] \arrow[rr, bend left, "0"] & A^n \arrow[r, "d_A^n"] \arrow[d, two heads, "p_n^A"'] \arrow[dr, dashed, "\widetilde{d_A^n}"] & A^{n+1} \arrow[r] & \dots \\
 & A^n/I^n \arrow[r, "\overline{d_A^n}"'] & Z^{n+1}(A^\bullet) \arrow[u, hook, "i^{n+1}"'] &
\end{tikzcd}
. \tag{4.24}
$$

Then, since $i^{n+1} \circ \widetilde{d_A^n} \circ i_A^n = d_A^n \circ i_A^n = 0$ and i^{n+1} is a mono, we obtain that $\widetilde{d_A^n}$ factors through $\operatorname{coker} i_A^n \simeq \operatorname{coker} d_A^{n-1} = A^n/I^n$ (by lemma 2.13, tk: fix the reference). Then, patching together the information from the snake lemma and the above construction, we obtain the following commutative diagram, with exact rows:

$$\begin{tikzcd} & A^n/I^n(A^\bullet) \arrow[r] \arrow[d, "\overline{d_A^n}"] & B^n/I^n(B^\bullet) \arrow[r] \arrow[d, "\overline{d_B^n}"] & C^n/I^n(C^\bullet) \arrow[r] \arrow[d, "\overline{d_C^n}"] & 0 \\ 0 \arrow[r] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(A^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) & \end{tikzcd} . \tag{4.25}$$

Now we claim that

$$\ker \overline{d_A^n} \simeq H^n\left(A^\bullet\right) \tag{4.26}$$

$$\operatorname{coker} \overline{d_A^n} \simeq H^{n+1}\left(A^\bullet\right). \tag{4.27}$$

(This will be proved after the theorem). Applying the snake lemma now grants the desired sequence:

$$\begin{tikzcd} \dots \arrow[r] & H^n\left(A^\bullet\right) \arrow[r] & H^n\left(B^\bullet\right) \arrow[r] & H^n\left(C^\bullet\right) \arrow[dll, rounded corners, "\delta^n"] \\ & H^{n+1}\left(A^\bullet\right) \arrow[r] & H^{n+1}\left(B^\bullet\right) \arrow[r] & H^{n+1}\left(C^\bullet\right) \arrow[r] & \dots \end{tikzcd} . \tag{4.28}$$

Then functoriality follows from functoriality of snake lemma and functoriality of the claim, which is proved thanks to the following lemma. ■

Proof of the claim. We have the commutative diagram (as seen before)

$$\begin{tikzcd} A^n \arrow[rr, "\widetilde{d_A^n}"] \arrow[dr, "p_A^n"'] & & Z^{n+1} \\ & A^n/I^n \arrow[ur, "\overline{d_A^n}"'] & \end{tikzcd} . \tag{4.29}$$

Which means that $\widetilde{d_A^n} = \overline{d_A^n} \circ p_A^n$. Applying the ker-coker sequence lemma to this composition we obtain the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker p_A^n \arrow[r] & \ker \widetilde{d_A^n} \arrow[r] & \ker \overline{d_A^n} \arrow[dll, rounded corners] \\ & \operatorname{coker} p_A^n \arrow[r] & \operatorname{coker} \widetilde{d_A^n} \arrow[r] & \operatorname{coker} \overline{d_A^n} \arrow[r] & 0 \end{tikzcd} . \tag{4.30}$$

Since p_A^n is an epimorphism it has zero cokernel, moreover its kernel, by definition is I^n. Then, substituting in the above we obtain

$$\begin{tikzcd} 0 \arrow[r] & I^n \arrow[r] & \ker \widetilde{d_A^n} \arrow[r] & \ker \overline{d_A^n} \arrow[dll, rounded corners] \\ & 0 \arrow[r] & \operatorname{coker} \widetilde{d_A^n} \arrow[r] & \operatorname{coker} \overline{d_A^n} \arrow[r] & 0 \end{tikzcd} . \tag{4.31}$$

From exactness of the sequence we immediately obtain

$$\operatorname{coker} \overline{d^n} \simeq \operatorname{coker} \widetilde{d^n} \simeq Z^{n+1}/I^{n+1} = H^{n+1}\left(A^\bullet\right). \tag{4.32}$$

Then, since the inclusion $i_A^n\colon Z^n \hookrightarrow A^n$ is injective and $d_A^n = i_A^n \circ \widetilde{d_A^n}$, one obtains

$$Z^n = \ker d_A^n \simeq \ker \widetilde{d^n}. \tag{4.33}$$

Then, again, from exactness of (4.31) one can conclude with:

$$\ker \overline{d^n} \simeq Z^n/I^n = H^n\left(A^\bullet\right). \tag{4.34}$$

■

Definition 4.8: Nullhomotopic morphism.
Consider two cochain complexes $\left(A^\bullet, d_A\right), \left(B^\bullet, d_B\right) \in \mathrm{Ch}(\mathsf{C})$. A morphism $f^\bullet\colon A^\bullet \to B^\bullet$ is said to be *nullhomotopic* iff there exist a family $\{h^n\}_{n\in\mathbb{Z}}$ of morphisms

$$h^n\colon A^n \longrightarrow B^{n-1} \tag{4.35}$$

s.t. for all $n \in \mathbb{Z}$,

$$f^n = h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n. \tag{4.36}$$

In pictures:

$$\begin{tikzcd}
\dots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] \arrow[dl, "h^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] \arrow[dl, "h^{n+1}"] & \dots \\
\dots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"'] & \dots
\end{tikzcd} . \tag{4.37}$$

Proposition 4.9. *If $f^\bullet\colon A^\bullet \to B^\bullet$ is nullhomotopic, then, for any $n \in \mathbb{Z}$, it induces the zero morphism at the level of cohomologies, i.e.*

$$0 = H^n\left(f\right) : H^n\left(A^\bullet\right) \longrightarrow H^n\left(B^\bullet\right). \tag{4.38}$$

Proof. By definition (using Freyd-Mitchell, operating in a category of modules) we have

$$\begin{aligned} H^n\left(f\right) : Z^n(A^\bullet)/I^n(A^\bullet) &\longrightarrow Z^n(B^\bullet)/I^n(B^\bullet) \\ \overline{x} &\longmapsto \overline{f^n(x)}, \end{aligned} \tag{4.39}$$

for $x \in Z^n(A^\bullet)$. By assumption we have the following equality:

$$f^n(x) = h^{n+1} \circ d_A^n(x) + d^{n+1} \circ h^n(x). \tag{4.40}$$

Since $x \in Z^n(A^\bullet) = \ker d_A^n(x)$ the first summand is zero. Then $f^n(x) \in \operatorname{im} d_B^{n+1}$ and

$$0 = \overline{f^n(x)} \in Z^n(B^\bullet)/I^n(B^\bullet). \tag{4.41}$$

■

Definition 4.10: Homotopic morphisms.
Two morphisms $f^\bullet, g^\bullet\colon A^\bullet \to B^\bullet$ are called *homotopic* iff $f^\bullet - g^\bullet$ is nullhomotopic. The family $\left(h^n\right)_{n\in\mathbb{Z}}$ making $f^\bullet - g^\bullet$ nullhomotopic is called *homotopy*. More explicitly we have, for every $n \in \mathbb{Z}$

$$f^n - g^n = h^{n+1} \circ d_A^n + d_B^n \circ h^n. \tag{4.42}$$

Remark 4.11 If $f^\bullet, g^\bullet \colon A^\bullet \to B^\bullet$ are homotopic, then for all $n \in \mathbb{Z}$ we have

$$H^n(f^\bullet) = H^n(g^\bullet) : H^n(A^\bullet) \longrightarrow H^n(B^\bullet). \tag{4.43}$$

This is true, since H^n is an additive functor, hence

$$H^n(f^\bullet - g^\bullet) = H^n(f^\bullet) - H^n(g^\bullet). \tag{4.44}$$

Remark 4.12 Let $f, g \colon X \to Y$ be continuous maps of nice (e.g. CW-complexes) topological spaces. One can define the singular cochain complex $C^\bullet(X, \mathbb{Z})$. Suppose that f and g are homotopic, i.e. there is a continuous map

$$h \colon X \times [0,1] \longrightarrow Y \tag{4.45}$$

s.t. $h|_{X\times\{0\}} = f$ and $h|_{X\times\{1\}} = g$. Then they induce two maps of complexes

$$g^\bullet, f^\bullet : C^\bullet(X, \mathbb{Z}) \longrightarrow C^\bullet(Y, \mathbb{Z}) \tag{4.46}$$

which are homotopic as in the above definition.

4.1 Resolutions

Let C be an *abelian* category.

Definition 4.13: Resolution.
Consider $A \in \mathrm{Ob}(\mathsf{C})$, then a *resolution* of A is an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots \tag{4.47}$$

If, moreover, each I^i is *injective*, then we say that it is an *injective resolution*. More compactly we are going to denote it by $A \to I^\bullet$.

Proposition 4.14. *If C has enough injectives, then any $A \in \mathrm{Ob}(\mathsf{C})$ admits an injective resolution.*

Proof. Since C has enough injectives we have

$$0 \longrightarrow A \longrightarrow I^0 \tag{4.48}$$

for an injective object I^0. Set $I^0/A := \mathrm{coker}\left(A \hookrightarrow I^0\right)$. Again, C has enough injectives, then there is

$$0 \longrightarrow I^0/A \longrightarrow I^1 \tag{4.49}$$

for I^1 injective. We define

$$d^0 \colon I^0 \longrightarrow\!\!\!\!\!\rightarrow I^0/A \rightarrowtail I^1 \ . \tag{4.50}$$

The kernel of the above map is just A, since the second map is injective, hence it coincides with $A = \ker I^0 \twoheadrightarrow I^0/A$. Then, as above, we can choose I^2 injective and

$$0 \longrightarrow \mathrm{coker}\, d^0 \longrightarrow I^2 \ . \tag{4.51}$$

We define again

$$d^1 \colon I^1 \longrightarrow\!\!\!\!\!\rightarrow \mathrm{coker}\, d^0 \rightarrowtail I^2 \ . \tag{4.52}$$

Then, again reasoning as before, we obtain

$$\ker d^1 \simeq \ker\left(\mathrm{coker}\, d^0\right) =: \mathrm{im}\, d^0. \tag{4.53}$$

Then by induction one defines $(I^\bullet, d_I)$ s.t. $\ker d^n \simeq \mathrm{im}\, d^{n-1}$. ■

Remark 4.15: Another point of view.
Given $A \in \mathrm{Ob}\,(\mathsf{C})$, one can view an injective resolution of A as a morphism of complexes

$$A^\bullet \longrightarrow I^\bullet. \tag{4.54}$$

If we consider the complex $A^\bullet$ concentrated in degree 0, i.e.

$$\ldots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \ldots \tag{4.55}$$

$A^0 = A$ and $A^n = 0$ for all $n \neq 0$. Then the morphism of complexes

$$A^\bullet \longrightarrow I^\bullet \tag{4.56}$$

is given by the commutative

$$\begin{tikzcd} \ldots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d^0"'] & I^1 \arrow[r, "d^1"'] & I^2 \arrow[r] & \ldots \end{tikzcd} \tag{4.57}$$

such that have the following isomorphisms:

$$\begin{array}{ccc} A \simeq H^0\left(A^\bullet\right) & \xrightarrow{\sim} & H^0\left(I^\bullet\right) \simeq \ker d^0 \\ 0 = H^n\left(A^\bullet\right) & = & H^n\left(I^\bullet\right) \end{array} . \tag{4.58}$$

By exactness of $I^\bullet$ and triviality of $A^\bullet$ the above mean that $A^\bullet \to I^\bullet$ induces isomorphism for each n at the level of cohomologies. And we give a name to such maps.

Definition 4.16: Quasi-isomorphism.
Consider a morphism of complexes in $\mathrm{Ch}(\mathsf{C})$

$$f\colon X^\bullet \longrightarrow Y^\bullet. \tag{4.59}$$

It is a *quasi-isomorphism* iff, for all $n \in \mathbb{Z}$, it induces an isomorphism at the level of the n-th cohomology:

$$H^n\left(f\right) : H^n\left(X^\bullet\right) \xrightarrow{\sim} H^n\left(Y^\bullet\right) . \tag{4.60}$$

Remark 4.17: Resolution, take 2.
Actually an injective resolution of A is just the data of a quasi-isomorphism between $A^\bullet$ the complex concentrated in 0 defined above, and a complex $I^\bullet$ of injective objects, starting from degree 0. This point of view will allow to generalize the definition later on.

Definition 4.18: Extension of a morphism.
Let $A, B \in \mathsf{C}$, with a morphism $f\colon A \to B$. Consider two resolutions

$$A \longrightarrow I^\bullet \qquad \text{and} \qquad B \longrightarrow J^\bullet . \tag{4.61}$$

An extension of $f\colon A \to B$ is a morphism of complexes

$$f^\bullet : I^\bullet \longrightarrow J^\bullet \tag{4.62}$$

s.t. the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & \dots \\ 0 \arrow[r] & B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \dots \end{tikzcd} . \tag{4.63}$$

In other words the following diagram of morphisms of complexes commute:

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d, "f"'] & I^\bullet \arrow[d, "f^\bullet"] \\ B^\bullet \arrow[r] & J^\bullet \end{tikzcd} . \tag{4.64}$$

Proposition 4.19. *Consider $A, B \in \mathsf{C}$, $A \to I^\bullet$ a resolution of A and $B \to J^\bullet$ an injective resolution of B. Then, for any $f\colon A \to B$, there is an extension $f^\bullet\colon I^\bullet \to J^\bullet$, which is unique up to homotopy.*

Proof.

Uniqueness by homotopy: Suppose one has two extensions $f^\bullet, g^\bullet\colon I^\bullet \to J^\bullet$ of $f\colon A \to B$. Then $f^\bullet - g^\bullet\colon I^\bullet \to J^\bullet$ extends the zero map $f - f = 0\colon A \to B$. Then we only need to show that any extension of the zero map is nullhomotopic. Consider

$$\begin{tikzcd} A \arrow[r] \arrow[d, "0"'] & I^\bullet \arrow[d, "f^\bullet"] \\ B \arrow[r] & J^\bullet \end{tikzcd} . \tag{4.65}$$

In particular, at degree 0, we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "0"'] \arrow[dr, dashed] & I^0 \arrow[d, "f^0"] \\ 0 \arrow[r] & B \arrow[r, "i_B"'] & J^0 \end{tikzcd} . \tag{4.66}$$

By commutativity of the diagram the dashed arrow is the zero map, hence f^0 factorizes through $\operatorname{coker}(A \hookrightarrow I^0) = I^0/A$. This gives rise to the map

$$I^0/A \xrightarrow{\overline{f^0}} J^0 \ . \tag{4.67}$$

Then we recall that, since $A \to I^\bullet$ is a complex, $d_I^0 \circ i_A = 0$, hence also this map factors through I^0/A and we have $\ker d_I^0 \simeq \operatorname{im} i_A = \ker \operatorname{coker} i_A$. But then, applying lemma 4.6, we obtain that the map through which d_I^0 factorizes is a mono, i.e. $I^0/A \hookrightarrow I^1$. Moreover, since J^0 is injective, we obtain the diagram

$$\begin{tikzcd} I^0 \arrow[r, two heads] \arrow[rr, bend left, "d_I^0"] \arrow[dr, "f^0"'] & I^0/A \arrow[r, tail] \arrow[d, "\overline{f^0}"] & I^1 \arrow[dl, dashed, "\exists h^1"] \\ & J^0 & \end{tikzcd} . \tag{4.68}$$

More explicitly this diagram states that $f^0 = h^1 \circ d_I^0$. Suppose now, by strong induction on $n \geq 1$, that there exist $h^n \colon I^n \longrightarrow J^{n-1}$ and $h^{n+1} \colon I^{n+1} \longrightarrow J^n$ s.t.

$$f^n = h^{n+1} \circ d_I^n + d_J^{n-1} \circ h^n. \tag{4.69}$$

The above construction grants this for $n = 0$, with $h^0 = 0$. In fact

$$\begin{tikzcd}
0 \arrow[r] & I^0 \arrow[r, "d_I^0"] \arrow[d, "f^0"] \arrow[dl, "h^0"'] & I^1 \arrow[d, "f^1"] \arrow[dl, "h^1"'] \\
0 \arrow[r] & J^0 \arrow[r] & J^1
\end{tikzcd} . \tag{4.70}$$

By induction we obtain the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & I^{n-1} \arrow[r, "d_I^{n-1}"] \arrow[d, "f^1"] & I^n \arrow[r, "d_I^n"] \arrow[d, "f^n"] \arrow[dl, "h^n"'] & I^{n+1} \arrow[r, "d_I^{n+1}"] \arrow[d, "f^{n+1}"] \arrow[dl, "h^{n+1}"'] & I^{n+2} \arrow[r] \arrow[d, "f^{n+2}"] & \dots \\
\dots \arrow[r] & J^{n-1} \arrow[r, "d_J^{n-1}"'] & J^n \arrow[r, "d_J^n"'] & J^{n+1} \arrow[r, "d_J^{n+1}"'] & J^{n+2} \arrow[r] & \dots
\end{tikzcd} . \tag{4.71}$$

Consider now the following composition:

$$\begin{aligned}
(f^{n+1} - d_I^n \circ h^{n+1}) \circ d_I^n &= (f^{n+1} \circ d_I^n) - d_I^n \circ h^{n+1} \circ d_I^n \\
&= \left(f^{n+1} \circ d_I^n\right) - d_J^n \circ h^{n-1} = d_I^n \circ \left(f^n - h^{n+1} \circ d_I^n\right) \\
&= d_J^n \circ d_J^{n-1} \circ h^n = 0,
\end{aligned} \tag{4.72}$$

where the last equality holds from induction hypothesis. Hence the map $f^{n+1} - d_J^n \circ h^{n+1} = (*)$ factorizes through $\operatorname{coker} d_I^n$, giving rise to the following commutative diagram:

$$\begin{tikzcd}
I^{n+1} \arrow[r] \arrow[rr, bend left, "(*)"] & \operatorname{coker} d_I^n \arrow[r] & J^{n+1}
\end{tikzcd} . \tag{4.73}$$

Again, since d_I^{n+1} factors through $\operatorname{coker} d_I^n$, using lemma 4.6, we obtain that the factorization is injective. Then combining it with the above diagram we get

$$\begin{tikzcd}
I^{n+1} \arrow[r] \arrow[d, "(*)"'] & \operatorname{coker} d_I^n \arrow[r, tail] \arrow[dl] & I^{n+2} \arrow[dll, dashed, bend left, "\exists h^{n+2}"] \\
J^{n+1} & &
\end{tikzcd} . \tag{4.74}$$

And h^{n+2} exists since J^{n+1} is injective. Finally commutativity of the diagram grants that

$$f^{n+1} - d_J^n \circ h^{n+1} = h^{n+2} \circ d_I^{n+1}. \tag{4.75}$$

Existance of the extension: By hypothesis we are given the diagram, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"'] \arrow[dr, dashed] & I^0 \\
0 \arrow[r] & B \arrow[r, "i_B"] & J^0
\end{tikzcd} . \tag{4.76}$$

Since J^0 is injective and i_A a mono, there exists a map $f^0 \colon I^0 \to J^0$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & I^0 \arrow[d, "f^0"] \\ B \arrow[r, "i_B"] & J^0 \end{tikzcd} . \tag{4.77}$$

As proved before d_I^0 factors through $\operatorname{coker} i_A$, via a monomorphism. Moreover we can see that $(\operatorname{coker} i_B \circ f^0) \circ i_A = \operatorname{coker} i_B \circ i_B \circ f = 0$, hence f^0 factorizes via $\overline{f^0}$ through $\operatorname{coker} i_A$. This gives rise to the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"] & I^0/A \arrow[r, tail] \arrow[d, "\overline{f^0}"] \arrow[dr, dashed] & I^1 \\ B \arrow[r, "i_B"] & J^0 \arrow[r] & J^0/B \arrow[r, tail] & J^1 \end{tikzcd} . \tag{4.78}$$

Again, by injectivity of J^1 one obtains the map $f^1 \colon I^1 \to J^1$. Notice that commutativity of the whole diagram, in particular, grants commutativity of

$$\begin{tikzcd} I^0 \arrow[r, "d_I^0"] \arrow[d, "f^0"'] & I^1 \arrow[d, "f^1"] \\ J^0 \arrow[r, "d_J^0"'] & J^1 \end{tikzcd} . \tag{4.79}$$

Now we can procede by induction on $n \geq 1$. Assume that we have constructed f^{n-1} and f^n, then reasoning similarly to before we obtain the commutative diagram

$$\begin{tikzcd} I^{n-1} \arrow[r, "i_A"] \arrow[d, "f^{n-1}"] & I^n \arrow[r] \arrow[d, "f^n"] & \operatorname{coker} d_I^{n-1} \arrow[r, tail] \arrow[d, "\overline{f^n}"] \arrow[dr, dashed] & I^{n+1} \arrow[d, dashed, "f^{n+1}"] \\ J^{n-1} \arrow[r, "i_B"] & J^n \arrow[r] & \operatorname{coker} d_J^{n-1} \arrow[r, tail] & J^{n+1} \end{tikzcd} . \tag{4.80}$$

Again the arrow f^{n+1} exists by injectivity of J^{n+1}, and the following square commutes:

$$\begin{tikzcd} I^n \arrow[r, "d_I^n"] \arrow[d, "f^n"'] & I^{n+1} \arrow[d, "f^{n+1}"] \\ J^n \arrow[r, "d_J^n"'] & J^{n+1} \end{tikzcd} . \tag{4.81}$$

Then this construction gives rise to a morphism of complexes $f^\bullet \colon I^\bullet \to J^\bullet$, and by construction it is an extension of $f \colon A \to B$. ■

5 Derived functors

In the following assume that C and D are abelian categories s.t. C has enough injectives. We now want to define the family of right derived functors associated to a left exact functor $F \colon \mathsf{C} \to \mathsf{D}$.

Definition 5.1: Right derived functors.
Let $F \colon \mathsf{C} \to \mathsf{D}$ be a left exact functors, we define the family of morphisms

$$R^n F \colon \mathsf{C} \longrightarrow \mathsf{D} \tag{5.1}$$

in the following way: fix $A \in \mathrm{Ob}\,(\mathsf{C})$, chose any injective resolution

$$A \longrightarrow I^\bullet\ . \tag{5.2}$$

Then applying F to $I^\bullet$ one obtains the complex (which, in general, is no longer exact)

$$0 \longrightarrow F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} \dots \tag{5.3}$$

denoted by $F(I^\bullet) \in \mathrm{Ch}(\mathsf{D})$. One defines

$$R^n F(A) := H^n\left(F(I^\bullet)\right). \tag{5.4}$$

It is not obvious that this is a good definition, it could depend on the chosen injective resolution, but we will shortly prove this is not the case. Before doing so a couple of important remarks:

Remark 5.2 From this definition we immediately obtain that $R^0F \simeq F$. In fact F is left exact, hence it preserves kernels. In particular, since $I^\bullet$ is zero at any negative degree, and since $A \to I^\bullet$ is a quasi isomorphism, we obtain that

$$\ker d_I^0 \simeq H^0\left(I^\bullet\right) \simeq H^0\left(A\right) \simeq A. \tag{5.5}$$

Applying the definition of right derived functor we obtain

$$R^0F\left(A\right) := H^0\left(F(I^\bullet)\right) \simeq \ker(Fd_I^0) \simeq F(\ker d_I^0) \simeq F(A). \tag{5.6}$$

Remark 5.3: Case of exact functor.
Assume F is exact. Then $R^nF = 0$ for all $n > 0$.

Proof. Consider the injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots\ . \tag{5.7}$$

If we apply the functor F we obtain an exact sequence in D:

$$0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow \dots\ . \tag{5.8}$$

Then computing the cohomology we obtain that, for all $n > 0$, $H^n(F(I^\bullet)) = 0$. ■

These above remarks will actually be useful later on, since we want to prove that the family of morphism $\left(R^nF\right)_{n\in\mathbb{N}}$ is universal with respect to the families with $R^0F \simeq F$. Morever they give us the intuition that the right derived functors $(R^nF)_{n\in\mathbb{N}}$ measure how far is F from being exact.

Let's now prove good definition and functoriality of the construction of right derived functors:

Proposition 5.4 (Good definition of derived functors)**.** *Assume that* C *and* D *are abelian categories, and* C *has enough injectives. Consider a left exact functor* $F\colon \mathsf{C} \to \mathsf{D}$*.*

1. *Consider* $A \in \mathrm{Ob}\,(\mathsf{C})$ *and two injective resolutions*

$$0 \longrightarrow A \longrightarrow I^\bullet \qquad \textit{and} \qquad 0 \longrightarrow A \longrightarrow J^\bullet \tag{5.9}$$

of A*. Then for all* $n \geq 0$ *there is a canonical isomorphism*

$$H^n(F(I^\bullet)) \simeq H^n(F(J^\bullet)). \tag{5.10}$$

2. *Given injective resolutions of $A, B \in \mathrm{Ob}(\mathsf{X})$*

$$0 \longrightarrow A \longrightarrow I^\bullet \qquad \textit{and} \qquad 0 \longrightarrow B \longrightarrow J^\bullet \tag{5.11}$$

and a morphism

$$f\colon A \longrightarrow B, \tag{5.12}$$

we can construct a canonical morphism

$$H^n\left(F(f^\bullet)\right) : H^n(F(I^\bullet)) \longrightarrow H^n(F(J^\bullet)) \ . \tag{5.13}$$

3. *Given composable morphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{5.14}$$

and respective resolutions $A \to I^\bullet, B \to J^\bullet$ and $C \to K^\bullet$, then the traingle commutes:

$$\begin{tikzcd} H^n\left(F(I^\bullet)\right) \arrow[rr, "H^n(F(g^\bullet \circ f^\bullet))"] \arrow[dr, "H^n(F(f^\bullet))"'] & & H^n\left(F(K^\bullet)\right) \\ & H^n\left(F(J^\bullet)\right) \arrow[ur, "H^n(F(g^\bullet))"'] & \end{tikzcd} . \tag{5.15}$$

Proof.

2. Given the two resolutions, by proposition 4.19. there is a lift

$$f^\bullet\colon I^\bullet \longrightarrow J^\bullet \tag{5.16}$$

extending $f\colon A \to B$, unique up to homotopy. This induces, for all $n \in \mathbb{N}$ a map

$$H^n(F(f^\bullet))\colon H^n(F(I^\bullet)) \longrightarrow H^n(F(J^\bullet)) \ . \tag{5.17}$$

Uniqueness up to homotopy means that any other lift of f

$$g^\bullet\colon I^\bullet \longrightarrow J^\bullet \tag{5.18}$$

satisfies $f \sim g$. More explicitly there exists a homotopy $\{h^n\}_{n\in\mathbb{N}}$ s.t.

$$f^n - g^n = d^{n-1} \circ h^n + h^{n+1} \circ d^n. \tag{5.19}$$

Since F is left exact it is also additive, then it preserves sums and compositions, i.e.

$$F(f^n) - F(g^n) = F(d^{n-1}) \circ F(h^n) + F(h^{n+1}) \circ F(d^n). \tag{5.20}$$

Which means that $F(f^\bullet) \sim F(g^\bullet)$, hence F sends homotopic morphism of complexes to homotopic morphisms of complexes. Then $H^n(F(f)) = H^n(F(g))$ does not depend on the chosen extension of f.

3. Let $f^\bullet$ and $g^\bullet$ be lifts of f and g respectively. Then $g^\bullet \circ f^\bullet$ clearly is an extension of $g \circ f$. Since H^n, for any $n \geq 0$, and F are all functors, we have

$$H^n\left(F(g^\bullet \circ f^\bullet)\right) = H^n\left(F(g^\bullet) \circ F(f^\bullet)\right) = H^n\left(F(g^\bullet)\right) \circ H^n\left(\circ F(f^\bullet)\right). \tag{5.21}$$

1. Given the two resolutions $A \to I^\bullet$ and $A \to B^\bullet$, proposition 4.19 grants that we can lift id_A to two maps

$$\begin{aligned} f^\bullet \colon I^\bullet &\longrightarrow J^\bullet \\ g^\bullet \colon J^\bullet &\longrightarrow I^\bullet . \end{aligned} \tag{5.22}$$

Moreover, evidently, $id_{I^\bullet} \colon I^\bullet \to I^\bullet$ is and extension of id_A. By uniqueness up to homotopy, still from proposition 4.19, we obtain

$$H^n\left(F(g^\bullet \circ f^\bullet)\right) = H^n\left(F(id_{I^\bullet})\right). \tag{5.23}$$

Then functoriality grants

$$H^n\left(F(g^\bullet)\right) \circ H^n\left(F(f^\bullet)\right) = id_{H^n(F(I^\bullet))}. \tag{5.24}$$

Arguing in the same way for $f^\bullet \circ g^\bullet$ one obtains that $H^n\left(F(f^\bullet)\right)$ and $H^n\left(F(g^\bullet)\right)$ are isomorphism for all $n \geq 0$, hence

$$H^n\left(F(I^\bullet)\right) \simeq H^n\left(F(J^\bullet)\right). \tag{5.25}$$

■

Remark 5.5 This proposition grants that $R^n F$ is well defined, and moreover it is a functor. Actually this functor has quite a few interesting properties: let's investigate them.

tk: decide whether to repeat the now correct definition of right derived functor.

Proposition 5.6. *Let $F \colon \mathsf{C} \longrightarrow \mathsf{D}$ be a left exact functor between abelian categories, and let C have enough injectives. Then*

1. *$R^n F \colon \mathsf{C} \longrightarrow \mathsf{D}$ is additive for all $n \geq 0$.*
2. *If $I \in \mathrm{Ob}(\mathsf{C})$ is injective, then for all $n > 0$*

$$R^n F(I) = 0. \tag{5.26}$$

3. *Consider a short exact sequence in C:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \ . \tag{5.27}$$

Then there exist canonical maps $\delta^n \colon R^n F(C) \to R^{n+1} F(A)$ for all n s.t. the following is an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & R^0F(A) \arrow[r] & R^0F(B) \arrow[r] & R^0F(C) \arrow[dll, "\delta^0"] \\ & R^1F(A) \arrow[r] & R^1F(B) \arrow[r] & R^1F(C) \arrow[dll, "\delta^1"] \\ & R^2F(A) \arrow[r] & R^2F(B) \arrow[r] & R^2F(C) \arrow[r] & \dots \end{tikzcd} \ . \tag{5.28}$$

Moreover the above construction is functorial, i.e. given any morphism of short exact sequences in C:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd} \ , \tag{5.29}$$

one can define a morphism of long exact sequences:

$$\begin{tikzcd}
R^0F(A) \arrow[r] \arrow[d] & R^0F(B) \arrow[r] \arrow[d] & R^0F(C) \arrow[r, "\delta^0"] \arrow[d] & R^1F(A) \arrow[r] \arrow[d] & \dots \\
R^0F(A') \arrow[r] & R^0F(B') \arrow[r] & R^0F(C') \arrow[r, "\delta'^0"] & R^1F(A') \arrow[r] & \dots
\end{tikzcd} .$$

In particular the following square commutes for all $n \geq 0$*:*

$$\begin{tikzcd}
R^nF(C) \arrow[r, "\delta^n"] \arrow[d] & R^{n+1}F(A) \arrow[d] \\
R^nF(C') \arrow[r, "\delta'^n"'] & R^{n+1}F(A')
\end{tikzcd} . \tag{5.30}$$

Proof.

1. It is enough to prove that R^nF preserves preserves coproduct. To do so one need to check that given two resolutions

$$A \longrightarrow I^\bullet \qquad \text{and} \qquad B \longrightarrow J^\bullet . \tag{5.31}$$

One can combine them to obtain the following resolution of $A \oplus B$:

$$A \oplus B \longrightarrow I^\bullet \oplus J^\bullet . \tag{5.32}$$

In particular the direct sum of two injective objects is still injective. More generally the direct sum of two functors is exact iff each of the two is. Moreover direct sum is an exact functor, hence the above is an exact sequence. Moreover, by left exactness of F, one obtains

$$F(I^\bullet \oplus J^\bullet) \simeq F(I^\bullet) \oplus F(J^\bullet). \tag{5.33}$$

Then, since H^n is an additive functor (it is defined as a cokernel), one has:

$$R^nF(A \oplus B) \simeq H^n(F(I^\bullet) \oplus F(J^\bullet)) \tag{5.34}$$
$$\simeq H^n(F(I^\bullet)) \oplus H^n(F(J^\bullet)) \simeq R^nF(A) \oplus R^nF(B) . \tag{5.35}$$

2. Notice that the following is an injective resolution:

$$0 \longrightarrow I \xrightarrow{\sim} I \longrightarrow 0 \longrightarrow \dots . \tag{5.36}$$

More explicitly we have

$$I^\bullet := I \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots . \tag{5.37}$$

Then, for all $n > 0$, $F(I^\bullet)^n = 0$, hence

$$R^nF(I) := H^n(F(I^\bullet)) = 0. \tag{5.38}$$

In fact $F(I)$ appears at degree 0.

3. Consider the following injective resolutions for A and C:

$$A \longrightarrow I^\bullet \qquad\qquad \text{and} \qquad\qquad C \longrightarrow K^\bullet \ . \tag{5.39}$$

We now want to combine them into an injective resolution for B. Let's define $J^n := I^n \oplus K^n$. This is an injective object as remarked before. Moreover one can prove that in an additive category, the structural morphism for a product are epimorphisms, whereas the structural morphisms for coproducts are monomorphisms. Then, since direct sums in additive categories are both products and coproducts, one obtains, for all $n \geq 0$, the exact sequence:

$$0 \longrightarrow I^n \longrightarrow J^n \longrightarrow K^n \longrightarrow 0. \tag{5.40}$$

Then let's start at degree zero. We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl, "\alpha^0"'] \arrow[d, dashed] \arrow[dr, "\beta^0"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & J^0 \arrow[r] & K^0 \arrow[r] & 0
\end{tikzcd} \cdot \tag{5.41}$$

Here α^0 exists since I^0 is injective and $A \to B$ is a mono. The map β^0, instead, is just the composite $B \to C \to K^0$. Then the dashed arrow is $\left(\alpha^0, \beta^0\right)$, defined viewing $J^0 = I^0 \oplus K^0$ as a product. Notice that the above diagram has both exact rows and columns. This implies that we can apply the snake lemma and obtain

$$0 \longrightarrow 0 \longrightarrow \ker\left(\alpha^0, \beta^0\right) \longrightarrow 0 \longrightarrow I^0/A \longrightarrow J^0/B \longrightarrow K^0/C \longrightarrow 0. \tag{5.42}$$

As a consequence we obtain that $B \to J^0$ is a monomorphism, and moreover we have the short exact sequence

$$0 \longrightarrow I^0/A \longrightarrow J^0/B \longrightarrow K^0/C \longrightarrow 0. \tag{5.43}$$

Now, recalling the construction in proposition 4.19, we see that d_I^0 can be decomposed in $I^0 \twoheadrightarrow I^0/A \hookrightarrow I^1$. Analogously for d_K^0 and d_I^n, d_K^n, for all n, substituting I^0/A with coker d^{n-1}. Instead, for d_J^n we will define it to be such composition.

Then, combining the information we have obtained so far, we can construct the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & I^0/A \arrow[r] \arrow[d] & J^0/B \arrow[r] \arrow[dl, "\alpha^1"'] \arrow[d, dashed] \arrow[dr, "\beta^1"] & K^0/C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] & J^1 \arrow[r] & K^1 \arrow[r] & 0
\end{tikzcd} \cdot \tag{5.44}$$

Again α^1 exists since I^1 is injective, and the dashed arrow is constructed by universal property of products. Then, applying the snake lemma to the diagram, from exactness of the columns, we obtain that $J^0/B \to J^1$ is injective and

$$0 \longrightarrow \operatorname{coker}\left(I^0/A \to I^1\right) \longrightarrow \operatorname{coker}\left(J^0/B \to J^1\right) \longrightarrow \operatorname{coker}\left(K^0/C \to K^1\right) \longrightarrow 0 \ . \tag{5.45}$$

Then we can put $d_J^0 := J^0 \twoheadrightarrow J^0/B \hookrightarrow J^1$. Now it is important to notice that $\operatorname{coker}(\beta \circ \nu) \simeq \operatorname{coker} \nu$ for any composition of β epi and ν mono, and we are going to prove it right after this theorem. Then, in our situation, we obtain $\operatorname{coker}\left(I^0/A \to I^1\right) \simeq \operatorname{coker} d_I^0$, and analogously for all sequences. This translates in the exact sequence

$$0 \longrightarrow \operatorname{coker}\left(d_I^0\right) \longrightarrow \operatorname{coker}\left(d_J^0\right) \longrightarrow \operatorname{coker}\left(d_K^0\right) \longrightarrow 0\ . \tag{5.46}$$

This, combined with injectivity of all the sequences and the above mentioned factorization $d^n = \operatorname{coker} d^{n-1} \circ \nu^n$ for a mono ν^n, let's us procede by induction in this construction.

Now one can observe that kernels and cokernels commute with direc sums, granting that $B \to J^\bullet$ is exact, hence it is an injective resolution of B. Finally we can notice that F is left exact, hence additive. Then $F(J^\bullet) \simeq F(I^\bullet) \oplus F(K^\bullet)$ and this implies that F sends split exact seqeunces to split exact sequences. As a conseguence the following is an exact sequence of complexes:

$$0 \longrightarrow F(I^\bullet) \longrightarrow F(J^\bullet) \longrightarrow F(K^\bullet) \longrightarrow 0\ . \tag{5.47}$$

Then the fundamental theorem in cohomology let's us construct the desired exact sequence in cohomology. Notice that $H^{-1}\left(F(K^\bullet)\right) = 0$, hence the zero at the beginning. Moreover this whole construction is functorial in short exact sequence, hence the last statements hold. ■

Lemma 5.7. *Consider a preadditive category* C *and* f *a morphism in* C*. Assume that* $f = \beta \circ \nu$*, with* β *epi and* ν *mono. Then* $\ker f \simeq \ker \beta$ *and* $\operatorname{coker} f \simeq \operatorname{coker} \nu$*.*

Proof. We'll prove it only for kernels, for cokernels one just needs to dualize the construction. Let $i\colon \ker f \to A$ be a kernel for f, we want to prove that it is also a kernel for β. In fact $0 = f \circ i = \nu \circ \beta \circ i$ implies that $\beta \circ i = 0$, since ν is mono. Moreover, for any $g\colon D \to A$ s.t. $\beta \circ g = 0$, by bilinearity of composition we obtain $f \circ g = \nu \circ \beta \circ g = 0$, hence g factors uniquely through $\ker f$, i.e. $\ker f$ is a kernel for β. ■

We will now show that the family $\left(R^n F\right)_{n\in\mathbb{N}}$ together with $\left(\delta^n\right)_{n\in\mathbb{N}}$ satisfy a universal property among a class of similar functors. This point of view goes back to Grothendieck.

Definition 5.8: Cohomological functor.
Let C and D be abelian categories. We define a *cohomological* functor, sometimes called an exact δ-functor, from C to D, as:

1. a family of additive functors, for $n \in \mathbb{N}$,

$$T^n\colon \mathsf{C} \longrightarrow \mathsf{D}\ , \tag{5.48}$$

2. for all $n \geq 0$, and for all short exact sequences in C

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{5.49}$$

a family of morphisms

$$T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \tag{5.50}$$

making the following sequence exact in D:

$$\begin{array}{l} T^0(A) \longrightarrow T^0(B) \longrightarrow T^0(C) \xrightarrow{\delta^0} \\ \hookrightarrow T^1(A) \longrightarrow T^1(B) \longrightarrow T^1(C) \xrightarrow{\delta^1} \\ \hookrightarrow T^2(A) \longrightarrow \dots \end{array}\ . \tag{5.51}$$

Moreover we require this construction to be functorial, i.e. for all morphisms of short exact sequences in C

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array} \tag{5.52}$$

then, for all $n \geq 0$, the square commutes:

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \big\downarrow & & \big\downarrow \\ T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A') \end{array} . \tag{5.53}$$

Definition 5.9: Morphism of cohomological functors.
A morphism of cohomological functors is the data of a family of atural transformations:

$$(f^n)_{n\in\mathbb{N}} \colon (T^n, \delta^n)_{n\in\mathbb{N}} \longrightarrow (T'^n, \delta'^n) \tag{5.54}$$

s.t. for all $n \geq 0$, the following square commutes:

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \big\downarrow{\scriptstyle f^n(C)} & & \big\downarrow{\scriptstyle f^{n+1}(A)} \\ T^n(C') & \xrightarrow{\delta'^n} & T^{n+1}(A') \end{array} . \tag{5.55}$$

Remark 5.10 T^0 is not supposed to be left exact. Moreover we do not ask C to have enough injectives.

Definition 5.11: Universal cohomological functor.
Let C and D be abelian categories. Then a cohomological functor $(T^n, \delta^n)_{n\in\mathbb{N}}$ is said to be universal iff for any $(T'^n, \delta'^n)_{n\in\mathbb{N}}$ cohomological functor and any morphism of functors

$$g \colon T^0 \longrightarrow T'^0 \tag{5.56}$$

there exists a unique morphism of cohomological functors

$$f \colon T \longrightarrow T' \tag{5.57}$$

s.t. $f^0 = g$.

Lemma 5.12. *If $F \colon \mathsf{C} \to \mathsf{D}$ is an additive functor, then there exists at most one universal cohomological functor T, up to canonical isomorphism, s.t. $T^0 \simeq F$.*

Proof. Consider T and T' cohomological functors satisfying the above condition. In particular $T^0 \simeq T'^0 \simeq F$. Since T is universal, there is a unique morphism

$$f \colon T \longrightarrow T' \tag{5.58}$$

s.t. $f^0 = id_F$. Analogously there is a unique morphism

$$g \colon T' \longrightarrow T \tag{5.59}$$

s.t. $g^0 = id_F$. Then $g \circ f$ is the unique morphism of cohomological functors

$$g \circ f \colon T \longrightarrow T \tag{5.60}$$

s.t. $(g \circ f)^0 = g^0 \circ f^0 = id_F$. Then id_T also has this property, hence $g \circ f = id_T$. Analogously we prove that $f \circ g = id_{T'}$ and we are done. ■

Definition 5.13 Let C and D be abelian categories. We say that an additive functor $F \colon \mathsf{C} \to \mathsf{D}$ is *effaceable* iff for any $A \in \mathrm{Ob}\,(\mathsf{C})$ there exists a monomorphism

$$i \colon A \rightarrowtail M \tag{5.61}$$

s.t. $F(i) = 0$.

Proposition 5.14. *Suppose that* C *has enough injectives. Then* $F \colon \mathsf{C} \to \mathsf{D}$ *is effaceable iff* $F(I) = 0$ *for any injective* $I \in \mathrm{Ob}\,(\mathsf{C})$*.*

Proof. $\Leftarrow$ Let $A \in \mathrm{Ob}\,(\mathsf{C})$, then there exist $I \in \mathrm{Ob}\,(\mathsf{C})$ injective with a mono

$$0 \longrightarrow A \xrightarrow{\;i\;} I\ . \tag{5.62}$$

Then we obtain that

$$F(A) \xrightarrow{F(i)} F(I) = 0 \tag{5.63}$$

implies that $F(i) = 0$.

$\Rightarrow$ Let $I \in \mathrm{Ob}\,(\mathsf{C})$ injective. There is a mono

$$0 \longrightarrow I \xrightarrow{\;i\;} M \tag{5.64}$$

s.t. $F(i) = 0$. By injectivity of I we can obtain the commutative

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[d, "id_I"'] & M \arrow[dl, "r"] \\ I & \end{tikzcd} . \tag{5.65}$$

This means that $r \circ i = id_I$. Then $F(r) \circ F(i) = if_{F(I)}$, i.e. $id_{F(I)} = 0$. Check that this implies that $F(I) = 0 \in \mathrm{Ob}\,(\mathsf{D})$.

■

Remark 5.15 Consider $F \colon \mathsf{C} \to \mathsf{D}$ a left exact functor, from C with enough injectives. Then $R^n F \colon \mathsf{C} \to \mathsf{D}$ is effaceable for all $n > 0$, since $R^n F\,(I) = 0$ for all injectives I.

Theorem 5.16. *Let* C *and* D *be abelian categories. let* $T = (T^n, \delta^n)_{n \in \mathbb{N}}$ *be a cohomological functor from* C *to* D*. If* T^n *is effaceable for all* $n > 0$*, then* T *is a universal functor.*

Corollary 5.17. *Suppose that* C *has enough injectives, and* $F \colon \mathsf{C} \to \mathsf{D}$ *is a left exact functors. Then* $(R^n F, \delta^n)_{n \in \mathbb{N}}$ *is a universal cohomological functor.*

Remark 5.18 $(R^n F, \delta^n)_{n \in \mathbb{N}}$, together with $F \simeq R^0 F$ is an *initial* object in the category of cohomological functors

$$T = (T^n, \delta^n)_{n \in \mathbb{N}} \colon \mathsf{C} \longrightarrow \mathsf{D} \tag{5.66}$$

endowed with a morphism $f \to T^0$.

Proof. T^n is effaceable for all $n > 0$. Consider $T' := (T'^n, \delta'^n)_{n\in\mathbb{N}}$ another cohomological functor and $g\colon T^0 \to T'^0$ be a natural transformation. Let's define by induction the family

$$f^n\colon T^n \longrightarrow T'^n \ . \tag{5.67}$$

FOr $n = 0$ we already have the unique $f^0 = g$. Suppose we are given

$$f^i\colon T^i \longrightarrow T'^i \tag{5.68}$$

compatible with δ^i, δ'^i for all $i \leq n$. Let's define f^n: consider $A \in \mathrm{Ob}\,(\mathsf{C})$. Since T^n is effaceable, there is a mono

$$0 \longrightarrow A \xrightarrow{\ i\ } M \tag{5.69}$$

s.t. $T^n(i) = 0$. Then we can consider the exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow M/A \longrightarrow 0 \ . \tag{5.70}$$

Since T is a cohomological functor and by definition of f^i we get the commutative diagram with exact rows:

$$\begin{tikzcd}
& 0 \arrow[r] & T^n(M) \arrow[r] \arrow[d, "f^n(M)"] & T^n(M) \arrow[r] \arrow[d, "f^n(M/A)"] & T^n(M) \arrow[r] \arrow[d, dashed, "\exists!\, f^{n+1}(A)"] & 0 \\
T^n(A) \arrow[r] \arrow[rr, bend right, "0"'] & T^n(M) \arrow[r] & T^n(M) \arrow[r] & T^n(M) \arrow[r] & 0
\end{tikzcd} \ . \tag{5.71}$$

Notice that $f^{n+1}(A)$ exists by universal property of cokernels. Let's check that $f^{n+1}(A)$ does not depend on M: Let

$$0 \longrightarrow A \xrightarrow{\ i'\ } M' \tag{5.72}$$

with $F(i') = 0$. Take the pushout

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, tail, "i'"] & M \arrow[d] \\
M' \arrow[r] & M \coprod_A M'
\end{tikzcd} \ . \tag{5.73}$$

Then we have

$$\begin{tikzcd}
A & M \arrow[d, tail, "\alpha"] \\
& M \coprod_A M'
\end{tikzcd} \ , \tag{5.74}$$

which gives (see remarkable): ■

Definition 5.19 Consider $F\colon \mathsf{C} \to \mathsf{D}$ a left exact functor between abelian categories. Assume that C has enough injectives. We say that $C \in \mathrm{Ob}\,(\mathsf{C})$ is F-acyclic iff, for all $n \geq 1$,

$$R^n(F)(C) = 0_{\mathsf{D}}. \tag{5.75}$$

Remark 5.20 Any $I \in \mathrm{Ob}\,(\mathsf{C})$ injective is F-acyclic for any left exact functor.

Let's define F-acyclic resolutions, hoping to use the to compute derived functors.

Definition 5.21: F-acyclic resolution.
Let $A \in \mathrm{Ob}\,(\mathsf{C})$. An F-acyclic resolution of A is an exact sequence

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots \,. \tag{5.76}$$

s.t. C^i is F-acyclic for all $i \geq 0$.

Proposition 5.22. *Let $A \in \mathrm{Ob}\,(\mathsf{C})$. Consider an F-acyclic resolution and an injective resolution*

$$0 \longrightarrow A \longrightarrow C^\bullet \qquad \text{and} \qquad 0 \longrightarrow A \longrightarrow I^\bullet \,. \tag{5.77}$$

Then there is a morphism of complexes $C^\bullet \to I^\bullet$ extending the identity of $id_A \colon A \to A$, which is unique up to homotopy.

Proof. COnsider the morphism $F(C^\bullet) \to F(I^\bullet)$, in $\mathrm{Ch}(\mathsf{D})$. It is a quasi isomorphism. In other words it acts as an iso at the level of cohomologies: for all $n \geq 0$

$$H^n\left(F(C^\bullet)\right) \simeq H^n\left(F(I^\bullet)\right) = R^n F\left(A\right). \tag{5.78}$$

■

5.1 Spectral sequences

6 Cohomology of sheaves

Let $X \in \mathsf{Top}$. We denote by $\mathsf{PSh}\,(X)$ the category of abelian presheaves and by $\mathsf{Sh}\,(X)$ the category of abelian sheaves.

Proposition 6.1. *Limits and colimits in $\mathsf{PSh}\,(X)$ are computed component-wise. More explicitly, for a functor $\beta \colon \mathsf{I} \to \mathsf{PSh}\,(X)$, from a small category I, then we have*

$$\left(\varinjlim \beta\right)(\mathcal{U}) \simeq \varinjlim\left(\beta(\mathcal{U})\right) \tag{6.1}$$

$$\left(\varprojlim \beta\right)(\mathcal{U}) \simeq \varprojlim\left(\beta(\mathcal{U})\right) \tag{6.2}$$

$$. \tag{6.3}$$

In particular arbitrary limits and colimits exist in $\mathsf{PSh}\,(X)$, since they do in Ab.

Proof. Just a sketch: ■

Corollary 6.2. $\mathsf{PSh}\,(X)$ *is an abelian category, since it can be expressed in terms of limits and colimits.*

Proof. The zero presheaf is a zero object in the category of presheaves, by the above proposition. Then, again thanks to the above, one can define the coproduct and products. Then one obtains a morphism

$$F \coprod G \xrightarrow{\sim} F \prod G, \tag{6.4}$$

which is an iso, because it is sectionwise. Again one checks that the map

$$\mathrm{coim}\, f \xrightarrow{\sim} \mathrm{im}\, f \tag{6.5}$$

is an isomorphism, since it is sectionwise. ■

Corollary 6.3. *The category* $\mathsf{Sh}(X)$ *is an abelian category.*

Proof. Recall the adjunction $\left(\iota, (-)^{\#}\right)$

$$\mathsf{PSh}(X) \longrightarrow \mathsf{Sh}(X) \tag{6.6}$$

s.t. $(\iota\mathcal{F})^{\#} \simeq \mathcal{F}$. Recall that left adjoints preserve limits and right adjoint preserve colimits. Actually one functor is a right adjoint iff it preserves colimits. Consider a functor

$$\begin{aligned} \phi\colon \Lambda &\longrightarrow \mathsf{Sh}(X) \\ \lambda &\longmapsto F_\lambda. \end{aligned} \tag{6.7}$$

Then $\phi \circ \iota$ sends $\lambda \mapsto \iota F_\lambda$. Here we can compute limits, and one can check that $\varprojlim \iota F_\lambda$ is already a sheaf (since the definition of sheaf only requires limits, hence should commute). Then

$$\varprojlim F_\lambda := \left(\varprojlim \iota F_\lambda\right)^{\#} \in \mathsf{Sh}(X) \tag{6.8}$$

is the limit of the considered family. It follows that $\left(0_{\mathsf{PSh}(X)}\right)^{\#} = 0_{\mathsf{Sh}(X)}$ is a zero object (it is both initial and final, we are computing limits and colimits on appropriate categories). Then one can define

$$(\iota\mathcal{F} \oplus \iota\mathcal{G})^{\#} \in \mathsf{Sh}(X) \tag{6.9}$$

which is both a product and a coproduct in $\mathsf{Sh}(X)$ by the above remark. The same holds for cokernels and cokernel:

$$(\ker \iota f)^{\#} = \ker\left((\iota f)^{\#}\right) = \ker f, \tag{6.10}$$

since $(-)^{\#} \circ \iota = id_{\mathsf{Sh}(X)}$. Finally, since $\mathsf{Sh}(X) \subset \mathsf{PSh}(X)$ is a full subcategry and $\mathsf{PSh}(X)$ is abelian, then the hom set is always an abelian group.

For the parallel morphism is always the same thing. ■

Remark 6.4 One can prove this result in other way.

Proposition 6.5. *Let* $X \in \mathsf{Top}$ *and* $\mathcal{U} \subset X$ *an open subset. Then the functor*

$$\begin{aligned} \Gamma_{\mathcal{U}}\colon \mathsf{Sh}(X) &\longrightarrow \mathsf{Ab} \\ \mathcal{F} &\longmapsto \mathcal{F}(\mathcal{U}) \end{aligned} \tag{6.11}$$

is left *exact.*

Proof. Consider an exact sequence in $\mathsf{Sh}(X)$

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \ . \tag{6.12}$$

Then this induces an exact sequence in Ab, at the level of stalks, for all $x \in X$:

$$0 \longrightarrow \mathcal{F}'_x \xrightarrow{\alpha_x} \mathcal{F} \xrightarrow{\beta_x} \mathcal{F}''_x \ . \tag{6.13}$$

In particular we want to look at sections, i.e. prove exactness of

$$0 \longrightarrow \mathcal{F}'(\mathcal{U}) \xrightarrow{\alpha(\mathcal{U})} \mathcal{F}(\mathcal{U}) \xrightarrow{\beta(\mathcal{U})} \mathcal{F}''(\mathcal{U}). \tag{6.14}$$

We have proved that $\alpha(\mathcal{U})$ is injective. We are left to prove that

$$\mathrm{im}\,(\alpha(\mathcal{U})) \simeq \ker\,(\beta(\mathcal{U}))\,. \tag{6.15}$$

Since $\beta(\mathcal{U})\circ\alpha(\mathcal{U}) = 0$ we obtain that $\mathrm{im}\,(\alpha(\mathcal{U})) \subset \ker\,(\beta(\mathcal{U}))$. Consider now $s \in \ker\beta(\mathcal{U}) \subset \mathcal{F}(\mathcal{U})$. Let $x \in \mathcal{U}$, then $s_x = [\mathcal{U}_x, s] \in \ker\beta_x$. This, by exacntess of the sequence at the level of stalks, implies that there is $s'_x \in \mathcal{F}'_x$ s.t. $\alpha_x(s'_x) = s_x$. In particualr there is an open subset $x \in \mathcal{U}_x$ s.t. $s'_x = [\mathcal{U}_x, s']$ and

$$s|_{\mathcal{U}_x} = \alpha(\mathcal{U}_x)(s'). \tag{6.16}$$

Then we can find an open covering $\mathcal{U} = \bigcup_{i\in I}\mathcal{U}_i$ and $s'_i \in \mathcal{F}'(\mathcal{U}_i)$ s.t.

$$\alpha(\mathcal{U}_i)(s'_i) = s|_{\mathcal{U}_i}\,. \tag{6.17}$$

Let's look at intersections:

$$\alpha(\mathcal{U}_{ij})(\,s'_i|_{\mathcal{U}_{ij}}) = s|_{\mathcal{U}_{ij}} = \alpha(\mathcal{U}_{ij})(\,s'_j\,|_{\mathcal{U}_{ij}}) = . \tag{6.18}$$

By injectivity of $\alpha(\mathcal{U}_{ij})$ we obtain that the family $\{s'_i\}_{i\in I}$ is compatible on intersections, hence by sheaf properties one can find a unique $s' \in \mathcal{F}'(\mathcal{U})$ s.t. $s'|_{\mathcal{U}_i} = s'_i$ for all i. Then

$$\alpha(\mathcal{U})(s')|_{\mathcal{U}_i} = \alpha(\mathcal{U}_i)(\,s'|_{\mathcal{U}_i}) = \alpha(\mathcal{U}_i)(s'_i)\; s|_{\mathcal{U}_i}\,. \tag{6.19}$$

By sheaf properties of $\mathcal{F}$ we obtain that $\alpha(\mathcal{U})(s') = s$. ■

Remark 6.6: Another proof.
One can factor the above functor by

$$\begin{tikzcd} \mathsf{Sh}\,(X) \arrow[rr, "\Gamma_{\mathcal{U}}"] \arrow[dr, "\iota"'] & & \mathsf{Ab} \\ & \mathsf{PSh}\,(X) \arrow[ur, "\widetilde{\Gamma}_{\mathcal{U}}"'] & \end{tikzcd}\;. \tag{6.20}$$

One can conclude since, by construction, $\widetilde{\Gamma}_{\mathcal{U}}$ is exact (hence left exact), whereas ι is a right adjoint, hence a left exact functor.

Proposition 6.7. $\mathsf{Sh}\,(X)$ *has enough injectives.*

Definition 6.8: Cohomology.
Let $\mathcal{F} \in \mathsf{Sh}\,(X)$. The cohomology groups of X, with coefficients in $\mathcal{F}$, are defined as

$$H^n\,(X,\mathcal{F}) := R^n\Gamma_X\,(\mathcal{F}) \tag{6.21}$$

for all $n \geq 0$. More generally, given $\mathcal{U} \subset X$ open, one defines:

$$H^n\,(\mathcal{U},\mathcal{F}) := R^n\Gamma_{\mathcal{U}}\,(\mathcal{F})\,. \tag{6.22}$$

Remark 6.9 The cohomology of a t.s. X with coefficients in $\mathcal{F}$, $H^n\,(X,\mathcal{F})$ is defined as the homology of the complex

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots\,, \tag{6.23}$$

for an injective resolution $0 \to \mathcal{F} \to I^\bullet$ of $\mathcal{F}$. One could actually choose to take the cohomology of a $\Gamma_{\mathcal{U}}$-acyclic resolution of $\mathcal{F}$

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots\,. \tag{6.24}$$

Definition 6.10: Flasque sheaf.
A sheaf $\mathcal{F} \in \mathsf{Sh}(X)$ is said to be *flasque* or *falbby* iff, for all $\mathcal{V} \subset \mathcal{U} \subset X$ open subsets, then restriction map

$$\mathcal{F}(\mathcal{U}) \xrightarrow{\rho^{\mathcal{F}}_{\mathcal{U}\mathcal{V}}} \mathcal{F}(\mathcal{V}) \tag{6.25}$$

is an epimorphism.

Remark 6.11 We shall prove that flasque sheaves are Γ_X-acyclic. Moreover each sheaf admits a falsque resolution.

Proposition 6.12. *Consider an exact sequence in* $\mathsf{Sh}(X)$

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0\ , \tag{6.26}$$

where $\mathcal{F}'$ is flasque. Then, for all $\mathcal{U} \subset X$ open, the sequence

$$0 \longrightarrow \mathcal{F}'(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}''(\mathcal{U}) \longrightarrow 0 \tag{6.27}$$

is still exact.

Remark 6.13 Assuming the above remark one obtains the following exact sequence

$$0 \longrightarrow H^0(\mathcal{U}, \mathcal{F}') \longrightarrow H^0(\mathcal{U}, \mathcal{F}) \longrightarrow H^0(\mathcal{U}, \mathcal{F}'') \longrightarrow H^1(\mathcal{U}, \mathcal{F}') \longrightarrow `\dots \tag{6.28}$$

and $H^n(\mathcal{U}, \mathcal{F}) = 0$ for all n.

Proof. 1. *Let $s'' \in \mathcal{F}''(\mathcal{U})$. Consider the set $\mathcal{S}$ defined by*

$$\mathcal{S} := \left\{(\mathcal{U}_i, s_i) \,\middle|\, \mathcal{U}_i \subset \mathcal{U} \text{ is open, } s_i \in \mathcal{F}(\mathcal{U}_i) \text{ s.t. } g(\mathcal{U}_i)(s_i) = s''|_{\mathcal{U}_i}\right\}. \tag{6.29}$$

Define a partial order on $\mathcal{S}$ by

$$(\mathcal{U}_i, s_i) \leq (\mathcal{U}_j, s_j) \iff \mathcal{U}_i \subset \mathcal{U}_j \text{ and } s_j|_{\mathcal{U}_i} = s_i. \tag{6.30}$$

Show that $\mathcal{S}$ admits a maximal element.

Consider a totally ordered subset $T = \{(\mathcal{U}_i, s_i)\}_{i \in I} \subset \mathcal{S}$. We now want to apply Zorn's lemma to conclude, hence we need to construct an upper bound for T in $\mathcal{S}$. Then we define $\mathcal{V} := \bigcup_{i \in I} \mathcal{U}_i$, an open subset of X (union of open subsets of X). Moreover $\mathcal{V}$ is covered by $\mathcal{U}_i$. Then we can define $s \in \mathcal{F}(\mathcal{V})$ as the unique section s.t. $s|_{\mathcal{U}_i} = s_i$ by the gluing properties of the sheaf $\mathcal{F}$. This can be done since T is totally ordered, hence for any i, j either $\mathcal{U}_i \subset \mathcal{U}_j$ or the inverse. WLOG we assume the latter, hence $\mathcal{U}_i \cap \mathcal{U}_j = \mathcal{U}_j$. Then, since T is totally ordered, we have $s_i|_{\mathcal{U}_j} = s_j$, hence the family $\{s_i\}_{i \in I}$ defines a family compatible on intersections and we can glue it to s.

We are only left to prove that $g(\mathcal{V})(s) = s''|_{\mathcal{V}}$. Also this follows from the gluing properties of sheaves, in fact we know that, for each i,

$$g(\mathcal{V})(s)|_{\mathcal{U}_i} = g(\mathcal{U}_i)(s|_{\mathcal{U}_i}) = g(\mathcal{U}_i)(s_i) = s''|_{\mathcal{U}_i} = (s''|_{\mathcal{V}})|_{\mathcal{U}_i}. \tag{6.31}$$

Then $g(\mathcal{V})(s)$ and $s''|_{\mathcal{V}}$ coincide on an open cover of $\mathcal{V}$, hence they coincide on $\mathcal{V}$ ($\mathcal{F}''$ is a sheaf).

We have then constructed an upper bound to T in $\mathcal{S}$, then we can apply Zorn's lemma and conclude that $\mathcal{S}$ admits a maximal element.

2. *Assume that $\mathcal{F}'$ is flasque. Show that, for any $\mathcal{U} \subset X$ open subset, the sequence of abelian group*

$$0 \longrightarrow \mathcal{F}'(\mathcal{U}) \xrightarrow{f(\mathcal{U})} \mathcal{F}(\mathcal{U}) \xrightarrow{g(\mathcal{U})} \mathcal{F}''(\mathcal{U}) \longrightarrow 0 \tag{6.32}$$

is exact.

As proven in exercise 2 the functor $\Gamma_{\mathcal{U}}$ is left exact, then exactness of

$$0 \longrightarrow \mathcal{F}'(\mathcal{U}) \xrightarrow{f(\mathcal{U})} \mathcal{F}(\mathcal{U}) \xrightarrow{g(\mathcal{U})} \mathcal{F}''(\mathcal{U}) \tag{6.33}$$

follows from that. We are only left to prove that $g(\mathcal{U})$ is an epimorphism in Ab, i.e. that it is surjective. Let's now use the previous point: fix $s'' \in \mathcal{F}''(\mathcal{U})$, then the corresponding set $\mathcal{S}$ admits a maximal element

$$(\mathcal{V}, s)\,. \tag{6.34}$$

Our aim is to prove that $\mathcal{V} = \mathcal{U}$, in such case we have $g(\mathcal{U})(s) = s''$, hence surjectivity of $g(\mathcal{U})$, since we have not imposed any restrictions on $s'' \in \mathcal{F}''(\mathcal{U})$.

Assume, by contradiction, that this is not the case, i.e. that there exists $x \in \mathcal{U} \setminus \mathcal{V}$. Since g is an epimorphism, we obtain that there is $t_x \in \mathcal{F}_x$ s.t.

$$t_x \xmapsto{g_x} s''_x\;. \tag{6.35}$$

More explicitly there exist $\mathcal{U}_x \subset X$ open, and $t \in \mathcal{F}(\mathcal{U}_x)$ s.t. $g(\mathcal{U}_x)(t) = s''|_{\mathcal{U}_x}$. Then we can concentrate on $\mathcal{U}_x \cap \mathcal{V}$. In case this is empty we can clearly glue together t and s to a section $u \in \mathcal{F}(\mathcal{V} \cup \mathcal{U}_x)$, getting mapped to $s''|_{\mathcal{V}\cup\mathcal{U}_x}$ (reasoning as in the previous point, $\mathcal{F}''$ is a sheaf). This would make

$$(\mathcal{V} \cup \mathcal{U}_x, u) > (\mathcal{V}, s)\,, \tag{6.36}$$

contradicting maximality of the latter. Then we have $\mathcal{V}_x := \mathcal{U}_x \cap \mathcal{V} \neq \emptyset$. In particular

$$g(\mathcal{V}_x)\left(s|_{\mathcal{V}_x} - t|_{\mathcal{V}_x}\right) = g(\mathcal{V}_x)\left(s|_{\mathcal{V}_x}\right) - g(\mathcal{V}_x)\left(t|_{\mathcal{V}_x}\right) = s''|_{\mathcal{V}_x} - s''|_{\mathcal{V}_x} = 0. \tag{6.37}$$

(The above follows from linearity of $g(\mathcal{V}_x)$, morphism in Ab). Then $s|_{\mathcal{V}_x} - t|_{\mathcal{V}_x} \in \ker g(\mathcal{V}_x)$. By left exactness of $\Gamma_{\mathcal{V}_x}$ we obtain that this is in the image of $f(\mathcal{V}_x)$, i.e. there is $\tilde{w} \in \mathcal{F}'(\mathcal{V}_x)$ s.t. $f(\mathcal{V}_x)(\tilde{w}) = s|_{\mathcal{V}_x} - t|_{\mathcal{V}_x}$. But $\mathcal{F}'$ is flasque, hence $\tilde{w}$ can be extended to the whole $\mathcal{U}_x$, i.e. there is $w \in \mathcal{F}'(\mathcal{U}_x)$ s.t. $w|_{\mathcal{V}_x} = \tilde{w}$. Then, again by left exactness of $\Gamma_{\mathcal{U}_x}$, we obtain that $f(\mathcal{U}_x)(w) \in \ker g(\mathcal{U}_x)$, hence for $\tilde{t} := t + f(\mathcal{U}_x)(w)$, by linearity of $g(\mathcal{U}_x)$, we have

$$g(\mathcal{U}_x)(\tilde{t}) = g(\mathcal{U}_x)(t). \tag{6.38}$$

Moreover we can look at the restriction of $\tilde{t}$ to $\mathcal{V}_x$ and obtain

$$\tilde{t}|_{\mathcal{V}_x} = t|_{\mathcal{V}_x} + f(\mathcal{U}_x)(w)|_{\mathcal{V}_x} = t|_{\mathcal{V}_x} + f(\mathcal{V}_x)(w|_{\mathcal{V}_x}) \tag{6.39}$$

$$= t|_{\mathcal{V}_x} + f(\mathcal{V}_x)(\tilde{w}) = t|_{\mathcal{V}_x} + s|_{\mathcal{V}_x} - t|_{\mathcal{V}_x} = s|_{\mathcal{V}_x}\,. \tag{6.40}$$

Then we can glue together s and $\tilde{t}$ to a section in $\mathcal{F}(\mathcal{V} \cup \mathcal{U}_x)$. As before this section will get mapped to $s''|_{\mathcal{V}\cup\mathcal{U}_x}$ since both s and $\tilde{t}$ do on their respective domains and $\mathcal{F}''$ is a sheaf. Then, again, we have contradicted maximality of $(\mathcal{V}, s)$. We can conclude since it can only happen that $\mathcal{V} = \mathcal{U}$. ■

Corollary 6.14. *Consider a sequence exact in* $\mathsf{Sh}(X)$*:*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \tag{6.41}$$

and $\mathcal{F}, \mathcal{F}'$ *are flasque, then also* $\mathcal{F}''$ *is flasque.*

Proof. Consider $\mathcal{V} \subset \mathcal{U}$ open. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'(\mathcal{U}) \arrow[r] \arrow[d, two heads] & \mathcal{F}(\mathcal{U}) \arrow[r] \arrow[d, two heads] & \mathcal{F}''(\mathcal{U}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}'(\mathcal{V}) \arrow[r] & \mathcal{F}(\mathcal{V}) \arrow[r] & \mathcal{F}''(\mathcal{V}) \arrow[r] & 0 \end{tikzcd}. \tag{6.42}$$

Moreover the first two vertical arrows are epimorphisms. Then it trivially follows that also $\rho_{\mathcal{U}\mathcal{V}}^{Z^1(\mathcal{F})}$ is an epimorphism. In fact, for any composable $f \circ \rho_{\mathcal{U}\mathcal{V}}^{Z^1(\mathcal{F})} = 0$, we obtain

$$0 = f \circ \rho_{\mathcal{U}\mathcal{V}}^{Z^1(\mathcal{F})} \circ \operatorname{coker} i^0(\mathcal{U}) = f \circ \operatorname{coker} i^0(\mathcal{V}) \circ \rho_{\mathcal{U}\mathcal{V}}^{C^0(\mathcal{F})} = f, \tag{6.43}$$

since the last two maps are epimorphisms and composition is bilinear. This holds for any $\mathcal{V} \subset \mathcal{U}$, i.e. also $Z^1(\mathcal{F})$ is flasque. Then, iterating by induction, we obtain that $Z^n(\mathcal{F})$ is flasque. (tk: fix the names). ■

tk: copy the definition of $C^)(\mathcal{F})$

Proposition 6.15. *For any* $\mathcal{F} \in \mathsf{Sh}(X)$*, there is a canonical monomorphism*

$$\mathcal{F} \rightarrowtail C^0(\mathcal{F}). \tag{6.44}$$

into $C^0(\mathcal{F})$ *a* flasque *sheaf.*

Proof. Consider the étalé space associated to $\mathcal{F}$:

$$\widetilde{\mathcal{F}} = \coprod_{x \in X} \xrightarrow{\pi} X. \tag{6.45}$$

By definition $C^0(\mathcal{F})$ is the sheaf of not necessairily continuous sections of π. Consider $\mathcal{U} \subset X$ open, then

$$C^0(\mathcal{F})(\mathcal{U}) = \left\{ \sigma \colon \mathcal{U} \to \widetilde{\mathcal{F}} \,\middle|\, \pi \circ \sigma = id_{\mathcal{U}} \right\} \simeq \prod_{x \in \mathcal{U}} \mathcal{F}_x. \tag{6.46}$$

For all $\mathcal{V} \subset \mathcal{U}$ we than ahve a surjective map

$$\begin{tikzcd} \prod_{x \in \mathcal{U}} \mathcal{F}_x \arrow[r, two heads] \arrow[d, "\sim"] & \prod_{x \in \mathcal{V}} \mathcal{F}_x \arrow[d, "\sim"] \\ C^0(\mathcal{F})(\mathcal{U}) \arrow[r] & C^0(\mathcal{F})(\mathcal{V}) \end{tikzcd}. \tag{6.47}$$

Then $C^0(\mathcal{F})$ is flasque. Let's construct the canonical monomorphism:

$$\begin{aligned} \alpha \colon \mathcal{F} &\longrightarrow C^0(\mathcal{F}) \\ \mathcal{F}(\mathcal{U}) &\longmapsto C^0(\mathcal{F})(\mathcal{U}) = \prod_{x \in X} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x \in X} \,. \end{aligned} \tag{6.48}$$

α is injective iff $\alpha(\mathcal{U})$ is injective for all $\mathcal{U} \subset X$. In fact consider $s \in \ker(\alpha(\mathcal{U}))$, iff $s_x = 0$ for all $x \in \mathcal{U}$. But this means that $s = 0$. ■

Proposition 6.16. *Let $\mathcal{I} \in \mathsf{Sh}(X)$ be an* injective *sheaf. Then $\mathcal{I}$ is* flasque*.*

Proof. By the above we have a canonical monomorphism

$$0 \longrightarrow \mathcal{I} \longrightarrow C^0(\mathcal{I}). \tag{6.49}$$

By injectivity of $\mathcal{I}$ there is a map r making the diagram commute:

$$\begin{tikzcd} \mathcal{I} \arrow[r, tail, "\alpha"] \arrow[d, "id_{\mathcal{I}}"'] & C^0(\mathcal{I}) \arrow[dl] \\ \mathcal{I} & \end{tikzcd} . \tag{6.50}$$

Then $r \circ \alpha = id_{\mathcal{I}}$. Moreover, for any $\mathcal{U} \subset X$ open, we have

$$r(\mathcal{U}) \circ \alpha(\mathcal{U}) = id_{\mathcal{I}(\mathcal{U})}. \tag{6.51}$$

But this means that $r(\mathcal{U})$ is surjective. Since r is a morphism of sheaves, and $C^0(\mathcal{I})$ is flasque, we have the following diagram

$$\begin{tikzcd} C^0(\mathcal{I})(\mathcal{U}) \arrow[r, two heads, "r(\mathcal{U})"] \arrow[d, "\rho^{C^0(\mathcal{I})}_{\mathcal{U}\mathcal{V}}"] & \mathcal{I}(\mathcal{U}) \arrow[d, "\rho^{\mathcal{I}}_{\mathcal{U}\mathcal{V}}"] \\ C^0(\mathcal{I})(\mathcal{V}) \arrow[r, two heads, "r(\mathcal{V})"] & \mathcal{I}(\mathcal{V}) \end{tikzcd} . \tag{6.52}$$

But this implies that also $\rho^{\mathcal{I}}_{\mathcal{U}\mathcal{V}}$ is an epimorphism (tk: argue, but it seems easy). ■

Proposition 6.17. *Let $\mathcal{F} \in \mathsf{Sh}(X)$ be a flasque sheaf on X. Then $\mathcal{F}$ is $\Gamma_{\mathcal{U}}$-acyclic for any $\mathcal{U} \subset X$ open.*

Proof. Consider a monomorphism

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}, \tag{6.53}$$

for $\mathcal{I}$ injective. This can be extended to a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}/\mathcal{F} \longrightarrow 0, \tag{6.54}$$

in which both $\mathcal{F}$ and $\mathcal{I}$ are flasque. Then also $\mathcal{I}/\mathcal{F}$ is flasque. From the fundamental theorem in cohomology we obtain the long exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{U}, \mathcal{F}) \longrightarrow H^0(\mathcal{U}, \mathcal{I}) \longrightarrow H^0(\mathcal{U}, \mathcal{I}/\mathcal{F}) \longrightarrow \dots . \tag{6.55}$$

In particular we obtain

$$H^0(\mathcal{U}, \mathcal{I}) \twoheadrightarrow H^0(\mathcal{U}, \mathcal{I}/\mathcal{F}) \longrightarrow H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^0(\mathcal{U}, \mathcal{I}) \longrightarrow \dots . \tag{6.56}$$

Since the first map is surjective, exactness implies that $H^1(\mathcal{U}, \mathcal{F}) = 0$. Let's now argue by induction:we obtain that $H^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i \leq n$. Then, from the exact sequence (tk: copy it from someone). Then we are done, since $\mathcal{I}/\mathcal{F}$ is also flasque. ■

Construction: Godement resolution. It is a canonical resolution by flasque sheaves.

Definition 6.18 Given a sheaf $\mathcal{F}$, one defines

$$Z^1(\mathcal{F}) := C^0(\mathcal{F})/\mathcal{F} \qquad \text{and} \qquad C^1(\mathcal{F}) := C^0(Z^1(\mathcal{F})). \tag{6.57}$$

Then, by induction, one defines

$$Z^n(\mathcal{F}) := C^{n-1}(\mathcal{F})/Z^{n-1}(\mathcal{F}) \qquad \text{and} \qquad C^n(\mathcal{F}) := C^0(Z^n(\mathcal{F})). \tag{6.58}$$

Moreover, by induction, one defines the differential

$$d^n \colon C^n(\mathcal{F}) \xrightarrow{\operatorname{coker} i^n} Z^{n+1}(\mathcal{F}) \xrightarrow{i^{n+1}} C^{n+1}(\mathcal{F}). \ . \tag{6.59}$$

Remark 6.19 For $n \geq 0$ we have $\ker d^n = Z^n(\mathcal{F})$, where $Z^0(\mathcal{F}) := \mathcal{F}$. Moreover $\operatorname{im} d^n = Z^n(\mathcal{F})$. (tk: copy the proof from the homework: factorization in mono and epi).

It follows that the sequence

$$0 \longrightarrow C^0(\mathcal{F}) \longrightarrow C^1(\mathcal{F}) \longrightarrow C^2(\mathcal{F}) \longrightarrow \dots \tag{6.60}$$

is exact in $\mathsf{Sh}\,(X)$. In particular we have constructed a flasque resolution of $\mathcal{F}$.

Moreover, since each part of the construction is functorial, then also the Godement resolution is. In particular any morphism $\alpha \colon \mathcal{F} \to \mathcal{G}$ of sheaves induces a morphism in $\mathrm{Ch}(\mathsf{Sh}\,(X))$ $\alpha^\bullet \colon C^\bullet(\mathcal{F}) \to C^\bullet(\mathcal{G})$.

One can apply to the cochain complex, degree-wise, the functor $\Gamma_{\mathcal{U}}$, hence one defines the functor

$$\Gamma_{\mathcal{U}} \colon \mathrm{Ch}(\mathsf{Sh}\,(X)) \longrightarrow \mathrm{Ch}(\mathsf{Ab}) \ . \tag{6.61}$$

Proposition 6.20. *The composition of the above two functors is exact. tk: write it better, copy proof from the homework, correct it as on the remarkable.*

Remark 6.21 One applies the fundamental theorem of cohomology and obtains the long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{U},\mathcal{F}') \arrow[r] & H^0(\mathcal{U},\mathcal{F}) \arrow[r] & H^0(\mathcal{U},\mathcal{F}'') \arrow[dll] \\
& H^1(\mathcal{U},\mathcal{F}') \arrow[r] & H^1(\mathcal{U},\mathcal{F}) \arrow[r] & H^1(\mathcal{U},\mathcal{F}'') \arrow[dll] \\
& {} & \vdots \arrow[dl] & \\
& H^n(\mathcal{U},\mathcal{F}') \arrow[r] & H^n(\mathcal{U},\mathcal{F}) \arrow[r] & H^n(\mathcal{U},\mathcal{F}'') \arrow[dll] \\
& H^{n+1}(\mathcal{U},\mathcal{F}') \arrow[r] & \dots &
\end{tikzcd} \ , \tag{6.62}$$

in which we recall that

$$H^n(\mathcal{U},\mathcal{F}) := R^n\Gamma_{\mathcal{U}}(\mathcal{F}) \simeq H^n(C^\bullet(\mathcal{F})(\mathcal{U}))\,. \tag{6.63}$$

We have recovered the long exact sequence of cohomology groups associated to the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \ . \tag{6.64}$$