ORBIFOLD DIAGRAMS

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ABSTRACT. We study alternating strand diagrams on the disk with an orbifold point. These are quotients by rotation of Postnikov diagrams on the disk, and we call them orbifold diagrams. We associate a quiver with potential to each orbifold diagram, in such a way that its Jacobian algebra and the one associated to the covering Postnikov diagram are related by a skew-group algebra construction. We moreover realise this Jacobian algebra as the endomorphism algebra of a certain explicit cluster-tilting object. This is similar to (and relies on) a result by Baur-King-Marsh for Postnikov diagrams on the disk.

1. INTRODUCTION

In this article we study *orbifold diagrams*, i.e. alternating strand diagrams on the disk with an orbifold point. These are collections of oriented arcs satisfying certain properties, which we define as quotients by rotation of alternating strand diagrams on the disk (also called Postnikov diagrams). The latter have been used in the study of the coordinate ring of the Grassmannian: they give rise to clusters of the Grassmannian cluster algebras, [Sco06], or to cluster tilting objects of the Grassmannian cluster categories [JKS16], [BKM16]. On the other hand, orbifolds have also been related to cluster structures, [PS19], [CS14]. In [AP18], Amiot and Plamondon construct cluster algebras on surfaces with orbifold points of order 2, and in their construction skew-group algebras appear naturally. Here we associate quivers with potentials to orbifold diagrams in such a way that skew-group algebras play a major role.

Skew group construction have been used in representation theory, for example in the seminal work of Reiten and Riedtmann [RR85] and of Asashiba [Asa11]. In [LFV19], the authors consider the triangulated disk with one orbifold point of order three. However in their set-up, the authors do not need skew group constructions because the action considered is free. The authors obtain a generalized cluster algebra from the Jacobian algebra associated to each triangulation of the aforementioned triangulated orbifold. Let us point out there is a well-known relation between triangulated surfaces and certain Postnikov diagram, see [BKM16, Section 13], first described for the disk by Scott in her work [Sco06, Section 3]. This relation allow us to expect a generalized cluster structure from the constructions we give in this paper. We will investigate this in future work.

Our set-up is the following. We start with Postnikov diagrams with rotational invariance, i.e. with an action of a cyclic group G of order d, and take the quotient with respect to this action. We also give an intrinsic definition of such a quotient as a new combinatorial datum associated to a disk with an orbifold point, and call this an orbifold diagram. We associate a quiver with potential $(Q_{\mathcal{O}}, W_{\mathcal{O}})$ to every orbifold diagram $\mathcal{O}$, with a construction that depends on whether the orbifold point corresponds to a vertex of the quiver or not. In particular, we give a construction in case the action is not free on vertices. In Proposition 6.1 we prove that the frozen Jacobian algebras $A_{\mathcal{O}}$ of this new quiver and the one of the associated Postnikov diagram are related by a skew-group construction.

We then restrict to the case where the permutation induced by the strands of the associated Postnikov diagram on the cover is of Grassmannian type (k, n), to use results from [JKS16, BKM16]. As for Postnikov diagrams, there is an idempotent subalgebra $B(\mathcal{O})$ of the frozen Jacobian algebra $A(\mathcal{O})$ that only depends on (k, n, d).

Our aim is to realise the frozen Jacobian algebra as an endomorphism algebra of a cluster tilting object as in the statement in [BKM16, Theorem 10.3]. To do this, we construct modules over the idempotent algebra $B(\mathcal{O})$ of an orbifold diagram in such a way that they are the images of the rank 1 modules from [JKS16] under a canonical functor.

Any orbifold diagram determines a collection of such modules whose direct sum is a cluster-tilting objects in a Frobenius, stably 2-Calabi-Yau category. Our main result, Theorem 6.23, is that the endomorphism ring of this cluster-tilting object is isomorphic to $A(\mathcal{O})$.

Conventions. We always consider finitely generated left modules and we compose arrows from right to left. The base field is the complex numbers.

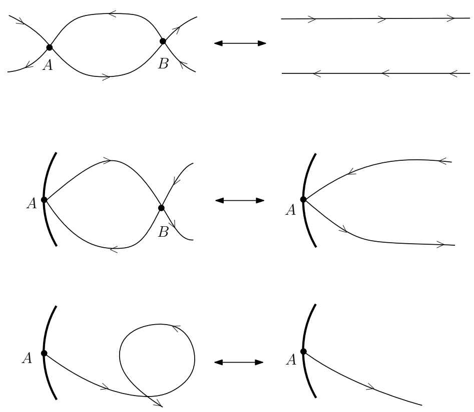

FIGURE 1. Pulling strands in order to reduce a Postnikov diagram.

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2. ORBIFOLD DIAGRAMS

In this section, we define orbifold diagrams on the disk with an orbifold point. Informally, these are quotients of rotation-invariant Postnikov diagrams, also called alternating strand diagrams. We start by defining these, following [Pos06].

We write S_n for the symmetric group of permutations of n elements.

Definition 2.1. A *Postnikov diagram of type* $\sigma \in S_n$ is a collection of n oriented curves γ_i, called *strands*, on a disk with n marked points on the boundary (clockwise labeled $1, \dots, n$), such that

(1) The strand γ_i connects the boundary point i with $\sigma(i)$, starting at i. The strand γ_i intersects the boundary only in those two (possibly coinciding) points.
(2) There are a finite number of crossings, all between two strands, all transverse.
(3) Following a strand, the strands crossing it come alternatingly from the left and from the right. This includes strands crossing at boundary points.
(4) If two strands cross in two points A and B, then one is oriented from A to B and the other is oriented from B to A. This also applies to crossings at boundary points.
(5) If a strand crosses itself other than at a boundary point, then consider the disk determined by the loop. No strand intersects the interior of this disk.

A *Grassmannian Postnikov diagram of type* (k, n) is a Postnikov diagram satisfying the additional condition

(6) The permutation $\sigma \in S_n$ is given by $\sigma(i) = i + k \pmod{n}$.

Postnikov diagrams are considered modulo isotopy fixing the boundary.

Remark 2.2. Postnikov diagrams can be reduced as follows, see Figure 1:
(i) If two strands cross in points A and B such that the region formed by A and B is simply connected then we can reduce by "pulling the strands" in a way to remove the two crossings. Note that one of the points A and B may be a marked point on the boundary; in that case, only one crossing gets removed.
(ii) If a strand crosses itself and if the disk determined by the loop contains no other strands, the strand can be straightened, i.e. the crossing removed.

Diagrams reduced in this way retain many properties of the original diagram, and so we will often assume in the following that Postnikov diagrams are reduced.

Since we plan to take quotients by rotations of the disk, an important role is played by the Postnikov diagrams which are rotation-invariant. These were first studied in [Pas20] in relation to self-injective Jacobian algebras.

Definition 2.3. A Postnikov diagram of type $\sigma \in S_n$ is *d-symmetric* if it is (up to isotopy) invariant under rotation by $\frac{2\pi}{d}$.

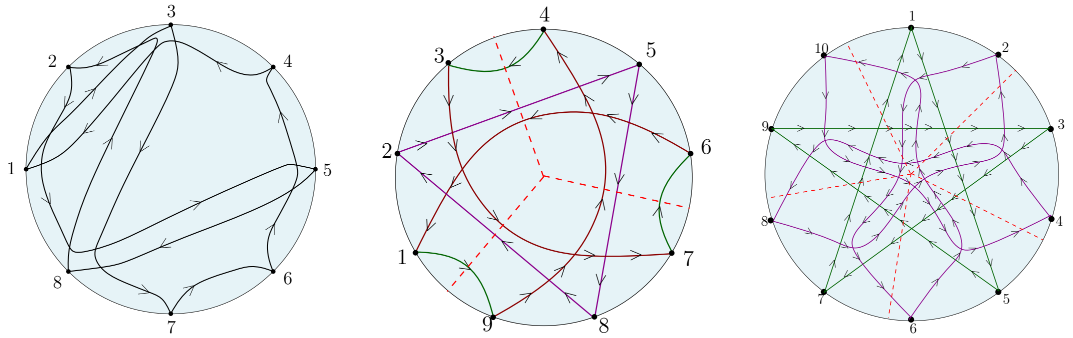

FIGURE 2. Examples of Postnikov diagrams. Symmetry axes indicated by dashed lines.

Observe that in this case $\mathbb{Z}_d$ must act freely on $\{1, \dots, n\}$, and so $d \mid n$.

Example 2.4. Figure 2 shows examples of Postnikov diagrams. The first is of type $\sigma = (13764)(258) \in S_8$, the second is a symmetric Postnikov diagram of type $\sigma = (194276)(258) \in S_9$ and the last is a symmetric Grassmannian Postnikov diagram of type $(4, 10)$.

If we start with a d-symmetric Postnikov diagram of order $d > 1$, we can construct its (topological) quotient by the cyclic group of order d acting by rotations. This will be a collection of curves on a disk with an orbifold point of order d. The resulting diagram is what we will call an “orbifold diagram”.

We first give an abstract definition of a (weak) orbifold diagram and introduce orbifold diagrams in Definition 2.12. We will then show that orbifold diagrams as defined through this are the same as quotients of symmetric Postnikov diagrams (Proposition 2.14).

Notation 2.5. We will use the usual notion of winding number for a closed curve with respect to a point, but the clockwise direction is for us positive. This is because in the literature the boundary points are usually labeled clockwise.

Let Σ be a disk with n_0 marked points on the boundary (clockwise labeled $1, \dots, n_0$) and an orbifold point Ω of order $d > 1$.

Definition 2.6. A *weak orbifold diagram of type* $\tau \in S_{n_0}$ *on* Σ is a collection of n_0 oriented curves γ_i, called *strands*, on Σ, such that

(1) The strand γ_i connects the boundary point i with $\tau(i)$, starting from i. The strand γ_i intersects the boundary only in those two (possibly coinciding) points, and does not go through Ω.
(2) There is a finite number of crossings, all between two strands, all transverse.
(3) Following a strand, the strands crossing it come alternatingly from the left and from the right. This includes strands crossing at boundary points.
(4) If two strands cross in two points A and B and both are oriented from A to B, then consider the closed curved formed by following a strand from A to B and then following the other strand in the opposite direction from B to A. The winding number of this closed curve with respect to Ω is not 0 (for an example, see the curves between the points Q_1 and Q_2 in both pictures in Figure 7).
(5) If a strand crosses itself, then consider the closed curve formed by following the strand from a point of intersection to itself. Either this has nonzero winding number with respect to Ω, or it is a simple loop not intersecting any other strand (and thus can be reduced as for Postnikov diagrams).

Weak orbifold diagram are considered up to isotopy fixing the boundary and the center of the disk. Weak orbifold diagrams can be reduced like Postnikov diagrams, provided that strands do not need to be moved across the orbifold point when doing so.

Figure 3 shows examples of weak orbifold diagrams. Observe that the order d of the orbifold point Ω does not appear in the axioms: it is part of the datum of the surface. So we can have the same diagram (picture) for varying orders d.

Remark 2.7. To any weak orbifold diagram $\mathcal{O}$ on a disk Σ we will consider a symmetrized version of $\mathcal{O}$ on the universal cover of Σ. This depends on the order of Ω, in particular, the same strand configuration (picture) leads to a symmetrized version for every $d > 1$.

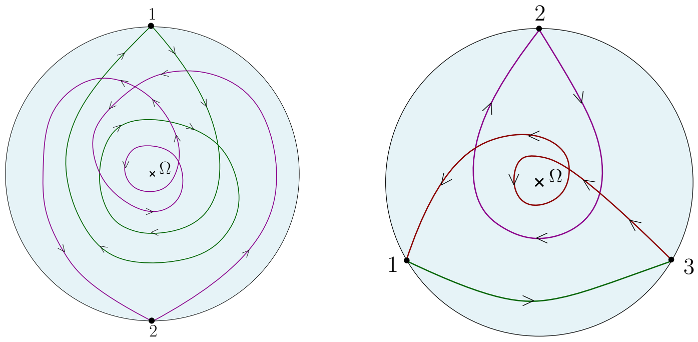

Figure 3. Two weak orbifold diagrams; $\tau = \mathrm{id} \in S_2$ on the left, $\tau = (13) \in S_3$ on the right.

Definition 2.8. Let $\mathcal{O}$ be a (reduced) weak orbifold diagram on Σ, assume that Ω has order d. Draw a simple curve joining Ω to the boundary arc between n_0 and 1. Then $\mathrm{sym}_d(\mathcal{O})$ is the collection of $n_0 d$ strands obtained from taking d copies of $\mathcal{O}$ and gluing them along the copies of the simple curve. We draw the resulting surface as a disk and label the marked points by $1, 2, \ldots, n_0, n_0 + 1, \ldots, dn_0$ clockwise around the boundary.

By construction, the image $\mathrm{sym}_d(\mathcal{O})$ is a collection of $n = n_0 d$ strands on a disk (without orbifold points) which is symmetric under rotation by $\frac{2\pi}{d}$. The image $\mathrm{sym}_d(\mathcal{O})$ corresponds to taking the universal cover of the orbifold diagram $\mathcal{O}$ for the surface Σ with Ω a point of order d.

The result is not a Postnikov diagram in general, as it may have "lenses" (pairs of twice-crossing parallel strands) and self-crossings (compare Definition 2.6 and Defintion 2.1). This is illustrated in Example 2.9 below. If d is large enough, the symmetrized version $\mathrm{sym}_d(\mathcal{O})$ of $\mathcal{O}$ is a Postnikov diagram, which is d-symmetric by construction, see Proposition 2.14.

Note that the the quotient of $\mathrm{sym}_d(\mathcal{O})$ under the rotation by $\frac{2\pi}{d}$ is $\mathcal{O}$. We will write $\mathcal{P}/d$ to denote the quotient of a d-symmetric Postnikov diagram under the rotation by $\frac{2\pi}{d}$.

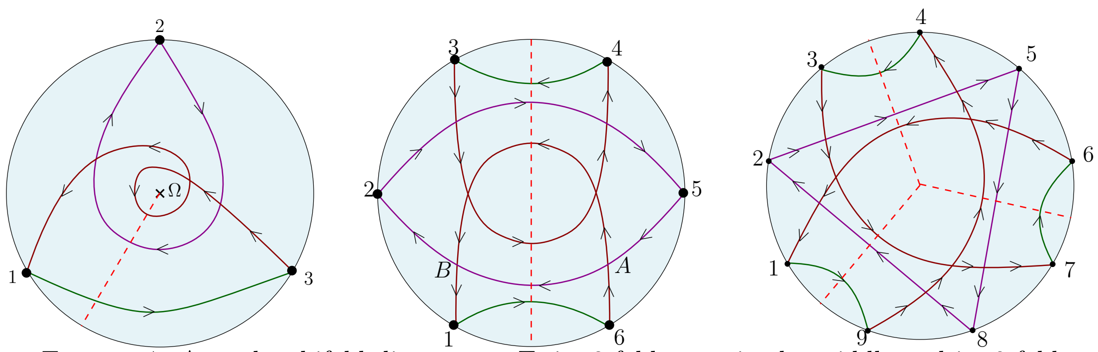

Figure 4. A weak orbifold diagram on Σ, its 2-fold cover in the middle and its 3-fold cover on the right. Red dashed lines indicate the fundamental domains/symmetry axes.

Example 2.9. Here we start with a weak orbifold diagram $\mathcal{O}$ for $\tau = (13) \in S_3$ on Σ, with orbifold point Ω of order d, see first picture in Figure 4. We consider $\mathrm{sym}_d(\mathcal{O})$ for $d = 2$ and $d = 3$.

Let us consider the 2-fold cover in Figure 4. This is not a good cover of $\mathcal{O}$ for two independent reasons. First, it is not a Postnikov diagram, since it violates condition (4) of Definition 2.1: the strands crossing at A and B are both oriented from A to B. This is because the order of the orbifold point (i.e. 2) is too small compared to how much the strands wind around it. In Definition 2.12 we will precisely quantify how large d needs to be for the d-fold cover to be a Postnikov diagram.

The second issue is more subtle: the diagram of the 2-fold cover is not reduced, in the sense that we can apply a reduction move as in Remark 2.2. However, the quotient of the reduced diagram by the rotation of order 2 is not the same as $\mathcal{O}$ (it corresponds to applying a forbidden reduction move that goes through Ω). This issue arises because the order of the orbifold point is exactly 2. Indeed, the reduction moves are applied to digons, and those arise precisely from covers of order 2. To avoid this, we will stipulate that the order of orbifold diagrams is at least 3, which ensures that if $\mathcal{O}$ is reduced then its cover is also reduced.

Finally, the 3-fold cover $\mathrm{sym}_3(\mathcal{O})$ is a 3-symmetric Postnikov diagram: the problems disappear, since 3 is large enough (as per Definition 2.12) and is not equal to 2.

Let us point out that if we start from a d-symmetric Postnikov diagram on a disk with $n = n_0 d$ marked points and take its quotient under the rotation by $\frac{2\pi}{d}$, we obtain a weak orbifold diagram on a disk Σ with n_0 points with additional properties, see Example 2.10.

Example 2.10. We start with a 5-symmetric Grassmannian Postnikov diagram of type $(4, 10)$, see Figure 5. When we quotient by the 5-fold symmetry we get a weak orbifold diagram on a disk Σ with a point Ω of order 5 and with 2 marked points. The type of the image is $\tau = \mathrm{id}$.

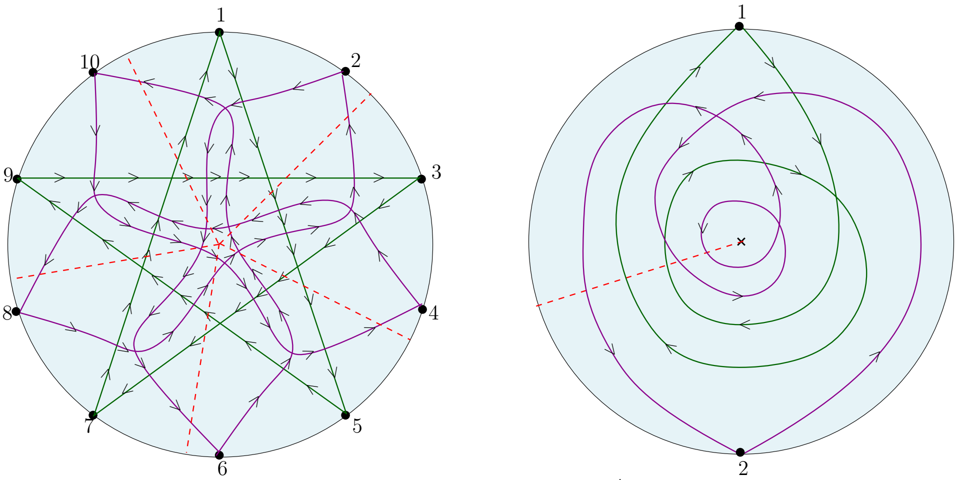

FIGURE 5. Taking the quotient under rotation by $2\pi/5$ gives a weak orbifold with a point of order 5; red dashed lines indicate symmetry axes/fundamental domains.

Example 2.11. Here we have a 3-symmetric Postnikov diagram of type $(3, 9)$. Its quotient by the 3-fold symmetry, on the right, is a weak orbifold diagram on a disk Σ with Ω of order 3 and 3 marked points, of type $\tau = \mathrm{id}$.

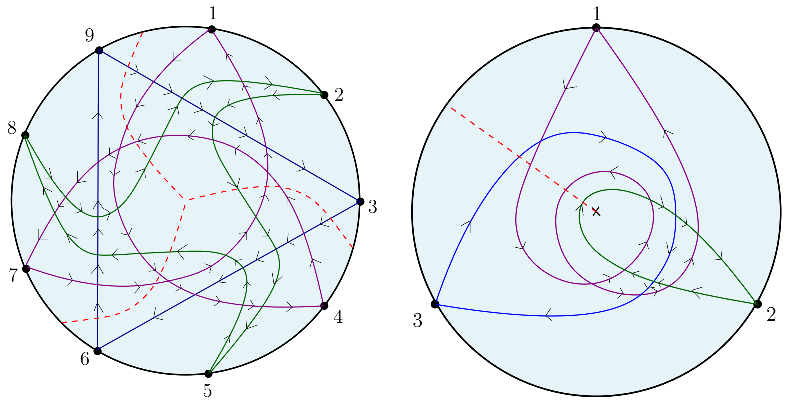

FIGURE 6. A symmetric orbifold diagram $\mathcal{P}$ with its quotient $\mathcal{P}/3$ on the right.

We would like to upgrade our definition of weak orbifold diagram by including the value of d in the datum of the picture, as well as guaranteeing that the d-fold cover is a Postnikov diagram. The only properties that might fail are (4) and (5) in Definition 2.1. Since for sufficiently large d these properties hold, we pick the smallest such d.

Let us define some notation. For a strand γ in a weak orbifold diagram, consider its points of self-intersection (including at the boundary). Each of these points P determines a closed subcurve of γ (going from P to itself), which has a winding number $w(P)$ with respect to Ω. We define $S(\gamma)$ to be the maximum of the absolute values of $w(P)$, where P varies in the set of self-intersections of γ. If γ does not intersect itself we set $S(\gamma) = 0$.

Similarly, let γ_1 and γ_2 be two strands in an orbifold diagram. Assume that they meet in two points A and B, and that they are both oriented from A to B. Then consider the curve formed by following γ_1 from A to B and then γ_2 against the orientation from B to A. This is a closed loop and it has a winding number $w(A, B)$ with respect to Ω. Strictly speaking, this is not well-defined as the sign of $w(A, B)$ depends on the choice of the curve that is taken against the orientation. But we are only interested in the absolute value of $w(A, B)$: We define $L(\gamma_1, \gamma_2)$ to be the maximum of the absolute values of $w(A, B)$ for all pairs A, B as above. We set $L(\gamma_1, \gamma_2) = 0$ if γ_1 and γ_2 do not meet as above.

Definition 2.12. Let Σ be a disk with an orbifold point Ω of order $d > 1$. A weak orbifold diagram $\mathcal{O}$ on Σ is an *orbifold diagram (of order d)* if

$$d > \max\{\max_{\gamma} S(\gamma), \max_{\gamma_1 \neq \gamma_2} L(\gamma_1, \gamma_2)\}.$$

An orbifold diagram on Σ is *Grassmannian* if $\tau = \mathrm{id}$ and there is an integer $0 < w_+ < d$ such that every strand has winding number w_+ or $w_+ - d$. In this case, we say that the orbifold diagram is *of type* (k, n), where $n = n_0 d$ and $k = n_0 w_+$.

Example 2.13. We consider the weak orbifold diagrams from Examples 2.9 and 2.10.

(1) We first take the weak orbifold diagram $\mathcal{O}$ on the left of Figure 7. We want to see whether the conditions of Definition 2.12 hold. The strand γ_1 does not have self-intersection points, $w(P_2) = 1$ and $w(P_3) = -1$, so $S(\gamma_1) = 0$, $S(\gamma_2) = S(\gamma_3) = 1$. For the second condition: we have $w(Q_1, Q_2) = 2$ and so $L(\gamma_2, \gamma_3) = 2$. Since $d = 3$, $\mathcal{O}$ is indeed an orbifold diagram.

(2) Now we look at the diagram $\mathcal{O}$ on the right of Figure 7. This is a weak orbifold diagram of order $d = 5$ and we want to see whether the conditions of Definition 2.12 hold. We get $w(P_{11}) = 2$, $w(P_{12}) = 1$ $w(P_{21}) = -3$, $w(P_{22}) = -2$ and $w(P_{23}) = -1$. So $S(\gamma_1) = 2$ and $S(\gamma_2) = 3$. We have $w(Q_1, Q_2) = 1$, $w(Q_1, Q_3) = 3$, $w(Q_1, Q_4) = 4$, $w(Q_2, Q_3) = 2$, $w(Q_2, Q_4) = 3$ and $w(Q_3, Q_4) = 1$. In this case $L(\gamma_1, \gamma_2) = 4$. Since $d = 5$, $\mathcal{O}$ is also an orbifold diagram.

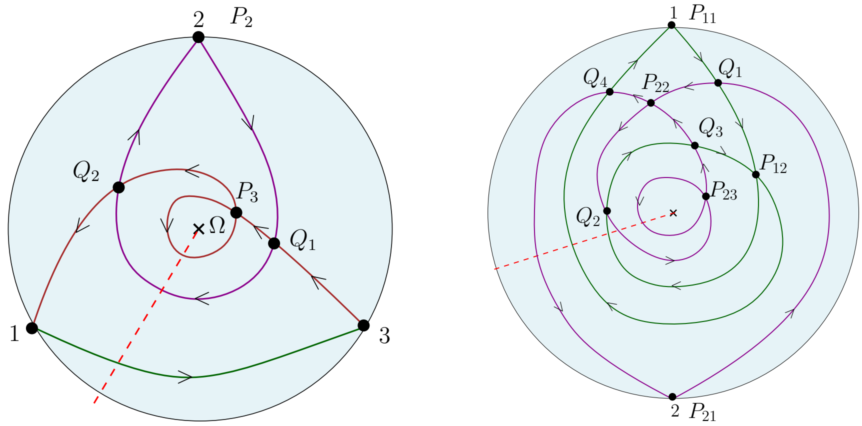

FIGURE 7. Computing the values of the winding numbers, see Definition 2.12.

Proposition 2.14. *Let $\mathcal{O}$ be an orbifold diagram of order $d > 2$, and let $\mathcal{P}$ be an s-symmetric Postnikov diagram (for some $s > 1$). Then:*

(1) $\mathrm{sym}_d(\mathcal{O})$ *is a d-symmetric Postnikov diagram.*
(2) $\mathcal{P}/s$ *is an orbifold diagram on a disk with an orbifold point Ω of order s.*
(3) $\mathrm{sym}_d(\mathcal{O})/d = \mathcal{O}$ *and, if $s > 2$,* $\mathrm{sym}_s(\mathcal{P}/s) = \mathcal{P}$.

Proof. Let us begin with (1). First, at every boundary point of $\mathrm{sym}_d(\mathcal{O})$ there is exactly one outgoing and one incoming strand, as this is true for $\mathcal{O}$. Moreover, since every point on a strand can be reached by walking along a strand on $\mathcal{O}$, the same is true for $\mathrm{sym}_d(\mathcal{O})$, which means that every piece of curve in $\mathrm{sym}_d(\mathcal{O})$ is indeed part of a strand coming from the boundary (i.e. there are no closed strands inside the interior of the disk). So condition (1) of Definition 2.1 is satisfied. Condition (2) holds by construction, as does condition (3) since it is local. Let us examine condition (5). Every self-crossing of a strand γ on $\mathrm{sym}_d(\mathcal{O})$ comes from a self-crossing of a strand η of $\mathcal{O}$. Let η' be the subcurve of η defined by such a crossing, i.e. we start from a crossing point Q and follow η until we reach Q again. Thus η' is a closed curve in $\mathcal{O}$. By condition (5) of Definition 2.6, this either can be reduced or has nonzero winding number. If it can be reduced, so can its images in $\mathrm{sym}_d(\mathcal{O})$ and condition (5) is satisfied. So assume that η' cannot be reduced. Then by Definition 2.12, the absolute value of the winding number of η' is strictly less than d. Without loss of generality, this winding number w is positive, so $1 < w < d$.

Call c the curve connecting Ω to the boundary of the disk in $\mathcal{O}$ which we chose to construct $\mathrm{sym}_d(\mathcal{O})$. We may choose c in a way to minimise the crossings with η'. Then η' crosses c exactly w times.

So if we follow the image of η' in $\mathrm{sym}_d(\mathcal{O})$ starting from a chosen lift of the crossing point Q, it will reach another lift of Q in the w-th copy of the fundamental domain, counting clockwise from the copy where Q is. Since $w < d$, these two lifts are in different regions so the lifts of the starting segment and of the ending segment of η' belong to different strands in $\mathrm{sym}_d(\mathcal{O})$. In particular, the lifts of η' do not violate condition (5) of Definition 2.1.

An analogous argument applied to the closed subcurve associated to two strands crossing in two points (as in condition (4) of Definition 2.6) shows that condition (4) must hold as well. We conclude that $\mathrm{sym}_d(\mathcal{O})$ is a Postnikov diagram, which is also invariant under rotation by $\frac{2\pi}{d}$ by construction.

Now to prove claim (2). First, conditions (1)–(5) in Definition 2.6 follow each from the corresponding condition in Definition 2.1. The inequality of Definition 2.12 follows from the (converse of) the argument we used for claim (1): the points in $\mathcal{P}$ mapping down to a self-intersection in $\mathcal{P}/s$ must be distinct since $\mathcal{P}$ is a Postnikov diagram, and so the order s is large enough. The same holds for two strands crossing in two points, and so $\mathcal{P}/s$ is an orbifold diagram.

Claim (3) is clear by definition of the operations sym_d and $/s$. □

3. Labels on orbifold diagrams

We will now explain how to associate equivalence classes of subsets of $\{1,\dots,n\}$ to alternating regions of an orbifold diagram, in a way that corresponds to the construction for Postnikov diagrams from [Pos06].

Let $\mathcal{O}$ be an orbifold diagram of order d and of type $\tau \in S_{n_0}$. To it we associate the d-symmetric Postnikov diagram $\mathrm{sym}_d(\mathcal{O})$ as explained before. The latter has $n = n_0 d$ marked points on the boundary and has type σ for some σ depending on τ and $\mathcal{O}$. For Postnikov diagrams, the rule in [Pos06] can be used to assign a k-element subset of $\{1,\dots,n\}$ to certain regions delimited by the strands (for a certain k depending on σ). We recall this construction here.

First observe that the complement of the strands of $\mathrm{sym}_d(\mathcal{O})$ in the disk is a disjoint union of topological disks, which can each be of one of three kinds. There are *boundary regions*, whose boundary contains a segment (of positive length) of the boundary of the disk, and there are *cyclical* and *alternating* regions, depending on whether the strands adjacent to them give their boundary a cyclic orientation or not. The strands adjacent to the boundary regions are alternatingly oriented and so we count these regions as alternating regions. We will assign to each alternating region a label, which is a subset of $\{1,\dots,n\}$. We do this as follows: every strand divides the disk into two pieces, one on its left and one on its right (when following the strand in its orientation). A number i is part of the label of an alternating region if and only if the region is in the left piece determined by the strand starting at vertex i. This procedure assigns a subset of some constant cardinality to every boundary and every alternating region as when we move from one alternating region to a neighbouring alternating region we always exchange one label for another one. If the Postnikov diagram is Grassmannian of type (k,n), then this cardinality is equal to k. Examples of labels on (symmetric) Postnikov diagrams are on the left hand side of Figure 10 and on the left of Figure 11.

If the Postnikov diagram is d-symmetric, then the labels of two regions related by rotation by $\frac{2\pi}{d}$ differ by adding $n_0 = n/d$ (addition on sets is meant pointwise). We use the labels of $\mathrm{sym}_d(\mathcal{O})$ to associate labels to the alternating regions of $\mathcal{O}$ by taking equivalence classes of sets of labels under adding n_0 pointwise (that is, $\{i_1,\dots,i_k\} \sim_{n_0} \{h_1,\dots,h_k\}$ if there is j such that $\{i_1 + jn_0,\dots,i_k + jn_0\} = \{h_1,\dots,h_k\}$). We use square brackets to denote the equivalence classes of sets of labels: $[i_1,\dots,i_k]_{n_0}$ for the set $\{\{i_1+jn_0, i_2+jn_0,\dots,i_k+jn_0\} \mid 1 \le j \le d\}$ where all labels i_l are from $\{1,2,\dots,n_0 d\}$. Every alternating region of $\mathcal{O}$ corresponds to d different alternating regions of $\mathrm{sym}_d(\mathcal{O})$ in general (see Remark 3.2) and as such to an equivalence class $[i_1,\dots,i_k]_{n_0}$ of labels. We assign this equivalence class to the alternating region, and do this for all alternating regions of $\mathcal{O}$.

Definition 3.1. Let $\mathcal{O}$ be an orbifold diagram with n_0 boundary points. Let $\mathcal{I}$ be the collection of labels of alternating regions of $\mathrm{sym}_d(\mathcal{O})$ and let $\sim_{n_0}$ be the equivalence relation on $\mathcal{I}$ described above. We define $\mathcal{I}_{\mathcal{O}} = \mathcal{I}/\sim_{n_0}$. By the previous discussion, $\mathcal{I}_{\mathcal{O}}$ is the set of labels attached to the alternating regions of $\mathcal{O}$.

Remark 3.2. The equivalence classes $[i_1,\dots,i_k]_{n_0}$ usually contain d elements, corresponding to the d different regions of $\mathrm{sym}_{\mathcal{O}}$ mapping down to a given region of $\mathcal{O}$. A possible exception is the central region of $\mathcal{O}$, in case it happens to be alternating: its label is a single subset which is invariant under adding n_0 to its elements. In this case we have $[i_1, \ldots, i_k]_{n_0} = \{i_1, \ldots, i_k\}$.

We now give a way to obtain the labels directly from the orbifold diagram, without going through the associated symmetric Postnikov diagram. We illustrate this algorithm in Examples 3.6, 3.7 and 3.8.

Algorithm 3.3. Step 1: Let $\mathcal{O}$ be an orbifold diagram of order d on a disk with n_0 marked points. Let $n = dn_0$. Draw a curve γ from the orbifold point Ω to the boundary of the disk which ends between n_0 and 1 (see Remark 3.4 (1)) such that all crossings of this curve with the strands γ_i are transversal, of multiplicity 2.

Step 2: The curve γ divides (some of) the strands into different connected components which we call *segments*. We now label these different segments as follows. The strand γ_i gets the label i from its starting point to the first intersection with γ. If γ_i leaves i clockwise (i.e. when leaving i, it appears to follow the boundary in a clockwise way and the orbifold point is to the right of γ_i when it crosses γ), we subtract n_0 from the label, reducing integers modulo n. If γ_i leaves i counterclockwise, we add n_0 to the label, reducing modulo n. The segment between the first crossing and the second crossing is then $i \mp n_0$ accordingly. We iterate this until all segments of each γ_i are labeled. The labels on the segments of γ_i are in $\{1, 2, \ldots, n\}$ since we reduce modulo n.

Step 3: Every strand divides the surface into two regions, one on its left and one of its right (when following the strand in its orientation). Furthermore, the complement of all strands is a union of faces, one of them containing the orbifold, where the boundary of each face is formed by parts of the strands and where each face is either cyclical or alternating. To every alternating region which is not incident with the curve γ, we associate the label $i + mn_0$ if the alternating region is to the left of the strand segment with label $i + mn_0$ (for some $m \in \mathbb{Z}$).

Step 4: Observe that the alternating regions through which γ goes are *cut in two* if we open the disc Σ along γ. We only associate labels to the region which is counterclockwise from γ (see Remark 3.4(2) below) as in Step 3: such an alternating region gets label $i+mn_0$ if it is to the left of the strand segment with label $i + mn_0$ (for some $m \in \mathbb{Z}$).

Step 5: "Add missing labels": After steps 3 and 4, every alternating region has a certain number of labels. This number is constant as whenever we go from one alternating region to a neighbouring alternating region, we cross exactly two strands, one in each direction, so one label gets added and one removed, keeping the number of labels constant. However, certain elements of $\{1, 2, \ldots, n\}$ do not appear as segment labels (Step 2). Let j be such a label and let j_0 be its reduction modulo n_0. If the orbifold point Ω is to the left of strand γ_{j_0}, we associate the label j to every alternating region of $\mathcal{O}$. If not, the label j does not appear in any of the regions.

Remark 3.4. (1) The curve γ breaks $\mathcal{O}$ open so that it can be viewed to be a copy of the fundamental domain of $\mathrm{sym}_d(\mathcal{O})$, with marked points $i, i+1, \ldots, i+n_0-1$ along the boundary (for some $i \in \{1, \ldots, dn_0\}$). It is important that γ links the orbifold point Ω with the boundary segment between n_0 and 1, in order for the algorithm to agree with Definition 3.1 without further adjustments.
(2) Note that the collection of the labels on all the segments of the strands γ_i is multiplicity-free. In general, it is a proper subset of $\{1, 2, \ldots, n\}$.
(3) Consider an alternating region which is "cut" by γ. Associate labels to the two halves of this region (under the cut by γ) according to steps 4 and 5. Let $\{i_1, \ldots, i_r\}$ be the labels of the region clockwise from γ. Then the labels of the other half are $\{i_1 + n_0, \ldots, i_r + n_0\}$.

Comparing the above construction with the definition of labels for orbifold diagrams, we get:

Lemma 3.5. *The set of labels for $\mathcal{O}$ obtained through Algorithm 3.3 is a system of representatives for $I_{\mathcal{O}}$.*

We illustrate the algorithm on the three running examples to show how we associate labels to orbifold diagrams.

Example 3.6. In Figure 8, we apply Algorithm 3.3 to the orbifold diagram of Example 2.13 (2). Recall that $d = 5$ and $n = 10$. The labels to consider in Step 5 are $3, 5, 10$. Only 10 satisfies the condition of Step 5 and will get added to all alternating regions.

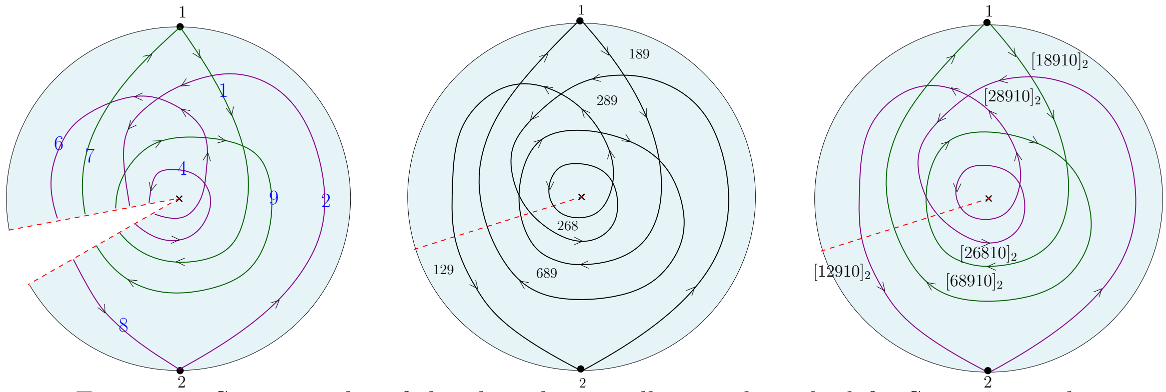

FIGURE 8. Steps 1 and 2 of the algorithm are illustrated on the left. Steps 4,5 in the middle and the resulting set of labels is drawn on the orbifold diagram on the right.

Example 3.7. In Figure 9, we apply the Algorithm 3.3 to Example 2.13 (1) with $d = 3$, $n = 9$. The labels to consider in Step 5 are $5, 7, 9$. Both 7 and 9 satisfy the condition and will get added to all alternating regions.

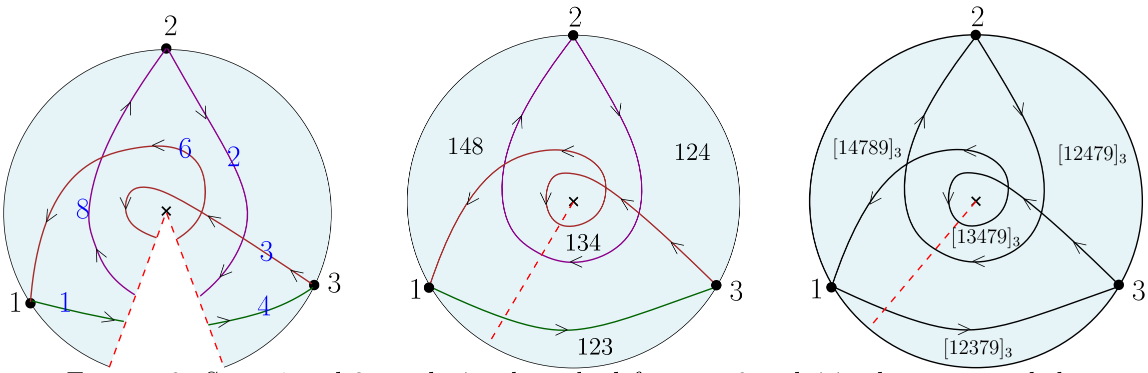

FIGURE 9. Steps 1 and 2 are depicted on the left, steps 3 and 4 in the center and the last step on the right.

Example 3.8. We consider the orbifold diagram $\mathcal{O}$ of order 3 from Example 2.11 and want to determine its labels. In this example we have an alternating region around the orbifold point of order 3. We are going to use Definition 3.1. On the left hand in Figure 10 we have the labels for the Postnikov diagram $\mathrm{sym}_3(\mathcal{O})$, the dashed lines indicate three fundamental regions for the action, namely the rotation by $\frac{2\pi}{3}$. For each alternating region of the diagram on the left in Figure 10, we assign an equivalence class of 3-element subsets of $\{1, 2, \dots, 9\}$ by considering a representative given by the label associated with the fundamental region containing the vertices $\{2, 3, 4\}$ of $\mathrm{sym}_3(\mathcal{O})$ on the right of the same Figure. Note that all but the region containing the orbifold point have an equivalence class with three elements, one for each copy of the fundamental region. The region containing the orbifold point has an equivalence class with just one element because the corresponding region in $\mathrm{sym}_3(\mathcal{O})$ is fixed by the action.

4. QUIVERS WITH POTENTIALS

Now we shall define a quiver with potential (QP for short) associated to an orbifold diagram, in order to compare it with the one associated to its cover as in [BKM16, Section 3].

Definition 4.1. Let $\mathcal{P}$ be a Postnikov diagram. We associate to it a quiver with potential $(Q_{\mathcal{P}}, W_{\mathcal{P}})$ as follows. The vertices of $Q_{\mathcal{P}}$ are given by the alternating regions of $\mathcal{P}$ (recall that we treat boundary regions as alternating). For any two alternating regions sharing a crossing, there is an arrow in $Q_{\mathcal{P}}$ going through that crossing following the orientation of the strands. Observe that $Q_{\mathcal{P}}$ is naturally a quiver with faces, with fundamental cycles corresponding to cyclical regions of $\mathcal{P}$. The potential $W_{\mathcal{P}}$ is defined as the sum of these cycles, with signs depending on their orientations.

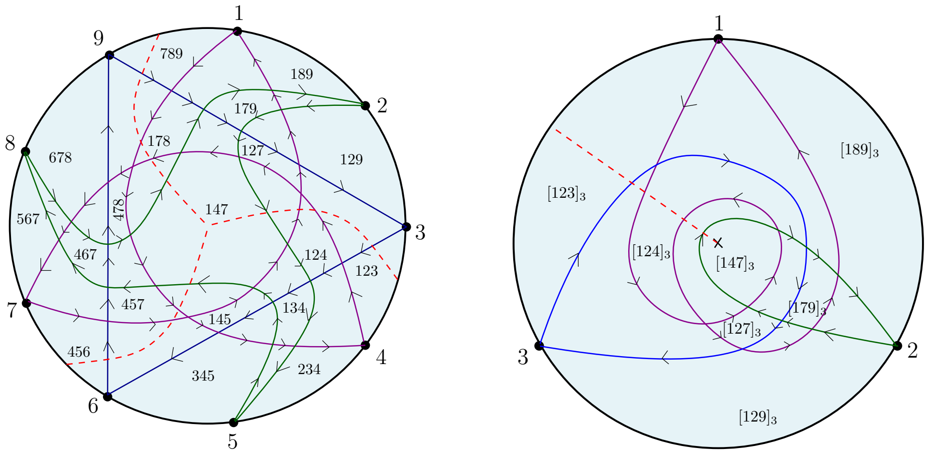

FIGURE 10. The first figure shows the labels of $\mathrm{sym}_3(\mathcal{O})$, the second figure shows the equivalence classes of the labels for $\mathcal{O}$ under the equivalence relation $\sim_3$

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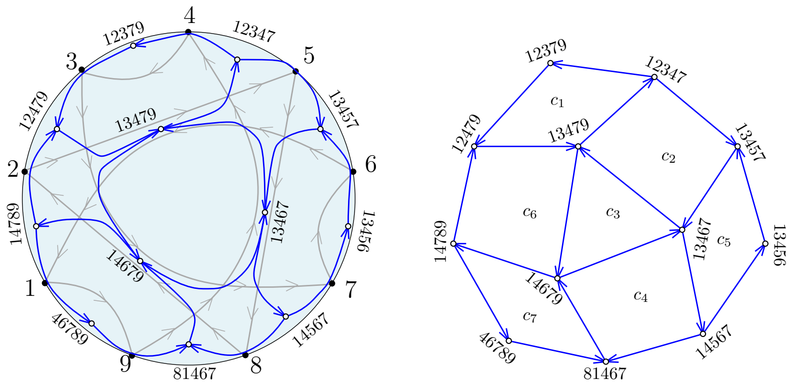

FIGURE 11. The quiver $Q_\mathcal{P}$ for $\mathcal{P} = \mathrm{sym}_3(\mathcal{O})$ from Example 2.9, straightened on the right.

Example 4.2. In Figure 11, we have the quiver $Q_\mathcal{P}$ for the Postnikov diagram (with labels) from Figure 2. On the right, the quiver is drawn with straight arrows. The potential $W_\mathcal{P}$ is

$$\sum_{i=1}^{7} \mathrm{sgn}(c_i)c_i = c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + c_7,$$

for c_i the fundamental cycle of the face indicated by c_i, taken with $+$ if and only if c_i is counterclockwise.

The quiver Q_P is called a dimer model with boundary in [BKM16]. We recall the definition of the (frozen) Jacobian algebra associated to a quiver with potential:

Definition 4.3. Let $(Q_\mathcal{P}, W_\mathcal{P})$ be the quiver with potential associated to the Postnikov diagram $\mathcal{P}$. The *frozen Jacobian algebra associated to the QP* $(Q_\mathcal{P}, W_\mathcal{P})$ is the completed path algebra of $Q_\mathcal{P}$ modulo the closure of the relations given by the cyclic derivatives of the potential with respect to *internal arrows* (arrows incident with two faces): Let α be internal and let c_1 and c_2 be the two fundamental cycles containing α, with $\mathrm{sgn}\, c_1 = +$, $\mathrm{sgn}\, c_2 = -$. We write $c_i = \alpha c_i'$ for $i = 1, 2$. Then $\partial_\alpha W_\mathcal{P} = c_1' - c_2'$. In other words, for any internal arrow α, the two paths completing α to a fundamental cycle agree. For example, in Figure 11, the arrow $14679 \to 12467$ induces as a relation that the path of length 2 from 12467 to 14679 is equal to the path of length 3 from 12467 to 14679.

Definition 4.4. Now assume that $\mathcal{P}$ is reduced, i.e. no reduction moves such as in Figure 1 are possible. Then we define the *algebra* $A(\mathcal{P})$ *of* $\mathcal{P}$ as the (completed) frozen Jacobian algebra of $(Q_\mathcal{P}, W_\mathcal{P})$, where the frozen vertices correspond to the boundary regions. If e is the idempotent corresponding to the vertices of the boundary regions, we also define the *boundary algebra* $B(\mathcal{P})$ *of* $\mathcal{P}$ to be the idempotent subalgebra $eA(\mathcal{P})e$.

We will give an analogous definition of quiver with potential for orbifold diagrams, in such a way that the frozen Jacobian algebras are related to each other by a skew group algebra construction. This requires some work.

Definition 4.5. Let $\mathcal{O}$ be an orbifold diagram of order d. We associate to it a quiver $Q_{\mathcal{O}}$ as follows. The vertices of $Q_{\mathcal{O}}$ are given by the alternating regions of $\mathcal{O}$ (including the regions on the boundary). If the orbifold point Ω is contained in an alternating region, we associate to that region d vertices $v_1, \ldots, v_d$ of $Q_{\mathcal{O}}$. We imagine the v_i as lying on a line orthogonal to the disk above the orbifold point. For any two vertices which are separated by a crossing of oriented strands, there is an arrow in $Q_{\mathcal{O}}$ going through that crossing following the orientation of the strands. In the case of vertices $v_1, \ldots, v_d$ (if present), we draw arrows between each of them and all the neighbouring regions but no arrows between these these vertices.

In case the region containing Ω is alternating, with an even number $r \geq 2$ of arrows incident with it, then each of the vertices v_i has r fundamental cycles incident with it.

The quiver $Q_{\mathcal{O}}$ is also naturally a quiver with faces: Its fundamental cycles correspond to cyclical regions in $\mathcal{O}$ which do not involve the orbifold point, together with with d copies of the r cycles corresponding to cyclical regions adjacent to the central region containing Ω, if this region is alternating. Seen as a CW-complex, this quiver with faces consists in this case of an annulus where the boundaries are given by non-oriented cycles, together with d disks. These disks are all isomorphic as quivers with faces and their boundary cycle (which is in general not oriented) is identical to the inner boundary cycle of the annulus. These boundary cycles are identified with the inner boundary of the annulus, i.e. the d disks are all glued along one of the boundary components of this annulus, see [GP19, Proposition 7.7]. If the region containing Ω is cyclical then $Q_{\mathcal{O}}$, as quiver with faces, is a tiling of the disk.

In what follows, if Ω belongs to an alternating region, we write $c_i^{(1)}, \ldots, c_i^{(r)}$ for the r fundamental cycles in $Q_{\mathcal{O}}$ through v_i, for $i = 1, \ldots, d$.

Note that all the fundamental cycles come with an orientation and hence with a sign: We set $\operatorname{sgn}(c)$ to be 1 if c is a counterclockwise fundamental cycle and -1 if c is clockwise. Then we can define a potential for the quiver $Q_{\mathcal{O}}$.

Definition 4.6. Let $\mathcal{O}$ be an orbifold diagram of order d and let $Q_{\mathcal{O}}$ be its quiver. Let $\mathcal{C}$ be the set of the fundamental cycles of $Q_{\mathcal{O}}$. We define a potential $W_{\mathcal{O}}$ on $Q_{\mathcal{O}}$ as follows.

- Assume that the orbifold point Ω lies in a cyclical region and let c be the corresponding fundamental cycle. We set
$$W_{\mathcal{O}} = \frac{1}{d}\operatorname{sgn}(c)c^d + \sum_{c' \in \mathcal{C}\setminus\{c\}} \operatorname{sgn}(c')c'.$$
- Assume that Ω lies in an alternating region and let $\mathcal{C}'$ the set of all the $c_i^{(j)}$. Fix a primitive d-th root of unity ζ. We set
$$W_{\mathcal{O}} = \sum_{c \in \mathcal{C}\setminus\mathcal{C}'} \operatorname{sgn}(c)c + \sum_{j \neq r}\sum_{i=1}^{d} \operatorname{sgn}(c_i^{(j)})c_i^{(j)} + \sum_{i=1}^{d} \zeta^i \operatorname{sgn}(c_i^{(r)})c_i^{(r)}.$$

Remark 4.7. The above definition of $W_{\mathcal{O}}$ depends on the choice of ζ. However, Theorem 6.23 shows that the frozen Jacobian algebras corresponding to different choices are isomorphic.

Remark 4.8. The potential $W_{\mathcal{O}}$ of Definition 4.6 is equal, for suitable choices (see the proof of Proposition 6.1), to the potential W_G of [GP19, Notation 3.18] divided by d. It is also equal to the potential W_G of [GPP19, Definition 5.3].

Definition 4.9. For an orbifold diagram $\mathcal{O}$, define the *algebra* $A(\mathcal{O})$ *of* $\mathcal{O}$ as the frozen Jacobian algebra of $(Q_{\mathcal{O}}, W_{\mathcal{O}})$, with frozen vertices the boundary vertices. If e is the idempotent corresponding to the boundary vertices, we define the *boundary algebra* $B(\mathcal{O})$ *of* $\mathcal{O}$ to be the algebra $B(\mathcal{O}) = eA(\mathcal{O})e$.

Note that whenever for $\mathcal{O}$, the orbifold point Ω is contained in a cyclic region, we have a new type of terms in the relations for $A(\mathcal{O})$: In this case, the cycle (or loop) appears as term c^d in the potential. Taking derivatives with respect to arrows of this cycle (with respect to the arrow of the loop) gives a d-fold term in the relations for $A(\mathcal{O})$ (with this d cancelling out the $\frac{1}{d}$ coefficient). For the quiver with potential in Figure 12, taking the derivative with respect to the loop arrow c gives $c_1' = c^2$, where c_1' is the path $[14679]_3 \to [14789]_3 \to [12479]_3$.

Example 4.10. We illustrate Definitions 4.5 and 4.6 on the orbifold diagram $\mathcal{O}_1$ of order 3 from Example 2.13 (1) with labels in Example 3.7 and on the orbifold diagram $\mathcal{O}_2$ from Example 2.13 (2) with labels in Example 3.6. The quivers $Q_{\mathcal{O}_1}$ and $Q_{\mathcal{O}_2}$ are depicted in Figure 12 and Figure 13, respectively.

$$W_{\mathcal{O}_1} = -c_1 + c_2 + \frac{1}{3}c^3,$$
$$W_{\mathcal{O}_2} = -c_1 + c_2 + c_3 - c_4 + \frac{1}{5}c^5.$$

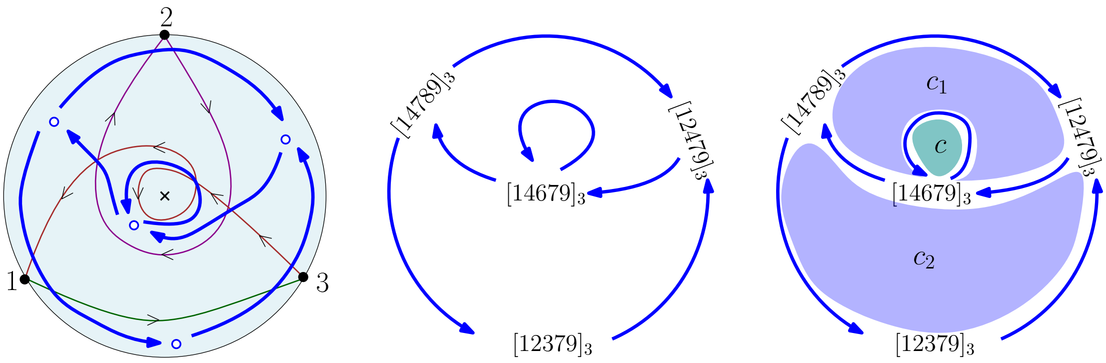

FIGURE 12. The quiver $Q_{\mathcal{O}_1}$ associated to the orbifold diagram of Examples 2.13(1).

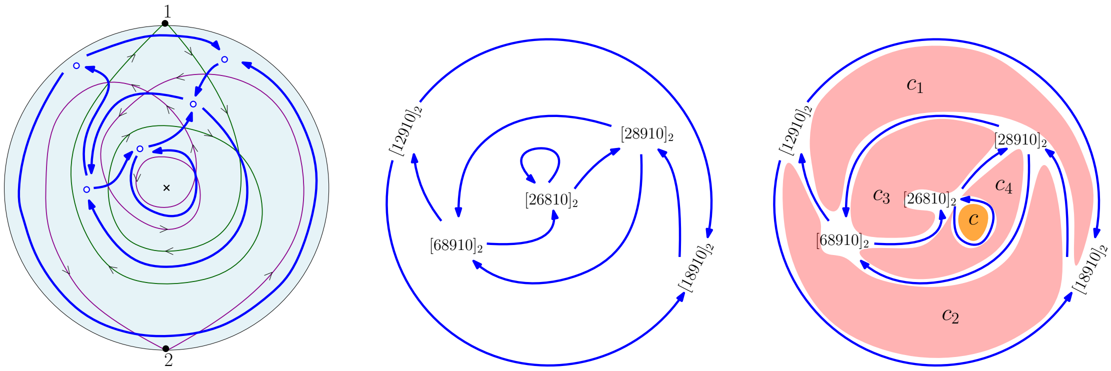

FIGURE 13. The quiver $Q_{\mathcal{O}_2}$ associated to the orbifold diagram of Examples 2.13(2).

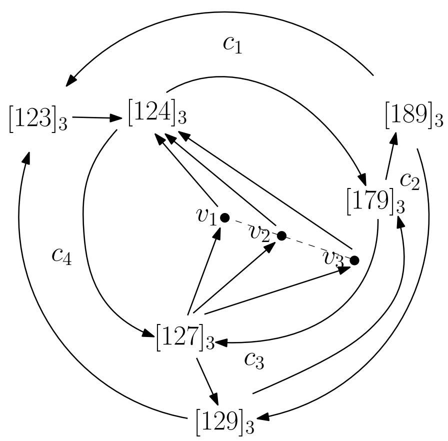

FIGURE 14. The quiver $Q_{\mathcal{O}}$ associated to an orbifold diagram $\mathcal{O}$ of order 3 with the orbifold point inside an alternating region and the fundamental cycles c_i far away from the orbifold point.

Example 4.11. Let us consider the quiver with potential for the orbifold diagram $\mathcal{O}$ given in Example 3.8. Recall that this example is particularly interesting because it has an alternating region containing the orbifold point. We are going to follow Definition 4.5. The quiver $Q_{\mathcal{O}}$ is depicted in Figure 14. Note that on the right of Figure 10 the alternating region containing the orbifold point got the label $[147]_3$. Following Definition 4.5, the label $[147]_3$ yields three vertices called v_1, v_2 and v_3 on $Q_{\mathcal{O}}$, see Figure 14. Since the alternating region containing Ω is given by 2 arrows, we have two cyclic regions around this region /with label $[147]_3$). One is a triangle given by $\{[124]_3, [127]_3, [147]_3\}$, the other one is a quadrilateral given by $\{[127]_3, [147]_3, [124]_3, [179]_3\}$. We write c_i^1 $(i = 1, 2, 3)$ to denote the three fundamental cycles arising from the triangle at $[147]_3$ and c_i^2 for the three fundamental cycles arising from the quadrilateral. These six faces are illustrated in Figure 15 by different shadings. The labeling of the other faces is given in Figure 14. We thus have the potential

$$W_{\mathcal{O}} = +c_1 - c_2 + c_3 - c_4 + c_1^{(1)} + c_2^{(1)} + c_3^{(1)} - \zeta c_1^{(2)} - \zeta^2 c_2^{(2)} - c_3^{(2)},$$

where ζ is a primitive third root of the unity.

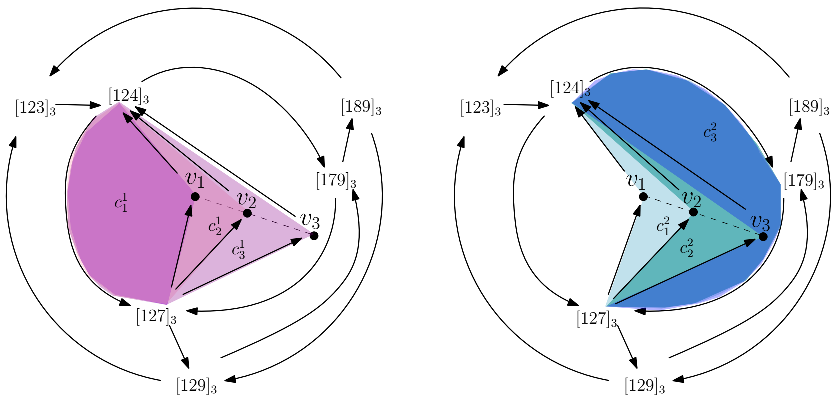

FIGURE 15. The fundamental cycles around the orbifold point on $Q_{\mathcal{O}}$.

5. Skew group algebras from orbifold diagrams

In this section we recall the notion of a skew group algebra and prove properties which we will need later. We want to relate the algebras $A(\mathcal{O})$ and $B(\mathcal{O})$ of an orbifold diagram $\mathcal{O}$ with the algebras $A(\mathrm{sym}_d(\mathcal{O}))$ and $B(\mathrm{sym}_d(\mathcal{O}))$ of the associated symmetric Postnikov diagram. In particular, we want to describe endomorphism algebras (see Lemma 5.3 and Lemma 5.12). This will be a key ingredient for Section 6 where we study the associated module categories.

Definition 5.1. Let S be an algebra with an action of a finite group G by automorphisms. The *skew group algebra* $S * G$ is $S \otimes_{\mathbb{C}} \mathbb{C}G$ as a vector space, with multiplication linearly induced by

$$(s \otimes g)(t \otimes h) = sg(t) \otimes gh$$

(where $s, t \in S$, $g, h \in G$).

The group G acts on the category $\mathrm{mod}(S)$ (left modules) by twists, that is $g(L) = {}^gL$ which is L as a set, with S-action given by $s \cdot_{{}^gL} l = g(s)l$. This gives an autofunctor of $\mathrm{mod}(S)$ by letting G act trivially on morphisms. To simplify the notation, we will write morphisms as f instead of ${}^gf$.

There is an induction functor F from $\mathrm{mod}\, S$ to $\mathrm{mod}(S * G)$ sending M to $FM := (S * G) \otimes_S M$.

The *category of G-equivariant S-modules* $\mathrm{mod}(S)^G$ is defined to have as objects the pairs $\left(L, (\varphi_g)_{g\in G}\right)$, where L is an object of $\mathrm{mod}(S)$ and where the $\varphi_g : L \to {}^gL$ are isomorphisms satisfying the following:

(1) $\varphi_{gh} = \varphi_h \circ \varphi_g$,
(2) $\varphi_1 = \mathrm{id}_L$.

Morphisms in $\mathrm{mod}(S)^G$ are morphisms of S-modules which intertwine the φ_g. Precisely, if $\left(L, (\varphi_g)_{g\in G}\right)$ and $\left(N, (\psi_g)_{g\in G}\right)$ are in $\mathrm{mod}(S)^G$, then a map $f : L \to N$ of S-modules is a morphism in $\mathrm{mod}(S)^G$ if there is a commutative diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\varphi_g"'] & N \arrow[d, "\psi_g"] \\ {}^gL \arrow[r, "f"] & {}^gN \end{tikzcd}$$

for every $g \in G$.

There is a functor $E : \mathrm{mod}(S * G) \to \mathrm{mod}(S)^G$ that takes a module L to $\Big(L, (1 \otimes g \cdot -)_{g\in G}\Big)$. The functor E is in fact an equivalence: informally, this is because an $(S * G)$-module is the same as an S-module with a compatible G-action.

Let L be an S-module. Its image under EF is

$$EFL = \left(\bigoplus_{g\in G} {}^gL, (\varphi_h)_{h\in G}\right)$$

where φ_h permutes the summands in the natural way.

From now on, let us assume that G is a cyclic group of order d. We will use the skew-group construction for G and the algebras associated to orbifold diagrams (of order d). On one hand, we will have modules which are invariant under the group action and on the other hand modules, which are cyclically permuted by the elements of G. In Section 6, we will use the effect of the composition EF on modules which are built up from these two types, i.e. we will need to study the functor EF on a direct sum $M_0 \otimes M$ where M_0 is invariant under the group action and where M is a sum of isomorphic summands for each group element (see Lemma 6.14).

Let $M_0 \in \mathrm{mod}(S)$ be a module such that ${}^gM_0 = M_0$ for all $g \in G$.

Then

$$EFM_0 = \left(\bigoplus_{g\in G} M_0, (\varphi_h)_{h\in G}\right) = \left(\bigoplus_{\sigma\in G^*} M_0, (\lambda_h)_{h\in G}\right) = \bigoplus_{\sigma\in G^*} (M_0, (\sigma(h)\,\mathrm{id})_{h\in G}),$$

where $(\lambda_h)_{\sigma,\sigma} = \sigma(h)\,\mathrm{id}_{M_0}$ and the other entries of λ_h are zero. This is because the subset

$$\left\{ x \in \bigoplus_{g\in G} M_0 \;\mid\; h(x) = \sigma(h)x \right\} \subseteq \bigoplus_{g\in G} M_0$$

is isomorphic to M_0 as an S-module (intuitively, we are decomposing the regular representation into the irreducible characters of G).

Let $M_1 \in \mathrm{mod}(S)$ be any module. Define $M_g = {}^gM_1$ for every $g \in G$, and $M = \bigoplus_{g\in G} M_g$. Then define $\varphi_g : M \to {}^gM$ to be the canonical isomorphism that permutes the summands, i.e. $\varphi_g(M_h) = {}^gM_{g^{-1}h}$. This makes $\Big(M, (\varphi_g)_{g\in G}\Big)$ an object in $\mathrm{mod}(S)^G$. A preimage of $\Big(M, (\varphi_g)_{g\in G}\Big)$ under E is M as a set, with $b \otimes g$ acting by $\varphi_g \circ b \cdot \circ \varphi_g^{-1}$. We will just write M for this object of $\mathrm{mod}(S * G)$ as well as for the S-module M.

Remark 5.2. Observe that we really mean M as an $(S * G)$-module, and not FM. Indeed, we have $FM \cong M^{\oplus d}$, and in fact $M = FM_1$ naturally.

Let now $X = M_0 \oplus M$ for M_0 and M defined as above. We associate two algebras R_1 and R_2 to X. First we define

$$R_1 = \mathrm{End}_{\mathrm{mod}(S)^G}(EFM_0 \oplus EM) = \mathrm{End}_{\mathrm{mod}(S)^G}(EF(M_0 \oplus M_1)).$$

To define R_2, consider the ($\mathbb{C}$-linear) action of the group G on $\mathrm{End}_S(X)$:

(1) If $f : M_0 \to M_0$, then $g(f) = f$.
(2) If $f : M_0 \to M$, then $g(f) = \varphi_g \circ f : M_0 \to {}^gM$.
(3) If $f : M \to M_0$, then $g(f) = f \circ \varphi_g^{-1} : {}^gM \to M_0$.
(4) If $f : M \to M$, then $g(f) = \varphi_g \circ f \circ \varphi_g^{-1} : {}^gM \to {}^gM$.

With this action, we can construct the skew group algebra $\mathrm{End}_S(X) * G$. For $g \in G$, let $\pi_g \in \mathrm{End}_S(X)$ be the projection onto M_g, and let π_0 be the projection onto M_0. Then $h(\pi_0) = \pi_0$ and $h(\pi_g) = \pi_{gh^{-1}}$ for any $h \in G$. We write $e = \pi_1 \otimes 1 + \pi_0 \otimes 1 \in \mathrm{End}_S(X) * G$. Then we define the idempotent subring $R_2 = e(\mathrm{End}_S(X) * G)e$. Our main technical lemma is:

Lemma 5.3. *With this setup, the algebras R_1 and R_2 are isomorphic. In particular, R_1 is Morita equivalent to $\operatorname{End}_S(X) * G$.*

The rest of the section is devoted to the proof of Lemma 5.3. To study $R_1 = \operatorname{End}_{\operatorname{mod}(S)^G}(EFM_0 \oplus EM)$, we will consider its four direct summands, i.e. $\operatorname{End}(EFM_0)$, $\operatorname{Hom}(EFM_0, EM)$, $\operatorname{Hom}(EM, EFM_0)$, and End(EM) as well as the algebra multiplication in R_1.

We begin by analysing which endomorphisms of $EFM_0 \oplus EM$ as an S-module are actually endomorphisms of $EFM_0 \oplus EM$ in $\operatorname{mod}(S)^G$.

Lemma 5.4. *A map of S-modules $f : EFM_0 \to EFM_0$ is a map in $\operatorname{mod}(S)^G$ if and only if $f_{\sigma,\tau} = 0$ whenever $\sigma \neq \tau$.*

Proof. The only condition is $f \circ \lambda_g = \lambda_g \circ f$, which yields $f_{\sigma,\tau}\sigma(g) = f_{\sigma,\tau}\tau(g)$ for all $\sigma, \tau \in G^*$. □

Lemma 5.5. *A map of S-modules $f : EFM_0 \to EM$ is a map in $\operatorname{mod}(S)^G$ if and only if*

$$f_{g,\sigma} = \sigma(g) \left(\varphi_g^{-1}\right)_{g,1} \circ f_{1,\sigma}$$

for all $g \in G, \sigma \in G^*$.

Proof. We need to impose the condition $\varphi_g \circ f = f \circ \lambda_g$ for all g. This gives $f_{gh,\sigma} = \sigma(g) \left(\varphi_g^{-1}\right)_{gh,h} \circ f_{h,\sigma}$, which is equivalent to the condition of the statement. □

Lemma 5.6. *A map of S-modules $f : EM \to EFM_0$ is a map in $\operatorname{mod}(S)^G$ if and only if*

$$f_{\sigma,g} = \sigma^{-1}(g) f_{\sigma,1} \circ (\varphi_g)_{1,g}$$

for all $g \in G, \sigma \in G^*$.

Proof. We need to impose the condition $f \circ \varphi_g = \lambda_g \circ f$ for all g. This gives $f_{\sigma,h} = \sigma^{-1}(g) f_{\sigma,g^{-1}h} \circ (\varphi_g)_{g^{-1}h,h}$, which is equivalent to the condition of the statement. □

Lemma 5.7. *A map of S-modules $f : EM \to EM$ is a map in $\operatorname{mod}(S)^G$ if and only if*

$$f_{g^{-1},g^{-1}h} = (\varphi_g)_{g^{-1},1} \circ f_{1,h} \circ \left(\varphi_g^{-1}\right)_{h,g^{-1}h}$$

for all $g, h \in G$.

Proof. We need to impose the condition $f \circ \varphi_g = \varphi_g \circ f$ for all g. This gives $f_{l,g^{-1}h} = (\varphi_g)_{l,gl} \circ f_{gl,h} \circ \left(\varphi_g^{-1}\right)_{h,g^{-1}h}$, which is equivalent to the condition of the statement. □

Now we analyse the ring R_2. By construction, it is linearly generated by the four subspaces

$$\begin{aligned} V_1 &= (\pi_0 \otimes 1)(\operatorname{End}_S(X) * G)(\pi_0 \otimes 1), \\ V_2 &= (\pi_1 \otimes 1)(\operatorname{End}_S(X) * G)(\pi_0 \otimes 1), \\ V_3 &= (\pi_0 \otimes 1)(\operatorname{End}_S(X) * G)(\pi_1 \otimes 1), \text{ and} \\ V_4 &= (\pi_1 \otimes 1)(\operatorname{End}_S(X) * G)(\pi_1 \otimes 1). \end{aligned}$$

We will construct linear bijections

$$\begin{aligned} V_1 &\to \operatorname{End}_{\operatorname{mod}(S)^G}(EFM_0), \\ V_2 &\to \operatorname{Hom}_{\operatorname{mod}(S)^G}(EFM_0, EM), \\ V_3 &\to \operatorname{Hom}_{\operatorname{mod}(S)^G}(EM, EFM_0), \text{ and} \\ V_4 &\to \operatorname{End}_{\operatorname{mod}(S)^G}(EM), \end{aligned}$$

and then prove that multiplication in R_1 and in R_2 agrees. This will prove that $R_1 \cong R_2$.

Map from the subspace V_1. By definition V_1 can be identified with $\operatorname{End}_S(M_0) * G$ (in particular, it is a subalgebra of $\operatorname{End}_S(X) * G$). It is generated (as a vector space) by

$$\{f \otimes g \mid f : M_0 \to M_0, g \in G\},$$

so it also generated by

$$\{f \otimes e_\sigma \mid f : M_0 \to M_0, \sigma \in G^*\},$$

where e_σ is the idempotent of $\mathbb{C}G$ given by $e_\sigma = \frac{1}{d}\sum_{g\in G}\sigma(g)g$. Note that $he_\sigma = \sigma(h^{-1})e_\sigma$ for any $h \in G$.

Let us define a linear map $\psi : V_1 \to \operatorname{End}_{\operatorname{mod}(S)^G}(EFM_0)$ by setting $\psi(f \otimes e_\sigma)$ to have f in component (σ, σ) and zeros everywhere else. By Lemma 5.4, this is indeed an element of $\operatorname{End}_{\operatorname{mod}(S)^G}(FM_0)$, and ψ thus defined is clearly a linear bijection.

Lemma 5.8. *The map ψ is a ring isomorphism $V_1 \cong \mathrm{End}_{\mathrm{mod}(S)^G}(EFM_0)$.*

Proof. On the one side, we compute $\psi(f' \otimes e_{\sigma'})\psi(f \otimes e_\sigma)$. By definition this has $f' \circ f$ in component σ, σ if $\sigma = \sigma'$, and is zero otherwise. On the other side,

$$\psi((f' \otimes e_{\sigma'})(f \otimes e_\sigma)) = \frac{1}{d} \sum_{g' \in G} \sigma'(g') \psi\left(f' \circ g'(f) \otimes g' e_\sigma\right).$$

This has as only nonzero entry the σ, σ component, which equals

$$\frac{1}{d} \sum_{g' \in G} \sigma'(g') \sigma^{-1}(g') f' \circ f$$

which in turn is (by Schur orthogonality) $f' \circ f$ if $\sigma = \sigma'$ and zero else. This proves that ψ is multiplicative. By definition, $\mathrm{id}_{M_0} \otimes 1$ is mapped to id_{M_0}, so we are done. □

Map from the subspace V_2. Let us now analyse V_2. This is not a ring, but can instead be identified as a vector space with $\mathrm{Hom}_S(M_0, M_1) \otimes \mathbb{C}G$. So it is generated by elements of the form $f \otimes e_\sigma$ for $f : M_0 \to M_1$, and we can define $\psi : V_2 \to \mathrm{Hom}_{\mathrm{mod}(S)^G}(EFM_0, EM)$ by setting

$$\psi(f \otimes e_\sigma)_{g,\sigma} = \sigma(g) \left(\varphi_g^{-1}\right)_{g,1} \circ f$$

for all $g \in G$, and the other entries to be zero.

Lemma 5.9. *The map ψ is a linear bijection $V_2 \to \mathrm{Hom}_{\mathrm{mod}(S)^G}(EFM_0, EM)$.*

Proof. The map ψ is linear by definition. Let us check that its image inside $\mathrm{Hom}_S(EFM_0, EM)$ does lie inside $\mathrm{Hom}_{\mathrm{mod}(S)^G}(EFM_0, EM)$. This follows directly from Lemma 5.5, by observing that by definition $\psi(f \otimes e_\sigma)_{1,\sigma} = f$. Moreover, every map from EFM_0 to EM satisfying the condition of Lemma 5.5 arises as a linear combination of $\psi(f \otimes e_\sigma)$ for some f's and σ's, so ψ is surjective. Finally, $\psi(f \otimes e_\sigma) = 0$ implies $f = 0$, so ψ is injective and we are done. □

Map from the subspace V_3. Let us now look at V_3. This is also not a ring, and can be identified with $(\mathrm{Hom}_S(M, M_0) \otimes \mathbb{C}G)(\pi_1 \otimes 1)$. Since $g(\pi_1) = \pi_{g^{-1}}$, V_3 is generated as a vector space by

$$\left\{f \otimes g \mid f : M_{g^{-1}} \to M_0,\ g \in G\right\}.$$

Now, for a fixed $f_{0,1} : M_1 \to M_0$, the subspace of V_3 generated by

$$\left\{f_{0,1} \circ \left(\varphi_{g^{-1}}\right)_{1,g^{-1}} \otimes g \mid g \in G\right\}$$

is the same as the one generated by

$$\left\{\sum_{g \in G} \sigma(g) f_{0,1} \circ \left(\varphi_{g^{-1}}\right)_{1,g^{-1}} \otimes g \mid \sigma \in G^*\right\}$$

which in turn can be written as

$$\begin{aligned}&\left\{f \otimes e_\sigma \mid \sigma \in G^*, f : M \to M_0 \text{ such that } f_{0,g} = f_{0,1} \circ \left(\varphi_{g^{-1}}\right)_{1,g^{-1}} \text{ for all } g \in G\right\}\\ &= \{f \otimes e_\sigma \mid \sigma \in G^*, f : M \to M_0 \text{ such that } g(f) = f \text{ for all } g \in G\}\end{aligned}$$

Since any map $M_{g^{-1}} \to M_0$ is of the form $f_{0,1} \circ \left(\varphi_{g^{-1}}\right)_{1,g^{-1}}$ for some $f_{0,1}$, in order to define a linear map $\psi : V_3 \to \mathrm{Hom}_{\mathrm{mod}(S)^G}(EM, EFM_0)$ it is enough to specify $\psi(f \otimes e_\sigma)$ for $f : M \to M_0$ as above. We define $\psi(f \otimes e_\sigma)_{\sigma,g} = \sigma^{-1}(g) f_{0,g}$ and the other entries to be zero.

Lemma 5.10. *The map ψ is a linear bijection $V_3 \to \mathrm{Hom}_{\mathrm{mod}(S)^G}(EM, EFM_0)$.*

Proof. The map ψ is linear by definition. Its image is indeed in $\mathrm{Hom}_{\mathrm{mod}(S)^G}(EM, EFM_0)$ by Lemma 5.6, since for every f which appears in our chosen generating set for V_3 we have $f_{0,g} = f_{0,1} \circ \left(\varphi_{g^{-1}}\right)_{1,g^{-1}}$. Moreover, Lemma 5.6 also implies that ψ is surjective. As above, $\psi(f \otimes e_\sigma) = 0$ implies $f = 0$, so we are done. □

Map from the subspace V_4. Let us finally look at V_4. This is a subalgebra of $\mathrm{End}_S(X) * G$, which can be identified with

$$(\pi_1 \otimes 1)(\mathrm{End}_S(M) * G)(\pi_1 \otimes 1).$$

It is thus generated by

$$\left\{ f \otimes g \mid f : M_{g^{-1}} \to M_1,\ g \in G \right\}.$$

Let us define a linear map $\psi : V_4 \to \mathrm{End}_{\mathrm{mod}(S)^G}(EM)$ by setting

$$(\psi(f \otimes g))_{h^{-1},h^{-1}g^{-1}} = (\varphi_h)_{h^{-1},1} \circ f \circ \left(\varphi_h^{-1}\right)_{g^{-1},g^{-1}h^{-1}}$$

for all $h \in G$, and the other entries to be zero. By Lemma 5.7, the image of ψ is indeed in $\mathrm{End}_{\mathrm{mod}(S)^G}(EM)$, and moreover ψ is surjective. It is also clearly injective, so a linear bijection.

Lemma 5.11. *The map ψ is a ring isomorphism $V_4 \cong \mathrm{End}_{\mathrm{mod}(S)^G}(EM)$.*

Proof. We need to check multiplicativity. Looking at a generic component h, l we have on the one hand

$$(\psi(f' \otimes g')\psi(f \otimes g))_{h,l} = (\psi(f' \otimes g'))_{h,hg'^{-1}}\,(\psi(f \otimes g))_{hg'^{-1},hg'^{-1}g^{-1}}$$

when $l = hg'^{-1}g^{-1}$, in which case the above equals

$$(\varphi_{h^{-1}})_{h,1} \circ f' \circ \left(\varphi_{h^{-1}}^{-1}\right)_{g'^{-1},hg'^{-1}} \circ \left(\varphi_{h^{-1}g'}\right)_{hg'^{-1},1} \circ f \circ \left(\varphi_{h^{-1}g'}^{-1}\right)_{g^{-1},hg'^{-1}g^{-1}}.$$

On the other hand,

$$(\psi((f' \otimes g')(f \otimes g)))_{h,l} = (\psi(f' \circ g'(f) \otimes gg'))_{h,l}$$

which is zero unless $l = hg'^{-1}g^{-1}$, in which case it equals

$$(\varphi_{h^{-1}})_{h,1} \circ f' \circ (\varphi_{g'})_{g'^{-1},1} \circ f \circ \left(\varphi_{g'}^{-1}\right)_{g^{-1},g^{-1}g'^{-1}} \circ \left(\varphi_{h^{-1}}^{-1}\right)_{g^{-1}g'^{-1},hg'^{-1}g^{-1}},$$

so we conclude that ψ is indeed multiplicative since the assumption that $\varphi_{gh} = \varphi_h \circ \varphi_g$ implies that the components of φ_g are multiplicative. Finally, $\psi(\mathrm{id}_M \otimes 1) = \mathrm{id}_{EM}$ by definition and we are done. $\square$

5.1. **Putting the isomorphisms together.** We can extend by linearity the maps ψ of the four subsections to a linear map $\psi : R_2 \to R_1$.

Lemma 5.12. *The map ψ is an algebra isomorphism $R_2 \to R_1$.*

Proof. We already proved that ψ is a unital linear bijection (in Lemmas 5.8, 5.9, 5.10 and 5.11). It remains to prove multiplicativity. Let $f_1 : M_0 \to M_0$, $f_2 : M_0 \to M_1$, $f_3 : M \to M_0$ and $f_4 : M_{g_4^{-1}} \to M_1$. As in the definition of ψ on V_3, we assume that $g(f_3) = f_3$ for all $g \in G$. Moreover let $\sigma_1, \sigma_2, \sigma_3 \in G^*$ and $g_4 \in G$. It is enough to prove:

1. $\psi((f_1 \otimes e_{\sigma_1})(f_3 \otimes e_{\sigma_3})) = \psi(f_1 \otimes e_{\sigma_1})\psi(f_3 \otimes e_{\sigma_3})$,
2. $\psi((f_2 \otimes e_{\sigma_2})(f_1 \otimes e_{\sigma_1})) = \psi(f_2 \otimes e_{\sigma_2})\psi(f_1 \otimes e_{\sigma_1})$,
3. $\psi((f_3 \otimes e_{\sigma_3})(f_4 \otimes g_4)) = \psi(f_3 \otimes e_{\sigma_3})\psi(f_4 \otimes g_4)$, and
4. $\psi((f_4 \otimes g_4)(f_2 \otimes e_{\sigma_2})) = \psi(f_4 \otimes g_4)\psi(f_2 \otimes e_{\sigma_2})$.

We start by proving (1). We have

$$\begin{aligned}\psi((f_1 \otimes e_{\sigma_1})(f_3 \otimes e_{\sigma_3})) &= \frac{1}{d}\sum_{g\in G} \sigma_1(g)\psi(f_1 \circ g(f_3) \otimes g e_{\sigma_3}) \\ &= \frac{1}{d}\sum_{g\in G} \sigma_1\sigma_3^{-1}(g)\psi(f_1 \circ f_3 \otimes e_{\sigma_3}).\end{aligned}$$

Now the map $f_1 \circ f_3 : M \to M_0$ is G-invariant, so we can compute

$$(\psi(f_1 \circ f_3 \otimes e_{\sigma_3}))_{\sigma_3,h} = \sigma_3^{-1}(h) f_1 \circ (f_3)_{0,h}$$

for $h \in G$, and its other entries are zero. On the other hand, $\frac{1}{d}\sum_{g\in G}\sigma_1\sigma_3^{-1}(g)$ is 1 if $\sigma_1 = \sigma_3$ and zero otherwise, so in the end the only nonzero entries of $\psi((f_1 \otimes e_{\sigma_1})(f_3 \otimes e_{\sigma_3}))$ are the ones in position (σ, h) for $\sigma = \sigma_1 = \sigma_3$ and $h \in G$, and they are given by $\sigma^{-1}(h) f_1 \circ (f_3)_{0,h}$.

On the other hand, we have that

$$\psi(f_1 \otimes e_{\sigma_1})\psi(f_3 \otimes e_{\sigma_3})$$

has nonzero entries (σ, h) only for $\sigma = \sigma_1 = \sigma_3$. In this case we also obtain

$$(\psi(f_1 \otimes e_{\sigma_1})\psi(f_3 \otimes e_{\sigma_3}))_{\sigma,h} = \sigma^{-1}(h) f_1 \circ (f_3)_{0,h}.$$

This concludes the proof of (1).

Let us now prove (2). We have

$$\psi((f_2 \otimes e_{\sigma_2})(f_1 \otimes e_{\sigma_1})) = \frac{1}{d}\sum_{g\in G}\sigma_2(g)\psi(f_2 \circ g(f_1) \otimes ge_{\sigma_1}).$$

By a similar computation as above this expression equals $\psi(f_2 \circ f_1 \otimes e_{\sigma_1})$ if $\sigma_1 = \sigma_2$, and zero otherwise. Now $f_2 \circ f_1$ is a map $M_0 \to M_1$, so we can compute its (h, σ_1) entry which equals $\sigma_1(h)\left(\varphi_h^{-1}\right)_{h,1} \circ f_2 \circ f_1$ for every $h \in G$. The other entries are zero.

On the other hand,

$$\psi(f_2 \otimes e_{\sigma_2})\psi(f_1 \otimes e_{\sigma_1})$$

also has as only nonzero entries the (h, σ) for $\sigma = \sigma_1 = \sigma_2$, and they by definition are also equal to $\sigma_1(h)\left(\varphi_h^{-1}\right)_{h,1} \circ f_2 \circ f_1$. This proves (2).

Let us now prove (3). We have

$$\begin{aligned}\psi((f_3 \otimes e_{\sigma_3})(f_4 \otimes g_4)) &= \psi\left(\frac{1}{d}\sum_{g\in G}\sigma_3(g)f_3 \circ g(f_4) \otimes gg_4\right)\\ &= \sigma_3(g_4^{-1})\psi\left(\frac{1}{d}\sum_{g'\in G}\sigma_3(g')g'g_4^{-1}(f_3 \circ f_4) \otimes g'\right).\end{aligned}$$

Here the argument of ψ is of the form

$$\frac{1}{d}\sum_{g'\in G}\sigma_3(g')f'_{0,g'^{-1}} \otimes g'$$

since $g(f_4)$ starts in $M_{g_4^{-1}g^{-1}}$. Moreover, by multiplicativity of the components of φ we obtain that $f'_{0,h} = f'_{0,1} \circ (\varphi_h)_{1,h}$ for every h. The argument of ψ is therefore equal to $f' \otimes e_{\sigma_3}$ and we can apply the formula defining ψ to conclude that the (σ_3, h) entry of $\psi((f_3 \otimes e_{\sigma_3})(f_4 \otimes g_4))$ equals

$$\sigma_3(g_4^{-1})\sigma_3^{-1}(h)\,(f_3)_{0,1} \circ f_4 \circ \left(\varphi^{-1}_{g_4^{-1}h^{-1}}\right)_{g_4^{-1},h}$$

for every h and the other entries are zero.

On the other hand, the (σ_3, h) entry of $\psi(f_3 \otimes e_{\sigma_3})\psi(f_4 \otimes g_4)$ equals

$$(\psi(f_3 \otimes e_{\sigma_3}))_{\sigma_3,g_4h}\,(\psi(f_4 \otimes g_4))_{g_4h,h},$$

for every h, the other terms being zero. The above equals in turn

$$\begin{aligned}&\sigma_3^{-1}(g_4h)\,(f_3)_{0,g_4h} \circ \left(\varphi_{g_4^{-1}h^{-1}}\right)_{g_4h,1} \circ f_4 \circ \left(\varphi^{-1}_{g_4^{-1}h^{-1}}\right)_{g_4^{-1},h}\\ &= \sigma_3^{-1}(g_4h)\,(f_3)_{0,1} \circ (\varphi_{g_4h})_{1,g_4h} \circ \left(\varphi_{g_4^{-1}h^{-1}}\right)_{g_4h,1} \circ f_4 \circ \left(\varphi^{-1}_{g_4^{-1}h^{-1}}\right)_{g_4^{-1},h}\\ &= \sigma_3^{-1}(g_4h)\,(f_3)_{0,1} \circ f_4 \circ \left(\varphi^{-1}_{g_4^{-1}h^{-1}}\right)_{g_4^{-1},h}\end{aligned}$$

which proves (3).

Let us finally look at (4). On the one hand we have

$$\psi((f_4 \otimes g_4)(f_2 \otimes e_{\sigma_2})) = \sigma_2^{-1}(g_4)\psi\left(f_4 \circ (\varphi_{g_4})_{g_4^{-1},1} \circ f_2 \otimes e_{\sigma_2}\right)$$

which by definition has as its nonzero entries the ones indexed by (g, σ_2) for $g \in G$, and they equal

$$(\psi((f_4 \otimes g_4)(f_2 \otimes e_{\sigma_2})))_{g,\sigma_2} = \sigma_2^{-1}(g_4)\sigma_2(g)\left(\varphi_g^{-1}\right)_{g,1} \circ f_4 \circ (\varphi_{g_4})_{g_4^{-1},1} \circ f_2.$$

On the other hand, the product $\psi(f_4 \otimes g_4)\psi(f_2 \otimes e_{\sigma_2})$ also has nonzero entries only for (g, σ_2), and they equal

$$(\psi(f_4 \otimes g_4)(f_2 \otimes e_{\sigma_2}))_{g,\sigma_2} = \left(\varphi_{g^{-1}}\right)_{g,1} \circ f_4 \circ \left(\varphi^{-1}_{g^{-1}}\right)_{g_4^{-1},gg_4^{-1}} \circ \sigma_2(gg_4^{-1})\left(\varphi^{-1}_{gg_4^{-1}}\right)_{gg_4^{-1},1} \circ f_2,$$

so once more we conclude using multiplicativity of entries of φ_g. This completes the proof. □

Proof of Lemma 5.3. By Lemma 5.12, there is an isomorphism $R_1 \to R_2$. Moreover, the idempotent $e \in \mathrm{End}_S(X) * G$ is a full idempotent: since $(1 \otimes g)(\pi_1 \otimes 1)(1 \otimes g^{-1}) = \pi_{g^{-1}} \otimes 1$, the ideal generated by e contains the identity. So R_2 is an idempotent subalgebra Morita equivalent to $\mathrm{End}_S(X) * G$, and we are done. □

6. Characterising the algebras arising from orbifold diagrams

In this section we combine the results of the previous sections to characterise the algebras $A(\mathcal{O})$ and $B(\mathcal{O})$. From now on, we assume that $\mathcal{O}$ is a reduced orbifold diagram and that its cover $\mathrm{sym}_d(\mathcal{O})$ is a (reduced) Postnikov diagram. By Proposition 2.14, this is the case as soon as $d > 2$. There are also examples of orbifold diagrams of order 2 where $\mathrm{sym}_2(\mathcal{O})$ is also a Postnikov diagram and the results in this section hold in this case.

Let $\mathcal{P}$ be a d-symmetric Postnikov diagram. Let G be the cyclic group generated by clockwise rotation by $\frac{2\pi}{d}$. This group acts on both $A(\mathcal{P})$ and $B(\mathcal{P})$ by automorphisms in a natural way, and exactly this group and its action that we fix when we take skew group algebras. Before restricting to the Grassmannian setting, we present a general result.

We will use the construction of the quiver with potential of a skew group algebra given in [GP19]. For convenience of the reader, we summarize here some points about this construction. If (Q, W) is a QP and G is a finite group acting on Q fixing W, it is known by the work of Le Meur, [LM20], that the skew group algebra of the Jacobian algebra of (Q, W) is Morita equivalent to the Jacobian algebra of a new QP (Q_G, W_G). (We will use $\sim$ below to denote Morita equivalence). Under certain assumptions which are satisfied in the case of a symmetric Postnikov diagram with G acting by rotations, one can describe (Q_G, W_G) explicitly. The quiver Q_G was constructed in [RR85], and the potential W_G is defined in [GP19, Notation 3.18] after making certain choices, notably: a set of representatives of vertices of Q, and a suitable set of representatives of cycles appearing in W. The potential W_G depends on these choices (and on the choice of a primitive root of unity), but the resulting Jacobian algebras are isomorphic.

Proposition 6.1. *Let $\mathcal{O}$ be an orbifold diagram of order d. Then we have the Morita equivalences*

$$A(\mathcal{O}) \sim A(\mathrm{sym}_d(\mathcal{O})) * G$$

and

$$B(\mathcal{O}) \sim B(\mathrm{sym}_d(\mathcal{O})) * G.$$

Proof. We will use Theorem 3.20 of [GP19], with Λ being $A(\mathrm{sym}_d(\mathcal{O}))$. We remark this theorem still works if we replace the usual definition of the Jacobian ideal by any ideal generated by cyclic derivatives with respect to arrows (see Definition 4.3) provided that these arrows are closed under the G-action. In particular, the statement immediately extends to the case of frozen arrows, which we are considering in our definition of $A(\mathcal{O})$. In our case, the frozen arrows are the boundary arrows, which indeed form a set closed under the G-action.

Moreover, the G-orbits of the boundary arrows for $A(\mathrm{sym}_d(\mathcal{O}))$ correspond exactly (in the sense of [GP19, Notation 3.13]) to the boundary arrows of $A(\mathcal{O})$. It follows that even in our case, it is enough to show that the QP (Q_G, W_G) of [GP19] is equal to $(Q_{\mathcal{O}}, W_{\mathcal{O}})$, if we make appropriate choices. The fact that $Q_G = Q_{\mathcal{O}}$ is clear, as the two constructions both agree with the general construction presented in [RR85, Section 2]. This is also illustrated in Examples 8.1 and 8.3 of [GP19].

If the central region is cyclical, we are in the special case where G acts freely on the whole quiver Q of $\Lambda = A(\mathrm{sym}_d(\mathcal{O}))$ (and hence on Λ), which means that $\Lambda * G$ is Morita equivalent to the quotient Λ/G. In particular, the potential $W_{\mathcal{O}}$ we have defined in this case makes $A(\mathcal{O})$ isomorphic to this quotient and we are done.

It remains to check that the potential $W_{\mathcal{O}}$ equals the potential W_G of [GP19, Notation 3.18] (for appropriate choices) in the case where the central region is alternating. Following [GP19, §3.2], we choose a set $\mathcal{E}$ of representatives of vertices of Q. In order to get the simple formulas we gave for $W_{\mathcal{O}}$, we should be careful in how we choose the set $\mathcal{E}$. Let us pick a simple curve joining Ω to the boundary in $\mathcal{O}$, draw its d preimages under the quotient in $\mathrm{sym}_d(\mathcal{O})$, and consider one of the regions bounded by two consecutive copies, see for example Figure 10. We pick $\mathcal{E}$ to consist of exactly the vertices in this region. If the curve cuts an alternating region in two, we pick the part which is clockwise from the two copies of the simple curve and inside this region. We call the cycles appearing in the potential W on $\mathrm{sym}_d(\mathcal{O})$ are of type (i) if they are far from the center. The cycles going through the central vertex are said to be of type (ii), there an even number r of such. The contributions of the cycles of type (i) to both W_G and $W_{\mathcal{O}}$ (where they correspond to $\mathcal{C} \setminus \mathcal{C}'$) are easily seen to agree, so it remains to check what happens with the cycles going through the middle (those giving rise to the cycles in $\mathcal{C}'$).

The region between the two curves we chose on $\mathrm{sym}_d(\mathcal{O})$ contains $\frac{r}{2}$ outgoing and $\frac{r}{2}$ incoming arrows to the central vertex, so that $r - 1$ cycles $\hat{c}$ corresponding to the cycles of type (ii) have $p(\hat{c}) = 0$, for p as in [GP19, §3.2]. The contribution of these cycles to W_G is then precisely the same as the part of the sum with no roots of unity in our definition of $W_{\mathcal{O}}$. There is exactly one cycle $\hat{c}$ missing, which has $p(\hat{c})$

equal to ± 1 depending on whether it is clockwise or not. By possibly choosing ζ^{-1} instead of ζ, we can make the remaining terms in W_G and $W_{\mathcal{O}}$ be equal, proving the first statement.

The second statement follows directly from the first and [RR85, Lemma 2.2]. □

Let us recall a construction of [JKS16]. Let Π_n be the complete preprojective algebra of type $\widetilde{\mathrm{A}_{n-1}}$, with vertices $1, 2, \ldots, n$ around the cycle and arrows labeled $x_i : (i-1) \to i$ and $y_i : i \to (i-1)$.

Definition 6.2 ([JKS16])**.** Let $B = B(k, n)$ be the quotient of Π_n by the closure of the ideal generated by the relations $x^k - y^{n-k}$.

This algebra gives rise to an additive categorification of Scott's cluster algebra structure of the coordinate ring of the affine cone over the Grassmannian variety of k-spaces in $\mathbb{C}^n$, by taking the category $\mathcal{F}_{k,n}$ of maximal Cohen-Macaulay modules over B, [JKS16]. Furthermore, every Postnikov diagram of type (k, n) gives rise to a cluster-tilting object for this category, [BKM16] and from the boundary of its endomorphism algebra we recover the algebra B:

Proposition 6.3 ([BKM16])**.** *Let $\mathcal{P}$ be a Grassmannian Postnikov diagram of type (k, n). Then $B(\mathcal{P}) \cong B^{op}$.*

Recall that $n_0 = n/d$. We define a quotient of Π_{n_0} similarly as above. It will give us a basic Morita equivalent version of $B * G$.

Definition 6.4. Let $B_G = B_G(n_0, k, n)$ be the quotient of Π_{n_0} by the ideal generated by the relations $x^k - y^{n-k}$.

The group G acts on B by letting the generator act by the quiver automorphism rotating i to $i + \mathrm{GCD}(n, k)$. Denote this automorphism by g.

Proposition 6.5. *Let $e = e_1 + \cdots + e_{n_0}$ be the idempotent in B corresponding to the first n_0 vertices of Π_n. Then $e \otimes 1$ is a Morita idempotent in $B * G$, and there is an isomorphism $B_G \cong (e \otimes 1)B * G(e \otimes 1)$ mapping e_i to $e_i \otimes 1$, x_i and y_i to $x_i \otimes 1$ and $y_i \otimes 1$ for $i \neq 1$, x_1 to $x_1 \otimes g^{-1}$, and y_1 to $y_{n_0+1} \otimes g$.*

Proof. The first assertion follows from [RR85] since the first n_0 vertices form a cross-section of vertices of the quiver of B under the action of G. The explicit isomorphism is a direct application of [GPP19, Lemma 4.6], observing that $y_{n_0+1} \otimes g = (1 \otimes g)(y_1 \otimes 1)$. □

From now on we will freely identify B_G with $(e \otimes 1)B * G(e \otimes 1)$ using this isomorphism.

Corollary 6.6. *Let $n = dn_0$, let $\mathcal{O}$ be a Grassmannian orbifold diagram of order d and of type (k, n). Then we have that $B(\mathcal{O}) \cong (B_G)^{op}$.*

Proof. We have $B(\mathcal{O}) \sim B(\mathcal{P}) * G \sim B^{op} * G \sim (B_G)^{op}$, and both algebras are basic. □

6.1. **Modules for the skew group algebra B_G.** For the rest of the paper, we assume that $\mathcal{O}$ is a Grassmannian orbifold diagram of type (k, n) and of order d. Thus its universal cover $\mathcal{P} = \mathrm{sym}_d(\mathcal{O})$ is a d-symmetric Grassmannian Postnikov diagram of type (k, n). Our goal is to explain the relationship between the boundary algebras $B(\mathcal{P})$ and $B(\mathcal{O})$ of the two diagrams. By the above results (Proposition 6.3 and Corollary 6.6), these algebras are isomorphic to (the opposites of) B and B_G respectively, independently of $\mathcal{O}$ and its symmetrized version, so we will focus our attention on the algebras B and B_G, for which we have a quiver description.

The algebra B and its singularity category have been thoroughly studied, see for instance [JKS16], [DL16], [BBGE19]. We are interested in carrying out a similar study for B_G.

The element $t = \sum_i x_i y_i$ is central in B and in fact, $Z(B) = \mathbb{C}[[t]]$, [JKS16]. Its image $(e\otimes 1)(t\otimes 1)(e\otimes 1)$ is a central element of B_G.

We are interested in special B-modules, namely the rank one Cohen-Macaulay B-modules. They give rise to cluster-tilting objects in the category $\mathcal{F}_{k,n}$ of maximal Cohen-Macaulay modules over B. Furthermore, every object in $\mathcal{F}_{k,n}$ has a filtration by such modules. These modules are constructed as follows.

Definition 6.7. Let I be a k-element subset of $\{1, \ldots, n\}$. Let L_I be the B-module given as a representation by:

- A copy of $Z(B)$ at every vertex. Call $\mathbf{1}_i$ the identity of $Z(B)$ at vertex i.
- The arrow x_i maps $\mathbf{1}_{i-1}$ to $\mathbf{1}_i$ if $i \in I$, maps $\mathbf{1}_{i-1}$ to $t\mathbf{1}_i$ otherwise.
- The arrow y_i maps $\mathbf{1}_i$ to $t\mathbf{1}_i$ if $i \in I$, maps $\mathbf{1}_i$ to $\mathbf{1}_{i-1}$ otherwise.

Remark 6.8. The module L_I is free of rank n over $Z(B)$. It is in fact Cohen-Macaulay of rank one, and all Cohen-Macaulay modules of rank one over B are of this form for some I, by [JKS16].

Remark 6.9. By construction, we have canonical isomorphisms ${}^gL_I = L_{I-n_0}$, where ${}^gM$ denotes the twist of M by g in $\mathrm{mod}(B)$.

Example 6.10. We recall how the modules L_I can be visualised as lattice diagrams ([JKS16, Section 5]) by presenting L_{127} for $\mathrm{sym}_3(\mathcal{O})$ of Example 3.8 in Figure 16.

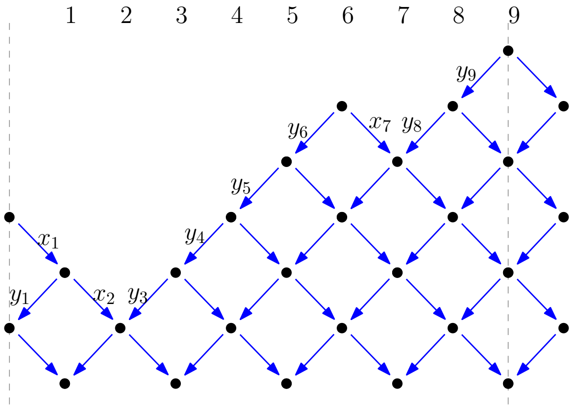

FIGURE 16. The lattice diagram for the module L_{127} for $n = 9$.

We want to find analogous modules as the rank 1 modules from Definition 6.7 for the algebra B_G. Let us write t for the element $(e \otimes 1)(t \otimes 1)(e \otimes 1) \in B_G$. Let $[I]_{n_0}$ be an equivalence class of k-element subsets of $\{1, \ldots, n\}$ under the equivalence $\sim_{n_0}$ of $2^{\{1,\ldots,n\}}$ (these are labels of regions of orbifold diagrams, as introduced in Section 3).

Definition 6.11. Let I be a k-subset of $\{1, 2, \ldots, n\}$ and let $[I]_{n_0}$ be its equivalence class, for $n = dn_0$. We define a B_G-module as follows. As a vector space, we define

$$L_{[I]_{n_0}} = \bigoplus_{l=0}^{d-1} \bigoplus_{i=1}^{n_0} \mathbb{C}[[t]],$$

where we denote the identities of the above power series rings by $\mathbf{1}_i^l$. It is enough to describe the action of the elements $e_i, x_i, y_i \in B_G$ on the elements $\mathbf{1}_j^l$. Addition on the superscripts l is always modulo n_0.

- The element e_i maps $\mathbf{1}_i^l$ to $\mathbf{1}_i^l$.
- The arrow x_1 maps $\mathbf{1}_{n_0}^{l-1}$ to $\mathbf{1}_1^l$ if $1 + ln_0 \in I$, it maps $\mathbf{1}_{n_0}^{l-1}$ to $t\mathbf{1}_1^l$ if $1 + ln_0 \notin I$.
- The arrow y_1 maps $\mathbf{1}_1^l$ to $\mathbf{1}_{n_0}^{l-1}$ if $1 + ln_0 \notin I$, it maps $\mathbf{1}_1^l$ to $t\mathbf{1}_{n_0}^{l-1}$ if $1 + ln_0 \in I$

For $i = \{2, \ldots, n_0\}$, the x_i and y_i act as follows:

- The arrow x_i, $i = \{2, \ldots, n_0\}$, maps $\mathbf{1}_{i-1}^l$ to $\mathbf{1}_i^l$ if $i + ln_0 \in I$, it maps $\mathbf{1}_{i-1}^l$ to $t\mathbf{1}_i^l$ if $i + ln_0 \notin I$.
- The arrow y_i maps $\mathbf{1}_i^l$ to $\mathbf{1}_{i-1}^l$ if $i + ln_0 \notin I$, it maps $\mathbf{1}_i^l$ to $t\mathbf{1}_{i-1}^l$ if $i + ln_0 \in I$.

Observe that the definition of $L_{[I]_{n_0}}$ does indeed depend on the choice of $I \in [I]_{n_0}$, see Example 6.12 below. So this seems not well-defined at first sight. However, we will show in Lemma 6.14 that different choices of representatives for $[I]_{n_0}$ result in modules which are canonically isomorphic, cf. also Remarks 6.15 and 6.18.

Example 6.12. We illustrate Definition 6.11 on Example 3.8. Let $n_0 = d = 3$ and so $n = 9$. The quiver with potential of the above example is described in Example 4.11. Recall that $[147]_3 = \{\{1, 4, 7\}\}$ because the corresponding alternating region contains the orbifold point. All other equivalence classes contain three 3-subsets of 9. In Figure 17 we give the modules $L_{[I]_3}$ and $L_{[J]_3}$ for $I = \{1, 2, 7\}$ and $J = \{1, 4, 5\}$ as lattice diagrams (similarly as for the rank 1 modules for B in [JKS16, Section 5]). Note that $[127]_3 = [145]_3$. Figure 18 gives the top part of the three different modules $L_{[127]_3}$, $L_{[145]_3}$ and $L_{[478]_3}$ as lattices on the three columns for the three vertices of the algebra B_G, with the layers $l = 0, l = 1, l = 2$ in different colours. The vertices and arrows outside of the middle region are repeated as empty circles and dashed arrows to the left and right.

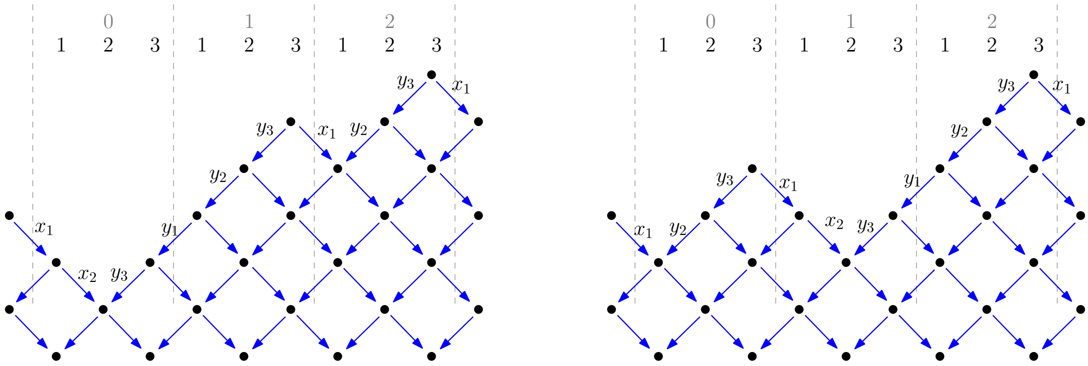

FIGURE 17. On the left the module $L_{[127]_3}$ and on the right $L_{[145]_3}$.

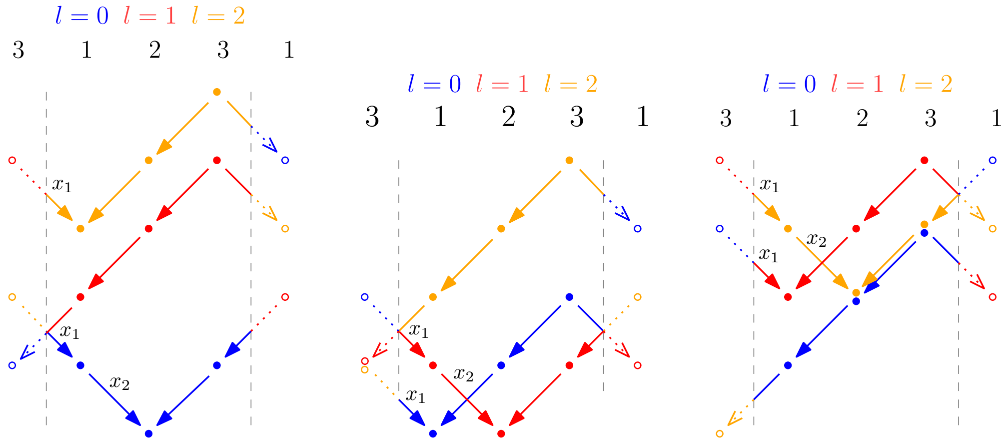

FIGURE 18. The modules $L_{[127]_3}$, $L_{[145]_3}$ and $L_{[478]_3}$ with their layers coloured.

We define a map φ from $L_{[I]_{n_0}}$ to $L_{[I+n_0]_{n_0}}$ where $I + n_0$ is the k-subset obtained from I by adding n_0 to each element of I. The map φ increases the label l by 1, i.e. we set: $\varphi(\mathbf{1}_j^l) = \mathbf{1}_i^{l+1}$ for all j.

We claim that this map φ is a B_G-module homomorphism:

Remark 6.13. Let I be a k-subset of $\{1, 2, \ldots, n\}$ and $[I]_{n_0}$ its equivalence class for $n = dn_0$. Note that $i + (l+1)n_0 \in [I + n_0]_{n_0}$ if and only if $i + ln_0 \in [I]_{n_0}$.

Lemma 6.14. *Let I be a k-subset of $\{1, 2, \ldots, n\}$ and $[I]_{n_0}$ its equivalence class for $n = dn_0$. The map φ induces an isomorphism $L_{[I]_{n_0}} \cong L_{[I+n_0]_{n_0}}$ of B_G-modules.*

Proof. First, we check φ is a homomorphism of B_G-modules. To ease the notation, let $J = [I + n_0]_{n_0}$, and so by Remark 6.13, $i + ln_0 \in J$ if and only if $i + (l-1)n_0 \in I$ for any $i \in \{1, \ldots, n_0\}$, $l = 0, \ldots, d-1$.

We have, following the list in the definition of $L_{[I]_{n_0}}$:

- For every $i \in \{1, \ldots, n_0\}$,

$$e_i\varphi(\mathbf{1}_j^l) = \begin{cases} \mathbf{1}_j^{l+1} & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi(e_i\mathbf{1}_j^l) = \begin{cases} \varphi(\mathbf{1}_j^l) = \mathbf{1}_j^{l+1} & \text{if } i = j; \\ \varphi(0) = 0 & \text{if } i \neq j. \end{cases}$$

- Since $x_1\mathbf{1}_j^l = 0$ unless $j = n_0$, it is enough to consider the effect of x_1 only on $\varphi(\mathbf{1}_{n_0}^{l-1})$:

$$\begin{aligned} x_1\varphi(\mathbf{1}_{n_0}^{l-1}) = x_1\mathbf{1}_{n_0}^l &= \begin{cases} \mathbf{1}_1^{l+1} & \text{if } 1 + ln_0 \in J \\ t\mathbf{1}_1^{l+1} & \text{if } 1 + ln_0 \notin J \end{cases} \\ &= \begin{cases} \mathbf{1}_j^{l+1} & \text{if } 1 + (l-1)n_0 \in I \\ t\mathbf{1}_j^{l+1} & \text{if } 1 + (l-1)n_0 \notin I \end{cases} \\ &= \varphi(x_1\mathbf{1}_{n_0}^{l-1}), \end{aligned}$$

- and the effect of y_1 on $\varphi(\mathbf{1}_1^{l-1})$:

$$\begin{aligned} y_1\varphi(\mathbf{1}_1^{l-1}) = y_1\mathbf{1}_1^l &= \begin{cases} \mathbf{1}_{n_0}^{l-1} & \text{if } 1 + ln_0 \notin J \\ t\mathbf{1}_{n_0}^{l-1} & \text{if } 1 + ln_0 \in J \end{cases} \\ &= \begin{cases} \mathbf{1}_{n_0}^{l-1} & \text{if } i + (l-1)n_0 \notin I \\ t\mathbf{1}_{n_0}^{l-1} & \text{if } i + (l-1)n_0 \in I \end{cases} \\ &= \varphi(y_1\mathbf{1}_1^{l-1}). \end{aligned}$$

- For $i \in \{2, \ldots, n_0\}$, the effect of x_i on $\varphi(\mathbf{1}_{i-1}^l)$ is:

$$\begin{aligned} x_i\varphi(\mathbf{1}_{i-1}^l) = x_i\mathbf{1}_{i-1}^{l+1} &= \begin{cases} \mathbf{1}_i^{l+1} & \text{if } i + (l+1)n_0 \in J \\ t\mathbf{1}_i^{l+1} & \text{if } i + (l+1)n_0 \notin J \end{cases} \\ &= \begin{cases} \mathbf{1}_i^{l+1} & \text{if } i + ln_0 \in I \\ t\mathbf{1}_i^{l+1} & \text{if } i + ln_0 \notin I \end{cases} \\ &= \varphi(x_i\mathbf{1}_{i-1}^l). \end{aligned}$$

- For $i \in \{2, \ldots, n_0\}$, the effect of y_i on $\varphi(\mathbf{1}_i^l)$ is

$$\begin{aligned} y_i\varphi(\mathbf{1}_i^l) = y_i\mathbf{1}_i^{l+1} &= \begin{cases} \mathbf{1}_{i-1}^{l+1} & \text{if } i + (l+1)n_0 \notin J \\ t\mathbf{1}_{i-1}^{l+1} & \text{if } i + (l+1)n_0 \in J \end{cases} \\ &= \begin{cases} \mathbf{1}_{i-1}^{l+1} & \text{if } i + ln_0 \notin I \\ t\mathbf{1}_{i-1}^{l+1} & \text{if } i + ln_0 \in I \end{cases} \\ &= \varphi(y_i\mathbf{1}_i^l). \end{aligned}$$

For the bijectivity, we note that φ permutes the generators of the modules and that the generators freely generate the modules over the centre. □

Remark 6.15. By Lemma 6.14, the definition of B_G-modules is justified.

Remark 6.16. As we have mentioned, from the definition it is clear that x_iy_i maps $\mathbf{1}_i^l$ to $t\mathbf{1}_i^l$, so calling all the variables in the power series rings in Definition 6.11 t is justified.

We want to relate the B_G-modules $L_{[I]_{n_0}}$ to the B-modules L_I. For this, we need to introduce some notation. The map $B \to B * G$ given by $b \mapsto b \otimes 1$ induces a functor $F = (B * G) \otimes_B -$ from $\mathrm{mod}(B)$ to $\mathrm{mod}(B * G)$. There is an equivalence $j^* : \mathrm{mod}(B * G) \to \mathrm{mod}(B_G)$ given by

$$j^* = (e \otimes 1)B * G \otimes_{B*G} -$$

using the isomorphism of Proposition 6.5.

We aim to prove that $L_{[I]_{n_0}} \cong j^*F(L_I)$. Let us do some preparation. As a B-module, $(B * G) \otimes_B L_I$ is generated by the elements $(1 \otimes g^l) \otimes_B \mathbf{1}_h$. This in turn implies that the B_G-module $j^*F(L_I)$ is generated by elements of the form $(e_i \otimes g^l) \otimes_B \mathbf{1}_h$. However, these elements are nonzero (if and) only if $h = g^{-l}(i)$. We are left with considering the elements $(e_i \otimes g^l) \otimes_B \mathbf{1}_{g^{-l}(i)}$, for $i \in \{1, \ldots, n_0\}$ and $l \in \{0, \ldots, d-1\}$. Observe that t (which again we use as notation for $(e \otimes 1)(t \otimes 1)(e \otimes 1)$ as well as for $\sum_{i=1}^{n_0} x_iy_i$ in the quiver description of B_G) is in the center of B_G, so that as a $\mathbb{C}[[t]]$-module we have a decomposition $j^*F(L_I) = \bigoplus_{l=0}^{d-1} \bigoplus_{i=1}^{n_0} \mathbb{C}[[t]]$. It is therefore natural to define a map of $\mathbb{C}[[t]]$-modules $\psi : L_{[I]_{n_0}} \to j^*F(L_I)$ by setting

$$\psi(\mathbf{1}_i^l) = (e_i \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(i)}.$$

Lemma 6.17. *The map $\psi : L_{[I]_{n_0}} \to j^*F(L_I)$ is an isomorphism of B_G-modules.*

Proof. The strategy is the same as in the proof of Lemma 6.14. First of all, ψ permutes the free generators of the corresponding modules, giving the bijection.

To see that ψ is a B_G-module homomorphism, we check that e_i, x_i, y_i in B_G act on $\mathbf{1}_j^l$ in the same way as the corresponding elements in $(e \otimes 1)(B * G)(e \otimes 1)$ act on $(e_j \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(j)}$ by left multiplication. Note that $g^l(j) = j + ln_0$.

As before, for the action of x_i, we will restrict to $j = i-1$, for the action of y_i to $j = i$.

We have, following the list in the definition of $L_{[I]_{n_0}}$:

- For every $i \in \{1, \dots, n_0\}$,

$$\begin{aligned}
(e_i \otimes 1)\psi(\mathbf{1}_j^l) &= (e_i \otimes 1)(e_j \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(j)} \\
&= (e_i e_j \otimes g^{-l}) \otimes_B \mathbf{1}_j^l \\
&= \begin{cases} (e_i \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(i)} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \psi(\mathbf{1}_i^l) & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases} \\
&= \psi(e_i \mathbf{1}_j^l).
\end{aligned}$$

- For $i \in \{2, \dots, n_0\}$ (and $j = i-1$),

$$\begin{aligned}
(x_i \otimes 1)\psi(\mathbf{1}_{i-1}^l) &= (x_i \otimes 1)(e_{i-1} \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(i-1)} \\
&= (x_i \otimes g^{-l}) \otimes_B \mathbf{1}_{(i-1)+ln_0} \\
&= (e_i \otimes g^{-l})(g^l(x_i) \otimes 1) \otimes_B \mathbf{1}_{(i-1)+ln_0} (\text{using Definition 5.1}) \\
&\text{now } g^l(x_i) \otimes 1 = x_{i+ln_0} \otimes 1 \text{ is in } B \text{ and we can pull it across } \otimes_B \text{ to get} \\
&= (e_i \otimes g^{-l}) \otimes_B \begin{cases} \mathbf{1}_i^l & \text{if } i + ln_0 \in I; \\ t\mathbf{1}_i^l & \text{if } i + ln_0 \notin I \end{cases} \\
&= \begin{cases} \psi(\mathbf{1}_i^l) & \text{if } i + ln_0 \in I; \\ \psi(t\mathbf{1}_i^l) & \text{if } i + ln_0 \notin I \end{cases} \\
&= \psi(x_i \mathbf{1}_{i-1}^l).
\end{aligned}$$

- For $i \in \{2, \dots, n_0\}$ (and $j = i$),

$$\begin{aligned}
(y_i \otimes 1)\psi(\mathbf{1}_i^l) &= (y_i \otimes 1)(e_i \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(i)} = (y_i \otimes g^{-l}) \otimes_B \mathbf{1}_{i+ln_0} \\
&= (e_{i-1} \otimes g^{-l})(g^l(y_i) \otimes 1) \otimes_B \mathbf{1}_{i+ln_0} \\
&= (e_{i-1} \otimes g^{-l}) \otimes_B \begin{cases} \mathbf{1}_{(i-1)+ln_0} & \text{if } i + ln_0 \notin I \\ t\mathbf{1}_{(i-1)+ln_0} & \text{if } i + ln_0 \in I \end{cases} \\
&= \begin{cases} \psi(\mathbf{1}_{i-1}^l) & \text{if } i + ln_0 \notin I \\ \psi(t\mathbf{1}_{i-1}^l) & \text{if } i + ln_0 \in I \end{cases} \\
&= \psi(y_i \mathbf{1}_i^l).
\end{aligned}$$

- Finally, for $i = 1$ and $j = n_0$, recalling that $x_1 \in B_G$ maps to $x_1 \otimes g^{-1}$,

$$\begin{aligned}
(x_1 \otimes g^{-1})\psi(\mathbf{1}_{n_0}^l) &= (x_1 \otimes g^{-1})(e_{n_0} \otimes g^{-l}) \otimes_B \mathbf{1}_{g^l(n_0)} = x_1 \otimes g^{-l-1} \otimes_B \mathbf{1}_{(l+1)n_0} \\
&= (e_1 \otimes g^{-l-1})(g^{l+1}(x_1) \otimes 1) \otimes_B \mathbf{1}_{(l+1)n_0} \\
&= (e_1 \otimes g^{-l-1}) \otimes_B x_{1+(l+1)n_0} \mathbf{1}_{(l+1)n_0} \\
&= (e_1 \otimes g^{-l-1}) \otimes_B \begin{cases} \mathbf{1}_{g^{l+1}(1)} & \text{if } 1 + (l+1)n_0 \in I \\ t\mathbf{1}_{g^{l+1}(1)} & \text{if } 1 + (l+1)n_0 \notin I \end{cases} \\
&= \begin{cases} \psi(\mathbf{1}_1^{l+1}) & \text{if } 1 + (l+1)n_0 \in I \\ \psi(t\mathbf{1}_1^{l+1}) & \text{if } 1 + (l+1)ln_0 \notin I \end{cases} \\
&= \psi(x_1 \mathbf{1}_{n_0}^l)
\end{aligned}$$

- and to check y_1, we only consider $j=1$, recalling that $y_1 \mapsto y_{n_0+1}\otimes g$,

$$
\begin{aligned}
(y_{n_0+1}\otimes g)\psi(\mathbf{1}_1^l) &= (y_{n_0+1}\otimes g)(e_1\otimes g^{-l})\otimes_B \mathbf{1}_{g^l(1)} = (y_{n_0+1}\otimes g^{-l+1})\otimes_B \mathbf{1}_{1+ln_0}\\
&= (e_{n_0}\otimes g^{-l+1})(g^{l-1}(y_{n_0+1})\otimes 1)\otimes_B \mathbf{1}_{1+ln_0}\\
&= (e_{n_0}\otimes g^{-l+1})\otimes_B y_{1+ln_0}\mathbf{1}_{1+ln_0}\\
&= (e_{n_0}\otimes g^{-l+1})\otimes_B \begin{cases}\mathbf{1}_{g^{l-1}(n_0)} & \text{if } 1+ln_0\notin I\\ t\mathbf{1}_{g^{l-1}(n_0)} & \text{if } 1+ln_0\in I\end{cases}\\
&= \begin{cases}\psi(\mathbf{1}_{n_0}^{l-1}) & \text{if } 1+ln_0\notin I\\ \psi(t\mathbf{1}_{n_0}^{l-1}) & \text{if } 1+ln_0\in I\end{cases}\\
&= \psi(y_1\mathbf{1}_1^l).
\end{aligned}
$$

□

Remark 6.18. Since $F(M)=F({}^gM)$ for any $M\in \mathrm{mod}(B)$, we recover that the modules $L_{[I]_{n_0}}$ are well defined (cf. Remark 6.15).

Lemma 6.19. *If $[I]_{n_0}\neq [J]_{n_0}$ then $L_{[I]_{n_0}}\not\cong L_{[J]_{n_0}}$.*

Proof. Assume that $L_{[I]_{n_0}}\cong L_{[J]_{n_0}}$. By Lemma 6.17, we have that $j^*F(L_I)\cong j^*F(L_J)$ hence $F(L_I)\cong F(L_J)$. This implies that $L_I\cong {}^gL_J$ for some $g\in G$. We conclude that I and J differ by a multiple of n_0 and so $[I]_{n_0}=[J]_{n_0}$. □

Corollary 6.20. *Let I and J be k-subsets of $n=dn_0$. Then $L_{[I]_{n_0}}\cong L_{[J]_{n_0}}$ if and only if $[I]_{n_0}=[J]_{n_0}$.*

6.2. **The algebra $A(\mathcal{O})$ as endomorphism algebra.** Let now $T_{\mathcal{P}}$ be the B-module defined by

$$T_{\mathcal{P}}=\bigoplus_{I\in\mathcal{I}} L_I.$$

Since $\mathcal{P}$ is d-symmetric, it is invariant under rotation by n_0 steps. It follows that $T_{\mathcal{P}}={}^gT_{\mathcal{P}}$, since $L_{I-n_0}={}^gL_I$. As the labels correspond to regions on the disk, they either come in orbits of length d (i.e. are acted upon freely by G) or are fixed, and there can be at most one fixed label (the label of the central region, if it is alternating).

Definition 6.21. Let $\mathcal{O}$ be an orbifold diagram on a disk with n_0 points. Let $T_{\mathcal{O}}$ be the B_G-module defined by

$$T_{\mathcal{O}}=\bigoplus_{[I]_{n_0}\in\mathcal{I}_{\mathcal{O}}} L_{[I]_{n_0}}.$$

We need some more notation. Let $T_0=L_I$ for the label I of the central region of $\mathcal{P}$, if it is alternating, and $T_0=0$ otherwise. The group G acts freely on $T_{\mathcal{P}}\setminus T_0$, and we call $T'_{\mathcal{P}}$ a chosen cross-section of this action. Finally, we call $T_{\mathcal{P}}^{red}=T_0\oplus T'_{\mathcal{P}}$.

Lemma 6.22. *We have $T_{\mathcal{O}}=j^*F(T_{\mathcal{P}}^{red})$.*

Proof. Use Lemma 6.17 and Definition 3.1. □

Theorem 6.23. *With the above notation, we have $A(\mathcal{O})\cong \mathrm{End}_{B_G}(T_{\mathcal{O}})$.*

Proof. By Proposition 6.1, we know that $A(\mathcal{O})$ is Morita equivalent to $A(\mathcal{P})*G$, where the action of a generator is given by rotating $n_0=\mathrm{GCD}(k,n)$ steps clockwise. With the action by twists on $\mathrm{End}_B(T_{\mathcal{P}})$ as in SectionREF, the isomorphism $A(\mathcal{P})\cong \mathrm{End}_B(T_{\mathcal{P}})$ is G-invariant. We get then an isomorphism $A(\mathcal{P})*G\cong \mathrm{End}_B(T_{\mathcal{P}})*G$. Now we can apply Lemma 5.3 to $M_1=T'_{\mathcal{P}}, M_0=T_0$ and $X=T_{\mathcal{P}}$. We obtain a Morita equivalence

$$\mathrm{End}_B(T_{\mathcal{P}})*G\sim \mathrm{End}_{\mathrm{mod}(S)^G}\left(EF(T_0)\oplus E(T_{\mathcal{P}})\right)=\mathrm{End}_{\mathrm{mod}(S)^G}\left(EF(T_{\mathcal{P}}^{red})\right)$$

The latter is in turn Morita equivalent to $\mathrm{End}_{B*G}(F(T_{\mathcal{P}}^{red}))$ and then to $\mathrm{End}_{B_G}(j^*F(T_{\mathcal{P}}^{red}))$, since both E and j^* are equivalences. Finally, by Lemma 6.22, the latter equals $\mathrm{End}_{B_G}(T_{\mathcal{O}})$. We have proved that $A(\mathcal{O})\sim \mathrm{End}_{B_G}(T_{\mathcal{O}})$. The statement follows since both algebras are basic (the latter by Lemma 6.19). □

Remark 6.24. We conclude with a remark motivated by the following question: in [BKM16], the dimer algebra A is shown to be isomorphic to the endomorphism algebra of a module T, which is a cluster tilting object in a Frobenius, stably 2-Calabi-Yau category. Is the same true in our case? By results of Demonet ([Dem08, §2.2.4]), it is indeed the case that the skew group category $\mathrm{CM}(B)*G$ of the category of Cohen-Macaulay B-modules is Frobenius and stably 2-CY, and our module $T_{\mathcal{O}}$ does lie in it. Moreover, F maps G-invariant cluster tilting objects to cluster tilting objects, so indeed $T_{\mathcal{O}}$ is cluster tilting. We note however that we do not have a direct description of the category $\mathrm{CM}(B) * G$ as (equivalent to) a subcategory of $\mathrm{mod}(B_G)$.

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