Chapter 1

GCV

1.1 GCV-I-derivative

$\frac{dGCV}{d\lambda}(\lambda) = \frac{d(s\frac{SS_{res}(\lambda)}{dor^2(\lambda)})}{d\lambda} = s(\frac{dSS_{res}}{d\lambda}\frac{1}{dor^2} + SS_{res}\frac{d(dor^{-2})}{d\lambda})$

$[1] = \frac{dSS_{res}}{d\lambda} = \frac{d(<\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\varepsilon}}>)}{d\lambda} = 2 < \frac{d\hat{\boldsymbol{\varepsilon}}}{d\lambda}, \hat{\boldsymbol{\varepsilon}} >= 2 < \frac{d((\boldsymbol{Q}-\boldsymbol{QS})\boldsymbol{z})}{d\lambda}, \hat{\boldsymbol{\varepsilon}} >= -2 < \boldsymbol{Q}\frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} > \stackrel{\boldsymbol{Q}\in Sym}{=} -2 < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \boldsymbol{Q}\hat{\boldsymbol{\varepsilon}} > \stackrel{\boldsymbol{Q}^2=\boldsymbol{Q}}{=} -2 < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >$

$[2] = \frac{d(dor^{-2})}{d\lambda} = -2\frac{1}{dor^3}\frac{d(dor)}{d\lambda} = -2\frac{1}{dor^3}\frac{d(s-q-tr(\boldsymbol{S}))}{d\lambda} = 2\frac{1}{dor^3}tr(\frac{d\boldsymbol{S}}{d\lambda})$

$\implies \frac{dGCV}{d\lambda}(\lambda) = s(- < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} > \frac{2}{dor^2} + SS_{res}\frac{2}{dor^3}tr(\frac{d\boldsymbol{S}}{d\lambda})) = \frac{2s}{dor^2}(\hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}) - < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >)$

1.2 GCV-II-derivative

$\frac{d^2GCV}{d\lambda^2}(\lambda) = 2s\frac{d(dor^{-2}(\hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}) - <\frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z},\hat{\boldsymbol{\varepsilon}}>))}{d\lambda}$

$[1] = \frac{d(dor^{-2})}{d\lambda}(\hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}) - < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >) = 2\frac{1}{dor^3}tr(\frac{d\boldsymbol{S}}{d\lambda})(\hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}) - < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >)$

$[2] = \frac{1}{dor^2}\frac{d(\hat{\sigma}^2)}{d\lambda}tr(\frac{d\boldsymbol{S}}{d\lambda}) = \frac{1}{dor^2}\frac{\frac{d(SS_{res})}{d\lambda}dor - SS_{res}\frac{d(dor)}{d\lambda}}{dor^2}tr(\frac{d\boldsymbol{S}}{d\lambda}) = \frac{1}{dor^2}\frac{-2<\frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z},\hat{\boldsymbol{\varepsilon}}>dor + SS_{res}tr(\frac{d\boldsymbol{S}}{d\lambda})}{dor^2}tr(\frac{d\boldsymbol{S}}{d\lambda}) = \frac{1}{dor^3}(-2 < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} > + \hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}))tr(\frac{d\boldsymbol{S}}{d\lambda})$

$[3] = \frac{1}{dor^2}\hat{\sigma}^2 tr(\frac{d^2\boldsymbol{S}}{d\lambda^2})$

$[4] = -\frac{1}{dor^2} < \frac{d^2\boldsymbol{S}}{d\lambda^2}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >$

$[5] = -\frac{1}{dor^2} < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \frac{d\hat{\boldsymbol{\varepsilon}}}{d\lambda} >= \frac{1}{dor^2} < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \boldsymbol{Q}\frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z} >$

$\implies \frac{d^2GCV}{d\lambda^2}(\lambda) = \frac{2s}{dor^2}(\frac{1}{dor}[3\hat{\sigma}^2 tr(\frac{d\boldsymbol{S}}{d\lambda}) - 4 < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >]tr(\frac{d\boldsymbol{S}}{d\lambda}) + \hat{\sigma}^2 tr(\frac{d^2\boldsymbol{S}}{d\lambda^2}) + < \frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z}, \boldsymbol{Q}\frac{d\boldsymbol{S}}{d\lambda}\boldsymbol{z} > - < \frac{d^2\boldsymbol{S}}{d\lambda^2}\boldsymbol{z}, \hat{\boldsymbol{\varepsilon}} >)$

Chapter 2

Forcing term and nonhomogeneous Dirichlet

Define GCV_{homo} the computation of the GCV in the homogeneous case

2.1 GCV-I-derivative

$\frac{dGCV}{d\lambda}(\lambda) = \frac{d[GCV_{homo}+\frac{s}{dor^2}(||\mathbf{r}(\lambda)||^2-2<\hat{\varepsilon},\mathbf{r}(\lambda)>)]}{d\lambda} = \frac{dGCV_{homo}}{d\lambda} + s(||\mathbf{r}(\lambda)||^2 - 2 < \hat{\varepsilon}, \mathbf{r}(\lambda) >)\frac{d(dor^{-2})}{d\lambda} + \frac{2s}{dor^2}(< \mathbf{r}(\lambda), \frac{d\mathbf{r}}{d\lambda} > - < \frac{d\hat{\varepsilon}}{d\lambda}, \mathbf{r}(\lambda) > - < \hat{\varepsilon}, \frac{d\mathbf{r}(\lambda)}{d\lambda} >)$

$$V = \Psi^T \mathbf{A}\mathbf{Q}\Psi + \lambda \mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{R}_1 \tag{2.1}$$

$$\frac{d\mathbf{V}^{-1}}{d\lambda} = -\mathbf{V}^{-1}\mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{R}_1 \mathbf{V}^{-1} \tag{2.2}$$

$$\frac{d\mathbf{S}}{d\lambda} = -\Psi \frac{d\mathbf{V}^{-1}}{d\lambda} \Psi^T \mathbf{A}\mathbf{Q} \tag{2.3}$$

$$\frac{d\mathbf{r}}{d\lambda} = \frac{1}{\lambda}\mathbf{r}(\lambda) + \lambda \mathbf{Q}\Psi \frac{d\mathbf{V}^{-1}}{d\lambda} \mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{u}^* \tag{2.4}$$

2.2 GCV-II-derivative

$$\begin{aligned}
\frac{d^2GCV}{d\lambda^2} = & \frac{d^2GCV_{homo}}{d\lambda^2} + 2s(< \mathbf{r}(\lambda), \frac{d\mathbf{r}}{d\lambda} > - < \frac{d\hat{\varepsilon}}{d\lambda}, \mathbf{r}(\lambda) > - < \hat{\varepsilon}, \frac{d\mathbf{r}(\lambda)}{d\lambda} >)\frac{d(dor^{-2})}{d\lambda} + s(||\mathbf{r}(\lambda)||^2 \\
& - 2 < \hat{\varepsilon}, \mathbf{r}(\lambda) >)\frac{d^2}{d\lambda^2}(dor^{-2}) + 2\frac{s}{dor^2}[\ ||\frac{d\mathbf{r}}{d\lambda}||^2 + < \mathbf{r}, \frac{d^2\mathbf{r}}{d\lambda^2} > - < \frac{d^2\hat{\varepsilon}}{d\lambda}, \mathbf{r}(\lambda) > -2 < \frac{d\hat{\varepsilon}}{d\lambda}, \frac{d\mathbf{r}(\lambda)}{d\lambda} > \\
& - < \hat{\varepsilon}, \frac{d^2\mathbf{r}}{d\lambda^2} >] \\
& + 2s\frac{d(dor^{-2})}{d\lambda}[< \mathbf{r}(\lambda), \frac{d\mathbf{r}}{d\lambda} > - < \frac{d\hat{\varepsilon}}{d\lambda}, \mathbf{r}(\lambda) > - < \hat{\varepsilon}, \frac{d\mathbf{r}(\lambda)}{d\lambda} >]
\end{aligned}$$

$$\frac{d^2\hat{\varepsilon}}{d\lambda^2} = -\mathbf{Q}\frac{d^2\mathbf{S}}{d\lambda^2}\mathbf{z} \tag{2.5}$$

$$\frac{d^2\mathbf{r}}{d\lambda^2} = -\frac{1}{\lambda^2}\mathbf{r}(\lambda) + \frac{1}{\lambda}\frac{d\mathbf{r}}{d\lambda} + \mathbf{Q}\Psi(\frac{d\mathbf{V}^{-1}}{d\lambda} + \lambda\frac{d^2\mathbf{V}^{-1}}{d\lambda^2})\mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{u}^* \tag{2.6}$$

$$\frac{d^2\mathbf{V}^{-1}}{d\lambda^2} = -\frac{d\mathbf{V}^{-1}}{d\lambda}\mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{R}_1 \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{R}_1^T \mathbf{R}_0^{-1} \mathbf{R}_1 \frac{d\mathbf{V}^{-1}}{d\lambda} \tag{2.7}$$

$$\frac{d\ dor}{d\lambda} = -tr(\frac{d\mathbf{S}(\lambda)}{d\lambda}) \tag{2.8}$$

$$\frac{d^2 dor^{-2}}{d\lambda^2} = 2tr(\frac{d^2\mathbf{S}(\lambda)}{d\lambda^2})dor^{-3} + 6\ dor^{-4}\{tr(\frac{d\mathbf{S}(\lambda)}{d\lambda})\}^2 \tag{2.9}$$