KASHIWARA CONJUGATION AND THE ENHANCED RIEMANN–HILBERT CORRESPONDENCE

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Abstract. We study some aspects of conjugation and descent in the context of the irregular Riemann–Hilbert correspondence of D'Agnolo–Kashiwara. First, we prove the compatibility between Kashiwara's conjugation functor for holonomic D-modules and the enhanced De Rham functor. Afterwards, we give some complements Galois descent for enhanced ind-sheaves, slighly generalizing results obtained in previous joint work with Barco, Hien and Sevenheck. Finally, we show how Hukuhara–Levelt–Turrittin type decompositions of an enhanced ind-sheaf descend to structures over a smaller fields. This shows in particular that a structure of the enhanced solutions of a meromorphic connection over a subfield of the complex numbers has implications on its generalized monodromy data (in particular, the Stokes matrices), generalizing and significantly simplifying an argument given in our previous work.

1. Introduction

In general, a Riemann–Hilbert correspondence is an equivalence of categories between some category of differential systems (such as integrable connections, meromorphic connections, or D-modules) and some category of topological objects (such as local systems, perverse sheaves, Stokes-filtered local systems, or enhanced ind-sheaves). In the context of differential equations in one or several complex variables (i.e. in the theory of conections or D-modules on a complex algebraic variety or complex manifold), these topological objects are a priori defined over the field of complex numbers (as a field of coefficients).

Having said this, three questions immediately come to our mind:

(1) How does the Riemann–Hilbert functor behave with respect to the natural complex conjugation on the target?
(2) How can we detect if the target object is already defined over a subfield of $\mathbb{C}$?
(3) What implications does such a structure over a smaller field have?

The first question was studied by M. Kashiwara in [Kas86] in the case of regular holonomic D-modules. At the time, it was known that the De Rham functor induces an equivalence between the derived category of regular holonomic D-modules and the derived category of constructible sheaves, and in loc. cit. a conjugation functor on the category of D-modules was defined that corresponds to complex conjugation on constructible sheaves via this equivalence. More than 30 years later, in [DK16], A. D'Agnolo and M. Kashiwara generalized Kashiwara's equivalence to (not necessarily regular) holonomic D-modules. It is the aim of the first part (Section 3) of this article to show that the conjugation functor of [Kas86] is – not surprisingly – still compatible with this new equivalence of categories (Theorem 3.8).

In the second part (Section 4) of the paper, we want to give some complements on a study of the second question that has been initiated in [BHHS22]. In loc. cit.,

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together with D. Barco, C. Sevenheck and M. Hien, we studied the question of when topological data associated to hypergeometric differential systems are defined over a subfield of $\mathbb{C}$. For this, we used a technique called *Galois descent*: Given a finite Galois extension L/K and an object over L, the idea is that in order to find a structure of our object over K, it is enough to find isomorphisms to all its Galois conjugates. We developed some statements in this theory for sheaves and enhanced ind-sheaves there. In particular, we showed that Galois descent is possible for $\mathbb{R}$-constructible enhanced ind-sheaves concentrated in one degree on compact spaces. We will reformulate and slightly generalize this statement here, dropping the compactness assumption (Theorem 4.7).

Finally, in the last part (Section 5), we give a consequence of the existence of K-structures (for an arbitrary subfield $K \subset \mathbb{C}$) in the irregular setting, addressing thus Question (3): We show that a K-structure on an enhanced ind-sheaf associated to a meromorphic connection on a Riemann surface via the irregular Riemann–Hilbert correspondence implies that its Stokes matrices (and indeed all its generalized monodromy data) are defined over K. In the case of hypergeometric systems, this was already done in [BHHS22, §5]. Our proof is, however, valid in general and simplifies significantly the quite involved and a slightly artificial argument given in our previous work. To this purpose, we study the topological counterpart of meromorphic connections on complex curves (already introduced in [DK18]), which we call *of HLT type*, and prove that this property descends to a K-lattice (Theorem 5.11).

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2. Background and notation

In this work, we want to study objects related to Riemann–Hilbert correspondences for holonomic D-modules, and we will mainly use the approach via (enhanced) ind-sheaves here, great parts of which have been developed in [KS01] and [DK16]. We will therefore need some analytic and topological notions, which we briefly recall in the following, referring to the existing literature for further details.

Even if not strictly necessary in all places, we will assume all our topological spaces to be *good*, i.e. Hausdorff, locally compact, second countable and of finite flabby dimension. This is especially important when it comes to the construction of proper direct images and exceptional inverse images, and since we are mainly interested in all these objects in the context of the Riemann–Hilbert correspondence on complex manifolds, goodness is not a restriction.

2.1. **Holonomic D-modules.** Let X be a complex manifold with structure sheaf $\mathcal{O}_X$. We denote by $\mathcal{D}_X$ the sheaf of (non-commutative) rings of linear partial differential operators with coefficients in $\mathcal{O}_X$ on X. The category of (left) $\mathcal{D}_X$-modules is denoted by $\mathrm{Mod}(\mathcal{D}_X)$, and its bounded derived category by $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$. We denote by $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ the full subcategory of *holonomic* $\mathcal{D}_X$-modules and by $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ the full subcategory of *regular holonomic* $\mathcal{D}_X$-modules. We moreover denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$) the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ of complexes with cohomologies in $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ (resp. $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$).

We write $\mathbb{D}_X$ for the duality functor for $\mathcal{D}_X$-modules, and if $f\colon X \to Y$ is a morphism, we denote by $\mathrm{D}f_*$ (resp. $\mathrm{D}f^*$) the direct image image for $\mathcal{D}_X$-modules (resp. the inverse image for $\mathcal{D}_Y$-modules) along f.

If $D \subset X$ is a normal crossing divisor, we denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic function with poles on D at most. It is a left $\mathcal{D}_X$-module and for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ we denote its *localization at* D by $\mathcal{M}(*D) := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$. We call $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ a *meromorphic connection* along D if $\operatorname{sing\,supp}(\mathcal{M}) = D$ and $\mathcal{M}(*D) \simeq \mathcal{M}$, where $\operatorname{sing\,supp}(\mathcal{M})$ denotes the singular support of $\mathcal{M}$.

An important basic example of a meromorphic connection which is not regular holonomic is the following: Let X be a complex manifold, $D \subset X$ a normal crossing divisor and $f \in \Gamma(X; \mathcal{O}_X(*D))$, then the *exponential* $\mathcal{D}_X$*-module* $\mathcal{E}^f$ is defined by $\mathcal{E}^f := (\mathcal{D}_X/\mathrm{ann}(f))(*D)$, where $\mathrm{ann}(f))$ is the (left) ideal of $\mathcal{D}_X$ of operators annihilating f.

The classification of holonomic D-modules – in particular in the case of irregular singularities – has been a difficult problem. In the one-dimensional case, it turned out that the Stokes phenomenon is the key ingredient to achieve a complete description of connections with irregular singular points. Very roughly, the Stokes phenomenon in this context is the observation that a meromorphic connection decomposes as a direct sum of certain “elementary” connections (namely, exponential D-modules) in sufficiently small sectors with vertex at the singular point, but this decomposition might not exist globally in an open neighbourhood of the singularity. It was conjectured (see [Sab00]) that a similar statement holds for meromorphic connections in higher dimensions, and such local decompositions still exist, at least after suitable blow-ups of the singular locus and de-ramification. A proof was given for the two-dimensional case in [Sab00], and finally for the general case by K. Kedlaya [Ked10, Ked11] and T. Mochizuki [Moc09, Moc11].

This classification leads to a useful technique for proving statements about holonomic D-modules, and we will briefly recall it here for later use. If X is a complex manifold and $D \subset X$ a normal crossing divisor, we denote by $\widetilde{X}$ the real oriented blow-up of X along the components of D, and we denote by $\mathcal{A}_{\widetilde{X}}$ the sheaf of functions holomorphic on $\widetilde{X} \setminus \partial\widetilde{X}$ having moderate growth along $\partial\widetilde{X}$ (see e.g. [DK16, Notation 7.2.1] for a more precise definition, or [Moc14] where this sheaf is called $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$). For a $\mathcal{D}_X$-module $\mathcal{M}$, we write

$$\mathcal{M}^{\mathcal{A}} := \mathcal{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathcal{M}.$$

Definition 2.1 (cf. [DK16, Definition 7.3.3])**.** *Let X be a complex manifold and $D \subset X$ a normal crossing divisor. A holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ is said to have a* normal form *along D if*

- *$\mathcal{M} \simeq \mathcal{M}(*D)$,*
- *$\operatorname{sing\,supp}\mathcal{M} = D$,*
- *for any $x \in \partial\widetilde{X}$, there exists an open neighbourhood $V \subset \widetilde{X}$ of x such that*
$$(\mathcal{M}^{\mathcal{A}})|_V \simeq \big((\bigoplus \mathcal{E}^{\varphi_i})^{\mathcal{A}}\big)|_V$$
for some $\varphi_i \in \Gamma(U; \mathcal{O}_X(*D))$, where $U \subset X$ is an open neighbourhood of $\varpi(x)$.

From this classification of holonomic D-modules, one can deduce the following fundamental lemma (first stated in [DK16, Lemma 7.3.7]).

Lemma 2.2 ([DK16, Lemma 7.3.7])**.** *Let $\mathrm{P}_X(\mathcal{M})$ be a statement concerning a complex manifold X and an object $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Then, $\mathrm{P}_X(\mathcal{M})$ is true for any complex manifold X and any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ if all of the following conditions are satisfied:*

(a) *Locality: If* $X = \bigcup_{i \in I} U_i$ *is an open covering, then* $\mathrm{P}_X(\mathcal{M})$ *is true if and only if* $\mathrm{P}_{U_i}(\mathcal{M}|_{U_i})$ *is true for every* $i \in I$.
(b) *Stability by translation: If* $n \in \mathbb{Z}$ *and* $\mathrm{P}_X(\mathcal{M})$ *is true, then* $\mathrm{P}_X(\mathcal{M}[n])$ *is true.*
(c) *Stability in exact triangles: If* $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$ *is a distinguished triangle in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *and both* $\mathrm{P}_X(\mathcal{M}')$ *and* $\mathrm{P}_X(\mathcal{M}'')$ *are true, then* $\mathrm{P}_X(\mathcal{M})$ *is true.*
(d) *Stability by direct summands: If* $\mathcal{M}, \mathcal{M}' \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *and* $\mathrm{P}_X(\mathcal{M} \oplus \mathcal{M}')$ *is true, then* $\mathrm{P}_X(\mathcal{M})$ *is true.*
(e) *Stability by projective pushforward: If* $f\colon X \to Y$ *is a projective morphism,* $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *and* $\mathrm{P}_X(\mathcal{M})$ *is true, then* $\mathrm{P}_Y(\mathrm{D}f_*\mathcal{M})$ *is true.*
(f) *If* $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ *has a normal form along a normal crossing divisor* $D \subset X$*, then* $\mathrm{P}_X(\mathcal{M})$ *is true.*

2.2. **Sheaves.** Let $\Bbbk$ be a field and let X be a topological space. We denote by $\mathrm{Mod}(\Bbbk_X)$ the category of sheaves of $\Bbbk$-vector spaces on X and by $\mathrm{D}^{\mathrm{b}}(\Bbbk_X)$ its bounded derived category.

If X is a real analytic (resp. complex) manifold, we have the full subcategory $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\Bbbk_X)$ (resp. $\mathrm{Mod}_{\mathbb{C}\text{-c}}(\Bbbk_X)$) of $\mathrm{Mod}(\Bbbk_X)$ of $\mathbb{R}$-constructible (resp. $\mathbb{C}$-constructible) sheaves and the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\Bbbk_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\Bbbk_X)$) of $\mathrm{D}^{\mathrm{b}}(\Bbbk_X)$ consisting of complexes with $\mathbb{R}$-constructible (resp. $\mathbb{C}$-constructible) cohomologies.

The six Grothendieck operations for sheaves are denoted by $\mathrm{R}\mathcal{H}om$, $\otimes$, $\mathrm{R}f_*$, f^{-1}, $\mathrm{R}f_!$ and $f^!$ for $f\colon X \to Y$ a morphism.

For details on the theory of sheaves of vector spaces and constructibility, we refer to the standard literature, such as [KS90] or [Dim04].

2.3. **Ind-sheaves and subanalytic sheaves.** Let X be a good topological space. In [KS01], M. Kashiwara and P. Schapira introduced and studied the category $\mathrm{I}(\Bbbk_X)$ of *ind-sheaves* on X as a generalization of the category $\mathrm{Mod}(\Bbbk_X)$, and its bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_X)$. There is a fully faithful and exact embedding $\iota\colon \mathrm{Mod}(\Bbbk_X) \hookrightarrow \mathrm{I}(\Bbbk_X)$. Since this embedding does not commute with inductive limits in general, but the functor ι is sometimes suppressed in the notation, one uses the notation "$\varinjlim$" (instead of just $\varinjlim$) for the inductive limit in the category of ind-sheaves.

It was also shown that if X is a real analytic manifold, the category of "subanalytic ind-sheaves" (called "ind-$\mathbb{R}$-constructible ind-sheaves" in [KS01]) is equivalent to the category of sheaves on the (relatively compact) subanalytic site X^c_{sa} (see [KS01, Theorem 6.3.5] and cf. [Pre08] for a more detailed study of subanalytic sheaves).

Let us briefly recall this version of the subanalytic site: Open sets of X^c_{sa} are relatively compact open subanalytic subsets of X. A covering of an open subset U in X^c_{sa} is a covering $U = \bigcup_{i \in I} U_i$ by open sets U_i in X^c_{sa} admitting a finite subcover.

We denote the subcategory of $\mathrm{I}(\Bbbk_X)$ consisting of subanalytic ind-sheaves by $\mathrm{I}_{\mathrm{suban}}(\Bbbk_X)$.

In some regards, subanalytic sheaves behave differently from sheaves on a usual topology. For example, if we are given a filtrant inductive system $F_j \in \mathrm{I}_{\mathrm{suban}}(\Bbbk_X)$, $j \in J$, and $U \in X^c_{sa}$, we have

$$\Gamma(U; \text{``}\varinjlim_{j \in J}\text{''}\, F_j) \simeq \varinjlim_{j \in J} \Gamma(U; F_j).$$

The following lemma is also easily proved using the finiteness of the gluing procedure for subanalytic sheaves (see [KS01, Proposition 6.4.1]).

Lemma 2.3. *Let* $F \in \mathrm{I}_{\mathrm{suban}}(\Bbbk_X)$ *and let* A *be a* $\Bbbk$*-algebra. Then for* $U \subseteq X$ *open, we have* $\Gamma(U; A_X \otimes_{\Bbbk_X} F) \simeq A \otimes_{\Bbbk} \Gamma(U; F)$.

2.4. **Enhanced ind-sheaves.** In [DK16] and [DK19], A. D'Agnolo and M. Kashiwara extended the theory further, introducing and studying the category of *enhanced ind-sheaves* on a so-called *bordered space*. We recall very few basics here and refer to loc. cit. for more details (see also [KS16] for an exposition).

A bordered space $\mathcal{X} = (X, \widehat{X})$ is a pair of topological spaces such that $X \subseteq \widehat{X}$ is an open subspace. In particular, every topological space X can be considered a bordered space (X, X). We say that $\mathcal{X}$ is a *real analytic bordered space* if $\widehat{X}$ is a real analytic manifold and X is a subanalytic open subset. (Everything we do for real analytic manifolds or bordered spaces could also be done slightly more generally for subanalytic spaces or bordered spaces, see [KS90, Exercise 9.2] and [DK19, §3.1] for these notions.)

The category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}})$ of *enhanced ind-sheaves* on $\mathcal{X} = (X, \widehat{X})$ is a quotient category of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{X\times\overline{\mathbb{R}}})$, where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{\pm\infty\}$. The category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}})$ has six Grothendieck operations denoted by $\mathrm{R}\mathcal{I}hom^+$, $\overset{+}{\otimes}$, $\mathrm{E}f_*$, $\mathrm{E}f^{-1}$, $\mathrm{E}f_{!!}$, and $\mathrm{E}f^!$ (for morphisms f of bordered spaces). One also has a sheaf-valued hom functor

$$\mathrm{R}\mathcal{H}om^{\mathrm{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}}) \to \mathrm{D}^{\mathrm{b}}(\Bbbk_X)$$

and for any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\Bbbk_X)$ a tensor product functor

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}}), \quad H \mapsto \pi^{-1}\mathcal{F} \otimes H.$$

The natural t-structure on the derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\widehat{X}\times\overline{\mathbb{R}}})$ induces a t-structure on $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}})$, whose heart is denoted by $\mathrm{E}^0(\mathrm{I}\Bbbk_{\mathcal{X}})$ (these are therefore the objects represented by complexes of ind-sheaves concentrated in degree 0). Moreover, a notion of $\mathbb{R}$-constructibility is defined for enhanced ind-sheaves, yielding to the full subcategory $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_{\mathcal{X}}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_{\mathcal{X}})$. We write $\mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_{\mathcal{X}}) := \mathrm{E}^0(\mathrm{I}\Bbbk_{\mathcal{X}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_{\mathcal{X}})$.

An important object in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_X)$ is

$$\Bbbk^{\mathrm{E}}_{\mathcal{X}} := \text{“}\varinjlim_{a\to\infty}\text{”}\, \Bbbk_{\{t\geq a\}} \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_{\mathcal{X}}),$$

where $\{t \geq a\} := \{(x,t) \in \widehat{X} \times \overline{\mathbb{R}} \mid x \in X, t \in \mathbb{R}, t \geq a\} \subset \widehat{X} \times \overline{\mathbb{R}}$.

A fundamental class of objects are the so-called *exponential enhanced ind-sheaves*: Let us now consider a complex manifold X and assume (for simplicity and since it suffices for what we want to do below) $\widehat{X} = X$. If $W \subseteq X$ is open and $f \in \mathcal{O}_X(U)$ is a holomorphic function on U, we set[1]

$$\mathbb{E}^f_{W,\Bbbk} := \text{“}\varinjlim_{a\to\infty}\text{”}\, \Bbbk_{\{t\geq -\operatorname{Re} f + a\}} \simeq \Bbbk^{\mathrm{E}}_X \overset{+}{\otimes} \Bbbk_{\{t = -\operatorname{Re} f\}} \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_X),$$

where $\{t \geq -\operatorname{Re} f + a\} := \{(x,t) \in X \times \overline{\mathbb{R}} \mid x \in W, t \in \mathbb{R}, t \geq -\operatorname{Re} f(x) + a\}$. If $\Bbbk = \mathbb{C}$, we often omit the subscript $\mathbb{C}$ and just write $\mathbb{E}^f_W$ for this object.

Remark 2.4. The category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_X)$ can be viewed (via the left adjoint of the quotient functor) as a full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\Bbbk_{X\times\overline{\mathbb{R}}})$. In particular, the image of $\mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_X)$ lies in $\mathrm{I}_{\mathrm{suban}}(\Bbbk_{X\times\overline{\mathbb{R}}})$. Hence, the object $\mathbb{E}^f_{W,k}$ can be understood as a sheaf on the subanalytic site $(X \times \overline{\mathbb{R}})^c_{sa}$.

For example, if X is a disc around a point p in the complex plane, $W = X \setminus \{p\}$ and f has a pole (of order at least one) at 0, then $\mathbb{E}^f_W$ is described as follows:

Let $U \in (X \times \overline{\mathbb{R}})^c_{sa}$ and denote by $\tau \colon X \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ the projection.

Then we have

[1]Note that our notation is slightly less general and simplified here, compared to works like [DK18], for example: Our object $\mathbb{E}^f_W$ would be denoted by $\mathbb{E}^{\operatorname{Re} f}_{W|X}$ in loc. cit.

- If $U \subseteq W \times \mathbb{R}$,
$$\mathbb{E}^f_W(U) \simeq \varinjlim_{a\to\infty} \Gamma(U; \Bbbk_{\{t \geq -\operatorname{Re} f + a\}}) \simeq \varinjlim_{a\to\infty} \Bbbk^{\pi_0(U\cap\{t\geq -\operatorname{Re} f+a\})},$$
where $\pi_0(U \cap \{t \geq -\operatorname{Re} f + a\})$ is the set of connected components Y of the intersection $U \cap \{t \geq -\operatorname{Re} f + a\}$.
- More generally, if $U \subseteq X \times \overline{\mathbb{R}}$, then
$$\mathbb{E}^f_W(U) \simeq \varinjlim_{a\to\infty} \Bbbk^{\pi_0^{<\infty}(U\cap\{t\geq -\operatorname{Re} f+a\})},$$
where $\pi_0^{<\infty}(U \cap \{t \geq -\operatorname{Re} f + a\})$ is the set of connected components Y of the intersection $U \cap \{t \geq -\operatorname{Re} f + a\}$ such that $\overline{Y} \cap U$ does not intersect $X \times \{+\infty\}$ and $\{p\} \times \overline{\mathbb{R}}$.

In particular, we have $\mathbb{E}^f_W(W \times \mathbb{R}) = \Bbbk$ and $\mathbb{E}^f_W(X \times I) = 0$ for $I \subseteq \overline{\mathbb{R}}$ open, as well as $\mathbb{E}^f_W(S \times \overline{\mathbb{R}})$ for $S \subseteq X$ open. For $V \subseteq U$, the restriction maps $\mathbb{E}^f_W(U) \to \mathbb{E}^f_W(V)$ are given in the obvious way, by a combination of identities, zero maps or diagonal maps $\Bbbk \to \Bbbk^M$ (for M a set).

The sheaves $\mathbb{E}^f_U$ are therefore very similar to sheaves of the form k_Z on the usual topology, i.e. constant sheaves on a closed subset $Z \subseteq X$, extended by zero outside Z. For $\mathbb{E}^f_W$, the set Z is a half-space lying over the graph of some real-valued function and considered to be shifted towards “infinitely high real values”.

If $W \subseteq X$ is open and $f, g \in \mathcal{O}_X(W)$ are holomorphic functions on W, we write (similarly to [DK18], but again we deviate slightly from the convention in loc. cit.)

$$\begin{aligned}
f \sim_W g \quad &:\Leftrightarrow \quad \operatorname{Re}(f-g) \text{ is bounded (from above and below)}\\
f \preceq_W g \quad &:\Leftrightarrow \quad \operatorname{Re}(f-g) \text{ is bounded from below}\\
f \prec_W g \quad &:\Leftrightarrow \quad f \preceq g \text{ but not } f \sim g,\\
&\qquad\ \text{i.e. } \operatorname{Re}(f-g) \text{ is bounded from below and unbounded from above}
\end{aligned}$$

Note that if $f \sim_W g$, then $\mathbb{E}^f_W \simeq \mathbb{E}^g_W$.

Let us recall the following fundamental property (cf. e.g. [DK16, Lemma 9.8.1] or [DK18, Lemma 3.2.2]).

Lemma 2.5. *Let $W \subseteq X$ be open and let $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{O}_X(W)$ be holomorphic functions. Then*
$$\begin{aligned}
&\operatorname{Hom}\Big(\bigoplus_{k=1}^n \mathbb{E}^{f_k}_W, \bigoplus_{j=1}^n \mathbb{E}^{g_j}_W\Big)\\
&\qquad \simeq \Big\{A = (a_{jk}) \in \Bbbk^{m\times n} \Big| a_{jk} = 0 \textit{ if } g_j \prec_{W'} f_k \textit{ for some open } W' \subseteq W\Big\}
\end{aligned}$$

Morphisms between direct sums of exponentials can therefore – after numbering the direct summands – be represented by matrices (and composition corresponds to matrix multiplication). In particular, if $f_1 \prec_W f_2 \prec_W \ldots \prec_W f_n$, endomorphisms of $\bigoplus_{i=1}^n \mathbb{E}^{f_i}_W$ are represented by lower-triangular square matrices.

2.5. **Analytic Riemann–Hilbert correspondences.** It is a classical idea to ask if the functor associating to a differential system its solution space is an equivalence. (The question for surjectivity of such a functor for Fuchsian differential equations goes back at least to Hilbert’s 21st problem.)

Let X be a complex manifold of (complex) dimension d_X. Classically, in the theory of D-modules, one studies the following objects: Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$. Denote by Ω_X the sheaf of top-degree holomorphic differential forms on X. Then one defines the *holomorphic De Rham complex* of $\mathcal{M}$ by

$$\mathrm{DR}_X(\mathcal{M}) := \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M}$$

and the *holomorphic solution complex* of $\mathcal{M}$ by

$$\mathcal{S}ol_X(\mathcal{M}) := \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

In [Kas84], M. Kashiwara proved that the De Rham functor gives an equivalence

$$\mathrm{DR}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X).$$

It was then natural to search for a generalization of this result to holonomic D-modules (relaxing the regularity assumption). It was clear that the functor DR_X is no longer fully faithful on the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$, and hence the framework (the functor and the target category) would need to be modified.

Finally, A. D'Agnolo and M. Kashiwara were able to establish a fully faithful functor

$$\mathrm{DR}^{\mathrm{E}}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \hookrightarrow \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X).$$

Let us briefly give some details on the construction:

In [KS01], it was shown that tempered holomorphic functions (i.e. holomorphic functions with moderate growth near the boundary of their domain) form an ind-sheaf $\mathcal{O}^{\mathrm{t}}_X \in \mathrm{I}(\mathbb{C}_X)$. Then, in [DK16], the authors define the enhanced ind-sheaf of enhanced tempered holomorphic functions $\mathcal{O}^{\mathrm{E}}_X \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$. This is done by first defining enhanced tempered distributions $\mathcal{D}b^{\mathrm{E}}_{X_{\mathbb{R}}}$ and setting $\mathcal{O}^{\mathrm{E}}_X := \Omega_{X^c} \otimes_{\mathcal{D}_{X^c}} \mathcal{D}b^{\mathrm{E}}_{X_{\mathbb{R}}}$, in analogy to the classical fact that the Dolbeault complex with distribution coefficients is quasi-isomorphic to the sheaf of holomorphic functions. Roughly, a section of $\mathcal{D}b^{\mathrm{E}}_{X_{\mathbb{R}}}$ is of the form $e^t\delta(x)$ for a tempered distribution $\delta(x)$ on X. They then set $\Omega^{\mathrm{E}}_X := \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{O}^{\mathrm{E}}_X$ and define the functors

$$\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}) := \Omega^{\mathrm{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M}$$

and

$$\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}) := \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_X).$$

The framework is therefore set up to be similar to the regular case, and we refer to [DK16] and [KS16] for more details. (Let us in particular remark that the notation is not self-explanatory here, and quite some simplification in notation has happened between [KS01] and these works.)

These two functors are related by duality:

$$\mathrm{DR}^{\mathrm{E}}_X(\mathbb{D}_X\mathcal{M}) \simeq \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})[-\dim_{\mathbb{C}} X].$$

3. Hermitian duality and the enhanced De Rham functor

In this section, our aim is to generalize a result of [Kas86] to the context of the enhanced De Rham functor. Let us briefly recall the idea and argument of the main statement in [Kas86].

The Riemann–Hilbert correspondence for regular holonomic D-modules states that the De Rham functor induces an equivalence of categories

$$\mathrm{DR}_X \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X).$$

On the right-hand side, complex conjugation defines an auto-equivalence. It is therefore natural to ask what the corresponding operation on the left-hand side is. In other words, the question is the following: Given an object $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$, how is the object $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ satisfying $\mathrm{DR}_X(\mathcal{N}) \simeq \overline{\mathrm{DR}_X(\mathcal{M})}$ related to $\mathcal{M}$?

Indeed, M. Kashiwara is able to define a functor $c\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ such that the desired description is $\mathcal{N} = c(\mathcal{M})$, i.e.

$$\mathrm{DR}_X(c(\mathcal{M})) \simeq \overline{\mathrm{DR}_X(\mathcal{M})}. \tag{1}$$

The key to this result is an intermediate step, namely the definition and study of a functor

$$\begin{aligned} C_X\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) &\to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{X^c}), \\ \mathcal{M} &\mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_{X_\mathbb{R}}), \end{aligned}$$

later baptized the *Hermitian duality functor*. Here, $\mathcal{D}b_{X_\mathbb{R}}$ is the sheaf of Schwartz distributions on $X_\mathbb{R}$ (the real manifold underlying X and X^c).

One of the key observations in the proof of formula (1) is the fact that one has an isomorphism of functors

$$\mathrm{DR}_{X^c} \circ C_X[-d_X] \simeq \mathcal{S}ol_X \tag{2}$$

(note that $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X^c})$ are naturally identified), and this again follows directly from the fact that the Dolbeault complex with distribution coefficients is a resolution of the sheaf of holomorphic functions, i.e. $\mathrm{DR}_{X^c}(\mathcal{D}b_{X_\mathbb{R}}) \simeq \mathcal{O}_X$.

The idea in this section is now to establish a statement similar to (1) for the enhanced De Rham functor. More precisely, the idea is that the exact same relation holds if we replace DR_X by $\mathrm{DR}^{\mathrm{E}}_X$, so the functor c will remain unchanged. This is not surprising since the conjugation functor has already been studied in the context of irregular singularities and preserves, for example, the property of being holonomic. On the other hand, it is not directly obvious that (1) is still true in the context of the enhanced De Rham functor since the argument for a statement analogous to (2) is not as direct as in the classical case: We would need a statement like $\mathrm{DR}^{\mathrm{E}}_{X^c}(\mathcal{D}b_{X_\mathbb{R}}) \simeq \mathcal{O}^{\mathrm{E}}_X$, but what we have by definition is rather "$\mathrm{DR}_X(\mathcal{D}b^{\mathrm{E}}_{X_\mathbb{R}}) \simeq \mathcal{O}^{\mathrm{E}}_X$".

We therefore apply a different method of proof here, using the classification of general holonomic D-modules due to C. Sabbah, K. Kedlaya and T. Mochizuki, to establish the generalization of (2).

In the next subsection, we review very briefly the Hermitian dual of a D-module, together with its main properties that we will apply later.

3.1. **The Hermitian dual.** Let X be a complex manifold. In [Kas86], M. Kashiwara introduced the functor $C_X\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{X^c})$ given by

$$C_X(\mathcal{M}) := \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_{X_\mathbb{R}}).$$

Here, $\mathcal{D}b_{X_\mathbb{R}}$ is the sheaf of Schwartz distributions on the underlying real manifold $X_\mathbb{R}$ of X.

This functor has been studied further by other authors, in particular by C. Sabbah and T. Mochizuki, and we recall two important properties that we will use below.

The following lemma is shown in [Moc15, §12.5.2].

Lemma 3.1 ([Moc15, §12.5.2])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module such that f is proper on* $\operatorname{supp}\mathcal{M}$*. Then there is a natural morphism*

$$\mathrm{D}f^c_* C_X(\mathcal{M}) \longrightarrow C_Y(\mathrm{D}f_*\mathcal{M}) \tag{3}$$

and this morphism is an isomorphism.

Lemma 3.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and $D \subset X$ a hypersurface. Then there is an isomorphism*

$$C_X(\mathcal{M}(*D)) \simeq C_X(\mathcal{M})(!D).$$

Proof. Set $\mathcal{N} = \overline{C_X(\mathcal{M})}$ and denote by $C\colon \mathcal{M} \times \overline{\mathcal{N}} \to \mathcal{D}b_{X_{\mathbb{R}}}$ the canonical pairing. Then $(\mathcal{M}, \mathcal{N}, C)$ is a non-degenerate $\mathcal{D}$-triple (in the sense of [Moc15], for example). Then by [Moc15, §12.2.2], the $\mathcal{D}$-triple $(\mathcal{M}(*D), \mathcal{N}(!D), C(!D))$ is still non-degenerate. In particular, $C_X(\mathcal{M}(*D)) \simeq \overline{\mathcal{N}(!D)} \simeq C_X(\mathcal{M})(!D)$, as desired. □

Recall the notation on the real oriented blow-up from Section 2.1. If X is a complex manifold and $D \subset X$ is a normal crossing divisor, we also have the normal crossing divisor $D^c \subset X^c$ (if D is locally given by $\{z_1 \cdot \ldots \cdot z_k = 0\}$, then D^c is locally given by $\{\overline{z_1} \cdot \ldots \cdot \overline{z_k} = 0\}$) and the real blow-up space $\varpi_c\colon \widetilde{X^c} \to X^c$. (It has the same underlying topological space as $\widetilde{X}$.) If $\mathcal{M}$ is a $\mathcal{D}_{X^c}$-module, we write

$$\mathcal{M}^{\mathcal{A}^c} := \mathcal{A}_{\widetilde{X^c}} \otimes_{\varpi_c^{-1}\mathcal{O}_{X^c}} \varpi^{-1}\mathcal{M}$$

to emphasize that we are taking the tensor product with $\mathcal{A}_{\widetilde{X^c}}$, the sheaf of *anti-holomorphic* functions with moderate growth at $\partial\widetilde{X^c}$.

Lemma 3.3. *There is an isomorphism*

$$(C_X(\mathcal{E}^{\varphi}))^{\mathcal{A}^c} \simeq (\mathcal{E}^{-\overline{\varphi}})^{\mathcal{A}^c}.$$

Proof. Recall the notation in [Sab00], in particular we are going to use the functor $C_X^{\bmod D} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}b_X^{\bmod D})$. There are isomorphisms

$$(C_X(\mathcal{E}^{\varphi}))^{\mathcal{A}^c} \simeq (C_X(\mathcal{E}^{\varphi})(*D))^{\mathcal{A}^c} \simeq (C_X(\mathcal{E}^{\varphi}(!D)))^{\mathcal{A}^c} \simeq (C_X^{\bmod D}(\mathcal{E}^{\varphi}))^{\mathcal{A}^c}$$

where the second isomorphism follows from Lemma 3.2. On the other hand, we know from [Sab00, p. 69] that

$$(C_X^{\bmod D}(\mathcal{E}^{\varphi}))^{\mathcal{A}^c} \simeq C_{\widetilde{X}}^{\bmod D}((\mathcal{E}^{\varphi})^{\mathcal{A}}) \simeq (\mathcal{E}^{-\overline{\varphi}})^{\mathcal{A}^c}.$$

□

The following result about the local model of $C_X(\mathcal{M})$ is shown in [Sab00, p. 69] (see also Proposition 3.2.5 of loc. cit. and its proof). It describes that the Hermitian dual of a meromorphic connection with a normal form still has a normal form, and how its exponential factors change under Hermitian duality.

Proposition 3.4 ([Sab00])**.** *If $D \subset X$ is a normal crossing divisor and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module with a normal form along D, such that locally on $\widetilde{X}$, we have*

$$\mathcal{M}^{\mathcal{A}}|_V \simeq \Big(\bigoplus_{\varphi\in\Phi} \mathcal{E}^{\varphi}\Big)^{\mathcal{A}}|_V,$$

then $C_X(\mathcal{M})$ has a normal form along D^c, with local isomorphisms

$$C_X(\mathcal{M})^{\mathcal{A}^c}|_{V'} \simeq \Big(\bigoplus_{\varphi\in\Phi} \mathcal{E}^{-\overline{\varphi}}\Big)^{\mathcal{A}^c}|_{V'}.$$

3.2. **Conjugation of D-modules and the enhanced De Rham complex.** As mentioned above, the main step is a generalization of (2). Before we can prove an isomorphism, let us first show the existence of a canonical morphism.

Lemma 3.5. *There is a canonical morphism*

$$\mathrm{DR}^{\mathrm{E}}_{X^c}(C_X(\mathcal{M}))[-d_X] \longrightarrow \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}). \tag{4}$$

Proof. We have

$$\begin{aligned}\mathrm{DR}^{\mathrm{E}}_{X^c}(C_X(\mathcal{M})) &\simeq \Omega^{\mathrm{E}}_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_{X_{\mathbb{R}}})\\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Omega^{\mathrm{E}}_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}})\end{aligned}$$

and

$$\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^{\mathrm{E}}).$$

Therefore, it suffices to find a canonical morphism $\Omega_{X^c}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}}[-d_X] \to \mathcal{O}_X^{\mathrm{E}}$. Note further that

$$\begin{aligned}\Omega_{X^c}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}} &\simeq (\Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{O}_{X^c}^{\mathrm{E}}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}} \\ &\simeq \Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} (\mathcal{O}_{X^c}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}})\end{aligned}$$

and

$$\mathcal{O}_X^{\mathrm{E}} \simeq \Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}[-d_X],$$

which means that it suffices to find a morphism

$$\mathcal{O}_{X^c}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}} \to \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}.$$

The canonical morphism $\mathcal{D}_X \to \mathcal{O}_X$ (the quotient map with respect to the maximal ideal) clearly induces a morphism

$$\mathcal{O}_{X^c}^{\mathrm{E}} \simeq \mathrm{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{O}_X, \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}) \longrightarrow \mathrm{R}\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{D}_X, \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}) \simeq \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}.$$

We therefore get a chain of morphisms

$$\begin{aligned}\mathcal{O}_{X^c}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}} &\to \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}} \\ &\to H^{-1}(\mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X^c}} \mathcal{D}b_{X_{\mathbb{R}}})[1] \\ &= (H^{-1}(\mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}) \otimes_{\mathcal{O}_{X^c}} H^0(\mathcal{D}b_{X_{\mathbb{R}}}))[1] \to \mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}\end{aligned}$$

To understand the arrow in the second line, note that $\mathcal{D}b_{X_{\mathbb{R}}}^{\mathrm{E}}$ is concentrated in degree -1 and hence the derived tensor product is concentrated in degrees ≤ -1 (taking a projective resolution of the first factor). Then applying the functor $H^{-1}(-)[1]$ amounts to applying the truncation functor $\tau_{\geq -1}$ and there is a canonical morphism $\mathrm{id} \to \tau_{\geq -1}$. The equality in the third line follows from the right exactness of the tensor product, again considering a projective resolution of the first factor.

Here, the last morphism is given by $e^t v(x) \otimes \delta(x) \mapsto e^t v \cdot \delta(x)$, where $\delta(x)$ is a section of $\beta\pi^{-1}\mathcal{D}b_{X_{\mathbb{R}}}$.

□

The classification of holonomic D-modules now enables us to prove that this morphism is an isomorphism for any holonomic D-module $\mathcal{M}$.

Proposition 3.6. *The morphism* (4) *is an isomorphism for any* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$.

Proof. Consider the statement for any complex manifold X and any holonomic $\mathcal{D}_X$-module $\mathcal{M}$:

$$\mathrm{P}_X(\mathcal{M}) : \text{The morphism (4) is an isomorphism.}$$

To show that it holds for any X and $\mathcal{M}$, we will employ [DK16, Lemma 7.3.7], and we have to check conditions (a)–(f).

Condition (a) is clear by the compatibility of $\mathrm{DR}_X^{\mathrm{E}}$ and $\mathcal{S}ol_X^{\mathrm{E}}$ with inverse images. (Note that the a-priori existence and canonicity of the morphism (4) is essential here.) Condition (b) is also obvious. Condition (c) holds by the axioms of a triangulated category, and condition (d) is true since the morphism (4) is functorial with respect to direct sums.

The condition (e) is proved as follows: Assume $\mathrm{P}_X(\mathcal{M})$ holds, and let $f\colon X \to Y$ be a projective morphism. It also induces a morphism $f\colon X^c \to Y^c$ (which we will not distinguish in the notation). Then we have

$$\begin{aligned}\mathcal{S}ol^{\mathrm{E}}_Y(\mathrm{D}f_*\mathcal{M}) &\simeq \mathrm{E}f_*\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})[d_X - d_Y]\\ &\simeq \mathrm{E}f_*\mathrm{DR}^{\mathrm{E}}_{X^c}(C_X(\mathcal{M}))[-d_Y]\\ &\simeq \mathrm{DR}^{\mathrm{E}}_{Y^c}(\mathrm{D}f_*C_X(\mathcal{M}))[-d_Y]\\ &\simeq \mathrm{DR}^{\mathrm{E}}_{Y^c}(C_Y(\mathrm{D}f_*\mathcal{M}))[-d_Y],\end{aligned}$$

using in particular Lemma 3.1.

It remains hence to check condition (f): Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X)$ have normal form along a normal crossing divisor $D \subset X$.

Due to Lemma 3.7 below, there is a natural morphism similar to (4)

$$\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^c}(C_X(\mathcal{M})^{\mathcal{A}^c})[-d_X] \to \mathcal{S}ol^{\mathrm{E}}_{\widetilde{X}}(\mathcal{M}^{\mathcal{A}}).$$

We want to show that this is an isomorphism. Locally on $\widetilde{X} = \widetilde{X^c}$, $\mathcal{M}^{\mathcal{A}}$ decomposes as a direct sum of exponential D-modules $\mathcal{E}^{\varphi}$, and $C_X(\mathcal{M})$ decomposes accordingly into exponential D-modules $\mathcal{E}^{-\overline{\varphi}}$ (cf. Proposition 3.4), so it suffices to prove this result for $\mathcal{M} = \mathcal{E}^{\varphi}$. (Note that we have $(C_X(\mathcal{E}^{\varphi}))^{\mathcal{A}^c} \simeq (\mathcal{E}^{-\overline{\varphi}})^{\mathcal{A}^c}$ by Lemma 3.3).

Then we get

$$\begin{aligned}\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^c}(C_X(\mathcal{E}^{\varphi})^{\mathcal{A}^c}) &\simeq \mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^c}((\mathcal{E}^{-\overline{\varphi}})^{\mathcal{A}^c})\\ &\simeq \mathrm{E}\varpi^!\mathrm{DR}^{\mathrm{E}}_{X^c}(\mathcal{E}^{-\overline{\varphi}})\\ &\simeq \mathbb{C}^{\mathrm{E}}_{\widetilde{X}^c} \overset{+}{\otimes} \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathbb{C}_{\{t+\mathrm{Re}\,\overline{\varphi}(x)\geq 0\}})[d_X].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathcal{S}ol^{\mathrm{E}}_{\widetilde{X}}((\mathcal{E}^{\varphi})^{\mathcal{A}}) &\simeq \mathrm{E}\varpi^!\mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathcal{S}ol^{\mathrm{E}}_X(\mathcal{E}^{\varphi}))\\ &\simeq \mathbb{C}^{\mathrm{E}}_{\widetilde{X}} \overset{+}{\otimes} \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathbb{C}_{\{t+\mathrm{Re}\,\varphi(x)\geq 0\}})\end{aligned}$$

Since conjugation does not affect the real part of a holomorphic function, we see that both objects are isomorphic.

Applying $\mathrm{E}\varpi_*$ to both sides and noting that $\mathcal{M}$ is meromorphic (see [DK16, Corollary 9.2.3] and [IT20, §3]), while $C_X(\mathcal{M}) = C_X(\mathcal{M})(!D)$, we obtain the desired isomorphism. □

Lemma 3.7. *There is a canonical morphism*

$$\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^c}(C_X(\mathcal{M})^{\mathcal{A}^c})[-d_X] \to \mathcal{S}ol^{\mathrm{E}}_{\widetilde{X}}(\mathcal{M}^{\mathcal{A}}).$$

Proof. Again, we use the notation in [Sab00], in particular the functor $C^{\mathrm{mod}\,D}_{\widetilde{X}} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\widetilde{X}}}(-, \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}})$.

First, note that there is a canonical morphism $C_X(\mathcal{M})^{\mathcal{A}^c} \to C^{\mathrm{mod}\,D}_{\widetilde{X}}(\mathcal{M}_{\widetilde{X}})$ induced by the natural morphism $\varpi^{-1}\mathcal{D}b_{X_{\mathbb{R}}} \to \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}}$, and

$$\begin{aligned}C^{\mathrm{mod}\,D}_{\widetilde{X}}(\mathcal{M}_{\widetilde{X}}) &\overset{\mathrm{def}}{=} \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\widetilde{X}}}(\mathcal{M}_{\widetilde{X}}, \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}})\\ &\simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathcal{M}^{\mathcal{A}}, \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}})\end{aligned}$$

for coherent $\mathcal{M}$.

It therefore suffices to find a morphism

$$\mathrm{DR}^{\mathrm{E}}_{\widetilde{X}^c}(\mathrm{R}\mathcal{H}om_{\mathcal{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathcal{M}^{\mathcal{A}}, \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}_{\mathbb{R}}}))[-d_X] \to \mathcal{S}ol^{\mathrm{E}}_{\widetilde{X}}(\mathcal{M}^{\mathcal{A}}).$$

Similarly to the proof of Lemma 3.5, it is enough to give a morphism

$$\varpi^{-1}\Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_{X^c}} (\mathcal{O}^{\mathrm{E}}_{\widetilde{X}^c} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_{\widetilde{X}^c}} \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}_{\mathbb{R}}}) \longrightarrow \mathcal{O}^{\mathrm{E}}_{\widetilde{X}}.$$

Now, noting that

$$\mathcal{O}^{\mathrm{E}}_{\widetilde{X}} \simeq \varpi^{-1}\Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{D}_{X^c}} \mathrm{E}\varpi^! \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathcal{D}b^{\mathrm{E}}_{X_{\mathbb{R}}})$$

and writing for short $\mathcal{D}b^{\mathrm{E}}_{\widetilde{X}_{\mathbb{R}}} := \mathrm{E}\varpi^! \mathrm{R}\mathcal{I}hom(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathcal{D}b^{\mathrm{E}}_{X_{\mathbb{R}}})$, it suffices to give a morphism

$$\mathcal{O}^{\mathrm{E}}_{\widetilde{X}^c} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{O}_{X^c}} \mathcal{D}b^{\mathrm{mod}\,D}_{\widetilde{X}_{\mathbb{R}}} \to \mathcal{D}b^{\mathrm{E}}_{\widetilde{X}_{\mathbb{R}}}.$$

Such a morphism is naturally given by multiplication of distributions, as in Lemma 3.5, which is well-defined since the right-hand side is the ind-sheaf of enhanced tempered distributions on the blow-up and hence multiplication with a moderate growth distribution remains an element in this ind-sheaf (similarly to [DK16, Proposition 5.1.3]). □

With this in hand, we can now proceed completely analogously to [Kas86] to define the conjugation functor c, the functor corresponding to complex conjugation on constructible sheaves via the Riemann–Hilbert correspondence.

We define

$$\begin{aligned} c\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) &\to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \\ \mathcal{M} &\mapsto c(\mathcal{M}) := C_{X^c}(\mathbb{D}_{X^c}(\mathcal{M}^c)), \end{aligned}$$

where for a $\mathcal{D}_X$-module $\mathcal{M}$ the $\mathcal{D}_{X^c}$-module $\mathcal{M}^c$ is defined as follows: It is the same sheaf of additive abelian groups as $\mathcal{M}$, but with the action of $\mathcal{D}_{\mathcal{X}}$ induced by the natural morphism $\mathcal{D}_{X^c} \to \mathcal{D}_X$, sending an antiholonomrphic function $\overline{f}$ to f and a derivative $\partial_{\overline{x_i}}$ to ∂_{x_i} for a local coordinate x_i of X.

Theorem 3.8. *There is an isomorphism*

$$\mathrm{DR}^{\mathrm{E}}_X(c(\mathcal{M})) \simeq \overline{\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})}.$$

Proof. It is easy to check that

$$\begin{aligned} \mathrm{DR}^{\mathrm{E}}_X(C_{X^c}(\mathbb{D}_{X^c}(\mathcal{M}^c))) &\simeq \mathcal{S}ol^{\mathrm{E}}_{X^c}(\mathbb{D}_{X^c}(\mathcal{M}^c)) \\ &\simeq \mathrm{DR}^{\mathrm{E}}_{X^c}(\mathcal{M}^c) \simeq \overline{\mathrm{DR}^{\mathrm{E}}_X(\mathcal{M})}, \end{aligned}$$

which follows from Proposition 3.6. □

In [Sab00], the following theorem, equivalent to a conjecture of M. Kashiwara, is proved under the hypothesis that we have the classification of holonomic D-modules as conjectured by C. Sabbah, which is nowadays known from works by himself, K. Kedlaya and T. Mochizuki.

Theorem 3.9 (see [Kas86, Remark 3.5], [Sab00, p. 63 and Theorem 3.1.2])**.** *If $\mathcal{M}$ is holonomic, then $C_X(\mathcal{M}) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_{X_{\mathbb{R}}})$ is concentrated in degree 0 and holonomic.*

The original conjecture of M. Kashiwara contains the following claim. With the help of the above proposition, we give a different way to deduce it from Theorem 3.9.

Corollary 3.10. *The functor $C_X\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X^c})$ is an equivalence with inverse C_{X^c}.*

Proof. The first point is clear by induction on the amplitude of a holonomic complex. For the second point, let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then we have

$$\begin{aligned}\mathrm{DR}^{\mathrm{E}}_X(C_{X^c}(C_X(\mathcal{M}))) &\simeq \mathcal{S}ol^{\mathrm{E}}_{X^c}(C_X(\mathcal{M})) \simeq \mathrm{D}_{X^c}\mathrm{DR}^{\mathrm{E}}_{X^c}(C_X(\mathcal{M}))\\ &\simeq \mathrm{D}_{X^c}\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M}) \simeq \mathrm{DR}^{\mathrm{E}}_X(\mathcal{M}),\end{aligned}$$

and by the Riemann–Hilbert correspondence of [DK16], we get the isomorphism $C_{X^c}(C_X(\mathcal{M})) \simeq \mathcal{M}$. The analogous statement for complexes $\mathcal{M}^\bullet$ with holonomic cohomologies follows by noting that C_X commutes with taking cohomologies. □

4. Galois descent for enhanced ind-sheaves

In this section, we give some complements on the study of Galois descent done in [BHHS22]. We mostly reformulate what has been established there, slightly generalizing the descent statement by removing the compactness assumption that was present in [BHHS22, Proposition 2.15]. We mainly study $\mathbb{R}$-constructible enhanced ind-sheaves here.

The starting point is the functor of extension of scalars: Let $\mathcal{X}$ be a bordered space and let L/K be a field extension, then we have the functor

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}K_{\mathcal{X}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}L_{\mathcal{X}}), \quad H \mapsto \pi^{-1}L_X \otimes_{\pi^{-1}K_X} H.$$

Its compatibilities with direct and inverse images have been described in [BHHS22]. We restate them here, removing some restrictions that are not necessary.

Lemma 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bordered spaces and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism. Let $F, F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}K_{\mathcal{X}})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}K_{\mathcal{Y}})$. Then we have isomorphisms*

$$\begin{aligned}\pi^{-1}L_{\mathcal{Y}} \otimes_{\pi^{-1}K_{\mathcal{Y}}} \mathrm{E}f_{!!}F &\simeq \mathrm{E}f_{!!}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F)\\ \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathrm{E}f^{-1}G &\simeq \mathrm{E}f^{-1}(\pi^{-1}L_{\mathcal{Y}} \otimes_{\pi^{-1}K_{\mathcal{Y}}} G).\end{aligned}$$

If $\mathcal{X}$ and $\mathcal{Y}$ are real analytic bordered spaces and $F \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$, $G \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_{\mathcal{Y}})$, we also have isomorphisms

$$\begin{aligned}\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathrm{D}^{\mathrm{E}}_{\mathcal{X}}F &\simeq \mathrm{D}^{\mathrm{E}}_{\mathcal{X}}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F),\\ \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathrm{E}f^{!}G &\simeq \mathrm{E}f^{!}(\pi^{-1}L_{\mathcal{Y}} \otimes_{\pi^{-1}K_{\mathcal{Y}}} G).\end{aligned}$$

If $X \subseteq \widehat{X}$ and $Y \subseteq \widehat{Y}$ are relatively compact, we have an isomorphism

$$\pi^{-1}L_{\mathcal{Y}} \otimes_{\pi^{-1}K_{\mathcal{Y}}} \mathrm{E}f_{*}F \simeq \mathrm{E}f_{*}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F).$$

Remark 4.2. The condition for the isomorphism for the direct image $\mathrm{E}f_*$ means that the bordered spaces $\mathcal{X}$ and $\mathcal{Y}$ are of a particular type, namely they are b-analytic manifolds and F and G are b-constructible, notions that have been introduced in [Sch23].

Such functorialities are not restricted to the tensor product with a field extension. One could also try to perform a study of functorialities for more general sheaves L_X, similar to [HS23].

In order to understand homomorphisms between scalar extensions of enhanced ind-sheaves, we prove the following two statements.

Proposition 4.3. *Let L/K be a field extension. Let $\mathcal{X}$ be a real analytic bordered space and let $F, G \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$. Then there is an isomorphism in $\mathrm{D}^{\mathrm{b}}(L_X)$*

$$\mathrm{R}\mathcal{H}om^{\mathrm{E}}_{L_{\mathcal{X}}}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F, \pi^{-1}L_X \otimes_{\pi^{-1}K_X} G) \simeq L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om^{\mathrm{E}}_{K_{\mathcal{X}}}(F, G).$$

Proof. There exists a canonical morphism (from left to right), and it suffices to prove that it is an isomorphism locally on an open covering of X by relatively compact open subsets. In other words, it suffices to prove the isomorphism under the assumption that $F \simeq K^{\mathrm{E}}_{\mathcal{X}} \overset{+}{\otimes} \mathcal{F}$, $G \simeq K^{\mathrm{E}}_{\mathcal{X}} \overset{+}{\otimes} \mathcal{G}$ for some $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_{\mathcal{X}\times\mathbb{R}_\infty})$. Let us, without loss of generality, choose them such that $K_{\{t\geq 0\}} \overset{+}{\otimes} \mathcal{F} \simeq \mathcal{F}$ (and similarly for $\mathcal{G}$). Then

$$\begin{aligned}
&\mathrm{R}\mathcal{H}om^{\mathrm{E}}_{L_{\mathcal{X}}}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F, \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} G)\\
&\qquad \simeq \mathrm{R}\mathcal{H}om^{\mathrm{E}}_{L_{\mathcal{X}}}\big(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathcal{F}, \mathrm{R}\mathcal{I}hom^{+}(L^{\mathrm{E}}_{\mathcal{X}}, L^{\mathrm{E}}_{\mathcal{X}} \overset{+}{\otimes} (\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathcal{G}))\big)\\
&\qquad \simeq \alpha_{\mathcal{X}} \mathrm{R}\pi_{\mathcal{X}*} \mathrm{R}\mathcal{I}hom_{L_{\mathcal{X}}}\big(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathcal{F}, L^{\mathrm{E}}_{\mathcal{X}} \overset{+}{\otimes} (\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} \mathcal{G})\big)\\
&\qquad \simeq L_X \otimes_{K_X} \alpha_{\mathcal{X}} \mathrm{R}\pi_{\mathcal{X}*} \mathrm{R}\mathcal{I}hom_{K_{\mathcal{X}}}(\mathcal{F}, K^{\mathrm{E}}_{\mathcal{X}} \overset{+}{\otimes} \mathcal{G})\\
&\qquad \simeq L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om^{\mathrm{E}}_{K_{\mathcal{X}}}(F, G),
\end{aligned}$$

where the third isomorphism follows from [BHHS22, Lemma 2.7] and [DK16, Lemmas 3.3.12 and 3.3.7] (note that the extended map $\overline{\pi}\colon \widehat{X} \times \overline{\mathbb{R}} \to \widehat{X}$ is proper). □

Corollary 4.4. *Let L/K be a field extension. Let $\mathcal{X}$ be a real analytic bordered space and let $F, G \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$. If L/K is finite or X is compact, then the natural morphism*

$$L \otimes_K \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)}(F, G) \to \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_X)}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F, \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} G)$$

is an isomorphism.

Proof. This follows from Proposition 4.3, applying global sections and zeroth cohomology. Note that L is isomorphic to a (possibly infinite) direct sum of copies of K, by choosing a basis of L over K. In the case of a finite field extension, global sections commutes with extension of fields (since finite direct sums are finite direct products and direct images are right adjoints). In the case of a compact space X, global sections are a proper direct image and hence commute with arbitrary direct sums. □

Let us also state the following lemma, whose proof is completely analogous to [Ho23, Proposition 4.4].

Lemma 4.5. *Let L/K be a finite Galois extension with Galois group G. Let $\mathcal{X}$ be a real analytic bordered space and let $F, G \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$. Then the subspace of*

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_X)}(\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F, \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} G) \simeq L \otimes_K \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)}(F, G)$$

consisting of morphisms that fit into a commutative diagram

$$\begin{array}{ccc}
\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F & \xrightarrow{f} & \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} G\\
\big\downarrow{\scriptstyle g\otimes \mathrm{id}_F} & & \big\downarrow{\scriptstyle g\otimes \mathrm{id}_G}\\
\pi^{-1}\overline{L}^{g}_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} F & \xrightarrow{\overline{f}^{g}} & \pi^{-1}\overline{L}^{g}_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} G
\end{array}$$

for any $g \in G$ is exactly the subset $1 \otimes \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)}(F, G)$. If $1 \otimes f \in 1 \otimes \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)}(F, G)$ is an isomorphism, so is $f \in \mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)}(F, G)$.

We can now give a slightly more general version (without the compactness assumption) of [BHHS22, Proposition 2.15].

Proposition 4.6. *Let L/K be a finite Galois extension with Galois group G. Let $\mathcal{X}$ be a real analytic bordered space and let $H \in \mathrm{E}^{0}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$ equipped with a G-structure. Then there exists $H_K \in \mathrm{E}^{0}_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$ and an isomorphism $H \simeq \pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} H_K$ through which the given G-structure on H coincides with the natural one on $\pi^{-1}L_{\mathcal{X}} \otimes_{\pi^{-1}K_{\mathcal{X}}} H_K$.*

Proof. In [BHHS22, Proposition 2.15], we have proved the result in the case that $\mathcal{X} = (X, \widehat{X})$ is such that X is relatively compact in $\widehat{X}$. (The part of the statement about the G-structures has not explicitly been mentioned in loc. cit. but is clear from the construction.)

Now, if $X \subseteq \widehat{X}$ is not relatively compact, we can cover X by open subsets U_i, $i \in I$, all of which are relatively compact in $\widehat{X}$. On each $(U_i)_\infty = (U_i, \overline{U_i})$, we can perform the above construction. and obtain $H_{K,i} \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}K_{(U_i)_\infty})$ with $H|_{(U_i)_\infty} \simeq \pi^{-1}L_{(U_i)_\infty} \otimes_{\pi^{-1}K_{(U_i)_\infty}} H_{K,i}$. On any overlap $(U_{ij})_\infty := (U_i \cap U_j, \overline{U_i \cap U_j})$ we have an isomorphism

$$\pi^{-1}L_{(U_{ij})_\infty} \otimes_{\pi^{-1}K_{(U_{ij})_\infty}} H_{K,i}|_{(U_{ij})_\infty} \simeq \pi^{-1}L_{(U_{ij})_\infty} \otimes_{\pi^{-1}K_{(U_{ij})_\infty}} H_{K,j}|_{(U_{ij})_\infty}$$

induced by the identity on H (which is certainly compatible with the given G-structure). Since the natural G-structures on both sides correspond to the given one on H, we see that this morphism is compatible with the natural G-structures on both sides. Hence, by Lemma 4.5, it descends to an isomorphism $H_{K,i}|_{(U_{ij})_\infty} \simeq H_{K,j}|_{(U_{ij})_\infty}$. Since $U \mapsto \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}K_{U_\infty})$ is a stack, this yields an object $H_K \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}})$ as desired. □

With these results in hand, we can now state Galois descent for $\mathbb{R}$-constructible enhanced ind-sheaves concentrated in one degree as an equivalence of categories.

We denote by $\mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}L_{\mathcal{X}})^G$ the category of pairs $(H, (\varphi_g)_{g\in G})$ of objects $H \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}L_{\mathcal{X}})$ together with a G-structure. A morphism $(H, (\varphi_g)_{g\in G}) \to (H', (\varphi'_g)_{g\in G})$ is a morphism $f\colon H \to H'$ such that for any $g \in G$ the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "\varphi_g"] & H' \arrow[d, "\varphi'_g"] \\ \overline{H}^g \arrow[r, "\overline{f}^g"] & \overline{H'}^g \end{tikzcd}$$

Theorem 4.7. *Let L/K be a finite Galois extension. Let $\mathcal{X}$ be a real analytic bordered space. Then extension of scalars induces an equivalence of categories*

$$\mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}K_{\mathcal{X}}) \xrightarrow{\sim} \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}L_{\mathcal{X}})^G.$$

Proof. Full faithfulness follows from Lemma 4.5 and essential surjectivity follows from Proposition 4.6. □

5. K-structures and monodromy data

In this section, we are going to study the impact of a K-structure on enhanced solutions on generalized monodromy data (in particular Stokes matrices) associated to an enhanced ind-sheaf or a meromorphic connection.

Definition 5.1. *If $K \subset \mathbb{C}$ is a subfield and $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathcal{X}})$, then a K-lattice of H is an enhanced ind-sheaf $H_K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}K_{\mathcal{X}})$ such that $H \simeq \pi^{-1}\mathbb{C}_X \otimes_{K_X} H_K$. If such an H_K exists, we say that H has a K-structure.*

Example 5.2. It is obvious that $\mathbb{E}^f_X$ has a K-structure for any subfield $K \subset \mathbb{C}$. Similarly, consider the following situation: Let X and S be smooth algebraic varieties and consider morphisms

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"] & S \\ \mathbb{A}^1 & \end{tikzcd}$$

Consider the (algebraic) $\mathcal{D}_X$-module $\mathcal{E}^f$ on X and its direct image $g_+\mathcal{E}^f$ on S. Then, the enhanced ind-sheaf $\mathcal{S}ol^{\mathrm{E}}_{S_\infty}(g_+\mathcal{E}^f) \simeq \mathrm{E}g_{\infty!!}\mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathcal{E}^f)$ has a K-structure over any field $K \subset \mathbb{C}$.

Using Galois descent, we have also deduced certain K-structures of the enhanced solutions of hypergeometric systems in [BHHS22].

In the rest of this section, let X be a complex curve and $D \subset X$ a discrete set of points. We write $X^* := X \setminus D$. Given charts around every point $p \in D$ and $\varepsilon = (\varepsilon_p)_{p\in D}$, where $0 < \varepsilon_p << 1$ (we just write $\varepsilon << 1$), we write $X^*_\varepsilon := X \setminus \bigcup_{p\in D} B_{\varepsilon_p}(p)$, where $B_{\varepsilon_p}(p)$ is the ball around p of radius ε_p in the given chart around p.

Let $\Bbbk$ be a field. Recall the notation for enhanced exponentials from Section 2.4. We will simply write $\mathbb{E}^f_W$ instead of $\mathbb{E}^f_{W,\Bbbk}$ to simplify notation.

Descent of Hukuhara–Levelt–Turrittin form.

Definition 5.3. *Let $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\Bbbk_X)$. We say that H is of* Hukuhara–Levelt–Turrittin (HLT) type *with respect to D if the following conditions are satisfied:*

(a) $\pi^{-1}\Bbbk_{X^*} \otimes H \simeq H$,

(b) *For any $\varepsilon << 1$, we have $\pi^{-1}\Bbbk_{X^*_\varepsilon} \otimes H \simeq \Bbbk^{\mathrm{E}}_X \otimes \pi^{-1}\mathcal{L}$ for some local system $\mathcal{L}$ on X^*_ε (extended by 0 to X),*

(c) *For any $p \in D$, a local coordinate z in a neighbourhood U of p and any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, there exist $r, \delta \in \mathbb{R}_{>0}$, $n \in \mathbb{Z}_{>0}$, a determination of $z^{1/n}$ on the sector $S_\theta := \{z \in U \mid 0 < |z| < r, \theta - \delta < \arg z < \theta + \delta\}$, a finite set $\Phi_\theta \subset z^{-1/n}\mathbb{C}[z^{-1/n}]$ and an integer $r_f \in \mathbb{Z}_{>0}$ for any $f \in \Phi_\theta$ such that the $f \in \Phi_\theta$ define holomorphic functions on S_θ and one has an isomorphism*
$$\pi^{-1}\Bbbk_{S_\theta} \otimes H \simeq \bigoplus_{f\in\Phi_\theta} (\mathbb{E}^f_{S_\theta})^{r_f}.$$

In particular, H being of HLT type implies $H \in \mathrm{E}^0_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_X)$.

Remark 5.4. To make it more similar to [DK18], our condition (b) of Defintion 5.3 can be reformulated as follows:

(b') There is an isomorphism $H|_{X^*} \simeq \Bbbk^{\mathrm{E}}_{X^*} \otimes \pi^{-1}\mathcal{L}$ for a local system $\mathcal{L}$ on X^*.

This is equalent to (b) since $\mathrm{E}^0(\mathrm{I}\Bbbk_{X^*})$ is a stack and hence the isomorphisms on all the X^*_ε imply the isomorphism on the whole of X^*.

Since the circle of directions $\mathbb{R}/2\pi\mathbb{Z}$ is compact, part (c) of Definition 5.3 can equivalently be formulated as follows:

(c') *For any $p \in D$ and a local coordinate z in a neighbourhood U of p, there exists a finite number of sectors $S^p_1, \dots, S^p_{k_p}$ (for some $k_p \in \mathbb{Z}_{>0}$) covering a punctured neighbourhood of p such that for any $j \in \{1, \dots, k_p\}$ one has: a determination of $z^{1/n}$ on the sector S^p_j, a finite set $\Phi^p_j \subset z^{-1/n}\mathbb{C}[z^{-1/n}]$ and an integer $r_f \in \mathbb{Z}_{>0}$ for any $f \in \Phi^p_j$ such that the $f \in \Phi^p_j$ define holomorphic functions on S^p_j and one has an isomorphism*

$$\pi^{-1}\Bbbk_{S^p_j} \otimes H \simeq \bigoplus_{f\in\Phi^p_j} (\mathbb{E}^f_{S^p_j})^{r_f}. \tag{5}$$

We will assume that the sectors $S^p_1, \dots, S^p_{k_p}$ are ordered in a counter-clockwise sense around p (with respect to their central directions). Then, for any $j \in \{1, \dots, k_p\}$ we can consider the overlap $S^p_{j,j+1} := S^p_j \cap S^p_{j+1}$ (where, of course, we count modulo k_p, so that $k_p + 1 := 1$). On this overlap we have the two isomorphisms induced by the ones from (5)

$$\bigoplus_{f\in\Phi_j^p} (\mathbb{E}^f_{S^p_{j,j+1}})^{r_f} \simeq \pi^{-1}\Bbbk_{S^p_{j,j+1}} \otimes H \simeq \bigoplus_{g\in\Phi_{j+1}^p} (\mathbb{E}^g_{S^p_{j,j+1}})^{r_g}.$$

This implies that we can identify the index sets Φ_j^p and Φ_{j+1}^p (see [Moc22, Lemma 3.25], cf. also [DK18, Corollary 5.2.3]), and also $r_f = r_g$ via this identification. More precisely, this identification is given by holomorphic continuation of functions in counter-clockwise direction from S_j^p to S_{j+1}^p. We will hence consider all the direct sums in the isomorphisms (5) to be indexed by the same set $\Phi^p = \Phi_1^p$. In particular, this also means that Φ^p is closed under analytic continuation around the circle, i.e. if $f(z^{-1/n}) \in \Phi^p$, then $f(e^{\frac{2\pi i}{n}} z^{-1/n}) \in \Phi^p$.

It is also clear that $r = \sum_{f\in\Phi^p} r_f$ for any p.

Our goal is to show that the property of being of HLT type descends to a lattice. To do this, let us prepare some lemmas.

Remark 5.5. To prepare for the argument that follows, let us make the following easy observation: If S is a sector at p and we are given two holomorphic functions f, g on S such that $f \prec g$, i.e. $\operatorname{Re} f - g$ is bounded from below but unbounded from above near p. Then:

- For any relatively compact subanalytic $U \subseteq X \times \overline{\mathbb{R}}$ such that $\mathbb{E}^g_S(U) \neq 0$, we have $\mathbb{E}^f_S(U) \neq 0$.

To see this, note that $\mathbb{E}^g_S(U) \neq 0$ means in particular that U intersects any $\{t \geq -\operatorname{Re} g + a\}$ nontrivially. Now consider an arbitrary $b \in \mathbb{R}$, then there exists $a \in \mathbb{R}$ such that $\operatorname{Re}(f - g) > b - a$, so $\{t \geq -\operatorname{Re} f + b\} \supseteq \{t \geq -\operatorname{Re} g + a\}$. This means that also $\mathbb{E}^g_S(U) \neq 0$.

- There exists $U \subseteq X \times \overline{\mathbb{R}}$ such that $\mathbb{E}^f_S(U) = \mathbb{C}$ but $\mathbb{E}^g_S(U) = 0$.

Such a set can be given explicitly: Set $U := \{(x,t) \in X \times \overline{\mathbb{R}} \mid x \in S, t \in \mathbb{R}, t < -\operatorname{Re} g\}$. Certainly, U does not intersect $\{t \geq -\operatorname{Re} g + a\}$ for any $a > 0$, so $\mathbb{E}^g_S(U) = 0$. On the other hand, for any $a \in \mathbb{R}$ there exists $x \in S$ such that $\operatorname{Re}(f(x) - g(x)) > a$, and hence U intersects any $\{t \geq -\operatorname{Re} f + a\}$ (and the intersection eventually has one connected component), so $\mathbb{E}^f_S(U) = \mathbb{C}$. We can hence assume that all the isomorphisms

Lemma 5.6. *Let S be a sector at p. Then, up to an automorphism of $\bigoplus_{f\in\Phi}(\mathbb{E}^f_{S,L})^{r_f}$, a K-lattice $F_K \subset \bigoplus_{f\in\Phi}(\mathbb{E}^f_{S,L})^{r_f}$ with $F_K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)$ is of the form $F_K = \bigoplus_{f\in\Phi}(\mathbb{E}^f_{S,K})^{r_f}$.*

Proof. Let us assume for simplicity that $r_f = 1$ for any $f \in \Phi$, and let us think of the $\mathbb{E}^f_{S,L}$ as subanalytic sheaves on $X \times \overline{\mathbb{R}}$ (cf. Remark 2.4). Then for any open $W \subseteq X \times \overline{\mathbb{R}}$, the sections of F_K on W must be a K-lattice of the sections of $\bigoplus_{f\in\Phi}\mathbb{E}^f_{S,L}$ on W (see Lemma 2.3).

Choose an open part $T \subset S$ with p in its boundary such that all the $f \in \Phi$ are totally ordered. Let $n := |\Phi|$ and choose a numbering $\Phi = \{f_1, \ldots, f_n\}$ such that $f_1 \prec f_2 \prec \ldots \prec f_n$.

The sections of $\bigoplus_{f\in\Phi}\mathbb{E}^f_{S,L}$ on some $(U \cap S) \times \mathbb{R}$ (with $U \subset X$ an open neighbourhood of p) will be $\mathbb{C}^n$. A K-lattice for this space of sections is given by a vector space

$$\sum_{j=1}^{n} K \cdot v_j$$

for some $v_j \in \mathbb{C}^n$ (linearly independent over $\mathbb{C}$). Since restriction maps of $\bigoplus_{f\in\Phi} \mathbb{E}^f_{S,L}$ (recall that they mainly consist of projections) need to be compatible with those of F_K, this space also determines the spaces of sections of F_K on any other open set.

Now we see that there are several conditions imposed on these v_j: Since there is a region in S where $f_1 \prec \ldots \prec f_n$, we see (as in Remark 5.5) that there are open sets $W_j \subset X \times \overline{\mathbb{R}}$ such that $\mathbb{E}^{f_k}_{S,L}(W_j) = 0$ if $f_j \prec f_k$ and $\mathbb{E}^{f_k}_{S,L}(W_j) = L$ if $f_k \preceq f_j$, and restriction maps between these W_j are given by projections. Therefore, this means that for any j, the vectors $\mathrm{pr}_{\leq j}(v_1), \ldots, \mathrm{pr}_{\leq j}(v_n)$ span a K-vector space of dimension j. (Here, $\mathrm{pr}_{\leq j}$ is the projection to the first j components.)

In particular, we can without loss of generality assume that the invertible matrix $(v_1, \ldots, v_n)$ is lower-triangular.

Now, if S contains a Stokes direction for a pair (f_j, f_k), then there is another part T' of S (having p in its boundary and separated from T by a Stokes curve) with a different total ordering on Φ. There, similar arguments apply: Let us assume that we have $f_j \prec f_k$ in the chosen order (on T), but $f_j \succ f_k$ in the other order (on T'). Then, arguing similarly as above, we get the condition that $\mathrm{pr}_{\leq j-1,k}(v_1), \ldots, \mathrm{pr}_{\leq j-1,k}(v_n)$ span a K-vector space of dimension j. (Here $\mathrm{pr}_{\leq j-1,k}$ is the projection to the components $1, 2, \ldots, j-1, k$.) This means that we can assume that the kth components of $v_j, \ldots, v_{k-1}$ (and in particular that of v_j) vanish.

Similarly, continuing these arguments over all Stokes directions contained in S, we see that we choose $v_1, \ldots, v_n$ in such a way that the k-th component of v_j is zero if S contains a Stokes direction for (f_j, f_k), $j < k$ (in the chosen order on T). In other words, the matrix $(v_1, \ldots, v_n)^{-1}$ then defines an automorphism of $\bigoplus_{f\in\Phi} \mathbb{E}^f_{S,L}$, under which the K-structure F_K is $\bigoplus_{f\in\Phi} \mathbb{E}^f_{S,K}$, as desired.

The proof for the case $r_f \neq 1$ is analogous, thinking in terms of graded vector spaces and block matrices instead. $\square$

Lemma 5.7. *Let $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}L_X)$ and let $U \subseteq X$ be an open subset such that there is an isomorphism $\pi^{-1}L_U \otimes H \simeq L^{\mathrm{E}}_X \otimes \pi^{-1}\mathcal{L}$ for some local system $\mathcal{L}$ on U. Assume that H has a K-lattice $H_K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_X)$. Then, there exists a local system $\mathcal{L}_K$ over K on U with $\pi^{-1}K_U \otimes H_K \simeq K^{\mathrm{E}}_X \otimes \pi^{-1}\mathcal{L}_K$ and $\mathcal{L} \simeq L_U \otimes_{K_U} \mathcal{L}_K$.*

Proof. Since $\mathcal{L}$ is a local system, there exists, for each $x \in U$, an open neighbourhood V such that

$$\pi^{-1}L_V \otimes H \simeq L^{\mathrm{E}}_X \otimes \pi^{-1}L^r_V = (\mathbb{E}^0_{V,L})^r \tag{6}$$

for some $r \in \mathbb{Z}_{>0}$. By an argument similar to (but much simpler than) that of Lemma 5.6, we see that the lattice of $(\mathbb{E}^0_{V,L})^r$ given by the image of $\pi^{-1}K_V \otimes H_K$ is – up to an automorphism – isomorphic to $(\mathbb{E}^0_{V,K})^r$. Hence, by composing with a suitable automorphism of $(\mathbb{E}^0_{V,L})^r$, the isomorphism (6) comes from an isomorphism

$$\pi^{-1}K_V \otimes H_K \simeq (\mathbb{E}^0_{V,K})^r$$

by applying $\pi^{-1}L_X \otimes_{\pi^{-1}K_X} (-)$. Now, if we choose an open cover $U = \bigcup_i V_i$ such that for every i, we have

$$\pi^{-1}K_{V_i} \otimes H_K \simeq (\mathbb{E}^0_{V_i,K})^r \simeq K^{\mathrm{E}}_X \otimes \pi^{-1}K^r_{V_i}.$$

On the intersection of two sets V_i and V_j we therefore get an isomorphism

$$K^r_{V_i}|_{V_i\cap V_j} \simeq K^r_{V_j}|_{V_i\cap V_j}$$

since the functor $K^{\mathrm{E}}_X \otimes \pi^{-1}(-)$ is fully faithful (see [DK16, Proposition 4.7.15]), and via these the constant sheaves glue to a local system $\mathcal{L}_K$ over K on U such that $\pi^{-1}K_U \otimes H_K \simeq K_X^{\mathrm{E}} \otimes \pi^{-1}\mathcal{L}_K$. Therefore it follows

$$\begin{aligned} L_X^{\mathrm{E}} \otimes \pi^{-1}\mathcal{L} &\simeq \pi^{-1}L_U \otimes H \\ &\simeq \pi^{-1}L_X \otimes_{\pi^{-1}K_X} \pi^{-1}K_U \otimes H_K \\ &\simeq \pi^{-1}L_X \otimes_{\pi^{-1}K_X} K_X^{\mathrm{E}} \otimes \pi^{-1}\mathcal{L}_K \simeq L_X^{\mathrm{E}} \otimes \pi^{-1}(L_U \otimes \mathcal{L}_K) \end{aligned}$$

and hence the statement follows from the full faithfulness of the functor $L_X^{\mathrm{E}} \otimes \pi^{-1}(-)$. □

Theorem 5.8. *Let L/K be a field extension and let $H \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}L_X)$. Assume that there exists $H_K \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}K_X)$ which is a K-lattice for H, i.e. there is an isomorphism $H \simeq \pi^{-1}L_X \otimes_{\pi^{-1}K_X} H_K$. Then, if H is of HLT type, then H_K is of HLT type.*

Proof. Let us check that H_K is of HLT type:

The condition (a) from Definition 5.3 is easily checked: Consider the distinguished triangle

$$\pi^{-1}K_{X^*} \otimes H_K \longrightarrow H_K \longrightarrow \pi^{-1}K_D \otimes H_K \xrightarrow{+1}$$

Applying the functor $\pi^{-1}L_X \otimes_{\pi^{-1}K_X} (-)$ to it makes the first morphism an isomorphism and hence $\pi^{-1}L_X \otimes_{\pi^{-1}K_X} (\pi^{-1}K_D \otimes H_K) \simeq 0$. This implies $\pi^{-1}K_D \otimes H_K \simeq 0$ and then the desired isomorphism, since extension of scalars is faithful (by Corollary 4.4).

Property (b) is exactly what we proved in Lemma 5.7.

Property (c) follows directly from Lemma 5.6: Given an isomorphism

$$\alpha\colon \pi^{-1}L_{S_\theta} \otimes H \simeq \bigoplus_{f\in\Phi_\theta} (\mathbb{E}^f_{S_\theta,L})^{r_f}$$

as in Definition 5.3(c) for H, we see that, due to the L-linearity of α, the object $\alpha(\pi^{-1}K_{S_\theta} \otimes H_K) \subset \bigoplus_{f\in\Phi_\theta} (\mathbb{E}^f_{S_\theta,L})^{r_f}$ is a K-lattice. We therefore see that, by Lemma 5.6, the isomorphism above comes, after composing it with an automorphism of the right-hand side, from an isomorphism

$$\pi^{-1}K_{S_\theta} \otimes H_K \simeq \bigoplus_{f\in\Phi_\theta} (\mathbb{E}^f_{S_\theta,K})^{r_f}.$$

This completes the proof. □

Generalized monodromy data as gluing data for enhanced ind-sheaves. Given an enhanced ind-sheaf $H \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\Bbbk_X)$ of HLT type, we can describe it purely in terms of linear algebra data. Let us recall the decription of these so-called *generalized monodromy data* here (in the case $\Bbbk = \mathbb{C}$, they are what is often called *Stokes data*).

Since H is of HLT type, any $p \in D$ admits data as in Remark 5.4, in particular we can fix a finite collection of open sectors $S_1^p, \dots, S_{k_p}^p$ and isomorphisms

$$\alpha_j^p\colon \pi^{-1}\Bbbk_{S_j^p} \otimes H \simeq \bigoplus_{f\in\Phi^p} (\mathbb{E}^f_{S_j^p})^{r_f}. \tag{7}$$

Moreover, for any $p \in D$ we choose $\varepsilon_p \in \mathbb{R}_{>0}$ such that $\overline{B_{\varepsilon_p}(p)} \subset \bigcup_{j=1}^{k_p} S_j^p$. Then we can also choose an isomorphism by Definition 5.3(b)

$$\gamma\colon \pi^{-1}\Bbbk_{X_\varepsilon^*} \otimes H \simeq \Bbbk_X^{\mathrm{E}} \otimes \pi^{-1}\mathcal{L} \tag{8}$$

for a local system $\mathcal{L}$ on X_ε^*.

In view of wanting to use Lemma 2.5, we also fix the following data:

- For any p, a numbering on Φ^p and also, for any $f \in \Phi^p$, a numbering on the factors of the power $(\mathbb{E}^f_{S^p_j})^{r_f}$, so that there is a total order on the summands of $\bigoplus_{f\in\Phi^p}(\mathbb{E}^f_{S^p_j})^{r_f} = \bigoplus_{f\in\Phi^p}\bigoplus_{m=1}^{r_f}\mathbb{E}^f_{S^p_j}$.
- A point $x \in X^*_\varepsilon$, a basis of the stalk $\mathcal{L}_x$, and for each $p \in D$ a point $y_p \in S^p_1 \cap X^*_\varepsilon$ and a path in X^*_ε from x to y_p. Together with the above γ, this gives in particular an isomorphism $\pi^{-1}\Bbbk_{S^p_1\cap X^*_\varepsilon} \otimes H \simeq \bigoplus_{m=1}^{r}\mathbb{E}^0_{S^p_1\cap X^*_\varepsilon}$.

Then we can associate the following generalized monodromy data to H: Let $p \in D$, then for any $j \in \{1, \ldots, k_p\}$ we get two (in general different) isomorphisms on the overlap $S^p_{j,j+1}$

$$\alpha^p_j, \alpha^p_{j+1} \colon \pi^{-1}\Bbbk_{S^p_{j,j+1}} \otimes H \xrightarrow{\sim} \bigoplus_{f\in\Phi^p}(\mathbb{E}^f_{S^p_{j,j+1}})^{r_f}.$$

(These are induced by α^p_j and α^p_{j+1} above and we denote them by the same symbols.) From these we get an automorphism (a transition or gluing automorphism)

$$\sigma^p_j := \alpha^p_{j+1} \circ (\alpha^p_j)^{-1} \in \operatorname{End}\Big(\bigoplus_{f\in\Phi^p}(\mathbb{E}^f_{S^p_{j,j+1}})^{r_f}\Big),$$

which is represented by an invertible square matrix Σ^p_j by Lemma 2.5.

Similarly, on the overlap $U^p_\varepsilon := S^p_1 \cap X^*_\varepsilon$, we get two isomorphisms

$$\alpha^p_1, \gamma \colon \pi^{-1}\Bbbk_{U^p_\varepsilon} \otimes H \xrightarrow{\sim} (\mathbb{E}^0_{U^p_\varepsilon})^r$$

(note that $\mathbb{E}^f_{U^*_\varepsilon} \simeq \mathbb{E}^0_{U^*_\varepsilon}$). Then the composition

$$c_p := \gamma \circ (\alpha^p_1)^{-1} \in \operatorname{Hom}\big((\mathbb{E}^0_{U^p_\varepsilon})^r, (\mathbb{E}^0_{U^p_\varepsilon})^r\big)$$

is again represented by an invertible square matrix C_p.

The following statement is easily proved.

Lemma 5.9. *The collection of the sectors S^p_j and the matrices Σ^p_j for any $p \in D$ and any $j \in \{1, \ldots, k_p\}$, together with the matrices C_p for any $p \in D$ determines H up to isomorphism.*

Of course, given H of HLT type, the associated generalized monodromy data are not unique, but depend on concrete choices of sectors S^p_j and isomorphisms (7) and (8).

Remark 5.10. The matrices Σ^p_j are what is usually called “Stokes matrices” in the context of solutions of differential equations. They describe the behaviour of solutions around the singularity. On the other hand, the matrices C_p are usually referred to as “connection matrices”. They give the relation between solutions around a singularity with the generic solutions away from the singularities.

Let us also note that there are ways to make the choice and size of the sectors S^p_j as well as the isomorphisms of α^p_j more canonical, which makes the definition of Stokes data less ambiguous, but we will not insist on these choices here and use only the existence of these sectors and isomorphisms. Let us just remark that with the correct canonical choices of sectors and isomorpgisms, one gets Stokes matrices with a certain block-triangular structure and the diagonal entries of all but one Stokes matrix consists of identities.

Let us just remark that in general the sectors are chosen such that they contain at most one Stokes line for each pair of functions $f, f' \in \Phi_k$. In other words, the subset of S_k where $f < f'$ is connected. We will assume this here.

From what we proved above, we can now deduce the following statement.

Proposition 5.11. *Let L/K be a field extension, and let $H \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}L_X)$ be of HLT type. Assume that H has a K-structure. Then there exist generalized monodromy data for H with entries in K.*

Proof. By Theorem 5.8, if $H = \pi^{-1}L_X \otimes_{\pi^{-1}K_X} H_K$, then H_K is also of HLT type. Therefore, H admits generalized monodromy data that agrees with generalized monodromy data of H_K and hence the matrices Σ_j^p and C_p have entries in K. □

Meromorphic connections and Stokes data. Let now $\mathcal{M}$ be a meromorphic connection on X with poles at D, i.e. a holonomic $\mathcal{D}_X$-module such that $\mathcal{M}(*D) \simeq \mathcal{M}$ and $\operatorname{sing\,supp} \mathcal{M} = D$.

We recall briefly why $\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$ is of HLT type:

First of all, $\mathcal{M}(*D) \simeq \mathcal{M}$ implies that $\pi^{-1}\mathbb{C}_{X^*} \otimes \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M})$.

Since a meromorphic connection is generically an integrable connection, we also have, for any ε, an isomorphism

$$\mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \mathbb{C}_X^{\mathrm{E}} \otimes \pi^{-1}\mathcal{L},$$

where $\mathcal{L}$ is the local system of solutions $\mathcal{L} := \mathcal{S}ol_{X_\varepsilon^*}(\mathcal{M}|_{X_\varepsilon^*})$.

Let $p \in D$, then the classical Levelt–Turrittin theorem gives us a decomposition (up to ramification) of the formalization of the stalk of $\mathcal{M}$ at p

$$(\rho^*\mathcal{M})\hat{|}_p \simeq \big(\bigoplus_{i\in I} \mathcal{E}^{\varphi_i} \overset{\mathrm{D}}{\otimes} \mathcal{R}_i\big)\hat{|}_p,$$

where $\rho(t) = t^n$ is a ramification map in a small neighbourhood of p, the $\varphi_i \in z^{-1}\mathbb{C}[z^{-1}]$ are (pairwise distinct) polar parts of Laurent series in a local coordinate z at p and the $\mathcal{R}_i$ are regular holonomic.

By the Hukuhara–Turrittin theorem, locally on sufficiently small sectors around p, this decomposition lifts to an analytic decomposition of $\mathcal{M}$, which is usually formulated as a statement on the real blow-up space of X at the points of D. We will not go into too much detail here, and rather refer to the existing literature on Stokes phenomena.

What we need is the following consequence on the level of the enhanced ind-sheaf associated to $\mathcal{M}$: Let $p \in D$ and let z be a local coordinate of X at p. Then for any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, there exists a sufficiently small sector $S = \{z \in U \mid 0 < |z| < r, \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ and a finite set Φ of holomorphic functions on S such that there is an isomorphism

$$\pi^{-1}\mathbb{C}_S \otimes \mathcal{S}ol_X^{\mathrm{E}}(\mathcal{M}) \simeq \bigoplus_{f\in\Phi} (\mathbb{E}_S^f)^{r_f}.$$

Remark 5.12. The right-hand side is a short notation but can be described more explicitly: Concretely, Φ is the set of functions $\varphi_i(\zeta_n^j z^{1/n})$ for all $i \in I$ and $j \in \{1, \ldots, n-1\}$, where $\zeta_n := e^{2\pi i/n}$ is a primitive n-th root of unity and $z^{1/n}$ is the choice of an n-th root on S of a local coordinate z around p. In particular, the functions f have Puiseux series expansions with a pole at p. The r_f are the ranks of the corresponding $\mathcal{R}_i$. (Note that a function cannot appear multiple times in Φ since we consider it as a set, i.e. if multiple f are the same, we only count it once.)

Indeed, there is the following equivalence (see [DK18, Proposition 6.2.4]).

Proposition 5.13. *The category of meromorphic connections at D is equivalent to the subcategory of $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X)$ consisting of objects of HLT type.*

Proof. The functor is certainly fully faithful by [DK16, Theorem 9.5.3]. It therefore remains to show that it is essentially surjective. Let therefore H be of HLT type. The statement is proved on disks in [Moc22, Lemma 4.8], so on suitable small disks

B_p around any p, we can find $\mathcal{M}_p$ such that $\mathcal{S}ol^{\mathrm{E}}_{B_p}(\mathcal{M}_p) \simeq H|_{B_p}$. Furthermore, on X^*_ε we can certainly find (by the regular Riemann–Hilbert correspondence) a $\mathcal{D}_{X^*_\varepsilon}$-module $\mathcal{M}_\varepsilon$, locally free over $\mathcal{O}_{X^*_\varepsilon}$, such that $\mathcal{S}ol^{\mathrm{E}}_{X^*_\varepsilon}(\mathcal{M}_\varepsilon) \simeq H|_{X^*_\varepsilon}$. All these D-modules glue to a single meromorphic connection on X with poles at D (cf., e.g., the argument in [Ho20, Proposition 2.17]). □

We can now draw two direct consequences of our studies above. The first one follows directly from Proposition 5.11.

Corollary 5.14. *If $\mathcal{S}ol^{\mathrm{E}}_X(\mathcal{M})$ has a K-structure, then its Stokes matrices around any point of D are defined over K, i.e. there are representatives of the generalized monodromy data (Stokes and connection matrices) with entries in K.*

As a kind of upshot of all our considerations in this article, we get the following consequence of Theorem 3.8, Theorem 4.7 and Corollary 5.14. Note, however, that our results from Sections 4 and 5 are valid in greater generality and not restricted to the case of the field extension $\mathbb{C}/\mathbb{R}$ (and not even to finite Galois extensions in the present section).

Corollary 5.15. *Let $\mathcal{M}$ be a meromorphic connection with an isomorphism $\varphi\colon \mathcal{M} \to c(\mathcal{M})$ such that $c(\varphi) \circ \varphi = \mathrm{id}$, then $\mathcal{M}$ admits generalized monodromy data with entries in $\mathbb{R}$.*

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