Field extensions and Galois descent for sheaves of vector spaces

Andreas Hohl*

We study extension of scalars for sheaves of vector spaces, assembling results that follow from well-known statements about vector spaces, but also developing some complements. In particular, we formulate Galois descent in this context, and we also discuss the case of derived categories and establish Galois descent for perverse sheaves. On the way, we prove interesting compatibilities between the six Grothendieck operations and extension of scalars, carefully distinguishing the cases of finite and infinite extensions, and we make use of some gluing techniques for $\mathbb{R}$-constructible and perverse sheaves.

Contents

*The author's research is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Projektnummer 465657531.

1 Introduction

Given a field extension L/K, it is easy to produce an L-vector space V out of a K-vector space W by "extending scalars" (or "change of rings"), i.e. setting $V := L \otimes_K W$. The case of vector spaces is, of course, the most basic example, but similar constructions are performed in more advanced contexts: For instance, one can "upgrade" a scheme over K to a scheme over L. While it is certainly interesting to study such an extension functor itself, one might also wonder if it is possible to describe more precisely its essential image and a possible way of reconstructing an object over K from its associated object over L and some extra data. The last question has nice answers mainly in the case where L/K is a Galois extension, and the machinery behind its solution is commonly referred to as *Galois descent*. In the case of vector spaces (where Galois descent is actually a special case of faithfully flat descent for modules), such questions have, for example, been studied in classical literature such as [Win74], [Jac62] [Wat79], [Bor99]. We also like to mention the nice surveys [Con] and [Jah00].

If k is a field, k-vector spaces are nothing but sheaves (modules) over the constant sheaf k on the one-point space, which is the same as the structure sheaf if we consider the one-point space as $\operatorname{Spec} k$. They therefore have two natural generalizations: One can think about $\mathcal{O}_X$-modules on more general varieties or schemes X over k, or one can think about modules over the constant sheaf k_X on more general topological spaces X. New questions arise in these contexts, since one can, for instance, ask the question of compatibility of extension of scalars with operations on sheaves (such as direct and inverse images, duality, etc.). The first viewpoint seems to be widely established (see e.g. [Jah00] for an overview and [Stacks, Tag 0CDQ]).

In this work, we are going to take the second viewpoint and study extension and descent questions in the case of sheaves of vector spaces and related categories. Most of the basic results from the theory of extension of scalars and Galois descent for vector spaces imply (more or less directly) similar results for sheaves of vector spaces, and indeed our main reference will be classical sheaf theory (see, for example, [KS90]). However, the technical subtleties of certain statements are not always obvious, and it is difficult to keep the overview of the conditions that are required for every single statement: Some assertions hold for arbitrary field extensions and sheaves, others require finite or Galois extensions or certain constructibility assumptions on the sheaves. Although the main results in this direction are probably known to experts or follow from more general frameworks, a unique reference for the details in the concrete setting of sheaves of vector spaces does not seem to exist. These concepts play, however, a crucial role in theories like that of mixed Hodge modules, where extension of scalars is used for perverse sheaves.

The aim of this work is therefore twofold: Firstly, we like to collect and present the main definitions and statements about extension of scalars and Galois descent for sheaves of vector spaces, together with the arguments needed to deduce them from the well-known statements in linear algebra. Great parts are therefore meant to be rather expository and we do not claim originality for all of these statements, but we hope that this overview will serve as a useful reference.

Secondly, in the course of this detailed presentation, we establish some complementary technical results: We describe compatibilities between the six Grothendieck operations (and in particular the functor $\mathrm{R}\mathcal{H}om$) and extension of scalars in the (derived) category of sheaves of vector spaces. Moreover, we investigate Galois descent for complexes in the derived category of sheaves of vector spaces as well as for perverse sheaves. All these considerations involve in particular some interesting arguments using results on the structure of $\mathbb{R}$-constructible and perverse sheaves. Even for readers already familiar with the basic ideas, these statements may be an enlightening illustration of gluing techniques for constructible and perverse sheaves.

We originally got interested in this subject during the preparation of our joint article with Davide Barco, Marco Hien and Christian Sevenheck [BHHS22], where we studied certain differential equations from a topological viewpoint. The basic idea is the following: A Riemann–Hilbert correspondence is an equivalence between certain categories of differential systems and categories of topological objects (such as local systems and perverse sheaves, for example). These topological objects are a priori defined over the field of complex numbers, so one can ask under which conditions such an object "descends" to one over a subfield of $\mathbb{C}$, and it turned out that Galois descent serves as a useful technique there. Although this was our original motivation for studying Galois descent for sheaves of vector spaces, we will not make any reference to the concrete application in loc. cit. here, nor to the more general framework of enhanced ind-sheaves we were studying in there. We rather consider this an independent, self-contained and accessible exposition of the subject, providing more details for a broader readership familiar with Galois and sheaf theory.

In our common work, we established some Galois descent results for sheaves (and more general objects), some of which we will reformulate and complement in this work. We will also investigate the case of complexes of sheaves.

We will mostly distinguish between results for finite Galois extensions (where we get the best results) and results for arbitrary (in particular infinite) field extensions. Let us note that the classical theory of Galois descent for vector spaces can also be adapted to the case of infinite Galois extensions, where again one gets better results than in the case of arbitrary infinite extensions. We will not discuss this case explicitly in this article, although it is certainly interesting to study this also in the context of sheaves.

Outline After reviewing some basics of sheaf theory (mainly to set notation and terminology) in Section 2, we set up the concepts of G-structures, extension of scalars, K-structures and Galois descent in a quite abstract categorical framework in Section 3. On the one hand, this allows us to define these notions for multiple categories at the same time, on the other hand, it might also serve as a useful framework for studying them in different examples later. After each definition, we directly give the explicit description in our categories of interest, and the main immediate properties in the case of sheaves of vector spaces. We remark that several compatibilities between extension of scalars and the six Grothendieck operations remain open there (and are, in general, not true), cf. Lemma 3.12. We address them in Section 4, where we mainly study the compatibility of extension of scalars with the functor $\mathrm{R}\mathcal{H}om$. On the way, we will also discover compatibilities with the remaining functors. We distinguish the cases of finite and infinite field extensions since the constructions involved have different flavours. In the latter case, we will in particular work with $\mathbb{R}$-constructible sheaves and perform some interesting constructions using simplicial complexes.

In Section 5, we first describe explicitly Galois descent for sheaves and formulate it as an equivalence of categories, using in particular the results of the previous section to obtain full faithfulness. We then briefly discuss descent in derived categories of sheaves of vector spaces. We do not obtain an equivalence as in the abelian case, but we will give some background and explanations on the problems that arise. Finally, the last subsection is devoted to the study of Galois descent for perverse sheaves. We use a construction of A. Beilinson [Bei87] – which we will briefly recall and study in the context of extension of scalars – to realize perverse sheaves by "gluing data" and inductively reduce to the case of sheaves proved before.

Acknowledgements I would like to thank Claude Sabbah for invaluable discussions, which in particular helped me to work out Galois descent for perverse sheaves using Beilinson's construction. I also thank Davide Barco, Marco Hien and Christian Sevenheck who, through our common work, inspired me to work out more details on this subject. Finally, I am particularly grateful to Takuro Mochizuki for answering my questions and providing some ideas during the preparation of [BHHS22], which helped me to better understand the theory. In particular, the proofs of Lemmas 4.8 and 4.9 go back to his suggestions.

2 A very short review of sheaf theory

We assume the readers to be familiar with the theory of sheaves of vector spaces. We will recall here some basic facts and notations, and refer to the standard literature such as [KS90] or [Dim04] for details.

Although not strictly necessary in all places, we will assume all our topological spaces to be *good*, i.e. Hausdorff, locally compact, second countable and of finite cohomological dimension. (This is particularly important for the construction of the functors $f_!$ and $f^!$.)

Presheaves and sheaves Let X be a topological space and let $\mathrm{Op}(X)$ be the category of open subsets of X, where $\mathrm{Hom}(U, V)$ has one element if $U \subseteq V$ and is empty otherwise. Let k be a field.

A presheaf of k-vector spaces on X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Vect}_k$. It is a sheaf if for every open $U \subseteq X$ qnd every open covering $U = \bigcup_{i\in I} U_i$ the natural sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is exact. If $\mathcal{P}$ is a presheaf, its sheafification will be denoted by $\mathcal{P}^{\#}$. The sheafification functor is left adjoint to the natural inclusion of sheaves into presheaves.

We denote the category of sheaves of k-vector spaces on X by $\mathrm{Mod}(k_X)$, and its bounded derived category by $\mathrm{D}^{\mathrm{b}}(k_X)$. There are the six Grothendieck operations $\mathrm{R}f_*$, $\mathrm{R}f_!$, f^{-1}, $f^!$, $\otimes$ and $\mathrm{R}\mathcal{H}om$ (and the underived versions of all these functors except $f^!$). We will sometimes write $\otimes_{k_X}$ or $\mathrm{R}\mathcal{H}om_{k_X}$ if we want to emphasize the field, but it will usually be clear from the context.

(Locally) constant sheaves If $V \in \mathrm{Vect}_k$, we denote by $V_X \in \mathrm{Mod}(k_X)$ the constant sheaf with stalk V on X. It is the sheafification of the constant presheaf V_X^{pre} defined by $V_X^{\mathrm{pre}}(U) = V$ for all $U \in \mathrm{Op}(X)$. If $p_X\colon X \to \{\mathrm{pt}\}$ is the map to the one-point space, we also have $V_X = p_X^{-1}V$ (note that sheaves of vector spaces on a one-point space are the same as vector spaces). In particular, we have the constant sheaf k_X, and if $f\colon X \to Y$ is a morphism of topological spaces, then $f^{-1}k_Y \simeq k_X$.

If $\mathcal{F} \in \mathrm{Mod}(k_X)$ and $Z \subseteq X$ is a locally closed subset with inclusion $j\colon Z \hookrightarrow X$, we write $\mathcal{F}_Z := j_!j^{-1}\mathcal{F}$. This is the restriction of $\mathcal{F}$ to Z, extended again to X by zero outside of Z. We will also sometimes denote by $k_Z \in \mathrm{Mod}(k_X)$ the sheaf $j_!k_Z$ if there is no risk of confusion.

We moreover denote the duality functor by $\mathrm{D}_X := \mathrm{R}\mathcal{H}om(-, \omega_X)$, where $\omega_X = p_X^!k$ is the dualizing complex.

We call a sheaf $\mathcal{F} \in \mathrm{Mod}(k_X)$ *locally constant* (or a *local system*) if every point $x \in X$ has an open neighbourhood $U \subseteq X$ such that $\mathcal{F}|_U$ is isomorphic to a constant sheaf V_X for some $V \in \mathrm{Vect}_k$. It is *locally constant of finite rank* if all the V are finite-dimensional.

Constructibility and perversity If X is a real analytic manifold, $\mathcal{F} \in \mathrm{Mod}(k_X)$ is called $\mathbb{R}$-*constructible* if there exists a locally finite covering $X = \bigcup_{\alpha \in A} X_\alpha$ by subanalytic subsets such that $\mathcal{F}|_{X_\alpha}$ is locally constant of finite rank for every α. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X)$ of complexes with $\mathbb{R}$-constructible cohomologies. We also note that this category is in fact equivalent to the bounded derived category of the category of $\mathbb{R}$-constructible sheaves.

If X is a complex manifold, $\mathcal{F} \in \mathrm{Mod}(k_X)$ is called $\mathbb{C}$-*constructible* if there exists a locally finite covering $X = \bigcup_{\alpha \in A} X_\alpha$ by $\mathbb{C}$-analytic subsets such that $\mathcal{F}|_{X_\alpha}$ is locally constant of finite rank for every α. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X)$ of complexes with $\mathbb{C}$-constructible cohomologies.

An object $\mathcal{F}^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ is called a *perverse sheaf* if $\dim \operatorname{supp} \mathrm{H}^{-i}(\mathcal{F}^\bullet) \leq i$ for any $i \in \mathbb{Z}$ (we say that $\mathcal{F}^\bullet$ satisfies the *support condition*) and $\dim \operatorname{supp} \mathrm{H}^{-i}(\mathrm{D}_X\mathcal{F}^\bullet) \leq i$ for any $i \in \mathbb{Z}$ (i.e. $\mathrm{D}_X\mathcal{F}^\bullet$ also satisfies the support condition). We denote by $\mathrm{Perv}(k_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ consisting of perverse sheaves. We refer in particular to [BBD82] for the theory of perverse sheaves. The category $\mathrm{Perv}(k_X)$ is an abelian category. Let us note that, in particular, a perverse sheaf has no nontrivial cohomologies in degrees less than $-\dim_{\mathbb{C}} X$. (This follows easily from [BBD82, p. 56], for example.)

A remark on operations and sheafification Before entering the main topic of the article, let us remark the following elementary fact that we are going to use throughout the article: If $\mathcal{P}$ is a presheaf and $\mathcal{F}$ is a sheaf, then the tensor product sheaf $\mathcal{P}^\# \otimes \mathcal{F}$ is isomorphic to the sheafification of the (naïve) presheaf tensor product $\mathcal{P} \overset{\text{pre}}{\otimes} \mathcal{F}$. This is easy to show using tensor-hom adjunctions for presheaves and sheaves and the universal property of sheafification. It is, however, crucial that $\mathcal{F}$ is already a sheaf here. In particular, this means that if L/K is a field extension (i.e. L is a K-module) and $\mathcal{F} \in \mathrm{Mod}(K_X)$, then the sheaf $L_X \otimes \mathcal{F}$ is the sheafification of the presheaf $U \mapsto L \otimes_K \mathcal{F}(U)$.

Let us also note that sheafification does not commute with operations on sheaves in general: For example, it is not true in general that $f_*(\mathcal{P}^\#) \simeq (f_*^{\text{pre}}\mathcal{P})^\#$ for a presheaf $\mathcal{P}$. Indeed, if this were true, we would not have examples as in Remark 4.6.

3 Linear categories and field extensions

The general philosophy of Galois descent is the following: Given a Galois extension L/K with Galois group G and an object F over L, then the existence of a G-structure (i.e. a suitable collection of isomorphisms between F and its Galois conjugates, often formulated as a suitable action of the Galois group on F) should guarantee the existence of a K-structure of F, i.e. an object over K which is isomorphic to F after extension of scalars. (Even more, a G-structure should in some sense determine a particular K-structure, since in the case where "object" means "finite-dimensional vector space", the pure existence of such a structure is not big news.)

We first set up a very general framework for the concepts of G- and K-structures, motivated by the notions set up in [BHHS22], but immediately describe these notions explicitly in our categories of interest. Recall that, given a field k, a k-linear category is a category whose hom spaces are k-vector spaces and composition of morphisms is k-linear. Note that if L/K is a field extension, any L-linear category is automatically also K-linear. We will assume any functor between two linear categories to be linear (meaning that the induced map on hom spaces is linear).

3.1 G-structures

Let L/K be a field extension and denote by $G = \mathrm{Aut}(L/K)$ the group of field automorphisms $g\colon L \to L$ such that $g|_K = \mathrm{id}_K$.

Definition 3.1. Let $\mathcal{C}(L)$ be an L-linear category. A *G-conjugation* on $\mathcal{C}(L)$ is a collection of auto-equivalences

$$\gamma_g = \overline{(\bullet)}^g \colon \mathcal{C}(L) \xrightarrow{\sim} \mathcal{C}(L)$$

(one for each $g \in G$) and natural isomorphisms

$$I_{g,h} \colon \gamma_h \circ \gamma_g \xrightarrow{\sim} \gamma_{gh}$$

for any $g, h \in G$ such that for any $g, h, k \in G$ the following diagram is commutative:

$$
\begin{tikzcd}
& \gamma_{hk} \circ \gamma_g \arrow[dr, "I_{g,hk}"] & \\
\gamma_k \circ \gamma_h \circ \gamma_g \arrow[ur, "I_{h,k} \circ \gamma_g"] \arrow[dr, "\gamma_k \circ I_{g,h}"'] & & \gamma_{ghk} \\
& \gamma_k \circ \gamma_{gh} \arrow[ur, "I_{gh,k}"'] &
\end{tikzcd}
$$

Note that this implies in particular that $\gamma_{\mathrm{id}_L} \simeq \mathrm{id}_{\mathcal{C}(L)}$, where $\mathrm{id}_L \in G$ is the identity element of the G. In the categories we use, it will actually be equal to the identity functor. More precisely, all the $I_{g,h}$ will be equalities rather than isomorphisms and hence the γ_g are actually automorphisms of $\mathcal{C}(L)$.

Example 3.2. Here are the categories we are interested in in this article: The classical example is that of vector spaces, and it easily generalizes to presheaves and sheaves.

(a) Let Vect_L be the category of L-vector spaces. Then, for an object $V \in \mathrm{Vect}_L$ and an element $g \in G$, the L-vector space $\overline{V}^g$ is defined as follows: As K-vector spaces (or sets), we set $\overline{V}^g = V$, and the action of L on $\overline{V}^g$ is given by

$$\ell \cdot v := g(\ell) v$$

for $\ell \in L$ and $v \in \overline{V}^g$, where the right-hand side is the given scalar multiplication on V.

Given a morphism $V \to W$ in Vect_L, then for any $g \in G$ the same set-theoretic map defines an L-linear morphism $\overline{f}^g \colon \overline{V}^g \to \overline{W}^g$.

Altogether, this gives a functor

$$\overline{(\bullet)}^g \colon \mathrm{Vect}_L \to \mathrm{Vect}_L,$$

and it is easy to see that the functor $\overline{(\bullet)}^{g^{-1}}$ is a quasi-inverse, hence the above functor is indeed an auto-equivalence.

Moreover, given $g, h \in G$, the identification $\overline{\overline{V}^g}^h = \overline{V}^{gh}$ is immediate by the definition, as is compatibility of these identifications (that is, commutativity of the diagram in the above definition).

(b) Let $\mathcal{C}$ be a category and consider the category $\mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$ of (covariant) functors from $\mathcal{C}$ to Vect_L. Then there is a G-conjugation on this category given as follows: Let $F \in \mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$ and $g \in G$, then we define $\overline{F}^g$ by setting

$$\overline{F}^g(A) := \overline{F(A)}^g$$

for any $A \in \mathcal{C}$. A morphism $A \to B$ in $\mathcal{C}$ is sent to the morphism $F(A) \to F(B)$, considered as a morphism $\overline{F(A)}^g \to \overline{F(B)}^g$, as remarked in (a). Thus, this clearly defines an element $\overline{F}^g \in \mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$.

Given a morphism $F \to G$ in $\mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$, it is equally easy to see that this induces a morphism $\overline{F}^g \to \overline{G}^g$ for any $g \in G$, and hence we obtain an auto-equivalence $\overline{(\bullet)}^g$ of $\mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$ with the desired compatibilities.

(c) Consider the category $\mathrm{Mod}(L_X)$ of sheaves of L-vector spaces on a topological space X. It is a subcategory of $\mathrm{Funct}(\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Vect}_L)$, where $\mathrm{Op}(X)$ is the category of open subsets of X (with inclusions as morphisms). Hence, by (b), to a sheaf $\mathcal{F} \in \mathrm{Mod}(L_X)$ we can a priori associate a presheaf $\overline{\mathcal{F}}^g \in \mathrm{Funct}(\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Vect}_L)$ for any $g \in G$. It is, however, clear that this presheaf is automatically a sheaf: The unique gluing condition required for sheaves is indeed independent of the action of L, it can be checked on the level of sheaves of K-vector spaces (or even sets), and on this level $\overline{\mathcal{F}}^g$ and $\mathcal{F}$ are the same object. We therefore get an auto-equivalence

$$\overline{(\bullet)}^g \colon \mathrm{Mod}(L_X) \xrightarrow{\sim} \mathrm{Mod}(L_X)$$

satisfying the required compatibilities. Moreover, since this equivalence is exact, it also induces a functor

$$\overline{(\bullet)}^g \colon \mathrm{D}^{\mathrm{b}}(L_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(L_X)$$

on the level of derived categories, equipping the latter with a G-conjugation.

In the sequel, when we work in one of these categories, we will always use the G-conjugations described in Example 3.2.

Since we are mainly concerned with sheaves in this note, let us state the main properties of G-conjugation for sheaves of vector spaces.

Lemma 3.3 (cf. [BHHS22, Lemma 2.1]). *Let L/K be a field extension and $G := \mathrm{Aut}(L/K)$. Let $g \in G$. Let $f\colon X \to Y$ be a continuous map between topological spaces.*

(a) Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(L_X)$. Then $\overline{\mathrm{R}f_*\mathcal{F}}^g \simeq \mathrm{R}f_*\overline{\mathcal{F}}^g$ and $\overline{\mathrm{R}f_!\mathcal{F}}^g \simeq \mathrm{R}f_!\overline{\mathcal{F}}^g$.

(b) Let $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(L_Y)$. Then $\overline{f^{-1}\mathcal{G}}^g \simeq f^{-1}\overline{\mathcal{G}}^g$ and $\overline{f^!\mathcal{G}}^g \simeq f^!\overline{\mathcal{G}}^g$.

(c) Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^{\mathrm{b}}(L_X)$. Then $\overline{\mathrm{R}\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)}^g \simeq \mathrm{R}\mathcal{H}om(\overline{\mathcal{F}_1}^g, \overline{\mathcal{F}_2}^g)$ and $\overline{\mathcal{F}_1 \otimes \mathcal{F}_2}^g \simeq \overline{\mathcal{F}_1}^g \otimes \overline{\mathcal{F}_2}^g$.

(c) Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(L_X)$. Then $\overline{\mathrm{D}_X\mathcal{F}}^g \simeq \mathrm{D}_X\overline{\mathcal{F}}^g$.

(e) If $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(L_X)$ and $\mathrm{H}^i(\mathcal{F})$ is locally constant on $Z \subseteq X$, so is $\mathrm{H}^i(\overline{\mathcal{F}}^g)$.

In particular, if X is a real analytic manifold and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)$ (resp. X is a complex manifold and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(L_X)$), then $\overline{\mathcal{F}}^g \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)$ (resp. $\overline{\mathcal{F}}^g \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(L_X)$).

(f) If X is a complex manifold and $\mathcal{F} \in \mathrm{Perv}(L_X)$, then $\overline{\mathcal{F}}^g \in \mathrm{Perv}(L_X)$.

Proof. This proof of (a)–(c) is taken from [BHHS22, Lemma 2.1].

By the definitions of G-conjugation on sheaves and the direct image functor, we have $(\overline{f_*\mathcal{F}}^g)(U) = \overline{(f_*\mathcal{F})(U)}^g = \overline{\mathcal{F}(f^{-1}(U))}^g = \overline{\mathcal{F}}^g(f^{-1}(U)) = f_*\overline{\mathcal{F}}^g(U)$, which shows $\overline{f_*\mathcal{F}}^g \simeq f_*\overline{\mathcal{F}}^g$. Moreover, we get $\overline{f_!\mathcal{F}}^g \simeq f_!\overline{\mathcal{F}}^g$, since the proper direct image sheaf is a subsheaf of the direct image sheaf, containing the sections on U whose support is proper over U, and the notion of proper support does not depend on the action of L, so it is the same subsheaf in both cases. f For L-vector spaces V and W, it is clear that we have an isomorphism of L-vector spaces

$$\overline{\mathrm{Hom}_L(V,W)}^g \simeq \mathrm{Hom}_L(\overline{V}^g, \overline{W}^g)$$

(sending a morphism on the left to the same set-theoretic map on the right). Then for any open $U \subseteq X$, we have

$$\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2)(U) = \mathrm{Hom}(\mathcal{F}_1|_U,\mathcal{F}_2|_U) \subset \prod_{V\subset U \text{ open}} \mathrm{Hom}(\mathcal{F}_1(V),\mathcal{F}_2(V))$$

(namely the subset of families of morphisms compatible with restriction maps of $\mathcal{F}_1$ and $\mathcal{F}_2$). This implies $\overline{\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2)}^g \simeq \mathcal{H}om(\overline{\mathcal{F}_1}^g,\overline{\mathcal{F}_2}^g)$ since conjugation is an equivalence and hence commutes with products.

Finally, (a) and the first part of (c) follow by deriving functors (noting that conjugation is exact), and the other statements follow by adjunction, using

$$\mathrm{Hom}_{\mathrm{D^b}(L_X)}(\overline{\mathcal{F}_1}^g,\mathcal{F}_2) \simeq \mathrm{Hom}_{\mathrm{D^b}(L_X)}(\mathcal{F}_1,\overline{\mathcal{F}_2}^{g^{-1}}).$$

For (d), note that

$$\begin{aligned}\mathrm{D}_X\overline{\mathcal{F}}^g &= \mathrm{R}\mathcal{H}om(\overline{\mathcal{F}}^g,\omega_X) \simeq \mathrm{R}\mathcal{H}om(\overline{\mathcal{F}}^g,p_X^!L)\\ &\simeq \mathrm{R}\mathcal{H}om(\overline{\mathcal{F}}^g,p_X^!\overline{L}^g) \simeq \overline{\mathrm{R}\mathcal{H}om(\mathcal{F},p_X^!L)}^g \simeq \overline{\mathrm{D}_X\mathcal{F}}^g.\end{aligned}$$

Here, $p_X\colon X\to\{\mathrm{pt}\}$ is the map to the one-point space, and $\omega_X := p_X^!L$ is the dualizing complex. We have used an isomorphism of L-vector spaces $L\simeq\overline{L}^g$, which is given by $\ell\mapsto g(\ell)$, as well as (b) and (c).

We now prove (e). Let $U\subseteq Z$ be open such that $\mathrm{H}^i(\mathcal{F})|_U$ is a constant sheaf, i.e. $\mathrm{H}^i(\mathcal{F})|U \simeq V_U \simeq p_U^{-1}V$ for some L-vector space V (where $p_U\colon U\to\{\mathrm{pt}\}$ denotes the map to the one-point space). Then clearly

$$\mathrm{H}^i(\overline{\mathcal{F}}^g)|_U \simeq \overline{\mathrm{H}^i(\mathcal{F}|_U)}^g \simeq \overline{p_U^{-1}V}^g \simeq p_U^{-1}\overline{V}^g$$

is a constant sheaf (with stalk $\overline{V}^g$), by exactness of conjugation and (b).

Finally, to show (f), note first that, by (e), the support of the cohomologies of an object $\mathcal{F}\in\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(L_X)$ does not change under conjugation. Therefore, if $\mathcal{F}$ satisfies the support condition, so does $\overline{\mathcal{F}}^g$. Moreover, by (d), the support condition for $\mathrm{D}_X\mathcal{F}$ implies that for $\mathrm{D}_X\overline{\mathcal{F}}^g$. Consequently, $\overline{\mathcal{F}}^g$ is perverse if $\mathcal{F}$ is.

□

Definition 3.4. Let $\mathcal{C}(L)$ be an L-linear category with G-conjugation. Let $F\in\mathcal{C}(L)$ be an object. A G-structure on F is given by a family $(\varphi_g)_{g\in G}$ of isomorphisms $\varphi_g\colon F\xrightarrow{\sim}\overline{F}^g$

such that for any $g, h \in G$ the following diagram is commutative:

$$
\begin{tikzcd}
F \arrow[d, "\varphi_h"] \arrow[rr, bend left, "\varphi_{gh}"] & & \\
\overline{F}^h \arrow[r, "\gamma_h(\varphi_g)"] & \overline{\overline{F}^g}^h \arrow[r, "I_{g,h}"] & \overline{F}^{gh}
\end{tikzcd}
$$

Example 3.5. Consider again the category Vect_L as in Example 3.2(a). The trivial one-dimensional vector space $L \in \mathrm{Vect}_L$ has a natural G-structure: 'Any element $g \in G$ defines an isomorphism of L-vector spaces $\varphi_g = g \colon L \xrightarrow{\sim} \overline{L}^g$ given by $\ell \mapsto g(\ell)$ and satisfying the compatibilities from Definition 3.4. (We will often just denote it by the letter g.) The same works, of course, for any n-dimensional vector space of the form L^n for some $n \in \mathbb{Z}_{>0}$, applying g componentwise.

Similarly, in the category $\mathrm{Mod}(L_X)$ of sheaves of L-vector spaces on a topological space, one gets a canonical G-structure on the constant sheaf L_X. Indeed, the constant sheaf L_X is the sheafification of the constant presheaf L_X^{pre} defined by $L_X^{\mathrm{pre}}(U) = L$ for any open $U \subseteq X$. On L_X^{pre}, we can define the G-structure just as in the example of the trivial vector space L above. Then we conclude via Lemma 3.6 below.

Lemma 3.6. *Assume that $\mathcal{P} \in \mathrm{Funct}(\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Vect}_L)$ is a presheaf with a G-structure $(\phi_g)_{g \in G}$. Then its sheafification $\mathcal{F} := \mathcal{P}^{\#}$ has an induced G-structure $(\varphi_g)_{g \in G}$.*

Proof. For any $g \in G$, we are given

$$\phi_g \colon \mathcal{P} \xrightarrow{\sim} \overline{\mathcal{P}}^g,$$

and this induces, by the universal property of sheafification, isomorphisms

$$\mathcal{P}^{\#} \xrightarrow{\sim} (\overline{\mathcal{P}}^g)^{\#}.$$

It therefore remains to check that $(\overline{\mathcal{P}}^g)^{\#} \simeq (\overline{\mathcal{P}^{\#}})^g$.

To see this, note that there is a natural homomorphism $\mathcal{P} \to \mathcal{P}^{\#}$ and hence a natural morphism $\overline{\mathcal{P}}^g \to \overline{(\mathcal{P}\#)}^g$. By the universal property of sheafification, this induces a morphism $(\overline{\mathcal{P}}^g)^{\#} \to \overline{(\mathcal{P}\#)}^g$. We can check on stalks that it is an isomorphism: G-conjugation commutes with taking stalks, and stalks of a presheaf and its associated sheaf are the same. $\square$

The following is an easy corollary from Lemma 3.3. We leave it to the readers to verify that the necessary compatibilities are satisfied.

Corollary 3.7. *Let $f \colon X \to Y$ be a morphism of topological spaces. Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^{\mathrm{b}}(L_X)$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(L_Y)$ be equipped with G-structures. Then $\mathrm{R}f_*\mathcal{F}$, $\mathrm{R}f_!\mathcal{F}$, $f^{-1}\mathcal{G}$, $f^!\mathcal{G}$, $\mathcal{F}_1 \otimes \mathcal{F}_2$, $\mathrm{R}\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$, $\mathrm{D}_X\mathcal{F}$ and $\mathrm{H}^i(\mathcal{F})$ for any $i \in \mathbb{Z}$ are equipped with an induced G-structure.*

We now introduce the following category associated to a category with G-conjugation. It will serve as the target category for Galois descent statements later.

Definition 3.8. Let $\mathcal{C}(L)$ be an L-linear category with G-conjugation. Then we define the category $\mathcal{C}(L)^G$ as follows:

- An object of $\mathcal{C}(L)^G$ is a pair $(F, (\varphi_g)_{g\in G})$, where $F \in \mathcal{C}(L)$ is an object and $(\varphi_g)_{g\in G}$ is a G-structure on F.
- A morphism $(F, (\varphi_g)_{g\in G}) \to (F', (\varphi'_g)_{g\in G})$ is a morphism $f\colon F \to \widetilde{F}$ in $\mathcal{C}(L)$ compatible with the G-structures, i.e. such that for any $g \in G$ the diagram

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\varphi_g"', "\simeq"] & F' \arrow[d, "\simeq"', "\varphi'_g"] \\ \overline{F}^g \arrow[r, "\overline{f}^g"] & \overline{F'}^g \end{tikzcd}$$

commutes.

3.2 Extension of scalars

As before, let L/K be a field extension and set $G := \mathrm{Aut}(L/K)$.

Extension of scalars (change of fields) is a well-known principle for vector spaces, sheaves or schemes, for example. In these examples, it is formed by taking tensor products or fibre products of the object over K with an object determined by the field L. In the other direction, an object over L can naturally be considered as an object over K. (Note that these processes are not inverse to each other, but adjoint.) In an abstract setting, one could define such an extension/restriction datum as follows.

Definition 3.9. Let $\mathcal{C}(K)$ be a K-linear category and $\mathcal{C}(L)$ an L-linear category equipped with a G-conjugation $((\gamma_g)_{g\in G}, (I_{g,h})_{g,h\in G})$.

Consider a functor

$$\Phi_{L/K}\colon \mathcal{C}(K) \longrightarrow \mathcal{C}(L)$$

that factors as

$$\mathcal{C}(K) \xrightarrow{\Phi^G_{L/K}} \mathcal{C}(L)^G \longrightarrow \mathcal{C}(L),$$

where the second functor is just the one forgetting the G-structure. Consider furthermore a functor

$$\mathsf{for}_{L/K}\colon \mathcal{C}(L) \to \mathcal{C}(K)$$

together with natural isomorphisms of functors $J_g\colon \mathsf{for}_{L/K} \circ \gamma_g \xrightarrow{\sim} \mathsf{for}_{L/K}$ for any $g \in G$ such that for any $g, h \in G$ the diagram

$$\begin{tikzcd} \mathsf{for}_{L/K} \circ \gamma_h \circ \gamma_g \arrow[r, "\mathsf{for}_{L/K} \circ I_{g,h}"] \arrow[d, "J_h \circ \gamma_g"] & \mathsf{for}_{L/K} \circ \gamma_{gh} \arrow[d, "J_{gh}"] \\ \mathsf{for}_{L/K} \circ \gamma_g \arrow[r, "J_g"] & \mathsf{for}_{L/K} \end{tikzcd}$$

commutes.

If $\Phi_{L/K}$ is left adjoint to $\mathsf{for}_{L/K}$, i.e. there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}(K)}(Y, \mathsf{for}_{L/K}(X)) \simeq \mathrm{Hom}_{\mathcal{C}(L)}(\Phi_{L/K}(Y), X)$$

for any $X \in \mathcal{C}(L)$, $Y \in \mathcal{C}(K)$, we call $\Phi_{L/K}$ a functor of *extension of scalars* and $\mathsf{for}_{L/K}$ a functor of *restriction of scalars.*

If, in such a situation, one has objects $F \in \mathcal{C}(L)$ and $A \in \mathcal{C}(K)$ such that $\Phi_{L/K}(A) \simeq F$, we say that A is a K*-structure* (or K*-lattice*) of F.

For $F \in \mathcal{C}(L)$, we will often write $F^K := \mathsf{for}_{L/K}(F) \in \mathcal{C}(K)$.

Similar to what we saw above for the $I_{g,h}$ in Definition 3.1, the isomorphisms of functors J_g will be equalities in our examples of interest.

Example 3.10. The functor $\mathrm{Vect}_K \to \mathrm{Vect}_L$ defined by $W \mapsto L \otimes_K W$ is a functor of extension of scalars, and the functor $\mathrm{Vect}_L \to \mathrm{Vect}_K$ sending an L-vector space to itself (but only remembering the action of K) is a corresponding functor of restriction of scalars. Indeed, for $W \in \mathrm{Vect}_K$, the object $L \otimes_K W$ carries a natural G-structure: It is clear that $\overline{L \otimes_K W}^g = \overline{L}^g \otimes_K W$, so the G-structure is just given by the natural one on L (see Example 3.5). Moreover, it is clear that $V^K = (\overline{V}^g)^K$ for any $g \in G$, since $g|_K = \mathrm{id}_K$ and hence the action of K is the same on each $\overline{V}^g$. The property of adjointness of these two functors is classical.

Similarly, a functor of extension of scalars for sheaves is given by $\Phi_{L/K}\colon \mathrm{Mod}(K_X) \to \mathrm{Mod}(L_X), \mathcal{G} \mapsto L_X \otimes_{K_X} \mathcal{G}$ (with the obvious functor of restriction of scalars, which is defined as above for each space of sections $\mathcal{F}(U)$): The sheaf $L_X \otimes_{K_X} \mathcal{G}$ (a priori a tensor product of two K_X-modules) is the sheafification of the presheaf $U \mapsto L \otimes_K \mathcal{G}(U)$, and hence a sheaf of L-vector spaces. Again, the natural G-structure on $L_X \otimes_{K_X} \mathcal{G}$ is given by the natural G-structure on L_X (see Example 3.5). As above, it is clear that $(\overline{\mathcal{F}}^g)^K = \mathcal{F}^K$ for any $g \in G$. The fact that these two functors are adjoint follows easily from the case of vector spaces, and follows from Lemma 3.11 below.

Noting that both Φ_{L_K} and $\mathsf{for}_{L/K}$ for sheaves of vector spaces are exact, we also obtain extension/restriction of scalars functors between derived categories of sheaves.

When we work with these categories, we will always consider the functors of extension of scalars from the above example.

The following lemma describes the adjunction between extension and restriction of scalars in the case of sheaves of vector spaces. (This is rather classical, cf. e.g. [Stacks, Tag 0088].)

Lemma 3.11. *Let* $\mathcal{G} \in \mathrm{D^b}(K_X)$ *and* $\mathcal{F} \in \mathrm{D^b}(L_X)$*. Then there are isomorphisms*

$$\mathrm{R}\mathcal{H}om(\mathcal{G}, \mathsf{for}(\mathcal{F})) \simeq \mathsf{for}_{L/K}\big(\mathrm{R}\mathcal{H}om(L_X \otimes_{K_X} \mathcal{G}, \mathcal{F})\big)$$

in $\mathrm{D^b}(K_X)$ *and*

$$\mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{G}, \mathsf{for}(\mathcal{F})) \simeq \mathrm{Hom}_{\mathrm{D^b}(L_X)}(L_X \otimes_{K_X} \mathcal{G}, \mathcal{F})$$

as K*-vector spaces.*

Proof. For vector spaces $W \in \mathrm{Vect}_K$ and $V \in \mathrm{Vect}_L$, it is well-known that

$$\mathrm{Hom}_K(W, \mathsf{for}_{L/K}(V)) \simeq \mathrm{Hom}_L(L \otimes_K W, V)$$

(this is an isomorphism of K-vector spaces, i.e. there is a "hidden" functor $\mathsf{for}_{L/K}$ on the right-hand side), and the morphism from left to right is given by L-linear continuation.

Let now $\mathcal{G} \in \mathrm{Mod}(K_X)$ and $\mathcal{F} \in \mathrm{Mod}(L_X)$. Then we get

$$\mathrm{Hom}_{\mathrm{Mod}(K_X)}(\mathcal{G}, \mathsf{for}_{L/K}(\mathcal{F})) \simeq \mathrm{Hom}_{\mathrm{Mod}(L_X)}(L_X \otimes_{K_X} \mathcal{G}, \mathcal{F})$$

as follows: An element of the left hom space is a compatible collection of K-linear morphisms $\mathcal{G}(U) \to \mathcal{F}(U)$ for any open $U \subseteq X$. By the statement for vector spaces, it is easy to see that this corresponds to a compatible collection of L-linear morphisms $L \otimes_K \mathcal{G}(U) \to \mathcal{F}(U)$, which represent a morphism of presheaves from the presheaf given by $U \mapsto L \otimes_K \mathcal{G}(U)$ to $\mathcal{F}$. By the universal property of sheafification (and since $\mathcal{F}$ is already a sheaf), this is equivalent to a morphism $L_X \otimes_{K_X} \mathcal{G} \to \mathcal{F}$.

Finally, we obtain an isomorphism

$$\mathcal{H}om(\mathcal{G}, \mathsf{for}_{L/K}(\mathcal{F})) \simeq \mathsf{for}_{L/K}\big(\mathcal{H}om(L_X \otimes_{K_X} \mathcal{G}, \mathcal{F})\big)$$

by defining it on sections over any open $U \subseteq X$, which means that we need a compatible collection of isomorphisms

$$\mathrm{Hom}_{\mathrm{Mod}(K_X)}(\mathcal{G}|_U, \mathsf{for}_{L/K}(\mathcal{F}|_U)) \simeq \mathrm{Hom}_{\mathrm{Mod}(L_X)}(L_U \otimes_{K_U} \mathcal{G}|_U, \mathcal{F}|_U)$$

for any such U, which is what we have constructed above.

The statements of the lemma now follows by deriving functors (noting that $\Phi_{L/K}$ and $\mathsf{for}_{L/K}$ are exact) and taking zeroth cohomology. $\square$

Let us state the main compatibilities of extension of scalars for sheaves. (Parts (a) and (b) were also mentioned just before [BHHS22, Lemma 2.4].)

Lemma 3.12. *Let L/K be a field extension. Let X and Y be topological spaces and let $f\colon X \to Y$ be a continuous map.*

(a) Let $\mathcal{G} \in \mathrm{D^b}(K_X)$. Then $\mathrm{R}f_!(L_X \otimes_{K_X} \mathcal{G}) \simeq L_Y \otimes_{K_Y} \mathrm{R}f_!\mathcal{G}$.

(b) Let $\mathcal{H} \in \mathrm{D^b}(L_X)$. Then $f^{-1}(L_Y \otimes_{K_Y} \mathcal{H}) \simeq L_X \otimes_{K_X} f^{-1}\mathcal{H}$.

(c) If $\mathcal{G} \in \mathrm{D^b}(K_X)$ and $\mathrm{H}^i(\mathcal{G})$ is locally constant on $Z \subseteq X$, then so is $\mathrm{H}^i(L_X \otimes_{K_X} \mathcal{G})$. In particular, if $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$ (resp. $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(K_X)$), then $L_X \otimes_{K_X} \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)$ (resp. $L_X \otimes_{K_X} \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(L_X)$).

Proof. These statements follow from basic properties of the tensor product: (a) follows from the projection formula, and (b) follows from the fact that tensor products commute with inverse images (see e.g. Proposition 2.6.6 and Proposition 2.6.5 of [KS90], respectively). Then (c) is easily deduced from (b) since constant sheaves are inverse images of vector spaces along the map to the one-point space. $\square$

Note that this statement is "less complete", compared to Lemma 3.3. For example, it contains no statement about compatibility between extension of scalars and direct images f_*. In fact, such a compatibility is not true in general. We will come back to this point in Corollary 4.5 and Remark 4.6 below. On the other hand, the majority of Section 4 is devoted to proving statements about the compatibility of extension of scalars with $\mathrm{R}\mathcal{H}om$. We will also give statements about the duality functor and the exceptional inverse image functor in Corollary 4.14.

3.3 Galois descent

Given a functor of extension of scalars as in the previous subsection immediately raises the question if there is a construction in the other direction: Given an object over L, can does it admit a K-structure? In general, this is certainly not the case, but the idea is that from a given G-structure on L we might indeed be able to construct a K-structure. The context in which we can expect such a descent statement is mainly that a Galois extension L/K, since in this case the group $G = \mathrm{Aut}(L/K)$ (which is nothing but the Galois group) "knows enough" about the subfield K. This idea is made more precise by the concept of Galois descent.

Definition 3.13. Let L/K be a Galois extension, $\mathcal{C}(K)$ a K-linear category, $\mathcal{C}(L)$ an L-linear category with a G-conjugation, and let these categories be equipped with extension and restriction of scalars as in Definition 3.9. Then we say that *Galois descent* is satisfied in this setting if $\Phi^G_{L/K}\colon \mathcal{C}(K) \to \mathcal{C}(L)^G$ is an equivalence of categories.

Let now L/K be a finite Galois extension with Galois group G. In this case, it is well-known that Galois descent is satisfied for vector spaces, see [Con] for an exposition (as well as the references therein and in the introduction above). We briefly recall the construction: The functor

$$\Phi_{L/K}\colon \mathrm{Vect}_K \longrightarrow \mathrm{Vect}_L, \qquad V \longmapsto L \otimes_K V$$

induces an equivalence between K-vector spaces and L-vector spaces equipped with a G-structure. The quasi-inverse of $\Phi^G_{L/K}\colon \mathrm{Vect}_K \to \mathrm{Vect}^G_L$ can be explicitly described:

Let V be an L-vector space equipped with a G-structure $(\varphi_g)_{g\in G}$. Recall that we write $V^K := \mathsf{for}_{L/K}(V)$. Since $V^K = (\overline{V}^g)^K$ for any $g \in G$, the L-linear isomorphisms $\varphi_g\colon V \xrightarrow{\sim} \overline{V}^g$ can be interpreted as K-linear automorphisms $\varphi_g^K := \mathsf{for}_{L/K}(\varphi_g)$ of V^K. Then one defines the space of invariants of all these automorphisms

$$\begin{aligned} V_K &:= \{v \in V^K \mid \varphi_g^K(v) = v \text{ for any } g \in G\} \\ &= \ker\left(\prod_{g\in G}(\varphi_g^K - \mathrm{id}_{V^K})\colon V^K \to \prod_{g\in G} V^K\right). \end{aligned}$$

This is a K-sub-vector space of V and one can show that the natural morphism $L \otimes_K V_K \to V, \sum_i \ell_i \otimes v_i \mapsto \sum_i \ell_i v_i$ is an isomorphism.

The aim of the rest of this paper is to investigate the extension of scalars functor for sheaves of vector spaces, and to study Galois descent in this framework. We will first establish (not only in the case of Galois extensions) some statements about homomorphisms between extensions of scalars. Then, we will illustrate the problems that occur when we want to set up Galois descent in derived categories, and we will finally prove Galois descent for perverse sheaves.

4 Homomorphisms between scalar extensions

Let L/K be a field extension. (We do not assume that L/K is finite or Galois, if not explicitly stated.)

In this section, we want to describe spaces of morphisms $L_X \otimes_{K_X} \mathcal{G}_1 \to L_X \otimes_{K_X} \mathcal{G}_2$ for $\mathcal{G}_1, \mathcal{G}_2 \in \mathrm{D}^{\mathrm{b}}(K_X)$ and relate them to spaces of morphisms $\mathcal{G}_1 \to \mathcal{G}_2$.

As a motivation, consider the following well-known fact from linear algebra: Let $V, W \in \mathrm{Vect}_K$. If L/K is a finite field extension or V and W are finite-dimensional, then

$$\mathrm{Hom}_L(L \otimes_K V, L \otimes_K W) \simeq L \otimes_K \mathrm{Hom}_K(V, W). \tag{1}$$

This implies that in the case of finite field extensions the above functor of extension of scalars is faithful. In the case of arbitrary field extensions, it is still faithful if we restrict to finite-dimensional vector spaces.

We will develop analogous statements about homomorphism spaces in the context of sheaves. Similarly to the above, we impose some additional assumption if we do not require the field extension to be finite. For sheaves, we will replace the above finiteness assumption by $\mathbb{R}$-constructibility.

In the rest of this section, we will develop relations between homomorphisms of sheaves and their scalar extensions. First of all, we remark that there are always the following natural morphisms, and it is the main aim of the following two subsections to give conditions under which they are isomorphisms.

Lemma 4.1. *Let L/K be a field extension, X a topological space and $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(K_X)$. There are natural morphisms*

$$L \otimes_K \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(K_X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G})$$

and

$$L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}).$$

Proof. Since $\Phi_{L/K} = L_X \otimes_{K_X} (-)$ is a functor, we have a natural map

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(K_X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}),$$

which can be extended L-linearly to a map

$$L \otimes_K \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(K_X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}).$$

For the second morphism in the assertion, consider first the case $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(K_X)$. In order to construct

$$L_X \otimes_{K_X} \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}),$$

we construct it on sections over any $U \subseteq X$. The left-hand side is the sheaf associated to the presheaf $U \mapsto L \otimes_K \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U)$, and hence by the universal property of sheafification it suffices to construct for any such U a morphism

$$L \otimes_K \mathcal{H}om(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}(L_U \otimes_{K_U} \mathcal{F}|_U, L_U \otimes_{K_U} \mathcal{G}|_U)$$

compatible with restrictions, which is what we did above.

The statement of the lemma follows now by deriving functors. $\square$

Whenever we prove isomorphisms showing some commutation between $\mathrm{R}\mathcal{H}om$ or Hom and extension of scalars in the following, we will always implicitly mean that it is this natural morphism that induces the isomorphism.

We will study separately the cases of finite and general (in particular infinite) field extensions. As a consequence, we get in particular the following property.

Proposition 4.2. *Let L/K be a field extension and X a topological space.*

(a) If L/K is finite, the functor $\Phi_{L/K}$ is faithful on $\mathrm{D^b}(K_X)$.

(b) If X is a compact real analytic manifold, then $\Phi_{L/K}$ is faithful on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$.

Proof. These statements will follow directly from Proposition 4.4 and Proposition 4.13, respectively. $\square$

4.1 Finite field extensions

In this subsection, we will study the case of finite field extensions, which will allow us to get results about homomorphisms between sheaves without any constructibility assumption.

First of all, sections of extensions of scalars along finite field extensions are particularly easily described, as the following lemma shows.

Lemma 4.3. *If L/K is finite and $\mathcal{G} \in \mathrm{Mod}(K_X)$, then*

$$(L_X \otimes_{K_X} \mathcal{G})(U) = L \otimes_K \mathcal{G}(U).$$

Proof. We know that $L_X \otimes_{K_X} \mathcal{G}$ is the sheafification of $U \mapsto L \otimes_K \mathcal{G}(U)$. It therefore suffices to show that the presheaf $\mathcal{F}(U) := L \otimes_K \mathcal{G}(U)$ is already a sheaf.

This follows directly since L/K is finite and hence $L \otimes_K (-)$ commutes with arbitrary products: If $U \subseteq X$ is open and $U = \bigcup_{i \in I} U_i$ is an open covering, then

$$0 \to \mathcal{G}(U) \to \prod_{i \in I} \mathcal{G}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{G}(U_i \cap U_j)$$

is exact since $\mathcal{G}$ is a sheaf. Applying $L \otimes_K (-)$, we obtain an exact sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j),$$

proving that $\mathcal{F}$ is a sheaf, as claimed. $\square$

We are now going to describe the homomorphisms between objects of the form $L_X \otimes_{K_X} \mathcal{F}$ more precisely and give a characterization of the image of morphisms in $\mathrm{D^b}(K_X)$ under the functor $\Phi_{L/K}$.

Proposition 4.4. *Let L/K be a finite field extension. Let X be a topological space, and let $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(K_X)$. Then*

$$\mathrm{R}\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

In particular, we have

$$\mathrm{Hom}_{\mathrm{D^b}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L \otimes_K \mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{F}, \mathcal{G}). \tag{2}$$

If L/K is finite Galois with Galois group G, then the subset of (2) consisting of morphisms f fitting into the natural diagram

$$\begin{array}{ccc} L_X \otimes_{K_X} \mathcal{F} & \xrightarrow{f} & L_X \otimes_{K_X} \mathcal{G} \\ {\scriptstyle\simeq}\downarrow{\scriptstyle g\otimes \mathrm{id}_{\mathcal{F}}} & & {\scriptstyle\simeq}\downarrow{\scriptstyle g\otimes \mathrm{id}_{\mathcal{G}}} \\ \overline{L_X}^g \otimes_{K_X} \mathcal{F} & \xrightarrow{\overline{f}^g} & \overline{L_X}^g \otimes_{K_X} \mathcal{G} \end{array}$$

for any $g \in G$ is exactly the subset of morphisms of the form $1 \otimes f_K$ for some $f_K \in \mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{F}, \mathcal{G})$.

Proof. Let us first prove that for sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(K_X)$, we have

$$\mathrm{Hom}_{L_X}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L \otimes_K \mathrm{Hom}_{K_X}(\mathcal{F}, \mathcal{G}).$$

An element in $\mathrm{Hom}_{K_X}(\mathcal{F}, \mathcal{G})$ is a family (f_U) of morphisms $f_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for any open $U \subseteq X$ such that for any inclusion $V \subset U$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

where the vertical arrows are the restriction morphisms of the sheaves.

In the same way, using to Lemma 4.3, an element in $\mathrm{Hom}_{L_X}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G})$ is a family $(\tilde{f}_U)$ of morphisms $\tilde{f}_U \colon L \otimes_K \mathcal{F}(U) \to L \otimes_K \mathcal{G}(U)$, compatible with restrictions. Choose a (finite) basis $\ell_1, \ldots, \ell_n$ of L over K. Then, due to the isomorphism (1) for vector spaces, for any U we can write $\tilde{f}_U = \sum_{i=1}^n \ell_i \otimes f_U^i$ for suitable $f_U^i \colon \mathcal{F}(U) \to \mathcal{G}(U)$. It is easy to check that for fixed i, the f_U^i are still compatible with the restriction maps, and hence we can identify $(\tilde{f}_U)$ with an element $\sum_{i=1}^n \ell_i \otimes (f_U^i) \in L \otimes_K \mathrm{Hom}_{K_X}(\mathcal{F}, \mathcal{G})$.

Now, let us prove

$$\mathcal{H}om_{L_X}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L_X \otimes_{K_X} \mathcal{H}om_{K_X}(\mathcal{F}, \mathcal{G}).$$

Again in view of Lemma 4.3, applying sections on some $U \subseteq X$ to both sides of this isomorphism, what we need is an isomorphism

$$\mathrm{Hom}_{L_U}(L_U \otimes_{K_U} \mathcal{F}|_U, L_U \otimes_{K_U} \mathcal{G}|_U) \simeq L \otimes_K \mathrm{Hom}_{K_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

for any $U \subseteq X$, compatible with restrictions. This is exactly what we just proved.

Noting that $L_X \otimes_{K_X} (-)$ and $L \otimes_K (-)$ are exact, the first two statements of the proposition now follow by deriving functors and taking zeroth cohomology.

Now, let us prove the second part of the proposition: Let $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(K_X)$. By what we proved above, an element of $\mathrm{Hom}_{\mathrm{D^b}(L_X)}(\mathcal{F} \otimes_{K_X} L_X, \mathcal{G} \otimes_{K_X} L_X)$ can therefore uniquely be written in the form $f = \sum_{k=1}^d \ell_k \otimes f_k$ for some $\ell_k \in L$ (understood as the map $L_X \to L_X$ given by multiplication with ℓ_k) and some $f_k \in \mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{F}, \mathcal{G})$. Assume that f fits in a diagram as above for any $g \in G$. The vertical isomorphisms are induced by the natural G-structure on the constant sheaf L_X. Commutation of this diagram therefore means that $\sum_{k=1}^d g(\ell_k \cdot (-)) \otimes f_k = \sum_{k=1}^d (g(\ell_k) \cdot g(-)) \otimes f_k$ (going right and then down in the diagram) and $\sum_{k=1}^d (\ell_k \cdot g(-)) \otimes f_k$ (going down and then right) coincide as morphisms in $\mathrm{Hom}_{\mathrm{D^b}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, \overline{L_X}^g \otimes_{K_X} \mathcal{G})$. Now there is an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D^b}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, \overline{L_X}^g \otimes_{K_X} \mathcal{G}) &\simeq \mathrm{Hom}_{\mathrm{D^b}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \\ &\simeq L \otimes_K \mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{F}, \mathcal{G})\end{aligned}$$

with the first isomorphism given by $\varphi \mapsto (g^{-1} \otimes \mathrm{id}) \circ \varphi$, hence the above means that for any $g \in G$ we have $\sum_{k=1}^d \ell_k \otimes f_k = \sum_{k=1}^d g(\ell_k) \otimes f_k$ in $L \otimes_K \mathrm{Hom}_{\mathrm{D^b}(K_X)}(\mathcal{F}, \mathcal{G})$. Then the result follows from the fact that the invariants of the Galois action on $L \otimes_K V$ (for a K-vector space V) are given by the K-subspace $1 \otimes V$ (cf. e.g. [Con, Corollary 2.17]). □

Before continuing with the case of infinite extensions, let us establish a simple consequence of Lemma 4.3, extending the compatibility properties of Lemma 3.12 in the case of a finite field extension.

Corollary 4.5 (of Lemma 4.3). *Let L/K be a finite field extension. Let $f \colon X \to Y$ be a continuous map of topological spaces and $\mathcal{G} \in \mathrm{D^b}(K_X)$. Then there is an isomorphism in $\mathrm{D^b}(L_X)$*

$$\mathrm{R}f_*(L_X \otimes_{K_X} \mathcal{G}) \simeq L_Y \otimes_{K_Y} \mathrm{R}f_*\mathcal{G}.$$

Proof. Let us first treat the non-derived case. If $\mathcal{G} \in \mathrm{Mod}(K_X)$, then, by the definition of the direct image functor and Lemma 4.3, we have

$$\big(f_*(L_X \otimes_{K_X} \mathcal{G})\big)(U) = (L_X \otimes_{K_X} \mathcal{G})(f^{-1}(U)) \simeq L \otimes_K \mathcal{G}(f^{-1}(U))$$

and
$$\big(L_Y \otimes_{K_Y} (f_*\mathcal{G})\big)(U) \simeq L \otimes_K (f_*\mathcal{G})(U) = L \otimes_K \mathcal{G}(f^{-1}(U))$$
for any open $U \subseteq Y$, and thus $f_*(L_X \otimes_{K_X} \mathcal{G}) \simeq L_Y \otimes_{K_Y} f_*\mathcal{G}$.

Since extension of scalars is exact, the derived statement follows. □

Remark 4.6. Let us remark that the assumption that L/K is finite is crucial here, in contrast to the corresponding isomorphisms for the proper direct image $\mathrm{R}f_!$ and the inverse image f^{-1} (see Lemma 3.12), which are proved without this finiteness assumption, using standard properties of the tensor product.

Let us give a counterexample for the statement of Lemma 4.5 in the case of an infinite field extension: Consider $X = \mathbb{R}_{>0} =]0, \infty[\subset \mathbb{R}$, $Y = \mathbb{R}$ and $f \colon X \to Y$ the inclusion. Moreover, consider the field extension $\mathbb{Q} \subset \mathbb{C}$. Define the sheaf
$$\mathcal{G} := \prod_{n \in \mathbb{Z}_{>0}} \mathbb{Q}_{\{\frac{1}{n}\}}.$$
It has stalk $\mathbb{Q}$ at any point $\frac{1}{n}$ and stalk 0 otherwise. Therefore, we have
$$\mathbb{C}_X \otimes_{\mathbb{Q}_X} \mathcal{G} \simeq \prod_{n \in \mathbb{Z}_{>0}} \mathbb{C}_{\{\frac{1}{n}\}}.$$
On the other hand, if U is a small neighbourhood of $0 \in Y$ (automatically containing infinitely many points $\frac{1}{n}$), we have
$$(f_*\mathcal{G})(U) \simeq \prod_{n \geq N} \mathbb{Q} \simeq t^N \mathbb{Q}[\![t]\!]$$
for some N, and hence the stalk is
$$(f_*\mathcal{G})_0 \simeq \varinjlim_{N \to \infty} t^N \mathbb{Q}[\![t]\!].$$
Similarly,
$$\big(f_*(\mathbb{C}_X \otimes_{\mathbb{Q}_X} \mathcal{G})\big)_0 \simeq \varinjlim_{N \to \infty} t^N \mathbb{C}[\![t]\!],$$
and this is not isomorphic to $\mathbb{C} \otimes_{\mathbb{Q}} \varinjlim_{N \to \infty} t^N \mathbb{Q}[\![t]\!] \simeq \varinjlim_{N \to \infty} (\mathbb{C} \otimes_{\mathbb{Q}} t^N \mathbb{Q}[\![t]\!])$: An element of the first vector space is represented by a formal power series that might have infinitely many linearly independent (over $\mathbb{Q}$) coefficients, while an element of the second object needs to be represented by a finite sum of $\mathbb{Q}$-power series multiplied by a complex number, and two such series are equivalent only if they differ in a finite number of coefficients.

4.2 Infinite field extensions

In this subsection, we will now study the case of infinite field extensions. We are mainly after a statement similar to (the first part of) Proposition 4.4. (We certainly cannot expect an analogue of the second part of its statement outside the Galois case.)

Clearly, a statement like Lemma 4.3 does not hold any more in this generality. To get the desired compatibility between hom spaces and extension of scalars, we will need to make some extra assumptions: Instead of general sheaves, we will work with $\mathbb{R}$-constructible sheaves (and hence on real analytic manifolds), since these can be modelled by constant sheaves on simplicial complexes, which gives the theory a combinatorial flavour. Moreover, for the final statement, we will assume that the manifold is compact, since this guarantees – together with $\mathbb{R}$-constructibility – that global sections are finite-dimensional, and we can thus get an analogue of Lemma 4.3 at least for global sections.

We start with two lemmas about sheaves on simplices of a simplicial complex. We will not recall the theory of simplicial complexes here, and an intuitive understanding of simplicial complexes is probably enough to follow our arguments. We mainly use the terminology and notation of [KS90, §8.1], where details can be found. We deviate from loc. cit. in the following notation: For an abstract simplex σ, we denote by Z_σ its geometric realization (this is the region not containing the lower-dimensional edges of the simplex, and it is denoted by $|\sigma|$ in [KS90]).

Lemma 4.7. *Let k be a field. Let $\mathbf{S}$ be a simplicial complex with geometric realization $\mathcal{S} := |\mathbf{S}|$, and let σ be a simplex and write $Z := Z_\sigma$. Consider the inclusion $j\colon Z \hookrightarrow |\mathbf{S}|$. Then*

$$\mathrm{R}j_* k_Z \simeq k_{\overline{Z}},$$

where the right-hand side denotes the constant sheaf on $\overline{Z}$, extended by zero on $\mathcal{S} \setminus \overline{Z}$.

Proof. We can decompose j as $Z \overset{j_Z}{\hookrightarrow} \overline{Z} \overset{i_Z}{\hookrightarrow} |\mathbf{S}|$, where j_Z is an open embedding, whereas i_Z is a closed embedding. It follows that $\mathrm{R}j_* k_Z \simeq i_{Z!}\mathrm{R}j_{Z*}k_Z$ (since i_Z is proper and proper direct images are exact). It therefore suffices to show that $\mathrm{R}j_{Z*}k_Z \simeq k_{\overline{Z}}$.

The constant sheaf k_Z is the sheaf of locally constant k-valued functions on Z, and it is characterized by the fact that on connected open subsets of Z its sections are k, and restriction maps $k_Z(U) \to k_Z(V)$ for connected open subsets $U, V \subseteq Z$ with $V \subseteq U$ are given by the identity. An analogous statement holds for the constant sheaf $k_{\overline{Z}}$.

By definition of the direct image, we have

$$(j_{Z*}k_Z)(U) = k_Z(U \cap Z)$$

for any open $U \in \overline{Z}$. It is easy to see that $U \cap Z$ is connected if $U \subseteq \overline{Z}$ is open and connected, and hence it follows that $j_{Z*}k_Z = k_{\overline{Z}}$.

It remains to show that the higher direct images vanish. It is known that $\mathrm{R}^k j_{Z*}k_Z$ is the sheaf associated to the presheaf given by

$$U \mapsto \mathrm{H}^k(U \cap Z; k)$$

for any open $U \subseteq \overline{Z}$. For any contractible U, $U \cap Z$ is still contractible and hence $\mathrm{H}^k(U \cap Z; k) = 0$ for $k \neq 0$. This shows the vanishing of $\mathrm{R}^k j_{Z*}k_Z$ since open contractible sets form a basis of the topology. □

Lemma 4.8. *Let k be a field. Let $\mathbf{S}$ be a simplicial complex with geometric realization $\mathcal{S} := |\mathbf{S}|$, and let σ, σ' be two simplices with inclusions $\iota_\sigma \colon Z_\sigma \hookrightarrow \mathcal{S}$, $\iota_{\sigma'} \colon Z_{\sigma'} \hookrightarrow \mathcal{S}$. Then we have*

$$\mathrm{R}\mathcal{H}om(\iota_{\sigma!}k_{Z_\sigma}, \iota_{\sigma'!}k_{Z_{\sigma'}}) \simeq \begin{cases} k_{\overline{Z_\sigma}}[\dim Z_\sigma - \dim Z_{\sigma'}] & \text{if } Z_\sigma \subseteq Z_{\sigma'} \\ 0 & \text{if } Z_\sigma \cap \overline{Z_{\sigma'}} = \emptyset \end{cases}$$

(Note that the first case covers in particular the case when $\sigma = \sigma'$. Note also that the two cases cover all possible situations by the definition of a simplicial complex.)

In particular, if L/K is a field extension, there is an isomorphism

$$\mathrm{R}\mathcal{H}om(L_\mathcal{S} \otimes_{K_\mathcal{S}} \iota_{\sigma!}K_{Z_\sigma}, L_\mathcal{S} \otimes_{K_\mathcal{S}} \iota_{\sigma'!}K_{Z_{\sigma'}}) \simeq L_\mathcal{S} \otimes_{K_\mathcal{S}} \mathrm{R}\mathcal{H}om(\iota_{\sigma!}K_{Z_\sigma}, \iota_{\sigma'!}K_{Z_{\sigma'}}).$$

Proof. By adjunction (see e.g. [KS90, Proposition 3.1.10]), we have

$$\mathrm{R}\mathcal{H}om(\iota_{\sigma!}k_{Z_\sigma}, \iota_{\sigma'!}k_{Z_{\sigma'}}) \simeq \mathrm{R}\iota_{\sigma*}\mathrm{R}\mathcal{H}om(k_{Z_\sigma}, \iota_\sigma^!\iota_{\sigma'!}k_{Z_{\sigma'}}).$$

Now, note that

$$\begin{aligned} \iota_\sigma^!\iota_{\sigma'!}k_{Z_{\sigma'}} &\simeq \mathrm{D}_{Z_\sigma}\iota_\sigma^{-1}\mathrm{D}_X\iota_{\sigma'!}k_{Z_{\sigma'}} \\ &\simeq \mathrm{D}_{Z_\sigma}\iota_\sigma^{-1}\iota_{\sigma'!}\mathrm{D}_{Z_{\sigma'}}k_{Z_{\sigma'}} \simeq \mathrm{D}_{Z_\sigma}(\iota_\sigma^{-1}\iota_{\sigma'*}k_{Z_{\sigma'}}[\dim Z_{\sigma'}]), \end{aligned}$$

where the last isomorphism follows from the fact that $Z_{\sigma'}$ is in particular an orientable differentiable manifold, so its dualizing complex is $\omega_{Z_{\sigma'}} \simeq k_{Z_{\sigma'}}[\dim Z_{\sigma'}]$ (see [KS90, Proposition 3.3.6(iii)]), and hence

$$\mathrm{D}_{Z_{\sigma'}}k_{Z_{\sigma'}} = \mathrm{R}\mathcal{H}om(k_{Z_{\sigma'}}, \omega_{Z_{\sigma'}}) \simeq \mathrm{R}\mathcal{H}om(k_{Z_{\sigma'}}, k_{Z_{\sigma'}})[\dim Z_{\sigma'}] \simeq k_{Z_{\sigma'}}[\dim Z_{\sigma'}].$$

If $Z_\sigma \cap \overline{Z_{\sigma'}} = \emptyset$, we have (using Lemma 4.7) $\iota_\sigma^{-1}\iota_{\sigma'*}k_{Z_{\sigma'}} \simeq \iota_\sigma^{-1}k_{\overline{Z_{\sigma'}}} \simeq 0$, which is easy to check on stalks. This proves the first part of the statement.

On the other hand, if $Z_\sigma \subset \overline{Z_{\sigma'}}$, we get $\iota_\sigma^{-1}\iota_{\sigma'*}k_{Z_{\sigma'}} = \iota_\sigma^{-1}k_{\overline{Z_{\sigma'}}} = k_{Z_\sigma}$, and hence (with a similar argument for the dual as above)

$$\begin{aligned} \iota_\sigma^!\iota_{\sigma'!}k_{Z_{\sigma'}} &\simeq \mathrm{D}_{Z_\sigma}(\iota_\sigma^{-1}\iota_{\sigma'*}k_{Z_{\sigma'}}[\dim Z_{\sigma'}]) \simeq (\mathrm{D}_{Z_\sigma}k_{Z_\sigma})[-\dim Z_{\sigma'}] \\ &\simeq k_{Z_\sigma}[\dim Z_\sigma - \dim Z_{\sigma'}]. \end{aligned}$$

Finally, this implies (if $Z_\sigma \subset \overline{Z_{\sigma'}}$)

$$\begin{aligned} \mathrm{R}\mathcal{H}om(\iota_{\sigma!}k_{Z_\sigma}, \iota_{\sigma'!}k_{Z_{\sigma'}}) &\simeq \mathrm{R}\iota_{\sigma*}\mathrm{R}\mathcal{H}om(k_{Z_\sigma}, k_{Z_\sigma}[\dim Z_\sigma - \dim Z_{\sigma'}]) \\ &\simeq \mathrm{R}\iota_{\sigma*}k_{Z_\sigma}[\dim Z_\sigma - \dim Z_{\sigma'}] \\ &\simeq k_{\overline{Z_\sigma}}[\dim Z_\sigma - \dim Z_{\sigma'}]. \end{aligned}$$

□

Nest, we first establish the following lemma, which is the "sheaf hom" version of [BHHS22, Lemma 2.6]. In loc. cit., we considered the functor Hom instead of $\mathcal{H}om$ for compactly supported $\mathbb{R}$-constructible sheaves. However, we did not go into the details of the proof, but rather just indicated the induction to be performed. Here, we will give more details on the technique. We are grateful to Takuro Mochizuki for providing us the idea for this statement and its proof (including for Lemma 4.8).

Lemma 4.9. *Let L/K be a field extension. Let X be a real analytic manifold and let $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(K_X)$. Assume that $\mathcal{F}$ and $\mathcal{G}$ have compact support. Then there is an isomorphism*

$$\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L_X \otimes_{K_X} \mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

Proof. We choose a locally finite subanalytic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ such that $\mathcal{F}$ and $\mathcal{G}$ are locally constant on each stratum X_α (this is a common refinement of the stratifications that exist for $\mathcal{F}$ and $\mathcal{G}$ since they are $\mathbb{R}$-constructible, cf. [KS90, Lemma 8.3.21]). By [KS90, Proposition 8.2.5], there exists then a simplicial complex $\mathbf{S}$ with a homeomorphism $i\colon |\mathbf{S}| \xrightarrow{\sim} X$ such that in particular any $i(Z_\sigma)$ is contained in some X_α, and hence the sheaves $i^{-1}\mathcal{F}$ and $i^{-1}\mathcal{G}$ are constant on Z_σ for any simplex σ of $\mathbf{S}$ (since simplices are contractible, local constancy already implies constancy). Let us write $\mathcal{S} := |\mathbf{S}|$ and $F := i^{-1}\mathcal{F}$, $G := i^{-1}\mathcal{G}$. The statement we need to prove is equivalent to proving

$$\mathcal{H}om(L_\mathcal{S} \otimes_{K_\mathcal{S}} F, L_\mathcal{S} \otimes_{K_\mathcal{S}} G) \simeq L_\mathcal{S} \otimes_{K_\mathcal{S}} \mathcal{H}om(F, G).$$

This will follow from the more general statement

$$\mathrm{R}\mathcal{H}om(L_\mathcal{S} \otimes_{K_\mathcal{S}} F, L_\mathcal{S} \otimes_{K_\mathcal{S}} G) \simeq L_\mathcal{S} \otimes_{K_\mathcal{S}} \mathrm{R}\mathcal{H}om(F, G). \tag{3}$$

that we will prove now. (Remember that we still take F, G to be sheaves, not elements of the derived category of sheaves.)

Assume first that $F = \iota_{\sigma !} K_{Z_\sigma}$ and G is an arbitrary compactly supported sheaf on $\mathcal{S}$ which is constant on any Z_σ. We will argue by induction on the dimension of the support of G:

Lemma 4.8 shows that the isomorphism (3) is true if $\dim \operatorname{supp} G = 0$, since in this case G is supported on finitely many points, i.e. $G \simeq \bigoplus_{i=1}^k \iota_{\{p_i\}!}(K_{\{p_i\}})^{r_i}$ for some points (0-dimensional simplices) $p_1, \dots, p_k \in \mathcal{S}$ and some $r_1, \dots, r_k \in \mathbb{Z}_{>0}$, and $\mathrm{R}\mathcal{H}om$ as well as tensor products commute with direct sums.

Now suppose the isomorphism (3) is proved for $\dim \operatorname{supp} G \leq n$ for some n, and consider a G with $\dim \operatorname{supp} G = n+1$. Then let Z_{n+1} be the (disjoint) union of all $n+1$-dimensional simplices in the support of G (these are finitely many since the support of G is compact), and denote by $Z_{\leq n}$ the union of all other simplices contained in the support of G, so that $\operatorname{supp} G = Z_{n+1} \sqcup Z_{\leq n}$. Note that $Z_{n+1} \subset \operatorname{supp} G$ is open. We therefore have a short exact sequence

$$0 \longrightarrow G_{Z_{n+1}} \longrightarrow G \longrightarrow G_{Z_{\leq n}} \longrightarrow 0.$$

Considering the right derived functors of $\mathcal{H}_F^L := \mathcal{H}om(L_X \otimes_{K_X} F, L_X \otimes_{K_X} (-))$ and ${}^L\mathcal{H}_F := L_X \otimes_{K_X} \mathcal{H}om(F, -)$ as well as the natural morphism from the second to the first from Lemma 4.1, we obtain a commutative diagram in $\mathrm{D}^{\mathrm{b}}(L_{\mathcal{S}})$ whose rows are distinguished triangles:

$$\begin{tikzcd}
\mathrm{R}\mathcal{H}_F^L(G_{Z_{n+1}}) \arrow[r] & \mathrm{R}\mathcal{H}_F^L(G) \arrow[r] & \mathrm{R}\mathcal{H}_F^L(G_{Z_{\leq n}}) \arrow[r, "+1"] & {} \\
\mathrm{R}^L\mathcal{H}_F(G_{Z_{n+1}}) \arrow[r] \arrow[u] & \mathrm{R}^L\mathcal{H}_F(G) \arrow[r] \arrow[u] & \mathrm{R}^L\mathcal{H}_F(G_{Z_{\leq n}}) \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

The first vertical arrow is an isomorphism by Lemma 4.8 since $G_{Z_{n+1}}$ is direct sum of sheaves of the form $\iota_{Z_\sigma !}(K_{Z_\sigma})^{r_\sigma}$ for some $(n+1)$-dimensional simplices σ (the connected components of Z_{n+1}) and some $r_\sigma \in \mathbb{Z}_{>0}$. On the other hand, the third vertical arrow is an isomorphism by the induction hypothesis. This implies that the middle arrow is an isomorphism by the axioms of a triangulated category.

Finally, by an analogous induction on the dimension of the support of F, using the statement just proved instead of Lemma 4.8, we conclude that the isomorphism (3) holds for general F and G that are constant on each simplex. $\square$

We can now deduce the following proposition. Its statement was also established in [BHHS22], where, however, this was deduced a posteriori from a similar result for ind-sheaves. We give a direct proof using the above Lemma 4.9.

Proposition 4.10 (see [BHHS22, Lemma 2.7])**.** *Let L/K be a field extension. Let X be a real analytic manifold and let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$. Then there is an isomorphism*

$$\mathrm{R}\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

Proof. Let us first assume that $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(K_X)$. Then we will prove

$$\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L_X \otimes_{K_X} \mathcal{H}om(\mathcal{F}, \mathcal{G}). \tag{4}$$

There is a natural morphism between these two objects (from right to left, cf. Lemma 4.1), and it suffices to prove locally on an open covering that it is an isomorphism. Choose therefore an open covering $X = \bigcup_{i \in I} U_i$ by relatively compact open subsets $U_i \subseteq X$.

Now note that, writing $j_i \colon U_i \hookrightarrow X$ for the inclusion, we have

$$\begin{aligned}
(L_X \otimes_{K_X} \mathcal{H}om(\mathcal{F}, \mathcal{G}))|_{U_i} &\simeq L_{U_i} \otimes_{K_{U_i}} \mathcal{H}om(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}) \\
&\simeq L_{U_i} \otimes_{K_{U_i}} \mathcal{H}om((\mathcal{F})_{U_i}|_{U_i}, (\mathcal{G})_{U_i}|_{U_i}) \\
&\simeq \big(L_X \otimes_{K_X} \mathcal{H}om((\mathcal{F})_{U_i}, (\mathcal{G})_{U_i})\big)|_{U_i}
\end{aligned}$$

because $(\mathcal{F})_{U_i}|_{U_i} \simeq j_i^{-1} j_{i!} j_i^{-1} \mathcal{F} \simeq j_i^{-1}\mathcal{F} \simeq \mathcal{F}|_{U_i}$. Similarly,

$$\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G})|_{U_i} \simeq \mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}_{U_i}, L_X \otimes_{K_X} \mathcal{G}_{U_i})|_{U_i}.$$

Since $\mathcal{F}_{U_i}$ and $\mathcal{G}_{U_i}$ have compact support (it is clear that $\operatorname{supp} \mathcal{F}_{U_i} = (\operatorname{supp} \mathcal{F}) \cap \overline{U_i}$, i.e. the intersection of a closed and a compact subset of X), it follows from Lemma 4.9 that the restriction of (4) to each U_i is an isomorphism, and hence that (4) itself is.

The statement of the proposition now follows by deriving functors. $\square$

To deduce a statement about homomorphism spaces, (i.e. global sections of the isomorphism in Proposition 4.10), we need to study the behaviour of $\mathbb{R}$-constructible sheaves with respect to global sections. This is done in the following lemma.

Lemma 4.11. *Let X be a compact real analytic manifold, let L/K be a (not necessarily finite) field extension, and let $\mathcal{G} \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(K_X)$ be an $\mathbb{R}$-constructible sheaf on X. Then there is an isomorphism*

$$(L_X \otimes_{K_X} \mathcal{G})(X) \simeq L \otimes_K \mathcal{G}(X).$$

We prove this lemma after showing an auxiliary statement about the local behaviour of $\mathbb{R}$-constructible sheaves.

Lemma 4.12. *Let k be a field and $\mathcal{G} \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. Then for any $p \in X$ and for every $\delta > 0$ there exists an open, contractible neighbourhood $U_p \subset B_\delta(p)$ of p such that*

$$\Gamma(U_p; \mathcal{G}) \to \mathcal{G}_p$$

is an isomorphism. In particular, any element of the stalk can be uniquely extended to a section in a sufficiently small neighbourhood.

Proof. Let $p \in X$ and $\delta > 0$, and set $V = B_\delta(p)$. Then $\mathcal{G}|_V$ is $\mathbb{R}$-constructible (see [KS90, Proposition 8.4.10]) and hence there exists a stratification $V = \bigsqcup_{\alpha \in A} V_\alpha$ such that $\mathcal{G}$ is locally constant on the strata ([KS90, Lemma 8.3.21]). We can, without loss of generality, assume that $\{p\}$ is a stratum, by passing to a refinement. There exists hence (see [KS90, Proposition 8.2.5]) a simplicial complex $\mathbf{S}$ with geometric realization $\mathcal{S} := |\mathbf{S}|$ and a homeomorphism $i\colon |\mathbf{S}| \xrightarrow{\sim} V$ such that $i^{-1}(\mathcal{G}|_V)$ is constant on all the Z_σ. (In other words, $i^{-1}(\mathcal{G}|_V)$ is $\mathbf{S}$-constructible.)

We write $q := i^{-1}(p)$, and this point is a vertex of the simplicial complex $\mathbf{S}$. Set $U := U(\{q\}) \subseteq \mathcal{S}$ (this is the union of all simplices with q in their boundary, and it is an open subset of $\mathcal{S}$) and $F := i^{-1}(\mathcal{G}|_V)$. Then, by [KS90, Proposition 8.1.4], the natural map $\Gamma(U; F) \to F_q$ is an isomorphism, and hence $\Gamma(U_p; \mathcal{G}) \to \mathcal{G}_p$ is an isomorphism if we set $U_p := i(U)$. □

Proof of Lemma 4.11. Let $p \in X$. Then, due to Lemma 4.12, for any $\delta > 0$, there exists $U_p \subset B_\delta(p)$ such that the morphisms

$$\Gamma(U_p; \mathcal{G}) \to \mathcal{G}_p \quad \text{and} \quad \Gamma(U_p; L_X \otimes_{K_X} \mathcal{G}) \to (L_X \otimes_{K_X} \mathcal{G})_p$$

is an isomorphism. In other words, an element of the stalk extends uniquely to a section on a sufficiently small neighbourhood, regardless of a chosen upper bound for this neighbourhood.

Now, we know that $(L_X \otimes_{K_X} \mathcal{G})_p \simeq L \otimes_K \mathcal{G}_p$ and hence $\Gamma(U_p; L_X \otimes_{K_X} \mathcal{G}) \simeq L \otimes_K \Gamma(U_p; \mathcal{G})$.

Now we choose such a neighbourhood U_p^δ for any $p \in X$ and $\delta > 0$. Note that $\mathcal{T} := \{U_p^\delta \mid p \in X, \delta > 0\}$ is a basis of the topology of X, in particular, it is an open covering of X. Since X is compact, there exists therefore a finite number of $U_i \in \mathcal{T}$, $i \in I$, that cover X, and there is an exact sequence

$$0 \to \Gamma(X;\mathcal{G}) \to \prod_{i\in I} \Gamma(U_i;\mathcal{G}) \to \prod_{i,j\in I} \Gamma(U_i \cap U_j;\mathcal{G}).$$

We apply the functor $L \otimes_K (-)$ to this sequence and obtain (noting that tensor products commute with finite products and using the above observations)

$$0 \to L \otimes_K \Gamma(X;\mathcal{G}) \to \prod_{i\in I} \Gamma(U_i; L_X \otimes_{K_X} \mathcal{G}) \to \prod_{i,j\in I} \Gamma(U_i \cap U_j; L_X \otimes_{K_X} \mathcal{G}).$$

On the other hand, due to the sheaf property of $L_X \otimes_{K_X} \mathcal{G}$, the first object of this sequence is isomorphic to $\Gamma(X; L_X \otimes_{K_X} \mathcal{G})$. □

With this in hand, we can now state the description of the space of homomorphisms between extensions of scalars.

Proposition 4.13. *Let L/K be a field extension and let X be a compact real analytic manifold. Let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}) \simeq L \otimes_K \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)}(\mathcal{F},\mathcal{G}).$$

Proof. By Lemma 4.11, we get the isomorphism of functors $\Gamma \circ (L_X \otimes_{K_X} (-)) \simeq L \otimes_K \Gamma(-)$ on $\mathrm{Mod}_{\mathbb{R}\text{-c}}(K_X)$, where $\Gamma = \Gamma(X;-)$ denotes the functor of global sections. By deriving functors, we get an isomorphism of functors $\mathrm{R}\Gamma \circ (L_X \otimes_{K_X} (-)) \simeq L \otimes_K \mathrm{R}\Gamma(-)$ on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$. We then apply this functor to the object $\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})$ (which is also in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$, cf. [KS90, Proposition 8.4.10]) and get

$$\begin{aligned} L \otimes_K \mathrm{RHom}(\mathcal{F},\mathcal{G}) &\simeq \mathrm{R}\Gamma(L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})) \\ &\simeq \mathrm{R}\Gamma(\mathrm{R}\mathcal{H}om(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G})) \\ &\simeq \mathrm{RHom}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \mathcal{G}). \end{aligned}$$

Here, the second isomorphism follows from Proposition 4.10. Then, the assertion follows by taking zeroth cohomology. □

Finally, we can use our results of this subsection to further "complete" the statement of Lemma 3.12 with some more compatibilities in the case where X is an oriented differentiable manifold by studying the duality functor. Note, however, that for the exceptional inverse image, we need an $\mathbb{R}$-constructibility assumption here.

Corollary 4.14 (of Proposition 4.10)**.** *Let L/K a field extension and let X be a real analytic manifold.*

(a) Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(K_X)$. If L/K is finite, or if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$, then there is an isomorphism

$$\mathrm{D}_X(L_X \otimes_{K_X} \mathcal{F}) \simeq L_X \otimes_{K_X} \mathrm{D}_X\mathcal{F}.$$

(b) *Let $f\colon X \to Y$ be a morphism of real analytic manifolds, and let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_Y)$. Then there is an isomorphism*

$$f^!(L_Y \otimes_{K_Y} \mathcal{F}) \simeq L_X \otimes_{K_X} f^!\mathcal{F}.$$

(c) *If X is a complex manifold and $\mathcal{G} \in \mathrm{Perv}(K_X)$, then $L_X \otimes_{K_X} \mathcal{G}$.*

Proof. (a) It follows from [KS90, Proposition 3.3.4] that we have the relation $\omega_X^L \simeq L_X \otimes_{K_X} \omega_X^K$ between the dualizing complexes in $\mathrm{D}^{\mathrm{b}}(K_X)$ and $\mathrm{D}^{\mathrm{b}}(L_X)$. Then we get

$$\begin{aligned} \mathrm{D}_X(L_X \otimes_{K_X} \mathcal{F}) &= \mathrm{R}\mathcal{H}om_{L_X}(L_X \otimes_{K_X} \mathcal{F}, \omega_X^L) \\ &\simeq \mathrm{R}\mathcal{H}om_{L_X}(L_X \otimes_{K_X} \mathcal{F}, L_X \otimes_{K_X} \omega_X^K) \\ &\simeq L_X \otimes_{K_X} \mathrm{R}\mathcal{H}om_{K_X}(\mathcal{F}, \omega_X^K) = L_X \otimes_{K_X} \mathrm{D}_X\mathcal{F}. \end{aligned}$$

Here, we have applied Proposition 4.4 (if L/K is finite) or Proposition 4.10 (if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$) in the third line.

(b) Since $\mathcal{F}$ is a complex of $\mathbb{R}$-constructible sheaves (and so is $L_X \otimes_{K_X} \mathcal{F}$), we have $f^! \simeq \mathrm{D}_X \circ f^{-1} \circ \mathrm{D}_Y$ (cf. [KS90, Proposition 8.4.9 and Exercise VIII.3]). Hence, the statement follows from (a) and Lemma 3.12(b), since $\mathbb{D}_Y$ and f^{-1} preserve $\mathbb{R}$-constructibility (see [KS90, Propositions 8.4.9 and 8.4.10]).

(c) It is clear by Lemma 3.12(c) that extension of scalars does not change the support of the cohomologies of $\mathcal{F}$. Therefore, $L_X \otimes_{K_X} \mathcal{F}$ and $\mathrm{D}_X(L_X \otimes_{K_X} \mathcal{F}) \overset{(a)}{\simeq} L_X \otimes_{K_X} \mathrm{D}_X(\mathcal{F})$ satisfy the support condition if $\mathcal{F}$ and $\mathrm{D}_X\mathcal{F}$ do.

□

4.3 K-structures and constructibility

Let L/K be a field extension, and let X be a topological space.

We have seen in Lemma 3.12 and Corollary 4.14 that extension of scalars preserves constructibility and perversity. Here, we will show that constructiblity of a sheaf of L-vector spaces also descends to a K-structure.

Lemma 4.15. *Let $\mathcal{F} \in \mathrm{Mod}(L_X)$ be a local system of L-vector spaces (of finite rank), and let $\mathcal{G} \in \mathrm{Mod}(K_X)$ be a K-structure of $\mathcal{F}$. Then $\mathcal{G}$ is a local system of K-vector spaces (of finite rank).*

Proof. We reproduce the proof in [BHHS22] slightly differently here for completeness.

A local system is characterized by the fact that each $x \in X$ has an open neighbourhood $U \subseteq X$ such that for any $y \in U$ there is an isomorphism $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_y$ such that the diagram

$$\begin{tikzcd} & \mathcal{F}(U) \arrow[dl] \arrow[dr] & \\ \mathcal{F}_x \arrow[rr, "\sim"] & & \mathcal{F}_y \end{tikzcd}$$

commutes. (This is a nice exercise in basic sheaf theory.) We are given this property for $\mathcal{F}$ and want to prove the analogous property for $\mathcal{G}$. This follows from the commutative diagram

$$\begin{tikzcd}
 & L \otimes_K \mathcal{G}(U) \arrow[dl] \arrow[dr] \arrow[ddd, bend left] & \\
L \otimes_K \mathcal{G}_x \arrow[rr, dashed] \arrow[d, "\sim"] & & L \otimes_K \mathcal{G}_y \arrow[d, "\sim"] \\
\mathcal{F}_x \arrow[rr, "\sim"] & & \mathcal{F}_y \\
 & \mathcal{F}(U) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

It remains to observe that the dashed morphism descends to a morphism $\mathcal{G}_x \to \mathcal{G}_y$ (we can shrink U if necessary and then an element in $\mathcal{G}_x$ lifts to one in $\mathcal{G}(U)$ and hence its image is in $\mathcal{G}_y$), and that this is an isomorphism if and only if the dashed is so. □

Corollary 4.16. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(L_X)$ and let $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(K_X)$ be a K-lattice of $\mathcal{F}$.*

(a) If $\mathrm{H}^i\mathcal{F}$ is locally constant on some $Z \subseteq X$, so is $\mathrm{H}^i(\mathcal{G})$.

In particular, if X is a real analytic manifold and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)$ (resp. X is a complex manifold and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(L_X)$), then $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(K_X)$ (resp. $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(K_X)$).

(b) If X is a complex manifold and $\mathcal{F} \in \mathrm{Perv}(L_X)$, then $\mathcal{G} \in \mathrm{Perv}(K_X)$.

Proof. (a) follows directly from Lemma 4.15, since restriction and cohomology commute with extension of scalars (see Lemma 3.12), so $\mathrm{H}^i(\mathcal{F})|_Z \simeq L_Z \otimes_{K_Z} \mathrm{H}^i(\mathcal{G})|_Z$ being a local system of finite rank implies that $\mathrm{H}^i(\mathcal{G})|_Z$ is a local system of finite rank.

(b) By (a), the sheaf $\mathcal{G}$ is locally constant and nontrivial on the same covering as $\mathcal{F}$ (and the according statement holds for the duals due to Corollary 4.14). Hence the supports of the cohomologies of $\mathcal{F}$ (resp. $\mathrm{D}_X\mathcal{F}$) are the same as those for $\mathcal{G}$ (resp. $\mathrm{D}_X\mathcal{G}$) and hence the support condition for $\mathcal{G}$ and $\mathrm{D}_X\mathcal{G}$ holds if it holds for $\mathcal{F}$ and $\mathrm{D}_X\mathcal{F}$.

□

5 Galois descent for sheaves and their complexes

Let L/K be a finite Galois extension with Galois group G. In this section, we will formulate Galois descent for sheaves of vector spaces. Afterwards, we are going to investigate a similar procedure for derived categories of sheaves of vector spaces, and we will describe what kind of problems arise there and prevent us from obtaining an equally nice result. Finally, we restrict ourselves to a particularly nice subcategory of the derived category of sheaves of vector spaces, namely that of perverse sheaves, and we establish Galois descent for them, using a construction performed by A. Beilinson in [Bei87].

5.1 Galois descent for sheaves of vector spaces

Galois descent for sheaves of vector spaces has already been studied in [BHHS22], but the statement was not formulated as an equivalence of categories in loc. cit. We reformulate it here to fit into the framework set up in Section 5, using our results from Section 4.

Proposition 5.1 (cf. [BHHS22, Lemma 2.13]). *Let L/K be a finite Galois extension. Let X be a topological space.*

For any $\mathcal{F} \in \mathrm{Mod}(L_X)$ equipped with a G-structure $(\varphi_g)_{g\in G}$, there exists a K-structure $\mathcal{G} \in \mathrm{Mod}(K_X)$ together with a natural isomorphism $\psi\colon L_X \otimes_{K_X} \mathcal{G} \xrightarrow{\sim} \mathcal{F}$ such that the natural G-structure on $L_X \otimes_{K_X} \mathcal{G}$ corresponds via this isomorphism to the given one on $\mathcal{F}$, i.e. such that, for any $g \in G$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi_g"] & \overline{\mathcal{F}}^g \\
L_X \otimes_{K_X} \mathcal{G} \arrow[u, "\psi"] \arrow[r, "g\otimes \mathrm{id}_{\mathcal{G}}"] & \overline{L_X}^g \otimes_{K_X} \mathcal{G} \arrow[u, "\overline{\psi}^g"']
\end{tikzcd}$$

In particular, the functor $\Phi_{L/K} = L_X \otimes_{K_X} (-)$ induces an equivalence

$$\Phi^G_{L/K}\colon \mathrm{Mod}(K_X) \longrightarrow \mathrm{Mod}(L_X)^G.$$

Proof. We proceed analogously to the construction for vector spaces. We consider $\mathcal{F}$ as a sheaf of K-vector spaces and each φ_g as a K-linear automorphism of $\mathcal{F}$. Then $\mathcal{F}_K$ is defined as

$$\mathcal{F}_K := \ker\Big(\prod_{g\in G}(\varphi_g - \mathrm{id})\colon \mathcal{F} \longrightarrow \prod_{g\in G}\mathcal{F}\Big) \in \mathrm{Mod}(K_X).$$

Since kernels of sheaves are computed sectionwise and sections of $L_X \otimes_{K_X} \mathcal{F}_K$ are also obtained by sectionwise applying $L \otimes_K (-)$ (see Lemma 4.3), the isomorphism $L_X \otimes_{K_X} \mathcal{F}_K \simeq \mathcal{F}$ is obtained from Galois descent for vector spaces. The commutation of the above square is also clear from the construction of $\mathcal{F}_K$ as a sheaf of invariants.

This proves in particular essential surjectivity of the functor $\Phi^G_{L/K}$. Full faithfulness follows from Proposition 4.4. □

Remark 5.2. We can also similarly establish a Galois descent statement for functor categories as in Example 3.2(b). They admit an obvious functor of extension of scalars $\mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_K) \to \mathrm{Funct}(\mathcal{C}, \mathrm{Vect}_L)$. This yields then in particular Galois descent for presheaves of vector spaces (if we take $\mathcal{C} = \mathrm{Op}(X)^{\mathrm{op}}$), and to deduce Proposition 5.1, one just needs to check that the descent of a sheaf is still a sheaf. (The extension of scalars for sheaves is indeed the same as the one for presheaves in the finite Galois case due to Lemma 4.3.)

Choosing more complicated categories $\mathcal{C}$ (so-called categories of exit paths), one can also express certain categories of constructible sheaves as functor categories. This technique is known as *exodromy* (see e.g. [Tre09] and [BGH18]), and it has also been set up in the framework of quasi-categories (see [Lur17], and [PT22] for a recent generalization). Such exodromy equivalences might serve as an alternative approach to our questions for certain constructible sheaves, and they might also lead to a clearer study of complexes of sheaves. We will, however, not take this viewpoint here. We are grateful to Jean-Baptiste Teyssier for drawing our attention to these constructions.

5.2 The case of derived categories of sheaves

Having established the above equivalence (Theorem 5.1) for sheaves, one would, of course, like to generalize such a statement to derived categories of sheaves. Indeed, for a finite Galois extension L/K with Galois group G, we still have the functor induced by extension of scalars

$$\Phi^G_{L/K}\colon \mathrm{D}^{\mathrm{b}}(K_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(L_X)^G.$$

The following is deduced directly – as above – from Proposition 4.4.

Proposition 5.3. *The functor $L_X \otimes_{K_X} (-)$ is fully faithful on $\mathrm{D}^{\mathrm{b}}(K_X)$.*

Remark 5.4. Essential surjectivity does not seem to hold in this more general situation. An indication for why this makes sense is the following: To an object $\mathcal{F}^\bullet = \ldots \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \ldots$, the functor associates the object $L_X \otimes_{K_X} \mathcal{F}^\bullet = \ldots \to L_X \otimes_{K_X} \mathcal{F}_i \to L_X \otimes_{K_X} \mathcal{F}_{i+1} \to \ldots$ (due to the exactness of the tensor product), together with the natural G-structure given by that on L_X, i.e.

$$\begin{tikzcd}
\ldots \arrow[r] & L_X \otimes_{K_X} \mathcal{F}_i \arrow[r] \arrow[d, "g\otimes\mathrm{id}"] & L_X \otimes_{K_X} \mathcal{F}_{i+1} \arrow[r] \arrow[d, "g\otimes\mathrm{id}"] & \ldots \\
\ldots \arrow[r] & \overline{L_X}^g \otimes_{K_X} \mathcal{F}_i \arrow[r] & \overline{L_X}^g \otimes_{K_X} \mathcal{F}_{i+1} \arrow[r] & \ldots
\end{tikzcd}$$

In particular, this G-structure is given by morphisms of complexes (rather than roofs, as is the general case for morphisms in the derived category). In general, morphisms φ_g in the derived category can – after choosing suitable resolutions – be represented as morphisms of complexes, but the compatibilities that the φ_g have to satisfy will only hold up to homotopy of morphisms of complexes, and hence in general such a G-structure will not be in the essential image.

A standard technique for proofs of statements in derived categories is by induction on the amplitude of a complex. At least for complexes concentrated in two successive degrees, we can use this approach to deduce some existence statement of a K-lattice from the existence of a G-structure.

Proposition 5.5. *Let X be a real analytic manifold and $\mathcal{F}^\bullet \in \mathrm{D}^{\mathrm{b}}(L_X)$ with a G-structure $(\varphi_g)_{g\in G}$. Assume that $\mathcal{F}^\bullet$ is concentrated in degrees a and $a+1$ (i.e. the only non-vanishing cohomologies are $H^a(\mathcal{F}^\bullet)$ and $H^{a+1}(\mathcal{F}^\bullet)$. Then there exists $\mathcal{F}^\bullet_K \in \mathrm{D}^{\mathrm{b}}(K_X)$ and an isomorphism $L_X \otimes_{K_X} \mathcal{F}^\bullet_K \simeq \mathcal{F}^\bullet$.*

Proof. We know the statement for sheaves (i.e. complexes concentrated in one degree), so we know that $H^a(\mathcal{F}^\bullet) \simeq L_X \otimes_{K_X} \mathcal{G}_a$ and $H^{a+1}(\mathcal{F}^\bullet) \simeq L_X \otimes_{K_X} \mathcal{G}_{a+1}$ such that under these isomorphisms the G-structures on $H^i(\mathcal{F}^\bullet)$ induced by the one on $\mathcal{F}^\bullet$ coincide with those given by the natural G-structure on L_X.

Using the standard truncation functors for complexes (with respect to the standard t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(L_X)$), there is a distinguished triangle

$$H^{a+1}(\mathcal{F}^\bullet)[-a-1] \longrightarrow H^a(\mathcal{F}^\bullet)[-a] \longrightarrow \mathcal{F}^\bullet \xrightarrow{+1}$$

and for any $g \in G$ the G-structure on $\mathcal{F}^\bullet$ induces an isomorphism of distinguished triangles

$$\begin{tikzcd}
H^{a+1}(\mathcal{F}^\bullet)[-1] \arrow[r, "f"] \arrow[d, "\simeq"'] & H^a(\mathcal{F}^\bullet) \arrow[r] \arrow[d, "\simeq"'] & \mathcal{F}^\bullet \arrow[r, "+1"] \arrow[d, "\simeq"', "\varphi_g"] & {} \\
\overline{H^{a+1}(\mathcal{F}^\bullet)[-1]}^g \arrow[r] & \overline{H^a(\mathcal{F}^\bullet)}^g \arrow[r] & \overline{\mathcal{F}^\bullet}^g \arrow[r, "+1"] & {}
\end{tikzcd} \tag{5}$$

(note that truncation and conjugation commute).

By Lemma 4.4, there is a morphism $\tilde{f}\colon \mathcal{G}_{a+1}[-1] \to \mathcal{G}_a$ such that $f = g \otimes 1$. We can complete it to a distinguished triangle

$$\mathcal{G}_{a+1}[-1] \xrightarrow{\tilde{f}} \mathcal{G}_a \longrightarrow \mathcal{G}^\bullet \xrightarrow{+1}$$

where is $\mathcal{G}^\bullet$ is unique up to (non-unique) isomorphism. Hence, there exists a (non-unique) isomorphism $\gamma\colon \mathcal{G}^\bullet \otimes_{K_X} L_X \xrightarrow{\sim} \mathcal{F}^\bullet$. In other words, we have completed (5) to a commutative diagram

$$\begin{tikzcd}
 & \mathcal{G}_{a+1}[-a-1] \otimes_{K_X} L_X \arrow[rr, "g\otimes 1"] \arrow[dl, "\simeq"'] \arrow[dd] & & \mathcal{G}_a[-a] \otimes_{K_X} L_X \arrow[rr] \arrow[dl, "\simeq"'] \arrow[dd] & & \mathcal{G}^\bullet \otimes_{K_X} L_X \arrow[r, "+1"] \arrow[dl, "\simeq"'] \arrow[dd, dashed] & {} \\
H^{a+1}(\mathcal{F}^\bullet)[-a-1] \arrow[rr, "f"] \arrow[dd, "\simeq"'] & & H^a(\mathcal{F}^\bullet)[-a] \arrow[rr] \arrow[dd, "\simeq"'] & & \mathcal{F}^\bullet \arrow[r, "+1"] \arrow[dd, "\simeq"', "\varphi_g"] & {} & \\
 & \mathcal{G}_{a+1}[-a-1] \otimes_{K_X} \overline{L_X}^g \arrow[rr] \arrow[dl, "\simeq"'] & & \mathcal{G}_a[-a] \otimes_{K_X} \overline{L_X}^g \arrow[rr] \arrow[dl, "\simeq"'] & & \mathcal{G}^\bullet \otimes_{K_X} \overline{L_X}^g \arrow[r, "+1"] \arrow[dl, "\simeq"'] & {} \\
\overline{H^{a+1}(\mathcal{F}^\bullet)[-a-1]}^g \arrow[rr] & & \overline{H^a(\mathcal{F}^\bullet)[-a]}^g \arrow[rr] & & \overline{\mathcal{F}^\bullet}^g \arrow[r, "+1"] & {} &
\end{tikzcd}$$

□

Remark 5.6. The construction in the proof above is very non-canonical. This is due to the fact that the third objects and morphisms in distinguished triangles are not unique up to unique isomorphism. In particular, although we find an object $\mathcal{G}^\bullet$ with $L_X \otimes_{K_X} \mathcal{G}^\bullet \simeq \mathcal{F}^\bullet$ here, it seems not clear that the given G-structure corresponds to the natural one on $L_X \otimes_{K_X} \mathcal{G}^\bullet$. In other words, it is not clear that the dashed arrow in the big diagram is given by $\mathrm{id}_{\mathcal{G}^\bullet} \otimes g$. This has two implications:

- First, this theorem does not show that the object $(\mathcal{F}, (\varphi_g)_{g\in G})$ is in the essential image of the functor $L_X \otimes_{K_X} (-)$. It just shows that $\mathcal{F}^\bullet$ can be realized as such a tensor product if there exists a G-structure on it (but not saying that exactly this G-structure comes through the tensor product).

- Secondly, we cannot proceed inductively to get similar results for complexes concentrated in more than two degrees since the fact that the G-structure corresponds to the natural one on the tensor product is crucial for descending the morphism f to $\tilde{f}$.

5.3 Galois descent for perverse sheaves

It is reasonable to expect that the results are closer to the statements for sheaves if we do not consider general objects of the derived category, but perverse sheaves.[1]

Let L/K be a finite Galois extension, and let X be a complex manifold. The notion of G-conjugation on $\mathrm{Perv}(L_X)$ as well as the functors of extension and restriction of scalars on are inherited from those between $\mathrm{D^b}(K_X)$ and $\mathrm{D^b}(L_X)$.

Given a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(L_X)$ together with a G-structure $(\varphi_g)_{g\in G}$, we can define the presumptive K-structure similarly to the case of vector spaces and sheaves, namely as an object of invariants. Contrarily to the case of derived categories, we dispose of the notion of kernels here, since perverse sheaves form an abelian category.

Consider the underlying K-perverse sheaf $\mathcal{F}^K \in \mathrm{Perv}(K_X)$ (which is the same for every $\overline{\mathcal{F}}^g$) with the automorphisms $\varphi_g^K \colon \mathcal{F}^K \to \mathcal{F}^K$ induced by the φ_g.

Then define

$$\mathcal{G} := \ker\Big(\prod_{g\in G}(\varphi_g^K - \mathrm{id}_{\mathcal{F}^K})\colon \mathcal{F}^K \longrightarrow \prod_{g\in G}\mathcal{F}^K\Big).$$

Clearly, we have a morphism $\mathcal{G} \to \mathcal{F}^K$, and hence by Lemma 3.11, we have a morphism

$$\mathcal{G} \otimes_{K_X} L_X \to \mathcal{F}. \tag{6}$$

We will prove that it is an isomorphism. For this, we will use a description of perverse sheaves due to Beilinson [Bei87] (see also [Rei10] for some more details and complements on Beilinson's article).

Beilinson's equivalence Let us recall the idea of Beilinson's "gluing" of perverse sheaves, and study the properties of his equivalence with respect to field extensions. We will not review all the details, for which we refer to [Bei87] and [Rei10]. From now on, we work in the context of complex analytic varieties, which is a generalization of (but often analogous to) the theory of complex manifolds.

[1] Thinking in terms of Algebraic Analysis, perverse sheaves are the counterparts of regular holonomic D-modules – objects concentrated in one degree – via the Riemann–Hilbert correspondence (see [Kas84]), so these are the objects to understand if one wants to understand topologically the category of regular holonomic D-modules.

Let k be a field, X a complex manifold and $f\colon X \to \mathbb{C}$ a holomorphic function. Write $Z := f^{-1}(0)$ and $U := X \setminus Z$ with inclusion $j\colon U \hookrightarrow X$. Moreover, we fix a generator t of the fundamental group $\pi_1(\mathbb{C} \setminus \{0\})$.

There is a functor $\Psi_f^{\mathrm{un}}\colon \mathrm{Perv}(k_U) \to \mathrm{Perv}(k_Z)$ of *unipotent nearby cycles.* By its construction, the fundamental group $\pi_1(\mathbb{C}\setminus\{0\})$ acts on $\Psi_f^{\mathrm{un}}(\mathcal{F}_U)$ for any $\mathcal{F}_U \in \mathrm{Perv}(k_U)$. In particular, the fixed generator t induces an endomorphism of any such $\Psi_f^{\mathrm{un}}(\mathcal{F}_U)$, which we will still denote by t.

There is also a functor $\Phi_f^{\mathrm{un}}\colon \mathrm{Perv}(k_X) \to \mathrm{Perv}(k_Z)$ of *unipotent vanishing cycles.*

One defines the category of *gluing data* $\mathcal{GD}_f(X,k)$ to be the category whose objects are tuples $(\mathcal{F}_U, \mathcal{F}_Z, u, v)$, where $\mathcal{F}_U \in \mathrm{Perv}(k_U)$, $\mathcal{F}_Z \in \mathrm{Perv}(k_Z)$ and u and v are morphisms

$$\Psi_f^{\mathrm{un}}(\mathcal{F}_U) \xrightarrow{u} \mathcal{F}_Z \xrightarrow{v} \Psi_f^{\mathrm{un}}(\mathcal{F}_U)$$

such that their composition coincides with the endomorphism $\mathrm{id} - t$ of $\Psi_f^{\mathrm{un}}(\mathcal{F}_U)$. A morphism in $\mathcal{GD}_f(X,k)$ is defined in the obvious way, as two morphisms of perverse sheaves on U and Z, respectively, making the natural diagram with the u and v commute.

It is shown (see [Bei87, Proposition 3.1] or [Rei10, Theorem 3.6]) that $\mathcal{GD}_f(X,k)$ is an abelian category and that there is an equivalence

$$\mathsf{F}_f\colon \mathrm{Perv}(k_X) \xrightarrow{\sim} \mathcal{GD}_f(X,k),$$

sending $\mathcal{F} \in \mathrm{Perv}(k_X)$ to the tuple $(j^{-1}\mathcal{F}, \Phi_f^{\mathrm{un}}(\mathcal{F}), u, v)$, where we will not go into detail with the construction of the maps u and v. Also the quasi-inverse is explicitly described.

Let us now study this equivalence in the context of a finite field extension L/K. The construction of the nearby and vanishing cycles functor as well as of the maps u and v are completely topological and do not depend on the coefficient field (they could as well be performed on sheaves of sets). Therefore, the forgetful functor (restriction of scalars) $\mathsf{for}_{L/K}\colon \mathrm{Perv}(L_X) \to \mathrm{Perv}(K_X)$ corresponds to a forgetful functor $\mathcal{GD}_f(X,L) \to \mathcal{GD}_f(X,K)$ (given by the ones on perverse sheaves on U and Z).

On the other hand, we have the following statement about extension of scalars.

Lemma 5.7. *For $\mathcal{A} \in \mathrm{Perv}(K_U)$, we have an isomorphism*

$$\Psi_f^{\mathrm{un}}(L_U \otimes_{K_U} \mathcal{A}) \simeq L_Z \otimes_{K_Z} \Psi_f^{\mathrm{un}}(\mathcal{A}).$$

For $\mathcal{B} \in \mathrm{Perv}(K_X)$, we have an isomorphism

$$\Phi_f^{\mathrm{un}}(L_X \otimes_{K_X} \mathcal{B}) \simeq L_Z \otimes_{K_Z} \Phi_f^{\mathrm{un}}(\mathcal{B}).$$

The scalar extension functor $\Phi_{L/K}\colon \mathrm{Perv}(K_X) \to \mathrm{Perv}(L_X)$ corresponds to the functor

$$\begin{aligned} \mathcal{GD}_f(X,K) &\longrightarrow \mathcal{GD}_f(X,L) \\ (\mathcal{F}_U, \mathcal{F}_Z, u, v) &\longmapsto (L_U \otimes_{K_U} \mathcal{F}_U, L_Z \otimes_{K_Z} \mathcal{F}_Z, \mathrm{id}_{L_Z} \otimes u, \mathrm{id}_{L_Z} \otimes v) \end{aligned}$$

via Beilinson's equivalence F_f.

Proof. The construction of $\Psi_f^{\mathrm{un}}(\mathcal{A})$ is performed as follows: One first defines the nearby cycles functor $\mathrm{R}\varphi_f = i^{-1}\mathrm{R}j_*\mathrm{R}v_*v^{-1}$ (where $v\colon U \times_{\mathbb{C}\setminus\{0\}} \widetilde{\mathbb{C}\setminus\{0\}} \to U$ is the canonical map, with $\widetilde{\mathbb{C}\setminus\{0\}}$ the universal covering; this is, however, not important for what follows). Then one notices that t acts naturally on $\mathrm{R}\varphi_f(\mathcal{A})$ and that there is a decomposition $\mathrm{R}\varphi_f(\mathcal{A}) \simeq \mathrm{R}\varphi_f^{\mathrm{un}}(\mathcal{A}) \oplus \mathrm{R}\varphi_f^{\neq 1}(\mathcal{A})$, where $\mathrm{id} - t$ is nilpotent on the first and an automorphism on the second summand. Then one sets $\Psi_f^{\mathrm{un}}(\mathcal{A}) := \mathrm{R}\varphi_f^{\mathrm{un}}(\mathcal{A})[-1]$.

It is clear that $\mathrm{R}\varphi_f$ commutes with extension of scalars (see Lemma 3.12 and Corollary 4.5). Moreover, the action of t is induced purely topologically, i.e. the action of t on $\mathrm{R}\varphi_f(L_U \otimes_{K_U} \mathcal{A}) \simeq L_Z \otimes_{K_Z} \mathrm{R}\varphi_f(\mathcal{A})$ is induced by the one on $\mathrm{R}\varphi_f(\mathcal{A})$. Hence, the part of $\mathrm{R}\varphi_f(L_U \otimes_{K_U} \mathcal{A})$ on which $\mathrm{id} - t$ is nilpotent will be exactly $L_Z \otimes_{K_Z} \mathrm{R}\varphi_f^{\mathrm{un}}(\mathcal{A})$. This proves the first statement.

The construction of $\Phi_f^{\mathrm{un}}(\mathcal{B})$ is roughly as follows: One first defines the *maximal extension functor* $\Xi_f\colon \mathrm{Perv}(k_U) \to \mathrm{Perv}(k_X)$ and a complex

$$j_! j^{-1}\mathcal{B} \to \Xi_f(j^{-1}\mathcal{B}) \oplus \mathcal{B} \to j_* j^{-1}\mathcal{B}.$$

Then one defines $\Phi_f^{\mathrm{un}}(\mathcal{B})$ as the cohomology of this complex and notes that it is supported on Z.

Without going too much into the details of the construction, let us just mention that the definition of Ξ_f and the morphisms in the above complex is again just topological, i.e. using operations that do not depend upon the exact field of coefficients (such as natural morphisms $j_! \to j_*$, inclusions/projections, kernels etc.). Therefore and due to Lemma 3.12 and Corollary 4.5, the complex associated to $L_X \otimes_{K_X} \mathcal{B}$ is

$$L_X \otimes_{K_X} j_! j^{-1}\mathcal{B} \to L_X \otimes_{K_X} \big(\Xi_f(j^{-1}\mathcal{B}) \oplus \mathcal{B}\big) \to L_X \otimes_{K_X} j_* j^{-1}\mathcal{B}$$

and finally, since extension of scalars is exact and hence commutes with taking cohomology, we get the second isomorphism of the lemma.

For the last statement, the arguments are similar, recognizing that the definition of the maps u and v is topological and can therefore be defined over the smaller field and just "upgraded" to L. $\square$

Now let L/K be a field extension and $G := \mathrm{Aut}(L/K)$ (for our purposes, it will be a finite Galois extension with Galois group G). There is then an obvious G-conjugation on $\mathcal{GD}_f(X, L)$ defined by

$$\overline{(\mathcal{F}_U, \mathcal{F}_Z, u, v)}^g := (\overline{\mathcal{F}_U}^g, \overline{\mathcal{F}_Z}^g, \overline{u}^g, \overline{v}^g)$$

for any $g \in G$, i.e. simply induced by the natural G-conjugations on $\mathrm{Perv}(L_U)$ and $\mathrm{Perv}(L_Z)$.

Lemma 5.8. *For $\mathcal{A} \in \mathrm{Perv}(K_U)$, we have an isomorphism*

$$\Psi_f^{\mathrm{un}}(\overline{\mathcal{A}}^g) \simeq \overline{\Psi_f^{\mathrm{un}}(\mathcal{A})}^g.$$

For $\mathcal{B} \in \mathrm{Perv}(K_X)$, we have an isomorphism

$$\Phi_f^{\mathrm{un}}(\overline{\mathcal{B}}^g) \simeq \overline{\Phi_f^{\mathrm{un}}(\mathcal{B})}^g.$$

Under Beilinson's equivalence $\mathsf{F}_f \colon \mathrm{Perv}(L_X) \xrightarrow{\sim} \mathcal{GD}_f(X, L)$, the natural G-conjugations on both categories correspond to each other, and hence a G-structure on $\mathcal{F} \in \mathrm{Perv}(L_X)$ induces a G-structure on $\mathsf{F}_f(\mathcal{F})$ and vice versa.

Proof. Similarly to the proof of Lemma 5.7, the first statement follows mainly from the fact that conjugation is compatible with direct and inverse image functors. Moreover, since conjugation is an autoequivalence, the endomorphism $\mathrm{id} - t$ is nilpotent if and only if $\mathrm{id} - \overline{t}^g = \overline{\mathrm{id} - t}^g$ is.

The second and third statements are again due to the fact that the whole construction of Φ_f^{un}, u and v do not depend on the field structure and hence are the same if defined before or after applying g-conjugation. □

Application to Galois descent of perverse sheaves We are now ready to prove that the object $\mathcal{G}$ constructed above is actually a K-structure of $\mathcal{F}$.

Proposition 5.9. *The morphism* (6) *is an isomorphism.*

Proof. Let $\mathcal{F} \in \mathrm{Perv}(L_X)$ be a perverse sheaf on a complex analytic variety X and let $(\varphi_g)_{g \in G}$ be a G-structure on it. Let $\mathcal{G} \in \mathrm{Perv}(K_X)$ be the invariant K-perverse subsheaf (defined analogously as above) with its natural morphism $\mathcal{G} \to \mathcal{F}$.

Since $\mathcal{F}$ is perverse, it is in particular a complex of sheaves with $\mathbb{C}$-constructible cohomologies. For each of the (finitely many) cohomology sheaves $\mathrm{H}^i(\mathcal{F})$, there exists a locally finite covering $X = \bigcup_\alpha X_\alpha^i$ by $\mathbb{C}$-analytic subsets[2] on which $\mathrm{H}^i(\mathcal{F})$ is locally constant. Moreover, the problem is local (and restriction to an open subset is exact), so we can assume that the set of all X_α^i is finite.

If all the X_α^i are of maximal dimension, this means that every cohomology sheaf if locally constant on X. By the definition and basic properties of perverse sheaves, $\mathrm{H}^i(\mathcal{F}) = 0$ for $i < -\dim X$ and $\dim \operatorname{supp} \mathrm{H}^{-i}(\mathcal{F}) \leq i$ for any $i \in \mathbb{Z}$. This implies that $\mathcal{F}$ is concentrated in cohomological degree $-\dim X$, i.e. $\mathcal{F} \simeq \mathcal{L}[\dim U]$ for some locally constant sheaf $\mathcal{L} \in \mathrm{Mod}(L_X)$. Hence, the statement follows from Proposition 5.1 and we are done.

Now, assume that there exist X_α^i of non-maximal dimension. Then each of the X_α^i not having maximal dimension is contained in the zero locus of an analytic function that is not identically zero (since $\overline{X_\alpha^i}$ is analytic and not equal to the whole space), yielding a finite family of functions $(f_k)_{k \in I}$, $I = \{1, \ldots, m\}$. We can multiply these functions to obtain $f := f_1 \cdot \ldots \cdot f_k$ whose zero locus contains all the X_α^i of non-maximal dimension.

[2]We follow the terminology in [KS90] here: A subset $Y \subset X$ is called complex analytic if for any point $x \in X$ there exists an open neighbourhood $U \subset X$ of x and (finitely many) holomorphic functions $f_1, \ldots, f_n \in \mathcal{O}_X(U)$ such that $A \cap U = \{f_1 = \ldots = f_n = 0\}$. Moreover, Y is called $\mathbb{C}$-analytic if $\overline{Y}$ and $\overline{Y} \setminus Y$ are complex analytic subsets of X.

Consider now $Z := f^{-1}(0)$, $U := X \setminus Z$ and the inclusion $j\colon U \hookrightarrow X$. By Beilinson's equivalence, the datum of $\mathcal{F}$ is equivalent to the tuple

$$(j^{-1}\mathcal{F}, \Phi_f^{\mathrm{un}}(\mathcal{F}), u, v) \in \mathcal{GD}_f(X, L).$$

By Lemma 5.8, it still comes equipped with a G-structure, which we denote by $(\widetilde{\varphi}_g)_{g\in G}$.

Accordingly, $\mathcal{G}$ corresponds to a tuple

$$(j^{-1}\mathcal{G}, \Phi_f^{\mathrm{un}}(\mathcal{G}), u_K, v_K) \in \mathcal{GD}_f(X, K),$$

Due to the compatibility of Beilinson's equivalence with forgetful functors (restriction of scalars) $\mathsf{for}_{L/K}$ and kernels (equivalences of abelian categories are exact), we see that this object is actually the kernel of

$$\prod_{g\in G}(\widetilde{\varphi}_g - \mathrm{id})\colon \mathsf{F}_f(\mathcal{F})^K \to \prod_{g\in G}\mathsf{F}_f(\mathcal{F})^K$$

in the category $\mathcal{GD}_f(X, K)$, and the morphism induced by $L_X \otimes_{K_X} \mathcal{G} \to \mathcal{F}$ via F_f is nothing but the natural morphism

$$(L_U \otimes_{K_U} j^{-1}\mathcal{G}, L_Z \otimes_{K_Z} \Phi_f^{\mathrm{un}}(\mathcal{G}), \mathrm{id}_{L_Z} \otimes u_K, \mathrm{id}_{L_Z} \otimes v_K) \to (j^{-1}\mathcal{F}, \Phi_f^{\mathrm{un}}(\mathcal{F}), u, v)$$

by Lemma 5.7. To prove that it is an isomorphism, we need to prove that $L_U \otimes_{K_U} j^{-1}\mathcal{G} \to j^{-1}\mathcal{F}$ and $L_Z \otimes_{K_Z} \Phi_f^{\mathrm{un}}(\mathcal{G}) \to \Phi_f^{\mathrm{un}}(\mathcal{F})$ are isomorphisms, where $j^{-1}\mathcal{G}$ (resp. $\Phi_f^{\mathrm{un}}(\mathcal{G})$) is the perverse sheaf of invariants of the induced G-structure on $j^{-1}\mathcal{F}$ (resp. $\Phi_f^{\mathrm{un}}(\mathcal{F})$), since j^{-1} (resp. Φ_f^{un}) is exact and hence commutes with kernels.

For the first isomorphism, note that $j^{-1}\mathcal{F}$ is a complex of sheaves on U whose cohomologies are all locally constant L_U-modules of finite rank. Then, with the same arguments as above, $j^{-1}\mathcal{F} \simeq \mathcal{L}[\dim U]$ for some locally constant sheaf $\mathcal{L} \in \mathrm{Mod}(L_U)$. Hence, the desired isomorphism follows from Proposition 5.1.

For the second isomorphism, note that $\Phi_f^{\mathrm{un}}(\mathcal{F})$ is a perverse sheaf on the complex analytic variety Z and $\dim Z = \dim X - 1$. Hence, we can apply the same technique (determining, at least locally, a suitable covering of Z, choosing a suitable function that vanishes on all the sets of non-maximal dimension and applying Beilinson's equivalence) to this perverse sheaf. We continue this recursively, and the procedure will end if all elements of the covering are of maximal dimension, which will be the case at the latest when $\dim Z = 0$, which shows that this inductive procedure terminates. This concludes the proof. $\square$

The statement just proved shows the essential surjectivity part of Galois descent for perverse sheaves. Full faithfulness is inherited from the derived category (Proposition 5.3). We have therefore proved the following statement. (We will formulate in a slightly more general setting than we did in the rest of this work, namely in the context of complex analytic varieties, since this is what we have actually proved.)

Theorem 5.10. *Let X be a complex analytic variety. Then the functor of extension of scalars $\Phi_{L/K}\colon \mathrm{Perv}(K_X) \to \mathrm{Perv}(L_X)$ induces an equivalence*

$$\Phi_{L/K}^G\colon \mathrm{Perv}(K_X) \to \mathrm{Perv}(L_X)^G.$$

References

[BHHS22] D. Barco, M. Hien, A. Hohl and C. Sevenheck, *Betti Structures of Hypergeometric Equations*, Int. Math. Res. Not. (2022), rnac095.

[BGH18] C. Barwick, S. Glasman and P. J. Haine, *Exodromy*, Preprint (2020), `arXiv:1807.03281v7`.

[Bei87] A. A. Beilinson, *How to glue perverse sheaves*, in: Y. I. Manin (Ed.), *K-Theory, Arithmetic and Geometry*, Lecture Notes in Math., vol. 1289, Springer, Berlin 1987.

[BBD82] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982).

[Bor99] A. Borel, *Algebraic Groups*, Grad. Texts in Math., vol. 126, Springer Science+Business Media, New York 1999.

[Con] K. Conrad, *Galois descent*, Lecture notes, available at `https://kconrad.math.uconn.edu/blurbs/galoistheory/galoisdescent.pdf`

[Dim04] A. Dimca, *Sheaves in Topology*, Universitext, Springer, Berlin 2004.

[Jac62] N. Jacobson, *Lie Algebras*, Interscience Publishers, London 1962.

[Jah00] J. Jahnel, *The Brauer-Severi variety associated with a central simple algebra: a survey*, Preprint (2000), available at `https://www.math.uni-bielefeld.de/lag/man/052.pdf`

[Kas84] M. Kashiwara, *The Riemann-Hilbert Problem for Holonomic Systems*, Publ. Res. Inst. Math. Sci. **20** (1984), 319–365.

[KS90] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren Math. Wiss., vol. 292, Springer, Berlin 1990.

[Lur17] J. Lurie, *Higher Algebra*, Preprint (2017), available at `https://www.math.ias.edu/~lurie/papers/HA.pdf`

[PT22] M. Porta and J.-B. Teyssier, *Topological exodromy with coefficients*, Preprint (2022), `arXiv:2211.05004`.

[Rei10] R. Reich, *Notes on Beilinson's "How to glue perverse sheaves"*, J. Singul. **1** (2010), 94–115.

[Stacks] The Stacks Project Authors, *Stacks Project*, 2023. `https://stacks.math.columbia.edu`

[Tre09] D. Treumann, *Exit paths and constructible stacks*, Compos. Math. **145** (2009), 1504–1532.

[Wat79] W. C. Waterhouse, *Introduction to Affine Group Schemes*, Grad. Texts in Math., vol. 66, Springer, New York 1979.

[Win74] D. J. Winter, *The Structure of Fields*, Grad. Texts in Math., vol. 16, Springer, New York 1974.

Andreas Hohl
Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France
andreas.hohl@imj-prg.fr