Inrobin
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Experiment two:

Detecting Spectral Content

Set the notation of this experiment as follows:
Ω is the set of M_{true} vectors of parameters where each vector is J dimensional
$\Omega = \left\{\Psi^1, \ldots, \Psi^{M_{true}}\right\}$
Let $\Psi^j = [\Psi^j_1, \ldots, \Psi^j_J]$ which is the jth element of Ω set
Ψ^j_h is hth element of Ψ^j
$k(\cdot, \cdot; \Psi)$ a kernel function from parameterized by the parameter Ψ, $k : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$
Let $\mathbf{t}$ be a $1 \times N$ vector of real numbers
$\mathbf{K} = k(\mathbf{t}, \mathbf{t}; \Psi = \Psi)$
$j \in \left\{1, \cdots, M_{true}\right\}$, $m \in \left\{1, \cdots, M\right\}$, $h \in \left\{1, \ldots, J\right\}$
$n_{j,m}$ number of IMFs

Algorithm 1: Algorithm

Input: Define $k, J, \Omega, \mathbf{t}, M_{true}$
Set m, j, J

1. Evaluate each Gram Matrix $\mathbf{K}^{(j)}_h = k(\mathbf{t}, \mathbf{t}; \Psi^{(j)}_h)$ parametrized by the jth parameter from Ω. There are J Gram Matrices since $h = 1, \ldots, J$
2. Simulate $\mathbf{y}^{(m)}_{j,h} \sim \mathcal{N}(0, \mathbf{K}^{(j)}_h)$ an N dimensional vector for each Gram Matrix $\mathbf{K}^{(j)}_h$, therefore we have J one dimensional vectors of length N.
3. Compute $\mathbf{x}^{(m)}_j = \sum_{h=1}^{J} \mathbf{y}^{(m)}_{j,h}$
4. Fit a spline through each $\mathbf{x}^{(m)}_j$ denoted as $\hat{\mathbf{x}}^{(m)}_j$.
5. Apply the EMD to $\hat{\mathbf{x}}^{(m)}_j$ to get the IMFs decomposition and collect all the IMFs generated up to the stopping criterion chosen denoted as $\boldsymbol{\gamma}^{(m)}_{j,1}, \boldsymbol{\gamma}^{(m)}_{j,2}, \ldots, \boldsymbol{\gamma}^{(m)}_{j,v_{j,m}}$. For each j and for each m we might have different number of IMFs denote by $v_{j,m}$.
6. Compute the Instantaneous Frequency of each IMF denoted as $\boldsymbol{f}^{(m)}_{j,1}, \boldsymbol{f}^{(m)}_{j,2}, \ldots, \boldsymbol{f}^{(m)}_{j,v_{j,m}}$.
7. Compare the frequencies with Spectral Component of the kernels

Algorithm 2: Algorithm

Input: Define $k, J, \Omega, \mathbf{t}, M_{true}$

for ($m = 1 : M$) {

  for ($h = 1 : J$) {

    Evaluate each Gram Matrix $\mathbf{K}_h^{(j)} = k(\mathbf{t}, \mathbf{t}; \Psi_h^{(j)})$ parametrized by the jth parameter from Ω. There are J Gram Matrices since $h = 1, \ldots, J$

    Simulate $\mathbf{y}_{j,h}^{(m)} \sim \mathcal{N}(0, \mathbf{K}_h^{(j)})$ an N dimensional vector for each Gram Matrix $\mathbf{K}_h^{(j)}$, therefore we have J one dimensional vectors of length N.

    Compute $\mathbf{x}_j^{(m)} = \sum_{h=1}^J \mathbf{y}_{j,h}^{(m)}$

  Fit a spline through each $\mathbf{x}_j^{(m)}$ denoted as $\hat{\mathbf{x}}_j^{(m)}$.

  Apply the EMD to $\hat{\mathbf{x}}_j^{(m)}$ to get the IMFs decomposition and collect all the IMFs generated up to the stopping criterion chosen denoted as $\boldsymbol{\gamma}_{j,1}^{(m)}, \boldsymbol{\gamma}_{j,2}^{(m)}, \ldots, \boldsymbol{\gamma}_{j,v_{j,m}}^{(m)}$. For each j and for each m we might have different number of IMFs denote by $v_{j,m}$.

  Compute the Instantaneous Frequency of each IMF denoted as $\boldsymbol{f}_{j,1}^{(m)}, \boldsymbol{f}_{j,2}^{(m)}, \ldots, \boldsymbol{f}_{j,v_{j,m}}^{(m)}$.

  Compare the frequencies with Spectral Component of the kernels

Example: Periodic Kernel: $\Omega = \{(l_1, p_1), (l_1, p_2), (l_1, p_3), (l_2, p_1), (l_2, p_2), (l_2, p_3)\}$, $j = 2$, $h = 1 : 6$, $J = 6$, $\Psi^j = [\Psi_1^j, \ldots, \Psi_6^j]$ and $\Psi_1^j = \{l_1, p_1\}$, ..., $\Psi_6^j = \{l_2, p_3\}$.

Stopping Criteria for the sifting procedure

- **Cauchy-Type Convergence (SD)**. [?] proposed such criterion to limit amplitude and frequency modulations through a certain threshold of the standard deviation of two consecutive sifting results.

$$SD = \sum_{t=0}^{T} \left[\frac{|\left(h_{1(k-1)}(t) - h_{1k}(t)\right)|^2}{h_{1(k-1)}^2(t)} \right] \tag{1}$$

- **Mean Fluctuations Threshold**. [?] proposed three different thresholds allowing for small fluctuations of the mean and locally large excursions. The thresholds are α, θ_1 and θ_2. For $(1-\alpha)$ data, the criterion keeps sifting if $\sigma(t) < \theta_1$, while for the remaining fraction if $\sigma(t) < \theta_2$. The quantities are defines as:

$$\sigma(t) := \left|\frac{m(t)}{a(t)}\right| \qquad a(t) := \frac{S(e_1^S) - S(e_2^S)}{2} \qquad m(t) := \frac{S(e_1^S) + S(e_2^S)}{2}$$

3

- **Energy Difference Tracking**. The assumption of this criterion is that the EMD provides IMFs and a residual mutually orthogonal. If c_1 is extracted from the signal, the sum of its energy and those of the residual signal (E_{tot}) is equal to the energy of the original signal (E_S). If not, there must be a difference called energy error (E_{err}). The sifting procedure stops when both E_{err} and the mean value are small enough. E_{err} for c_1 is defined as:

$$E_{err} = E_{tot} - E_S = \int c_1^2(t)dt - \int S(t)c_1(t)dt \tag{2}$$

- **Orthogonality Criterion**. [?] underline the fact that any IMF, in this case c_1 should satisfy:

$$\sum_{t=1}^{N} c_1(t)\left(x(t) - c_1(t)\right) = 0 \tag{3}$$

which states the orthogonality of the IMFs. They determine the following index and the sifting procedure stops once that it reaches a certain minimum value:

$$OC = \left| \sum_{t=1}^{N} \frac{m\,(t)\,S(t)}{m\,(t)\,(S(t) - m\,(t))} \right| \quad \text{where} \quad m\,(t) = \frac{S(e_1^S) + S(e_2^S)}{2} \tag{4}$$

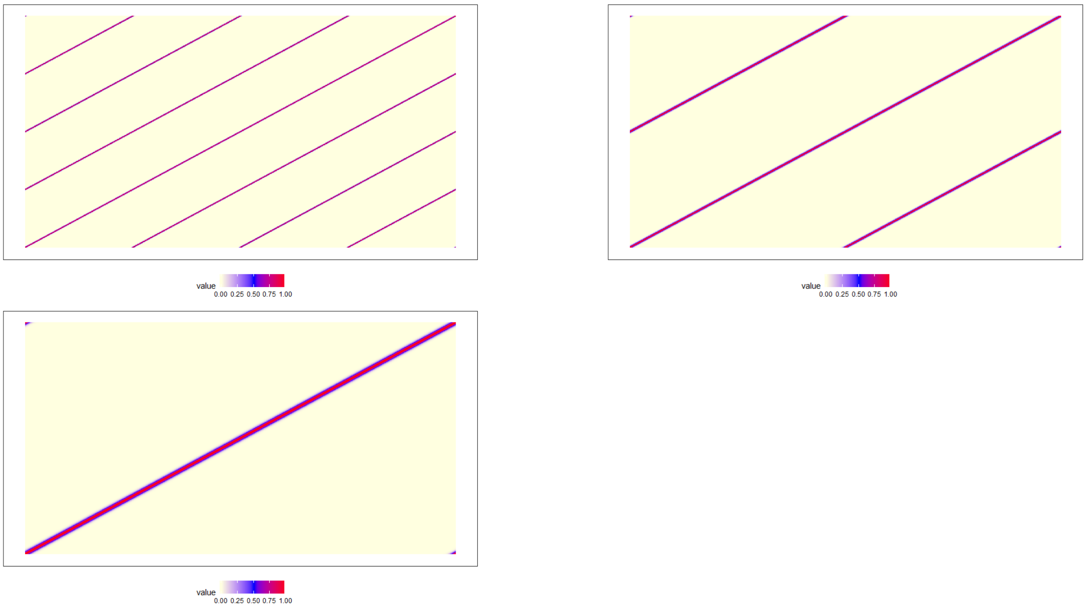


Figure 1: Heatmaps of Gram Matrices for the periodic Kernel with $N = 1000, l = 0.05, p = 0.25, 0.5, 1$

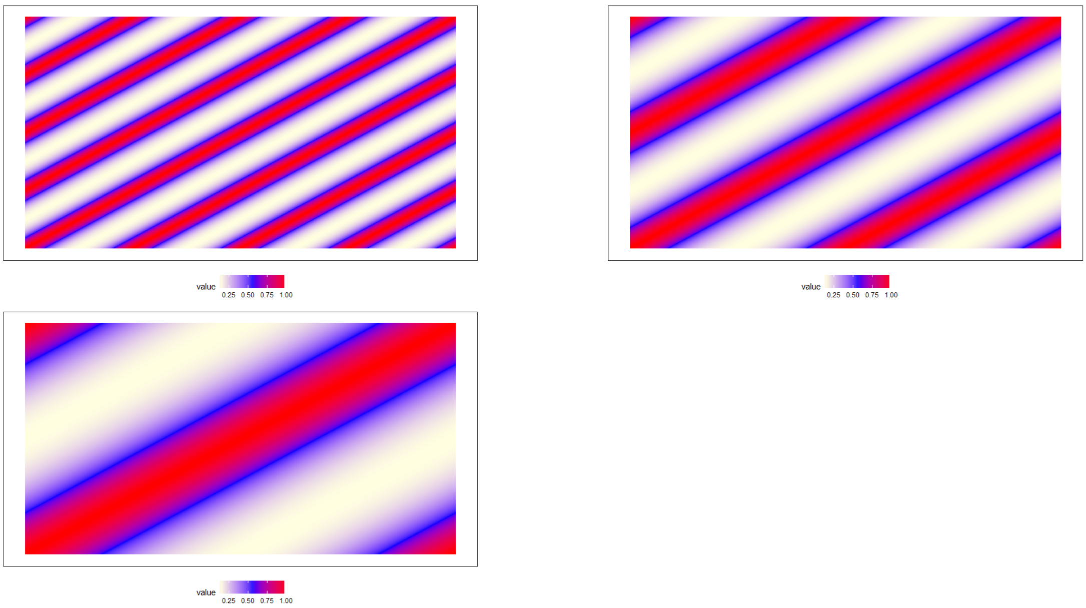


Figure 2: Heatmaps of Gram Matrices for the periodic Kernel with $N = 1000, l = 1, p = 0.25, 0.5, 1$

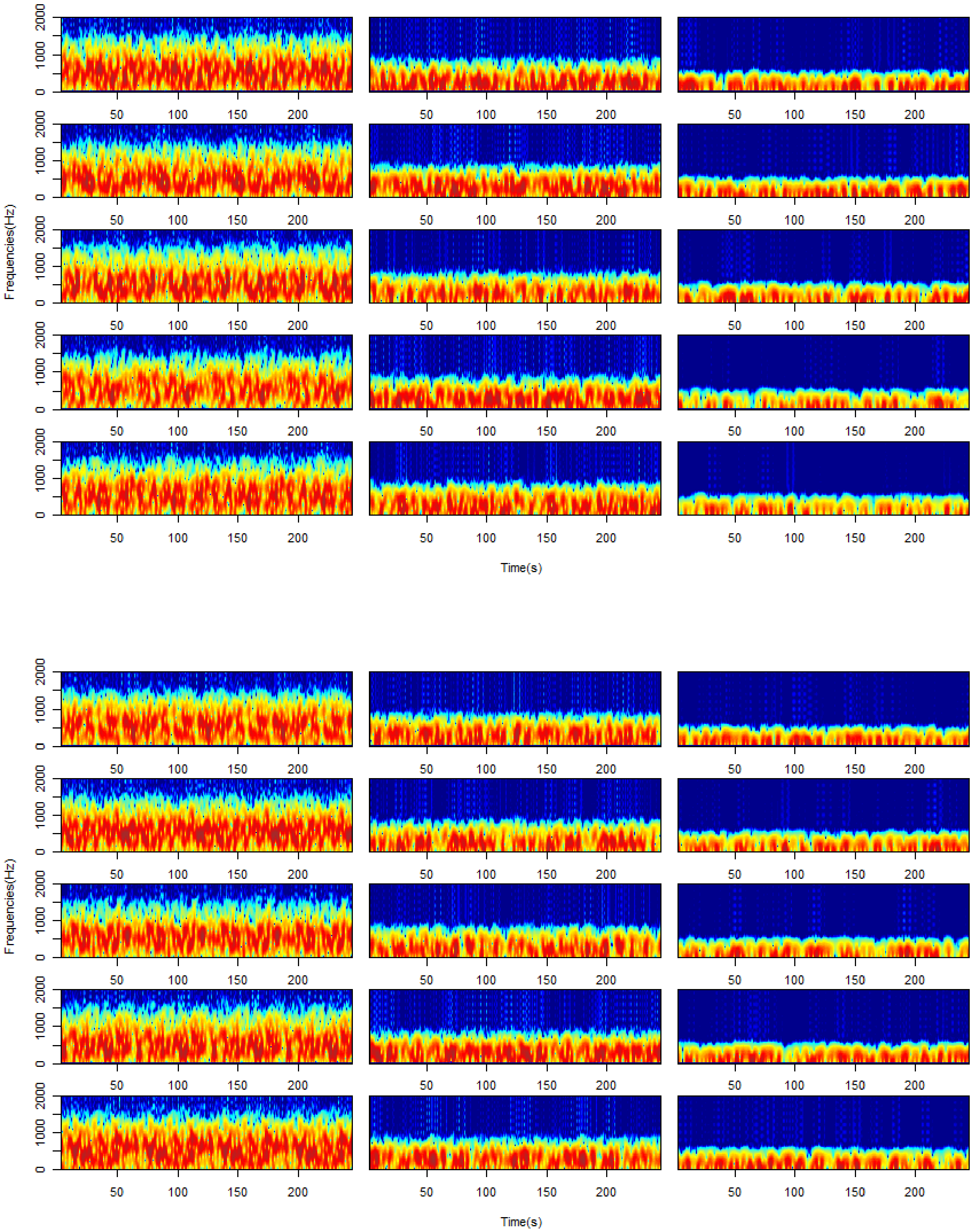


Figure 3: $N = 1000, l = 0.05$. Spectrograms of the $y^{(m)}$ constructed with the above matrices for $m = 1, \ldots, 10$.

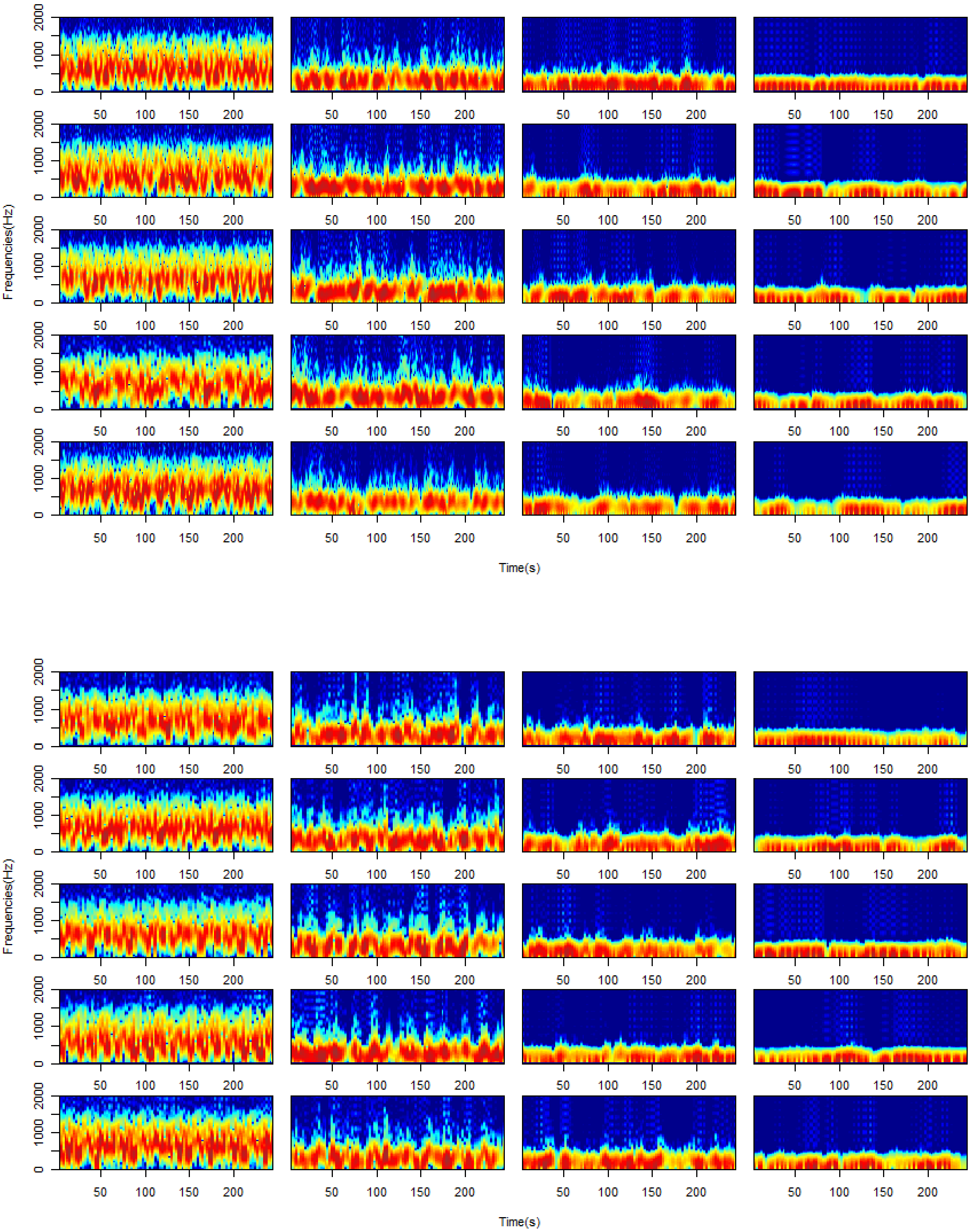


Figure 4: $N = 1000, l = 0.05, p = 0.25, 0.5, 1$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$ constructed with the above y_m for $m = 1, \ldots, 10$.

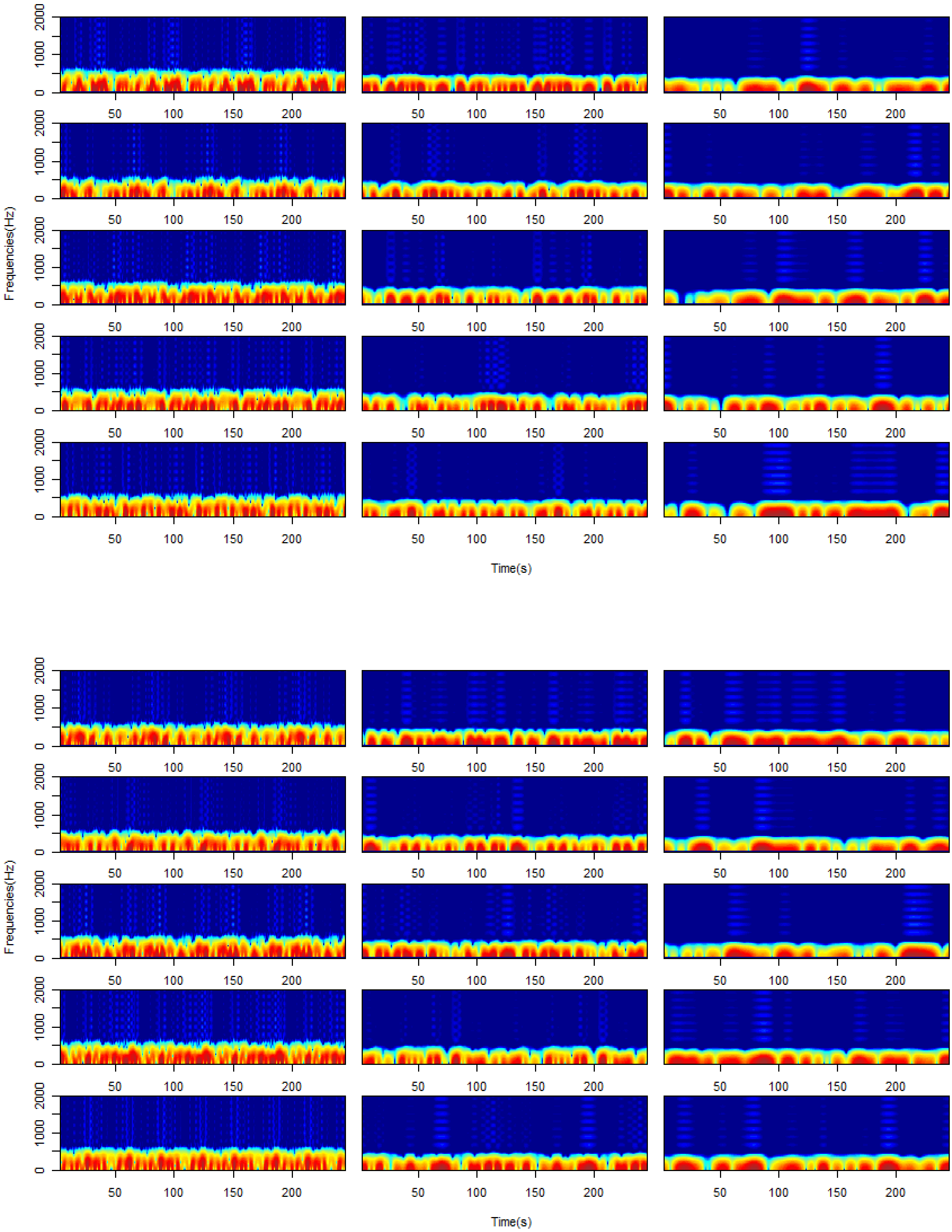


Figure 5: $N = 1000, l = 0.2, p = 0.25, 0.5, 1$. Spectrograms of the $y^{(m)}$ components constructed with the above matrices for $m = 1, \ldots, 10$.

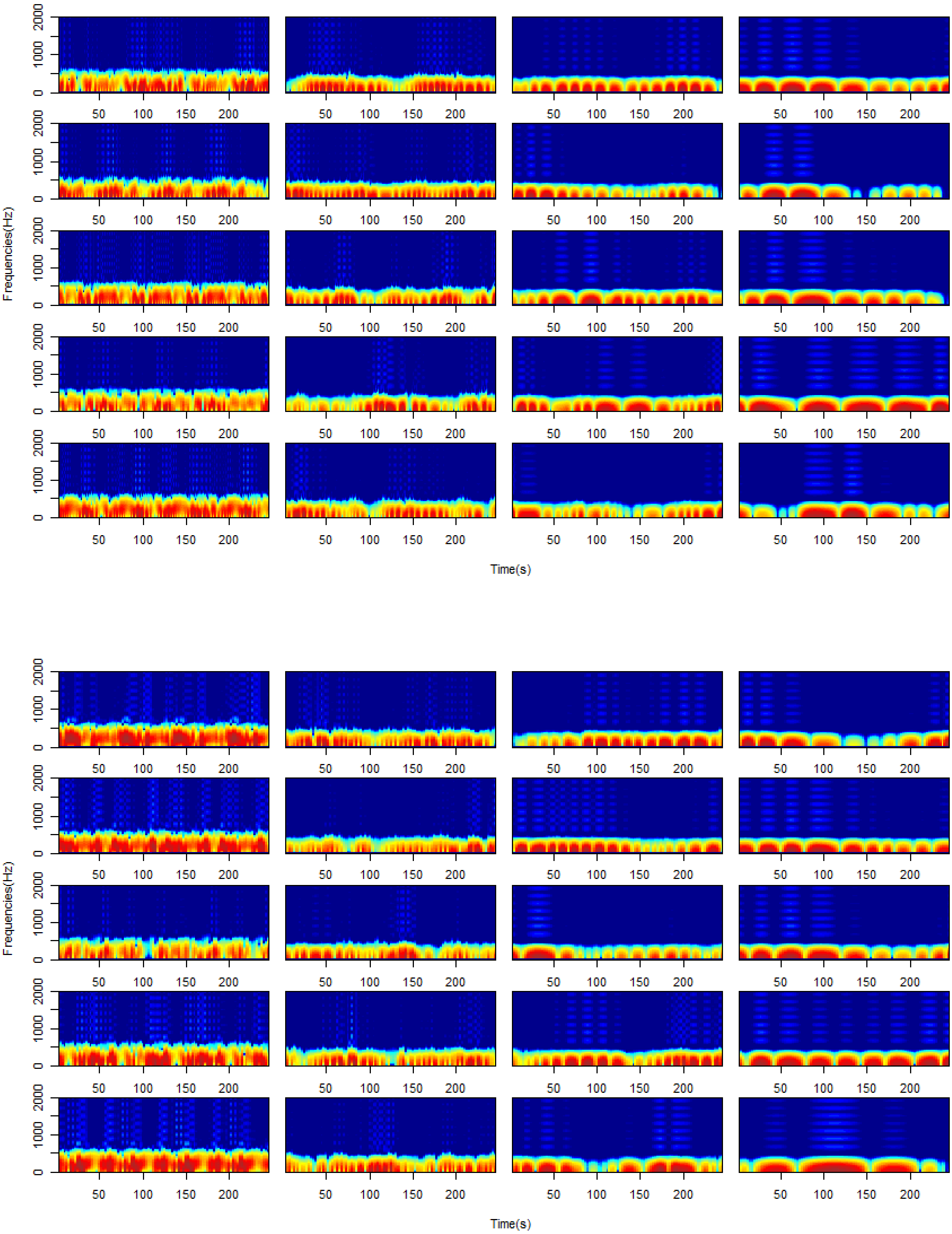


Figure 6: $N = 1000, l = 0.2, p = 0.25, 0.5, 1$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$ constructed with the above $y^{(m)}$ for $m = 1, \ldots, 10$.

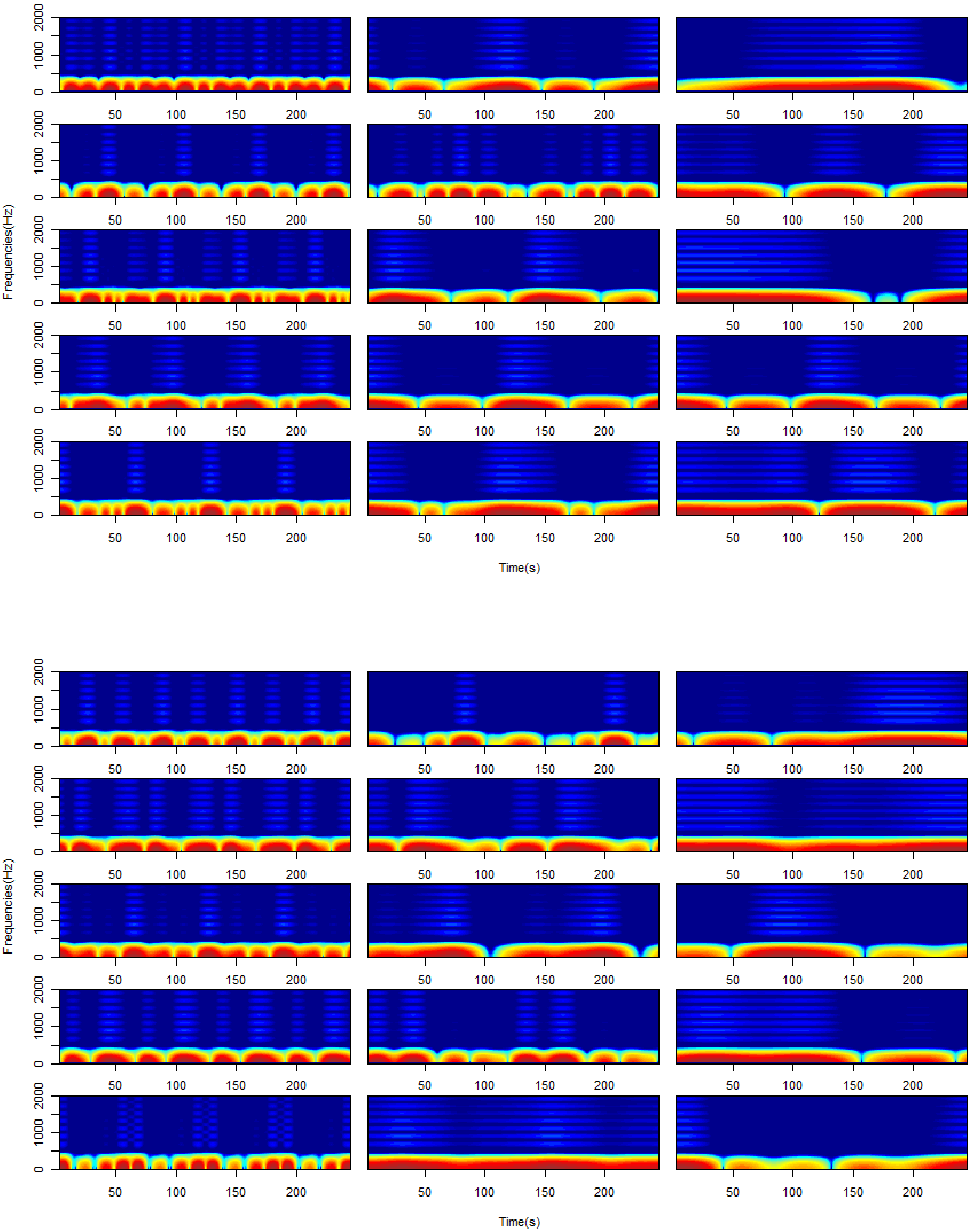
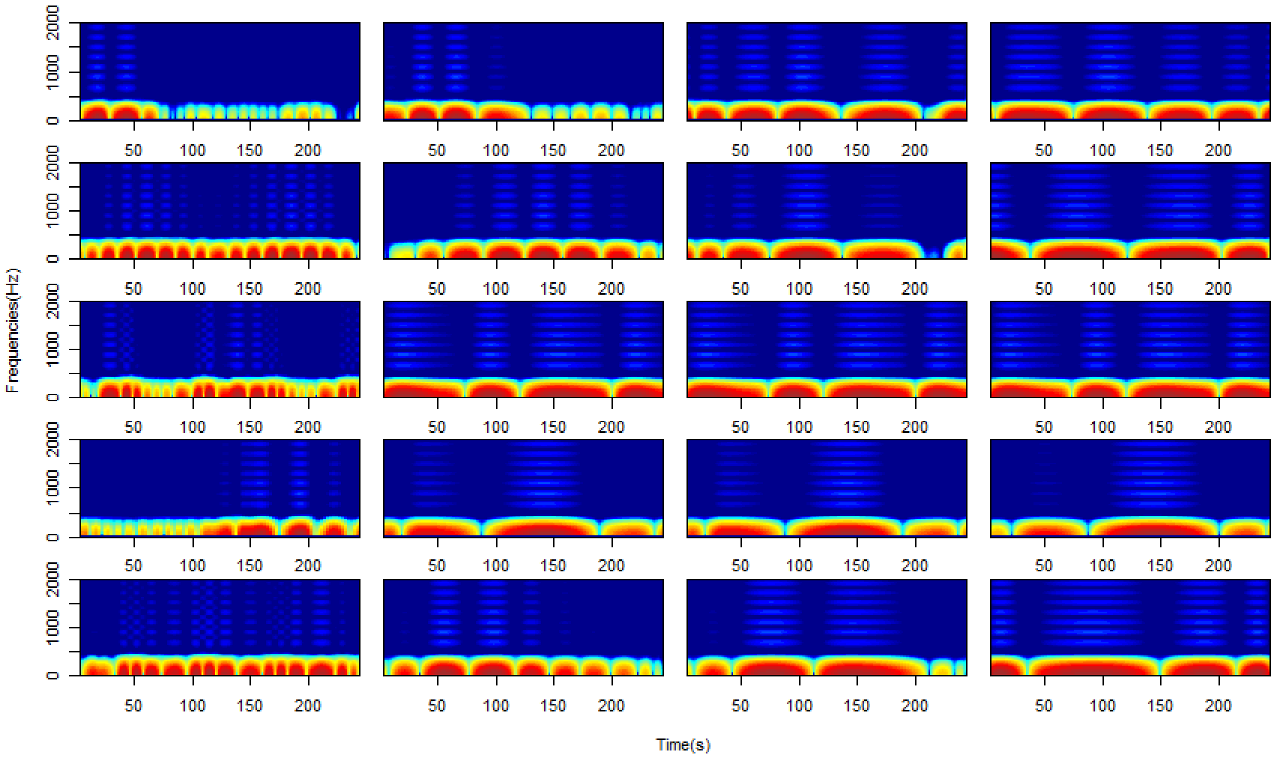


Figure 7: $N = 1000, l = 1, p = 0.25, 0.5, 1$. Spectrograms of the $y^{(m)}$ components constructed with the above matrices for $m = 1, \ldots, 10$



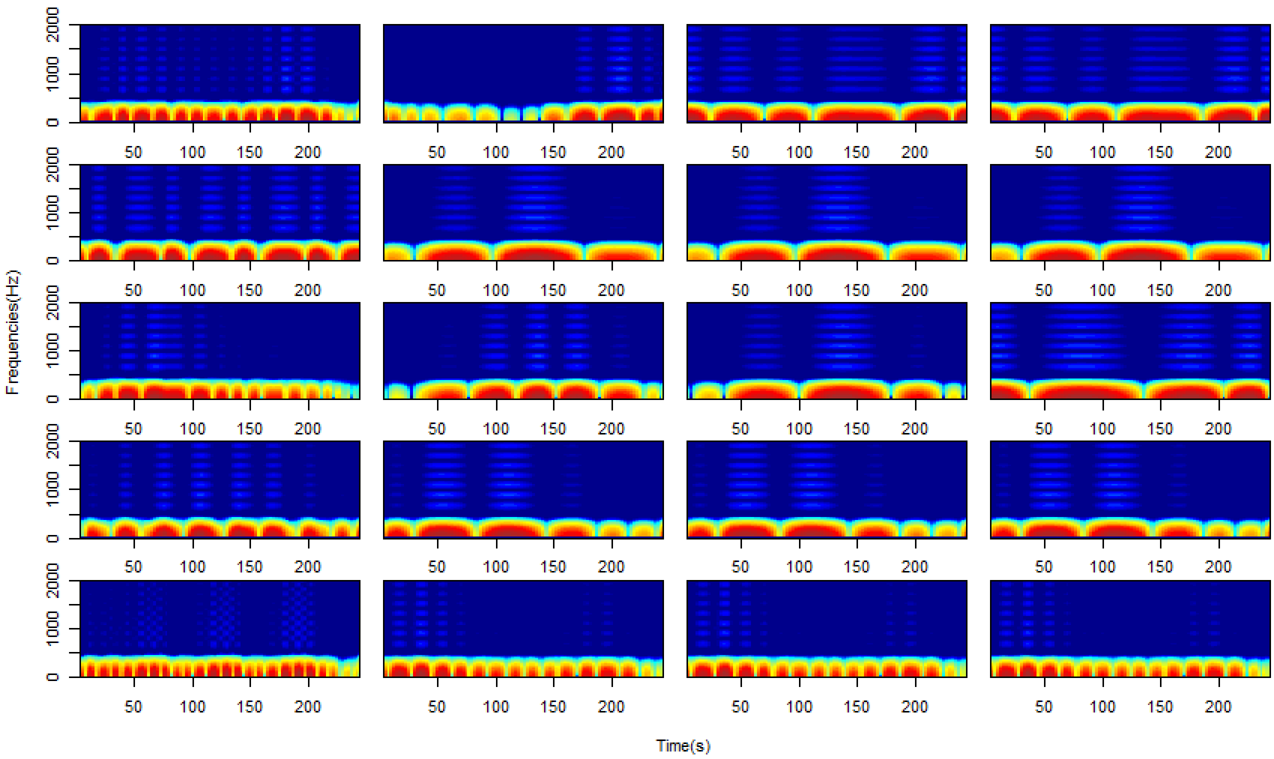
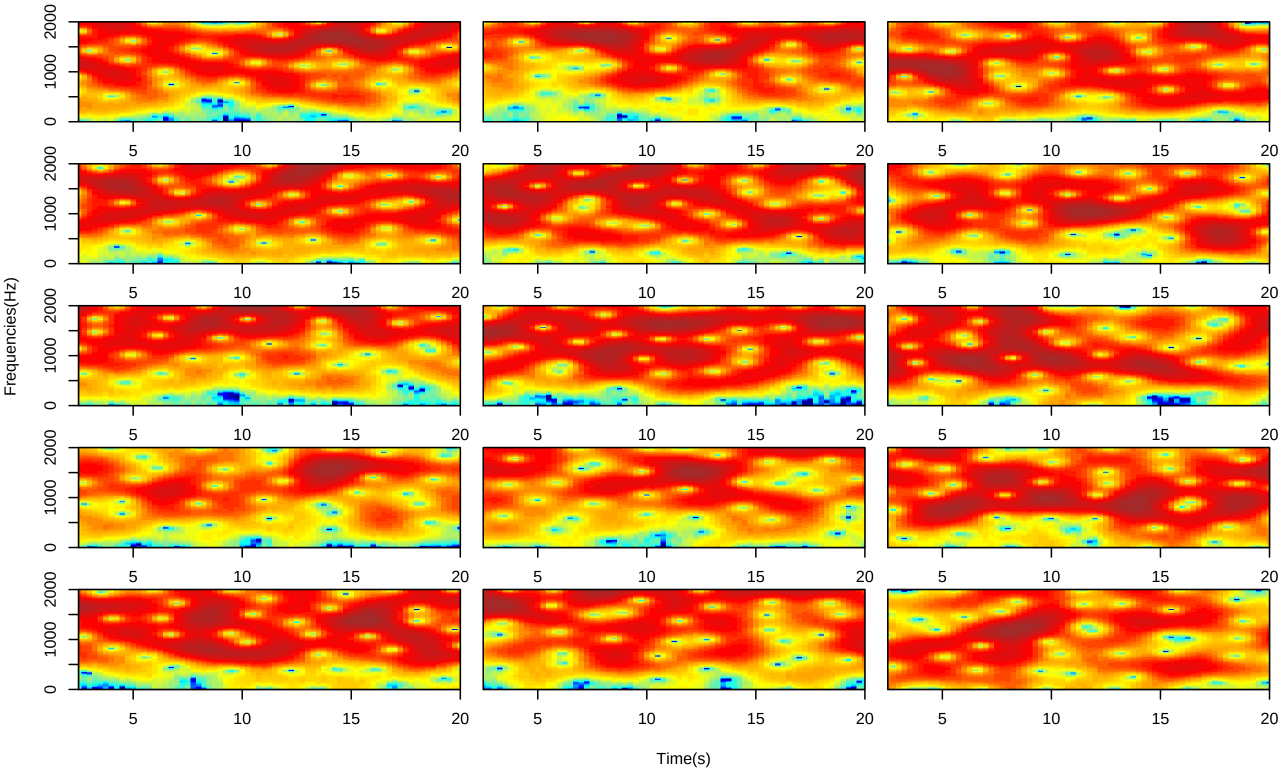


Figure 8: $N = 1000, l = 1, p = 0.25, 0.5, 1,\ m = 1, \cdots = 10$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$ constructed with the above $y^{(m)}$.



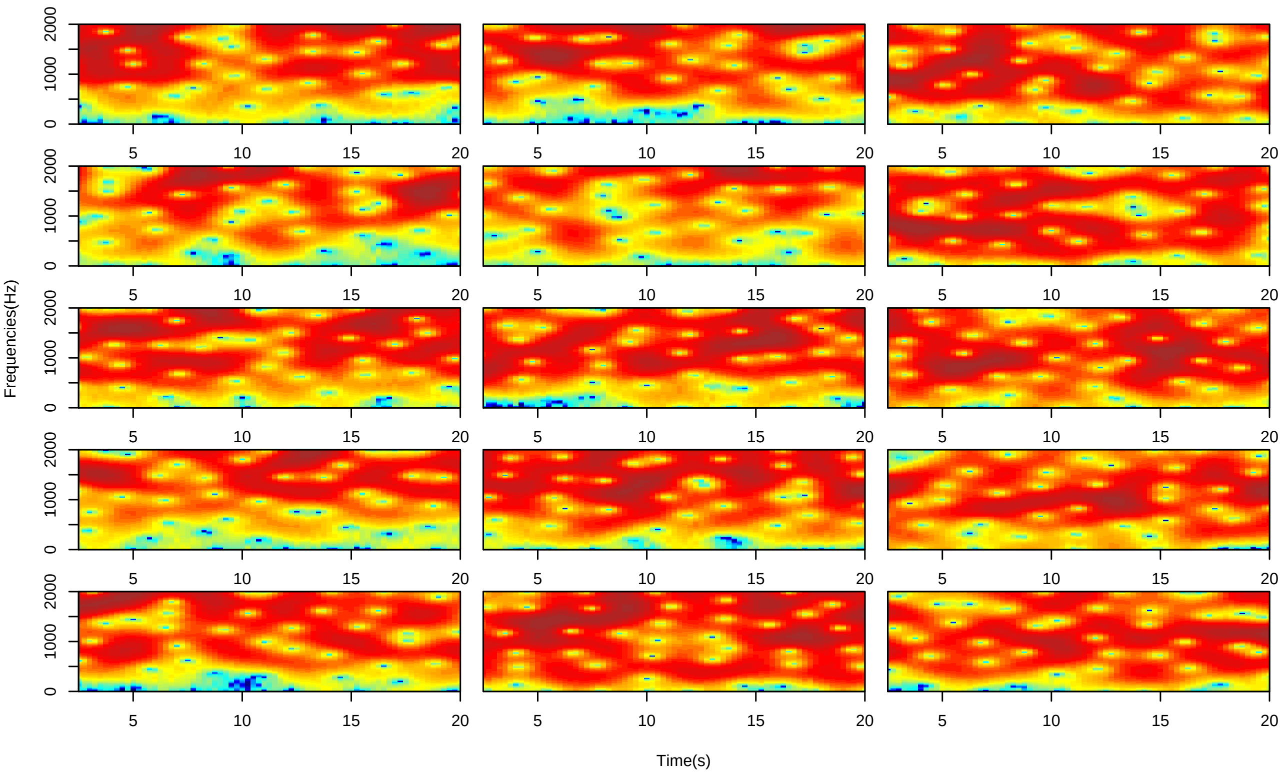


Figure 9: $N = 100$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Spectrograms of the $y^{(m)}$ components.

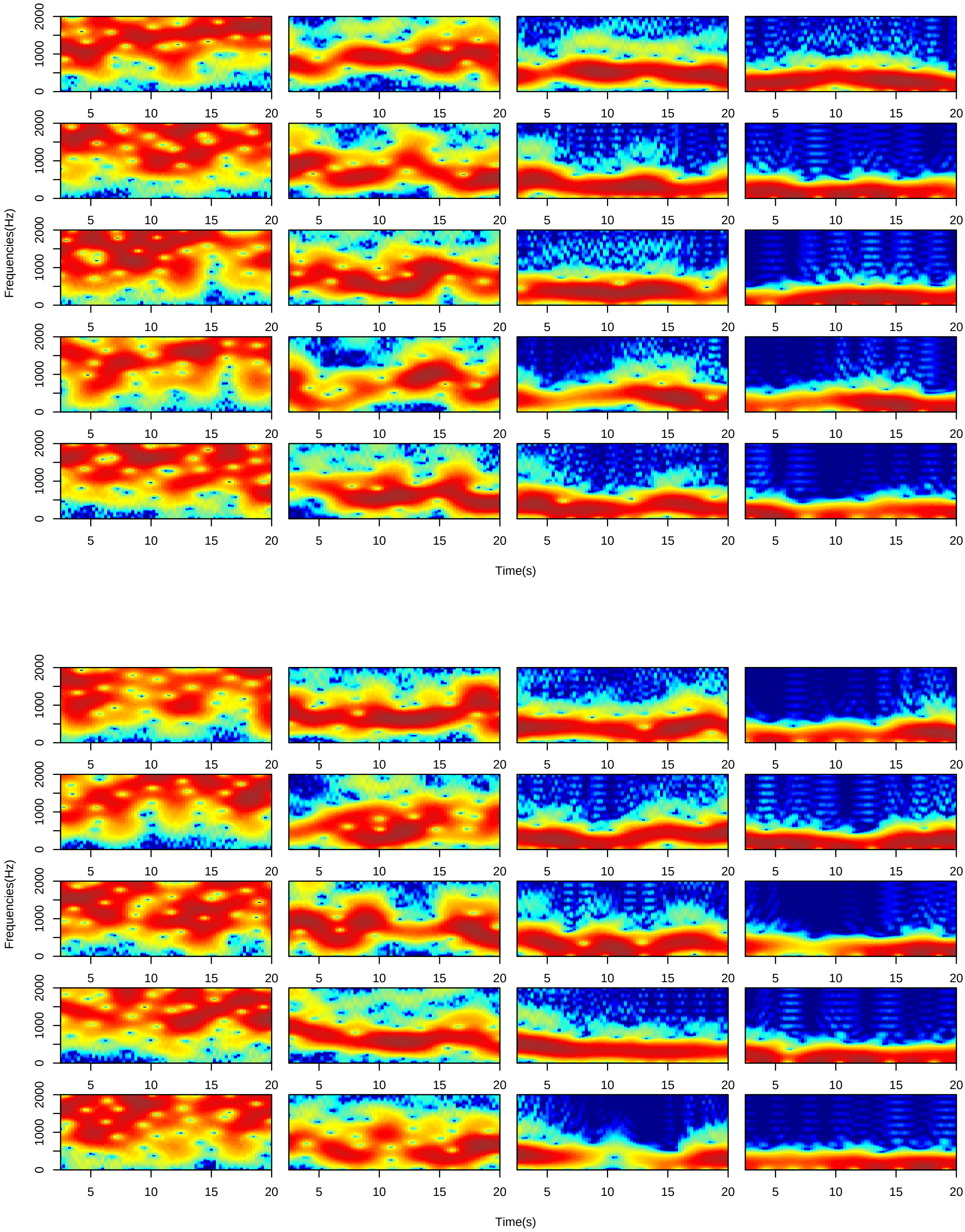


Figure 10: $N = 100$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$.

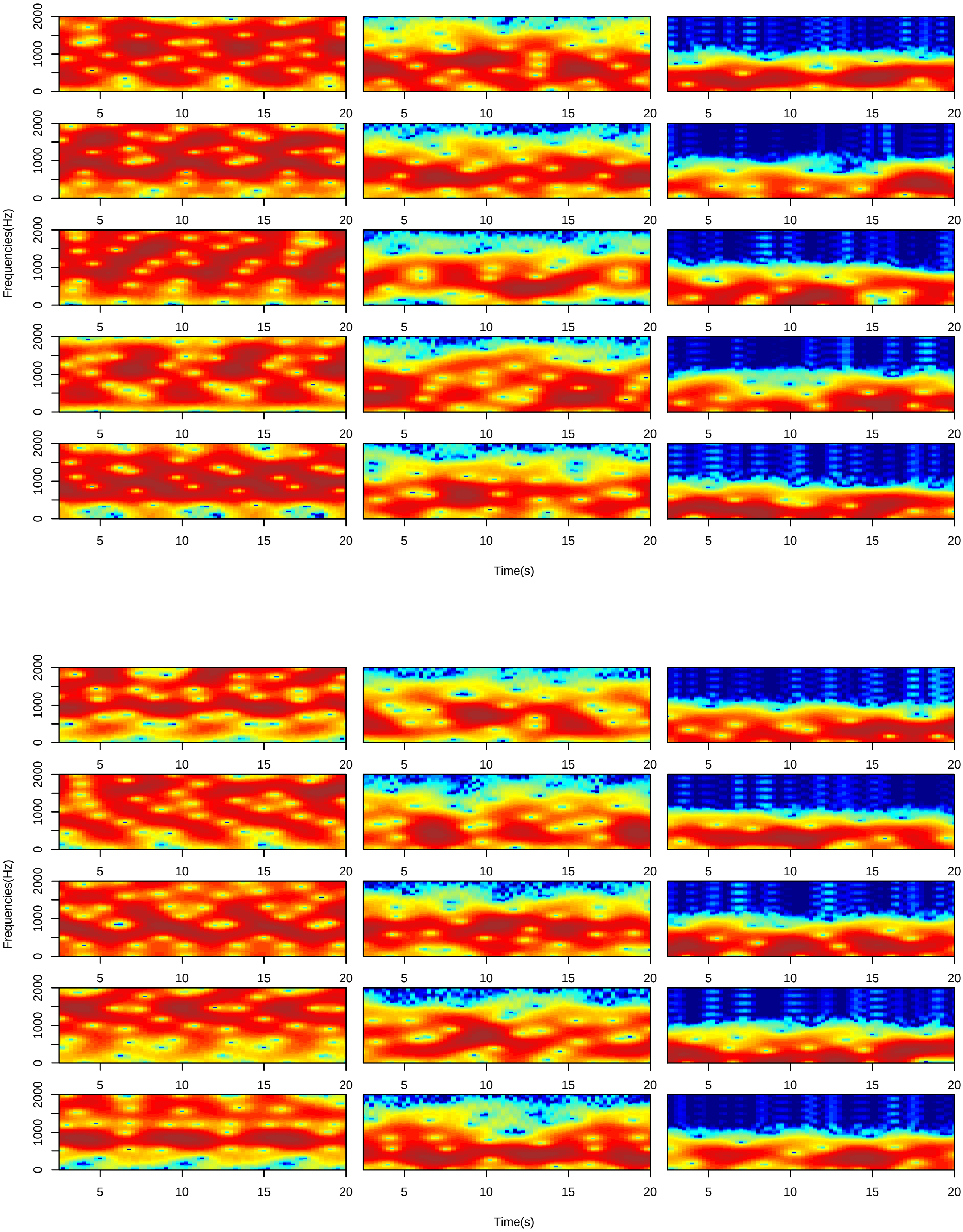


Figure 11: $N = 100$, $l = 0.2$, $p = 0.25, 0.5, 1.00$. Spectrograms of the $y^{(m)}$ components.

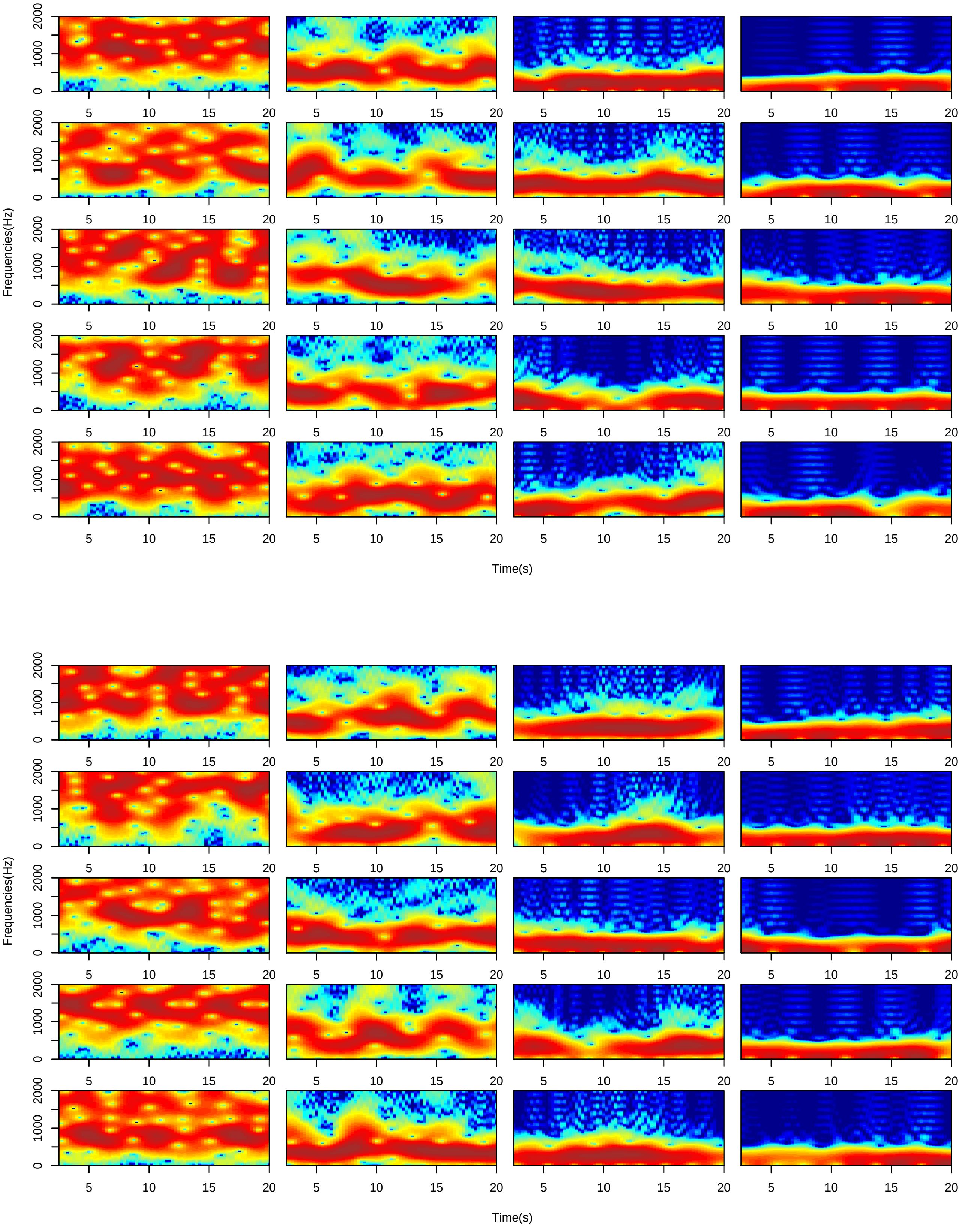
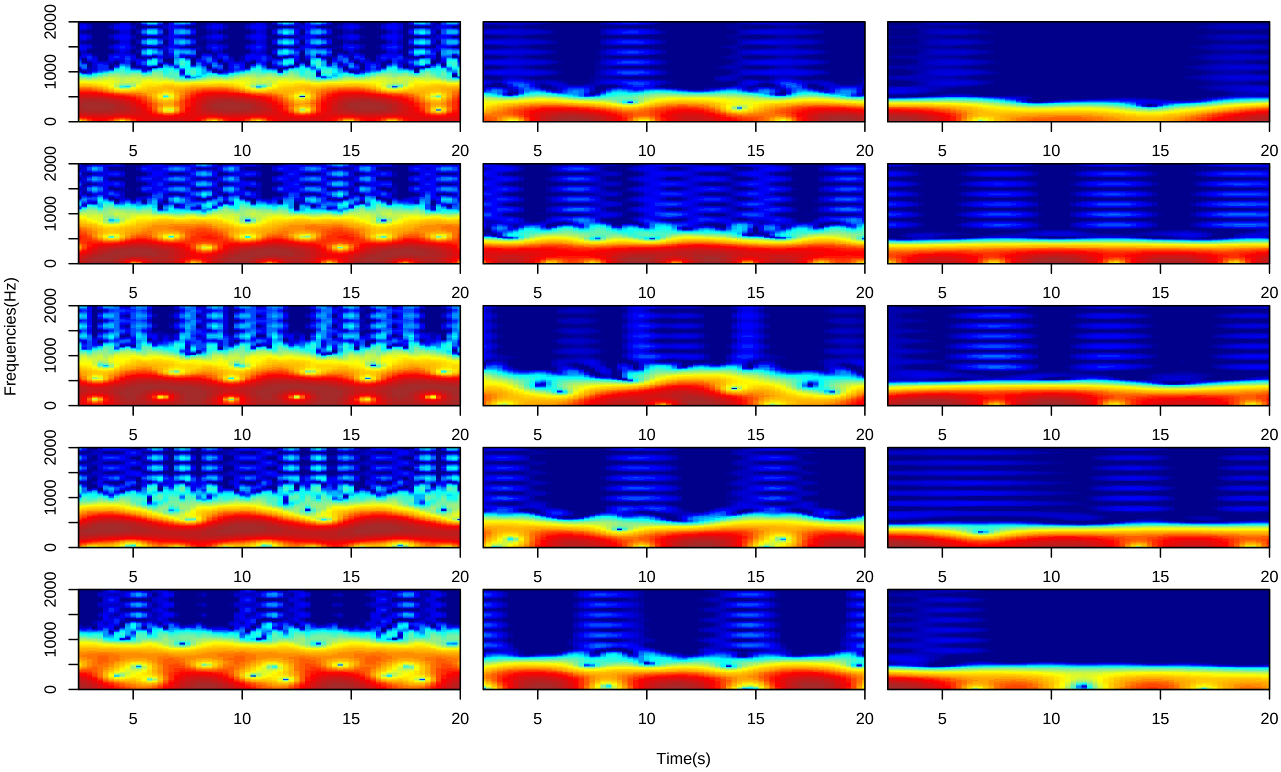


Figure 12: $N = 100$, $l = 0.2$, $p = 0.25, 0.5, 1.00$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$.



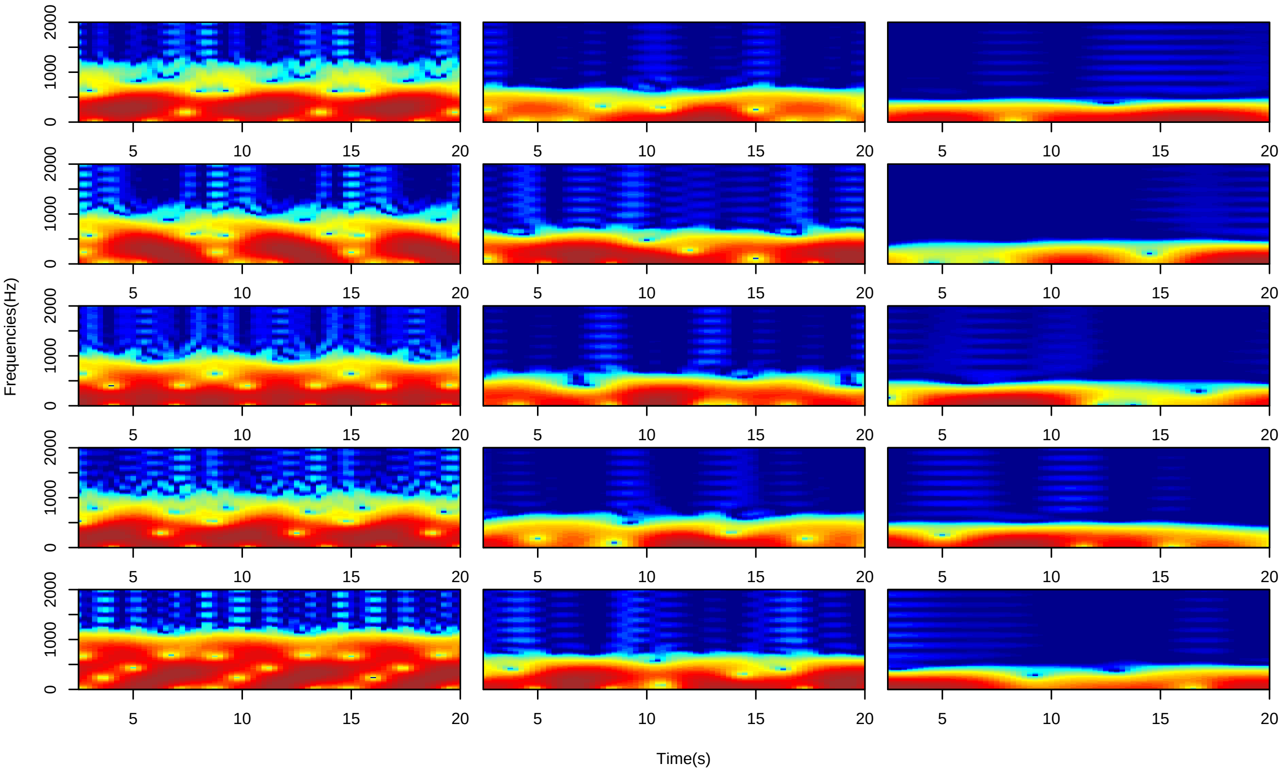


Figure 13: $N = 100$, $l = 1$, $p = 0.25, 0.5, 1.00$. Spectrograms of the $y^{(m)}$ components.

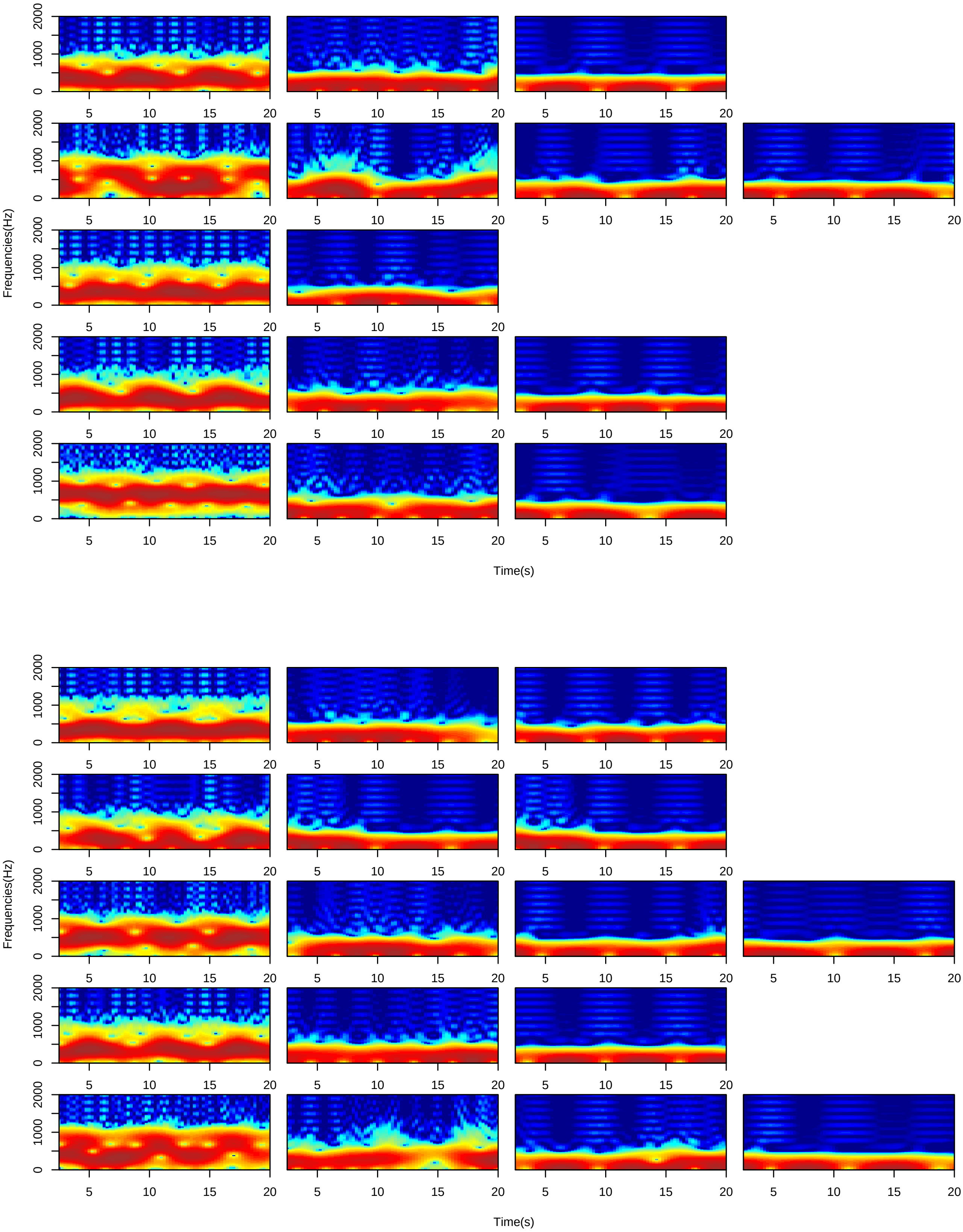


Figure 14: $N = 100$, $l = 1$, $p = 0.25, 0.5, 1.00$. Spectrograms of the IMFs extracted by $\hat{x}^{(m)}$.

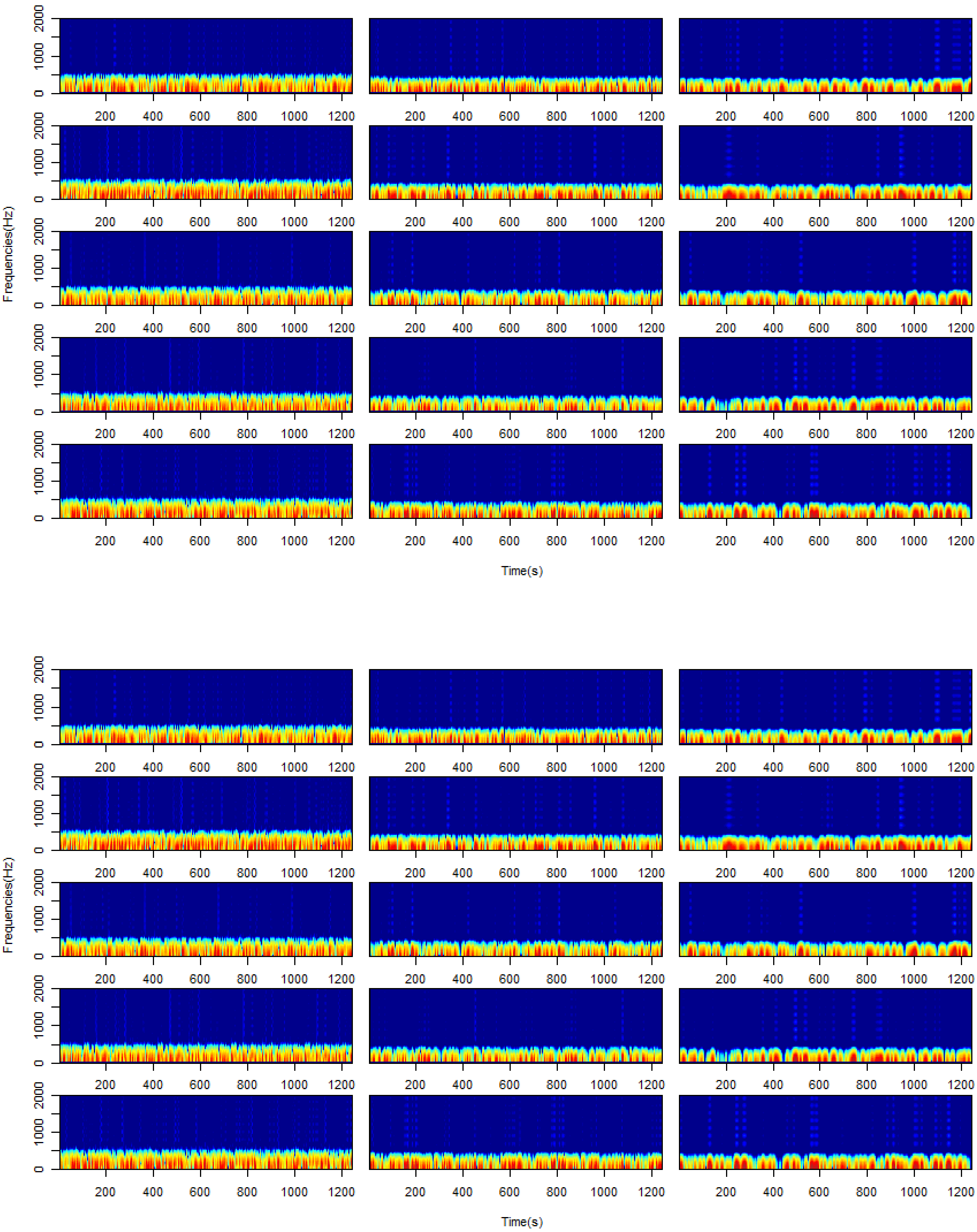


Figure 15: $N = 5000$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Spectrograms of the $y^{(m)}$ components and IMFs extracted by $\hat{x}^{(m)}$ for $m = 1, 2, 3, 4, 5$.

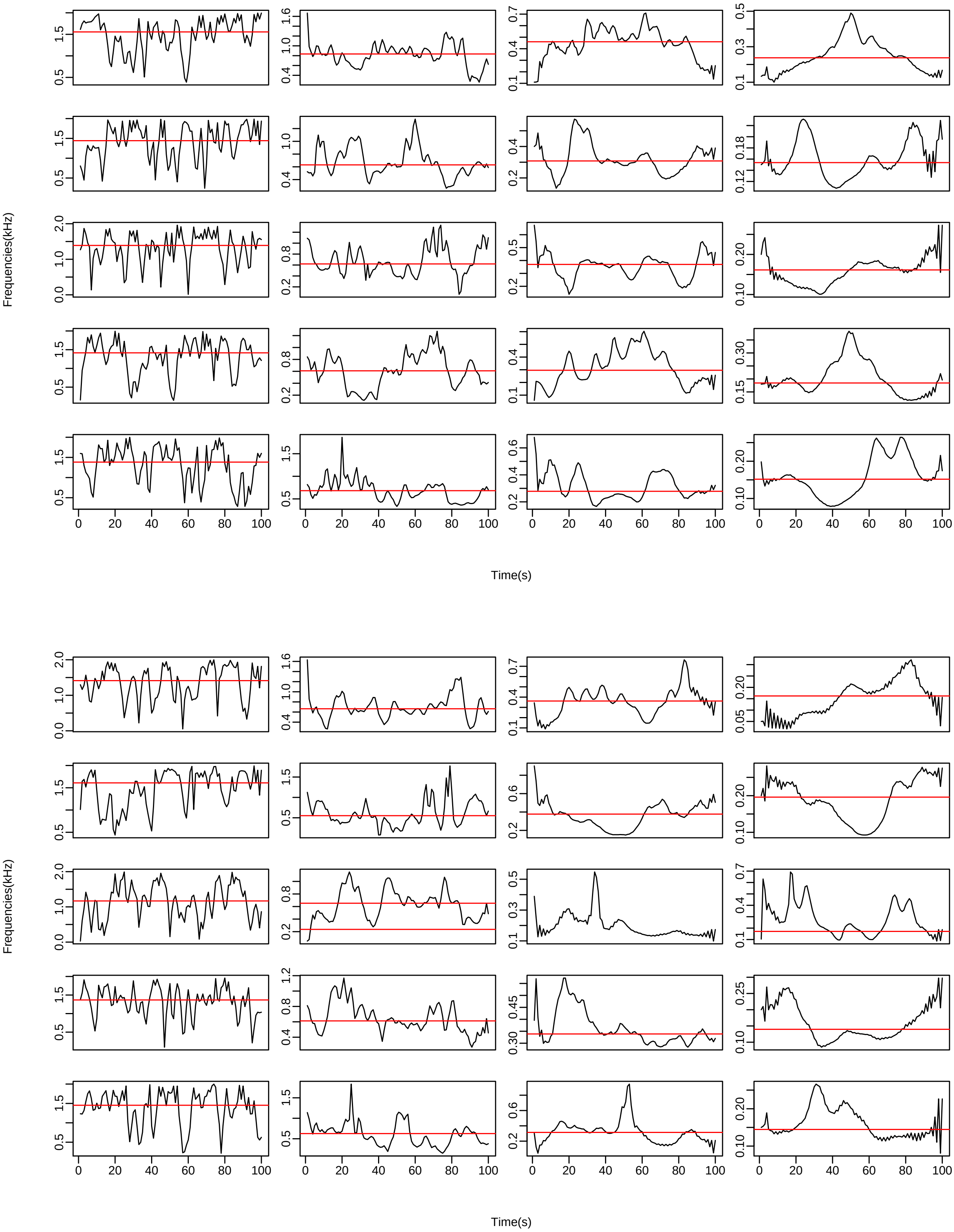


Figure 16: $N = 100$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.

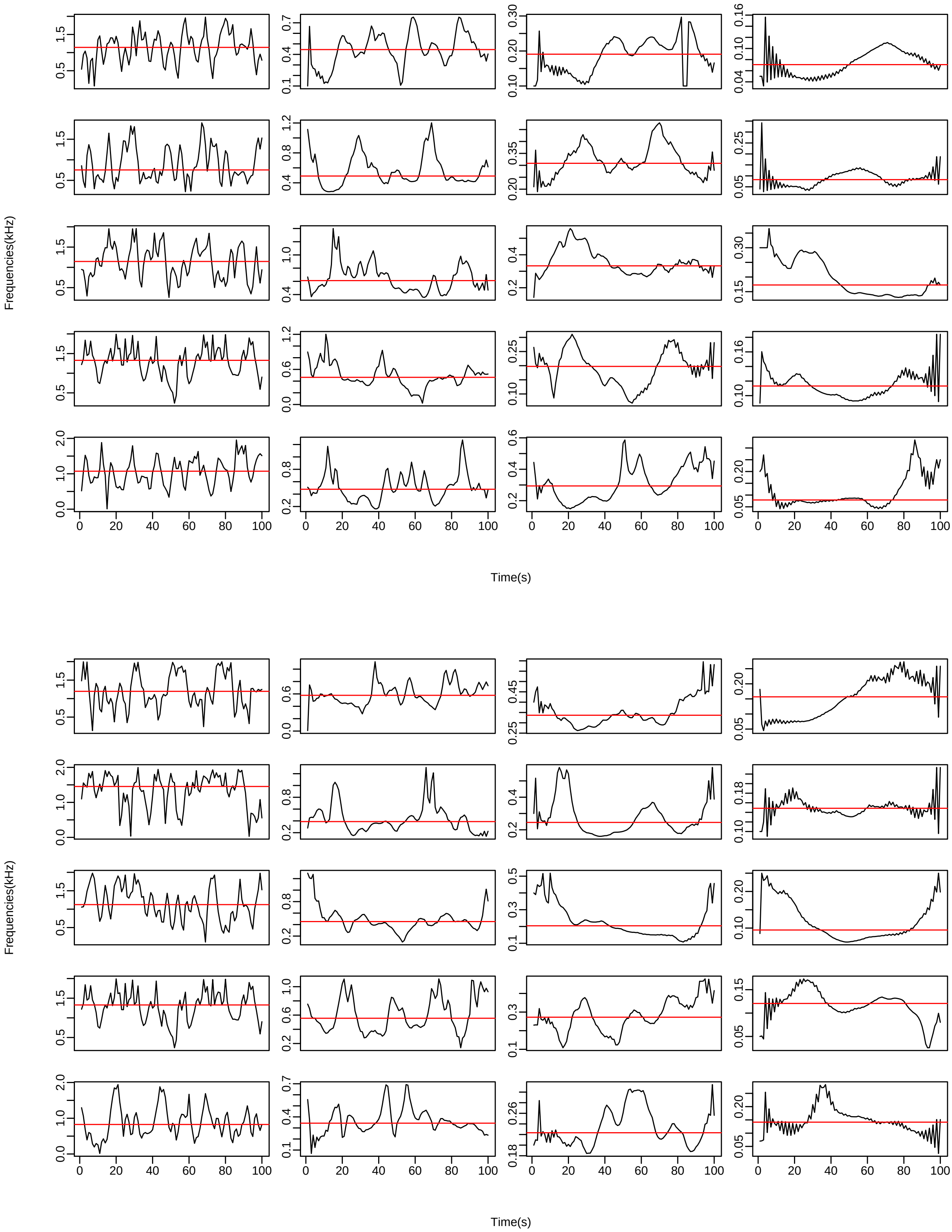


Figure 17: $N = 100$, $l = 0.2$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.

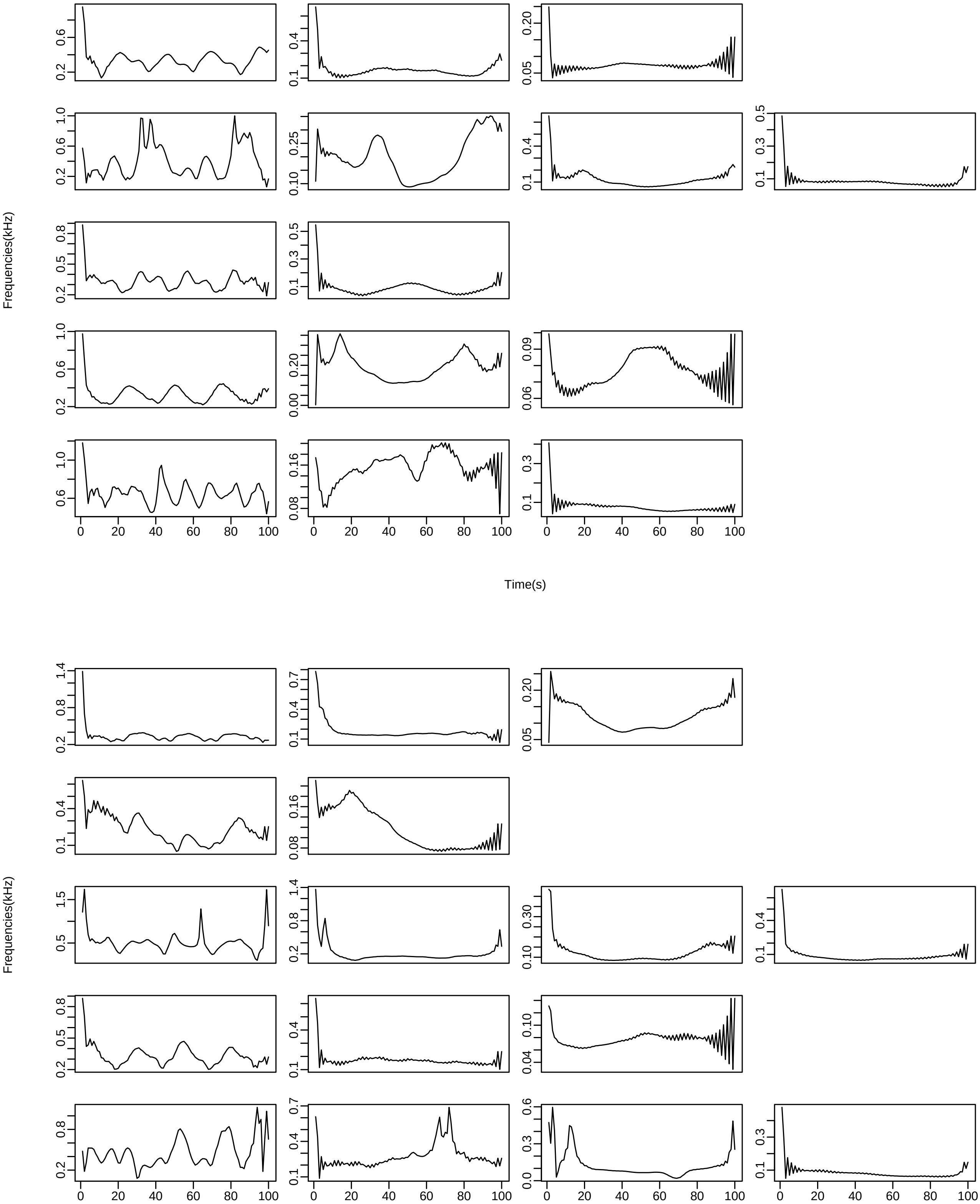
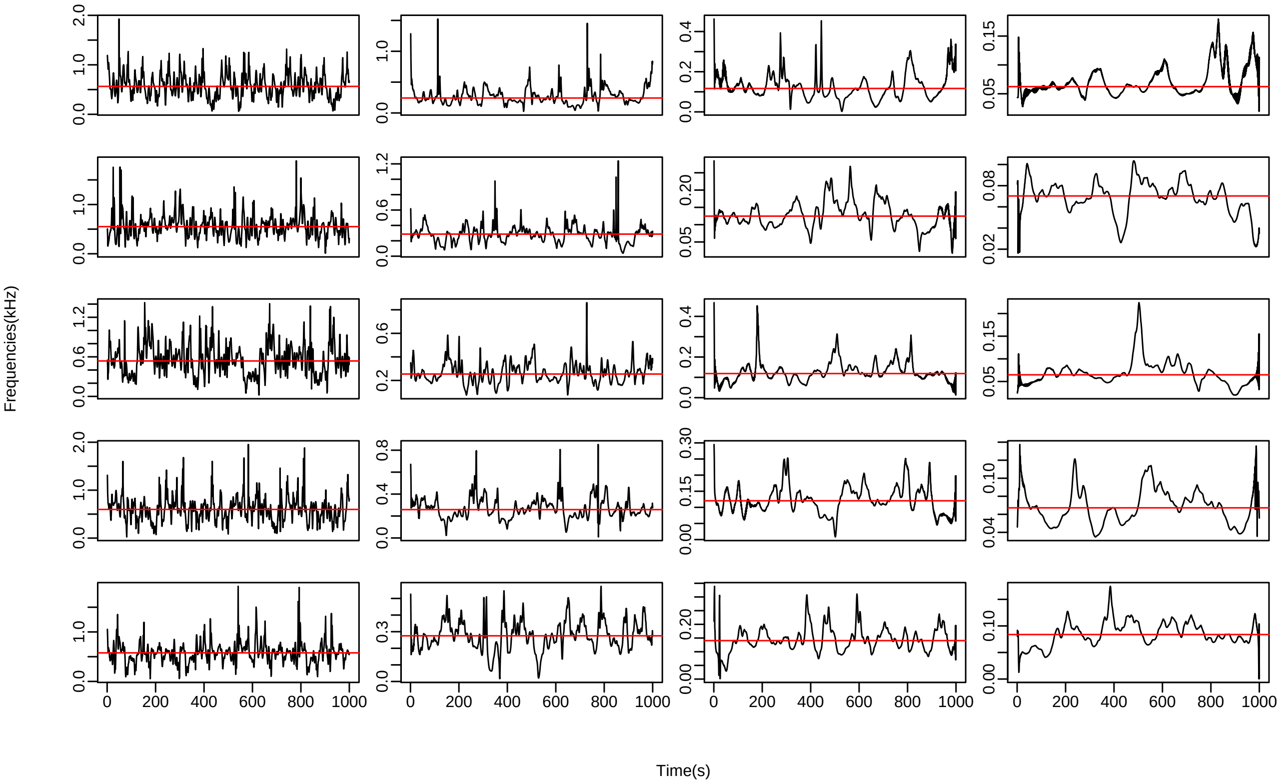


Figure 18: $N = 100$, $l = 1$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.



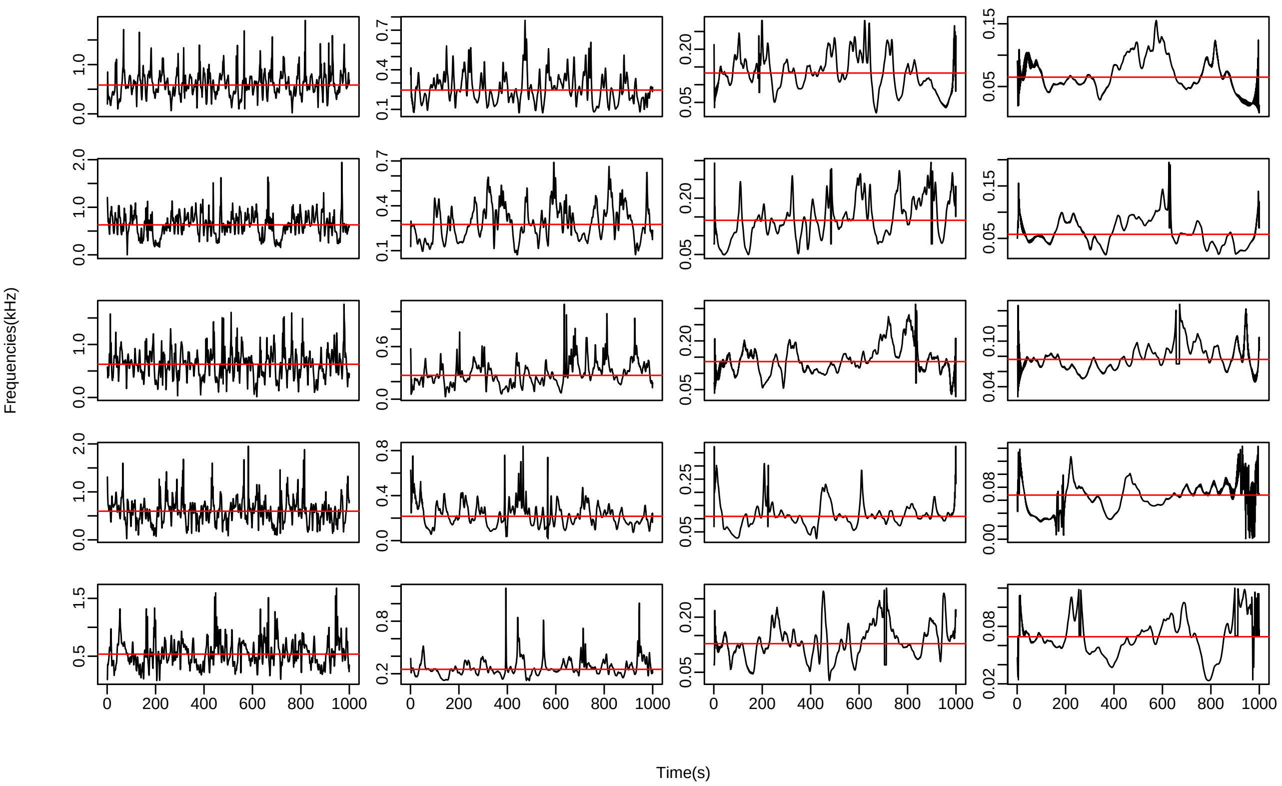


Figure 19: $N = 1000$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.

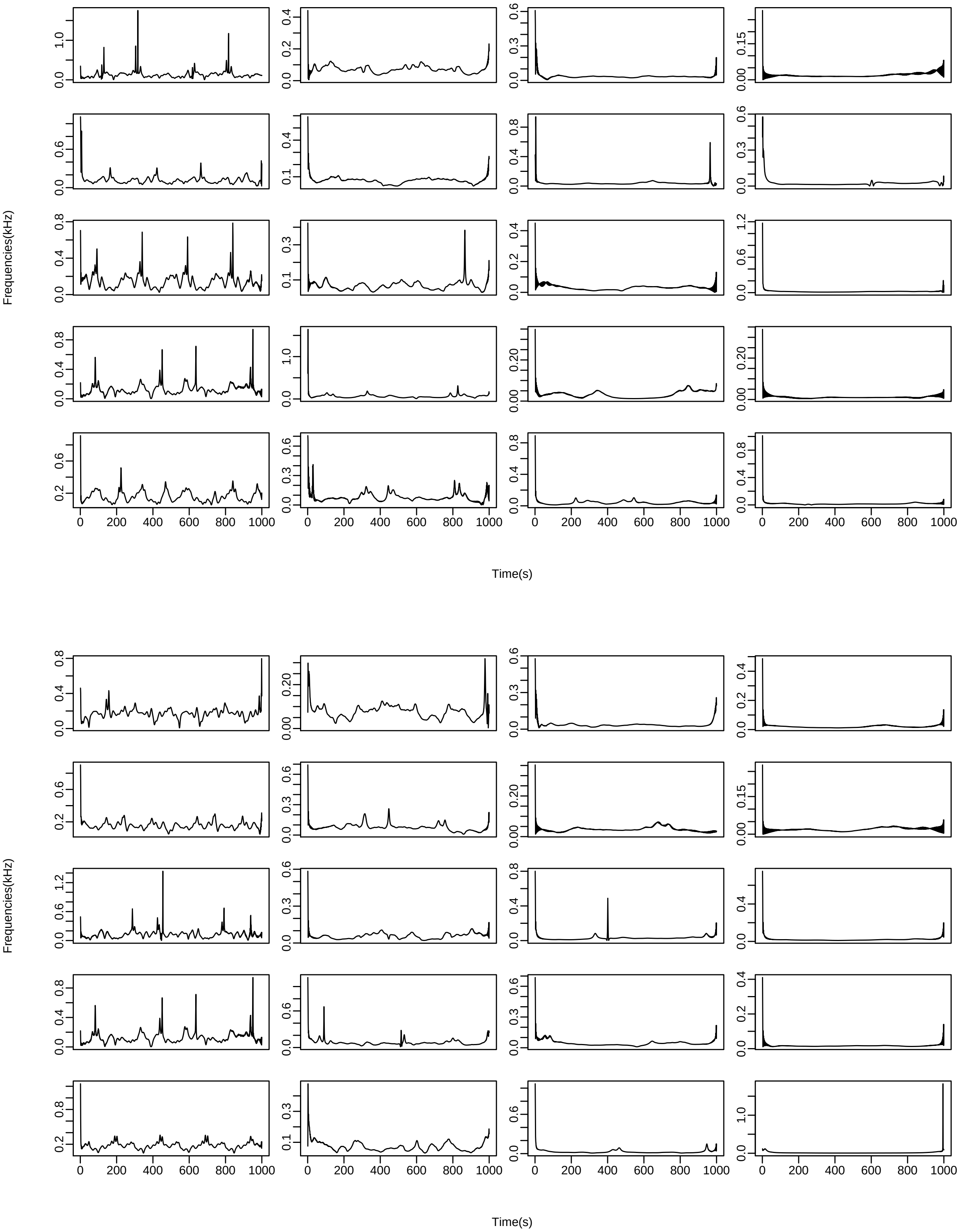


Figure 20: $N = 1000$, $l = 0.2$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.

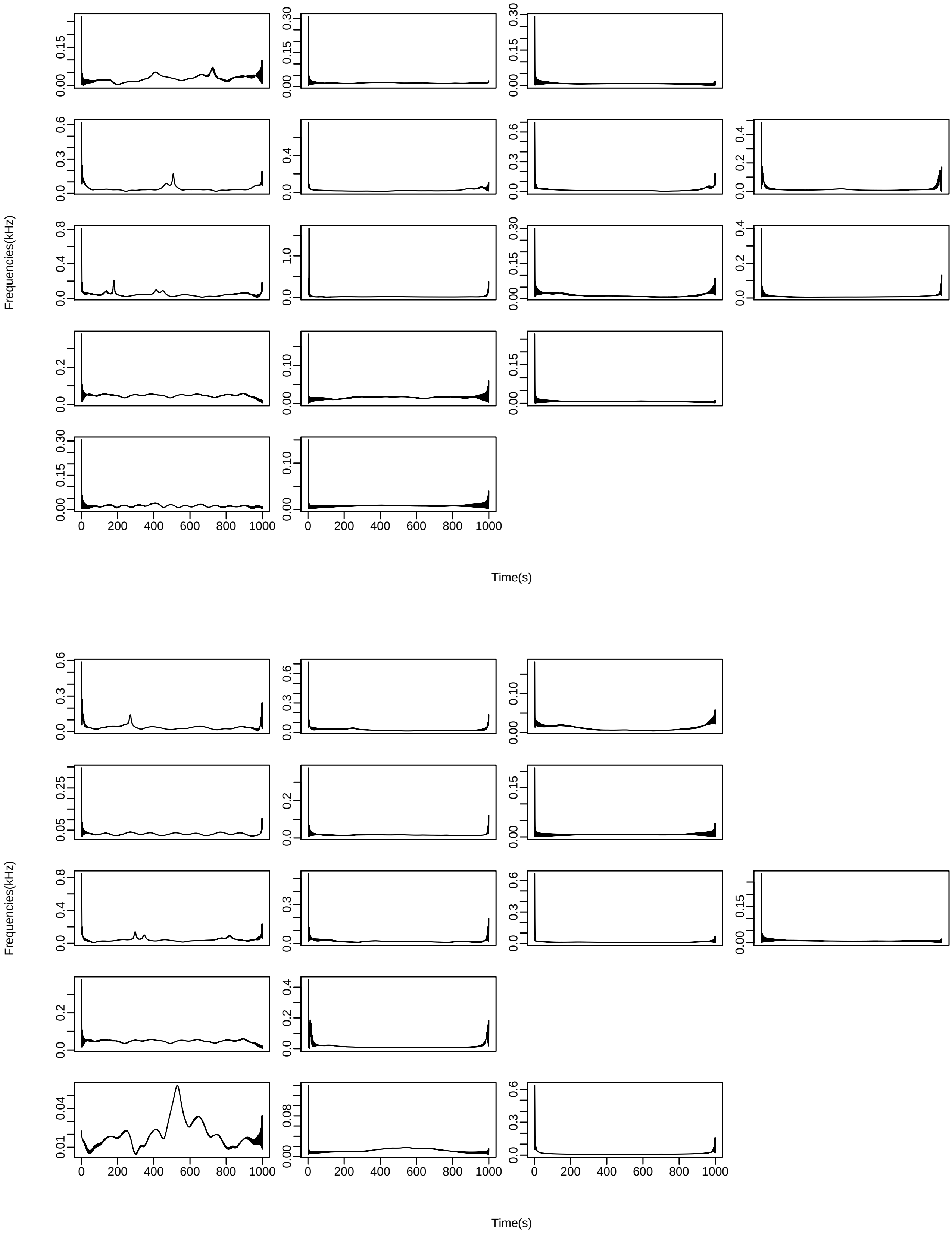


Figure 21: $N = 1000$, $l = 1$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$.

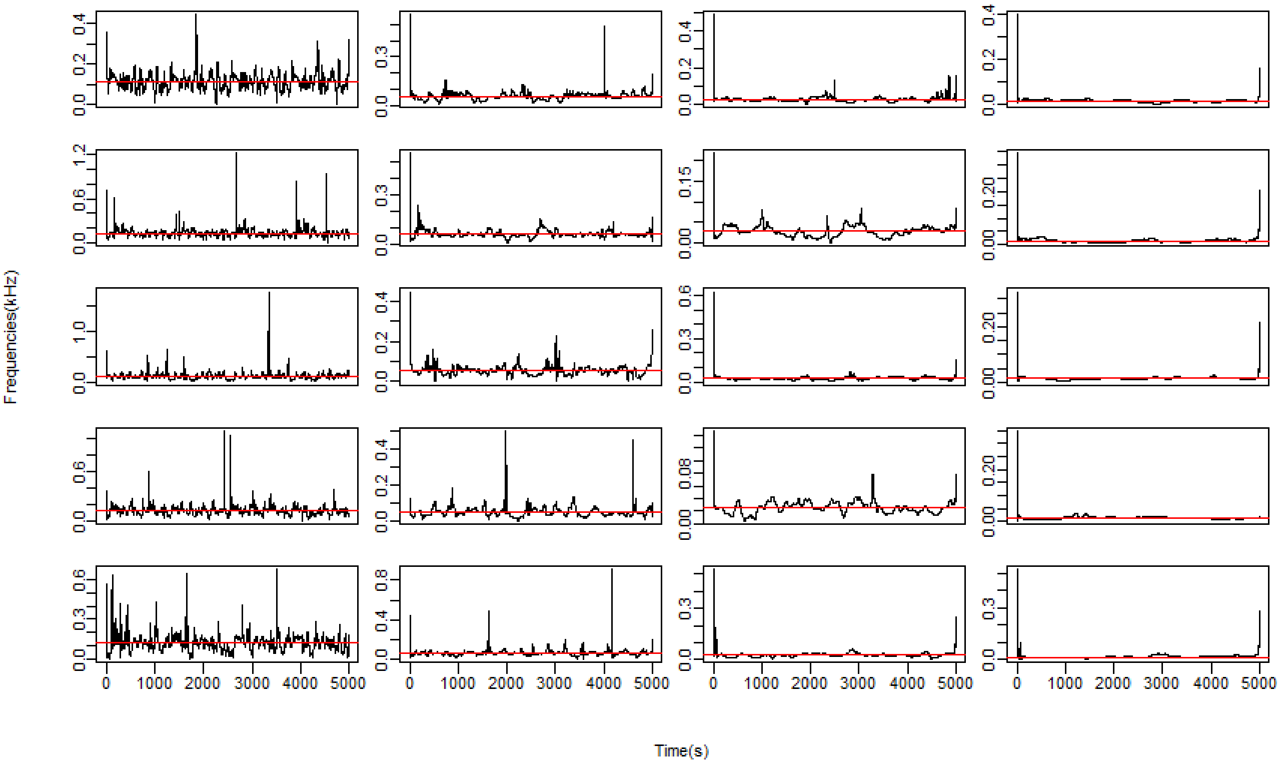


Figure 22: $N = 5000$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Instantaneous Frequencies of the IMFs extracted by $\hat{x}^{(m)}$, $m = 1, 2, 3, 4, 5$.

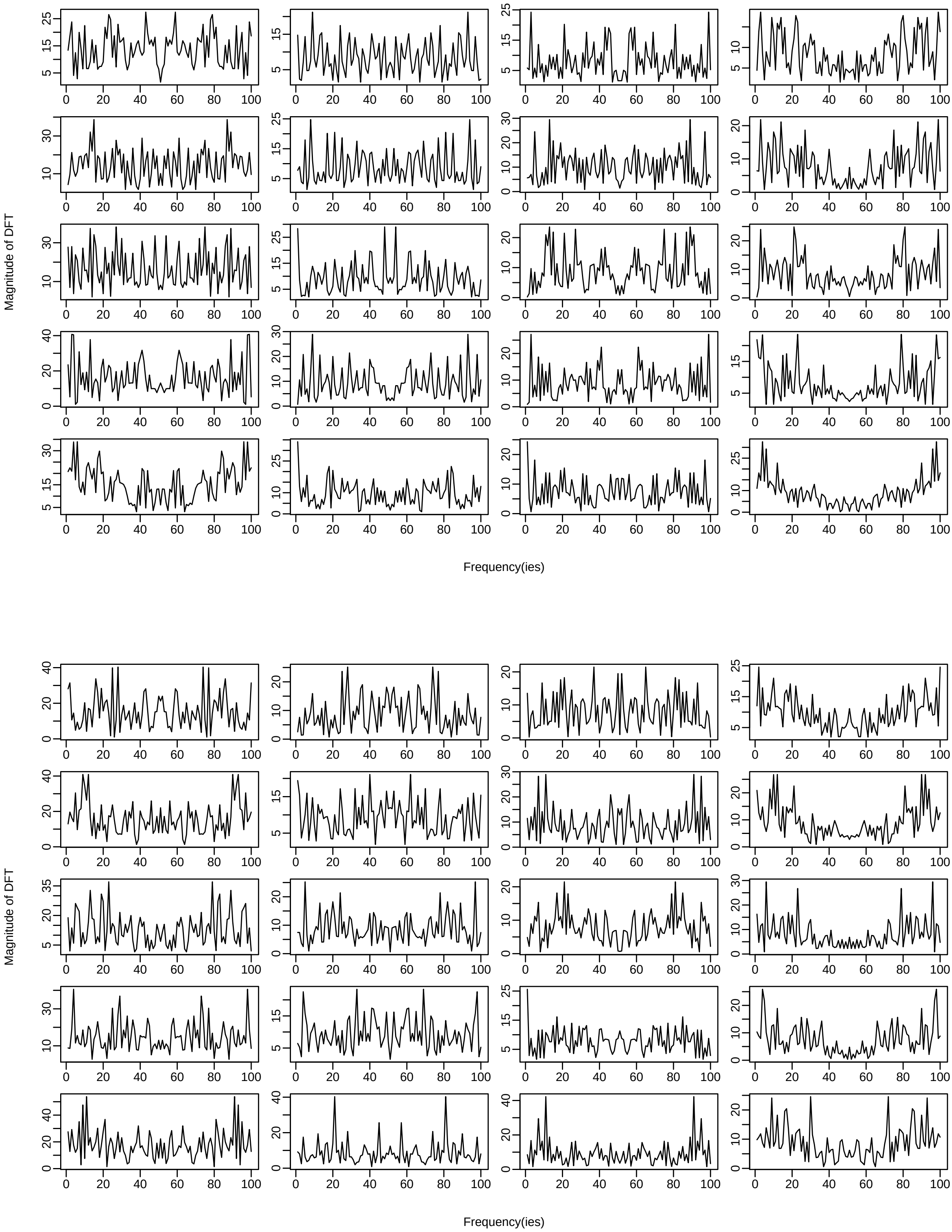


Figure 23: $N = 100$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.

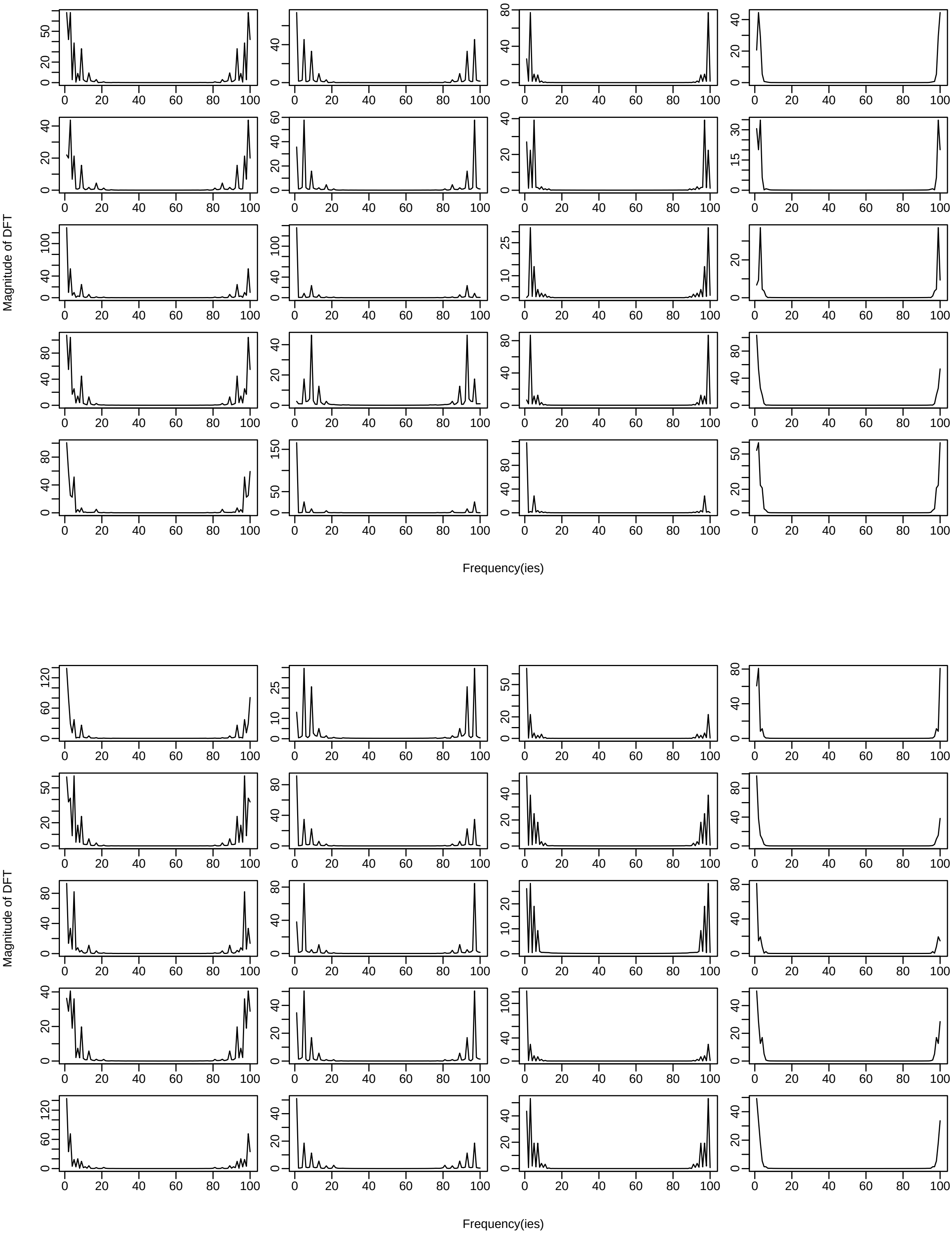
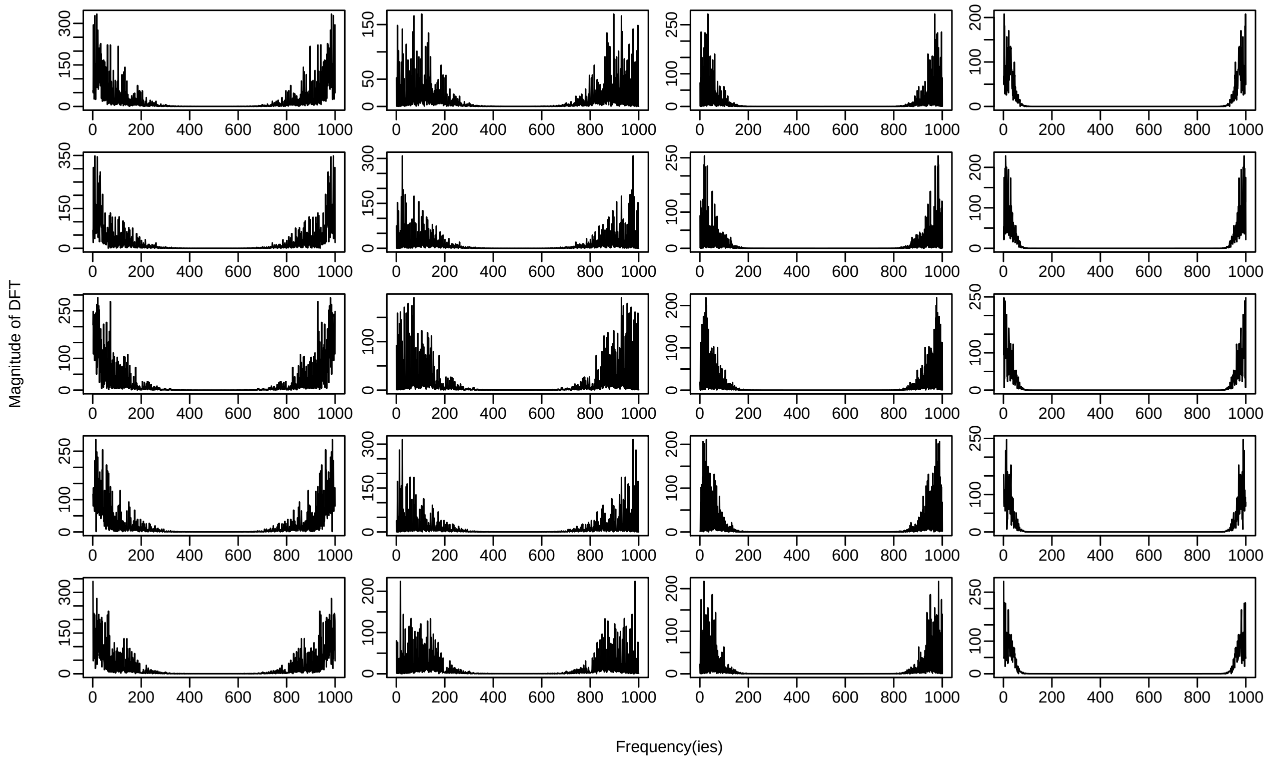


Figure 24: $N = 100$, $l = 1$, $p = 0.25, 0.5, 1.00$. Magnitudes of the DFT, $m = 1, 2, \dots, 10$.



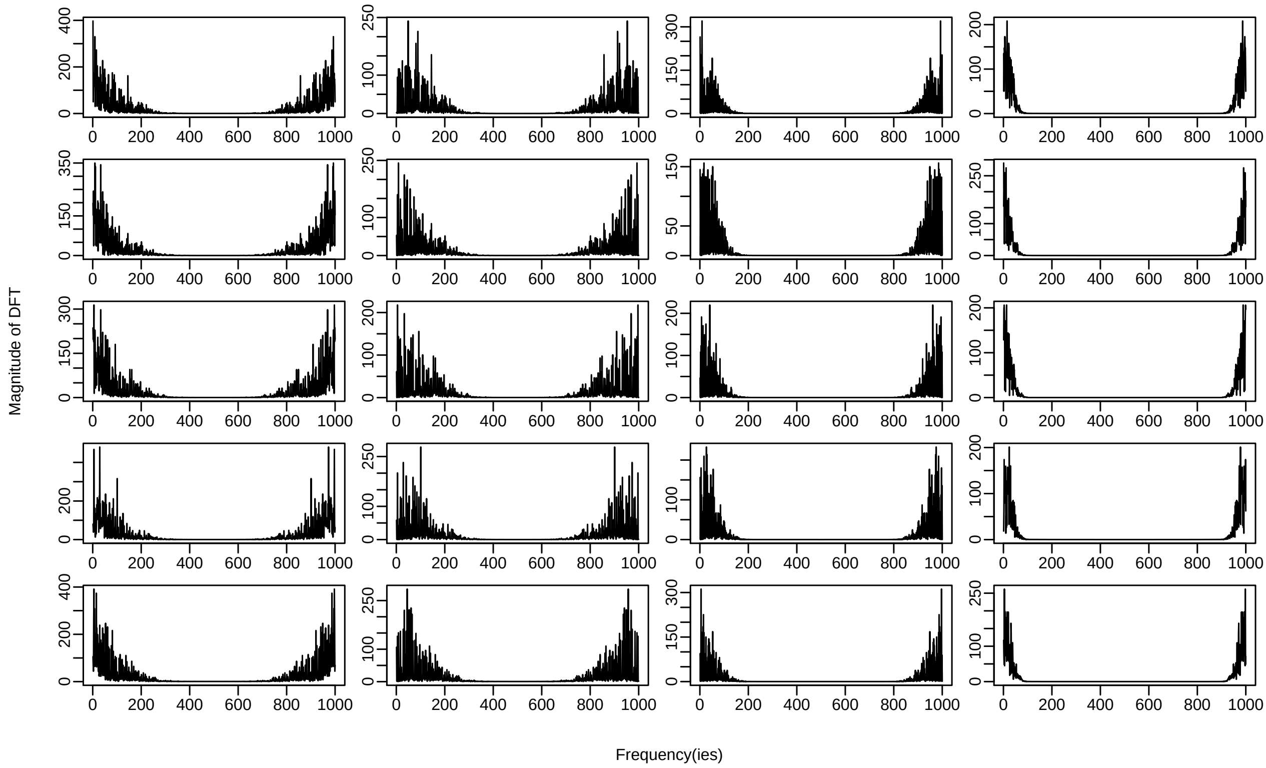
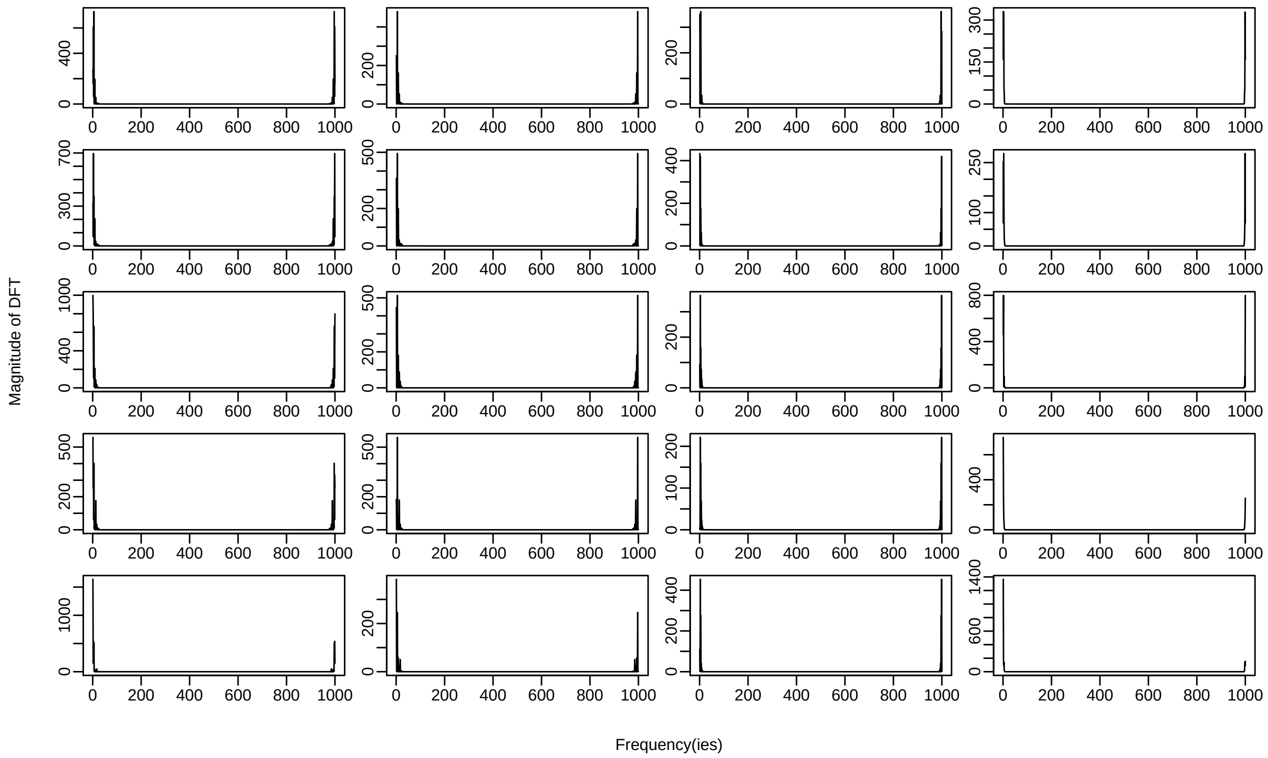


Figure 25: $N = 1000$, $l = 0.05$, $p = 0.25, 0.5, 1.00$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.



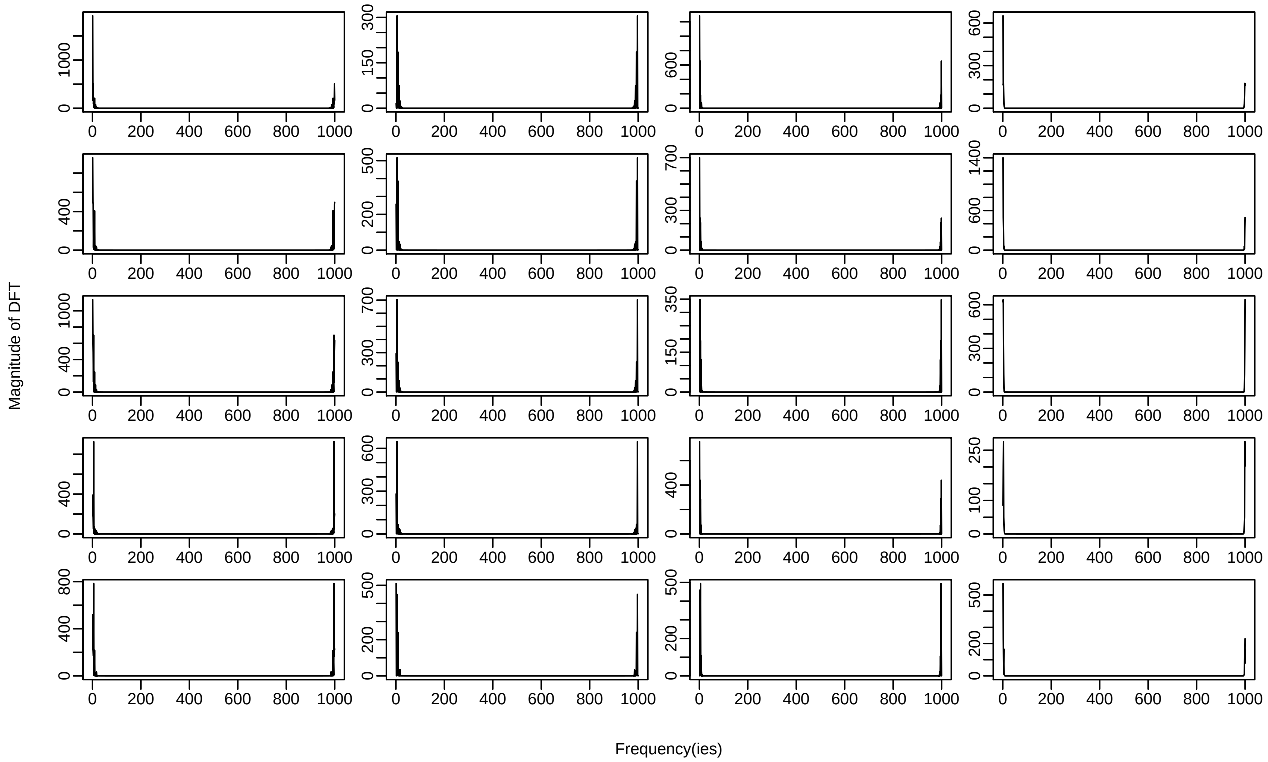


Figure 26: $N = 1000$, $l = 1$, $p = 0.25, 0.5, 1.00$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.

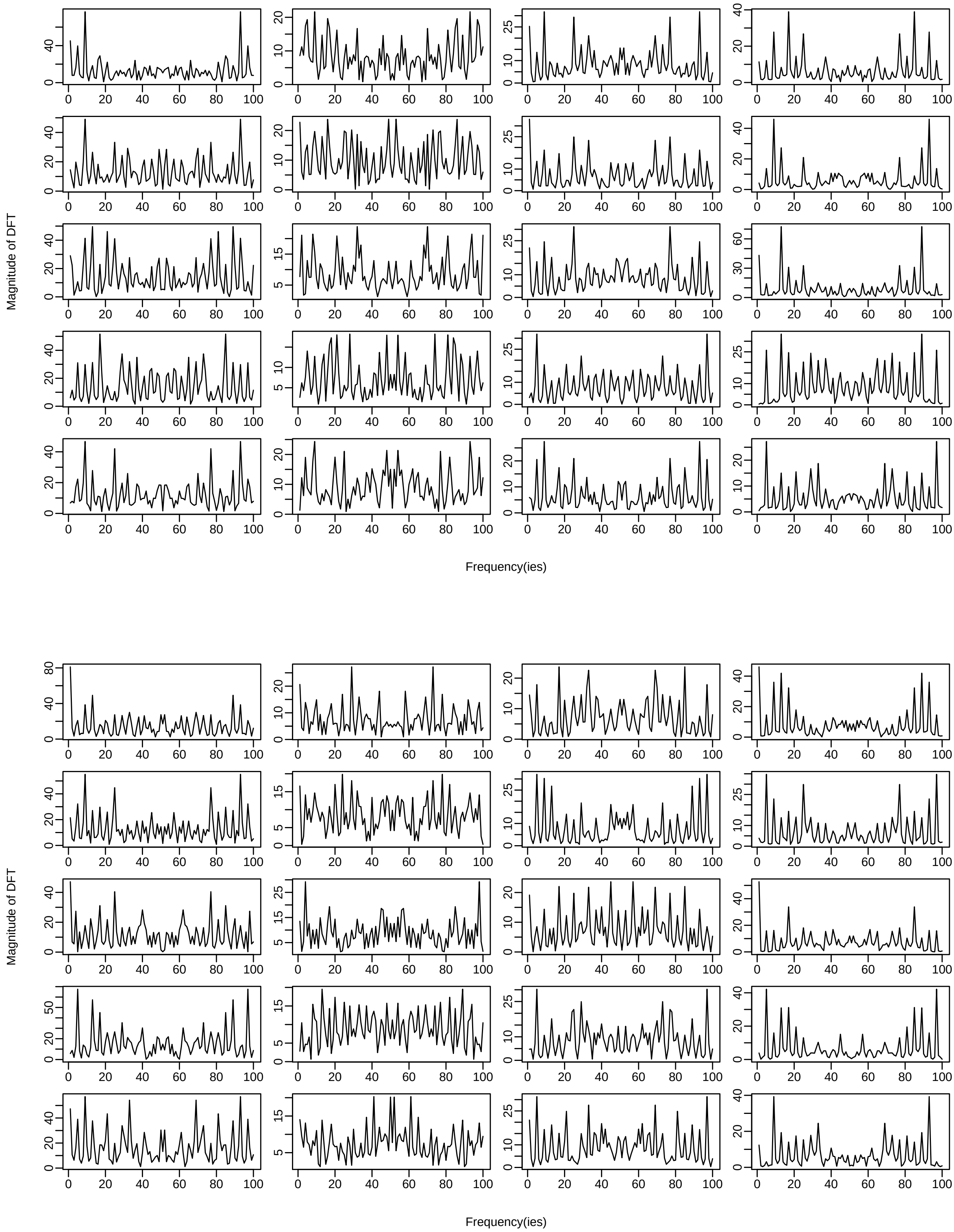


Figure 27: $N = 100$, $l = 0.05, 0.1, 0.15$, $p = 0.25$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.

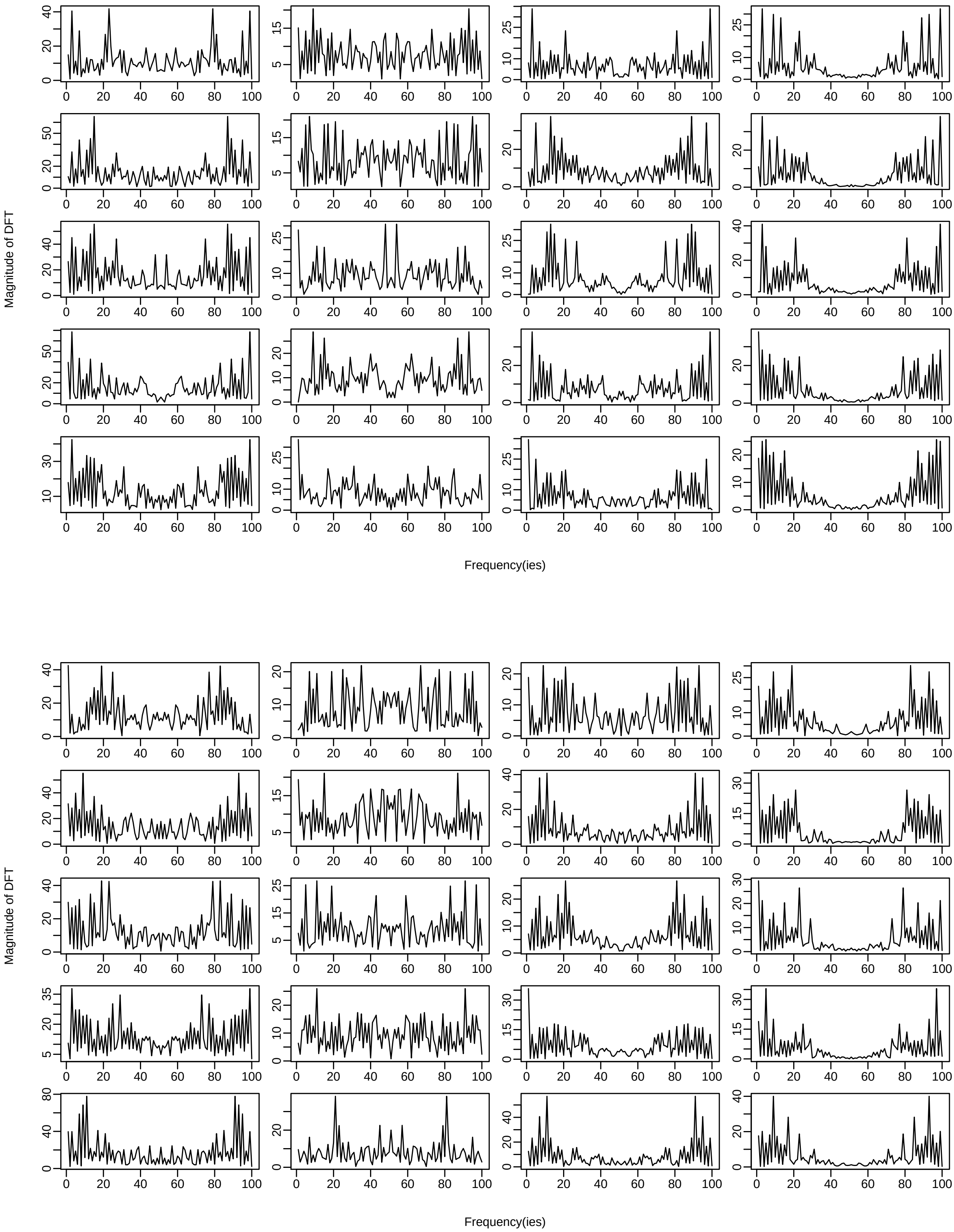


Figure 28: $N = 100$, $l = 0.05, 0.1, 0.15$, $p = 0.5$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.

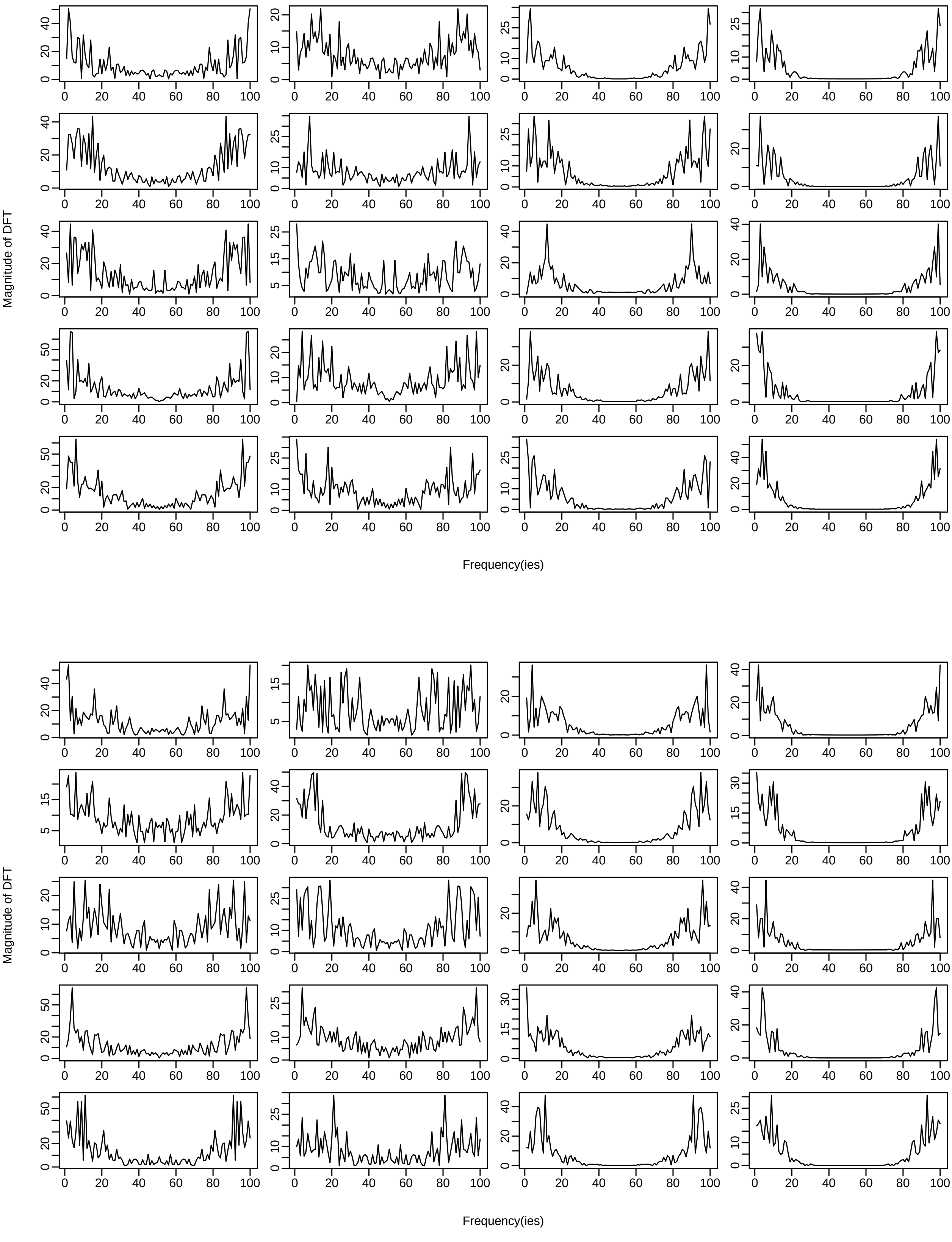


Figure 29: $N = 100$, $l = 0.05, 0.1, 0.15$, $p = 1$. Magnitudes of the DFT, $m = 1, 2, \ldots, 10$.