Applied and Computational Harmonic Analysis **6,** 1–52 (1999)
Article ID acha.1997.0242, available online at http://www.idealibrary.com on IDEAL®

The Wavelet Element Method

Part I. Construction and Analysis[1]

Claudio Canuto

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy, and Istituto di Analisi Numerica del C.N.R., 27100 Pavia, Italy

E-mail: ccanuto@polito.it

Anita Tabacco

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

E-mail: tabacco@polito.it

and

Karsten Urban[2]

Institut für Geometrie und Praktische Mathematik, RWTH Aachen, Templergraben 55, 52056 Aachen, Germany

E-mail: urban@igpm.rwth-aachen.de

Communicated by Guido L. Weiss

Received September 29, 1997

The Wavelet Element Method (WEM) combines biorthogonal wavelet systems with the philosophy of Spectral Element Methods in order to obtain a biorthogonal wavelet system on fairly general bounded domains in some $\mathbb{R}^n$. The domain of interest is split into subdomains which are mapped to a simple reference domain, here n-dimensional cubes. Thus, one has to construct appropriate biorthogonal wavelets on the reference domain such that mapping them to each subdomain and matching along the interfaces leads to a wavelet system on the domain. In this paper we use adapted biorthogonal wavelet systems on the interval in such a way that tensor products of these functions can be used for the construction of wavelet bases

[1] This work was partially supported by the following funds: in Italy, MURST ex-40% *Analisi Numerica* and CNR ST/74 *Progetto Strategico Modelli e Metodi per la Matematica e l'Ingegneria;* in Germany, DAAD Vigoni-Project *Multilevel-Zerlegungsverfahren für Partielle Differentialgleichungen* and DFG-Graduiertenkolleg *Analyse und Konstruktion in der Mathematik* at the RWTH Aachen.

[2] This author is in particular grateful to Wolfgang Dahmen. Moreover, he thanks Gero Nießen for helpful remarks concerning the presentation.

1063-5203/99 $30.00

on the reference domain. We describe the matching procedure in any dimension n in order to impose continuity and prove that it leads to a construction of a biorthogonal wavelet system on the domain. These wavelet systems characterize Sobolev spaces measuring both piecewise and global regularity. The construction is detailed for a bivariate example and an application to the numerical solution of second order partial differential equations is given.

1. INTRODUCTION

During the past years, wavelets have become a powerful tool in both pure and applied mathematics. For example, they allow us to extend classical results of Fourier Analysis to a much wider class of function spaces [31]. On the other hand, wavelet and multilevel systems are by now very widely used in many fields of science and technology such as signal analysis, data compression, and image processing [23, 35, 33]. More recently, starting from [4], they have shown promising features for the construction of efficient numerical schemes for solving operator equations, see, e.g., [17].

Many constructions of wavelets can be found in the literature. Each of them provide different features such as smoothness, arbitrary degree of exactness of approximation, compact support in physical or transformed space, etc. However, currently, most of these constructions are restricted to "simple domains," namely $\mathbb{R}^n$, the torus, the n-dimensional cube, or domains that can be easily mapped to these. This is a severe limitation to the successful use of wavelets in certain fields. Only in the last few years have papers appeared aimed at dealing with wavelets in general bounded domains [25, 10, 22].

In this paper, we propose a construction of biorthogonal wavelet systems on fairly general bounded domains, by following the philosophy that led A. T. Patera [32] to invent the *Spectral Element Method* (SEM). The SEM uses a global, high order polynomial basis on a closed interval, and, by tensor product, extends it on a n-dimensional cube. The construction on a bounded domain having complex geometry is then carried on by splitting the domain into subdomains and mapping these to a single reference domain, namely a cube. This has led to very efficient numerical solvers for partial differential equations, with significant applications also for "real life problems" [29]. The key for the efficiency of the SEM is the tensor product structure of the basis on the reference domain.

Because current mathematically sound and computationally efficient univariate wavelet systems are available on a closed interval, we propose to replace the global polynomial basis by such a multiscale basis; then we apply the above splitting-and-mapping approach, adding the advantage of multiscale decompositions to those of the SEM. The resulting construction provides a multilevel decomposition for function spaces built on the domain; so, it can be applied to any circumstance in which this is needed.

In particular, as for the SEM, the numerical approximation of operator equations can be a challenging field of application. The motivation for using wavelets here is at least twofold: they provide optimal preconditioning of the arising ill conditioned linear systems [24, 18, 21] and they allow the definition of efficient adaptive schemes [28, 30, 3, 13, 5, 14]. In addition, the flexibility in the construction of biorthogonal wavelets leaves some room which can be used to adapt these systems to special problems at hand, see [15, 34, 19], for example.

Biorthogonal wavelet systems on the unitary interval, which can be required to satisfy certain boundary conditions, are the initial point of our construction. The univariate systems are defined for instance as in [20, 27], starting from systems on the real line such as, e.g., Daubechies' compactly supported orthogonal wavelets [23] or the biorthogonal B-spline wavelets [11]. We recall that dealing with multiscale methods involves two different bases for the trial spaces, namely the *single scale* and the *multiscale* (or detail-) basis. The single scale basis is similar to finite elements on uniformly refined triangles or global polynomial bases on cubes. Hence, the matching of the single scale basis functions along the interfaces of the subdomains is similar to the matching in the SEM. The multiscale basis can be understood to span the details between succeeding trial spaces. For these functions (named *wavelets*) matching is more delicate. Moreover, preconditioning and adaptivity is based on certain stability properties of the wavelet bases which have to be valid also after the matching.

In this paper, we aim at designing bases that can be used, e.g., to build trial spaces in a Galerkin projection method for approximating second order partial differential equations; hence, we enforce a conformal C^0-matching. We prove that we can match wavelet functions in an appropriate way and give the construction independently of the spatial dimension. Other kinds of matchings with a different level of non-conformity will be considered elsewhere. A preliminary application and one particular example is given. In a forthcoming paper [7], we shall address many issues related to the actual realization in dimensions 2 and 3 and provide other applications and properties.

The paper is organized as follows: In Section 2, we review the main properties of biorthogonal wavelet systems on the interval and describe the possibilities to add certain boundary conditions to these systems. This latter topic is discussed in Appendix A in more detail for the convenience of the reader. By using tensor products one can then easily obtain wavelet systems on n-dimensional cubes. Section 3 is devoted to the description of this construction. Moreover, we recall that stability can easily be carried over from the corresponding univariate property. In Section 4, we use these multiscale bases on cubes to obtain a multiresolution decomposition on a general bounded n-dimensional domain partitioned into subdomains. The construction of biorthogonal wavelets is introduced in Section 5. The method is detailed for a bivariate example in Subsection 5.3 and the main results are collected in Subsection 5.4. Finally, Section 6 contains an application to the numerical solution of elliptic second order partial differential equations.

Some of the results of this paper (in Section 4 and Appendix A) overlap with similar ones by Dahmen and Schneider [22]. Although we stemmed the idea of the *Wavelet Element Method* (WEM) from the SEM independently of these authors, we are indebted to Wolfgang Dahmen for many discussions on multiscale methods over the past years, that have been quite influential for us. On the other hand, our construction of the WEM, in particular of the matching of the wavelet functions, differs from the results in [22].

2. BIORTHOGONAL SYSTEMS ON [0, 1]

In the literature, there is a whole variety of concrete examples of multiresolution analyses on the interval. All these constructions are based on scaling functions on the real line that are either orthogonal or biorthogonal. Then, these functions are modified near the boundaries in order to ensure the validity of this and other conditions on the interval, see [2, 12, 8, 20, 27], for example. In this section we collect the main properties of the biorthogonal wavelet systems on the interval, as constructed in [20, 27]. All the results we give are proven in these references, except a small number of them whose proofs will be provided in the Appendix. We first describe the general approach and then we detail the modifications for fulfilling boundary conditions.

We will frequently use the following notation: by $A \lesssim B$ we denote the fact that A can be bounded by a multiple constant times B, where the constant is independent of the various parameters A and B may depend on. Furthermore, $A \lesssim B \lesssim A$ (with different constants, of course) will be abbreviated by $A \sim B$.

2.1. *General Setting*

The starting point is two families of *scaling functions*

$$\Xi_j := \{\xi_{j,k}\colon k \in \Delta_j\}, \quad \tilde{\Xi}_j := \{\tilde{\xi}_{j,k}\colon k \in \Delta_j\} \subset L^2(0,1),$$

where Δ_j denotes an appropriate set of indices and $j \geq j_0$ can be understood as the scale parameter (with some j_0 denoting the coarsest scale). For subsequent convenience, these functions will not be labeled by integers as usual, but rather by a set of real indices

$$\Delta_j := \{\tau_{j,1}, \ldots, \tau_{j,K_j}\}, \qquad 0 = \tau_{j,1} < \tau_{j,2} < \cdots < \tau_{j,K_j} = 1. \tag{2.1}$$

In other words, each basis function is associated with a *node*, or grid point, in the interval [0, 1]; the actual position of the internal nodes $\tau_{j,2}, \ldots, \tau_{j,K_j-1}$ will be irrelevant in the sequel, except that it is required that $\Delta_j \subset \Delta_{j+1}$ (see (2.3.k)). It will be also convenient to consider Ξ_j as the column vector $(\xi_{j,k})_{k\in\Delta_j}$, and analogously for other set of functions.

The construction of these families Ξ_j, $\tilde{\Xi}_j$ guarantees that they are *dual generator systems* of a *multiresolution analysis* in $L^2(0, 1)$

$$S_j := \operatorname{span} \Xi_j, \qquad \tilde{S}_j := \operatorname{span} \tilde{\Xi}_j, \tag{2.2}$$

in the sense that the following conditions in (2.3) are fulfilled:

(2.3.a) The systems Ξ_j and $\tilde{\Xi}_j$ are *refinable*, i.e., there exist matrices M_j, $\tilde{M}_j$, such that

$$\Xi_j = M_j \Xi_{j+1}, \qquad \tilde{\Xi}_j = \tilde{M}_j \tilde{\Xi}_{j+1}.$$

This implies, in particular, that the induced spaces S_j, $\tilde{S}_j$ are *nested*, i.e., $S_j \subset S_{j+1}$, $\tilde{S}_j \subset \tilde{S}_{j+1}$.

(2.3.b) The functions have local support, in the sense

$$\operatorname{diam}(\operatorname{supp} \xi_{j,k}) \sim \operatorname{diam}(\operatorname{supp} \tilde{\xi}_{j,k}) \sim 2^{-j}.$$

(2.3.c) The systems are *biorthogonal*, i.e.,

$$(\xi_{j,k}, \tilde{\xi}_{j,k'})_{L^2(0,1)} = \delta_{k,k'}, \qquad \text{for all } k,\ k' \in \Delta_j.$$

(2.3.d) The functions are regular, i.e.,

$$\xi_{j,k} \in H^{\gamma}(0, 1), \qquad \tilde{\xi}_{j,k} \in H^{\tilde{\gamma}}(0, 1), \qquad \text{for some } \gamma,\ \tilde{\gamma} > 1,$$

where $H^s(0, 1)$, $s \geq 0$, denotes the usual Sobolev space on the interval as defined, e.g., in [1].

(2.3.e) The systems are exact of order L, $\tilde{L} \geq 1$, respectively, i.e., polynomials up to the degree $L - 1$, $\tilde{L} - 1$ are reproduced exactly,

$$\mathbb{P}_{L-1}(0, 1) \subset S_j, \qquad \mathbb{P}_{\tilde{L}-1}(0, 1) \subset \tilde{S}_j,$$

where $\mathbb{P}_r(0, 1)$ denotes the set of the algebraic polynomials of degree r at most, restricted to $[0, 1]$.

(2.3.f) The systems Ξ_j, $\tilde{\Xi}_j$ are uniformly stable, i.e.,

$$\|\sum_{k \in \Delta_j} c_k \xi_{j,k}\|_{L^2(0,1)} \sim \|\mathbf{c}\|_{\ell^2(\Delta_j)} \sim \|\sum_{k \in \Delta_j} c_k \tilde{\xi}_{j,k}\|_{L^2(0,1)},$$

where $\mathbf{c} := (c_k)_{k \in \Delta_j}$.

(2.3.g) The operators $\Pi_j\colon L^2(0, 1) \to S_j$ defined by

$$\Pi_j v := \sum_{k \in \Delta_j} (v, \tilde{\xi}_{j,k})_{L^2(0,1)} \xi_{j,k}$$

have the properties $\Pi_j \Pi_{j+1} = \Pi_j$, $\Pi_j^2 = \Pi_j$, $\|\Pi_j\| \lesssim 1$, and analogously for $\tilde{\Pi}_j$.

(2.3.h) The systems Ξ_j, $\tilde{\Xi}_j$ fulfill a *Jackson*-type inequality:

$$\inf_{v_j \in S_j} \|v - v_j\|_{L^2(0,1)} \lesssim 2^{-sj} \|v\|_{H^s(0,1)}, \qquad v \in H^s(0, 1),\ 0 \leq s \leq \min(L, \gamma),$$

$$\inf_{v_j \in \tilde{S}_j} \|v - v_j\|_{L^2(0,1)} \lesssim 2^{-sj} \|v\|_{H^s(0,1)}, \qquad v \in H^s(0, 1),\ 0 \leq s \leq \min(\tilde{L}, \tilde{\gamma}).$$

(2.3.i) The systems Ξ_j, $\tilde{\Xi}_j$ fulfill a *Bernstein*-type inequality:

$$\|v_j\|_{H^s(0,1)} \lesssim 2^{js} \|v_j\|_{L^2(0,1)}, \qquad v_j \in S_j,\ 0 \leq s \leq \gamma,$$

$$\|v_j\|_{H^s(0,1)} \lesssim 2^{js} \|v_j\|_{L^2(0,1)}, \qquad v_j \in \tilde{S}_j,\ 0 \leq s \leq \tilde{\gamma}.$$

(2.3.j) There exist complement spaces T_j and $\tilde{T}_j$ such that

$$S_{j+1} = S_j \oplus T_j, \qquad \tilde{S}_{j+1} = \tilde{S}_j \oplus \tilde{T}_j,$$
$$T_j \perp \tilde{S}_j, \qquad \tilde{T}_j \perp S_j.$$

(2.3.k) The spaces T_j and $\tilde{T}_j$ have biorthogonal, stable bases (in the sense of (2.3.f))

$$\Upsilon_j = \{\eta_{j,h}\colon h \in \nabla_j\}, \qquad \tilde{\Upsilon}_j = \{\tilde{\eta}_{j,h}\colon h \in \nabla_j\},$$

with

$$\nabla_j := \Delta_{j+1} \backslash \Delta_j = \{\nu_{j,1}, \ldots, \nu_{j,M_j}\}, \qquad 0 < \nu_{j,1} < \cdots < \nu_{j,M_j} < 1.$$

These basis functions are called *biorthogonal wavelets.*

(2.3.l) The collections of these functions for all $j \geq j_0$ form Riesz bases of $L^2(0, 1)$. Even more than that, these systems admit norm equivalences for a whole range in the Sobolev scale:

$$\Big\| \sum_{k \in \Delta_{j_0}} c_{j_0,k} \xi_{j_0,k} + \sum_{j=j_0}^{\infty} \sum_{h \in \nabla_j} d_{j,h} \eta_{j,h} \Big\|_{X^s}^2 \sim \sum_{k \in \Delta_{j_0}} |c_{j_0,k}|^2 + \sum_{j=j_0}^{\infty} \sum_{h \in \nabla_j} 2^{2sj} |d_{j,h}|^2,$$

where $s \in (-\min(\tilde{L}, \tilde{\gamma}), \min(L, \gamma))$ is related to the regularity and the exactness of the generator system, and $X^s = H^s(0, 1)$ if $s \geq 0$ or $X^s = (H^{-s}(0, 1))'$ if $s < 0$.

The following concept will be important in the sequel. The system Ξ_j is said to be *reflection invariant*, if Δ_j is invariant under the mapping $x \mapsto 1 - x$ and

$$\xi_{j,k}(1 - x) = \xi_{j,1-k}(x), \qquad \text{for all } x \in [0, 1] \text{ and } k \in \Delta_j, \tag{2.4}$$

which can also be abbreviated as

$$\Xi_j(1 - x) = \Xi_j^{\updownarrow}(x).$$

A similar definition can be given for the system Υ_j, as well as for the dual systems. If Ξ_j is reflection invariant, then Υ_j can be built to have the same property. This will be always implicitly assumed. For example, reflection invariant systems can be constructed from biorthogonal B-splines [11], whereas this is not possible starting from compactly supported Daubechies' scaling functions [23], as they lack symmetry.

2.2. *Systems Fulfilling Boundary Conditions*

One may want to incorporate boundary conditions in a multiresolution analysis, which will be crucial for the further construction of the WEM. To this end, let us introduce the following definitions.

DEFINITION 2.1. The systems Ξ_j and Υ_j are called *boundary adapted* if, at each boundary point:

(i) only one basis function in each system is not vanishing; precisely,

$$\xi_{j,k}(0) \neq 0 \Leftrightarrow k = 0, \qquad \xi_{j,k}(1) \neq 0 \Leftrightarrow k = 1, \tag{2.5}$$

$$\eta_{j,h}(0) \neq 0 \Leftrightarrow h = \nu_{j,1}, \qquad \eta_{j,h}(1) \neq 0 \Leftrightarrow h = \nu_{j,M_j}; \tag{2.6}$$

(ii) the nonvanishing scaling and wavelet functions take the same value; precisely, there exist constants c_0 and c_1 independent of j such that

$$\xi_{j,0}(0) = \eta_{j,\nu_{j,1}}(0) = c_0 2^{j/2}, \qquad \xi_{j,1}(1) = \eta_{j,\nu_{j,M_j}}(1) = c_1 2^{j/2}. \tag{2.7}$$

DEFINITION 2.2. The system Ξ_j is called *boundary symmetric* if

$$\xi_{j,0}(0) = \xi_{j,1}(1). \tag{2.8}$$

Note that if the systems Ξ_j and Υ_j are boundary adapted and if Ξ_j is boundary symmetric, then also Υ_j has the same property.

From now on, *we shall assume that the systems Ξ_j, Υ_j and $\tilde{\Xi}_j$, $\tilde{\Upsilon}_j$ are both boundary adapted and boundary symmetric.* As far as the former assumption is concerned, starting from generator and wavelet systems on the interval, one can indeed construct boundary adapted ones, for instance, for orthogonal systems and for systems arising from biorthogonal B-splines (see Proposition A.5 and Proposition A.6 in the Appendix, and also [22]). On the other hand, the latter assumption is not strictly necessary for carrying on our construction, yet it will greatly simplify the subsequent formalism. It holds for all reflection invariant systems, such as biorthogonal B-splines.

Boundary adapted generator and wavelet systems can be modified in order to fulfill homogeneous Dirichlet boundary conditions. For the scaling functions this is easily done by omitting those functions that do not vanish at those end points of the interval where boundary conditions are enforced. For the wavelets, the situation is a little bit more involved. To be specific, let us first introduce the following sets of the *internal* grid points:

$$\Delta_j^{int} := \Delta_j \backslash \{0, 1\}, \qquad \nabla_j^{int} := \nabla_j \backslash \{\nu_{j,1}, \nu_{j,M_j}\}. \tag{2.9}$$

Let us collect in the vector $\beta = (\beta_0, \beta_1) \in \{0, 1\}^2$ the information about where homogeneous boundary conditions are enforced, i.e., $\beta_d = 1$ means no boundary condition, whereas $\beta_d = 0$ denotes boundary condition at the point $d \in \{0, 1\}$. The corresponding set of indices is then given by

$$\Delta_j^\beta := \begin{cases} \Delta_j^{int}, & \text{if } \beta = (0, 0), \\ \Delta_j \backslash \{0\}, & \text{if } \beta = (0, 1), \\ \Delta_j \backslash \{1\}, & \text{if } \beta = (1, 0), \\ \Delta_j, & \text{if } \beta = (1, 1). \end{cases} \tag{2.10}$$

Let the generator systems be defined as

$$\Xi_j^\beta := \{\xi_{j,k}\colon k \in \Delta_j^\beta\}, \qquad \tilde{\Xi}_j^\beta := \{\tilde{\xi}_{j,k}\colon k \in \Delta_j^\beta\},$$

and let us define the multiresolution analyses

$$S_j^\beta := \operatorname{span} \Xi_j^\beta, \qquad \tilde{S}_j^\beta := \operatorname{span} \tilde{\Xi}_j^\beta. \tag{2.11}$$

The associated biorthogonal wavelet systems Υ_j^β, $\tilde{\Upsilon}_j^\beta$ are the same as the boundary adapted ones except that we possibly change the first and/or the last wavelet depending on β. More precisely, the wavelets can be chosen to vanish at each boundary point in which the corresponding component of β is zero. If the boundary condition is prescribed at 0, the first wavelets $\eta_{j,\nu_{j,1}}$ and $\tilde{\eta}_{j,\nu_{j,1}}$ are replaced by

$$\eta^D_{j,\nu_{j,1}} := \frac{1}{\sqrt{2}}\,(\eta_{j,\nu_{j,1}} - \xi_{j,0}), \qquad \tilde{\eta}^D_{j,\nu_{j,1}} := \frac{1}{\sqrt{2}}\,(\tilde{\eta}_{j,\nu_{j,1}} - \tilde{\xi}_{j,0}), \tag{2.12}$$

respectively. The wavelets $\eta^D_{j,\nu_j,Mj}$ and $\tilde{\eta}^D_{j,\nu_j,M_j}$ vanishing at 1 are defined similarly. We refer to Appendix A (see Corollary A.7) for the detailed construction of the new wavelet systems. Observe that the set of grid points ∇_j^β which labels the wavelets does not change, i.e., $\nabla_j^\beta = \nabla_j$ for all choices of β.

The new systems Ξ_j^β, Υ_j^β and $\tilde{\Xi}_j^\beta$, $\tilde{\Upsilon}_j^\beta$ fulfill the conditions in (2.3) stated above, provided the index β is appended to all symbols. To be more precise, in (2.3.e) the space of polynomials $\mathbb{P}_r^\beta(0, 1)$ is defined as

$$\mathbb{P}_r^\beta(0, 1) := \{p \in \mathbb{P}_r(0, 1)\colon p(d) = 0 \quad \text{if} \quad \beta_d = 0, \quad \text{for } d = 0, 1\};$$

in (2.3.g), the projection operators Π_j^β are defined as

$$\Pi_j^\beta v := \sum_{k \in \Delta_j^\beta} (v, \tilde{\xi}_{j,k})_{L^2(0,1)}\xi_{j,k};$$

finally, in (2.3.d) the Sobolev spaces $H^s_\beta(0, 1)$ are defined as

$$H^s_\beta(0, 1) := \{v \in H^s(0, 1)\colon v(d) = 0 \quad \text{if} \quad \beta_d = 0, \quad \text{for } d = 0, 1\}, \tag{2.13}$$

for $s \in \mathbb{N}\backslash\{0\}$, and by interpolation for $s \notin \mathbb{N}$, $s > 0$. Note that, unlike a common notation for Sobolev spaces with boundary conditions, we only require the vanishing of v, not of its derivatives, even in the case $s > \frac{3}{2}$.

Finally, suppose that the systems Ξ_j and $\tilde{\Xi}_j$ are reflection invariant, see (2.4). Then, the systems with boundary conditions can be built to be reflection invariant as well, in an obvious sense (i.e., the mapping $x \mapsto 1 - x$ induces a mapping of Ξ_j^β into itself if $\beta = (0, 0)$ or $\beta = (1, 1)$, while it produces an exchange of $\Xi_j^{(0,1)}$ with $\Xi_j^{(1,0)}$ in the other cases).

3. TENSOR PRODUCTS

The perhaps simplest way to build multivariate wavelets based on univariate ones is to employ tensor products. In this section we set up the notation for biorthogonal multiresolution analyses in $\bar{\hat{\Omega}}$, where $\hat{\Omega} = (0, 1)^n$, and we collect some properties that are well known. The notation in this section is already taylored to the kind of application of this material in the rest of the paper, namely, using $\hat{\Omega}$ as a reference domain.

Let us fix a vector $b = (\beta^1, \ldots, \beta^n)$ containing the information on the particular boundary conditions, where each $\beta^l \in \{0, 1\}^2$ for $1 \leq l \leq n$. Let us set, for all $j \geq j_0$,

$$V_j^b(\hat{\Omega}) := S_j^{\beta^1} \otimes \cdots \otimes S_j^{\beta^n},$$

and similarly for $\tilde{V}_j^b(\hat{\Omega})$. In order to construct a basis for these spaces, we define for $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \hat{\Omega}$ and $\hat{k} = (\hat{k}_1, \ldots, \hat{k}_n) \in \Delta_j^b := \Delta_j^{\beta^1} \times \cdots \times \Delta_j^{\beta^n}$

$$\hat{\varphi}_{j,\hat{k}}(\hat{x}) := (\xi_{j,\hat{k}_1} \otimes \cdots \otimes \xi_{j,\hat{k}_n})(\hat{x}) = \prod_{l=1}^{n} \xi_{j,\hat{k}_l}(\hat{x}_l);$$

we set

$$\hat{\Phi}_j := \{\hat{\varphi}_{j,\hat{k}} \colon \hat{k} \in \Delta_j^b\},$$

so that

$$V_j^b(\hat{\Omega}) = \operatorname{span}(\hat{\Phi}_j), \qquad \tilde{V}_j^b(\hat{\Omega}) = \operatorname{span}(\tilde{\hat{\Phi}}_j).$$

Similarly to the univariate case, let us introduce the function spaces $H_b^s(\hat{\Omega})$ by

$$H_b^s(\hat{\Omega}) := \{\hat{v} \in H^s(\hat{\Omega}) \colon \hat{v}_{|\{\hat{x}_l = d\}} \equiv 0 \quad \text{if} \quad \beta_d^l = 0, \quad \text{for } l = 1, \ldots, n,\ d = 0, 1\}, \tag{3.1}$$

for $s \in \mathbb{N} \backslash \{0\}$, and by interpolation for $s \notin \mathbb{N}$, $s > 0$. Then, we note that

$$V_j^b(\hat{\Omega}) \subset H_b^{\gamma}(\hat{\Omega}), \qquad \tilde{V}_j^b(\hat{\Omega}) \subset H_b^{\tilde{\gamma}}(\hat{\Omega}).$$

It is trivially seen that these spaces are nested, i.e., $V_j^b(\hat{\Omega}) \subset V_{j+1}^b(\hat{\Omega})$, $\tilde{V}_j^b(\hat{\Omega}) \subset \tilde{V}_{j+1}^b(\hat{\Omega})$. For the corresponding projectors, we define

$$\hat{P}_j^b \hat{v} := (\Pi_{j,1}^{\beta^1} \otimes \cdots \otimes \Pi_{j,n}^{\beta^n})\hat{v} = \sum_{\hat{k} \in \Delta_j^b} (\hat{v}, \tilde{\hat{\varphi}}_{j,\hat{k}})_{L^2(\hat{\Omega})} \hat{\varphi}_{j,\hat{k}}\,, \tag{3.2}$$

where $\Pi_{j,l}^{\beta^l}$ denotes the application of Π_j^{β} with respect to the direction l, $1 \leq l \leq n$ and $\beta = \beta^l$. Indeed, using the induction principle, (3.2) can be seen by the following reasoning

$$(\Pi_{j,1}^{\beta^1} \otimes \Pi_{j,2}^{\beta^2})\hat{v} = \Pi_{j,2}^{\beta^2}(\Pi_{j,1}^{\beta^1}\hat{v}(\hat{x}_1, \cdot)) = \Pi_{j,2}^{\beta^2}\Big(\sum_{\hat{k}_2 \in \Delta_j^{\beta^2}} (\hat{v}(\hat{x}_1, \cdot), \tilde{\xi}_{j,\hat{k}_2})_{L^2(0,1)}\xi_{j,\hat{k}_2}\Big)$$

$$= \sum_{\hat{k}_1 \in \Delta_j^{\beta^1}} \sum_{\hat{k}_2 \in \Delta_j^{\beta^2}} (\hat{v}, \tilde{\xi}_{j,\hat{k}_1} \otimes \tilde{\xi}_{j,\hat{k}_2})_{L^2((0,1)^2)}\xi_{j,\hat{k}_1} \otimes \xi_{j,\hat{k}_2}.$$

Moreover, the properties $\|\hat{P}_j^b\| \lesssim 1$, $\hat{P}_j^b\hat{P}_{j+1}^b = \hat{P}_j^b$, and $(\hat{P}_j^b)^2 = \hat{P}_j^b$ easily follow from the tensor product structure of $\hat{P}_j^b$. The polynomial exactness is trivial and the stability is implied by the biorthogonality

$$(\hat{\varphi}_{j,\hat{k}}, \hat{\tilde{\varphi}}_{j,\hat{k}'})_{L^2(\hat{\Omega})} = \delta_{\hat{k},\hat{k}'}, \qquad \hat{k}, \hat{k}' \in \Delta_j^b,$$

and the locality of the generators [16, 20]; it can also directly be checked by the following reasoning (here, for simplicity we set $n = 2$)

$$\|\sum_{\hat{k} \in \Delta_j^b} c_{\hat{k}}\hat{\varphi}_{j,\hat{k}}\|_{L^2((0,1)^2)} = \|\sum_{\hat{k}_2 \in \Delta_j^{\beta^2}} \Big(\sum_{\hat{k}_j \in \Delta_j^{\beta^1}} c_{\hat{k}}\xi_{j,\hat{k}_1}\Big)\xi_{j,\hat{k}_2}\|_{L^2(0,1;L^2(0,1))}$$

$$\sim \|\{\sum_{\hat{k}_1 \in \Delta_j^{\beta^1}} c_{\hat{k}}\xi_{j,\hat{k}_1}\}_{\hat{k}_2 \in \Delta_j^{\beta^2}}\|_{L^2(0,1;\ell^2(\Delta_j^{\beta^2}))} \sim \|\{c_{\hat{k}}\}_{\hat{k} \in \Delta_j^b}\|_{\ell^2(\Delta_j^b)}.$$

The Jackson and Bernstein inequalities, which extend in an obvious way (2.3.h) and (2.3.i), are well known to be implied by general principles [16, 20], but they can also be directly deduced by the univariate properties (here for simplicity for $n = 2$): for each $\hat{v} \in H_b^s((0, 1)^2)$ with $0 \le s \le \min(L, \gamma)$

$$\|\hat{v} - \hat{P}_j^b\hat{v}\|_{L^2((0,1)^2)} \le \|\hat{v} - \Pi_{j,1}^{\beta^1}\hat{v}\|_{L^2((0,1)^2)} + \|\Pi_{j,1}^{\beta^1}(\hat{v} - \Pi_{j,2}^{\beta^2}\hat{v})\|_{L^2((0,1)^2)}$$

$$\lesssim 2^{-sj}\|\hat{v}\|_{H^s(0,1;L^2(0,1))} + \|\hat{v} - \Pi_{j,2}^{\beta^2}\hat{v}\|_{L^2((0,1)^2)} \lesssim 2^{-sj}\|\hat{v}\|_{H^s((0,1)^2)}.$$

Let us now consider complement spaces $W_j^b(\hat{\Omega})$ and $\tilde{W}_j^b(\hat{\Omega})$ such that

$$V_{j+1}^b(\hat{\Omega}) = V_j^b(\hat{\Omega}) \oplus W_j^b(\hat{\Omega}), \qquad \tilde{V}_{j+1}^b(\hat{\Omega}) = \tilde{V}_j^b(\hat{\Omega}) \oplus \tilde{W}_j^b(\hat{\Omega}),$$

$$W_j^b(\hat{\Omega}) \perp \tilde{V}_j^b(\hat{\Omega}), \qquad \tilde{W}_j^b(\hat{\Omega}) \perp V_j^b(\hat{\Omega}).$$

Let us set $\nabla_j^b := \Delta_{j+1}^b \backslash \Delta_j^b$. Given any $\hat{h} = (\hat{h}_1, \ldots, \hat{h}_n) \in \nabla_j^b$, we define the corresponding wavelet

$$\hat{\psi}_{j,\hat{h}}(\hat{x}) := (\hat{\vartheta}_{\hat{h}_1} \otimes \cdots \otimes \hat{\vartheta}_{\hat{h}_n})(\hat{x}) = \prod_{l=1}^{n} \hat{\vartheta}_{\hat{h}_l}(\hat{x}_l),$$

where

$$\hat{\vartheta}_{\hat{h}_l} := \begin{cases} \xi_{j,\hat{h}_l}, & \text{if } \hat{h}_l \in \Delta_j^{\beta^l}, \\ \eta_{j,\hat{h}_l}, & \text{if } \hat{h}_l \in \nabla_j, \end{cases}$$

and we set

$$\hat{\Psi}_j := \{\hat{\psi}_{j,\hat{h}}: \hat{h} \in \nabla_j^b\}, \qquad W_j^b(\hat{\Omega}) := \operatorname{span} \hat{\Psi}_j.$$

A parallel construction is done for the dual complement space $\tilde{W}_j^b(\hat{\Omega})$. Due to the univariate properties, we have

$$(\hat{\varphi}_{j,\hat{k}}, \widehat{\tilde{\psi}}_{j,\hat{h}})_{L^2(\hat{\Omega})} = 0, \qquad (\hat{\psi}_{j,\hat{k}}, \hat{\tilde{\varphi}}_{j,\hat{h}})_{L^2(\hat{\Omega})} = 0, \qquad \forall \hat{k} \in \Delta_j^b,\ \forall \hat{h} \in \nabla_j^b,$$

as well as

$$(\hat{\psi}_{j,\hat{h}}, \widehat{\tilde{\psi}}_{j',\hat{h}'})_{L^2(\hat{\Omega})} = \delta_{j,j'}\delta_{\hat{h},\hat{h}'}, \qquad \forall j, j' \geq j_0,\ \forall \hat{h} \in \nabla_j^b,\ \forall \hat{h}' \in \nabla_{j'}^b.$$

Moreover, the wavelets form a Riesz basis in $L^2(\hat{\Omega})$ and the norm equivalences (2.3.l) extend to the multivariate case.

Finally, considering the boundary values, we note that, given any $l \in \{1, \ldots, n\}$ and $d \in \{0, 1\}$, we have

$$(\hat{\varphi}_{j,\hat{k}})_{|\hat{x}_l = d} \equiv 0 \qquad \text{iff } \hat{k}_l \neq d \text{ or } (\hat{k}_l = d \text{ and } \beta_d^l = 0)$$

and

$$(\hat{\psi}_{j,\hat{h}})_{|\hat{x}_l = d} \equiv 0 \qquad \text{iff } \hat{h}_l \neq \nu_d \text{ or } (\hat{h}_l = \nu_d \text{ and } \beta_d^l = 0),$$

with $\nu_d = \nu_{j,1}$ if $d = 0$, and $\nu_d = \nu_{j,M_j}$ if $d = 1$.

4. MULTIRESOLUTION ON GENERAL DOMAINS

Recently, various constructions of multilevel decompositions on general bounded domains have been introduced [10, 22]. Some aspects of the latter are closely related to our approach. The idea is to subdivide the domain of interest $\Omega \subset \mathbb{R}^n$ into subdomains Ω_i, which are images of the reference element $\hat{\Omega} = (0, 1)^n$. The multiresolution analysis on Ω is then obtained by transformations of properly matched systems on $\hat{\Omega}$.

Let us first set some notation, starting with the reference domain $\hat{\Omega}$. For $0 \leq p \leq n - 1$, a *p-face* of $\hat{\Omega}$ is a subset $\hat{\sigma} \subset \partial\hat{\Omega}$ defined by the choice of a set $\mathcal{L}_{\hat{\sigma}}$ of indices $l_1, \ldots, l_{n-p} \in \{1, \ldots, n\}$ and a set of integers $d_1, \ldots, d_{n-p} \in \{0, 1\}$ in the following way

$$\hat{\sigma} = \{(\hat{x}_1, \ldots, \hat{x}_n): \hat{x}_{l_1} = d_1, \ldots, \hat{x}_{l_{n-p}} = d_{n-p}, \text{ and } 0 \leq \hat{x}_l \leq 1 \text{ if } l \notin \mathcal{L}_{\hat{\sigma}}\} \tag{4.1}$$

(thus, e.g., in 3D, a 0-face is a vertex, a 1-face is a side, and a 2-face is a usual face of the reference cube). The coordinates $\hat{x}_l$ with $l \in \mathcal{L}_{\hat{\sigma}}$ will be termed the *frozen coordinates* of $\hat{\sigma}$, whereas the remaining coordinates will be termed the *free coordinates* of $\hat{\sigma}$.

Let $\hat{\sigma}$ and $\hat{\sigma}'$ be two p-faces of $\hat{\Omega}$, and let $H: \hat{\sigma} \to \hat{\sigma}'$ be a bijective mapping. We shall say that H is *order-preserving* if it is a composition of elementary permutations $(s, t) \mapsto$

(t, s) of the free coordinates of $\hat{\sigma}$. An order-preserving mapping is a particular case of an affine mapping, as made precise by the following simple lemma.

LEMMA 4.1. *H is affine if and only if it is a composition of elementary permutations $(s, t) \mapsto (t, s)$ and reflections $s \mapsto 1 - s$ of the free coordinates of $\hat{\sigma}$.*

Proof. If H is order-preserving it is trivially seen that H is affine. Conversely, by neglecting the frozen coordinates, we can assume that H: $[0, 1]^p \to [0, 1]^p$; then, we use induction on p. If $p = 1$, the result is obvious. Otherwise, set $H_0 := H([0, 1]^{p-1} \times \{0\})$ and $H_1 := H([0, 1]^{p-1} \times \{1\})$. If H is affine, then $H([0, 1]^p) = \{(1 - s)H_0 + sH_1$: $s \in [0, 1]\} = [0, 1]^p$, since H is bijective. Thus, necessarily, H_0 is a $(p - 1)$-face, say $H_0 = \{\hat{x}'_l = d\}$ for some $l \in \{1, \dots, p\}$ and some $d \in \{0, 1\}$, whereas $H_1 = \{\hat{x}'_l = 1 - d\}$. It follows that H maps $\hat{x}_p$ into $\hat{x}'_l = \hat{x}_p$ (if $d = 0$) or $\hat{x}'_l = 1 - \hat{x}_p$ (if $d = 1$). Since H: $[0, 1]^{p-1} \times \{0\} \to H_0$ is affine and bijective, we conclude by induction. ■

Let us now consider our domain of interest $\Omega \subset \mathbb{R}^n$, with Lipschitz boundary $\partial\Omega$. We assume that there exist N open disjoint subdomains $\Omega_i \subseteq \Omega$ $(i = 1, \dots, N)$ such that

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$$

and such that, for some $r \geq \gamma$ (see (2.3.d)), there exist r-time continuously differentiable mappings $F_i : \bar{\hat{\Omega}} \to \bar{\Omega}_i$ $(i = 1, \dots, N)$ satisfying

$$\Omega_i = F_i(\hat{\Omega}), \qquad \det(JF_i) > 0 \text{ in } \bar{\hat{\Omega}},$$

where JF_i denotes the Jacobian of F_i; in the sequel, it will be useful to set $G_i := F_i^{-1}$. The image of a p-face of $\hat{\Omega}$ under the mapping F_i will be termed a p-face of Ω_i; if $\Gamma_{i,i'} := \partial\Omega_i \cap \partial\Omega_{i'}$ is nonempty for some $i \neq i'$, then we assume that $\Gamma_{i,i'}$ is a p-face of both Ω_i and $\Omega_{i'}$ for some $0 \leq p \leq n - 1$. In addition, setting

$$\Gamma_{i,i'} = F_i(\hat{\sigma}) = F_{i'}(\hat{\sigma}'),$$

with two p-faces $\hat{\sigma}$ and $\hat{\sigma}'$ of $\hat{\Omega}$, we require that the bijection

$$H_{i,i'} := G_{i'} \circ F_i \colon \hat{\sigma} \to \hat{\sigma}'$$

fulfills the following Hypothesis (4.2):

(a) $H_{i,i'}$ is affine;
(b) in addition, if the systems of scaling functions and wavelets on $[0, 1]$ are *not* reflection invariant (see (2.4)), then $H_{i,i'}$ is order-preserving.

Remark 4.2. Suppose that Hypothesis (4.2) holds true. It is easily seen that, if $n = 2$, it is always possible to modify the mappings F_i in such a way that the new mappings $H_{i,i'}$ are all order-preserving. However, this is not true if $n = 3$, as the example of a 3D Moebius ring indicates.

The boundary $\partial\Omega$ is subdivided in two relatively open parts (with respect to $\partial\Omega$), the Dirichlet part Γ_D and the Neumann part Γ_N, in such a way that

$$\overline{\partial\Omega} = \bar{\Gamma}_D \cup \bar{\Gamma}_N, \qquad \Gamma_D \cap \Gamma_N = \emptyset,$$

where for $i = 1, \ldots, N$ we suppose that $\partial\Omega_i \cap \bar{\Gamma}_D$ and $\partial\Omega_i \cap \bar{\Gamma}_N$ are (possibly empty) unions of p-faces of Ω_i.

4.1. Multiresolution and Wavelets on the Subdomains

Let us now introduce multiresolution analyses on each Ω_i, $i = 1, \ldots, N$, by “mapping” appropriate multiresolution analyses on $\hat{\Omega}$.

To this end, let us define the vector $b(\Omega_i) = (\beta^1, \ldots, \beta^n) \in \{0, 1\}^{2n}$ as

$$\beta_d^l = \begin{cases} 0, & \text{if } F_i(\{\hat{x}_l = d\}) \subset \Gamma_D, \\ 1, & \text{otherwise}, \end{cases} \quad l = 1, \ldots, n, \quad d = 0, 1.$$

Moreover, let us introduce the one-to-one transformation

$$v \mapsto \hat{v} := v \circ F_i,$$

which maps functions defined in $\bar{\Omega}$ into functions defined in $\bar{\hat{\Omega}}$. Next, for all $j \geq j_0$, let us set

$$V_j(\Omega_i) := \{v \colon \hat{v} \in V_j^{b(\Omega_i)}(\hat{\Omega})\}.$$

If we introduce, for any $s \geq 0$, the Sobolev spaces

$$H_b^s(\Omega_i) = \{v \colon \hat{v} \in H^s_{b(\Omega_i)}(\hat{\Omega})\}$$

(see (3.1)), we observe that

$$V_j(\Omega_i) \subset H_b^\gamma(\Omega_i). \tag{4.3}$$

The projection operators $P_j^{\Omega_i} \colon L^2(\Omega_i) \to V_j(\Omega_i)$ are defined by the commutativity relation

$$(P_j^{\Omega_i} v)\hat{} := \hat{P}_j^{b(\Omega_i)} \hat{v}, \qquad \forall v \in L^2(\Omega_i).$$

This definition suggests to equip $L^2(\Omega_i)$ by the inner product

$$\langle u, v \rangle_{\Omega_i} := \int_{\Omega_i} u(x) v(x) |JG_i(x)| dx = \int_{\hat{\Omega}} \hat{u}(\hat{x}) \hat{v}(\hat{x}) d\hat{x}, \tag{4.4}$$

which, due to the properties of the transformation of the domains, induces an equivalent L^2-type norm

$$\|v\|^2_{L^2(\Omega_i)} \sim \langle v, v\rangle_{\Omega_i} = \|\hat{v}\|^2_{L^2(\hat{\Omega})}, \qquad \forall v \in L^2(\Omega_i).$$

Let us now define the single scale basis functions for the above defined multiresolution spaces. To this end, for $i = 1, \ldots, N$, let us consider the set $\Delta_j^{b(\Omega i)}$ of grid points in $\bar{\hat{\Omega}}$ and let us define

$$k^{(i)} := F_i(\hat{k}), \qquad \hat{k} \in \Delta_j^{b(\Omega_i)}$$

and

$$\mathcal{K}_j^i := \{k^{(i)} \colon \hat{k} \in \Delta_j^{b(\Omega_i)}\}.$$

In this way, we have a set of grid points in $\bar{\Omega}_i$. Each grid point can be associated to a basis function in $V_j(\Omega_i)$. Precisely, for each $k \in \mathcal{K}_j^i$ let us set $\hat{k} = \hat{k}(i) := G_i(k)$ and let us define the function

$$\varphi_{j,k}^{(i)} := \hat{\varphi}_{j,\hat{k}} \circ G_i,$$

i.e., $\widehat{\varphi_{j,k}^{(i)}} = \hat{\varphi}_{j,\hat{k}}$. The set of these functions will be denoted by Φ_j^i. This set and the dual set $\tilde{\Phi}_j^i$ form biorthogonal bases of $V_j(\Omega_i)$ and $\tilde{V}_j(\Omega_i)$, respectively, with respect to the inner product (4.4); indeed

$$\langle \varphi_{j,k}^{(i)}, \tilde{\varphi}_{j,k'}^{(i)}\rangle_{\Omega_i} = (\hat{\varphi}_{j,\hat{k}}, \hat{\tilde{\varphi}}_{j,\hat{k}'})_{L^2(\hat{\Omega})} = \delta_{k,k'}, \qquad \hat{k} = G_i(k),\ \hat{k}' = G_i(k'). \tag{4.5}$$

This yields the following representation of $P_j^{\Omega_i}$:

$$P_j^{\Omega_i} v = \sum_{k \in \mathcal{K}_j^i} \langle v, \tilde{\varphi}_{j,k}^{(i)}\rangle_{\Omega_i} \varphi_{j,k}^{(i)} = \sum_{\hat{k} \in \Delta_j^{b(\Omega_i)}} (\hat{v}, \hat{\tilde{\varphi}}_{j,\hat{k}})_{L^2(\hat{\Omega})} \hat{\varphi}_{j,\hat{k}} \circ G_i.$$

It is easily seen that the dual multiresolution analyses on Ω_i defined in this way inherit the properties of the multiresolution analyses on $\hat{\Omega}$ as far as stability of bases, properties of the biorthogonal projectors, and Jackson and Bernstein inequalities (and consequent characterization of function spaces) are concerned. Obviously, the property of exact reconstruction of polynomials has to be replaced by the property of exact reconstruction of the images of polynomials under the transformation F_i. For subsequent reference, we report the Jackson and Bernstein inequalities in Ω_i:

$$\|v - P_j^{\Omega_i} v\|_{L^2(\Omega_i)} \lesssim 2^{-sj} \|v\|_{H^s(\Omega_i)}, \qquad \forall v \in H_b^s(\Omega_i),\ 0 \le s \le \min(L, \gamma); \tag{4.6}$$

$$\|v\|_{H^s(\Omega_i)} \lesssim 2^{sj} \|v\|_{L^2(\Omega_i)}, \qquad \forall v \in V_j(\Omega_i),\ 0 \le s \le \gamma. \tag{4.7}$$

Finally, we come to the detail spaces. The biorthogonal complement of $V_j(\Omega_i)$ in $V_{j+1}(\Omega_i)$ is

$$W_j(\Omega_i) := \{w\colon \hat{w} \in W_j^{b(\Omega_i)}(\hat{\Omega})\}.$$

A biorthogonal basis in this space is associated to the grid

$$\mathcal{H}_j^i := \mathcal{K}_{j+1}^i \backslash \mathcal{K}_j^i = \{h = F_i(\hat{h})\colon \hat{h} \in \nabla_j^{b(\Omega_i)}\} \tag{4.8}$$

through the relation

$$\psi_{j,h}^{(i)} := \hat{\psi}_{j,\hat{h}} \circ G_i, \qquad \forall h \in \mathcal{H}_j^i. \tag{4.9}$$

The set of such functions will be denoted by Ψ_j^i, and the dual set by $\tilde{\Psi}_j^i$.

4.2. *Multiresolution on the Global Domain*

Now we describe the construction of dual multiresolution analyses on $\bar{\Omega}$. Let us define, for all $j \geq j_0$,

$$V_j(\Omega) := \{v \in C^0(\bar{\Omega})\colon v_{|\Omega_i} \in V_j(\Omega_i),\ i = 1, \ldots, N\}; \tag{4.10}$$

the dual spaces $\tilde{V}_j(\Omega)$ are defined in a similar manner, simply by replacing each $V_j(\Omega_i)$ by $\tilde{V}_j(\Omega_i)$. Then, nestedness is obvious from the analogous property in each Ω_i.

We shall now define an appropriate functional setting for the above family of spaces. To this end, let us introduce the Sobolev spaces $H_b^s(\Omega)$ by

$$H_b^s(\Omega) := \{v \in H^s(\Omega)\colon v = 0 \text{ on } \Gamma_D\} \tag{4.11}$$

for $s \in \mathbb{N}\backslash\{0\}$, and by interpolation for $s \notin \mathbb{N}$, $s > 0$. Furthermore, we introduce another scale of Sobolev spaces, depending upon the partition $\mathcal{P} := \{\Omega_i\colon i = 1, \ldots, N\}$ of Ω; precisely, we set

$$H_b^s(\Omega; \mathcal{P}) := \{v \in H_b^1(\Omega)\colon v_{|\Omega_i} \in H^s(\Omega_i),\ i = 1, \ldots, N\} \tag{4.12}$$

for any integer $s \in \mathbb{N}\backslash\{0\}$, and we extend the definition using interpolation for any $s \notin \mathbb{N}$, $s > 0$. Note that, for any real $s \geq 1$, $H_b^s(\Omega; \mathcal{P})$ can indeed be defined directly by (4.12); moreover,

$$\|v\|_{H_b^s(\Omega;\mathcal{P})} \sim \sum_{i=1}^{N} \|v_{|\Omega_i}\|_{H^s(\Omega_i)}, \qquad \forall v \in H_b^s(\Omega; \mathcal{P}).$$

In addition, $H^s_b(\Omega) \subseteq H^s_b(\Omega; \mathcal{P})$ for all $s \geq 0$, and $H^s_b(\Omega) = H^s_b(\Omega; \mathcal{P})$ for all s satisfying $0 \leq s < \frac{3}{2}$. Recalling (4.3), one has

$$V_j(\Omega) \subset H^\gamma_b(\Omega; \mathcal{P}). \tag{4.13}$$

In order to define a basis of $V_j(\Omega)$, let us introduce the set

$$\mathcal{K}_j := \bigcup_{i=1}^{N} \mathcal{K}^i_j \tag{4.14}$$

containing all the grid points in $\bar{\Omega}$. The following remark will be useful in the sequel.

Remark 4.3. Suppose that $\Gamma_{i,i'} = \partial\Omega_i \cap \partial\Omega_{i'} \neq \emptyset$ is a p-face, i.e., $\Gamma_{i,i'} = F_i(\hat{\sigma}) = F_{i'}(\hat{\sigma}')$ for two p-faces $\hat{\sigma}$ and $\hat{\sigma}'$ of $\hat{\Omega}$. If $k \in \Gamma_{i,i'} \cap \mathcal{K}_j$, there exist $\hat{k}^{(i)} \in \hat{\sigma} \cap \Delta_j^{b(\Omega_i)}$ and $\hat{k}^{(i')} \in \hat{\sigma}' \cap \Delta_j^{b(\Omega_{i'})}$ such that $k = F_i(\hat{k}^{(i)}) = F_{i'}(\hat{k}^{(i')})$ and the free coordinates of $\hat{k}^{(i')}$ are a permutation and (possibly) a reflection of the free coordinates of $\hat{k}^{(i)}$. This is a straightforward consequence of Hypothesis (4.2).

Each grid point of $\mathcal{K}_j$ can be associated to one single scale basis function of $V_j(\Omega)$, and conversely. To accomplish this, let us set

$$I(k) := \{i \in \{1, \ldots, N\} : k \in \bar{\Omega}_i\}, \qquad \forall k \in \mathcal{K}_j,$$

as well as

$$\hat{k}^{(i)} := G_i(k), \qquad \forall i \in I(k),\ \forall k \in \mathcal{K}_j.$$

Then, for any $k \in \mathcal{K}_j$ let us define the function $\varphi_{j,k}$ as

$$\varphi_{j,k|\Omega_i} := \begin{cases} |I(k)|^{-1/2}\varphi^{(i)}_{j,k}, & \text{if } i \in I(k), \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

This function belongs to $V_j(\Omega)$, since it is continuous across the interelement boundaries. This is a consequence of assumptions (2.5), (2.8), and Remark 4.3. Indeed, if $k \in \Omega_i$ (remember that Ω_i is open) for some i, then $I(k) = \{i\}$ and $\varphi_{j,k}$ vanishes on $\partial\Omega_i$, therefore it is continuous. Suppose now that k belongs to a common face of subdomains, i.e., as before, $k \in \Gamma_{i,i'} = \partial\Omega_i \cap \partial\Omega_{i'} = F_i(\hat{\sigma}) = F_{i'}(\hat{\sigma}')$ for two p-faces $\hat{\sigma}$ and $\hat{\sigma}'$ of $\hat{\Omega}$. Let x be any point of $\Gamma_{i,i'}$, and let $\hat{x} \in \hat{\sigma}$ and $\hat{x}' \in \hat{\sigma}'$ be such that $x = F_i(\hat{x}) = F_{i'}(\hat{x}')$. Then,

$$\varphi_{j,k|\Omega_i}(x) = |I(k)|^{-1/2}\xi_{j,\hat{k}^{(i)}_1}(\hat{x}_i)\cdot\,\cdot\,\cdot\xi_{j,\hat{k}^{(i)}_n}(\hat{x}_n) \tag{4.16}$$

and

$$\varphi_{j,k|\Omega_{i'}}(x) = |I(k)|^{-1/2}\xi_{j,\hat{k}^{(i')}_1}(\hat{x}'_1)\cdot\,\cdot\,\cdot\xi_{j,\hat{k}^{(i')}_n}(\hat{x}'_n). \tag{4.17}$$

Now, in (4.16) there are exactly $n - p$ factors of type $\xi_{j,\hat{k}_l^{(i)}}(\hat{x}'_l)$ corresponding to frozen coordinates in $\hat{\sigma}$; similarly, in (4.17), there are exactly $n - p$ factors of type $\xi_{j,\hat{k}_l^{(i')}}(\hat{x}'_l)$ corresponding to frozen coordinates in $\hat{\sigma}'$. By assumption (2.8), all these factors are equal. The remaining p factors in (4.16) appear in (4.17) as well, possibly in a different order, due to Hypothesis (4.2). Thus, $\varphi_{j,k|\Omega_i}(x) = \varphi_{j,k|\Omega_{i'}}(x)$ for all $x \in \Gamma_{i,i'}$. Finally note that $\varphi_{j,k}$ is identically zero on each face of subdomains which does not contain k.

Let us set $\Phi_j := \{\varphi_{j,k}\colon k \in \mathcal{K}_j\}$. The dual family $\tilde{\Phi}_j := \{\tilde{\varphi}_{j,k}\colon k \in \mathcal{K}_j\}$ is defined as in (4.15), simply by replacing each $\varphi_{j,k}^{(i)}$ by $\tilde{\varphi}_{j,k}^{(i)}$. Then, we have

$$V_j(\Omega) = \operatorname{span} \Phi_j, \qquad \tilde{V}_j(\Omega) = \operatorname{span} \tilde{\Phi}_j.$$

By defining the L^2-type inner product on Ω

$$\langle u, v\rangle_\Omega := \sum_{i=1}^{N} \langle u, v\rangle_{\Omega_i}, \tag{4.18}$$

it is easy to obtain the biorthogonality relations

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,k'}\rangle_\Omega = (|I(k)||I(k')|)^{-1/2} \sum_{i\in I(k)\cap I(k')} \langle \varphi_{j,k}^{(i)}, \tilde{\varphi}_{j,k'}^{(i)}\rangle_{\Omega_i} = \delta_{k,k'}, \tag{4.19}$$

from the analogous relations (4.5) in each Ω_i; indeed, $I(k) \cap I(k') \neq \emptyset$ if and only if there exists an index $i \in \{1, \dots, N\}$ such that $k, k' \in \bar{\Omega}_i$. It is easy to check that for each $k \in \mathcal{K}_j$, diam supp $\varphi_{j,k} \sim 2^{-j}$, $\|\varphi_{j,k}\|_{L^2(\Omega)} \lesssim 1$ and card$\{k'\colon \operatorname{supp} \varphi_{j,k} \cap \operatorname{supp} \varphi_{j,k'} \neq \emptyset\} \lesssim 1$. Similar results hold for the dual system $\tilde{\Phi}_j$. Thus, thanks to abstract results about the stability of biorthogonal bases (see, for example, [20]), we have

$$\Phi_j\ (\tilde{\Phi}_j, \text{ resp.}) \text{ is a stable basis in } V_j(\Omega)\ (\tilde{V}_j(\Omega), \text{ resp.}).$$

Let us introduce the biorthogonal projection operator upon $V_j(\Omega)$

$$P_j^\Omega v := \sum_{k\in\mathcal{K}_j} \langle v, \tilde{\varphi}_{j,k}\rangle_\Omega \varphi_{j,k}, \qquad \forall v \in L^2(\Omega).$$

The properties

$$P_j^\Omega v = v, \qquad \forall v \in V_j(\Omega),$$

$$P_j^\Omega P_{j+1}^\Omega v = P_j^\Omega v, \qquad \forall v \in L^2(\Omega),$$

$$\|P_j^\Omega\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \lesssim 1$$

(and the dual ones) are obvious by the construction of the spaces $V_j(\Omega)$ and their basis Φ_j.

It is useful to compare $P_j^\Omega v$ with $P_j^{\Omega_i} v$ in Ω_i $(i = 1, \dots, N)$. We have

$$
\begin{aligned}
P_j^\Omega v_{|\Omega_i} &= \sum_{k\in\mathcal{K}_j^i} |I(k)|^{-1/2}\langle v,\, \tilde\varphi_{j,k}\rangle_\Omega \varphi_{j,k}^{(i)} \\
&= \sum_{k\in\mathcal{K}_j^i} \Big(|I(k)|^{-1} \sum_{i'\in I(k)} \langle v,\, \tilde\varphi_{j,k}^{(i')}\rangle_{\Omega_{i'}}\Big)\varphi_{j,k}^{(i)}.
\end{aligned}
\tag{4.20}
$$

Thus, setting $R_j^{(i)}v := P_j^{\Omega_i}v - P_j^\Omega v_{|\Omega_i}$ and recalling that $|I(k)| = 1$ if $k \in \Omega_i$, we obtain

$$
R_j^{(i)}v = \sum_{k\in\mathcal{K}_j\cap\partial\Omega_i} r_{j,k}^{(i)}\varphi_{j,k}^{(i)}, \tag{4.21}
$$

where

$$
r_{j,k}^{(i)} := |I(k)|^{-1} \sum_{i'\in I(k)} [\langle v,\, \tilde\varphi_{j,k}^{(i)}\rangle_{\Omega_i} - \langle v,\, \tilde\varphi_{j,k}^{(i')}\rangle_{\Omega_{i'}}].
$$

We shall now establish a Jackson-type inequality for P_j^Ω. To this end, we need the three following lemmata.

LEMMA 4.4. *Suppose that, for some $l, m \in \{1, \ldots, N\}$, $\Gamma_{l,m} := \partial\Omega_l \cap \partial\Omega_m$ is an $(n-1)$-face. Then, for any $v \in H_b^1(\Omega)$ such that $v_{|\Omega_l} \in H^s(\Omega_l)$ and $v_{|\Omega_m} \in H^s(\Omega_m)$, with $1 \le s \le \min(L,\, \gamma)$, we have*

$$
\sum_{k\in\mathcal{K}_j\cap\Gamma_{l,m}} |\langle v,\, \tilde\varphi_{j,k}^{(l)}\rangle_{\Omega_l} - \langle v,\, \tilde\varphi_{j,k}^{(m)}\rangle_{\Omega_m}|^2 \lesssim 2^{-2sj}[\|v\|^2_{H^s(\Omega_l)} + \|v\|^2_{H^s(\Omega_m)}].
$$

Proof. By our Hypothesis (4.2), it is not restrictive to assume that $\Gamma_{l,m} = F_l(\hat\sigma) = F_m(\hat\sigma')$, where $\hat\sigma = \{(0, \hat x')\colon \hat x' \in [0,1]^{n-1}\}$ and $\hat\sigma' = \{(1, \hat x')\colon \hat x' \in [0,1]^{n-1}\}$. In addition, there is a set $\hat{\mathcal{K}}_j^* \subset [0,1]^{n-1}$ such that for all $k \in \mathcal{K}_j \cap \Gamma_{l,m}$, $k = F_l((0, \hat k^*)) = F_m((1, \hat k^*))$ for some $\hat k^* \in \hat{\mathcal{K}}_j^*$. Consequently, $\varphi_{j,k|\Omega_l} = \hat\varphi_{j,(0,\hat k^*)} \circ G_l$ with $\hat\varphi_{j,(0,\hat k^*)}(\hat x) = \xi_{j,0}(\hat x_1)\hat\varphi_{j,\hat k^*}(\hat x')$, where $\hat\varphi_{j,\hat k^*}$ is a tensor product of $(n-1)$-univariate scaling functions. Analogously, $\varphi_{j,k|\Omega_m} = \hat\varphi_{j,(1,\hat k^*)} \circ G_m$ with $\hat\varphi_{j,(1,\hat k^*)}(\hat x) = \xi_{j,1}(\hat x_1)\hat\varphi_{j,\hat k^*}(\hat x')$. Similar representations hold for the dual functions $\tilde\varphi_{j,k}$. Let us set $\hat v^{(l)} = v_{|\Omega_l} \circ G_l$, $\hat v^{(m)} = v_{|\Omega_m} \circ G_m$. With these notations

$$
\sum_{k\in\mathcal{K}_j\cap\Gamma_{l,m}} |\langle v,\, \tilde\varphi_{j,k}^{(l)}\rangle_{\Omega_l} - \langle v,\, \tilde\varphi_{j,k}^{(m)}\rangle_{\Omega_m}|^2 = \sum_{\hat k^*\in\hat{\mathcal{K}}_j^*} |(\hat v^{(l)},\, \hat{\tilde\varphi}_{j,(0,\hat k^*)})_{L^2(\hat\Omega)} - (\hat v^{(m)},\, \hat{\tilde\varphi}_{j,(1,\hat k^*)})_{L^2(\hat\Omega)}|^2. \tag{4.22}
$$

Now, we apply the inequality $\|f\|_{L^2(\partial\hat\Omega)} \le \|f\|^{1/2}_{L^2(\hat\Omega)}\|f\|^{1/2}_{H^1(\hat\Omega)}$, which holds for all $f \in H^1(\hat\Omega)$, to $f = \hat v^{(l)} - \hat P_j^b \hat v^{(l)}$, with $b = b(\Omega_l)$. Thanks to the characterization of $H^1(\hat\Omega)$ associated to the wavelet system in $\hat\Omega$, we have

$$
\|\hat v^{(l)} - \hat P_j^b \hat v^{(l)}\|_{H^1(\hat\Omega)} \lesssim 2^{-(s-1)j}\|\hat v^{(l)}\|_{H^s(\hat\Omega)};
$$

together with the Jackson inequality in $L^2(\hat\Omega)$, we obtain

$$
\|\hat v^{(l)} - \hat P_j^b \hat v^{(l)}\|_{L^2(\hat\sigma)} \lesssim 2^{-(s-1/2)j}\|\hat v^{(l)}\|_{H^s(\hat\Omega)}. \tag{4.23}
$$

Similarly,

$$\|\hat{v}^{(m)} - \hat{P}_j^b \hat{v}^{(m)}\|_{L^2(\hat{\sigma})} \lesssim 2^{-(s-1/2)j} \|\hat{v}^{(m)}\|_{H^s(\hat{\Omega})}. \tag{4.24}$$

Note that the functions $\hat{x}' \mapsto \hat{v}^{(l)}((0, \hat{x}'))$ and $\hat{x}' \mapsto \hat{v}^{(m)}((1, \hat{x}'))$ coincide (in the sense of $L^2((0, 1)^{n-1})$), since $v_{|\Omega_l}$ and $v_{|\Omega_m}$ have a common trace on $\Gamma_{l,m}$. Moreover, by (2.5),

$$\hat{P}_j^b \hat{v}^{(l)}((0, \hat{x}')) = \sum_{\hat{k}^* \in \hat{\mathcal{K}}_j^*} (\hat{v}^{(l)}, \hat{\tilde{\varphi}}_{j,(0,\hat{k}^*)})_{L^2(\hat{\Omega})} \xi_{j,0}(0) \hat{\varphi}_{j,\hat{k}^*}(\hat{x}'), \qquad \forall \hat{x}' \in [0, 1]^{n-1},$$

$$\hat{P}_j^b \hat{v}^{(m)}((1, \hat{x}')) = \sum_{\hat{k}^* \in \hat{\mathcal{K}}_j^*} (\hat{v}^{(m)}, \hat{\tilde{\varphi}}_{j,(1,\hat{k}^*)})_{L^2(\hat{\Omega})} \xi_{j,1}(1) \hat{\varphi}_{j,\hat{k}^*}(\hat{x}'), \qquad \forall \hat{x}' \in [0, 1]^{n-1},$$

with $\xi_{j,0}(0) = \xi_{j,1}(1) = c2^{j/2}$ by (2.7) and (2.8). Thus, from (4.23), (4.24), and the triangle inequality, we get

$$\Big\| \sum_{\hat{k}^* \in \hat{\mathcal{K}}_j^*} [(\hat{v}^{(l)}, \hat{\tilde{\varphi}}_{j,(0,\hat{k}^*)})_{L^2(\hat{\Omega})} - (\hat{v}^{(m)}, \hat{\tilde{\varphi}}_{j,(1,\hat{k}^*)})_{L^2(\hat{\Omega})}] \hat{\varphi}_{j,\hat{k}^*} \Big\|_{L^2((0,1)^{n-1})} \lesssim 2^{-sj} [\|\hat{v}^{(l)}\|_{H^s(\hat{\Omega})} + \|\hat{v}^{(m)}\|_{H^s(\hat{\Omega})}].$$

Then, the result follows from (4.22) and the stability of the system $\{\hat{\varphi}_{j,\hat{k}^*}\}_{\hat{k}^* \in \hat{\mathcal{K}}_j^*}$ in the space $L^2((0, 1)^{n-1})$. ■

LEMMA 4.5. *Let $k \in \mathcal{K}_j$ and set $C(k) := \{(l, m) \in I(k)^2 : \partial\Omega_l \cap \partial\Omega_m$ is an $(n-1)$-face$\}$. For any $i, i' \in I(k)$,*

$$|\langle v, \tilde{\varphi}_{j,k}^{(i)} \rangle_{\Omega_i} - \langle v, \tilde{\varphi}_{j,k}^{(i')} \rangle_{\Omega_{i'}}|^2 \lesssim |I(k)| \sum_{(l,m) \in C(k)} |\langle v, \tilde{\varphi}_{j,k}^{(l)} \rangle_{\Omega_l} - \langle v, \tilde{\varphi}_{j,k}^{(m)} \rangle_{\Omega_m}|^2.$$

Proof. Under our assumptions on the boundary $\partial\Omega$, there is a sequence of indices $i_1, i_2, \ldots, i_p \in I(k)$ such that $i_1 = i$, $i_p = i'$ and for $1 \le q < p$, $\partial\Omega_{i_q} \cap \partial\Omega_{i_{q+1}}$ is an $(n-1)$-face. Then, the result follows by a telescoping argument. ■

LEMMA 4.6. *Let $i \in \{1, \ldots, N\}$. Set $D(i) := \{i' : \partial\Omega_i \cap \partial\Omega_{i'} \neq \emptyset\}$. Assume that $v \in H_b^1(\Omega; \mathcal{P})$ for some nonnegative $s \le \min(L, \gamma)$. Then*

$$\|R_j^{(i)} v\|_{L^2(\Omega_i)} \lesssim 2^{-sj} \sum_{i' \in D(i)} \|v\|_{H^s(\Omega_{i'})}.$$

Proof. Let us first assume $s \geq 1$. By (4.21) and the L^2-stability of the basis Φ_j^i, we get

$$\begin{aligned}
\|R_j^{(i)} v\|_{L^2(\Omega_i)}^2 &\sim \sum_{k \in \mathcal{K}_j \cap \partial\Omega_i} \Big| |I(k)|^{-1} \sum_{i' \in I(k)} [\langle v, \tilde{\varphi}_{j,k}^{(i)} \rangle_{\Omega_i} - \langle v, \tilde{\varphi}_{j,k}^{(i')} \rangle_{\Omega_{i'}}] \Big|^2 \\
&\le \sum_{k \in \mathcal{K}_j \cap \partial\Omega_i} |I(k)|^{-1} \sum_{i' \in I(k)} |\langle v, \tilde{\varphi}_{j,k}^{(i)} \rangle_{\Omega_i} - \langle v, \tilde{\varphi}_{j,k}^{(i')} \rangle_{\Omega_{i'}}]|^2 \\
&\lesssim \sum_{k \in \mathcal{K}_j \cap \partial\Omega_i} |I(k)| \sum_{(l,m) \in C(k)} |\langle v, \tilde{\varphi}_{j,k}^{(l)} \rangle_{\Omega_l} - \langle v, \tilde{\varphi}_{j,k}^{(m)} \rangle_{\Omega_m}|^2
\end{aligned}$$

by Lemma 4.5. Define $E(i) := \{(l, m) \in D(i)^2 \colon \Gamma_{l,m} := \partial\Omega_l \cap \partial\Omega_m$ is an $(n-1)$-face$\}$. Recalling that $|I(k)| \lesssim 1$ and rearranging the last sum, we get

$$\|R_j^{(i)} v\|^2_{L^2(\Omega_i)} \lesssim \sum_{(l,m)\in E(i)} \sum_{k\in\mathcal{K}_j\cap\Gamma_{l,m}} |\langle v, \tilde{\varphi}^{(l)}_{j,k}\rangle_{\Omega_l} - \langle v, \tilde{\varphi}^{(m)}_{j,k}\rangle_{\Omega_m}|^2,$$

and the result for $s \geq 1$ follows from Lemma 4.4. For $s = 0$, the result is a consequence of the L^2-stability of P_j^Ω and $P_j^{\Omega_i}$, whereas for $0 < s < 1$ we conclude by interpolation. ∎

We are now ready to establish the Jackson inequality for P_j^Ω.

THEOREM 4.7. *Assume that $v \in H^s_b(\Omega; \mathcal{P})$ for some nonnegative $s \leq \min(L, \gamma)$. Then,*

$$\|v - P_j^\Omega v\|_{L^2(\Omega)} \lesssim 2^{-sj} \|v\|_{H^s_b(\Omega;\mathcal{P})}. \tag{4.25}$$

Proof. In each Ω_i, we use the triangle inequality for $v - P_j^\Omega v = (v - P_j^{\Omega_i} v) + R_j^{(i)} v$ and we conclude by (4.6) and Lemma 4.6. ∎

Remark 4.8. Note that (4.25) yields an optimal rate of decay of the approximation error even for those functions which are locally smooth in each subdomain, but not globally smooth in Ω (i.e., functions which do not belong to $H^s(\Omega)$). This feature turns out to be useful, for instance, in the numerical approximation of solutions of partial differential equations.

Finally, we consider the Bernstein inequality. Recalling the inclusion (4.13) and using (4.7), we easily get

$$\|v\|_{H^s_b(\Omega;\mathcal{P})} \lesssim 2^{sj} \|v\|_{L^2(\Omega)}, \qquad \forall v \in V_j(\Omega),\ 0 \leq s \leq \gamma. \tag{4.26}$$

This implies the possibility of characterizing the spaces $H^s_b(\Omega; \mathcal{P})$, as well as their duals, in terms of the L^2-norms of the detail operators $Q_j^\Omega := P_{j+1}^\Omega - P_j^\Omega$. The precise result will be given, after we provide a wavelet basis; see Theorem 5.6.

5. BIORTHOGONAL WAVELETS ON GENERAL DOMAINS

We now construct biorthogonal complement spaces $W_j(\Omega)$ and $\tilde{W}_j(\Omega)$ $(j \geq j_0)$ such that

$$\begin{aligned} V_{j+1}(\Omega) &= V_j(\Omega) \oplus W_j(\Omega), & \tilde{V}_{j+1}(\Omega) &= \tilde{V}_j(\Omega) \oplus \tilde{W}_j(\Omega), \\ V_j(\Omega) &\perp \tilde{W}_j(\Omega), & \tilde{V}_j(\Omega) &\perp W_j(\Omega), \end{aligned} \tag{5.1}$$

as well as the corresponding biorthogonal bases Ψ_j and $\tilde{\Psi}_j$, where the orthogonality is to be understood with respect to $\langle\cdot,\cdot\rangle_\Omega$. Here we detail the construction for the primal functions only, i.e., for Ψ_j, since the dual basis $\tilde{\Psi}_j$ is built in a completely analogous fashion.

To start with, let us define a set of grid points by

$$\mathcal{H}_j := \mathcal{K}_{j+1} \backslash \mathcal{K}_j = \bigcup_{i=1}^{N} \mathcal{H}_j^i$$

(see (4.14) and (4.8)). We shall associate to each $h \in \mathcal{H}_j$ a function $\psi_{j,h} \in V_{j+1}(\Omega)$ and a function $\tilde{\psi}_{j,h} \in \tilde{V}_{j+1}(\Omega)$ such that $\psi_{j,h}$ is orthogonal to $\tilde{V}_j(\Omega)$, $\tilde{\psi}_{j,h}$ is orthogonal to $V_j(\Omega)$, and the biorthogonality conditions $\langle \psi_{j,h}, \tilde{\psi}_{j,h'} \rangle_\Omega = \delta_{h,h'}$ hold. Then, setting $\Psi_j := \{\psi_{j,h}: h \in \mathcal{H}_j\}$, $\tilde{\Psi}_j := \{\tilde{\psi}_{j,h}: h \in \mathcal{H}_j\}$, it will be clear that the spaces $W_j(\Omega) :=$ span Ψ_j and $\tilde{W}_j(\Omega) :=$ span $\tilde{\Psi}_j$ satisfy (5.1) (see Theorem 5.5).

The construction will proceed as follows. Firstly, we build wavelets supported in the closure of only one subdomain. Next, we match wavelets and scaling functions across faces common to subdomains, starting from 0-faces and increasing the dimension of the face. Finally, the locally supported systems arising from the matching are biorthogonalized.

Let us fix $h \in \mathcal{H}_j$. By definition, there exists $i \in \{1, \ldots, N\}$ and $\hat{h} = \hat{h}^{(i)} \in \nabla_j^{b(\Omega_i)}$ such that $h = F_i(\hat{h})$, i.e., $\hat{h}$ is the corresponding grid point on the reference domain. Recalling the definition of internal grid points on the reference interval [0, 1] (see (2.9)), let $p = p(\hat{h}) \in \{0, \ldots, n\}$ be the number of components $\hat{h}_l$ of $\hat{h}$ belonging to $\Delta_j^{int} \cup \nabla_j^{int}$. Furthermore, let us define the auxiliary point $h^* := F_i(\hat{h}^*) \in \mathcal{K}_{j+1}$ by setting, for $1 \le l \le n$,

$$\hat{h}_l^* := \begin{cases} \hat{h}_l, & \text{if } \hat{h}_l \text{ is internal}, \\ 0, & \text{if } \hat{h}_l \in \{0, \nu_{j,1}\}, \\ 1, & \text{if } \hat{h}_l \in \{1, \nu_{j,M_j}\}. \end{cases} \tag{5.2}$$

The mapping $h \mapsto h^*$ will be denoted by $\mathcal{F}$. To be precise, we should write $\mathcal{F}_i$, but since $\mathcal{F}_i(h) = \mathcal{F}_{i'}(h)$ if $h \in \partial\Omega_i \cap \partial\Omega_{i'}$, we are allowed to drop the index of the subdomains and to consider $\mathcal{F}$ as a mapping from $\mathcal{H}_j$ to $\mathcal{K}_{j+1}$. It will be useful to consider the set

$$\mathcal{H}_j(h^*) := \{h \in \mathcal{H}_j : \mathcal{F}(h) = h^*\} = \mathcal{F}^{-1}(h^*). \tag{5.3}$$

The simplest situation occurs when $p = n$. In this case, $\mathcal{F}(h) = h \in \Omega_i$ and indeed $\mathcal{H}_j(h) = \{h\}$; moreover, the wavelet $\psi_{j,h}^{(i)} \in W_j(\Omega_i)$ (defined in (4.9)) vanishes identically on $\partial\Omega_i$; thus, we associate to h the function of $V_{j+1}(\Omega)$

$$\psi_{j,h}(x) := \begin{cases} \psi_{j,h}^{(i)}(x), & \text{if } x \in \Omega_i, \\ 0, & \text{elsewhere}. \end{cases} \tag{5.4}$$

If $p < n$, then h^* belongs to a p-face of Ω_i. Two situations may occur. If h^* does not belong to any $\partial\Omega_{i'}$ for $i' \ne i$, then it lies on the boundary of Ω and $\psi_{j,h}^{(i)}$ vanishes on $\partial\Omega_i \backslash \partial\Omega$. Thus, we associate to h the wavelet $\psi_{j,h}$ defined as in (5.4). Otherwise, h^* belongs to a face common to at least two subdomains, and we have to enforce a matching.

In the sequel, we construct a set of linearly independent functions in $V_{j+1}(\Omega)$ which will be associated to the set $\mathcal{H}_j(h^*)$.

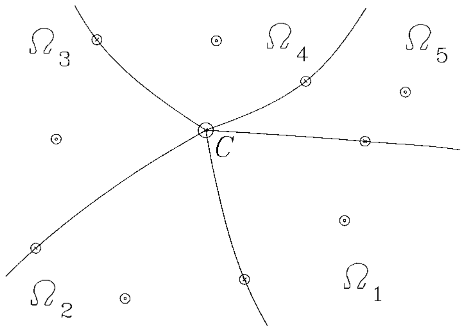

FIG. 1. Grid points around a cross point C surrounded by the subdomains Ω_i, $i = 1, \ldots, 5$.

5.1. *Matching at a Cross Point*

Let us start with the case in which $h^* =: C$ is a cross point, i.e., a 0-face common to N_C subdomains, that we assume to be (re-)labeled by $\Omega_1, \ldots, \Omega_{N_C}$. Let us first consider the case $C \in \Omega$ (see Fig. 1); next we shall indicate the modifications when $C \in \partial\Omega$.

5.1.1. Internal cross points. For each Ω_i, $i \in \{1, \ldots, N_C\}$, there are exactly $2^n - 1$ points $h \in \mathcal{H}_j^i$ such that $h^* = C$. Including C itself, we have 2^n points of the form $h = F_i(\hat{h})$, where $\hat{h} = (\hat{h}_l)_l$ is such that each component $\hat{h}_l$ ranges either in the set $\{0, \nu_{j,1}\}$ or in the set $\{\nu_{j,M_j}, 1\}$. This set of points can be identified with the set $E^n = \{0, 1\}^n$ by the mapping

$$h \mapsto e = (e_l)_l, \qquad \text{with } e_l := \begin{cases} 0, & \text{if } \hat{h}_l \in \{0, 1\}, \\ 1, & \text{if } \hat{h}_l \in \{\nu_{j,1}, \nu_{j,M_j}\}. \end{cases}$$

In turns, the vector e is associated with the function in $V_{j+1}(\Omega_i)$

$$\psi_e^{(i)}(x) = \hat{\psi}_e^{(i)}(\hat{x}) := \prod_{l=1}^{n} \vartheta_l^{(i)}(\hat{x}_l), \tag{5.5}$$

where $\hat{x} = G_i(x)$ and

$$\vartheta_l^{(i)} := \begin{cases} \xi_{j,0}, & \text{if } e_l = 0, \\ \eta_{j,\nu_{j,1}}, & \text{if } e_l = 1, \end{cases} \quad \text{if } (G_i(C))_l = 0$$

(i.e., if we are in a neighborhood of the left hand side of the interval [0, 1]), or

$$\vartheta_l^{(i)} := \begin{cases} \xi_{j,1}, & \text{if } e_l = 0, \\ \eta_{j,\nu_{j,M_j}}, & \text{if } e_l = 1, \end{cases} \quad \text{if } (G_i(C))_l = 1.$$

The set

$$V_{j+1}^C(\Omega_i) := \operatorname{span}\{\psi_e^{(i)}: e \in E^n\} \tag{5.6}$$

is a subset of $V_{j+1}(\Omega_i)$ of dimension 2^n. An element $v^{(i)} \in V^C_{j+1}(\Omega_i)$ is uniquely determined by the column vector $\boldsymbol{\alpha}^{(i)} = (\alpha^{(i)}_e)_{e\in E^n}$ by the relation

$$v^{(i)} = \sum_{e\in E^n} \alpha^{(i)}_e \psi^{(i)}_e. \tag{5.7}$$

Considering all the subdomains meeting at C, we have $2^n N_C$ free coefficients, which form the vector $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(i)})_{1\le i\le N_C}$.

Continuity. Since we are interested in continuous wavelets across interelements, we introduce the space

$$V^C_{j+1}(\Omega) := \{v \in C^0(\bar{\Omega}) \colon v_{|\Omega_i} \in V^C_{j+1}(\Omega_i), \text{ if } i \in \{1, \ldots, N_C\},\ v_{|\Omega_i} \equiv 0 \text{ elsewhere}\}.$$

By the representation (5.7), an element $v \in V^C_{j+1}(\Omega)$ is associated to a vector $\boldsymbol{\alpha}$, which belongs to the kernel of a certain matrix $\mathscr{C}$ representing an appropriate set of continuity conditions. We are now going to show a particular choice of such conditions and to construct the corresponding matrix.

The perhaps most natural approach would be to consider any $(n - 1)$-face $\Gamma_{i,i'}$ common to two subdomains Ω_i and $\Omega_{i'}$ with $i, i' \in \{1, \ldots, N_C\}$, and to impose the matching between the restriction to $\Gamma_{i,i'}$ of functions $v^{(i)} \in V^C_{j+1}(\Omega_i)$ and $v^{(i')} \in V^C_{j+1}(\Omega_{i'})$. (Note indeed that functions belonging to these spaces identically vanish on all $(n - 1)$-faces which do not contain C.) This would lead to 2^{n-1} conditions, linearly independent with respect to each other. However, certain matching conditions corresponding to different $(n - 1)$-faces are linearly dependent, and it is not obvious how to select a maximal set of linearly independent conditions. To avoid this problem, we consider all the p-faces, with $0 \le p \le n - 1$, which contain the cross point C, and we enforce one suitable matching condition along each face. We prove that all these conditions are linearly independent, and that they are equivalent to the matching conditions along all the $(n - 1)$-faces containing C.

To be precise, let σ be a p-face containing C, and let Ω_i be a subdomain having σ as a face. Then, $\sigma = F_i(\hat{\sigma})$, where $\hat{\sigma} \subset \partial\hat{\Omega}$ is defined as in (4.1) by a set $\mathscr{L}_{\hat{\sigma}}$ of frozen coordinates and corresponding values. The following notation will be useful. Given $t \in [0, 1]^n$, set

$$\dot{t} = \dot{\mathscr{D}}_{\hat{\sigma}}\, t := (t_l)_{l\notin\mathscr{L}_{\hat{\sigma}}} \in [0, 1]^p \tag{5.8}$$

(i.e., we delete the components of t corresponding to the frozen coordinates of $\hat{\sigma}$), and

$$\ddot{t} = \ddot{\mathscr{D}}_{\hat{\sigma}}\, t := (t_l)_{l\notin\mathscr{L}_{\hat{\sigma}}} \in [0, 1]^{n-p}.$$

Conversely, given $\dot{t} \in [0, 1]^p$ and $\ddot{t} \in [0, 1]^{n-p}$, let

$$t = \mathscr{R}_{\hat{\sigma}}(\dot{t}, \ddot{t}) \in [0, 1]^n \tag{5.9}$$

be the unique vector such that $\dot{t} = \dot{\mathcal{D}}_{\hat{\sigma}} t$ and $\ddot{t} = \ddot{\mathcal{D}}_{\hat{\sigma}} t$ (i.e., t is reconstructed from $\dot{t}$ and $\ddot{t}$ according to the position of the frozen coordinates of $\hat{\sigma}$). Moreover, given $\dot{e} \in E^p$ and $\hat{y} = (\hat{y}_l)_l \in [0, 1]^p$, we define $\hat{\psi}_{\dot{e}}^{(i)}(\hat{y})$ as in (5.5) with n replaced by p. Finally, recalling conditions (2.7) and (2.8), let us denote by

$$\begin{aligned} \lambda_j &:= \xi_{j,0}(0) = \eta_{j,\nu_{j,1}}(0) = \xi_{j,1}(1) = \eta_{j,\nu_{j,M_j}}(1), \\ \tilde{\lambda}_j &:= \tilde{\xi}_{j,0}(0) = \tilde{\eta}_{j,\nu_{j,1}}(0) = \tilde{\xi}_{j,1}(1) = \tilde{\eta}_{j,\nu_{j,M_j}}(1) \end{aligned} \tag{5.10}$$

the common value of the scaling and wavelet functions at the end points of the interval [0, 1]. Given any $v^{(i)} \in V_{j+1}^C(\Omega_i)$, represented as in (5.7), we have

$$\begin{aligned} v_{|\sigma}^{(i)}(x) &= \sum_{e \in E^n} \alpha_e^{(i)} \psi_e^{(i)}{}_{|\sigma}(x) \\ &= \lambda_j^{n-p} \sum_{e \in E^n} \alpha_e^{(i)} \hat{\psi}^{(i)}_{\dot{\mathcal{D}}_{\hat{\sigma}} e}(\dot{\mathcal{D}}_{\hat{\sigma}} \hat{x}), \qquad \hat{x} = G_i(x), \\ &= \lambda_j^{n-p} \sum_{\dot{e} \in E^{n-p}} \Big(\sum_{\ddot{e} \in E^{n-p}} \alpha_e^{(i)} \Big) \hat{\psi}_{\dot{e}}^{(i)}(\hat{y}), \qquad \hat{y} = \dot{\mathcal{D}}_{\hat{\sigma}}(\hat{x}),\ e = \mathcal{R}_{\hat{\sigma}}(\dot{e}, \ddot{e}), \\ &=: \lambda_j^{n-p} \sum_{\dot{e} \in E^p} b_{\dot{e}}^{(i)} \hat{\psi}_{\dot{e}}^{(i)}(\hat{y}). \end{aligned} \tag{5.11}$$

Let $\Omega_{i'}$ be another subdomain having σ as a face, and let $\sigma = F_{i'}(\hat{\sigma}')$. Given $v^{(i')} \in V_{j+1}^C(\Omega_{i'})$, we have as above

$$v_{|\sigma}^{(i')}(x) = \lambda_j^{n-p} \sum_{\dot{e} \in E^p} b_{\dot{e}}^{(i')} \hat{\psi}_{\dot{e}}^{(i')}(\hat{y}'), \tag{5.12}$$

where $b_{\dot{e}}^{(i')} = \sum_{\ddot{e} \in E^{n-p}} \alpha_{e'}^{(i')}$ with $e' = \mathcal{R}_{\hat{\sigma}'}(\dot{e}, \ddot{e})$, and $\hat{y}' = \dot{\mathcal{D}}_{\hat{\sigma}'}(G_{i'}(x))$. Now, by Hypothesis (4.2), the mapping $T\colon \hat{y} \mapsto \hat{y}'$ is a composition of reflections and permutations of coordinates, say $T = R \circ P$. Therefore, using (2.4) if there are reflections,

$$\begin{aligned} \hat{\psi}_{\dot{e}}^{(i')}(\hat{y}') &= \hat{\psi}_{\dot{e}}^{(i')}(T\hat{y}) = \prod_{l=1}^{p} \vartheta_l^{(i')}((T\hat{y})_l) \\ &= \prod_{l=1}^{p} \vartheta_l^{(i')}((R\hat{y})_{P(l)}) = \prod_{m=1}^{p} \vartheta_{P^{-1}(m)}^{(i)}((R\hat{y})_m) \\ &= \prod_{m=1}^{p} \vartheta_{P^{-1}(m)}^{(i)}(\hat{y}_m) = \hat{\psi}_{P^{-1}\dot{e}}^{(i)}(\hat{y}). \end{aligned}$$

Hence, (5.12) becomes

$$v_{|\sigma}^{(i')}(x) = \lambda_j^{n-p} \sum_{\dot{e}\in E^p} b_{\dot{e}}^{(i')} \hat{\psi}_{P^{-1}\dot{e}}^{(i)}(\hat{y}) = \lambda_j^{n-p} \sum_{\dot{e}\in E^p} b_{P\dot{e}}^{(i')} \hat{\psi}_{\dot{e}}^{(i)}(\hat{y}).$$

By the linear independence of the functions $\{\hat{\psi}_{\dot{e}}^{(i)}: \dot{e} \in E^p\}$, the matching condition $v_{|\sigma}^{(i)}(x) = v_{|\sigma}^{(i')}(x)$ is equivalent to the 2^p conditions

$$b_{\dot{e}}^{(i)} = b_{P\dot{e}}^{(i')}, \qquad \forall \dot{e} \in E^p. \tag{5.13}$$

We choose to enforce one particular combination of these conditions. This combination is uniquely associated to the face σ and the couple of subdomains Ω_i and $\Omega_{i'}$. For any $e \in E^q$, $q \geq 1$, let us set

$$\operatorname{sgn} e := (-1)^{|e|}, \qquad |e| := \sum_{l=1}^{q} e_l,$$

i.e., sgn e is $+1$ or -1, depending on the parity of the number of 1's in e; we also set sgn $e := 1$ when $q = 0$. Then, we require that

$$\sum_{\dot{e}\in E^p} (\operatorname{sgn} \dot{e}) b_{\dot{e}}^{(i)} = \sum_{\dot{e}\in E^p} (\operatorname{sgn} \dot{e}) b_{P\dot{e}}^{(i')}. \tag{5.14}$$

By observing that sgn $P\dot{e}$ = sgn $\dot{e}$ for any permutation P of components, (5.14) can be equivalently written as

$$\sum_{\dot{e}\in E^p} (\operatorname{sgn} \dot{e}) b_{\dot{e}}^{(i)} = \sum_{\dot{e}\in E^p} (\operatorname{sgn} \dot{e}) b_{\dot{e}}^{(i')}. \tag{5.15}$$

We want to express this condition in terms of the coefficients of the expansions (5.7) for $v^{(i)}$ and $v^{(i')}$. To this end, let us introduce the row vector

$$\mathbf{c}_\sigma^{(i)} := (\operatorname{sgn} \dot{\mathcal{D}}_{\hat{\sigma}} e)_{e\in E^n} \in \{-1, 1\}^{2^n}. \tag{5.16}$$

Then, (5.15) can be written as

$$\mathbf{c}_\sigma^{(i)} \cdot \boldsymbol{\alpha}^{(i)} = \mathbf{c}_\sigma^{(i')} \cdot \boldsymbol{\alpha}^{(i')}.$$

The following lemma will be crucial in the sequel.

LEMMA 5.1. *Let σ and ρ be two different faces of the same subdomain Ω_i containing C. Then*

$$\mathbf{c}_\sigma^{(i)} \cdot (\mathbf{c}_\rho^{(i)})^t = 0. \tag{5.17}$$

Proof. For convenience, we drop the index (i) throughout the proof. Let us assume that σ is a p-face, and ρ is a q-face with $p \geq q$. Then

$$\begin{aligned}
\mathbf{c}_\sigma \cdot (\mathbf{c}_\rho)^t &= \sum_{e \in E^n} (\operatorname{sgn} \dot{\mathcal{D}}_{\hat\sigma} e)(\operatorname{sgn} \dot{\mathcal{D}}_{\hat\rho} e) \\
&= \sum_{\dot e \in E^p} \sum_{\ddot e \in E^{n-p}} (\operatorname{sgn} \dot{\mathcal{D}}_{\hat\sigma} e)(\operatorname{sgn} \dot{\mathcal{D}}_{\hat\rho} e), \qquad \text{with } e = \mathcal{R}(\dot e, \ddot e), \\
&= \sum_{\dot e \in E^p} \operatorname{sgn} \dot e \sum_{\ddot e \in E^{n-p}} \operatorname{sgn} \dot{\mathcal{D}}_{\hat\rho} e.
\end{aligned}$$

Now, since σ and ρ are different, there exists an index l such that $\hat x_l$ is free on $\hat\sigma$ and frozen on $\hat\rho$. Given $\dot e \in E^p$, let us define $\dot f \in E^p$ as

$$\dot f_m := \begin{cases} 1 - \dot e_l, & \text{if } m = l \\ \dot e_m, & \text{if } m \neq l \end{cases} \quad (1 \leq m \leq p).$$

Then, $\operatorname{sgn} \dot f = -\operatorname{sgn} \dot e$; moreover, setting $f := \mathcal{R}_{\hat\sigma}(\dot f, \ddot e)$, we have $\dot{\mathcal{D}}_{\hat\rho} f = \dot{\mathcal{D}}_{\hat\rho} e$ for all $\ddot e \in E^{n-p}$. Thus

$$\mathbf{c}_\sigma \cdot (\mathbf{c}_\rho)^t = \frac{1}{2}\Big(\sum_{\dot e \in E^p} \operatorname{sgn} \dot e \sum_{\ddot e \in E^{n-p}} \operatorname{sgn} \dot{\mathcal{D}}_{\hat\rho} e + \sum_{\dot f \in E^p} \operatorname{sgn} \dot f \sum_{\ddot e \in E^{n-p}} \operatorname{sgn} \dot{\mathcal{D}}_{\hat\rho} f \Big) = 0,$$

which proves (5.17). ∎

Let us now prove that enforcing condition (5.15) on each face containing C is equivalent to enforcing the continuity across all subdomains meeting at C. More precisely, we prove the following result.

PROPOSITION 5.2. *Let σ be a p-face containing C, common to two subdomains Ω_i and $\Omega_{i'}$. Let $v^{(i)} \in V^C_{j+1}(\Omega_i)$ and $v^{(i')} \in V^C_{j+1}(\Omega_{i'})$ be given. The set of matching conditions* (5.13) *is equivalent to the following set: for any $q \in \{0, \ldots, p\}$ and any q-face ρ such that $C \in \rho \subseteq \sigma$,*

$$\mathbf{c}^{(i)}_\rho \cdot \boldsymbol{\alpha}^{(i)} = \mathbf{c}^{(i')}_\rho \cdot \boldsymbol{\alpha}^{(i')}. \tag{5.18}$$

Proof. Firstly, note that the number of q-faces containing C and contained in σ is $\binom{p}{p-q}$, since such faces are obtained by picking $p - q$ coordinates out of the p free coordinates of σ and freezing them. Since $\sum_{q=0}^{p} \binom{p}{p-q} = 2^p$,the number of conditions in (5.13) and in (5.18) is equal. Next, observe that

$$\mathbf{c}^{(i)}_\rho \cdot \boldsymbol{\alpha}^{(i)} = \sum_{\dot f \in E^q} \operatorname{sgn} \dot f \sum_{\ddot f \in E^{n-q}} \alpha^{(i)}_f, \qquad \text{with } f = \mathcal{R}_{\hat\rho}(\dot f, \ddot f).$$

Now,

$$\sum_{\dot f\in E^{n-q}} \alpha_f^{(i)} = \sum_{g\in E^{p-q}} \sum_{\ddot e\in E^{n-p}} \alpha_e^{(i)},$$

where $e = \mathcal{R}_{\hat\sigma}(\dot e, \ddot e)$ and $\dot e \in E^p$ is reconstructed from $\dot f \in E^q$ and $g \in E^{p-q}$. Thus,

$$\mathbf{c}_\rho^{(i)} \cdot \boldsymbol{\alpha}^{(i)} = \sum_{\dot e\in E^p} (\operatorname{sgn} \dot f) b_{\dot e}^{(i)},$$

in view of (5.11). Setting $\delta_{\dot e} := b_{\dot e}^{(i)} - b_{P\dot e}^{(i')}$ and defining the column vector $\dot{\boldsymbol{\delta}} := (\delta_{\dot e})_{\dot e\in E^p}$, (5.13) is equivalent to $\dot{\boldsymbol{\delta}} = \mathbf{0}$. On the other hand, recalling the equivalence of (5.14) and (5.15), we easily see that (5.18) is equivalent to

$$\dot{\mathscr{C}}\dot{\boldsymbol{\delta}} = \mathbf{0},$$

where the rows of the matrix ė are the vectors

$$\dot{\mathscr{C}}_\rho := (\operatorname{sgn} \dot f)_{\dot f\in E^p}$$

(when ρ varies among all the q-faces containing C and contained in σ). By Lemma 5.1 (here, in dimension p instead of n), the rows of $\dot{\mathscr{C}}$ are orthogonal to each other, so the matrix $\dot{\mathscr{C}}$ is regular and the result is proven. ■

COROLLARY 5.3. *For any $i \in \{1, \dots, N_C\}$, let $v^{(i)} \in V_{j+1}^C(\Omega_i)$ be given. Then, the function v defined as*

$$v_{|\Omega_i} := \begin{cases} v^{(i)}, & \text{if } i \in \{1, \dots, N_C\},\\ 0, & \text{if } i \notin \{1, \dots, N_C\},\end{cases} \tag{5.19}$$

belongs to $V_{j+1}^C(\Omega)$ if and only if (5.15) *holds for any p-face σ* $(0 \le p \le n-1)$ *containing C and any two subdomains Ω_i, $\Omega_{i'}$ having σ as a face.*

Proof. It is enough to apply Proposition 5.2 to all the $(n-1)$-faces containing C. ■

Finally, let us express the matching conditions in the compact form

$$\mathscr{C}\boldsymbol{\alpha} = \mathbf{0}, \tag{5.20}$$

where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(i)})_{1\le i\le N_C}$ and the matrix $\mathscr{C}$ has maximal rank. To this end, let us remark that obviously not all the conditions (5.15) associated to the same face σ are linearly independent. Thus, if N_σ denotes the number of subdomains having σ as a face and $\{\Omega_{i_m}\colon 1 \le m \le N_\sigma\}$ are such subdomains, we choose to enforce (5.18) between each pair Ω_{i_m}, $\Omega_{i_{m+1}}$ with $1 \le m \le N_\sigma - 1$. Obviously, these $N_\sigma - 1$ conditions imply that (5.18) is satisfied for any choice of indices $i, i' \in \{i_m\colon 1 \le m \le N_\sigma\}$.

It is convenient to consider $\mathscr{C}$ as a block matrix. The rows are grouped as follows: for each p between 0 and $n-1$, and for each p-face σ containing C, we have a block of N_σ

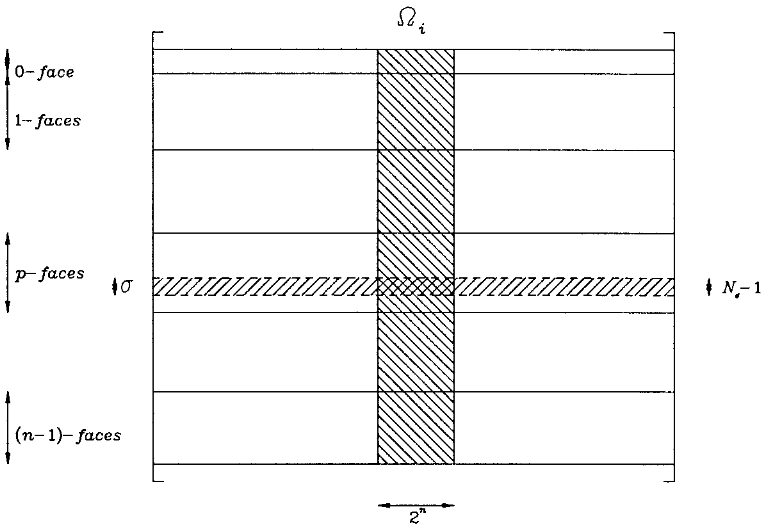

FIG. 2. Block structure of $\mathscr{C}$ containing the matching conditions for a p-face σ common to subdomain Ω_i.

$-$ 1 rows which correspond to the matching conditions described above. On the other hand, the columns are grouped by subdomains $\Omega_1, \ldots, \Omega_{N_C}$, see Fig. 2.

The row corresponding to condition (5.18) on the p-face σ between Ω_{i_m} and $\Omega_{i_{m+1}}$, will have the structure

$$\begin{array}{ccccc} \cdots 0 \cdots & \mathbf{c}_\sigma^{(i_m)} & \cdots 0 \cdots & -\mathbf{c}_\sigma^{(i_{m+1})} & \cdots 0 \cdots \\ & \uparrow & & \uparrow & \\ & \Omega_{i_m} & & \Omega_{i_{m+1}} & \end{array} \tag{5.21}$$

The following result proves that $\mathscr{C}$ has maximal rank.

PROPOSITION 5.4. *The matrix $\mathscr{C}\mathscr{C}^t$ is regular.*

Proof. Remember that each component of each vector $\mathbf{c}_\sigma^{(i)}$ is either $+1$ or -1, thus $\mathbf{c}_\sigma^{(i)} \cdot (\mathbf{c}_\sigma^{(i)})^t = 2^n$; moreover, remember Lemma 5.1. Let us fix our attention on one row of $\mathscr{C}$, say (5.21). The inner product with itself yields 2^{n+1}; the inner product with a row associated to the same face σ but to a different pair of subdomains is nonzero only if one of the blocks in that row corresponding to Ω_{i_m} or to $\Omega_{i_{m+1}}$ is nonzero; in these cases, the inner product yields -2^n; finally, the inner product with a row associated to a different face is always zero. It follows that the row of $\mathscr{C}\mathscr{C}^t$ corresponding to (5.21) has 2^{n+1} on the diagonal, two off-diagonal terms of the value -2^n if $1 < m < N_\sigma - 1$, or one off-diagonal term of value -2^n if $N_\sigma \geq 3$ and $m = 1$ or $m = N_\sigma - 1$; the remaining terms are zero.

By applying Gershgorin's Theorem, we conclude that all the eigenvalues of $\mathscr{C}\mathscr{C}^t$ are strictly positive. ■

Local dimension and association to grid points. The previous proposition implies that

$$\dim V_{j+1}^C(\Omega) = 2^n N_C - \dim \ker \mathscr{C} = 2^n N_C - \sum_{p=0}^{N_C} \sum_{\substack{p\text{-faces } \sigma \\ \text{containing } C}} (N_\sigma - 1). \tag{5.22}$$

On the other hand, the right-hand side is precisely the dimension of the set $\mathcal{H}_j(C) \cup \{C\}$, where $\mathcal{H}_j(C)$ is defined in (5.3). Indeed, in each subdomain $\Omega_1, \ldots, \Omega_{N_C}$, there are $2^n - 1$ grid points $h \in \mathcal{H}_j$ whose image under the mapping $\mathcal{F}$ defined in (5.2) is C; a grid point belonging to a face σ is common to exactly N_σ subdomains. We conclude that any basis in $V_{j+1}^C(\Omega)$ can be associated to these grid points by a one-to-one correspondence.

Dual system. The parallel construction of the space $\tilde{V}_{j+1}^C(\Omega)$ leads to the system

$$\tilde{\mathscr{C}}\tilde{\boldsymbol{\alpha}} = \mathbf{0}.$$

Note that the only difference in the construction described above is the presence of the factor $\tilde{\lambda}_j^{n-p}$ (see (5.10)) instead of λ_j^{n-p} in (5.11) and (5.12). When enforcing the matching conditions (5.13), we again drop this common factor on both sides: this leads to $\tilde{\mathscr{C}} = \mathscr{C}$. In other words, building a basis in $V_{j+1}^C(\Omega)$ or in $\tilde{V}_{j+1}^C(\Omega)$ amounts to solving the same problem, namely, finding a basis for the kernel of the *same* matrix. This will increase the efficiency of the method.

Biorthogonalization. We come now to orthogonality and biorthogonality. The condition $\langle v, \tilde{\varphi}_{j,C}\rangle_\Omega = 0$ for $v \in V_{j+1}^C(\Omega)$ is equivalent to the algebraic condition

$$\sum_{i=1}^{N_C} \alpha_e^{(i)} = 0, \qquad \text{for } e = (0, \ldots, 0). \tag{5.23}$$

In turns, this is equivalent to adding a row to the matrix $\mathscr{C}$, in which each block corresponding to a subdomain is constant and equal to $\mathbf{b} = (1, 0, \ldots, 0)$. It is easily seen that this row is orthogonal to all rows of $\mathscr{C}$. Denoting now by $\mathfrak{D}$ the new matrix, we have

$$\mathfrak{D}\mathfrak{D}^t = \left[\begin{array}{c|c} \mathscr{C}\mathscr{C}^t & \mathbf{0} \\ \hline \mathbf{0}^t & N_C \end{array}\right].$$

Thus, again $\mathfrak{D}\mathfrak{D}^t$ is regular. So, the space $W_j^C(\Omega) := \{v \in V_{j+1}^C(\Omega): \langle v, \tilde{\varphi}_{j,C}\rangle_\Omega = 0\}$ is a hyperplane in $V_{j+1}^C(\Omega)$; in terms of the coefficients $\boldsymbol{\alpha}$ of the representations (5.7), it corresponds to the condition

$$\mathfrak{D}\boldsymbol{\alpha} = \mathbf{0}. \tag{5.24}$$

Any basis in $W_j^C(\Omega)$ will be associated to the set of grid points $\mathcal{H}_j(C)$.

By Lemma A.8 applied to the matrices $\mathcal{M} = \tilde{\mathcal{M}} = \mathfrak{D}$, we conclude that we can construct a biorthogonal basis in $W_j^C(\Omega)$ and $\tilde{W}_j^C(\Omega)$. The basis functions constructed in this way are included in Ψ_j and $\tilde{\Psi}_j$, respectively. It is easily seen that $W_j^C(\Omega)$ is orthogonal to $\tilde{V}_j(\Omega)$, to all functions in $\tilde{\Psi}_j$ already constructed in (5.4), and to all functions that correspond possibly to grid points around other cross points. A similar observation holds for the dual space $\tilde{W}_j^C(\Omega)$.

5.1.2. Boundary cross points. Suppose that C belongs to $\partial\Omega$. Figure 3 indicates the possible cases, where it turns out that the last two cases can be treated together.

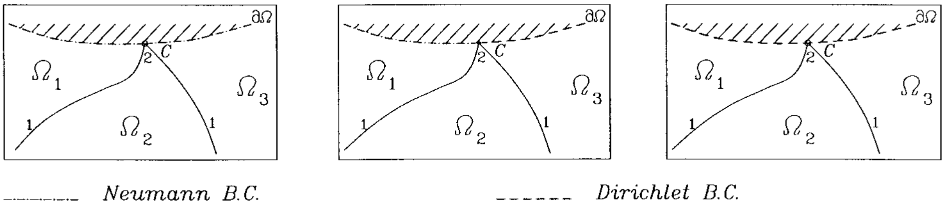

FIG. 3. Possible boundary cross point cases. The numbers denote the number of matching conditions enforced at the particular p-face.

Pure Neumann case. If $C \in \Gamma_N$, then, since Γ_N is relatively open in $\partial\Omega$, any p-face σ such that $C \in \sigma \subset \partial\Omega$ is contained in $\bar{\Gamma}_N$. So all the faces containing C are treated as in Subsection 5.1.1 (i.e., we enforce $(N_\sigma - 1)$ matching conditions if N_σ subdomains have σ as a face), and the construction of $W_j^C(\Omega)$ and $\tilde{W}_j^C(\Omega)$ is carried on similarly as before. By the same reasoning as above, any basis in $W_j^C(\Omega)$ is associated to the set of grid points $\mathcal{H}_j(C)$.

Dirichlet and mixed Dirichlet/Neumann case. Suppose now that $C \in \bar{\Gamma}_D$. Then, there exists at least one subdomain Ω_i, $i \in \{1, \ldots, N_C\}$, having at least one $(n-1)$-face containing C and contained in $\bar{\Gamma}_D$. We call such a face a *Dirichlet face*. Let $\delta = \delta^{(i)} \in \{1, \ldots, n-1\}$ be the number of Dirichlet faces of Ω_i. The local space $V_{j+1}^C(\Omega_i)$ is defined as follows. Going to the reference domain $\hat{\Omega}$, each Dirichlet face of Ω_i corresponds to freezing to 0 or 1 one particular coordinate $\hat{x}_l$ of $\hat{x} = G_i(x)$; denote by l_ν the index of the frozen coordinate of the νth Dirichlet face (in some arbitrary ordering), and call it a *Dirichlet direction*. Let $\mathcal{L}^{(i)} := \{l_1, \ldots, l_\delta\}$ be the set of all Dirichlet directions of Ω_i.

The following notation will be frequently used throughout the remainder of this subsection. For any $t \in \mathbb{R}^n$, we define the deletion operators

$$t^D = \mathcal{D}_D^{(i)} t := (t_l)_{l \notin \mathcal{L}^{(i)}} \in \mathbb{R}^{n-\delta}, \qquad t^* = \mathcal{D}_*^{(i)} t := (t_l)_{l \in \mathcal{L}^{(i)}} \in \mathbb{R}^{\delta},$$

and the reconstruction operator

$$t := \mathcal{R}^{(i)}(t^D, t^*),$$

which is uniquely defined by the conditions $t^D = \mathcal{D}_D^{(i)} t$, $t^* = \mathcal{D}_*^{(i)} t$ (note the analogy with (5.8)–(5.9)). Moreover, given $e^D \in E^{n-\delta}$, let us define $\tilde{e} := \mathcal{R}^{(i)}(e^D, f^*)$, where $f^* := (-1, \ldots, -1) \in \mathbb{Z}^\delta$.

Next, taking (2.12) into account, set

$$\psi_{e^D}^{(i)}(x) := \prod_{l=1}^{n} \vartheta_l^{(i)}(\hat{x}_l), \qquad \hat{x} = G_i(x), \tag{5.25}$$

where, if $(G_i(C))_l = 0$,

$$\vartheta_l^{(i)} := \begin{cases} \xi_{j,0}, & \text{if } \tilde{e}_l = 0, \\ \eta_{j,\nu_{j,1}}, & \text{if } \tilde{e}_l = 1, \\ \eta^D_{j,\nu_{j,1}}, & \text{if } \tilde{e}_l = -1, \end{cases}$$

whereas, if $(G_i(C))_l = 1$,

$$\vartheta_l^{(i)} := \begin{cases} \xi_{j,1}, & \text{if } \tilde{e}_l = 0, \\ \eta_{j,\nu_{j,M_j}}, & \text{if } \tilde{e}_l = 1, \\ \eta^D_{j,\nu_{j,M_j}}, & \text{if } \tilde{e}_l = -1. \end{cases}$$

Let us set

$$V^C_{j+1}(\Omega_i) := \operatorname{span}\{\psi^{(i)}_{e^D} : e^D \in E^{n-\delta}\} \tag{5.26}$$

and let us represent a function $v^{(i)} \in V^C_{j+1}(\Omega_i)$ as

$$v^{(i)} = \sum_{e^D \in E^{n-\delta}} \alpha^{(i)}_{e^D} \psi^{(i)}_{e^D},$$

i.e., $v^{(i)}$ is associated to the vector $\boldsymbol{\alpha}^{(i)} := (\alpha^{(i)}_{e^D})_{e^D \in E^{n-\delta}}$. We shall also need the representation of $v^{(i)}$ according to the basis $\{\psi^{(i)}_e : e \in E^n\}$ defined in (5.5). To this end, recalling the definition of η^D, we have

$$\psi^{(i)}_{e^D} = \sum_{e^* \in E^\delta} (-1)^{\delta - |e^*|} \psi^{(i)}_e, \tag{5.27}$$

where again $|e^*| = \sum_{l=1}^{\delta} e^*_l$ and $e = \mathcal{R}^{(i)}(e^D, e^*)$. It follows that

$$v^{(i)} = \sum_{e^D \in E^{n-\delta}} \alpha^{(i)}_{e^D} \sum_{e^* \in E^\delta} (-1)^{\delta - |e^*|} \psi^{(i)}_e = \sum_{e \in E^n} [(-1)^{\delta - |e^*|} \alpha^{(i)}_{e^D}] \psi^{(i)}_e =: \sum_{e \in E^n} \beta^{(i)}_e \psi^{(i)}_e.$$

Assume now that σ is a p-face of Ω_i containing C and let $\sigma = F_i(\hat{\sigma})$. If at least one of the $(n - p)$ frozen coordinates of $\hat{\sigma}$ is a Dirichlet direction, then $\psi^{(i)}_{e^D|\sigma} \equiv 0$ for all $e^D \in E^{n-\delta}$, hence $v^{(i)}_{|\sigma} \equiv 0$ for all $v^{(i)} \in V^C_{j+1}(\Omega_i)$. Conversely, if none of the frozen coordinates of $\hat{\sigma}$ is a Dirichlet direction, then necessarily $p \geq \delta$; moreover, functions of $V^C_{j+1}(\Omega_i)$ need not vanish identically on σ.

If $\Omega_{i'}$ is another subdomain having $\sigma = F_{i'}(\hat{\sigma}')$ as a face, a similar alternative occurs on $\hat{\sigma}'$. Therefore, if both $\hat{\sigma}$ and $\hat{\sigma}'$ have frozen coordinates which are Dirichlet directions, then

$$v^{(i)}_{|\sigma} \equiv 0 \equiv v^{(i')}_{|\sigma'} \qquad \forall v^{(i)} \in V^C_{j+1}(\Omega_i),\ \forall v^{(i')} \in V^C_{j+1}(\Omega'_i),$$

so no matching condition has to be enforced. If none of the frozen coordinates of $\hat{\sigma}$ is a Dirichlet direction, but some of the frozen coordinates of $\hat{\sigma}'$ is, then we enforce the condition

$$\sum_{\dot{e}\in E^p} \operatorname{sgn} \dot{e} \sum_{\ddot{e}\in E^{n-p}} \beta_e^{(i)} = 0, \tag{5.28}$$

where $e = \mathcal{R}_{\hat{\sigma}}(\dot{e}, \ddot{e})$. In the reverse situation, we enforce

$$\sum_{\dot{e}\in E^p} \operatorname{sgn} \dot{e} \sum_{\ddot{e}\in E^{n-p}} \gamma_{e'}^{(i')} = 0,$$

where $e' = \mathcal{R}_{\hat{\sigma}'}(\dot{e}, \ddot{e})$, and $\gamma_{e'}^{(i')} = \alpha_{e'}^{(i')}$ or $\gamma_{e'}^{(i')} = \beta_{e'}^{(i')}$, depending whether $\Omega_{i'}$ does not or does contain Dirichlet faces. Finally, if none of the frozen coordinates of both $\hat{\sigma}$ and $\hat{\sigma}'$ is a Dirichlet direction, we enforce

$$\sum_{\dot{e}\in E^p} \operatorname{sgn} \dot{e} \sum_{\ddot{e}\in E^{n-p}} \beta_e^{(i)} = \sum_{\dot{e}\in E^p} \operatorname{sgn} \dot{e} \sum_{\ddot{e}\in E^{n-p}} \gamma_{e'}^{(i')}. \tag{5.29}$$

Let us consider the left-hand side of (5.28) or (5.29), and let us write it as

$$\mathbf{c}_\sigma^{(i)} \cdot \boldsymbol{\alpha}^{(i)} = \sum_{e^D\in E^{n-\delta}} c_{e^D}^{(i)} \alpha_{e^D}^{(i)},$$

where, by (5.27),

$$c_{e^D}^{(i)} = \sum_{e^*\in E^\delta} (-1)^{\delta-|e^*|} \operatorname{sgn} \dot{e},$$

and $\dot{e} = \dot{\mathcal{D}}_{\hat{\sigma}} e$, with $e = \mathcal{R}^{(i)}(e^D, e^*)$. Recalling the fact that all the Dirichlet directions are among the p free coordinates of σ, i.e., $\mathcal{L}^{(i)} \subset \{1, \dots, n\} \backslash \mathcal{L}_{\hat{\sigma}}$, it is easily seen that $\mathcal{D}_*^{(i)} \dot{e} = \mathcal{D}_*^{(i)} e$ $(= e^* \in E^\delta)$, and that $\mathcal{D}_D^{(i)} \dot{e} = \dot{\mathcal{D}}_{\hat{\sigma}} e^D =: e^0 \in E^{p-\delta}$. Hence, we split $\dot{e}$ into the vectors e^* and e^0, and consequently,

$$(-1)^{\delta-|e^*|} \operatorname{sgn} \dot{e} = (-1)^{\delta-|e^*|}(-1)^{|e^*|+|e^0|} = (-1)^\delta(-1)^{|e^0|} = (-1)^\delta \operatorname{sgn} e^0,$$

whence,

$$c_{e^D}^{(i)} = (-2)^\delta \operatorname{sgn} e^0, \qquad e^0 = \dot{\mathcal{D}}_{\hat{\sigma}}(e^D).$$

Recalling the definition (5.16), we note that the vector $\mathbf{c}_\sigma^{(i)}$ is built as the analogous vector in the internal cross point case, except that the multiplicative factor $(-2)^\delta$ appears and the vector length n is replaced by $n - \delta$. So, by Lemma 5.1, vectors corresponding to different faces are orthogonal, whereas the inner product of each vector with itself is equal to 2^n.

From these facts, it is straightforward to check that the matrix $\mathcal{C}$ associated as in the previous subsection with the matching conditions around a boundary cross point has maximal rank, since again $\mathscr{C}\mathscr{C}^t$ is regular.

Local dimension and association to grid points. Let us first note that, for any $i \in \{1, \ldots, N_C\}$, $\dim V^C_{j+1}(\Omega_i) = 2^{n-\delta^{(i)}}$ according to (5.26). This is precisely 1 plus the cardinality of the set $\mathscr{H}^i_j(C) := \mathscr{H}_j(C) \cap \bar{\Omega}_i$ of the points $h \in \mathscr{H}^i_j$ mapped to C by the mapping $\mathscr{F}$ defined in (5.2); indeed, from the $2^n - 1$ grid points in $\bar{\Omega}_i$ that would be mapped to C in the absence of Dirichlet conditions, we remove all the points that belong to a Dirichlet face of Ω_i; equivalently, we remove one grid point per each p-face ($1 \le p \le n-1$) contained in a Dirichlet face of Ω_i.

The construction of $\mathscr{C}$ and its maximal rank property imply that

$$\dim V^C_{j+1}(\Omega) = \sum_{i=1}^{N_C} 2^{n-\delta^{(i)}} - \dim \ker \mathscr{C} = \sum_{i=1}^{N_C} 2^{n-\delta^{(i)}} - \sum_{p=0}^{N_C} \sum_{\sigma(p)} (N_\sigma - 1),$$

where the notation $\sum_{\sigma(p)}$ means summation over all the p-faces σ containing C and not contained in a Dirichlet face of some domain Ω_i ($i = 1, \ldots, N_C$). On the other hand,

$$\mathscr{H}_j(C) = \sum_{i=1}^{N_C} \mathscr{H}^i_j(C),$$

and a point $h \in \mathscr{H}^i_j(C)$, belonging to a p-face σ not contained in a Dirichlet face of Ω_i, also belongs to $N_\sigma - 1$ other sets $\mathscr{H}^{i'}_j(C)$. We conclude that

$$\dim V^C_{j+1}(\Omega) = \operatorname{card} \mathscr{H}_j(C). \tag{5.30}$$

Therefore, after biorthogonalization, we associate basis functions in $V^C_{j+1}(\Omega)$ to grid points in $\mathscr{H}_j(C)$. Note that $C \notin \mathscr{K}_{j+1}$, because of the Dirichlet boundary condition enforced at C. Hence, no $\tilde{\varphi}_{j,C} \in \tilde{V}_{j+1}(\Omega)$ exists; consequently, no orthogonalization is required, i.e., we are allowed to set $W^C_j(\Omega) := V^C_{j+1}(\Omega)$.

5.2. *Matching in the Interior of a p-Face*

We now consider the case in which $h^* =: X$, as defined in (5.2), has p internal components with $1 \le p \le n-1$. Then, there exists a unique p-face ρ such that $X \in \rho$. Note that $p = \min\{q \in \{1, \ldots, n-1\}: \exists$ a q-face containing $X\}$. Let $N_X \ge 2$ be the number of subdomains containing X, or, equivalently, containing ρ. As before, we assume for convenience that these subdomains are (re-)labeled by $\Omega_1, \ldots, \Omega_{N_X}$. Let us consider any one of these subdomains, say Ω_i. Then, $X = F_i(\hat{X})$ for some $\hat{X} \in \partial\hat{\Omega}$, and $\rho = F_i(\hat{\rho})$, where $\hat{\rho}$ is the p-face of $\hat{\Omega}$ whose set $\mathscr{L}_{\hat{\rho}}$ is precisely the set of noninternal components of X.

We now build the local subspace $V^X_{j+1}(\Omega_i)$ of $V_{j+1}(\Omega_i)$ associated to X. As opposed to Subsection 5.1, now the cases $X \in \Omega$ and $X \in \partial\Omega$ will not be treated separately. So, let $\delta = \delta^{(i)} \in \{0, \ldots, n-p-1\}$ be the number of Dirichlet faces of Ω_i containing X.

Let us set $\dot{\hat{\Omega}} := [0,1]^p$, and $\dot{\mathscr{D}}_{\hat{\rho}}(\hat{X}) =: \hat{\zeta} = (\hat{\zeta}_l)_{l=1,\ldots,p}$; then, for any $\hat{y} \in \dot{\hat{\Omega}}$, define

$$\dot{\hat{\psi}}^{(i)}(\hat{y}) := \prod_{l=1}^{p} \vartheta_{\hat{\zeta}_l}(\hat{y}_l),$$

where, recalling (2.9),

$$\vartheta_{\hat{\zeta}_l} := \begin{cases} \xi_{j,\hat{\zeta}_l}, & \text{if } \hat{\zeta}_l \in \Delta_j^{int}, \\ \eta_{j,\hat{\zeta}_l}, & \text{if } \hat{\zeta}_l \in \nabla_j^{int}. \end{cases}$$

Moreover, given $\ddot{e} \in E^{n-p-\delta}$, and $\hat{z} \in \ddot{\hat{\Omega}} := [0, 1]^{n-p}$, let $\ddot{\hat{\Psi}}^{(i)}_{\ddot{e}}(\hat{z})$ be defined as in (5.5) if $\delta = 0$ or in (5.25) if $\delta > 0$, but with n replaced by $n - p$, e or e^D replaced by $\ddot{e}$, $\hat{x}$ replaced by $\hat{z}$, and C replaced by X.

Given $x \in \Omega_i$, let us define

$$\dot{\psi}^{(i)}(x) = \dot{\hat{\psi}}^{(i)}(\hat{y}), \qquad \text{with } \hat{y} = \dot{\mathcal{D}}_{\hat{\rho}}\hat{x},$$

$$\ddot{\psi}^{(i)}_{\ddot{e}}(x) = \ddot{\hat{\psi}}^{(i)}_{\ddot{e}}(\hat{z}), \qquad \text{with } \hat{z} = \ddot{\mathcal{D}}_{\hat{\rho}}\hat{x},$$

and finally,

$$\psi^{(i)}_{\ddot{e}}(x) := \dot{\psi}^{(i)}(x)\ddot{\psi}^{(i)}_{\ddot{e}}(x).$$

Let $V^X_{j+1}(\Omega_i)$ be the space spanned by these functions, i.e., the space of functions

$$v^{(i)} = \sum_{\ddot{e} \in E^{n-p-\delta}} \alpha^{(i)}_{\ddot{e}} \psi^{(i)}_{\ddot{e}},$$

where $\boldsymbol{\alpha}^{(i)} = (\alpha^{(i)}_{\ddot{e}})_{\ddot{e} \in E^{n-p-\delta}}$ ranges in $\mathbb{R}^{2^{n-p-\delta}}$. Note that $v^{(i)}$ can be decomposed as

$$v^{(i)} = \dot{\psi}^{(i)}\ddot{v}^{(i)}, \qquad \text{where } \ddot{v}^{(i)} = \sum_{\ddot{e} \in E^{n-p-\delta}} \alpha^{(i)}_{\ddot{e}} \ddot{\psi}^{(i)}_{\ddot{e}}. \tag{5.31}$$

We want to construct a basis for the space

$$V^X_{j+1}(\Omega) := \{v \in C^0(\bar{\Omega}) \colon v_{|\Omega_i} \in V^X_{j+1}(\Omega_i), \text{ if } 1 \leq i \leq N_X,\ v_{|\Omega_i} \equiv 0 \text{ elsewhere}\}. \tag{5.32}$$

It is easily seen that a function v, such that $v_{|\Omega_i} =: v^{(i)} \in V^X_{j+1}(\Omega_i)$ for all $i \in \{1, \ldots, N_X\}$, belongs to $V^X_{j+1}(\Omega)$ if and only if the condition

$$v^{(i)}_{|\sigma} \equiv v^{(i')}_{|\sigma}$$

holds for any $(n - 1)$-face σ containing ρ and common to two subdomains Ω_i and $\Omega_{i'}$. Recalling (5.31), this is equivalent to

$$\dot{\psi}^{(i)}_{|\sigma}\ddot{v}^{(i)}_{|\sigma} = \dot{\psi}^{(i')}_{|\sigma}\ddot{v}^{(i')}_{|\sigma}. \tag{5.33}$$

But

$$\dot{\psi}^{(i)}_{|\sigma} \equiv \dot{\psi}^{(i')}_{|\sigma}. \tag{5.34}$$

Indeed, if $x \in \sigma$, then

$$\dot{\psi}^{(i)}(x) = \prod_{l=1}^{p} \vartheta_{\hat{\zeta}_l}(\hat{y}_l), \qquad \hat{y} = \dot{\mathcal{D}}_{\hat{\rho}}(G_i(x)),\ \hat{\zeta} = \dot{\mathcal{D}}_{\hat{\rho}}(G_i(X)),$$

$$\dot{\psi}^{(i')}(x) = \prod_{l=1}^{p} \vartheta_{\hat{\zeta}'_l}(\hat{y}'_l), \qquad \hat{y}' = \dot{\mathcal{D}}_{\hat{\rho}'}(G_{i'}(x)),\ \hat{\zeta}' = \dot{\mathcal{D}}_{\hat{\rho}'}(G_{i'}(X)).$$

By Hypotheses (4.2), the mapping T: $\hat{y} \mapsto \hat{y}'$ is such that $\hat{y}'_m = \tau(\hat{y}_{P(m)})$, $1 \leq m \leq n$, where τ is either the identity $s \mapsto s$ or the reflection $s \mapsto 1 - s$ (if the system on [0, 1] is reflection invariant) and P is a permutation of the set $\{1, \ldots, p\}$. It follows that

$$\vartheta_{\hat{\zeta}'_m}(\hat{y}'_m) = \vartheta_{\tau(\hat{\zeta}_{P(m)})}(\tau(\hat{y}_{P(m)})) = \vartheta_{\hat{\zeta}_{P(m)}}(\hat{y}_{P(m)}),$$

the last equality being again a consequence of the reflection invariance property. This implies (5.34). Thus, (5.33) amounts to satisfying

$$\ddot{v}^{(i)}_{|\sigma} = \ddot{v}^{(i')}_{|\sigma}. \tag{5.35}$$

Now, we show that this can be thought of as a matching condition around a cross point in dimension $\ddot{n} := n - p$ along an $(\ddot{n} - 1)$-face. Define the mapping $\ddot{F}_i$: $\operatorname{clos}\ddot{\hat{\Omega}}_{\mathrm{i}} \rightarrow \bar{\Omega}_i$ by setting $z = \ddot{F}_i(\hat{z}) := F_i(\mathcal{R}_{\hat{\rho}}(\hat{\zeta}, \hat{z}))$. Let $\ddot{\Omega}_{\mathrm{i}} := \ddot{\mathrm{F}}_{\mathrm{i}}(\ddot{\hat{\Omega}})$, which is an $\ddot{n}$-dimensional smooth manifold in Ω_i. Note that X is a 0-face of $\ddot{\Omega}_{\mathrm{i}}$, and that $\ddot{\sigma} := \sigma \cap \bar{\ddot{\Omega}}_{\mathrm{i}}$ is an $(\ddot{n} - 1)$-face of $\ddot{\Omega}_{\mathrm{i}}$. All these definitions are visualized in Fig. 4 for the case $n = 3$.

Furthermore, define the mapping $\mathcal{L}_i(x) := \ddot{F}_i(\ddot{\mathcal{D}}_{\hat{\rho}}\hat{x})$. Then it is easily seen that

$$\ddot{v}^{(i)}(x) = \ddot{v}^{(i)}(\mathcal{L}_i(x)), \qquad \forall x \in \bar{\Omega}_i.$$

This means that each function $\ddot{v}^{(i)}(x)$ defined in $\bar{\Omega}_i$ can be identified with its restriction $\ddot{v}^{(i)}(z)$ defined in $\bar{\ddot{\Omega}}_{\mathrm{i}}$. Let $V^X_{j+1}(\ddot{\Omega}_i)$ denote the space of these restrictions. We conclude that (5.35) can be rephrased as

$$\ddot{v}^{(i)}(z) = \ddot{v}^{(i')}(z), \qquad \forall z \in \ddot{\sigma}.$$

Define the compact, piecewise smooth $\ddot{n}$-dimensional manifold $\bar{\ddot{\Omega}} := \bigcup_{i=1}^{N_X} \bar{\ddot{\Omega}}_{\mathrm{i}}$, and let $\ddot{\Omega}$ be its (relative) interior. The previous considerations show that the problem of finding a basis in $V^X_{j+1}(\Omega)$ is reduced to that of finding a basis in the space $V^X_{j+1}(\ddot{\Omega})$, defined as in (5.32) with the obvious change of notation. To accomplish this task, we can apply the construction of

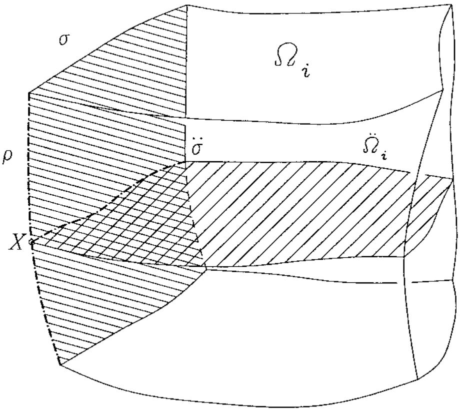

FIG. 4. Definition of the $\ddot{n}$-dimensional manifold $\ddot{\Omega}_i$, the $(\ddot{n} - 1)$-dimensional manifold $\ddot{\sigma}$ according to a point $X \in \partial\Omega_i$ common to the face σ. The face ρ is the uniquely defined face having the noninternal components of X as frozen coordinates.

Subsection 5.1, simply by replacing the dimension n by the dimension $\ddot{n}$ and the functions defined in Ω_i or Ω by analogous quantities defined in $\ddot{\Omega}_i$ or $\ddot{\Omega}$.

If $X \in \mathcal{K}_j$ (i.e., X is a grid point associated to a scaling function in $V_j(\Omega)$), then this construction yields a biorthogonal basis $\{\ddot{\psi}_{j,h}\}_{h\in\mathcal{K}_j(X)}$ in $W_j^X(\ddot{\Omega})$. Setting

$$\psi_{j,h}(x) := \begin{cases} \dot{\psi}^{(i)}(x)\ddot{\psi}_{j,h}(\mathcal{L}_i(x)), & \text{if } x \in \bar{\Omega}_i,\ 1 \leq i \leq N_X, \\ 0, & \text{elsewhere,} \end{cases} \tag{5.36}$$

we get a biorthogonal basis in $W_j^X(\Omega)$.

If $X \in \mathcal{H}_j$, then we apply a slight simplification to the construction of Subsection 5.1; namely, we need not to enforce the analog of the orthogonality condition (5.23), since the function $\dot{\psi}$ contains at least one 1D-wavelet function as a factor. Thus, using Lemma A.8 applied now to the matrices $\mathcal{M} = \tilde{\mathcal{M}} = \ddot{\mathcal{C}}$, we get a biorthogonal basis $\{\ddot{\psi}_{j,h}\}_{h\in\mathcal{H}_j(X)\cup\{X\}}$ in $V_{j+1}^X(\ddot{\Omega})$, which, again by (5.36), yields a biorthogonal basis in $W_j^X(\Omega)$.

5.3. *An Example*

We detail a simple 2D example for the construction of a biorthogonal basis in $L^2(\Omega)$. A comprehensive discussion of general 2D and 3D constructions, including implementation issues, will be given in a forthcoming paper [7].

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ with smooth boundary, partitioned into 5 quadrilateral subdomains as shown in Fig. 5. Let $\hat{\Omega} = (0, 1)^2$ be the reference square, and let us denote its corner points by $\hat{X}_{00} = (0, 0)$, $\hat{X}_{10} = (1, 0)$, $\hat{X}_{01} = (0, 1)$, $\hat{X}_{11} = (1, 1)$. We assume that the mappings F_i: $\bar{\hat{\Omega}} \to \bar{\Omega}_i$ ($i = 1, \ldots, 5$) are such that

$$X_3 = F_1(\hat{X}_{00}), \qquad X_4 = F_1(\hat{X}_{10}), \qquad X_2 = F_1(\hat{X}_{01}), \qquad X_1 = F_1(\hat{X}_{11}),$$

$$X_4 = F_2(\hat{X}_{00}), \qquad X_8 = F_2(\hat{X}_{10}), \qquad X_1 = F_2(\hat{X}_{01}), \qquad X_5 = F_2(\hat{X}_{11}),$$

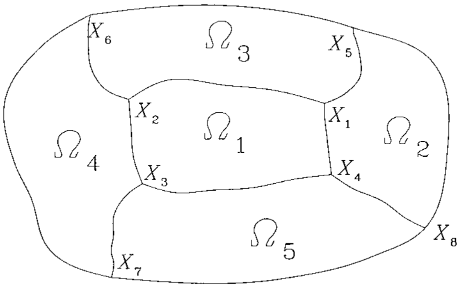

FIG. 5. Partition of the domain Ω into the subdomains Ω_i, $i = 1, \ldots, 5$, and the cross and boundary points X_l, $l = 1, \ldots, 8$.

$$\begin{array}{llll} X_2 = F_3(\hat{X}_{00}), & X_1 = F_3(\hat{X}_{10}), & X_6 = F_3(\hat{X}_{01}), & X_5 = F_3(\hat{X}_{11}), \\ X_3 = F_4(\hat{X}_{00}), & X_2 = F_4(\hat{X}_{10}), & X_7 = F_4(\hat{X}_{01}), & X_6 = F_4(\hat{X}_{11}), \\ X_3 = F_5(\hat{X}_{00}), & X_7 = F_5(\hat{X}_{10}), & X_4 = F_5(\hat{X}_{01}), & X_8 = F_5(\hat{X}_{11}). \end{array}$$

It is easily seen that these mappings satisfy Hypothesis (4.2); more precisely, each $H_{i,i'}$ is order-preserving.

We assume that the Dirichlet condition is prescribed on the whole of $\partial\Omega$, so that $W_j(\Omega)$ is a subspace of $H_0^1(\Omega)$. Let us now describe how the wavelets are constructed in different cases.

To start with, let $h \in \mathcal{H}_j$ be an internal grid point of a subdomain, say for instance $h = F_1(\hat{h})$, with $\hat{h} = (\hat{h}_1, \hat{h}_2)$ and $\hat{h}_1 \in \Delta_j^{int}$, $\hat{h}_2 \in \nabla_j^{int}$. Then, h is associated with the wavelet

$$\psi_{j,h}(x) = \begin{cases} \xi_{j,\hat{h}_1}(\hat{x}_1)\eta_{j,\hat{h}_2}(\hat{x}_2), & \text{if } x = F_1(\hat{x}) \in \Omega_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Next, suppose that h is close to the physical boundary, say $h_2 = F_2(\hat{h})$, with $\hat{h}_1 = \nu_{j,M_j}$ and $\hat{h}_2 \in \nabla_j^{int}$. Then, h is associated with the wavelet

$$\psi_{j,h}(x) = \begin{cases} \eta^D_{j,\hat{h}_1}(\hat{x}_1)\eta_{j,\hat{h}_2}(\hat{x}_2), & \text{if } x = F_2(\hat{x}) \in \Omega_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Suppose now that $X \in \mathcal{H}_{j+1}$ is internal to the face $\Gamma_{1,2}$ common to Ω_1 and Ω_2. For instance, we may have $X = F_1(\hat{X}) = F_2(\hat{X}') \in \mathcal{H}_j$, with $\hat{X} = (\hat{X}_1, \hat{X}_2)$ and $\hat{X}_1 = 1$, $\hat{X}_2 \in \Delta_j^{int}$, $\hat{X}' = (\hat{X}_1', \hat{X}_2')$ with $\hat{X}_1' = 0$ and $\hat{X}_2' = \hat{X}_2$. In this case we have $\mathcal{H}_j(X) = \{h^{(1)}, h^{(2)}\}$, where $h^{(1)} = F_1(\hat{h}^{(1)})$ with $\hat{h}^{(1)} = (\nu_{j,M_j}, \hat{X}_2)$, and $h^{(2)} = F_2(\hat{h}^{(2)})$ with $\hat{h}^{(2)} = (\nu_{j,1}, \hat{X}_2)$. The local space $V^X_{j+1}(\Omega_1)$ is then such that

$$v^{(1)} \in V^X_{j+1}(\Omega_1) \qquad \text{iff } v^{(1)}(x) = [\alpha_0^{(1)}\xi_{j,1}(\hat{x}_1) + \alpha_1^{(1)}\eta_{j,\nu_{j,M_j}}(\hat{x}_1)]\xi_{j,\hat{X}_2}(\hat{x}_2),$$

for $\boldsymbol{\alpha}^{(1)} = (\alpha_0^{(1)}, \alpha_1^{(1)}) \in \mathbb{R}^2$; similarly, $V^X_{j+1}(\Omega_2)$ is such that

$$v^{(2)} \in V^X_{j+1}(\Omega_2) \qquad \text{iff } v^{(2)}(x) = [\alpha_0^{(2)}\xi_{j,0}(\hat{x}_1) + \alpha_1^{(2)}\eta_{j,\nu_{j,1}}(\hat{x}_1)]\xi_{j,\hat{X}_2}(\hat{x}_2),$$

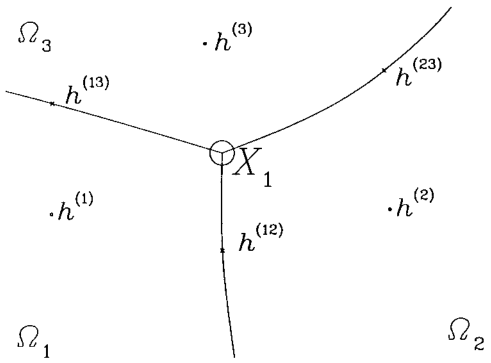

FIG. 6. Grid points around the internal cross point X_1. Here, $h^{(i)} \in \Omega_i$, $i = 1, \ldots, 3$, and $h^{(ii')} \in \Gamma_{i,i'}$, $i, i' = 1, \ldots, 3$, $i \neq i'$.

for $\boldsymbol{\alpha}^{(2)} = (\alpha_0^{(2)}, \alpha_1^{(2)}) \in \mathbb{R}^2$. Since both families of functions share a common factor, we are reduced to matching and biorthogonalizing at a 0-face in dimension 1. Setting $\boldsymbol{\alpha} = (\alpha_0^{(1)}, \alpha_1^{(1)}, \alpha_0^{(2)}, \alpha_1^{(2)})$, the matching condition (5.20) becomes

$$\mathscr{C}\boldsymbol{\alpha} = \mathbf{0},$$

where $\mathscr{C}$ is the row vector $(1, 1, -1, -1)$. Adding the orthogonality condition (5.5), we get the kernel relation

$$\mathscr{D}\boldsymbol{\alpha} = \mathbf{0},$$

where $\mathscr{D} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Thus $\mathscr{D}\mathscr{D}^t = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. Imposing biorthogonality, we end up with the two linearly independent vectors $\boldsymbol{\alpha}_I$ and $\boldsymbol{\alpha}_{II}$ which define two functions in $W_j^X(\Omega)$. They are associated with the grid points $h^{(1)}$ and $h^{(2)}$, respectively.

If we have $\hat{X}_2 = \hat{X}_2' \in \nabla_j^{int}$ instead, then $X \in \mathscr{H}_j$ and $\xi_{j,\hat{X}_2}(\hat{x}_2)$ is replaced by $\eta_{j,\hat{X}_2}(\hat{x}_2)$ in the definition of $v^{(1)}$ and $v^{(2)}$. In this case, we do not need to enforce condition (5.23), hence, after biorthogonalization we end up with three vectors $\boldsymbol{\alpha}_I$, $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_{II}$ which define three functions in $W_j^X(\Omega)$. They are associated with the grid points $h^{(1)}$, X, $h^{(2)}$, respectively.

Finally, consider the cross point $X_1 = F_1(\hat{X}_{11}) = F_2(\hat{X}_{01}) = F_3(\hat{X}_{10})$ common to the subdomains Ω_1, Ω_2, Ω_3. Then, $\mathscr{H}_j(X_1)$ contains 6 points (see Fig. 6):

$$h^{(1)} = F_1((\nu_{j,M_j}, \nu_{j,M_j})), \qquad h^{(2)} = F_2((\nu_{j,1}, \nu_{j,M_j})), \qquad h^{(3)} = F_3((\nu_{j,M_j}, \nu_{j,1})),$$

$$h^{(12)} = F_1((1, \nu_{j,M_j})) = F_2((0, \nu_{j,M_j})),$$

$$h^{(23)} = F_2((\nu_{j,1}, 0)) = F_3((1, \nu_{j,M_j})),$$

$$h^{(13)} = F_1((\nu_{j,M_j}, 1)) = F_3((\nu_{j,M_j}, 0)).$$

For the sake of simplicity, set here $\xi_0 := \xi_{j,0}$, $\xi_1 := \xi_{j,1}$, $\eta_0 := \eta_{j,\nu_{j,1}}$, $\eta_1 := \eta_{j,\nu_{j,M_j}}$. Then the local spaces are

$$v^{(1)} \in V_{j+1}^{X_1}(\Omega_1) \qquad \text{iff}$$

$$v^{(1)}(x) = \alpha_{00}^{(1)}\xi_1(\hat{x}_1)\xi_1(\hat{x}_2) + \alpha_{01}^{(1)}\xi_1(\hat{x}_1)\eta_1(\hat{x}_2) + \alpha_{10}^{(1)}\eta_1(\hat{x}_1)\xi_1(\hat{x}_2) + \alpha_{11}^{(1)}\eta_1(\hat{x}_1)\eta_1(\hat{x}_2)$$

with $\boldsymbol{\alpha}^{(1)} = (\alpha_{00}^{(1)}, \alpha_{01}^{(1)}, \alpha_{10}^{(1)}, \alpha_{11}^{(1)}) \in \mathbb{R}^4$;

$$v^{(2)} \in V_{j+1}^{X_1}(\Omega_2) \qquad \text{iff}$$

$$v^{(2)}(x) = \alpha_{00}^{(2)}\xi_0(\hat{x}_1)\xi_1(\hat{x}_2) + \alpha_{01}^{(2)}\xi_0(\hat{x}_1)\eta_1(\hat{x}_2) + \alpha_{10}^{(2)}\eta_0(\hat{x}_1)\xi_1(\hat{x}_2) + \alpha_{11}^{(2)}\eta_0(\hat{x}_1)\eta_1(\hat{x}_2)$$

with $\boldsymbol{\alpha}^{(2)} = (\alpha_{00}^{(2)}, \alpha_{01}^{(2)}, \alpha_{10}^{(2)}, \alpha_{11}^{(2)}) \in \mathbb{R}^4$;

$$v^{(3)} \in V_{j+1}^{X_1}(\Omega_3) \qquad \text{iff}$$

$$v^{(3)}(x) = \alpha_{00}^{(3)}\xi_1(\hat{x}_1)\xi_0(\hat{x}_2) + \alpha_{01}^{(3)}\xi_1(\hat{x}_1)\eta_0(\hat{x}_2) + \alpha_{10}^{(3)}\eta_1(\hat{x}_1)\xi_0(\hat{x}_2) + \alpha_{11}^{(3)}\eta_1(\hat{x}_1)\eta_0(\hat{x}_2)$$

with $\boldsymbol{\alpha}^{(3)} = (\alpha_{00}^{(3)}, \alpha_{01}^{(3)}, \alpha_{10}^{(3)}, \alpha_{11}^{(3)}) \in \mathbb{R}^4$.

Firstly, we have 2 matching conditions at the 0-face X_1. The corresponding vectors $\mathbf{c}_{123}^{(i)}$, as defined in (5.16), are

$$\mathbf{c}_{123}^{(1)} = \mathbf{c}_{123}^{(2)} = \mathbf{c}_{123}^{(3)} = (1, 1, 1, 1).$$

Secondly, we have 1 matching condition at each of the 1-faces $\Gamma_{1,2}$, $\Gamma_{2,3}$, $\Gamma_{3,1}$. The corresponding vectors, as defined in (5.16), are

$$\mathbf{c}_{12}^{(1)} = \mathbf{c}_{12}^{(2)} = \mathbf{c}_{23}^{(3)} = (1, -1, 1, -1), \qquad \mathbf{c}_{13}^{(1)} = \mathbf{c}_{23}^{(2)} = \mathbf{c}_{13}^{(3)} = (1, 1, -1, -1).$$

Finally, we have the orthogonality condition (5.23), which corresponds to the vector $\mathbf{b} = (1, 0, 0, 0)$. Thus, the matrix $\mathfrak{D}$ defined in (5.24) has the form

$$\mathfrak{D} = \begin{array}{|c|c|c|} \hline \mathbf{c}_{123}^{(1)} & -\mathbf{c}_{123}^{(2)} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{c}_{123}^{(2)} & -\mathbf{c}_{123}^{(3)} \\ \hline \mathbf{c}_{12}^{(1)} & -\mathbf{c}_{12}^{(2)} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{c}_{23}^{(2)} & -\mathbf{c}_{23}^{(3)} \\ \hline -\mathbf{c}_{13}^{(1)} & \mathbf{0} & \mathbf{c}_{13}^{(3)} \\ \hline \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \hline \end{array}.$$

It follows that

$$\mathfrak{D}\mathfrak{D}^t = \left[\begin{array}{ccccc|c} 8 & -4 & & & & \\ -4 & 8 & & & & \\ & & 8 & & & \mathbf{0} \\ & & & 8 & & \\ & & & & 8 & \\ \hline & & & \mathbf{0}^t & & 3 \end{array}\right].$$

Recalling that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}, \boldsymbol{\alpha}^{(3)}) \in \mathbb{R}^{12}$, by imposing biorthogonality we end up with 6 vectors $\boldsymbol{\alpha}_I, \ldots, \boldsymbol{\alpha}_{VI}$ which define as many biorthogonal functions in $W_j^{X_1}(\Omega)$. They are associated with the points in $\mathcal{H}_j(X_1)$.

In [7], we will discuss particular choices of the biorthogonal bases leading to a minimal localization of their supports.

5.4. Conclusions: A Characterization Theorem

For convenience and completeness, hereafter we collect the main results of the previous sections.

THEOREM 5.5. *Let us assume that the following conditions hold:*

(i) $\Omega \subset \mathbb{R}^n$ *is a bounded domain with Lipschitz boundary* $\Gamma = \partial\Omega$ *which is subdivided in two relatively open* (*with respect to* Γ) *disjoint parts* Γ_D *and* Γ_N *representing the Dirichlet and Neumann boundary part, respectively.*

(ii) Ω *is subdivided into* N *disjoint subdomains* $\Omega_i \subset \Omega$, $i = 1, \ldots, N$, *such that* $\bar{\Omega}$ *is the union of all* $\bar{\Omega}_i$. *In addition, the subdomains are the image of the reference domain* $\hat{\Omega} = (0, 1)^n$ *under certain r-time continuously differentiable mappings* F_i: $\bar{\hat{\Omega}} \to \bar{\Omega}_i$, *such that* $\det(JF_i) > 0$ *in* $\bar{\hat{\Omega}}$. *The decomposition is conformal, in the following sense: any nonempty interface* $\partial\Omega_i \cap \partial\Omega_{i'}$ *is a p-face of both subdomains for some* $0 \le p \le n - 1$; *any nonempty intersection* $\partial\Omega_i \cap \bar{\Gamma}_D$ *is a p-face of* Ω_i.

(iii) *On the interval* $[0, 1]$, *for all* $j \ge j_0$ (*for some* j_0), *dual systems of scaling functions* Ξ_j^β, $\tilde{\Xi}_j^\beta$ *and corresponding biorthogonal wavelet systems* Υ_j^β *and* $\tilde{\Upsilon}_j^\beta$ *are given, whose functions may vanish at one or both endpoints of the interval, depending on* β. *These systems satisfy the conditions in* (2.3) *listed in Section* 2 (*with the index* β *appended to all symbols*), *and they are boundary adapted for* $\beta = (0, 0)$. *In particular, the following inclusions hold* $\mathbb{P}_{L-1}^\beta(0, 1) \subset S_j^\beta := \operatorname{span} \Xi_j^\beta \subset H^\gamma(0, 1)$, $\mathbb{P}_{\tilde{L}-1}^\beta(0, 1) \subset \tilde{S}_j^\beta := \operatorname{span} \tilde{\Xi}_j^\beta \subset H^{\tilde{\gamma}}(0, 1)$, *for some* $L, \tilde{L} \ge 1$ *and some* $\gamma, \tilde{\gamma}$ *satisfying* $1 < \gamma, \tilde{\gamma} \le r$.

(iv) *The univariate scaling systems and the set of mappings* F_i *fulfill Hypothesis* (4.2).

Then we have:

(a) *The systems of locally supported scaling functions* $\Phi_j := \{\varphi_{j,k}: k \in \mathcal{K}_j\}$ *and* $\tilde{\Phi}_j := \{\tilde{\varphi}_{j,k}: k \in \mathcal{K}_j\}$, *defined by* (4.15) *for all* $j \ge j_0$, *form a dual multiresolution analysis in* $L^2(\Omega)$ *with respect to the inner product* $\langle \cdot, \cdot \rangle_\Omega$ *defined in* (4.18).

(b) *The functions in* Φ_j *and* $\tilde{\Phi}_j$ *are continuous across the interelement boundaries. Hence,* Φ_j *is contained in the Sobolev space* $H_b^\gamma(\Omega; \mathcal{P})$ *defined in* (4.12); *similarly* $\tilde{\Phi}_j \subset H_b^{\tilde{\gamma}}(\Omega; \mathcal{P})$.

(c) *The Jackson estimate* (4.25) *is valid for* $0 \le s \le \min(L, \gamma)$, $0 \le s \le \min(\tilde{L}, \tilde{\gamma})$, *respectively.*

(d) *The Bernstein inequality* (4.26) *is valid for* $0 \le s \le \gamma$, $0 \le s \le \tilde{\gamma}$, *respectively.*

(e) *The systems of locally supported wavelets* $\{\psi_{j,h}: j \ge j_0, h \in \mathcal{H}_j = \mathcal{K}_{j+1} \backslash \mathcal{K}_j\}$ *and* $\{\tilde{\psi}_{j,h}: j \ge j_0, h \in \mathcal{H}_j\}$ *constructed in Section* 5, *form biorthogonal bases in* $L^2(\Omega)$ *with respect to the inner product* $\langle \cdot, \cdot \rangle_\Omega$.

Proof. Claims (a) to (d) have been proven in Subsection 4.2, see (4.19), Theorem 4.7, and (4.26). As far as (e) is concerned, let us first observe that

$$\mathcal{H}_j = \bigcup_{h^* \in \mathcal{F}(\mathcal{H}_j)} \mathcal{H}_j(h^*),$$

where the disjoint sets $\mathcal{H}_j(h^*)$ are defined in (5.3). In Section 5, for each $h^* \in \mathcal{F}(\mathcal{H}_j)$, we have constructed a set of mutually biorthogonal wavelets $\{\psi_{j,h}: h \in \mathcal{H}_j(h^*)\}$ and $\{\tilde{\psi}_{j,h}: h \in \mathcal{H}_j(h^*)\}$. Precisely, if $h^* \in \Omega_i$ for some $i \in \{1, \ldots, N\}$, or if $h^* \in \partial\Omega \cap \partial\Omega_i$ for exactly one $i \in \{1, \ldots, N\}$, the wavelets are defined in (5.4); on the other hand, when h^* belongs to a common face of two or more subdomains, the wavelets are defined as in Subsection 5.1 if h^* is a cross point (i.e., a 0-face), whereas they are defined as in Subsection 5.2 if h^* is internal to some p-face (for $1 \le p \le n - 1$). In each step of the construction, the newly defined functions are orthogonal to all scaling functions and all previously defined wavelets of the dual family. It follows that the elements of the sets Ψ_j and $\tilde{\Psi}_j$ are mutually biorthogonal, hence, in particular, linearly independent. Thus, setting span $\Psi_j =: W_j(\Omega)$,

$$\dim W_j(\Omega) = \operatorname{card} \mathcal{H}_j = \operatorname{card} \mathcal{K}_{j+1} - \operatorname{card} \mathcal{K}_j = \dim V_{j+1}(\Omega) - \dim V_j(\Omega);$$

moreover $W_j(\Omega) \subset V_{j+1}(\Omega)$ and $W_j(\Omega) \perp \tilde{V}_j(\Omega)$, which implies $W_j(\Omega) \cap V_j(\Omega) = \{0\}$. We conclude that

$$V_{j+1}(\Omega) = V_j(\Omega) \oplus W_j(\Omega).$$

A similar result holds for span $\tilde{\Psi}_j =: \tilde{W}_j(\Omega)$, whence the claim (e) is proven. ■

At last, let us state a characterization theorem for Sobolev spaces satisfying boundary conditions, which is based on our biorthogonal multilevel decomposition $\{V_j(\Omega), \tilde{V}_j(\Omega)\}_{j \ge j_0}$. To this end, let us identify $L^2(\Omega)$ (equipped by the inner product (4.18)) to its dual. Then, the continuous inclusions $H^s_b(\Omega; \mathcal{P}) \hookrightarrow L^2(\Omega)$ $(s \ge 0)$ imply $L^2(\Omega) \hookrightarrow (H^s_b(\Omega; \mathcal{P}))'$, again with continuous injection. It will be convenient to set

$$X^s := \begin{cases} H^s_b(\Omega; \mathcal{P}), & \text{if } s \ge 0, \\ (H^{-s}_b(\Omega; \mathcal{P}))', & \text{if } s < 0, \end{cases}$$

so that $X^{s_2} \hookrightarrow X^{s_1}$ for all real $s_1 < s_2$. Suppose now that a distribution $D \in \mathcal{D}'(\Omega)$ belongs to some X^{s^*} for $s^* \ge -\tilde{\gamma}$; recalling the statement (b) of the previous Theorem 5.5, it follows that the quantities

$$d_{j,h} := \langle D, \tilde{\psi}_{j,h} \rangle_\Omega, \qquad j \ge j_0,\ h \in \mathcal{H}_j, \tag{5.37}$$

are well defined. Indeed, the right-hand side is precisely the inner product (4.18) if $s^* \ge 0$, whereas it is the duality pairing between X^{s^*} and X^{-s^*} if $s < 0$. Similarly, we can define the quantities

$$c_{j_0,k} := \langle D, \tilde{\varphi}_{j_0,k} \rangle_\Omega, \qquad k \in \mathcal{K}_{j_0}. \tag{5.38}$$

THEOREM 5.6. *Assume that* $s \in S := (-\min(\tilde{L}, \tilde{\gamma}), \min(L, \gamma))$. *Then*

$$X^s = \{D \in \mathcal{D}'(\Omega)\colon D \in X^{s^*} \textit{ for some } s^* \in S \textit{ and } \sum_{j=j_0}^{\infty} \sum_{h \in \mathcal{K}_j} 2^{2sj} |d_{j,h}|^2 < \infty\}.$$

In addition, if $D \in X^s$, *then*

$$D = \sum_{k \in \mathcal{K}_{j_0}} c_{j_0,k} \varphi_{j_0,k} + \sum_{j=j_0}^{\infty} \sum_{h \in \mathcal{K}_j} d_{j,h} \psi_{j,h},$$

the series being convergent in the norm of X^s, *and*

$$\|D\|_{X^s}^2 \sim \sum_{k \in \mathcal{K}_{j_0}} |c_{j_0,k}|^2 + \sum_{j=j_0}^{\infty} \sum_{h \in \mathcal{K}_j} 2^{2sj} |d_{j,h}|^2. \tag{5.39}$$

A dual statement holds if we exchange the roles of $V_j(\Omega)$ *and* $\tilde{V}_j(\Omega)$.

Proof. The result follows from the properties of our multilevel decomposition stated in the Theorem 5.5, using abstract results on the characterization of scales of Hilbert spaces [16, Corollary 5.2] (see also [6, Theorems 3.1 and 5.1]). ■

Recalling the identity $H_b^s(\Omega; \mathcal{P}) = H_b^s(\Omega)$ for $0 \leq s < \frac{3}{2}$, we obtain the following result.

COROLLARY 5.7. *Define now* $S := (-\min(\frac{3}{2}, \tilde{L}, \tilde{\gamma}), \min(\frac{3}{2}, L, \gamma))$ *and let* $s \in S$. *The conclusions of Theorem* 5.6 *hold if* $H_b^s(\Omega; \mathcal{P})$ *is replaced by* $H_b^s(\Omega)$ *in the definition of* X^s. ■

6. AN APPLICATION TO ELLIPTIC BOUNDARY VALUE PROBLEMS

As an application of our construction, let us consider the numerical approximation of a boundary value problem for a second order elliptic operator. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the condition described at the beginning of Section 4. Given $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, we want to approximate the solution of the mixed Dirichlet/Neumann boundary value problem

$$\begin{aligned} -\sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta} \frac{\partial u}{\partial x_\beta}\right) + a_0 u = f \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n_a} := \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} n_\alpha = g \qquad & \text{on } \Gamma_N. \end{aligned} \tag{6.1}$$

Here, a_0 and $a_{\alpha\beta} = a_{\beta\alpha}$ belong to $C_0(\bar{\Omega})$ for $\alpha, \beta = 1, \ldots, n$; n_α are the components of the outward normal unitary vector to $\partial\Omega$. We assume that the uniform ellipticity condition

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(x)\xi_\beta\xi_\alpha \geq \mu\|\xi\|^2, \qquad \forall\xi \in \mathbb{R}^n, \ \forall\, x \in \Omega,$$

holds with a constant $\mu > 0$, and the inequality $a_0(x) \geq \mu_0$, $\forall x \in \Omega$, holds with a constant $\mu_0 \geq 0$ if $\Gamma_D \neq \emptyset$, $\mu_0 > 0$ if $\Gamma_D = \emptyset$.

We set $V = H^1_b(\Omega)$, equipped with the norm $\|\cdot\|_V = \|\cdot\|_{H^1(\Omega)}$, and

$$a\colon V \times V \to \mathbb{R}, \qquad a(u, v) := \int_\Omega \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta}\frac{\partial v}{\partial x_\alpha} + a_0 uv,$$

$$F\colon V \to \mathbb{R}, \qquad F(v) := \int_\Omega fv + \int_{\Gamma_N} gv.$$

The linear form $F(\cdot)$ is continuous on V; the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous, and coercive on V, hence it satisfies

$$a(v, v) \sim \|v\|^2_{H^1(\Omega)}, \qquad \forall v \in V. \tag{6.2}$$

Problem (6.1) is formulated in the usual variational form: find $u \in V$ such that

$$a(u, v) = F(v), \qquad \forall v \in V.$$

Existence and uniqueness of the solution follow from the properties of the forms $a(\cdot,\cdot)$ and $F(\cdot)$.

In this framework, the Galerkin projection method provides a natural way of approximating the problem, by using the family of subspaces $V_j(\Omega)$ defined in Section 4. For any $J > j_0$, let $u_J \in V_J(\Omega)$ be the solution of the finite dimensional variational problem

$$a(u_J, v_J) = F(v_J), \qquad \forall v_J \in V_J(\Omega). \tag{6.3}$$

We recall the following classical stability and convergence result for a Galerkin approximation, which is a consequence of the properties of the forms $a(\cdot,\cdot)$ and $F(\cdot)$, the Jackson inequality (4.25), and the characterization result in Theorem 5.6 (see, e.g., [9]).

PROPOSITION 6.1. *For any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, and for any $J > j_0$ we have*

$$\|u_J\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)};$$

$$\|u - u_J\|_{H^1(\Omega)} \lesssim \inf_{v_J \in V_J(\Omega)} \|u - v_J\|_{H^1(\Omega)} \to 0, \qquad \text{as } J \to +\infty;$$

if $u \in H^s_b(\Omega; \mathcal{P})$ *for some* $s \leq \min(L, \gamma)$, *then*

$$\|u - u_J\|_{H^1(\Omega)} \lesssim 2^{-(s-1)J} \|u\|_{H^s(\Omega; \mathcal{P})}.$$

An application of the multilevel decomposition of Section 5 is the construction of an efficient preconditioner for the discrete problem (6.3), in view of its solution by an iterative (e.g., Conjugate Gradient) method [18]. Another application is the adaptive selection of a subspace of

$$V_{J_{max}} = V_{j_0}(\Omega) \oplus \left(\bigoplus_{j=j_0}^{J_{max}-1} W_j(\Omega) \right), \qquad \text{for some } J_{max} > j_0,$$

in which to seek the approximate solution, with the aim of optimizing the number of active degrees of freedom for a prescribed tolerance of the error [13]. In this paper, we shall be concerned with the first application only.

We recall that any $v_J \in V_J(\Omega)$ can be represented in the two equivalent forms

$$v_J = \sum_{k \in \mathcal{K}_J} c_{J,k} \varphi_{J,k} = \sum_{k \in \mathcal{K}_{j_0}} c_{j_0,k} \varphi_{j_0,k} + \sum_{j=j_0}^{J-1} \sum_{h \in \mathcal{H}_j} d_{j,h} \psi_{j,h}. \tag{6.4}$$

Thus, we identify v_J with the vectors in $\mathbb{R}^{N_J}$ (where $N_J = \text{card } \mathcal{K}_J$)

$$\mathbf{v}_J = \{c_{J,k} : k \in \mathcal{K}_J\},$$

or

$$\hat{\mathbf{v}}_J = \{\{c_{j_0,k} : k \in \mathcal{K}_{j_0}\}, \{d_{j,h} : j_0 \leq j < J, h \in \mathcal{H}_j\}\},$$

and we denote by $\hat{\mathbf{v}} = \mathbf{S}\mathbf{v}$ the wavelet transform implied by the change of basis (6.4). Problem (6.3) can be written in matrix form with respect to the scaling function basis as

$$\mathbf{A}_J \mathbf{u}_J = \mathbf{f}_J,$$

where

$$\mathbf{A}_J = \{a(\varphi_{J,k}, \varphi_{J,l}) : k, l \in \mathcal{K}_J\}, \qquad \mathbf{f}_J = \{F(\varphi_{J,l}) : l \in \mathcal{K}_J\}.$$

On the other hand, it can also be expressed in terms of the wavelet basis as

$$\hat{\mathbf{A}}_J \hat{\mathbf{u}}_J = \hat{\mathbf{f}}_J, \tag{6.5}$$

where

$$\hat{\mathbf{A}}_J = \begin{pmatrix} \mathbf{A}_{j_0} & \mathbf{A}_{\varphi\psi} \\ \mathbf{A}_{\varphi\psi}^t & \mathbf{A}_{\psi\psi} \end{pmatrix},$$

with $\mathbf{A}_{j_0}$ defined in a way similar to $\mathbf{A}_J$ and

$$\mathbf{A}_{\varphi\psi} = \{a(\varphi_{j_0,k}, \psi_{j',h'}) : k \in \mathcal{K}_{j_0}, j_0 \le j < J, h' \in \mathcal{H}_j\},$$

$$\mathbf{A}_{\psi\psi} = \{a(\psi_{j,h}, \psi_{j',h'}) : j_0 \le j, j' < J, h, h' \in \mathcal{H}_j\},$$

and

$$\hat{\mathbf{f}}_J = \{\{F(\varphi_{j_0,k}) : k \in \mathcal{K}_{j_0}\}, \{F(\psi_{j,h}) : j_0 \le j < J, h \in \mathcal{H}_j\}\}.$$

Obviously, $\mathbf{A}_J = \mathbf{S}^t\hat{\mathbf{A}}_J\mathbf{S}$ and $\mathbf{f}_J = \mathbf{S}^t\hat{\mathbf{f}}_J$. The matrix $\mathbf{A}_J$ is in general ill-conditioned, with $\text{cond}_2(\mathbf{A}_J) \sim 2^{2J}$, as a consequence of the fact that diam(supp $\varphi_{J,k}) \sim 2^{-J}$, $\|\varphi_{J,k}\|_{L^2(\Omega)} \sim 1$ for all $k \in \mathcal{K}_J$. Conversely, the matrix $\hat{\mathbf{A}}_J$ can be easily preconditioned by a block diagonal scaling, leading to a $\mathcal{O}(1)$-condition number. Indeed, for any $\hat{\mathbf{v}} \in \mathbb{R}^{N_J}$,

$$\begin{aligned} \hat{\mathbf{v}}^t\hat{\mathbf{A}}_J\hat{\mathbf{v}} = a(v, v) \sim \|v\|^2_{H^1(\Omega)} & \qquad \text{(by (6.2))} \\ \sim \|P^{\Omega}_{j_0}v\|^2_{H^1(\Omega)} + \sum_{j=j_0}^{J-1}\sum_{h\in\mathcal{H}_j} 2^{2j}|d_{j,h}|^2 & \qquad \text{(by Theorem 5.6)} \\ \sim \mathbf{v}^t_{j_0}\mathbf{A}_{j_0}\mathbf{v}_{j_0} + \sum_{j=j_0}^{J-1}\sum_{h\in\mathcal{H}_j} 2^{2j}|d_{j,h}|^2 & \qquad \text{(by (6.2) again)}, \end{aligned}$$

with $\mathbf{v}_{j_0} = \{c_{j_0,k} : k \in \mathcal{K}_{j_0}\}$. Setting

$$\mathbf{D} = \begin{pmatrix} \mathbf{A}_{j_0} & \mathbf{0} \\ \mathbf{0}^t & \hat{\mathbf{D}} \end{pmatrix},$$

with $\hat{\mathbf{D}} = \text{diag}\{\lambda_{j,h} := 2^{2j} : j_0 \le j < J, h \in \mathcal{H}_j\}$, we conclude that

$$\hat{\mathbf{v}}^t\hat{\mathbf{A}}_J\hat{\mathbf{v}} \sim \hat{\mathbf{v}}^t\,\mathbf{D}\hat{\mathbf{v}}, \qquad \forall\, \hat{\mathbf{v}} \in \mathbb{R}^{N_J},$$

i.e.,

$$\text{cond}_2(\mathbf{D}^{-1/2}\hat{\mathbf{A}}_J\mathbf{D}^{-1/2}) \sim 1, \qquad \text{as } J \to +\infty.$$

Thus, $\mathbf{D}$ can be used as a preconditioner while solving system (6.5) iteratively. Since j_0 is usually fairly small, $\mathbf{A}_{j_0}$ can be Cholesky-factorized once and for all in a preprocessing stage. The numerical solution of (6.5) can be computed with an optimal

$\mathcal{O}(N_J)$-operation count, despite the matrix $\hat{\mathbf{A}}_J$ is not sparse (its coefficients $a(\psi_{j,h}, \psi_{j',h'})$ involve interactions among all scales). Indeed, any product of the form $\hat{\mathbf{A}}_J\hat{\mathbf{z}}$ can be implemented as $\mathbf{S}^{-t}\mathbf{A}_J\mathbf{S}^{-1}\hat{\mathbf{z}}$; only the sparse matrix $\mathbf{A}_J$ need actually be formed, whereas the wavelet transform $\mathbf{S}$ and its inverse are accomplished by a recursion in $\mathcal{O}(N_J)$-operations [18].

APPENDIX

A. Boundary Conditions for Biorthogonal Wavelets on [0, 1]

In this appendix, we prove some new results, concerning the constructions of biorthogonal wavelet systems on the interval as done in [20, 27], which are not contained in these papers.

Let us start by proving that these systems can be built to be boundary adapted (see Definition 2.1). Firstly, let us recall that the conditions

$$\begin{aligned} &\xi_{j,k}(0) \neq 0 \Leftrightarrow k = 0, \qquad \xi_{j,k}(1) \neq 0 \Leftrightarrow k = 1, \\ &\tilde{\xi}_{j,k}(0) \neq 0 \Leftrightarrow k = 0, \qquad \tilde{\xi}_{j,k}(1) \neq 0 \Leftrightarrow k = 1 \end{aligned} \tag{A.1}$$

are certainly satisfied for these constructions (see [22, Proposition 2.1.3; 27, Section 4]). Next, let us explicitly write the relations (2.3.a) as

$$\xi_{j,k} = \sum_{l\in\Delta_{j+1}} a^j_{k,l}\xi_{j+1,l}, \qquad \tilde{\xi}_{j,k} = \sum_{l\in\Delta_{j+1}} \tilde{a}^j_{k,l}\tilde{\xi}_{j+1,l}. \tag{A.2}$$

LEMMA A.1. *There exist constants c, $\tilde{c} > 0$, such that*

$$\xi_{j,d}(d) = c2^{j/2}, \qquad \tilde{\xi}_{j,d}(d) = \tilde{c}2^{j/2}, \qquad d = 0, 1. \tag{A.3}$$

As a consequence, the following property holds

$$\xi_{j+1,d}(d)\tilde{\xi}_{j+1,d}(d) = 2\xi_{j,d}(d)\tilde{\xi}_{j,d}(d), \qquad d = 0, 1. \tag{A.4}$$

Proof. For the systems in [27], one has $\xi_{j,d}(d) = 2^{(j-j_0)/2}\xi_{j_0,d}(d)$ and $\tilde{\xi}_{j,d}(d) = 2^{(j-j_0)/2}\tilde{\xi}_{j_0,d}(d)$ (Definition 3.20), which implies (A.4). For the systems in [20], see [22, Proposition 2.1.3, formula (2.1.34)]. Moreover, for both constructions, one has $c = 1$. ■

We need some auxiliary results, which we will collect now.

Remark A.2. Let H be some Hilbert space equipped with an inner product $(\cdot\,,\cdot)_H$. For some countable set of indices I, we consider

$$V := \mathrm{clos}_H\,\mathrm{span}\{\phi_k : k \in I\}, \qquad W := \mathrm{clos}_H\,\mathrm{span}\{\zeta_k : k \in I\},$$

for ϕ_k, $\zeta_k \in H$. The two sets of functions can be biorthogonalized, if and only if the generalized Gramian matrix

$$G := ((\phi_k, \zeta_l)_H)_{k,l\in I}$$

is regular. This however is equivalent to one of the following statements: either $V \cap W^\perp = \{0\}$ or $V^\perp \cap W = \{0\}$.

Remark A.3. Let E, $\tilde{E}$ and F, $\tilde{F}$ be (different) biorthogonal bases for some subspaces X, $\tilde{X}$ in a Hilbert space H. If we denote the matrices for the change of bases by K and H, respectively, i.e.,

$$F = KE, \qquad \tilde{F} = H\tilde{E},$$

one has $H = K^{-t}$. In fact, let us consider the bases as column vectors. Clearly, K and H are regular transformations and hence biorthogonality implies

$$Id = (F, \tilde{F})_H = (KE, H\tilde{E})_H = K(E, \tilde{E})_H H^t = KH^t.$$

LEMMA A.4. *Let $E = \{\eta_k : k \in I\}$ and $\tilde{E} = \{\tilde{\eta}_k : k \in I\}$ be biorthogonal bases for some subspaces X, $\tilde{X}$ in a Hilbert space H. Given two linear forms ℓ: $X \to \mathbb{R}$ and $\tilde{\ell}$: $\tilde{X} \to \mathbb{R}$, let us define the column vectors $\ell E := (\ell\eta_k)_{k\in I}$ and $\tilde{\ell}\tilde{E} := (\tilde{\ell}\tilde{\eta}_k)_{k\in I}$. Then, the inner product $(\ell E)^t \cdot \tilde{\ell}\tilde{E}$ is invariant under any biorthogonal change of bases.*

Proof. Let $F = \{\rho_k : k \in I\}$ and $\tilde{F} = \{\tilde{\rho}_k : k \in I\}$ be another couple of biorthogonal bases for the subspaces X, $\tilde{X}$. Then, using Remark A.3, we have

$$(\ell F)^t \cdot \tilde{\ell}\tilde{F} = (K\ell E)^t \cdot K^{-t}\tilde{\ell}\tilde{E} = (\ell E)^t K^t K^{-t}\tilde{\ell}\tilde{E} = (\ell E)^t \cdot \tilde{\ell}\tilde{E}. \quad \blacksquare$$

PROPOSITION A.5. *Let $\check{\Upsilon}_j = \{\check{\eta}_{j,h} : h \in \nabla_j\}$, $\tilde{\check{\Upsilon}}_j = \{\tilde{\check{\eta}}_{j,h} : h \in \nabla_j\}$ be any biorthogonal wavelet systems arising from scaling function systems satisfying* (A.1) *and* (A.4). *Then, there exist biorthogonal wavelet systems $\Upsilon_j = \{\eta_{j,h} : h \in \nabla_j\}$, $\tilde{\Upsilon}_j = \{\tilde{\eta}_{j,h} : h \in \nabla_j\}$ satisfying the conditions*

$$\begin{aligned} &\eta_{j,h}(0) \neq 0 \Leftrightarrow h = \nu_{j,1}, \qquad \eta_{j,h}(1) \neq 0 \Leftrightarrow h = \nu_{j,M_j}, \\ &\tilde{\eta}_{j,h}(0) \neq 0 \Leftrightarrow h = \nu_{j,1}, \qquad \tilde{\eta}_{j,h}(1) \neq 0 \Leftrightarrow h = \nu_{j,M_j}. \end{aligned} \tag{A.5}$$

Proof. Since the construction is carried on independently at each end point of the interval, we will only treat the left boundary at $x = 0$. We show first, that there exists an index $\tau \in \nabla_j$, such that

$$\check{\eta}_{j,\tau}(0)\tilde{\check{\eta}}_{j,\tau}(0) \neq 0. \tag{A.6}$$

Indeed, we have

$$
\begin{aligned}
2\xi_{j,0}(0)\tilde{\xi}_{j,0}(0) &= \xi_{j+1,0}(0)\tilde{\xi}_{j+1,0}(0) && \text{(by (A.4))} \\
&= \sum_{k\in\Delta_{j+1}} \xi_{j+1,k}(0)\tilde{\xi}_{j+1,k}(0) && \text{(by (A.1))}.
\end{aligned}
$$

Applying Lemma A.4 with $X = S_{j+1}$, $\tilde{X} = \tilde{S}_{j+1}$, $\ell v = \tilde{\ell} v = v(0)$ and $E = \Xi_{j+1}$, $\tilde{E} = \tilde{\Xi}_{j+1}$, $F = \Xi_j \cup \check{Y}_j$, $\tilde{F} = \tilde{\Xi}_j \cup \tilde{\check{Y}}_j$, yields

$$
\begin{aligned}
2\xi_{j,0}(0)\tilde{\xi}_{j,0}(0) &= \sum_{k\in\Delta_j} \xi_{j,k}(0)\tilde{\xi}_{j,k}(0) + \sum_{h\in\nabla_j} \check{\eta}_{j,h}(0)\tilde{\check{\eta}}_{j,h}(0) \\
&= \xi_{j,0}(0)\tilde{\xi}_{j,0}(0) + \sum_{h\in\nabla_j} \check{\eta}_{j,h}(0)\tilde{\check{\eta}}_{j,h}(0) && \text{(by (A.1))}.
\end{aligned}
$$

Thus

$$
\sum_{h\in\nabla_j} \check{\eta}_{j,h}(0)\tilde{\check{\eta}}_{j,h}(0) = \xi_{j,0}(0)\tilde{\xi}_{j,0}(0) \neq 0, \tag{A.7}
$$

which, in particular, proves (A.6). Without loss of generality, we set $\tau = \nu_{j,1}$. For convenience, we will frequently use the abbreviations

$$
\varrho_j := \check{\eta}_{j,\nu_{j,1}}(0), \qquad \tilde{\varrho}_j := \tilde{\check{\eta}}_{j,\nu_{j,1}}(0). \tag{A.8}
$$

For $h \in \nabla_j^0 := \nabla_j \backslash \{\nu_{j,1}\}$ we define

$$
\eta^*_{j,h} := \check{\eta}_{j,h} - c_{j,h}\check{\eta}_{j,\nu_{j,1}}, \qquad \tilde{\eta}^*_{j,h} := \tilde{\check{\eta}}_{j,h} - \tilde{c}_{j,h}\tilde{\check{\eta}}_{j,\nu_{j,1}},
$$

where

$$
c_{j,h} := \frac{\check{\eta}_{j,h}(0)}{\varrho_j}, \qquad \tilde{c}_{j,h} := \frac{\tilde{\check{\eta}}_{j,h}(0)}{\tilde{\varrho}_j}
$$

and $\eta^*_{j,\nu_{j,1}} := \check{\eta}_{j,\nu_{j,1}}$, $\tilde{\eta}^*_{j,\nu_{j,1}} := \tilde{\check{\eta}}_{j,\nu_{j,1}}$. Obviously, this system of functions is boundary adapted, so that we have to prove that the generalized Gramian

$$
G := ((\eta^*_{j,h}, \tilde{\eta}^*_{j,l})_{L^2(0,1)})_{h,l\in\nabla_j^0}
$$

is regular. Then we could first biorthogonalize $\eta^*_{j,h}$, $\tilde{\eta}^*_{j,h}$, $h \in \nabla_j^0$, so that the resulting functions all vanish at the left boundary. Finally, biorthogonalization of the remaining functions $\eta^*_{j,\nu_{j,1}}$, $\tilde{\eta}^*_{j,\nu_{j,1}}$ with respect to all the others would lead to the seeked system.

Using the biorthogonality of $\check{\eta}_{j,h}$ and $\tilde{\check{\eta}}_{j,h}$ it is readily seen that the entries of G are of the form

$$g_{h,l} = \delta_{h,l} + c_h \tilde{c}_l,$$

which means $G = \mathrm{Id} + \mathbf{c}\tilde{\mathbf{c}}^t$, where $\mathbf{c} := ((c_h)_{h\in\nabla_j^0})^t$. This shows that G has the simple eigenvalue $1 + \tilde{\mathbf{c}}^t\mathbf{c}$ (corresponding to the eigenvector $\mathbf{c}$) and the eigenvalue 1 with multiplicity $|\nabla_j^0| - 1$. That means that G is regular, if and only if

$$0 \neq 1 + \tilde{\mathbf{c}}^t\mathbf{c} = 1 + \sum_{h\in\nabla_j^0} \frac{\check{\eta}_{j,h}(0)\,\tilde{\check{\eta}}_{j,h}(0)}{\varrho_j\tilde{\varrho}_j}.$$

But this condition is fulfilled in view of (A.7), which proves the theorem. ■

As in (A.8), we define

$$\begin{aligned} \lambda_j &:= \xi_{j,0}(0), & \tilde{\lambda}_j &:= \tilde{\xi}_{j,0}(0), \\ \varrho_j &:= \eta_{j,\nu_{j,1}}(0), & \tilde{\varrho}_j &:= \tilde{\eta}_{j,\nu_{j,1}}(0). \end{aligned} \tag{A.9}$$

Proposition A.6. *Under the hypotheses of Theorem* A.5*, one has in addition*

(i) $\lambda_j\tilde{\lambda}_j = \varrho_j\tilde{\varrho}_j$,
(ii) $\lambda_j = \varrho_j > 0$, $\tilde{\lambda}_j = \tilde{\varrho}_j$.

Proof. Using the refinement Eqs. (A.2) and (A.1) one has

$$\tilde{\lambda}_j = \sum_{k\in\Delta_{j+1}} \tilde{a}^j_{0,k}\tilde{\xi}_{j+1,k}(0) = \tilde{a}^j_{0,0}\tilde{\xi}_{j+1,0}(0) = \tilde{a}^j_{0,0}\tilde{\lambda}_{j+1},$$

hence, $\tilde{a}^j_{0,0} = \tilde{\lambda}_j/\tilde{\lambda}_{j+1}$. By using the expression of the wavelets in terms of the scaling functions on the next higher level we obtain

$$\tilde{\varrho}_j = \tilde{\eta}_{j,\nu_{j,1}}(0) = \sum_{k\in\Delta_{j+1}} \tilde{b}^j_{\nu_{j,1},k}\tilde{\xi}_{j+1,k}(0) = \tilde{b}^j_{\nu_{j,1},0}\tilde{\xi}_{j+1,0}(0) = \tilde{b}^j_{\nu_{j,1},0}\tilde{\lambda}_{j+1},$$

which implies $\tilde{b}^j_{\nu_{j,1},0} = \tilde{\varrho}_j/\tilde{\lambda}_{j+1}$. Finally, we use the reconstruction formula

$$\lambda_{j+1} = \sum_{k\in\Delta_j} \tilde{a}^j_{0,k}\xi_{j,k}(0) + \sum_{h\in\Delta_j} \tilde{b}^j_{h,0}\eta_{j,h}(0) = \tilde{a}^j_{0,0}\lambda_j + \tilde{b}^j_{\nu_{j,1},0}\varrho_j = \frac{\lambda_j\tilde{\lambda}_j + \varrho_j\tilde{\varrho}_j}{\tilde{\lambda}_{j+1}},$$

which implies, by (A.4),

$$\lambda_{j+1}\tilde{\lambda}_{j+1} = 2\lambda_j\tilde{\lambda}_j = \lambda_j\tilde{\lambda}_j + \varrho_j\tilde{\varrho}_j.$$

Thus, (i) follows.

To show (ii), we can assume $\lambda_j > 0$. Indeed, if $\lambda_j < 0$, one defines a new biorthogonal basis by $\xi^{\#}_{j,k} := (-1)\xi_{j,k}$ and $\tilde{\xi}^{\#}_{j,k} := (-1)\tilde{\xi}_{j,k}$, where now $\xi^{\#}_{j,0}(0) > 0$.

If we have $\tilde{\lambda}_j \neq \tilde{\varrho}_j$, then there exists a $c \notin \{0, 1\}$, such that $\tilde{\lambda}_j = c\tilde{\varrho}_j$. Now, in view of Remark A.3 we may define a new biorthogonal basis by setting

$$\eta^{\#}_{j,h} := c\eta_{j,h}, \qquad \tilde{\eta}^{\#}_{j,h} := c^{-1}\tilde{\eta}_{j,h},$$

and these functions fulfill $\tilde{\lambda}_j = \tilde{\varrho}^{\#}_j = \tilde{\eta}^{\#}_{j,\nu_{j,1}}(0)$. Hence, the remaining claim in (ii) follows by (i). ■

Propositions A.5 and A.6, together with Lemma A.1, guarantee that the systems Ξ_j, Υ_j and $\tilde{\Xi}_j$, $\tilde{\Upsilon}_j$ are boundary adapted.

Finally, let us enforce boundary conditions to the scaling systems. The construction of the corresponding biorthogonal system is now obvious.

COROLLARY A.7. *Let Ξ_j, Υ_j and $\tilde{\Xi}_j$, $\tilde{\Upsilon}_j$ be boundary adapted systems on $[0, 1]$ and denote by*

$$\Xi^{\beta}_j := \{\xi_{j,k} : k \in \Delta^{\beta}_j\}, \qquad \tilde{\Xi}^{\beta}_j := \{\tilde{\xi}_{j,k} : k \in \Delta^{\beta}_j\} \tag{A.10}$$

(see (2.10)) the scaling function bases satisfying zero boundary conditions corresponding to the vector $\beta = (\beta^0, \beta^1)$.

Then, setting

$$\eta^{d}_{j,h} := \begin{cases} \eta_{j,h}, & \text{if } d = 1, \\ \eta^{D}_{j,h}, & \text{if } d = 0, \end{cases} \qquad \tilde{\eta}^{d}_{j,h} := \begin{cases} \tilde{\eta}_{j,h}, & \text{if } d = 1, \\ \tilde{\eta}^{D}_{j,h}, & \text{if } d = 0, \end{cases}$$

for $h \in \{\nu_{j,1}, \nu_{j,M_j}\}$ and

$$\begin{aligned} \eta^{D}_{j,\nu_{j,1}} &:= \frac{1}{\sqrt{2}}(\eta_{j,\nu_{j,1}} - \xi_{j,0}), \qquad & \eta^{D}_{j,\nu_{j,M_j}} &:= \frac{1}{\sqrt{2}}(\eta_{j,\nu_{j,M_j}} - \xi_{j,1}), \\ \tilde{\eta}^{D}_{j,\nu_{j,1}} &:= \frac{1}{\sqrt{2}}(\tilde{\eta}_{j,\nu_{j,1}} - \tilde{\xi}_{j,0}), \qquad & \tilde{\eta}^{D}_{j,\nu_{j,M_j}} &:= \frac{1}{\sqrt{2}}(\tilde{\eta}_{j,\nu_{j,M_j}} - \tilde{\xi}_{j,1}), \end{aligned} \tag{A.11}$$

the families

$$\Upsilon^{\beta}_j := \{\eta_{j,h} : h \in \nabla^{int}_j\} \cup \{\eta^{\beta^0}_{j,\nu_{j,1}}, \eta^{\beta^1}_{j,\nu_{j,M_j}}\}, \qquad \tilde{\Upsilon}^{\beta}_j := \{\tilde{\eta}_{j,h} : h \in \nabla^{int}_j\} \cup \{\tilde{\eta}^{\beta^0}_{j,\nu_{j,1}}, \tilde{\eta}^{\beta^1}_{j,\nu_{j,M_j}}\},$$

are biorthogonal systems according to Ξ^{β}_j, $\tilde{\Xi}^{\beta}_j$.

Proof. Let us consider the case $\beta = (0, 0)$; the remaining cases are dealt with in a completely analogous fashion. The biorthogonality is easily checked by the analogous properties of the original system. Hence, it remains to show that $\eta^{D}_{j,d} \in S^{int}_{j+1}$, $\tilde{\eta}^{D}_{j,d} \in \tilde{S}^{int}_{j+1}$ for $d \in \{\nu_{j,1}, \nu_{j,M_j}\}$, where these spaces are defined as the span of Ξ^{β}_{j+1} and $\tilde{\Xi}^{\beta}_{j+1}$, respectively. It is obvious that these functions are contained in $S_{j+1} = \operatorname{span} \Xi_{j+1}$ and $\tilde{S}_{j+1} = \operatorname{span} \tilde{\Xi}_{j+1}$, respectively. Because of the stability of the single scale basis functions and (2.5), we have

$$v \in S_{j+1}^{int} \Leftrightarrow v \in S_{j+1} \qquad \text{and} \qquad v(0) = v(1) = 0. \tag{A.12}$$

Since, in view of (A.1)–(A.5), $\eta_{j,d}^D(0) = \eta_{j,d}^D(1) = 0$, $d \in \{\nu_{j,1}, \nu_{j,M_j}\}$ (and analogously for the dual functions) this proves our claim. ■

Finally, let us make some useful observations which are an easy consequence of Remark A.2.

LEMMA A.8. *Let $\mathcal{M}$, $\tilde{\mathcal{M}}$ be $p \times q$-matrices (with $p \leq q$) such that $\mathcal{M}\tilde{\mathcal{M}}^t$ is regular. Then, there exist $q \times (q - p)$-matrices $\mathcal{A}$, $\tilde{\mathcal{A}}$ such that*

$$\mathcal{M}\mathcal{A} = 0, \qquad \tilde{\mathcal{M}}\tilde{\mathcal{A}} = 0, \qquad \mathcal{A}^t\tilde{\mathcal{A}} = I. \tag{A.13}$$

Proof. Because of (A.13), $\dim \ker \mathcal{M} = \dim \ker \tilde{\mathcal{M}} = q - p$. Now, $\{0\} = \ker \mathcal{M}\tilde{\mathcal{M}} = \ker \mathcal{M} \cap \operatorname{im} \tilde{\mathcal{M}}^t = \ker \mathcal{M} \cap (\ker \tilde{\mathcal{M}})^\perp$, and the result follows by taking into account Remark A.2. ■

Remark A.9. Let X and Y be some normed spaces of the same dimension and b: $X \times Y \to \mathbb{R}$ a bilinear form. Then X and Y are biorthogonalizable with respect to $b(\cdot\,,\cdot)$ (i.e., there exist Riesz bases $\{x_i\}_{i \in I}$, $\{y_j\}_{j \in I}$, such that $b(x_i, y_j) = \delta_{i,j}$), if and only if an *inf-sup-condition* is valid,

$$\inf_{x \in X} \sup_{y \in Y} \frac{b(x, y)}{\|x\|_X \|y\|_Y} \geq \beta > 0. \tag{A.14}$$

Indeed, this is a direct consequence of Remark A.2, because (A.14) is equivalent to $X \cap Y^\perp = \{0\}$, where here the orthogonal complement is to be understood with respect to $b(\cdot\,,\cdot)$.

REFERENCES

1. R. A. Adams, "Sobolev Spaces," Academic Press, San Diego, 1978.
2. L. Andersson, N. Hall, B. Jawerth, and G. Peters, Wavelets on closed subsets of the real line, *in* "Topics in the Theory and Applications of Wavelets" (L. L. Schumaker and G. Webb, Eds.), pp. 1–61, Academic Press, Boston, 1994.
3. S. Bertoluzza, A posteriori error estimates for the wavelet Galerkin method, *Appl. Math. Lett.* **8,** No. 5 (1995), 1–6.
4. G. Beylkin, R. Coifman, and V. Rokhlin, Fast wavelet transforms and numerical algorithms, *Comm. Pure Appl. Math.* **44** (1991), 141–183.
5. C. Canuto and I. Cravero, A wavelet-based adaptive finite element method for the advection-diffusion equations, *Math. Mod. Methods Appl. Sci.* **7** (1997), 265–289.
6. C. Canuto and A. Tabacco, Multilevel decompositions of functional spaces, *J. Fourier Anal. Appl.* **3** (1997), 715–742.
7. C. Canuto, A. Tabacco, and K. Urban, The wavelet element method. Part II. Realization and additional features in 2D and 3D, preprint, No. 28, Dipartimento di Matematica, Politecnico di Torino, 1997.
8. G. Chiavassa and J. Liandrat, On the effective construction of compactly supported wavelets satisfying homogeneous boundary conditions on the interval, *Appl. Comput. Harmon. Anal.* **4**(1) (1997), 62–73.
9. Ph. G. Ciarlet, Basic error estimates for elliptic problems, *in* "Handbook of Numerical Analysis, Vol. II. Finite Element Methods" (Ph. G. Ciarlet and J. L. Lions, Eds.), pp. 17–351, North-Holland, Amsterdam, 1991.

10. A. Cohen, W. Dahmen, and R. A. DeVore, Multiscale decompositions on bounded domains, *Trans. Amer. Math. Soc.,* in press.
11. A. Cohen, I. Daubechies, and J.-C. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **45** (1992), 485–560.
12. A. Cohen, I. Daubechies, and P. Vial, Wavelets on the interval and fast wavelet transforms, *Appl. Comput. Harmon. Anal.* **1** (1993), 54–81.
13. S. Dahlke, W. Dahmen, R. Hochmuth, and R. Schneider, Stable multiscale bases and local error estimation for elliptic problems, *Appl. Numer. Math.* **23**(1) (1997), 21–48.
14. S. Dahlke, W. Dahmen, and R. DeVore, Nonlinear approximation and adaptive techniques for solving elliptic operator equations, *in* "Multiscale Wavelet Methods for PDEs" (W. Dahmen, A. Kurdilla, and P. Oswald, Eds.), Academic Press, San Diego, pp. 237–283.
15. S. Dahlke and I. Weinreich, Wavelet bases adapted to pseudo-differential operators, *Appl. Comput. Harmon. Anal.* **1** (1994), 267–283.
16. W. Dahmen, Stability of multiscale transformations, *J. Fourier Anal. Appl.* **4** (1996), 341–362.
17. W. Dahmen, Wavelet and multiscale methods for operator equations, *Acta Numer.* (1997), 55–228.
18. W. Dahmen and A. Kunoth, Multilevel preconditioning, *Numer. Math.* **63** (1992), 315–344.
19. W. Dahmen, A. Kunoth, and K. Urban, A Wavelet–Galerkin method for the Stokes problem, *Computing* **56** (1996), 259–302.
20. W. Dahmen, A. Kunoth, and K. Urban, Biorthogonal spline-wavelets on the interval—Stability and moment conditions, *Appl. Comput. Harmon. Anal.,* in press.
21. W. Dahmen, S. Prössdorf, and R. Schneider, Multiscale methods for pseudo-differential equations on smooth manifolds, *in* "Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications" (C. K. Chui, L. Montefusco, and L. Puccio, Eds.), pp. 385–424, Academic Press, San Diego, 1994.
22. W. Dahmen and R. Schneider, Composite wavelet bases for operator equations, preprint, IGPM No. 133, RWTH Aachen, 1996.
23. I. Daubechies, Ten lectures on wavelets, *in* CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 61, Soc. for Industr. & Appl. Math., Philadelphia, 1992.
24. S. Jaffard, Wavelet methods for fast resolution of elliptic problems, *SIAM J. Numer. Anal.* **29** (1992), 965–986.
25. A. Jouini and P.-G. Lemarié-Rieusset, Analyses multirésolutions biorthogonales et applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 453–476.
26. P. G. Lemarié-Rieusset, Analyses multi-résolutions non orthogonales, commutation entre projecteurs et derivation et ondelettes vecteurs à divergence nulle, *Rev. Mat. Iberoamericana* **8** (1992), 221–236.
27. L. Levaggi and A. Tabacco, Wavelets on the interval and related topics, preprint, No. 11, Dipartimento di Matematica, Politecnico di Torino, 1997.
28. J. Liandrat and Ph. Tchamitchian, Resolution of the 1D regularized Burgers equation using a spatial wavelet approximation, ICASE Report No. 90-83, NASA Langley, 1990.
29. Y. Maday and A. T. Patera, Spectral element methods for the incompressible Navier–Stokes equations, *in* "State of the Art Surveys in Computational Mechanics" (A. K. Noor and J. T. Oden, Eds.), pp. 71–144, ASME, New York, 1989.
30. Y. Maday, V. Perrier, and J. C. Ravel, Adaptativité dynamique sur bases d'ondelettes pour l'approximation d'équations aux dérivées partielles, *C. R. Acad. Sci. Paris Sér. I* **312** (1991), 405–410.
31. Y. Meyer, "Ondelettes. Vol. I. Ondelettes et Opèrateurs," Hermann, Paris, 1990.
32. A. T. Patera, A spectral element method for fluid dynamics: Laminar flow in a channel expansion, *J. Comput. Phys.* **54** (1984), 468–488.
33. G. Strang and T. Nguyen, "Wavelets and Filter Banks," Wellesley Cambridge Press, Cambridge, 1996.
34. K. Urban, On divergence-free wavelets, *Adv. Comput. Math.* **4** (1995), 51–82.
35. M. V. Wickerhauser, "Adapted Wavelet Analysis from Theory to Software," AK Peters, Wellesley, 1994.