Various Methods for Estimating Volatility

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The GARCH Process

The GARCH(p,q) process is given by

$$\begin{aligned} \epsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2, \end{aligned}$$

with ϵ_t being errors around a mean process, and z_t is standard Brownian motion.

The most common set of parameters are $p = 1$ and $q = 1$. Hansen and Lunde [4] show that GARCH(1,1) does as well, or better, than more complicated models in forecasting volatility. As a result, it is useful to focus on the properties of GARCH(1,1) as the time-step in the finite difference stochastic difference equation approaches zero.

Small-step asymptotics for GARCH(1,1)

If one is to use (regularly spaced) high-frequency data, it is useful to develop theory for ARCH processes where the time-step in the stochastic difference equation approaches zero. Nelson [5] develops such asymptotic relations. Most notably, with respect to estimation of parameters, we expect to observe

$$\begin{aligned} \alpha_h + \beta_h &\to 1 \text{ as } h \to 0 \\ \alpha_h &\to 0 \\ \beta_h &\to 1 \end{aligned}$$

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Empirical evidence for high-frequency data does not support this. What could cause theoretical deviations?

Reasons for GARCH failure at higher frequencies

1. intraday volatility patterns
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3. other microstructure noise

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The most commonly used method to model an intraday volatility process is Andersen and Bollerslev's sequential estimation approach [1], [2]

Realized Volatility

Assuming that logarithmic price of a financial asset is given by the diffusion process

$$p_t = \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s), \tag{1}$$

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From [3], $\frac{\sqrt{m}(RV^{(m)}-IV)}{\sqrt{2\int_0^1 \sigma^4(s)ds}} \to N(0,1)$

Microstructure Noise

The very high-frequency prices are contaminated by market microstructure effects (noise), which lead to biases in realized volatility. Realized volatility estimators are derived based on the assumptions for the microscruture noise:

$$r_i^{(m)} = r_i^{*(m)} + \epsilon_i^{(m)}, i = 1, 2, \ldots, m, \quad (4)$$

where $r_i^{*(m)}$ is the true log-difference of prices, $\epsilon_i^{(m)}$ is the contamination noise, and $r^{(m)}$ is the observed log-difference price.

RV decomposition in the presence of microstructure noise

$$RV^{(m)} = RV^{*(m)} + 2\sum_{i=1}^{m} r_i^{*(m)} \epsilon_i^{(m)} + \sum_{j=1}^{m} \epsilon_j^{(m)^2}$$

Under the assumptions that *a*) the noise process is independent and identically distributed with mean zero and finite variance ω^2 and finite fourth moments, and *b*) the noise is independent of the true price.

$$E[RV^{(m)}] = IV + 2m\omega^2 \tag{5}$$

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Sampling at lower frequencies reduces bias but leads to an increase in variance (bias-variance trade-off).

Averaging and Subsampling

Average a number of IV estimators for different sampling frequencies over the high-frequency samples:

$$TTSRV^{(m,m_1,\ldots,m_K,K)} = \frac{1}{K}\sum_{k=1}^{K} RV^{(k,m_k)} - \frac{\bar{m}}{m} RV^{(all)} \qquad (6)$$

Under independent noise assumptions, the estimator is consistent. Under equidistant observations and under regular allocation of the grids, there is an asymptotic distribution result.

Kernel-Based Estimation

$$KRV^{(m,H)} = RV^{(m)} + 2\sum_{h=1}^{H} \frac{m}{m-h}\gamma_h, \qquad (7)$$

with $\gamma_h = \sum_{i=1}^{m} r_i^{(m)} r_{i+h}^{(m)}$.

Stochastic Volatility

We model the log-price $y^*(t)$ follows the solution to the stochastic differential equation

$$dy^*(t) = \{\mu + \beta\sigma^2(t)\}dt + \sigma(t)dw(t) \tag{8}$$

Here μ is the familiar drift and β is the *risk-premium*. Over an interval $\Delta > 0$ returns are defined as

$$\begin{aligned} y_n &= y^*(\Delta n) - y^*\{(n-1)\Delta\} \\ y_n|\sigma_n^2 &\sim N(\mu\Delta + \beta\sigma_n^2, \sigma_n^2) \\ \sigma_n^2 &= \sigma^{2*}(n\Delta) - \sigma^{2*}\{(n-1)\Delta\} \end{aligned}$$

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Models is subject to microstructure noise. We therefore fit the models to realized volatility over the trading periods Δ.

References I

[1] Torben G Andersen and Tim Bollerslev. Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns. *The journal of Finance*, 52(3):975–1005, 1997.

[2] Torben G Andersen and Tim Bollerslev. Intraday periodicity and volatility persistence in financial markets. *Journal of empirical finance*, 4(2):115–158, 1997.

[3] Ole E Barndorff-Nielsen and Neil Shephard. Estimating quadratic variation using realized variance. *Journal of Applied Econometrics*, 17(5):457–477, 2002.

[4] Peter R Hansen and Asger Lunde. A forecast comparison of volatility models: does anything beat a garch (1, 1)? *Journal of applied econometrics*, 20(7):873–889, 2005.

References II

[5] Daniel B Nelson. Arch models as diffusion approximations. *Journal of econometrics*, 45(1):7–38, 1990.