Some Reflections on Analysis of High-Frequency Data

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Finance is arguably the most empirically oriented of all the social sciences. This is in part due to the deliberate practical orientation and the ready availability of high-quality financial market data. In recent years, the ever lower costs of data recording and storage have driven the phenomenon to the ultimate limit for some markets: We may have access to time-stamped observations on all quotes and transactions, denoted *ultra-high-frequency* data by Engle (in press). These data are of direct interest for market microstructure issues dealing with the price discovery process, the market infrastructure, the strategic behavior of market participants, and the modeling of real-time market dynamics. Moreover, the advent of ultra-high-frequency data poses interesting challenges to empirical work, and it has inspired the developments of new econometric and statistical tools dealing with such complications as a random daily number of time series observations with a random time between arrivals. In addition, the discrete price grid can distort inference concerning the asset price dynamics at the highest frequencies. Finally, any price series now has an associated string of natural conditioning variables—for example, volume, news, time of day, time between transactions or quotes, concurrent prices of similar assets, and so forth. Although current work is making strides on all these fronts, how to deal with the majority of these jointly is still an open question. Interesting surveys of the literature were provided by Hasbrouck (1996), Goodhart and O'Hara (1997), and Engle (in press). As such, the ability of high-frequency data to provide information about market microstructure issues is well recognized. Because the other participants in this discussion undoubtedly will discuss these developments, I shall instead concentrate my brief remarks on a different use of high-frequency data that is perhaps, as of yet, less appreciated—namely, the ability of such data to shed new light on issues concerning the distribution of lower-frequency speculative returns.

Although the large number of intraday observations allows for a string of real-time conditioning variables, our ability to estimate time variation in expected returns is hardly improved by the availability of high-frequency data. First, many of the key theoretical variables—for example, innovations to consumption and investment decisions—remain unavailable at this frequency. Second, the ability to measure expected returns is linked to the time span of the data rather than the frequency of observation, as noted by Merton (1980). In contrast, we achieve potentially huge gains in our ability to monitor variation in return volatility, or the second moments of returns, as argued forcefully by, for example, Nelson (1990, 1992). Recently, empirical work with high-frequency returns has started to substantiate these theoretical promises. It is increasingly evident that access to intraday observations from liquid financial markets such as the foreign-exchange, bond, or equity-index markets afford vastly improved ex post volatility measurement and forecast evaluation. Moreover, one may obtain invaluable new evidence on a variety of hypotheses regarding the long-memory features of volatility that are near impossible to disentangle from daily or lower-frequency data alone. My main objective is to provide illustrative evidence on the usefulness of high-frequency returns regarding these issues. Furthermore, I shall be brief, very informal, and highly subjective and selective. More detailed and formal discussions of the themes discussed here were provided by Andersen and Bollerslev (1998a).

1. DATA

For concreteness and brevity, I focus exclusively on the most active component of the foreign-exchange interbank market—namely, the spot Deutsche mark–U.S. dollar (DM–dollar) market. It serves as a near ideal setting for illustration of the potential benefits that can be garnered from high-frequency data because its high volume and liquidity imply frequent trading and quoting. Moreover, its 24-hour trading cycle within an unregulated over-the-counter market minimizes the loss of information associated with periodic closures, which effectively occur only on weekends and during regional holidays. Finally, the quoted spreads are low and the minimal tick size lower still, implying that problems induced by the existence of a discrete price grid are manageable. This does not mean that modeling intraday DM–dollar returns is straightforward. The time series properties are complicated by features that are unique to the high-frequency setting, such as (a) pronounced intraday periodicity, often termed an intraday seasonal; (b) repercussions from news releases, and in particular regularly scheduled macroeconomic announcements, that induce distinct short-lived volatility dynamics; (c) distinct regional "market opening" effects; (d) a persistent volatility factor that appears largely at the daily frequency; and (e) noise in the series arising from both recording errors and breakdowns in data transmission. In combination, these factors induce low-frequency dependencies and rather extreme outliers.

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These complications were extensively discussed by, for example, Andersen and Bollerslev (1998b). The following illustrations are based on the data from that study; that is, the intraday returns are computed from the DM–dollar log quote-midpoints appearing on the interbank Reuters network over October 1, 1992–September 30, 1993. To avoid dealing with the random time between quotes, I rely on linearly interpolated, continuously compounded 5-minute returns. The exact procedure for computation of the returns was detailed by Müller et al. (1990) and Dacorogna, Müller, Nagler, Olsen, and Pictet (1993). The 5-minute frequency is about the highest at which the properties of the return series are not seriously distorted by the irregular quoting, the discreteness of prices, and the tendency of foreign-exchange dealers to position their quotes with a view toward inventory control; see, for example, Andersen, Bollerslev, and Das (1998) for a thorough discussion of this issue. Furthermore, because trading is decidedly slower over weekends, all returns from Friday 21:00 Greenwich Mean Time (GMT) through Sunday 21:00 GMT were excluded. Thus, our sample covers 288 five-minute intervals over 260 days, for a total of 74,880 five-minute return observations; that is, $r_{t,n}, n = 1, \ldots, N, t = 1, \ldots, 260$, where $N = 288$ is the number of daily intervals. When necessary, these are complemented by a daily DM–dollar series covering March 14, 1979–September 29, 1993, excluding weekends and holidays, for a total of 3,649 observations; that is, $r_t, t = 1, \ldots,$ 3,649.

2. ON MODELING STRATEGY

Once the significance of both the intraday features and low-frequency components of the DM–dollar return series is recognized, the natural and efficient approach is to model all aspects of the time series jointly. Unfortunately, this is a rather challenging proposition and has yet to be implemented in convincing fashion, so most studies invoke some simplifications. A straightforward approach that accommodates the intraday pattern is to apply a time-of-day volatility dummy for each return observation, which is akin to a nonparametric fit without averaging across nearby observations. Unfortunately, this method fails to exploit the generally smooth nature of the pattern that renders it overly sensitive to outliers and generally overparameterized, resulting in inefficient estimates. More elaborate procedures portend to capture the systematic intraday properties in a first step that may then be removed from the series, leaving a residual series to represent the high-frequency returns in which the confounding intraday patterns have been eradicated. Such first-step methods include the "vega-time" transformation of Dacorogna et al. (1993) and the "Fourier flexible form" (FFF) regression approach of Andersen and Bollerslev (1997a, 1998b). For illustration, Figure 1 provides the quite satisfactory FFF fit to the average periodic pattern in absolute returns across the 24-hour trading cycle in the DM–dollar market [winter period only to avoid distortions from the summer time regime in the United States—the corresponding summer period fit is qualitatively similar (see Andersen and Bollerslev 1998b)]. Note that the systematic variation in absolute returns exceeds 250% across the daily cycle. This is much more dramatic than the typical changes in daily volatility estimates extracted from autoregressive conditional heteroscedasticity (ARCH) or stochas-

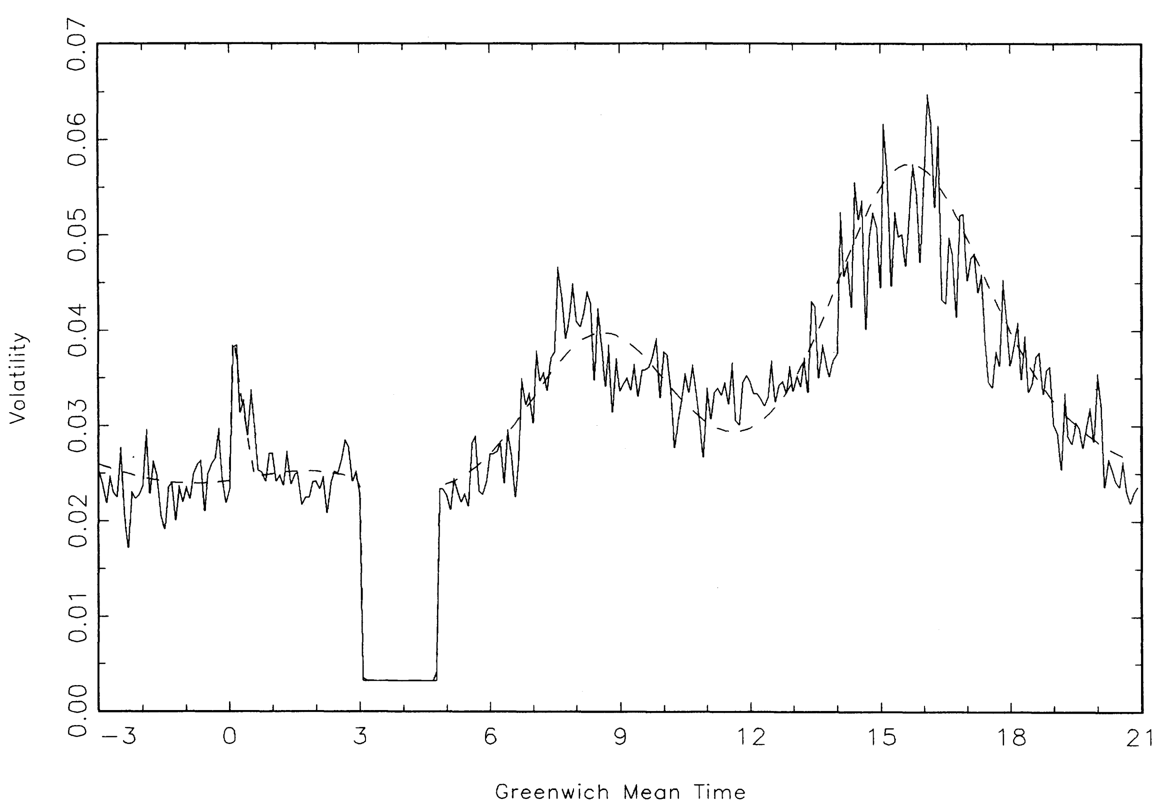

Figure 1. FFF Fit for Average Intraday Absolute Returns: ——, Absolute 5-Minute Return; – – –, Fitted.

tic volatility models, which rarely indicate changes of one-tenth that amount. In other words, the intraday periodic volatility fluctuations dwarf the interdaily factor that is of more economic interest from a portfolio-allocation or risk-measurement perspective. The relevance of this concern is confirmed by the absolute-return correlogram in Figure 2. The long-run features of the correlogram are distorted by the recurrent daily U-shaped correlation structure. These patterns are entirely alien to the standard volatility models, and it is evident that any sensible inference procedure applied directly to the high-frequency return series must accommodate both the strong daily periodicity and the longer-run slow decay in the serial dependence or rely on some less standard techniques that are largely immune to these high-frequency complications. The following sections provide brief illustrations of some useful approaches to the problem.

3. ANALYSIS OF FILTERED HIGH-FREQUENCY RETURNS

This section exemplifies the potential benefits to accommodating the intraday features directly, filtering them out, and treating the residuals as the time series of interest. Because only high-frequency features are removed, the low-frequency behavior should stand out more clearly and be amenable to direct analysis. As an illustration, I consider the FFF-filtered series, extracted from the FFF fit discussed in Section 2. Figure 3 provides the correlogram corresponding to Figure 2. The daily periodicity is largely annihilated, and a very slow hyperbolic rate of decay is evident. This is, of course, inconsistent with the eventual geometric decay imposed by standard volatility models and points instead toward fractional integration in the volatility process. Recall that, if $\rho(|r|, j)$ denotes the jth-order autocorrelation coefficient for the absolute-returns process, the fractionally integrated, or long-memory, volatility process then has the correlogram die out according to $\rho(|r|, j) \sim j^{2d-1}$, for $0 < d < \frac{1}{2}$ and $j \to \infty$. Fitting a hyperbolic decay rate to the correlogram leads to the heuristic estimate of $d = .387$. Although this procedure is by no means rigorous, it turns out to be highly suggestive of the estimates extracted by more formal methods. This represents a novel approach to inference regarding the long-memory features of volatility: Because the degree of fractional integration generally survives aggregation, I may estimate the underlying long-memory features reliably from high-frequency data spanning as little as one calendar year. Andersen and Bollerslev (1997b) noted that the preceding estimate is fully consistent with the parameter estimates obtained from daily samples spanning much longer periods. The problem with the latter is that it is hard to rule out the confounding impact of occasional structural breaks in the volatility process that spuriously induce long-memory type behavior. The ability to estimate these parameters frequently—for example, annually from samples like the previous one—allows me to monitor this aspect closely and directly assess the genuine fractional integration hypothesis versus the structural break hypothesis. It is striking how clearcut the high-frequency evidence in favor of long-memory features in volatility appears to be.

4. SPECTRAL ANALYSIS OF HIGH-FREQUENCY RETURNS

An alternative approach to inference regarding low-

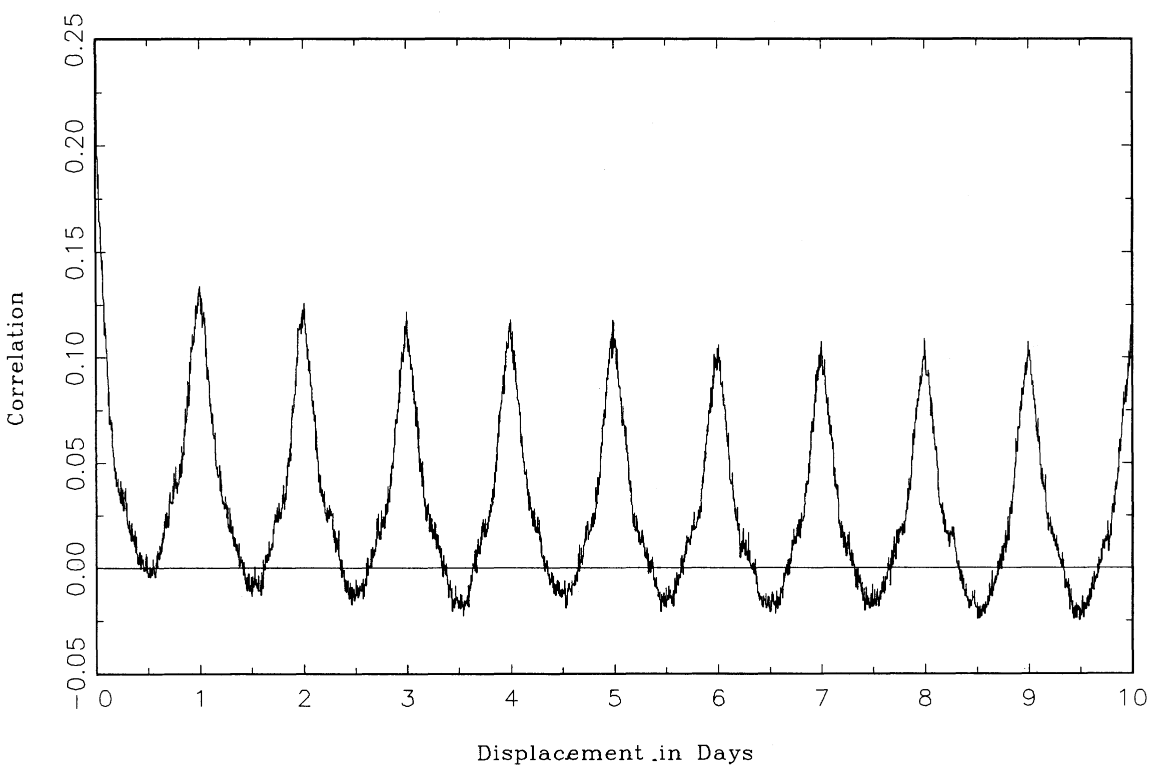

Figure 2. Absolute-Return Correlogram.

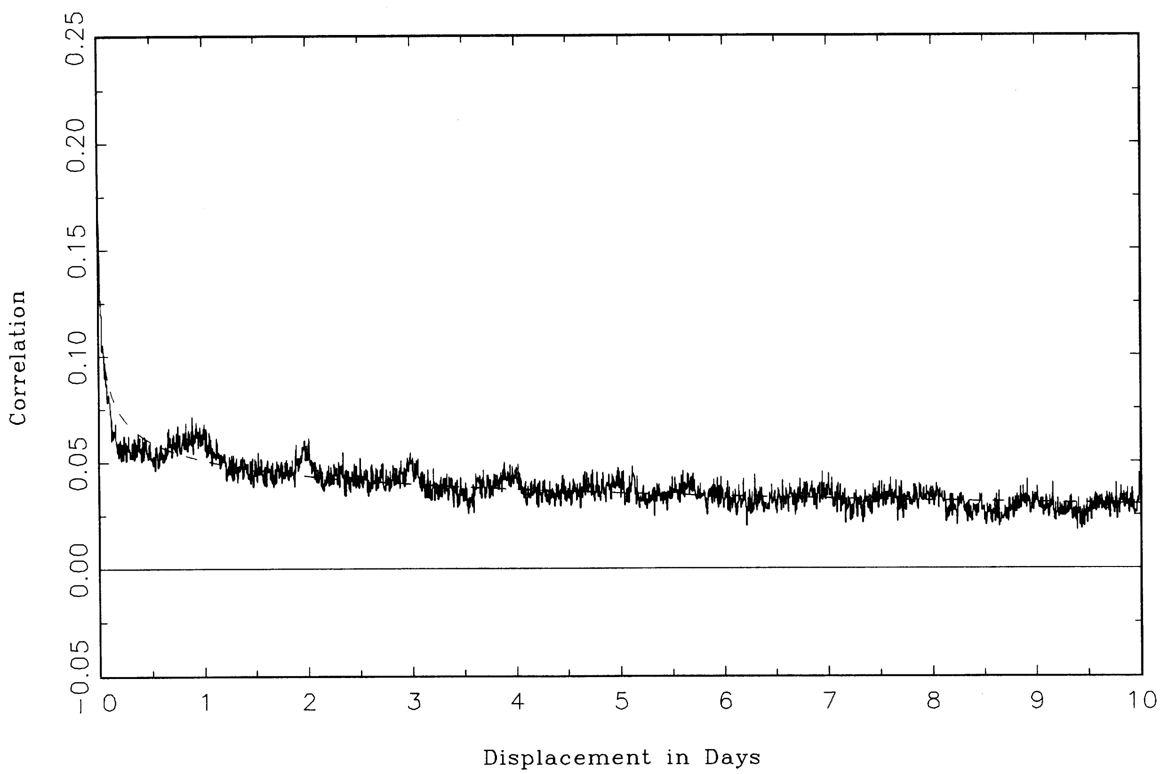

Figure 3. Absolute Filtered-Return Correlogram: ——, Absolute Filtered Returns; - - -, Hyperbolic Decay.

frequency properties from a high-frequency series is to select procedures that are immune to the intraday periodicities. Analysis in the spectral domain allows us to study features directly at specific frequencies of interest. Although this is a somewhat inconvenient approach for understanding the general time series properties of returns, it allows me to formally investigate an important implication of the long-memory hypothesis—namely that the spectrum should be approximately loglinear at the lowest frequencies with a slope of $-2d$. As is evident from Figure 4, the spectrum of the high-frequency DM–dollar returns is approximately loglinear close to the origin. Two formal estimators of d, inspired by this relation, were proposed by Geweke and Porter-Hudak (1983) and Robinson (1994), with the former being further clarified by Hurvich, Deo, and Brodsky (1998). For the 5-minute DM–dollar absolute return series these two estimates are .321 and .385, respectively. Moreover, Andersen and Bollerslev (1997b) showed that these estimates for d are remarkably stable across the different return horizons—that is, 5-minute, ..., 12-hour—confirming yet another important long-memory characteristic. Again, this analysis of the long-memory features is conducted from a sample spanning only one calendar year. Later analysis by Andersen, Bollerslev, Diebold, and Labys (1998) confirmed the robustness of these results over much longer 10-year samples of 5-minute returns.

5. THE NOTION OF INTEGRATED VOLATILITY

The second approach to inference regarding low-frequency volatility features is perhaps both the most straightforward and the most promising, but it does require a bit of motivation. The fact that returns can be observed over almost arbitrarily short intervals raises the question of internally consistent modeling across the return frequencies. The most natural approach is arguably to specify a continuous-time process and then derive the implied properties of the corresponding time-aggregated series. This is particularly appealing in finance, where portfolio-allocation and asset-pricing theories traditionally are cast in a diffusion framework. In the context of the foreign-exchange market and the major currencies, where the expected returns are essentially 0 over short periods and there is no presumption of an asymmetry in the return-volatility relation, a convenient continuous-time specification for the log-asset price, $p(t)$, takes the form

$$dp(t) = \sigma(t) \cdot dW(t), \qquad (1)$$

where $W(t)$ denotes a standard Brownian motion and the volatility process $\sigma(t)$ follows a separate diffusion. Letting the unit interval correspond to one day, the daily return is then given as

$$r_t = p(t) - p(t-1). \qquad (2)$$

This setting leads to a slight generalization of the notion of volatility entertained in the traditional discrete-time models. For example, for this particular diffusion, we have

$$r_t | \{\sigma^2(u); 0 \le u \le t\} \sim \mathrm{N}\left(0, \int_{t-1}^{t} \sigma^2(u)\, du\right). \qquad (3)$$

The term $\int_{t-1}^{t} \sigma^2(u)\, du$ is obviously the key ingredient and is denoted *integrated volatility*. Once this quantity is known, the entire return distribution is characterized. Of course,

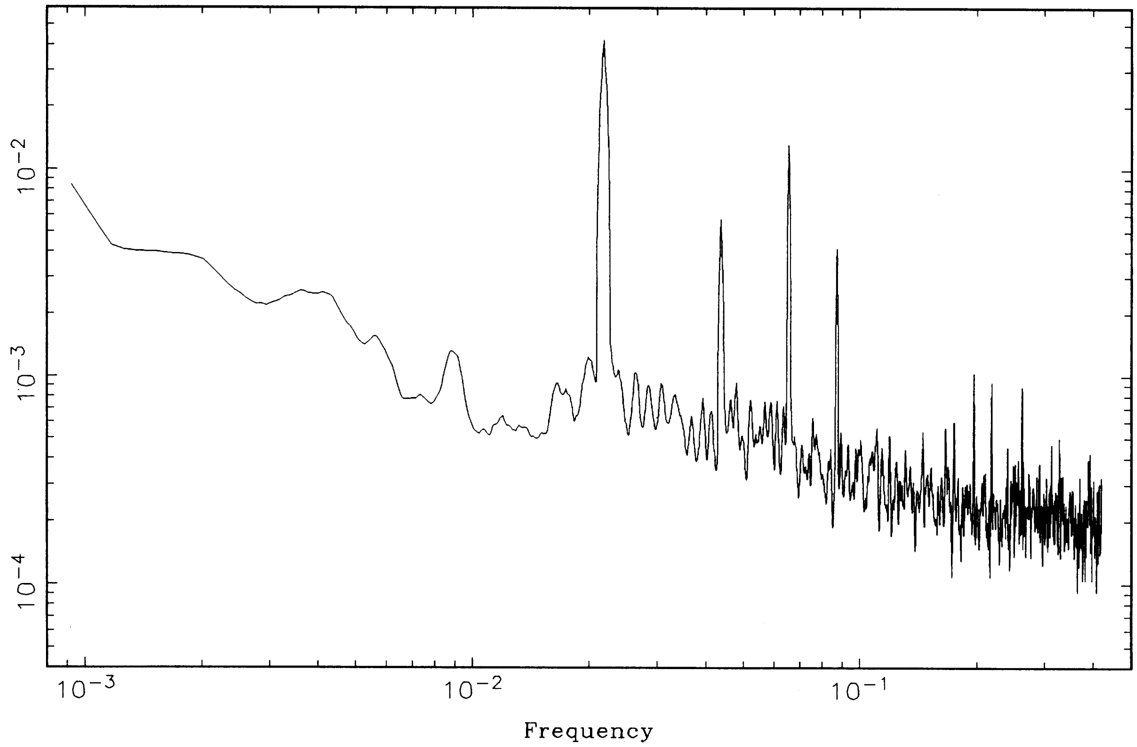

Figure 4. Spectrum for the Absolute Intraday Returns.

for any discrete interval there is genuine randomness in the volatility process, so it is impossible ex ante to predict the ex post integrated volatility perfectly. Instead the distribution in Equation (3) is, a priori, more appropriately viewed as a mixture of normals. Furthermore, it is clearly impossible to directly observe the volatility process from discretely sampled data because $\sigma^2(t)$ is a genuine latent stochastic volatility representation (Andersen 1992, 1994; Ghysels, Harvey, and Renault 1996). There are several implications for longer-horizon volatility forecasting and prediction. First, we observe that $W(t)$ is absent from the volatility definition so that idiosyncratic return variation should be excluded from the notion of volatility. In particular, it is clearly not optimal to use the daily squared return as an indicator of ex post volatility because this inevitably involves a large idiosyncratic return component. To convey the point, I invoke standard notation for daily volatility models; that is,

$$r_t = \sigma_t z_t, \qquad z_t \sim \text{iid } D(0,1), \tag{4}$$

and thus

$$r_t^2 = \sigma_t^2 z_t^2. \tag{5}$$

The only relevant predictable component is σ_t^2, and although r_t^2 provides an unbiased estimator of this quantity, it is a very noisy estimator because the idiosyncratic variation is much larger on a day-to-day basis than the systematic variation in the volatility process. Another way of stating the point is that the daily squared returns will display strong outliers and inliers, even over periods in which the underlying volatility is very smooth. Andersen and Bollerslev (1998c) studied the issue in depth. Second, because ex post volatility is unobserved and random, it is critical to extract efficient estimates of "realized" volatility both for evaluation of volatility forecasting and for portfolio selection and risk management. Third, the volatility diffusion renders discrete-time strong-form ARCH models invalid because it is impossible for a discrete return to serve as a sufficient statistic for the innovation to the volatility diffusion. In other words, the distributional assumptions invoked by standard ARCH models cannot apply simultaneously at two distinct frequencies, implying that the class is not closed to temporal aggregation. Instead, as shown by Drost and Nijman (1993) and Drost and Werker (1996), certain specifications of the system (1)–(2) imply so-called weak-form generalized ARCH (GARCH). Hence, modeling can be made internally consistent but only through newly developed theoretical concepts.

It turns out that the ex post volatility measurement problem for the daily or lower frequencies has an intriguingly simple solution. From the theory of quadratic variation, we have

$$\text{plim}_{n\to\infty}(r_{t,1}^2 + \cdots + r_{t,n}^2) = \int_{t-1}^{t} \sigma^2(u)\, du. \tag{6}$$

The result applies in generality for any jumpless diffusion process. It suggests that we can measure the realized integrated volatility arbitrarily well, as long as we have sufficiently finely sampled intraday returns. In reality, the lack of continuous trading, discrete prices, bid–ask type bounces, and other market microstructure issues prevents us from pushing this intuition to the limit. To the extent that our 288 daily 5-minute returns provide observations that are relatively free of such problems, however, as argued earlier, the cumulative sum of these intraday squared returns should provide a vastly improved volatility measure relative to one based on daily squared returns. In effect, the former approximates the actual sample-path variation of the price process over the day, rather than simply providing a measure based on two endpoints.

In principle, a similar argument applies to any discrete frequency, but the aggregation is best performed over a full multiple of days to avoid confounding the evidence with the systematic variation in volatility within the day. Although there are ways to accommodate the latter feature directly, the approach quickly loses some of its attractive simplicity if results become dependent on first-stage estimates of the intraday features. Here, the approach endogenously accommodates the intraday pattern by aggregating, or smoothing, across the entire day. At the same time, the full intraday variation in high-frequency returns is reflected in the resulting volatility measure.

Finally, note that the integrated volatility concept has arisen as a natural volatility concept in the earlier literature; for example, Hull and White (1987) found the quantity to be the exact measure needed for closing their original stochastic-volatility option-pricing model. The importance of the concept economically and as a key determinant of the return distribution, as shown in Equation (3), has inspired, among others, Meddahi and Renault (1997), Andersen and Bollerslev (1997a, 1998b), Taylor and Xu (1997), Gallant, Hsu, and Tauchen (1998), and Barndorff-Nielsen and Shephard (1998) to study its properties.

6. ANALYSIS OF CUMULATIVE SQUARED RETURNS

The cumulative sum of intraday squared returns should provide more accurate and robust measures of volatility. As such, both volatility forecasting and evaluation should potentially benefit greatly. To illustrate the use of the technique for volatility forecast evaluation, assume that we are interested in a simple measure of the reliability of daily GARCH(1, 1) forecasts. One particular response to this question has been to report the degree of "explained variation" or R^2 from the regression of actual daily squared returns on the corresponding daily volatility forecasts. Presumably, if the volatility forecasts are good, they will predict a large portion of the actual return fluctuations. The relevant studies invariably report R^2 measures of below 10%, however, with many being much lower. Does this imply that the GARCH volatility forecasts are poor? This does not necessarily follow, as can be grasped intuitively from inspecting Figure 5. The absolute returns are evidently extremely noisy with a large number of both outliers and inliers. In contrast, the GARCH forecasts are very smooth—a property shared by virtually all popular volatility models. The obvious explanation is that the idiosyncratic return component is large relative to the systematic volatility component. In other words, perhaps it is the realized daily squared returns that are poor indicators of "realized" volatility. One direct check of this hypothesis is to compare the identical GARCH forecasts to the cumulative squared 5-minute returns for the identical one-year sample period. Figure 6 shows the vastly improved coherence between the volatility forecasts and the ex post volatility measures. Concretely, the R^2 measures associated with the squared return regression described previously are .047 and .479 for the two figures, respectively. Hence, the identical volatility forecasts are assessed dramatically differently by the

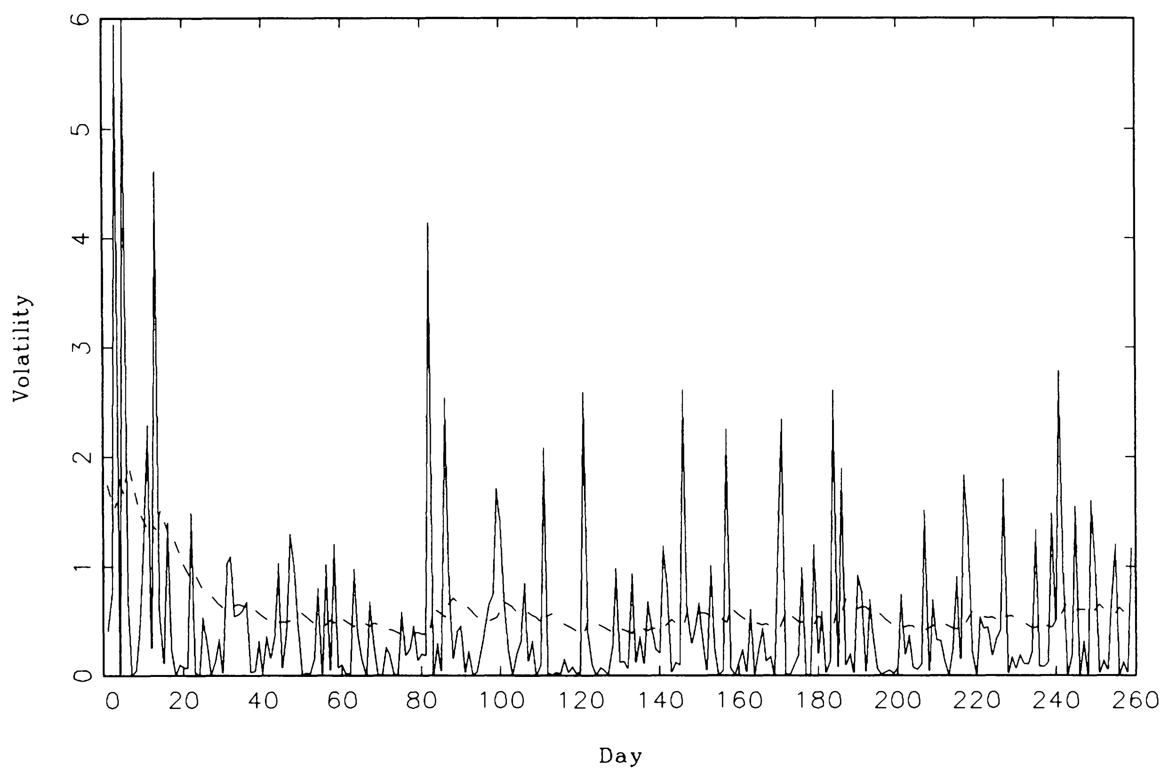

Figure 5. Daily Squared Returns Versus GARCH Forecasts: ——, Daily Squared Returns; - - -, GARCH Forecasts.

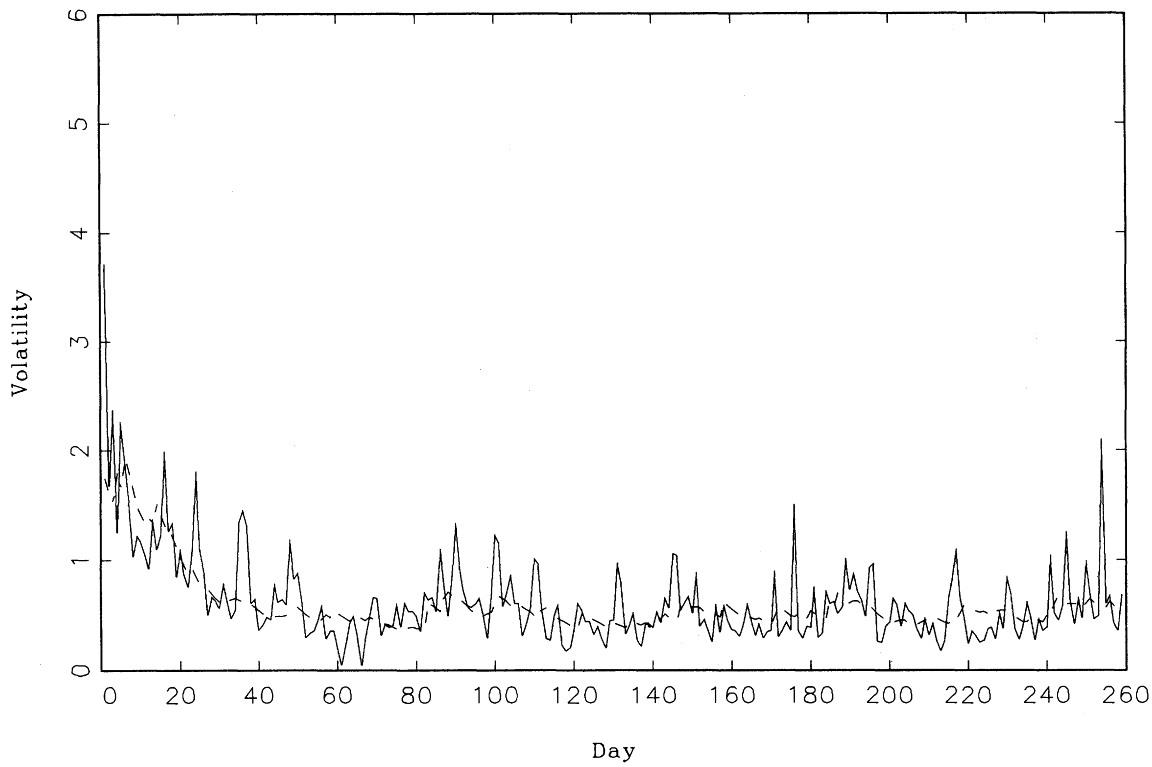

Figure 6. Cumulative Intraday Squared Returns Versus GARCH Forecasts: ——, Cumulative 5-Minute Squared Returns; - - -, GARCH(1, 1) Volatility Forecast.

two measures of realized return variability over the identical horizon. One may show theoretically that these findings are no coincidence. Even if the GARCH(1, 1) model constitutes the true data-generating process, the R^2 associated with the preceding regression will often be substantially below 10%. In contrast, the corresponding figure for the high-frequency-based volatility measure is typically around 50%. Thus, the GARCH model actually performs admirably over this one-year sample period, explaining about as much of the variation in volatility as one could ever hope for. Of course, the R^2 measure is quite arbitrary and in several respects not particularly useful. Andersen, Bollerslev, and Lange (in press) expanded the analysis to a string of alternative loss functions and considered volatility forecast horizons up to one month ahead. The results are all qualitatively similar. Standard volatility models provide very decent forecasts. On the other hand, it is also shown that the incorporation of high-frequency data in the volatility forecasting stage can improve performance quite significantly. Thus, both better forecasts and better assessment of forecasts are readily available when high-quality intraday data are on hand. Finally, it is documented that there is a substantial amount of predictable volatility even at the monthly horizon. This type of conclusion is very hard to attain with daily or monthly data because the strong idiosyncratic component tends to overwhelm the relatively less significant systematic features at this horizon.

7. SOME TOPICS FOR FUTURE WORK

Research in this area is progressing rapidly, even if it remains a specialized object of study. Several additional issues can readily be addressed. It is straightforward to allow for jumps in the price path so that the restrictive nature of the framework in Equations (1)–(2) may be relaxed. Another set of questions relates to multivariate extensions of the integrated volatility concept because applications to the risk-management area require such features. Work in these directions was initiated by Andersen, Bollerslev, Diebold, and Labys (1998). Furthermore, the exploration of the impact of the predictable volatility movements for a variety of practical financial and economic problems—for example, portfolio choice and derivatives pricing—remain a high priority, as pursued by, for example, Fleming, Kirby, and Ostdiek (1998). In addition, it should also be possible to perform meaningful and focused "horseraces" between alternative models, given the vastly improved measurements of the underlying realized volatility, although careful selection of the economic criteria for evaluation, or loss functions, is called for. Finally, there are some theoretical challenges in devising the optimal volatility measure when the returns and volatility process are correlated, as the pronounced "leverage effect" seems to imply for U.S. equity indexes. Moreover, if volatility risk is priced, it is no longer trivial to map the volatility measure into a unique measure of relevance for, say, option pricing. Rather than viewing these observations as stumbling blocks, I expect that the increasing availability of high-frequency data will allow us to explore such features in much more depth than hitherto.

In conclusion, I will simply reiterate that the ability of intraday returns to provide near irrefutable evidence for long-memory features in the volatility process and to confirm strong theoretical predictions regarding standard volatility forecasts at a variety of horizons has opened a new era in volatility research. I would predict that any serious investigation of volatility features for markets with limited market microstructure distortions in a few years will involve the constructive use of intraday observations. We have only just started to contemplate how this may be done optimally, but the payoff has already been very high. The future looks exciting.

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