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Tracking Volatility

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Abstract

This paper is concerned with nonlinear filtering of the volatility coefficient in a Black-Scholes type model that allows stochastic volatility. More specifically we assume that the asset price process $S = (S_t)_{t\geq 0}$ is given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t dB_t,$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and v_t is the (stochastic) volatility process. Moreover, it is assumed that $v_t = v(\theta_t)$ where v is a nonnegative function and $\theta = (\theta_t)_{t\geq 0}$ is a homogeneous Markov jump process, taking values in the finite alphabet $\{a_1, \ldots, a_M\}$, with the intensity matrix $\Lambda = \|\lambda_{ij}\|$ and the initial distribution $p_q = \underset{\sim}{P}(\theta_0 = a_q)$, $q = 1, \ldots, M$.

The random process θ is unobservable. Following to Frey and Runggaldier [4], we assume also that the asset price S_t is measured only at random times $0 < \tau_1 < \tau_2 < \ldots,$. This assumption is designed to reflect the discrete nature of high frequency financial data (e.g. tick-by-tick stock prices). The random time moments τ_k "represent instances at which a large trade occurs or at which a market maker updates his quotes in reaction to new definition."

In the above setting the problem of volatility estimation is reduced naturally to a special nonlinear filtering problem. We remark that while quite natural, the latter problem does not fit into the "standard" framework and requires new technical tools.

In this paper, we derive a mean-square optimal recursive Bayesian filter for θ_t based on the observations of $S_{\tau_1}, S_{\tau_2}, \ldots$ for all $\tau_k \leq t$. In addition we derive Duncan-Mortensen-Zakai and Wonham-Kushner type equations for posterior distributions of θ_t and prove uniqueness of their solutions.

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1 Introduction

In the classical Black-Scholes model for financial markets, the stock price S_t is modeled as a Geometric Brownian motion, namely with diffusion coefficient equal to σS_t, where "volatility" σ is assumed to be constant. The volatility parameter is the most important one when it comes to option pricing, so, naturally, many researchers generalized the constant volatility model to so-called stochastic volatility models, where σ_t is itself random and time dependent. There are two basic classes of models - complete and incomplete. In complete models the volatility is assumed to be a functional of the stock price, while in incomplete models it is driven by some other source of noise, possibly correlated with the original Brownian motion. In this paper we study a particular incomplete model in which the volatility process is independent of the driving Brownian motion process. This has the economic interpretation of the volatility being influenced by market, political, financial and other factors which are independent of the systematic risk (the Brownian motion process) associated to the particular stock price under study. It is also close in spirit to the way traders think about volatility - as a parameter that changes with time, and whose future value in a given period of interest has to be estimated/predicted. They need the estimate of the volatility to decide how they will trade in financial markets, especially derivatives markets. In fact, the notion of volatility is so important to traders that they even quote option prices in volatility units rather than in dollars (or some other currency). It is also important for investment banks who need to know the model for the future volatility in order to be able to price custom-made financial products, whose payoff depends on the future path of the underlying stock price. Very recently new contracts have been developed, which directly trade the volatility itself (volatility swaps, for example). We plan to address the issue of pricing options within the framework of our model in future research.

Estimation of volatility from observed stock prices is not a trivial task in either complete or incomplete models, in part because the prices are observed at discrete, possibly random time points. Since volatility itself is not observed, it is natural to apply filtering methods to estimate the volatility process from the historical stock price observations. Nevertheless, this has only recently been investigated in continuous-time models, in particular by Frey and Runggaldier [4]. Most of the earner research was concentrated on either the time series approach (ARCH-GARCH models) or on calibration methods. The latter do not use historical data to estimate the volatility, but try instead to find the volatility process that matches best (in some appropriately defined sense) the observed present prices of frequently traded option contracts. In other words, methods have been developed to solve the "inverse problem" of finding the volatility process such that the corresponding theoretical prices of options become close, in some sense, to the observed market prices of options. In this paper we adopt the approach of filtering the volatility from the observed historical stock prices. In the future work we plan to combine the two approaches, thereby estimating the model by taking into account both the historical behavior of volatility (as indicated through past stock prices), as well as the market opinion about its future behavior (as indicated through present option prices).

More precisely, we are concerned with nonlinear filtering of the volatility coefficient in a Black-Scholes type model in which the asset price process $S = (S_t)_{t\geq 0}$ is given by

$$dS_t = rS_t dt + \sqrt{v_t} S_t dB_t, \tag{1}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and v_t is the (stochastic) volatility process (see [3]). Moreover, it is assumed that $v_t = v(\theta_t)$, $v(x)$ is bounded positive (known) function, and $\theta = (\theta_t)_{t\geq 0}$ is a homogeneous Markov jump process taking values in the finite alphabet $\{a_1, \dots, a_M\}$ with the intensity matrix $\Lambda = ||\lambda_{ij}(t)||$ and the initial distribution $p_q = \mathrm{P}(\theta_0 = a_q)$, $q = 1, \dots, M$.

The random process θ is unobservable. Following Frey and Runggaldier [4], we assume also that the asset price S_t is observed only at random times $0 := \tau_0 < \tau_1 < \tau_2 < \dots,$. This assumption is designed to reflect the discrete nature of high frequency financial data (e.g. tick-by-tick stock prices). The random time moments τ_k "represent instances at which a large trade occurs or at which a market maker updates his quotes in reaction to new information" (see [3]).

In the above setting the problem of volatility estimation is reduced naturally to a special nonlinear filtering problem. We remark that while quite natural, the latter problem does not fit into the "standard" framework and requires new technical tools.

Frey and Runggaldier [4] derived a Kallianpur-Striebel type formula for the optimal mean-square filter for θ_t based on the observations of $S_{\tau_1}, S_{\tau_2}, \dots$ for all $\tau_k \leq t$ and investigated Markov Chain approximations for this formula. In this paper, we extend their result in that we derive exact Duncan-Mortensen-Zakai and Wonham-Kushner type filters for θ_t.

2 Preliminaries

In this Section we introduce additional notation and further specialize the mathematical model.

To begin with let us consider the observation process. As in Frey and Runggaldier [4], we assume that the price process is observed only at random time moments $0 < \tau_1 < \tau_2 < \dots$. More specifically, observation process is the discrete-time stochastic process $(\tau_k, S_{\tau_k})_{k\geq 1}$. By technical reasons it is more convenient to deal with the Gaussian process $U_t = \log S_t$ ruther then the original price process S_t. Obviously,

$$U_t - U_{\tau_{k-1}} = \int_{\tau_{k-1}}^{t} \left(r - \frac{1}{2} v(\theta_s)\right) ds + \int_{\tau_{k-1}}^{t} v^{1/2}(\theta_s) dB_s.$$

Write $U_{k-1}(t) := U_t - U_{\tau_{k-1}}$ and set

$$Y_t = I_{\{\tau_{k-1} < t \leq \tau_k\}} U_{k-1}(t)$$

where $\tau_0 = 0$. Of course, the sequences $(\tau_k, S_{\tau_k})_{k\geq 1}$and $((\tau_k, Y_{\tau_k})_{k\geq 1})$ provide the same information regarding the volatility process θ_t. Thus, without loss of generality, we can assume that the observation is given by the process

$$G_k = \{(\tau_k, Y_{\tau_k}); (\tau_{k-1}, Y_{\tau_{k-1}}); \dots; (\tau_1, Y_{\tau_1})\}.$$

The counting process

$$N_t = \sum_{k=1}^{\infty} I(\tau_k \geq t)$$

and the random integer-valued measure

$$\mu([0,t] \times \Gamma) = \int_0^t I(Y_s \in \Gamma) dN_s$$

will play an important role in the future.

We assume that the triple (B_t, θ_t, N_t) are defined on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ subject to standard conditions. Let $(\mathcal{G}_t)$ be the right continuous complete filtration generated by μ, that is by the processes $\int_0^t \int_{\mathbb{R}} f(y) \mu(ds, dy)$ with bounded (measurable) functions f. Obviously, $(\mathcal{G}_t) \subset (\mathcal{F}_t)$.

The following assumptions will be in force everywhere below:

A The Brownian motion (B_t) is independent of (θ_t, N_t).

B The counting process (N_t) is a double Poisson (Cox) process with the stochastic intensity $n(\theta_t)$, where n is bounded and strictly positive function. The jumps of the processes (θ_t) and (N_t) are disjoint.

The first part of **B** means that $N_t - \int_0^t n(\theta_s)ds$ is a martingale with respect to $(\mathcal{F}_t)$ or ,equivalently, that $\widehat{N}_t = \int_0^t n(\theta_s)ds$ is an $(\mathcal{F}_t)$ −adapted compensator of N_t

Note also, that assumption **A** yields that the conditional distribution

$$P\left(U_{k-1}(t) \le y | \mathcal{G}_{t-}; \theta_{[0,\tau_k]}\right)$$

is Gaussian with the density

$$p_{(k-1),\theta}(y) = \frac{1}{\sqrt{2\pi\sigma^2_{k-1}(\theta,t)}} e^{-\frac{(y-m_{k-1}(\theta,t))^2}{2\sigma^2_{k-1}(\theta,t)}}$$

where

$$m_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} \left(r - \frac{1}{2}v(\theta_s)\right)ds$$
$$\sigma^2_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} v(\theta_s)ds.$$

It is also convinient to introduce the vector process $\mathbf{I}_t$ with the components

$$I(\theta_t = a_1), I(\theta_t = a_2), \ldots, I(\theta_t = a_M).$$

Obviously, $\mathbf{I}_t$ is just another representation of θ_t.

Set

$$\mathfrak{v} = v(a_1), v(a_2), \ldots, v(a_M)$$
$$\mathfrak{n} = n(a_1), n(a_2), \ldots, n(a_M)$$

It is readily checked that $v(\theta_t) = \mathfrak{v}\mathbf{I}_t$, $n(\theta_t) = \mathfrak{n}\mathbf{I}_t$, and the process $\mathbf{I}_t$ is a semimartingale (with respect to $(\mathcal{F}_t)$) given by the Itô equation

$$\mathbf{I}_t = \mathbf{I}_0 + \int_0^t \Lambda \mathbf{I}_s ds + \mathfrak{I}_t \tag{2}$$

where $(\mathfrak{I}_t)$ is a purely discontinuous vector martingale. The paths of every component of $(\mathfrak{I}_t)$ is right continuous with unit size jumps, and admits left-hand limits (see Lemma 9.2 in [?]).

Note that the disjointness of jumps of (θ_t) and (N_t) is equivalent to the disjointness of jumps of $(\mathfrak{I}_t)$ and (N_t).

Write

$$\varrho_{k-1}(t,y) = \frac{E\left(\mathsf{diag}(\mathbf{I}_t)\mathfrak{n}^* p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\right)}{E\left(\mathfrak{n}\mathbf{I}_t p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\right)}$$

and

$$\varrho(t,y) = \sum_{k=1}^{\infty} I_{\{\tau_{k-1} < t \le \tau_k\}} \varrho_{k-1}(t,y)$$

where $p_{(k-1),\theta}(y)$ is the Gaussian density introduced above, * is the transposition symbol, and $\mathsf{diag}(\mathbf{I}_t)$ is the scalar matrix with the diagonal $\mathbf{I}_t$.

Note that the integrand in $\varrho_k(t,y)$ is a functional of $(\mathbf{I}_t)$. Indeed, the parameters of the density $p_{(k-1),\theta}(y)$ can be rewritten as follows:

$$m_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} \left(r - \frac{1}{2}\mathfrak{v}\mathbf{I}_s\right)ds$$
$$\sigma^2_{k-1}(\theta,t) = \int_{\tau_{k-1}}^{t\wedge\tau_k} \mathfrak{v}\mathbf{I}_s ds.$$

3 Filters

3.1 The Wonham-Kushner filter

Set $\pi_t(\mathbf{I}) = E(\mathbf{I}_t|\mathcal{G}_t)$. The process $\pi_t(\mathbf{I})$ describes the dynamics of posterior conditional distributions

$$P(\theta_t = a_1|\mathcal{G}_t), P(\theta_t = a_2|\mathcal{G}_t), \ldots, P(\theta_t = a_M|\mathcal{G}_t)$$

Of course, this posterior distribution fully defines the mean-square optimal nonlinear filter.

The main result of this paper is formulated in the following Theorem.

1. The posterior distribution $\pi_t(\mathbf{I})$ verifies the following equation

$$\begin{aligned} \pi_t(\mathbf{I}) &= \pi_0(\mathbf{I}) + \textstyle\int_0^t \Lambda\pi_s(\mathbf{I})ds \\ &+ \textstyle\sum_{k:\tau_k \le t} \left(\varrho_{k-1}(\tau_k, Y_{\tau_k}) - \pi_{\tau_k-}(\mathbf{I})\right) \\ &- \textstyle\int_0^t \left(\mathsf{diag}(\pi_s(\mathbf{I})) - \pi_s(\mathbf{I})\pi_s^*(\mathbf{I})\right)\mathfrak{n}^* ds. \end{aligned} \tag{3}$$

Equations for posterior distributions of jump processes is often referred to as Wonham-Kushner equations and we will follow this tradition. It is redily checked that Wonham-Kushner equation can be rewritten as follows:

$$\begin{aligned} \pi_t(\mathbf{I}) &= \pi_0(\mathbf{I}) + \int_0^t \Lambda\pi_s(\mathbf{I})ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \left(\varrho(s,y) - \pi_{s-}(\mathbf{I})\right)(\mu - \widetilde{\nu})(ds,dy)) \end{aligned}$$

where

$$\widetilde{\nu}(dt,dy) = \sum_{k=1}^{\infty} I_{\{\tau_{k-1} < t \le \tau_k\}} E\left(\mathfrak{n}\mathbf{I}p_{(k-1),\theta}(y)|\mathcal{G}_t\right)dtdy$$

is the compensator of μ with respect to the "observation". Moreover

$$\int_0^t \int_{\mathbb{R}} (\mu - \widetilde{\nu})(ds,dy)$$

is a martingale and "$\mu - \widetilde{\nu}$" plays the role of the innovation process. The letter form of the Kushner-Wonham equation is quite similar to its well known version developed for diffusion type observation process (see [8], [12]).

The proof of the Theorem is based on a general filtering result for semimartingales - Theorem 4.10.1 in Liptser and Shiryaev [9].

Continuous time filtering equation with discontinuous observation were addressesed by many authors (see e.g. Grigelionis [5], [6]; Elliott, Aggoun and Moore [2]; Krylov and Zatezalo [7] etc. Unfortunately, these works do not cover our setting.

3.2 Duncan-Mortensen-Zakai equation

If $n(x) \equiv 1$, i.e. N_t is the Poisson process with the unit intensity, the jump moments $\tau_1, \tau_2, \ldots$, do not carry any "information" regarding the volatility and the observation process $\mathcal{G}$ can be reduced to the sequence $Y_{\tau_1}, Y_{\tau_2}, \ldots$ It is readily checked that in this case the filtering equation takes a much simpler form:

$$\pi_t(\mathbf{I}) = \pi_0(\mathbf{I}) + \int_0^t \Lambda\pi_s(\mathbf{I})ds + \sum_{k:\tau_k\le t} \varrho_{k-1}(\tau_k, Y_{\tau_k})$$

where

$$\varrho_{k-1}(t,y) = \frac{E\big(\mathbf{I}_t p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}{E\big(p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}$$

This fact inspires the idea to find a new measure P' that is absolutely continuous with respect to the original measure P and such that (N_t, P') is the Poisson process with the unit intensity. Such change of the measure is possible, if the filtering problem is treated on a finite time interval $[0,T]$. Specifically, the new measure is defined by

$$dP' = \mathfrak{z}_T dP_T$$

where P_T is the restriction of P to $\mathcal{F}_T$, and $t \le T$,

$$\mathfrak{z}_t = \exp\Big(\int_0^t -\log n(\theta_{s-})dN_s - \int_0^t \frac{1-n(\theta_s)}{n(\theta_s)}d\widehat{N}s\Big) \tag{4}$$

(see e.g. Section 19.4 in Ch. 19 [8]).

In the future, an expectation with respect to the measure P' will be denoted E'. Write $\pi'_t(\mathfrak{z}\mathbf{I}) = E'(\mathfrak{z}_t^{-1}\mathbf{I}_t|\mathbf{G}_t)$, and $\pi'_t(\mathfrak{z}) = E'(\mathfrak{z}_t^{-1}|\mathbf{G}_t))$. By the Kallianpur-Striebel formula, we have

$$\pi_t(\mathbf{I}_t) = \frac{\pi'_t(\mathfrak{z}^{-1}\mathbf{I})}{\pi'_t(\mathfrak{z}^{-1})}.$$

The equation for $\pi'_t(\mathfrak{z}^{-1}\mathbf{I})$ in the case of diffusion type observation (usually referred to as the Duncan-Mortensen-Zakai equation) is well known (see e.g. [12]). To derive an analog of this equation in the present setting, we start with the Kushner type equations for the time evolution of $\pi_t(\mathfrak{z}^{-1}\mathbf{I})$ and $\pi_t(\mathfrak{z}^{-1})$. To this end, let us discuss briefly some important features of our model with respect to the new measure P'. Since (θ_t) and (N_t) have disjoint jumps, the distribution of the process θ remains the same under the new measure. Similarly, since the predictable covariance of (B_t) and $(N_t - \widehat{N}_t)$ is zero, it ie readily checked that the process (B_t) is a standard Brownian motion independent of (θ_t) and (N_t). Further, since (N_t) is the Poisson process with the unit intensity, we have

$$\varrho'_{k-1}(t,y) = \frac{E'\big(\mathbf{I}_t p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}{E'\big(p_{(k-1),\theta}(y)|\mathcal{G}_{t-}\big)}$$

and $\varrho'(t,y) = \sum_{k=1}^{\infty} \varrho'_{k-1}(t,y)$. The compensator $\widetilde{\nu}$ is is given now by the formula

$$\widetilde{\nu}'(dt,dy) = \sum_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}} E'\big(p_{(k-1),\theta}(y)|\mathcal{G}_t\big)dtdy.$$

Making use of Theorem 4.10.1 from [9], one can show that the following representation for $\pi'_t(\mathfrak{z}^{-1})$ holds true:

$$\pi'_t(\mathfrak{z}^{-1}) = 1 + \int_0^t \int_{\mathbb{R}} \varrho'(s,y)(\mu - \widetilde{\nu}')(ds,dy)$$

Write

$$w(s,y) = \frac{E(\mathbf{n}\mathbf{I}p_{(k-1),\theta}(y)|G_{s-})}{E(p_{(k-1),\theta}(y)|G_{s-})}. \tag{5}$$

It is readily checked that

$$\frac{\varrho'(s,y)}{\pi'_{s-}(\mathfrak{z}^{-1})} = w(s,y) - 1 \tag{6}$$

Theorem 4.10.1 in [9], (5) , and (6) yield the following linear equation for $\pi'_t(\mathfrak{z}^{-1})$:

$$\pi'_t(\mathfrak{z}^{-1}) = 1 + \int_0^t \int_{\mathbb{R}} \pi'_{s-}(\mathfrak{z}^{-1})\big(w(s,y)-1\big)(\mu - \widetilde{\nu}')(ds,dy).$$

Obviously the Dolean-Dade exponent is the unique solution of this equation. Thus we have

$$\begin{aligned}\pi'_t(\mathfrak{z}^{-1}) \;&=\; \exp\Big(\int_0^t \int_{\mathbb{R}} \log w(s,y)\mu(ds,dy) - \\ &\quad - \int_0^t \int_{\mathbb{R}} (w(s,y)-1)\widetilde{(\nu)}(ds,dy)\Big)\end{aligned}$$

To derive the Duncan-Mortensen-Zakai filter for $\pi'_t(\mathfrak{z}^{-1}\mathbf{I})$, we begin with deriving the semimartingale decomposition for $\mathfrak{z}_t^{-1}\mathbf{I}_t$. From (4) by the Itô formula, we have

$$\mathfrak{z}_t^{-1} = 1 + \int_0^t \mathfrak{z}_{s-}^{-1}(n(\theta_{s-}) - 1)(dN_s - ds).$$

Now, taking into account (2) and applying Ito's formula to $\mathfrak{z}_t\mathbf{I}_t$ we get

$$\begin{aligned}\mathfrak{z}_t\mathbf{I}_t &= \mathbf{I}_0 + \int_0^t \Lambda\mathfrak{z}_s\mathbf{I}_s ds + \int_0^t \mathfrak{z}_{s-}\mathbf{I}_{s-}d\mathfrak{I}_s \\ &\quad + \int_0^t \mathfrak{z}_{s-}\mathbf{I}_{s-}(n(\theta_{s-})-1)(dN_s - ds).\end{aligned}$$

Now, applying Theorem 4.10.1 in [9], we find

$$\begin{aligned}&\pi'_t(\mathfrak{z}^{-1}\mathbf{I}) \\ =\ & \pi'_0(\mathbf{I}) + \int_0^t \Lambda\pi'_s(\mathfrak{z}^{-1}\mathbf{I})ds \\ +\ & \int_0^t\int_{\mathbb{R}} \big(\varphi'(s,y) - \pi'_{s-}(\mathfrak{z}^{-1}\mathbf{I})\big)(\mu - \widetilde{\nu}')(ds,dy),\end{aligned}$$

where $\varphi'(t,y) = \sum\limits_{k=1}^{\infty} I_{\{\tau_{k-1}<t\le\tau_k\}}\varphi'_{k-1}(t,y)$ and

$$\varphi'_{k-1}(t,y) = \frac{E'\big(\mathsf{diag}(\mathfrak{z}_t^{-1}\mathbf{I}_t)\mathsf{n}^* p_{(k-1),\theta}(y)|\mathcal{G}'_{t-}\big)}{E'\big(p_{(k-1),\theta}(y)|\mathcal{G}'_{t-}\big)}.$$

The above results can be reformulated as follows:

Theorem. *The Duncan-Mortensen-Zakai filter is given by the equations*

$$\begin{aligned}\pi'_t(\mathfrak{z}^{-1}) &= 1 + \sum_{k:\tau_k\le t} \pi'_{\tau_k-}(\mathfrak{z}^{-1})\big(w(\tau_k, Y_{\tau_k}) - 1\big) \\ &\quad - \int_0^t \pi'_s(\mathfrak{z}^{-1})\big(\pi_s(\mathsf{n}\mathbf{I}) - 1\big)ds\end{aligned}$$

and

$$\begin{aligned}&\pi'_t(\mathfrak{z}^{-1}\mathbf{I}) \\ =\ & \pi'_0(\mathbf{I}) + \int_0^t \Lambda\pi'_s(\mathfrak{z}^{-1}\mathbf{I})ds \\ & + \sum_{k:\tau_k\le t} \big(\varphi'(\tau_k, Y_{\tau_k}) - \pi'_{\tau_k-}(\mathfrak{z}^{-1}\mathbf{I})\big) \\ & - \int_0^t \Big(\mathsf{diag}(\pi'_s(\mathfrak{z}^{-1}\mathbf{I})\mathsf{n}^* - \pi'_s(\mathfrak{z}^{-1}\mathbf{I})\Big)ds.\end{aligned}$$

The proof follows from the following relations:

$$\begin{aligned}&\int_{\mathbf{R}} \widetilde{\nu}'(ds,dy) = ds \\ &\int_{\mathbf{R}} w(s,y)\widetilde{\nu}'(ds,dy) = \pi_s(\mathbf{nI})ds \\ &\int_{\mathbf{R}} \varphi'(s,y)\widetilde{\nu}'(ds,dy) = \mathsf{diag}\big(\pi'_s(\mathfrak{z}^{-1})\big)ds.\end{aligned}$$

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