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High accuracy cubic spline approximation for two dimensional quasi-linear elliptic boundary value problems

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ABSTRACT

We report a new 9 point compact discretization of order two in y- and order four in x-directions, based on cubic spline approximation, for the solution of two dimensional quasi-linear elliptic partial differential equations. We describe the complete derivation procedure of the method in details and also discuss how our discretization is able to handle Poisson's equation in polar coordinates. The convergence analysis of the proposed cubic spline approximation for the nonlinear elliptic equation is discussed in details and we have shown under appropriate conditions the proposed method converges. Some physical examples and their numerical results are provided to justify the advantages of the proposed method.

1. Introduction

When a physical system depends on more than one variable a general description of its behavior often leads to partial differential equation. These equations arise in such diverse subjects as meteorology, electromagnetic theory, heat transfer, nuclear physics and elasticity to name just a few. Often, a system of differential equations of various order with different boundary conditions occur in the study of obstacle, unilateral, moving and free boundary value problems, problems of deflection of beams and in a number of other scientific applications. Most of these equations arising in applications are not solvable analytically and one is obliged to devise techniques for the determination of approximate solutions.

In the present paper, we aim to discuss the application of cubic spline functions to solve these boundary value problems. Use of cubic spline functions in the solution of nonlinear boundary value problems has been a challenging task for academic researchers. For the two dimensional linear elliptic equations, a number of constant mesh fourth order finite difference schemes have been designed by [1–5]. These linear systems have good numerical stability and provide high accuracy approximations. Later, using 9 point fourth order discretization for the solution of two dimensional non linear boundary value problems have been developed by Jain et al. [6,7], Mohanty [8], Mohanty and Singh [9] and Mohanty et al. [10]. Theory of splines and their applications to two point linear boundary value problems have been studied in [11–17]. Jain and Aziz [18] have derived fourth order cubic spline method for solving nonlinear two point boundary value problems with significant first derivative terms. Al-Said [19,20] has used cubic splines in the numerical solution of the second order boundary value problems. In the recent past, Mohanty et al. [21,22] have discussed fourth order accurate cubic spline Alternate Group Explicit

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method for the solution of two point boundary value problems. Khan et al. [23,24] have analyzed the use of parametric cubic spline for the solution of linear two point boundary value problems. Later, Rashidinia et al. [25,26] have derived the cubic spline method for the nonlinear singular two point boundary value problems. Houstis et al. [27] have first used point iterative cubic spline collocation method for the solution of linear elliptic equations. Later, Hadjidimos et al. [28] have extended their technique and used line iterative cubic spline collocation method for the solution of elliptic partial differential equation. Recently, Mohanty et al. [29–32] have derived high accuracy finite difference methods for the numerical solution of non-linear elliptic, hyperbolic and parabolic partial differential equations. To the author's knowledge, no high order cubic spline method for the solution of two dimensional quasi linear elliptic partial differential equations is known in the literature so far.

We now develop a new high accuracy 9-point (see Fig. 1) cubic spline discretization for the solution of two dimensional quasi-linear elliptic partial differential equation of the form:

$$A(x,y,u)\frac{\partial^2 u}{\partial x^2} + B(x,y,u)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right), \quad (x,y) \in \Omega \tag{1}$$

defined in the domain $\Omega = \{(x,y)\colon 0 < x,y < 1\}$ with boundary $\partial\Omega$, where $A(x,y,u) > 0$ and $B(x,y,u) > 0$ in Ω. The value of u is being given on the boundary of the solution domain Ω.

The corresponding Dirichlet boundary conditions are prescribed by

$$u(x,y) = \psi(x,y), (x,y) \in \partial\Omega. \tag{2}$$

We assume that for $0 < x$, $y < 1$,

(i) $f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$ is continuous,

(ii) $\frac{\partial f}{\partial u},\frac{\partial f}{\partial u_x},\frac{\partial f}{\partial u_y}$ exist and are continuous,

(iii) $\frac{\partial f}{\partial u} \geqslant 0, \left|\frac{\partial f}{\partial u_x}\right| \leqslant G$ and $\left|\frac{\partial f}{\partial u_y}\right| \leqslant H$

where G and H are positive constants (see [6,7]). Further, we may also assume that the coefficients $A(x,y,u)$ and $B(x,y,u)$ are sufficiently smooth and their required higher order partial derivatives exist in the solution domain Ω.

The main aim of this work is to use cubic spline functions and their certain consistency relations, which are then used to develop a numerical method for computing smooth approximations to the solution of Eq. (1). Note that each discretization of the elliptic differential Eq. (1) at an interior grid point is based on just 3 evaluations of the function f. The paper is designed as follows. The next section of this paper presents the high accuracy numerical method based on cubic spline approximations. Third section discusses the derivation procedure of the scheme developed. Section 4 concerns with establishing the convergence analysis of the proposed method. In Section 5, we discuss the application of proposed method to polar coordinate problems. Section 6 gives the results of the numerical experiments to verify the accuracy and computational efficiency of the proposed method. Finally, the paper concludes with some brief remarks on the present work.

2. The cubic spline approximation and numerical scheme

In this section, we first aim to discuss a numerical method based on cubic spline approximation for the solution of nonlinear elliptic equation

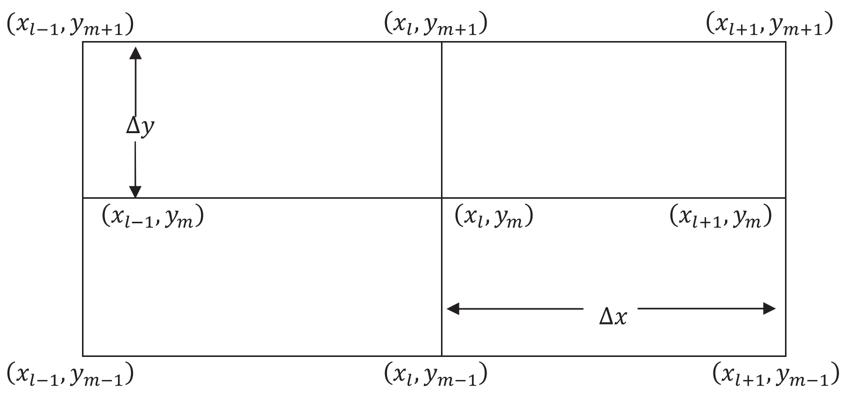

Fig. 1. 9-Point computational network.

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial y^2}=f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\quad 0<x,y<1. \tag{3}$$

We consider our region of interest, a rectangular domain $\Omega=[0,1]\times[0,1]$. A grid with spacing $\Delta x>0$ and $\Delta y>0$ in the directions x- and y- respectively are first chosen, so that the mesh points (x_l,y_m) denoted by (l,m) are defined as $x_l=l\Delta x$ and $y_m=m\Delta y$, $l=0,1,\ldots,N+1$, $m=0,1,\ldots,M+1$, where N and M are positive integers such that $(N+1)\Delta x=1$ and $(M+1)\Delta y=1$.

Let us denote the mesh ratio parameter by $p=(\Delta y/\Delta x)$. For convergence of the numerical scheme it is essential that our parameter remains in the range $0<\sqrt{6}p<1$. Let $U_{l,m}$ and $u_{l,m}$ be the exact and approximation solution values of $u(x,y)$ at the grid point (x_l,y_m), respectively. Similarly, let $A_{l,m}=A(x_l,y_m)$ and $B_{l,m}=B(x_l,y_m)$ be the exact values of $A(x,y)$ and $B(x,y)$ at the grid point (x_l,y_m), respectively.

At the grid point (x_l,y_m), we use the notation

$$W_{pq}=\frac{\partial^{p+q}W(x_l,y_m)}{\partial x^p\partial y^q}\ \text{for}\ W=U,A,B.$$

Let $S_m(x)$ be the cubic spline interpolating polynomial of the value $u_{l,m}$ at the grid point (x_l,y_m), and is given by

$$S_m(x)=\frac{(x_l-x)^3}{6\Delta x}M_{l-1,m}+\frac{(x-x_{l-1})^3}{6\Delta x}M_{l,m}+\left(u_{l-1,m}-\frac{\Delta x^2}{6}M_{l-1,m}\right)\left(\frac{x_l-x}{\Delta x}\right)+\left(u_{l,m}-\frac{\Delta x^2}{6}M_{l,m}\right)\left(\frac{x-x_{l-1}}{\Delta x}\right),$$
$$x_{l-1}\leqslant x\leqslant x_l;\quad l=1,2,\ldots,N+1,m=0,1,\ldots,M+1 \tag{4}$$

which satisfies at mth-line parallel to x-axis the following properties:

(i) $S_m(x)$ coincides with a polynomial of degree three on each $[x_{l-1},x_l]$, $l=1,2,,\ldots,N+1$, $m=0,1,\ldots,M+1$;
(ii) $S_m(x)\in C^2[0,1]$, and
(iii) $S_m(x_l)=u_{l,m}$, $l=0,1,\ldots,N+1$, $m=0,1,\ldots,M+1$.

and where

$$m_{l,m}=S'_m(x_l)=U_{xl,m}\quad\text{and}$$
$$M_{l,m}=S''_m(x_l)=U_{xxl,m}=\frac{1}{A_{l,m}}[-B_{l,m}U_{yyl,m}+f(x_l,y_m,U_{l,m},m_{l,m},U_{yl,m})],\quad l=0,1,\ldots,N+1,\quad m=0,1,\ldots,M+1$$

We consider the following approximations:

$$\overline{U}_{yl,m}=(U_{l,m+1}-U_{l,m-1})/(2\Delta y), \tag{5.1}$$

$$\overline{U}_{yl+1,m}=(U_{l+1,m+1}-U_{l+1,m-1})/(2\Delta y), \tag{5.2}$$

$$\overline{U}_{yl-1,m}=(U_{l-1,m+1}-U_{l-1,m-1})/(2\Delta y), \tag{5.3}$$

$$\overline{U}_{yyl,m}=(U_{l,m+1}-2U_{l,m}+U_{l,m-1})/\Delta y^2, \tag{5.4}$$

$$\overline{U}_{yyl+1,m}=(U_{l+1,m+1}-2U_{l+1,m}+U_{l+1,m-1})/\Delta y^2, \tag{5.5}$$

$$\overline{U}_{yyl-1,m}=(U_{l-1,m+1}-2U_{l-1,m}+U_{l-1,m-1})/\Delta y^2, \tag{5.6}$$

$$\bar{m}_{l,m}=\overline{U}_{xl,m}=(U_{l+1,m}-U_{l-1,m})/(2\Delta x), \tag{6.1}$$

$$\bar{m}_{l+1,m}=\overline{U}_{xl+1,m}=(3U_{l+1,m}-4U_{l,m}+U_{l-1,m})/(2\Delta x), \tag{6.2}$$

$$\bar{m}_{l-1,m}=\overline{U}_{xl-1,m}=(-3U_{l-1,m}+4U_{l,m}-U_{l+1,m})/(2\Delta x), \tag{6.3}$$

$$U_{xxl,m}=(U_{l+1,m}-2U_{l,m}+U_{l-1,m})/\Delta x^2, \tag{6.4}$$

$$\overline{F}_{l,m}=f(x_l,y_m,U_{l,m},\bar{m}_{l,m},\overline{U}_{yl,m}), \tag{7.1}$$

$$\overline{F}_{l+1,m}=f(x_{l+1},y_m,U_{l+1,m},\bar{m}_{l+1,m},\overline{U}_{yl+1,m}), \tag{7.2}$$

$$\overline{F}_{l-1,m}=f(x_{l-1},y_m,U_{l-1,m},\bar{m}_{l-1,m},\overline{U}_{yl-1,m}), \tag{7.3}$$

$$\overline{M}_{l,m} = \frac{1}{A_{00}}[-B_{00}\overline{U}_{yyl,m} + \overline{F}_{l,m}], \tag{8.1}$$

$$\overline{M}_{l+1,m} = \frac{1}{A_{00}}\left(1 - \frac{\Delta x A_{10}}{A_{00}}\right)[-B_{l+1,m}\overline{U}_{yyl+1,m} + \overline{F}_{l+1,m}], \tag{8.2}$$

$$\overline{M}_{l-1,m} = \frac{1}{A_{00}}\left(1 + \frac{\Delta x A_{10}}{A_{00}}\right)[-B_{l-1,m}\overline{U}_{yyl-1,m} + \overline{F}_{l-1,m}], \tag{8.3}$$

$$\bar{\bar{m}}_{l+1,m} = \overline{\overline{U}}_{xl+1,m} = \frac{U_{l+1,m} - U_{l,m}}{\Delta x} + \frac{\Delta x}{6}[\overline{M}_{l,m} + 2\overline{M}_{l+1,m}], \tag{9.1}$$

$$\bar{\bar{m}}_{l-1,m} = \overline{\overline{U}}_{xl-1,m} = \frac{U_{l,m} - U_{l-1,m}}{\Delta x} - \frac{\Delta x}{6}[\overline{M}_{l,m} + 2\overline{M}_{l-1,m}], \tag{9.2}$$

$$\widehat{U}_{xl,m} = \overline{U}_{xl,m} - \frac{\Delta x}{12A_{00}}[\overline{F}_{l+1,m} - \overline{F}_{l-1,m}] + \frac{\Delta x}{12A_{00}}B_{00}[\overline{U}_{yyl+1,m} - \overline{U}_{yyl-1,m}] + \frac{\Delta x^2}{6}\frac{A_{10}}{A_{00}}\overline{U}_{xxl,m} + \frac{\Delta x^2}{6}\frac{B_{10}}{A_{00}}\overline{U}_{yyl,m}, \tag{9.3}$$

$$\overline{\overline{F}}_{l+1,m} = f(x_{l+1}, y_m, U_{l+1,m}, \bar{\bar{m}}_{l+1,m}, \overline{U}_{yl+1,m}), \tag{10.1}$$

$$\overline{\overline{F}}_{l-1,m} = f(x_{l-1}, y_m, U_{l-1,m}, \bar{\bar{m}}_{l-1,m}, \overline{U}_{yl-1,m}), \tag{10.2}$$

$$\widehat{\widehat{F}}_{l,m} = f(x_l, y_m, U_{l,m}, \widehat{U}_{xl,m}, \overline{U}_{yl,m}). \tag{10.3}$$

The cubic spline approximations (8.1)–(9.2) are discussed in details in [18]. Then at each internal grid point (x_l, y_m), the cubic spline method with accuracy of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ for the solution of non-linear elliptic partial differential Eq. (3) may be written as

$$\begin{aligned} L_u \equiv{}& p^2\left[A_{00} - \frac{\Delta x^2}{6}\frac{A_{10}}{A_{00}}A_{10} + \frac{\Delta x^2}{12}A_{20}\right]\delta_x^2 U_{l,m} \\ &+ \frac{\Delta y^2}{12}\left[\left(1 - \frac{\Delta x A_{10}}{A_{00}}\right)B_{l+1,m}\overline{U}_{yyl+1,m} + \left(1 + \frac{\Delta x A_{10}}{A_{00}}\right)B_{l-1,m}\overline{U}_{yyl-1,m} + 10B_{l,m}\overline{U}_{yyl,m}\right] \\ ={}& \frac{\Delta y^2}{12}\left[\left(1 - \frac{\Delta x A_{10}}{A_{00}}\right)\overline{\overline{F}}_{l+1,m} + \left(1 + \frac{\Delta x A_{10}}{A_{00}}\right)\overline{\overline{F}}_{l-1,m} + 10\widehat{\widehat{F}}_{l,m}\right] + \widehat{T}_{l,m}, \quad l = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, M, \end{aligned} \tag{11}$$

where, $\delta_x U_l = \left(U_{l+\frac{1}{2}} - U_{l-\frac{1}{2}}\right)$ and $\mu_x U_l = \frac{1}{2}\left(U_{l+\frac{1}{2}} + U_{l-\frac{1}{2}}\right)$ are the central and average difference operators with respect to x-direction and the local truncation error $\widehat{T}_{l,m} = O(\Delta y^4 + \Delta y^4\Delta x^2 + \Delta y^2\Delta x^4)$.

3. Derivation of the cubic spline scheme

For the derivation of the numerical method (11) for the solution of partial differential Eq. (3), we follow the ideas given by Jain and Aziz [18].

At the grid point (x_l, y_m), we may write the differential equation (3) as

$$A_{l,m}U_{xxl,m} + B_{l,m}U_{yyl,m} = f(x_l, y_m, U_{l,m}, U_{xl,m}, U_{yl,m}) \equiv F_{l,m} \quad \text{(say)}. \tag{12}$$

It is easy to verify using Taylor series expansion about the grid point (x_l, y_m), that from Eq. (12) we obtain

$$\begin{aligned} L_u = \frac{\Delta y^2}{12}\left[\left(1 - \frac{\Delta x A_{10}}{A_{00}}\right)F_{l+1,m} + \left(1 + \frac{\Delta x A_{10}}{A_{00}}\right)F_{l-1,m} + 10F_{l,m}\right] + O(\Delta y^4 + \Delta y^4\Delta x^2 + \Delta y^2\Delta x^4); \quad l = 1, 2, \ldots, N, \\ m = 1, 2, \ldots M, \end{aligned} \tag{13}$$

$$\text{Let us denote : } \alpha_{l,m} = \left(\frac{\partial f}{\partial U_x}\right)_{l,m} \quad \text{and} \quad \beta_{l,m} = \left(\frac{\partial f}{\partial U_y}\right)_{l,m}. \tag{14}$$

With the help of the approximations (5.1)–(5.3), (6.1)–(6.3), from (7.1)–(7.3), we obtain

$$\overline{F}_{l,m} = F_{l,m} + \frac{\Delta x^2}{6}U_{30}\alpha_{l,m} + O(\Delta y^2 + \Delta x^4), \tag{15.1}$$

$$\overline{F}_{l+1,m} = F_{l+1,m} - \frac{\Delta x^2}{3}U_{30}\alpha_{l,m} + O(\Delta x^3 + \Delta x^4 + \Delta y^2 + \Delta y^2\Delta x + \Delta y^2\Delta x^2), \tag{15.2}$$

$$\overline{F}_{l-1,m} = F_{l-1,m} - \frac{\Delta x^2}{3} U_{30}\alpha_{l,m} + O(-\Delta x^3 + \Delta x^4 + \Delta y^2 - \Delta y^2\Delta x + \Delta y^2\Delta x^2). \tag{15.3}$$

Using the approximations (5.4)–(5.6), (15.1)–(15.3), and simplifying (9.1) and (9.2), we get

$$\bar{\bar{m}}_{l+1,m} = m_{l+1,m} + O(\Delta x^3 + \Delta x^4 + \Delta y^2 + \Delta y^2\Delta x + \Delta y^2\Delta x^2), \tag{16.1}$$

$$\bar{\bar{m}}_{l-1,m} = m_{l-1,m} + O(-\Delta x^3 + \Delta x^4 + \Delta y^2 - \Delta y^2\Delta x + \Delta y^2\Delta x^2). \tag{16.2}$$

Now we need $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$–approximation to $\widehat{U}_{xl,m}$.

Let

$$\widehat{U}_{xl,m} = \overline{U}_{xl,m} + a\Delta x[\overline{F}_{l+1,m} - \overline{F}_{l-1,m}] + b\Delta x B_{00}[\overline{U}_{yyl+1,m} - \overline{U}_{yyl-1,m}] + c\Delta x^2\overline{U}_{xxl,m} + d\Delta x^2\overline{U}_{yyl,m}, \tag{17}$$

where 'a', 'b', 'c' and 'd ' are free parameters to be determined.

By the help of the approximations (15.1)–(15.3), (5.4)–(5.6) and (6.4), from (17), we get

$$\begin{aligned}\widehat{U}_{xl,m} &= m_{l,m} + \frac{\Delta x^2}{6}[(1 + 12aA_{00})U_{30} + 12(aB_{00} + b)U_{12} + (6c + 12aA_{10})U_{20} + (12aB_{10} + 6d)U_{02}] \\ &\quad + O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4).\end{aligned} \tag{18}$$

From (18), it is easy to verify that, $a = -\frac{1}{12A_{00}}, b = \frac{B_{00}}{12A_{00}}, c = \frac{A_{10}}{6A_{00}}$ and $d = \frac{B_{10}}{6A_{00}}$, so that (17) can now be rewritten as

$$\begin{aligned}\widehat{U}_{xl,m} &= \overline{U}_{xl,m} - \frac{\Delta x}{12A_{00}}[\overline{F}_{l+1,m} - \overline{F}_{l-1,m}] + \frac{\Delta x}{12A_{00}}B_{00}[\overline{U}_{yyl+1,m} - \overline{U}_{yyl-1,m}] + \frac{\Delta x^2}{6}\frac{A_{10}}{A_{00}}\overline{U}_{xxl,m} + \frac{\Delta x^2}{6}\frac{B_{10}}{A_{00}}\overline{U}_{yyl,m} \\ &= m_{l,m} + O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4).\end{aligned} \tag{19}$$

Now, with the help of the approximations (5.1)–(5.3), (16.1), (16.2) and (19) , from (10.1)–(10.3), we obtain

$$\overline{\overline{\overline{F}}}_{l+1,m} = F_{l+1,m} + O(\Delta x^3 + \Delta x^4 + \Delta y^2 + \Delta y^2\Delta x + \Delta y^2\Delta x^2), \tag{20.1}$$

$$\overline{\overline{\overline{F}}}_{l-1,m} = F_{l-1,m} + O(-\Delta x^3 + \Delta x^4 + \Delta y^2 - \Delta y^2\Delta x + \Delta y^2\Delta x^2), \tag{20.2}$$

$$\widehat{\widehat{F}}_{l,m} = F_{l,m} + O(\Delta x^4 + \Delta y^2 + \Delta y^2\Delta x^2). \tag{20.3}$$

Finally, using the approximations (20.1)–(20.3), from (11) and (13), we obtain the local truncation error $\widehat{T}_{l,m} = O(\Delta y^4 + \Delta y^4\Delta x^2 + \Delta y^2\Delta x^4)$.

Now we describe the numerical method of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ for the solution of the quasi-linear elliptic Eq. (1). Whenever, the coefficients A and B are functions of x, y and u, the difference scheme (11) needs to be modified. For this purpose, we use the following central differences.

Let

$$A_{10} = (A_{l+1,m} - A_{l-1,m})/(2\Delta x), \tag{21.1}$$

$$A_{20} = (A_{l+1,m} - 2A_{l,m} + A_{l-1,m})/(\Delta x^2), \tag{21.2}$$

$$B_{10} = (B_{l+1,m} - B_{l-1,m})/(2\Delta x), \tag{21.3}$$

where $A_{00} = A_{l,m} = A(x_l, y_m, U_{l,m})$, $A_{l\pm1,m} = A(x_{l\pm1,m}, y_m, U_{l\pm1,m})$ *etc.*

With the help of the approximations (21.1)–(21.3), we see that

$$A_{00} - \frac{\Delta x^2}{6}\frac{A_{10}}{A_{00}}A_{10} + \frac{\Delta x^2}{12}A_{20} = A_{l,m} - \frac{1}{24A_{l,m}}(A_{l+1,m} - A_{l-1,m})^2 + \frac{1}{12}(A_{l+1,m} - 2A_{l,m} + A_{l-1,m}) + O(\Delta x^4).$$

Thus substituting the central difference approximations (21.1)–(21.3) into (11), we obtain the required numerical method of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ based on cubic spline approximation for the solution of quasi-linear elliptic partial differential Eq. (1).

Note that, the Dirichlet boundary conditions are given by (2). Incorporating the boundary conditions, we can write the cubic spline method (11) in a tri-block diagonal matrix form. If the differential Eq. (1) is linear, we can solve the linear system using Gauss-elimination (tri-diagonal solver) method; in the nonlinear case, we can use Newton–Raphson iterative method to solve the non-linear system (see [33–36]).

4. Convergence analysis

We consider the nonlinear elliptic differential equation

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \tag{22}$$

subject to the boundary condition $u(x,y) = \psi(x,y)$, $(x,y) \in \partial\Omega$.

The difference scheme (11) for which, at each (l,m), $[l = 1(1)N,\ m = 1(1)M]$ may be written as

$$\lambda_1(U_{l+1,m}+U_{l-1,m})+\lambda_2(U_{l,m+1}+U_{l,m-1})+\lambda_3(U_{l+1,m+1}+U_{l+1,m-1}+U_{l-1,m+1}+U_{l-1,m-1}-(24p^2+20)U_{l,m})$$
$$=\frac{\Delta y^2}{12}[\overline{\overline{F}}_{l+1,m}+\overline{\overline{F}}_{l-1,m}+10\hat{\hat{F}}_{l,m}]+\widehat{T}_{l,m}, \tag{23}$$

where $\lambda_1 = p^2-\frac{2}{12}, \lambda_2=\frac{10}{12}, \lambda_3=\frac{1}{12}$ and $\widehat{T}_{l,m}=O(\Delta y^4+\Delta y^4\Delta x^2+\Delta y^2\Delta x^4)$.

We next show that, under appropriate conditions, difference method (23) for elliptic Eq. (22) is $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-convergent. The condition which is usually imposed on Eq. (23) is that $\lambda_1 > 0$.

Let $\boldsymbol{U} = [U_{11},U_{21},\ldots,U_{N1},U_{12},U_{22},\ldots,U_{N2},\ldots,U_{1M},U_{2M},\ldots,U_{NM}]^T$ be the solution vector and $\boldsymbol{T}=[\widehat{T}_{11},\widehat{T}_{21},\ldots,\widehat{T}_{N1},\widehat{T}_{12},\widehat{T}_{22},\ldots,\widehat{T}_{N2},\ldots,\widehat{T}_{1M},\widehat{T}_{2M},\ldots,\widehat{T}_{NM}]^T$ be the local truncation error vector.

Let $\phi_{l,m}=\frac{\Delta y^2}{12}[\overline{\overline{F}}_{l+1,m}+\overline{\overline{F}}_{l-1,m}+10\hat{\hat{F}}_{l,m}]$ and

$$\boldsymbol{\phi}(\boldsymbol{U})=[\phi_{11},\phi_{21},\ldots,\phi_{N1},\phi_{12},\phi_{22},\ldots,\phi_{N2},\ldots,\phi_{1M},\phi_{2M},\ldots,\phi_{NM}]^T.$$

Then the finite difference scheme in the matrix form can be written as

$$\boldsymbol{DU}+\boldsymbol{\phi}(\boldsymbol{U})+\boldsymbol{T}=\boldsymbol{0} \tag{24}$$

where,

$$\begin{aligned}&\boldsymbol{D}=[\boldsymbol{D}_1,\boldsymbol{D}_2,\boldsymbol{D}_1]_{NM\times NM}\ \text{is a tri-blockdiagonal matrix,}\quad \text{where,}\\ &\boldsymbol{D}_1=[-\lambda_3,-\lambda_2,-\lambda_3]_{N\times N},\\ &\boldsymbol{D}_2=[-\lambda_1,(24p^2+20)\lambda_3,-\lambda_1]_{N\times N}\ \text{denote the } N\times N \text{ tridiagonal matrices.}\end{aligned}$$

The method consists in finding approximations $\boldsymbol{u}$ for $\boldsymbol{U}$ by solving the $NM \times NM$ system

$$\boldsymbol{Du}+\boldsymbol{\phi}(\boldsymbol{u})=\boldsymbol{0}. \tag{25}$$

Let $\in_{l,m} = u_{l,m} - U_{l,m}$, $l = 1(1)N$, $m = 1(1)M$ and let $\boldsymbol{E} = \boldsymbol{u} - \boldsymbol{U} = [\in_{11},\in_{21},\ldots,\in_{N1},\ldots,\in_{1M},\in_{2M},\ldots,\in_{NM}]^T$ be the error vector.

Also, let

$$\begin{aligned}\bar{\bar{f}}_{l\pm1,m}&=f(x_{l\pm1},y_m,u_{l\pm1,m},\bar{\bar{u}}_{xl\pm1,m},\bar{u}_{yl\pm1,m})\simeq\overline{\overline{F}}_{l\pm1,m},\\ \hat{\hat{f}}_{l,m}&=f(x_l,y_m,u_{l,m},\hat{u}_{xl,m},\hat{u}_{yl,m})\simeq\hat{\hat{F}}_{l,m}.\end{aligned}$$

Then we may write

$$\bar{u}_{xl\pm1,m}-\overline{U}_{xl\pm1,m}=(\pm3\in_{l\pm1,m}\mp4\in_{l,m}\pm\in_{l\mp1,m})/(2\Delta x), \tag{26.1}$$

$$\bar{u}_{xl,m}-\overline{U}_{xl,m}=(\in_{l+1,m}-\in_{l-1,m})/(2\Delta x), \tag{26.2}$$

$$\bar{u}_{yl\pm1,m}-\overline{U}_{yl\pm1,m}=(\in_{l\pm1,m+1}-\in_{l\pm1,m-1})/(2\Delta y), \tag{26.3}$$

$$\bar{u}_{yl,m}-\overline{U}_{yl,m}=(\in_{l,m+1}-\in_{l,m-1})/(2\Delta y), \tag{26.4}$$

$$\bar{u}_{yyl\pm1,m}-\overline{U}_{yyl\pm1,m}=(\in_{l\pm1,m+1}-2\in_{l\pm1,m}+\in_{l\pm1,m-1})/\Delta y^2, \tag{26.5}$$

$$\bar{f}_{l+1,m}-\overline{F}_{l+1,m}=\in_{l+1,m}P^{(1)}_{l+1,m}+(\bar{u}_{xl+1,m}-\overline{U}_{xl+1,m})Q^{(1)}_{l+1,m}+(\bar{u}_{yl+1,m}-\overline{U}_{yl+1,m})R^{(1)}_{l+1,m}, \tag{26.6}$$

$$\bar{f}_{l-1,m}-\overline{F}_{l-1,m}=\in_{l-1,m}P^{(1)}_{l-1,m}+(\bar{u}_{xl-1,m}-\overline{U}_{xl-1,m})Q^{(1)}_{l-1,m}+(\bar{u}_{yl-1,m}-\overline{U}_{yl-1,m})R^{(1)}_{l-1,m} \tag{26.7}$$

$$\bar{\bar{f}}_{l+1,m}-\overline{\overline{F}}_{l+1,m}=\in_{l+1,m}P^{(1)}_{l+1,m}+(\bar{\bar{u}}_{xl+1,m}-\overline{\overline{U}}_{xl+1,m})Q^{(1)}_{l+1,m}+(\bar{u}_{yl+1,m}-\overline{U}_{yl+1,m})R^{(1)}_{l+1,m}, \tag{26.8}$$

$$\bar{\bar{f}}_{l-1,m}-\overline{\overline{F}}_{l-1,m}=\in_{l-1,m}P^{(1)}_{l-1,m}+(\bar{\bar{u}}_{xl-1,m}-\overline{\overline{U}}_{xl-1,m})Q^{(1)}_{l-1,m}+(\bar{u}_{yl-1,m}-\overline{U}_{yl-1,m})R^{(1)}_{l-1,m}, \tag{26.9}$$

$$\text{and}\quad \hat{\hat{f}}_{l,m}-\hat{\hat{F}}_{l,m}=\in_{l,m}P^{(2)}_{l,m}+(\hat{u}_{xl,m}-\widehat{U}_{xl,m})Q^{(2)}_{l,m}+(\bar{u}_{yl,m}-\overline{U}_{yl,m})R^{(2)}_{l,m} \tag{26.10}$$

for suitable $I^{(1)}_{l\pm1,m}$ and $I^{(2)}_{l,m}$ where $I = P$, Q and R. Also, we may write

$$P^{(1)}_{l\pm1,m} = P^{(1)}_{l,m} \pm O(\Delta x), \tag{27.1}$$
$$Q^{(1)}_{l\pm1,m} = Q^{(1)}_{l,m} \pm \Delta x Q^{(1)}_{xl\pm1,m} + O(\Delta x^2), \tag{27.2}$$
$$R^{(1)}_{l\pm1,m} = R^{(1)}_{l,m} \pm \Delta x R^{(1)}_{xl\pm1,m} + O(\Delta x^2) \tag{27.3}$$

Subtracting (24) from (25), we get

$$\boldsymbol{DE} + \boldsymbol{\phi}(\boldsymbol{u}) - \boldsymbol{\phi}(\boldsymbol{U}) = \boldsymbol{T}. \tag{28}$$

Now,

$$\boldsymbol{\phi}(\boldsymbol{u}) - \boldsymbol{\phi}(\boldsymbol{U}) = \frac{\Delta y^2}{12}\left[(\bar{\bar{f}}_{l+1,m} - \bar{\bar{F}}_{l+1,m}) + (\bar{\bar{f}}_{l-1,m} - \bar{\bar{F}}_{l-1,m}) + 10(\hat{\hat{f}}_{l,m} - \hat{\hat{F}}_{l,m})\right],$$

Using Eqs. (26)–(27), we may obtain

$$\boldsymbol{\phi}(\boldsymbol{u}) - \boldsymbol{\phi}(\boldsymbol{U}) = \boldsymbol{P}\cdot\boldsymbol{E} \tag{29}$$

where $\boldsymbol{P} = (p_{r,s})$, $r = 1(1)NM$, $s = 1(1)NM$ is a tri-blockdiagonal matrix with matrix elements

$$p_{(m-1)N+l,(m-1)N+l} = \frac{\Delta y^2}{12}\left[10P^{(2)}_{l,m} - \frac{8}{3}Q^{(1)}_{xl,m} - \frac{4}{3}Q^{(1)}_{l,m}Q^{(1)}_{l,m} + \frac{2}{3}Q^{(1)}_{l,m}Q^{(2)}_{l,m}\right] + O(\Delta y^2\Delta x^2); \quad l = 1(1)N, \quad m = 1(1)M$$

$$p_{(m-1)N+l,(m-1)N+l\pm1} = \frac{\Delta y}{12}\left[\pm pQ^{(1)}_{l,m} \pm \frac{10}{3}pQ^{(2)}_{l,m}\right] + \frac{\Delta y^2}{12}\left[P^{(1)}_{l,m} + \frac{4}{3}Q^{(1)}_{xl,m} + \frac{2}{3}p^2Q^{(1)}_{xl,m} + \frac{2}{3}Q^{(1)}_{l,m}Q^{(1)}_{l,m} - \frac{40}{24}Q^{(1)}_{l,m}Q^{(2)}_{l,m}\right] + O(\pm\Delta y^2\Delta x + \Delta y^2\Delta x^2); \quad l = 1(1)N-1, 2(1)N; \quad m = 1(1)M$$

$$p_{(m-1)N+l,(m-1\pm1)N+l} = \pm\frac{\Delta y}{12}\left[5R^{(2)}_{l,m}\right]; \quad l = 1(1)N; \quad m = 1(1)M-1, 2(1)M$$

$$p_{(m-1)N+l,mN+l\pm1} = \frac{\Delta y}{12}\left[\mp\frac{1}{3}pQ^{(1)}_{l,m} \pm \frac{10}{12}pQ^{(2)}_{l,m} + \frac{1}{2}R^{(1)}_{l,m}\right] + \frac{\Delta y^2}{12}\left[-\frac{1}{3}p^2Q^{(1)}_{xl,m} \pm \frac{p}{6}R^{(1)}_{l,m}Q^{(1)}_{l,m} \mp \frac{10}{24}pR^{(1)}_{l,m}Q^{(2)}_{l,m} \pm \frac{p}{2}R^{(1)}_{xl,m}\right] + O(\Delta y^2\Delta x + \Delta y^2\Delta x^2); \quad l = 1(1)N-1, 2(1)N; \quad m = 1(1)M-1,$$

$$p_{(m-1)N+l,(m-2)N+l\pm1} = \frac{\Delta y}{12}\left[\mp\frac{1}{3}pQ^{(1)}_{l,m} \pm \frac{10}{12}pQ^{(2)}_{l,m} - \frac{1}{2}R^{(1)}_{l,m}\right] + \frac{\Delta y^2}{12}\left[-\frac{1}{3}p^2Q^{(1)}_{xl,m} \mp \frac{p}{6}R^{(1)}_{l,m}Q^{(1)}_{l,m} \pm \frac{10}{24}pR^{(1)}_{l,m}Q^{(2)}_{l,m} \mp \frac{p}{2}R^{(1)}_{xl,m}\right] + O(\Delta y^2\Delta x + \Delta y^2\Delta x^2); \quad l = 1(1)N-1, 2(1)N; \quad m = 2(1)M.$$

We obtain the error equation, in the absence of round-off errors, using the relation (29) in Eq. (28) as

$$(\boldsymbol{D} + \boldsymbol{P})\boldsymbol{E} = \boldsymbol{T} \tag{30}$$

Let,

$$P_* = \min_{(x,y)\in\overline{\Omega}} \frac{\partial f}{\partial U} \quad \text{and} \quad P^* = \max_{(x,y)\in\overline{\Omega}} \frac{\partial f}{\partial U} \quad \text{where } \overline{\Omega} = \Omega + \partial\Omega$$

Then,

$$0 < P_* \leqslant P^{(1)}_{l\pm1,m}, P^{(2)}_{l,m} \leqslant P^*$$

and for $L = Q \text{ and } R$, let

$$0 < |L^{(1)}_{l\pm1,m}|, |L^{(2)}_{l,m}| \leqslant L \quad \text{and} \quad |L^{(1)}_{xl,m}| \leqslant L^{(1)}$$

for some positive constant $L^{(1)}$.

For sufficiently small Δy, we have

$$|p_{(m-1)N+l,(m-1)N+l\pm1}| < \lambda_1, \quad [l = 1(1)N-1, 2(1)N; m = 1(1)M],$$
$$|p_{(m-1)N+l,(m-1\pm1)N+l}| < \lambda_2, \quad [l = 1(1)N; m = 1(1)M-1, 2(1)M],$$
$$|p_{(m-1)N+l,mN+l\pm1}| < \lambda_3, \quad [l = 1(1)N-1, 2(1)N; m = 1(1)M-1],$$
$$|p_{(m-1)N+l,(m-2)N+l\pm1}| < \lambda_3, \quad [l = 1(1)N-1, 2(1)N; m = 2(1)M].$$

We first show that the matrix $\boldsymbol{D} + \boldsymbol{P}$ is irreducible. This can be followed from the directed graph of $\boldsymbol{D} + \boldsymbol{P}$ (see Fig. 2):

The arrows indicate paths $i \to j$ for every non zero entry $(\boldsymbol{D} + \boldsymbol{P})_{i,j}$ of $\boldsymbol{D} + \boldsymbol{P}$. For $\lambda_1 > 0$, there exists a direct path $\overrightarrow{(i, l_1)}, \overrightarrow{(l_1, l_2)}, \ldots, \overrightarrow{(l_k, j)}$ connecting i to j for any ordered pair of nodes i and j. Hence the graph is strongly connected and thus $\boldsymbol{D} + \boldsymbol{P}$ is irreducible.

Now let S_k denote the sum of the elements in the kth row of $\boldsymbol{D}+\boldsymbol{P}$, then for $k=1$ & N,

$$S_k = (p^2 + 11\lambda_3) + \frac{\Delta y}{12}\left[b_k + \frac{\Delta y}{3}c_k\right] + \frac{\Delta y^2}{12}\left[10P^{(2)}_{k,1} + P^{(1)}_{k,1}\right] + O(\Delta y^2 \Delta x) \tag{31.1}$$

where

$$b_k = \pm\frac{4p}{3}Q^{(1)}_{k,1} \pm \frac{25p}{6}Q^{(2)}_{k,1} + 5R^{(2)}_{k,1} + \frac{1}{2}R^{(1)}_{k,1}; \quad c_k = (p^2-4)Q^{(1)}_{xk,1} - 2Q^{(1)}_{k,1}Q^{(1)}_{k,1} - 3Q^{(1)}_{k,1}Q^{(2)}_{k,1} \pm \frac{p}{2}R^{(1)}_{k,1}Q^{(1)}_{k,1} \pm \frac{3p}{2}R^{(1)}_{k,1} - \frac{5p}{4}R^{(1)}_{k,1}Q^{(2)}_{k,1}$$

$$S_{(M-1)N+k} = (p^2 + 11\lambda_3) + \frac{\Delta y}{12}\left[b_{(M-1)N+k} + \frac{\Delta y}{3}c_{(M-1)N+k}\right] + \frac{\Delta y^2}{12}\left[10P^{(2)}_{k,N} + P^{(1)}_{k,N}\right] + O(\Delta y^2 \Delta x) \tag{31.2}$$

where

$$b_{(M-1)N+k} = \mp\frac{4p}{3}Q^{(1)}_{k,N} \mp \frac{25p}{6}Q^{(2)}_{k,N} - 5R^{(2)}_{k,N} - \frac{1}{2}R^{(1)}_{k,N};$$
$$c_{(M-1)N+k} = (p^2-4)Q^{(1)}_{xk,1} - 2Q^{(1)}_{k,N}Q^{(1)}_{k,N} - 3Q^{(1)}_{k,N}Q^{(2)}_{k,N} \mp \frac{p}{2}R^{(1)}_{k,N}Q^{(1)}_{k,N} \mp \frac{3p}{2}R^{(1)}_{xk,N} \pm \frac{5p}{4}R^{(1)}_{k,N}Q^{(2)}_{k,N}$$

For $r = 2(1)M-1$:

$$S_{(r-1)N+k} = p^2 + \frac{\Delta y}{12}\left[b_{(r-1)N+k} + \frac{\Delta y}{6}c_{(r-1)N+k}\right] + \frac{\Delta y^2}{12}\left[10P^{(2)}_{k,r} + P^{(1)}_{k,r}\right] + O(\Delta y^2 \Delta x) \tag{31.3}$$

where

$$b_{(r-1)N+k} = \pm\left(pQ^{(1)}_{k,r} + 5Q^{(2)}_{k,r}\right); \quad c_{(r-1)N+k} = -8Q^{(1)}_{xk,r} - 4Q^{(1)}_{k,r}Q^{(1)}_{k,r} - 6Q^{(1)}_{k,r}Q^{(2)}_{k,r} \pm 3pR^{(1)}_{k,r} \mp \frac{p}{2}R^{(1)}_{xk,r}$$

For $q = 2(1)N-1$, $k = 1$ & M:

$$S_{(k-1)N+q} = 1 + \frac{\Delta y}{12}\left[b_{(k-1)N+q} + \frac{\Delta y}{3}c_{(k-1)N+q}\right] + \frac{\Delta y^2}{12}\left[10P^{(2)}_{q,k} + 2P^{(1)}_{q,k}\right] + O(\Delta y^2 \Delta x) \tag{31.4}$$

where

$$b_{(k-1)N+q} = \pm\left(R^{(1)}_{q,k} + 5R^{(2)}_{q,k}\right); \quad c_{(k-1)N+q} = 2p^2Q^{(1)}_{xq,k} - 8Q^{(1)}_{q,k}Q^{(2)}_{q,k}$$

and finally,

$$S_{(r-1)N+q} = 1 + \frac{\Delta y}{12}\left[\frac{\Delta y}{3}(-8Q^{(1)}_{q,r}Q^{(2)}_{q,r})\right] + \frac{\Delta y^2}{12}\left[10P^{(2)}_{q,r} + 2P^{(1)}_{q,r}\right] + O(\Delta y^2 \Delta x) \tag{31.5}$$

for $q = 2(1)N-1$, $r = 2(1)M-1$

Now, with the help of Eqs. (31), for $k = 1, N$, $(M-1)N+1$ & NM

$$|b_k| \leqslant \frac{11}{2}(pQ+R); |c_k| \leqslant (p^2-4)Q^{(1)} + 5Q^2 + \left(\frac{2p+5}{4}\right)RQ + \frac{3}{2}pR^{(1)}$$

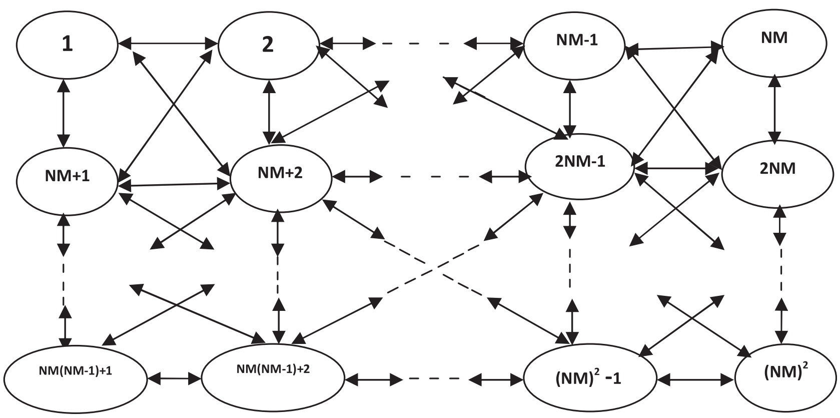

Fig. 2. Directed graph.

For $k = q$ & $(M-1)N+q$, $q = 2(1)N-1$

$$|b_k| \leqslant 6R; |c_k| \leqslant 8Q^2 + 2p^2Q^{(1)}$$

and for $k = (r-1)N+1$, & rN, $r = 2(1)M-1$

$$|b_k| \leqslant 6pQ; |c_k| \leqslant 8Q^{(1)} + 10Q^2 + 3pR + \frac{1}{2}pR^{(1)}$$

From the above set of equations, it follows that for sufficiently small Δy,

$$S_k > \frac{11}{12}\Delta y^2 P_*; \quad k = 1, N, (M-1)N+1 \ \& \ NM \tag{32.1}$$

$$S_k > \Delta y^2 P_*; \quad k = q \ \& \ (M-1)N+q, q = 2(1)N-1 \tag{32.2}$$

$$S_k > \frac{11}{12}\Delta y^2 P_*; \ k = (r-1)N+1, \ \& \ rN, r = 2(1)M-1 \tag{32.3}$$

$$S_{(r-1)N+q} \geqslant \Delta y^2 P_*; \quad q = 2(1)N-1, r = 2(1)M-1 \tag{32.4}$$

Thus for sufficiently small Δy, $\boldsymbol{D}+\boldsymbol{P}$ is monotone. Hence, $\boldsymbol{D}+\boldsymbol{P}$ is invertible and let $(\boldsymbol{D}+\boldsymbol{P})^{-1} = \boldsymbol{J} > 0$,

Here $J = (J_{r,s})[r = 1(1)NM, s = 1(1)NM]$.

Since, $\sum_{r=1}^{NM} J_{p,r}S_r = 1, p = 1(1)NM$, using the above equations with $p = 1(1)NM$, we obtain

$$J_{p,k} \leqslant \frac{1}{S_k} \leqslant \frac{12}{11\Delta y^2 P_*} \ \ k = 1, N, (M-1)N+1 \ \& \ NM \tag{33.1}$$

$$\sum_{q=2}^{N-1} J_{p,k} \leqslant \frac{1}{\min_{2\leqslant q\leqslant N-1} S_k} \leqslant \frac{1}{\Delta y^2 P_*} \quad k = q \ \& \ (M-1)N+q \tag{33.2}$$

$$\sum_{r=2}^{M-1} J_{p,k} \leqslant \frac{1}{\min_{2\leqslant r\leqslant M-1} S_k} \leqslant \frac{12}{11\Delta y^2 P_*} \quad k = (r-1)N+1, \ \& \ rN \tag{33.3}$$

$$\sum_{q=2}^{N-1}\sum_{r=2}^{M-1} J_{p,k} \leqslant \frac{1}{\min_{\substack{\Delta 2\leqslant q\leqslant N-1 \\ \Delta 2\leqslant r\leqslant M-1}} S_k} \leqslant \frac{1}{\Delta y^2 P_*} \quad k = (r-1)N+q \tag{33.4}$$

We may write error Eq. (30) as

$$\|\boldsymbol{E}\| \leqslant \|\boldsymbol{J}\| \cdot \|\boldsymbol{T}\| \tag{34}$$

where,

$$\|\boldsymbol{J}\| = \max_{1\leqslant p\leqslant NM}\left[\left(J_{p,1} + \sum_{q=2}^{N-1} J_{p,q} + J_{p,N}\right) + \left(\sum_{r=2}^{M-1} J_{p,(r-1)N+1} + \sum_{q=2}^{N-1}\sum_{r=2}^{M-1} J_{p,(r-1)N+q} + \sum_{r=2}^{M-1} J_{p,rN}\right) + \left(J_{p,(M-1)N+1} + \sum_{q=2}^{N-1} J_{p,(M-1)N+q} + J_{p,NM}\right)\right] \tag{35}$$

Substituting Eqs. (33) in (35), from Eq. (34) we obtain, for sufficiently small Δy,

$$\|\boldsymbol{E}\| \leqslant O(\Delta y^2 + \Delta y^2 \Delta x^2 + \Delta x^4). \tag{36}$$

Hence, $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ convergence is established.

5. Application to singular equations

Consider the two dimensional elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + B(x)\frac{\partial^2 u}{\partial y^2} = D(x)\frac{\partial u}{\partial x} + g(x, y), \quad 0 < x, y < 1 \tag{37}$$

subject to appropriate Dirichlet boundary conditions prescribed. The coefficients $B(x)$, $D(x)$ and function $g(x,y) \in C^2(\Omega)$, where $C^m(\Omega)$ denotes the set of all functions of x and y with continuous partial derivatives up to order m, in Ω.

On applying formula (11) to the elliptic Eq. (37), we obtain the following difference scheme

$$p^2\delta_x^2 U_{l,m} + \frac{\Delta y^2}{12}\left[B_{l+1}\overline{U}_{yyl+1,m} + B_{l-1}\overline{U}_{yyl-1,m} + 10B_l\overline{U}_{yyl,m}\right] = \frac{\Delta y^2}{12}\left[D_{l+1}\overline{\overline{U}}_{xl+1,m} + D_{l-1}\overline{\overline{U}}_{xl-1,m} + 10D_l\widehat{U}_{xl,m}\right] + \frac{\Delta y^2}{12}\left[g_{l+1,m} + g_{l-1,m} + 10g_{l,m}\right] + \widehat{T}_{l,m} \tag{38}$$

Note that scheme (38) is of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$. However, this scheme fails to compute at $l = 1$, when the coefficients $B(x)$, $D(x)$ and/ or $g(x,y)$ involves terms like $1/x$, $1/x^2$, $1/xy^3$ and so forth. Take for example, if $D(x) = 1/x$, then $D_{l-1} = 1/x_{l-1}$ which blows to infinity at $l = 1$. So, in order to handle the singularity at $x = 0$, we modify the scheme (38) such that the order and accuracy of the solution is retained in the solution region.

We need the following approximations:

$$D_{l\pm1} = D_{00} \pm \Delta x D_{10} + \frac{\Delta x^2}{2} D_{20} \pm O(\Delta x^3), \tag{39.1}$$

$$B_{l\pm1} = B_{00} \pm \Delta x B_{10} + \frac{\Delta x^2}{2} B_{20} \pm O(\Delta x^3), \tag{39.2}$$

$$g_{l\pm1,m} = g_{00} \pm \Delta x g_{10} + \frac{\Delta x^2}{2} g_{20} \pm O(\Delta x^3), \tag{39.3}$$

where $g_{l,m} = g_{00} = g(x_l, y_m)$ *etc.*

Now, substituting the approximations (39.1)–(39.3) in the difference scheme (38) and merging the higher order terms in local truncation error, we obtain the modified scheme as

$$\begin{aligned}
&\left[-12p^2 + \frac{4\Delta y^2}{3} D_{10} - \Delta y^2 D_{00}^2\right]\delta_x^2 U_{l,m} + \left[p\Delta y\left(6D_{00} + \frac{\Delta x^2}{2} D_{20}\right) - \frac{\Delta y^2 \Delta x}{6} D_{00} D_{10}\right](2\mu_x\delta_x)U_{l,m} \\
&+ \left[-12B_{00} - \Delta x^2 B_{20} - \frac{2\Delta x^2}{3} B_{00} D_{10} + \Delta x^2 B_{10} D_{00}\right]\delta_y^2 U_{l,m} - [B_{00}]\delta_x^2\delta_y^2 U_{l,m} \\
&+ \left[-\Delta x B_{10} + \frac{\Delta x}{2} B_{00} D_{00}\right]\delta_y^2(2\mu_x\delta_x)U_{l,m} = -\Delta y^2\left[12g_{00} + \Delta x^2\left(g_{20} - D_{00}g_{10} + \frac{2}{3}D_{10}g_{00}\right)\right].
\end{aligned} \tag{40}$$

Note that, the modified scheme (40) is of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ accurate and applicable to both singular and non-singular elliptic differential equations of the form (37).

Consider the elliptic equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = [4 - \pi^2]\cos(\pi\theta), \quad 0 < r, \theta < 1. \tag{41}$$

The above equation represents 2-D Poisson's equation in cylindrical polar coordinates in r–θ plane. Replacing the variables (x,y) by (r,θ) and substituting $B_{00} = 1/r_l^2, B_{10} = -2/r_l^3, B_{20} = 6/r_l^4, D_{00} = -1/r_l, D_{10} = B_{00}, D_{20} = B_{10}$ in (40), we obtain $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ scheme for the solution of elliptic Eq. (41).

Similarly, consider the 2-D Poisson's equation in cylindrical polar coordinates in r–z plane

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial u}{\partial r} = cosh z\left[2cosh r + \frac{1}{r} sinh r\right], \quad 0 < r, z < 1. \tag{42}$$

Replacing the variables (x,y) by (r,z) and setting $B_{00} = 1, B_{10} = 0 = B_{20}, D_{00} = -1/r_l, D_{10} = 1/r_l^2, D_{20} = -2/r_l^3$ in (40), we can get $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ scheme for the solution of elliptic Eq. (42).

Next consider the Convection–Diffusion equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \beta\frac{\partial u}{\partial x}, \tag{43}$$

Table 1.1
Example 1: The maximum absolute errors ($p = \frac{\Delta y}{\Delta x} = 0.8$).

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method			$O(\Delta y^2 + \Delta x^2)$-method		
	$\beta = 20$	$\beta = 50$	$\beta = 100$	$\beta = 20$	$\beta = 50$	$\beta = 100$
1/16	0.2439E−02	0.3662E−01	0.1658E+00	0.7424E−01	0.3062E+00	–
1/32	0.2253E−03	0.3789E−02	0.3349E−01	0.1614E−01	0.1016E+00	0.2872E+00
1/64	0.3683E−04	0.2410E−03	0.3430E−02	0.4056E−02	0.2305E−01	0.9479E−01
1/128	0.8396E−05	0.1769E−04	0.2122E−03	0.1005E−02	0.5496E−02	0.2145E−01

Table 1.2
Example 1: The maximum absolute errors ($\gamma = \frac{\Delta y}{\Delta x^2} = 20$).

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method			$O(\Delta y^4 + \Delta y^2\Delta x^2 + \Delta x^4)$-method discussed in [10]		
	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 10$	$\beta = 20$	$\beta = 30$
$\frac{1}{10}$	0.1062E−01	0.1823E−01	0.4618E−01	0.7918E−01	0.8220E−01	0.8998E−01
$\frac{1}{20}$	0.6971E−03	0.1213E−02	0.3922E−02	0.4919E−02	0.5155E−02	0.5676E−02
$\frac{1}{40}$	0.4352E−04	0.7360E−04	0.2312E−03	0.3102E−03	0.3214E−03	0.3511E−03

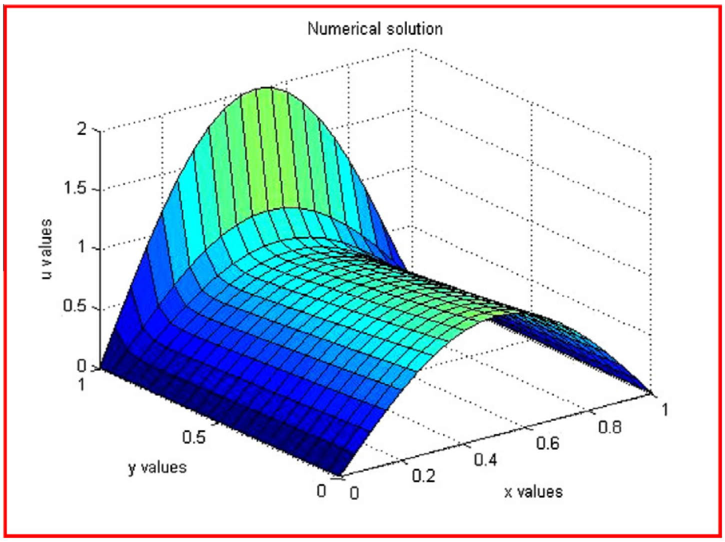

Fig. 3.1. Convection–Diffusion equation $\gamma = 20$, $\beta = 30$ [Numerical Solution].

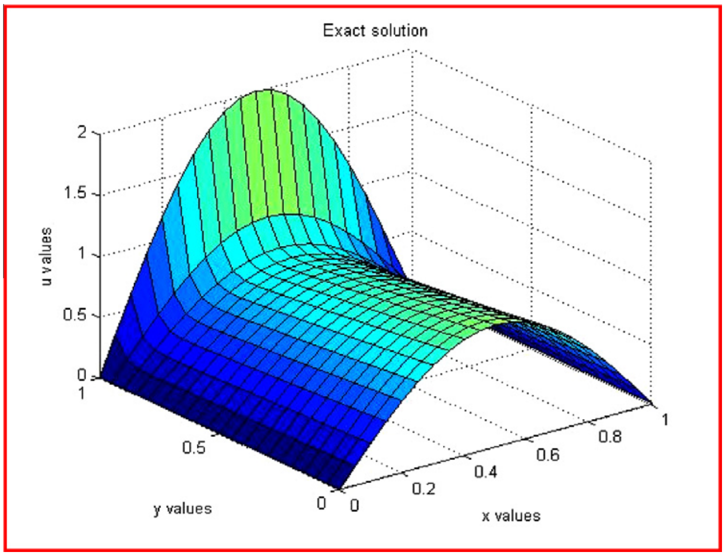

Fig. 3.2. Convection–Diffusion equation $\gamma = 20$, $\beta = 30$ [Exact Solution].

Table 2.1
Example 2: The maximum absolute errors $\left(p = \frac{\Delta y}{\Delta x} = 0.8\right)$.

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method		$O(\Delta y^2 + \Delta x^2)$-method	
	Eq. (41)	Eq. (42)	Eq. (41)	Eq. (42)
$\frac{1}{16}$	0.1970E−03	0.2183E−04	0.1963E−03	0.1147E−03
$\frac{1}{32}$	0.4870E−04	0.5668E−05	0.4865E−04	0.3034E−04
$\frac{1}{64}$	0.1212E−04	0.1405E−05	0.1231E−04	0.7789E−05
$\frac{1}{128}$	0.2975E−05	0.1322E−06	0.2968E−05	0.1762E−05

where $\beta > 0$ is a constant and magnitude of β determines the ratio of convection to diffusion. Substituting $B(x) = 1$, $D(x) = \beta$ and $g(x,y) = 0$ in the difference scheme (40) and simplifying, we obtain a difference scheme of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ accuracy for the solution of the convection–diffusion Eq. (43).

Table 2.2
Example 2: The maximum absolute errors ($\gamma = \frac{\Delta y}{\Delta x^2} = 20$).

Δx	Proposed $O(\Delta y^2 + \Delta y^2 \Delta x^2 + \Delta x^4)$-method		$O(\Delta y^4 + \Delta y^2 \Delta x^2 + \Delta x^4)$-method discussed in [10]	
	Eq. (41)	Eq. (42)	Eq. (41)	Eq. (42)
$\frac{1}{10}$	0.2976E−02	0.3574E−03	0.5018E−02	0.6662E−03
$\frac{1}{20}$	0.1917E−03	0.2343E−04	0.3072E−03	0.4155E−04
$\frac{1}{40}$	0.1202E−04	0.1448E−05	0.1911E−04	0.2614E−05

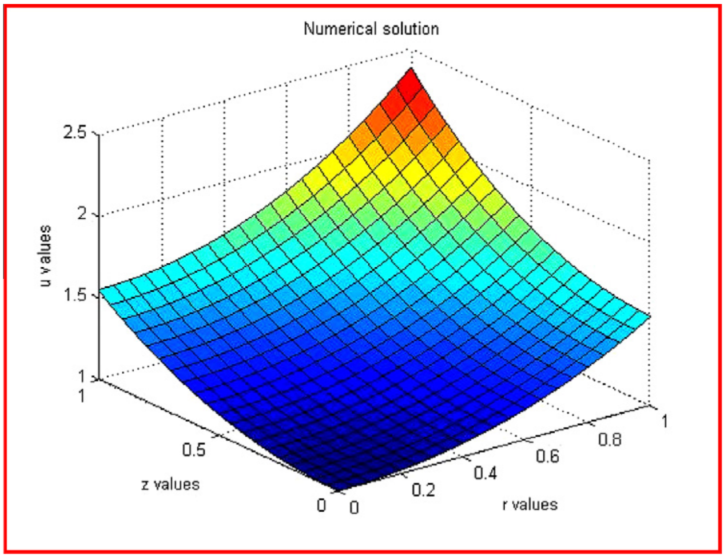

Fig. 4.1. Poisson's equation (r–z plane) $\gamma = 20$ [Numerical Solution].

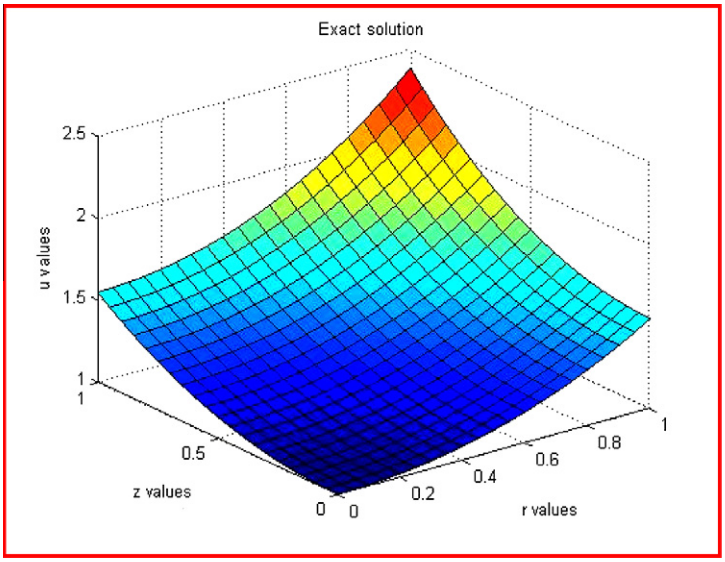

Fig. 4.2. Poisson's equation (r–z plane) $\gamma = 20$ [Exact Solution].

6. Numerical illustrations

Substituting the approximations (5.1), (5.4), (6.1), (6.4) in the differential Eq. (1), we obtain a central difference scheme of $O(\Delta y^2 + \Delta x^2)$ of the form

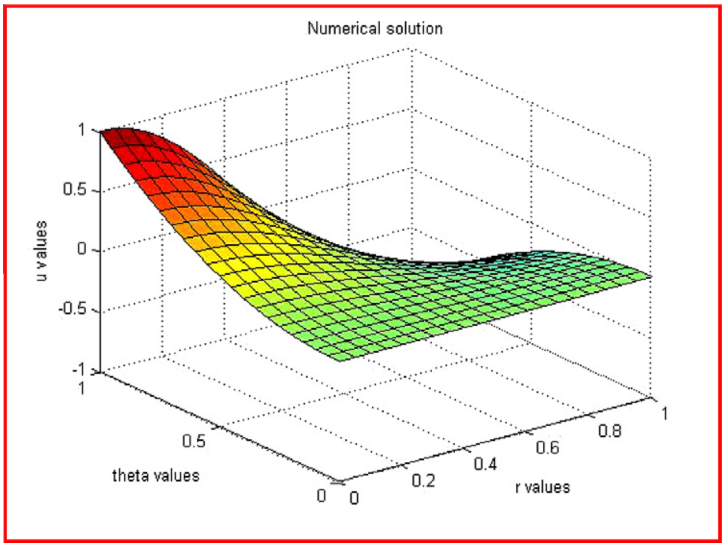

Fig. 5.1. Poisson's equation (r–θ plane) $\gamma = 20$ [Numerical Solution].

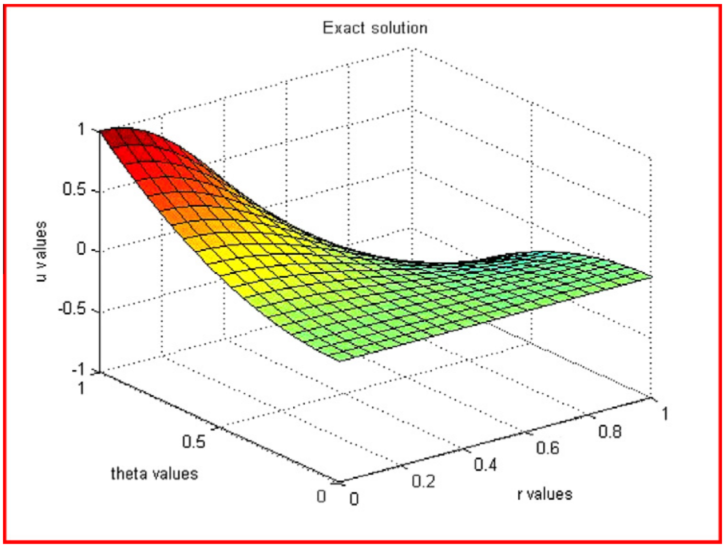

Fig. 5.2. Poisson's equation (r–θ plane) $\gamma = 20$ [Exact Solution].

Table 3.1
Example 3: The maximum absolute errors $(p = \frac{\Delta y}{\Delta x} = 0.8)$.

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method		$O(\Delta y^2 + \Delta x^2)$-method	
	$R_e = 50$	$R_e = 100$	$R_e = 50$	$R_e = 100$
$\frac{1}{32}$	0.1744E−03	0.1944E−03	Over Flow	Over Flow
$\frac{1}{64}$	0.4441E−04	0.4527E−04	0.4077E−04	Over Flow
$\frac{1}{128}$	0.1111E−04	0.1142E−04	0.9635E−05	0.9954E−05

$$A_{l,m}\overline{U}_{xxl,m} + B_{l,m}\overline{U}_{yyl,m} = f(x_l, y_m, U_{l,m}, \overline{U}_{xl,m}, \overline{U}_{yl,m}) + O(\Delta y^2 + \Delta x^2). \quad (44)$$

Numerical experiments are carried out to illustrate our method and to demonstrate computationally its convergence. We solve the following two dimensional elliptic boundary value problems on unequal mesh both on rectangular and cylindrical

Table 3.2
Example 3: The maximum absolute errors $\left(\gamma = \frac{\Delta y}{\Delta x^2} = 20\right)$.

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method		$O(\Delta y^4 + \Delta y^2\Delta x^2 + \Delta x^4)$-method discussed in [10]	
	$R_e = 10$	$R_e = 100$	$R_e = 10$	$R_e = 100$
$\frac{1}{10}$	0.1022E−01	0.8190E−02	0.4242E−01	0.1244E−01
$\frac{1}{20}$	0.5887E−03	0.7330E−03	0.2510E−02	0.7711E−03
$\frac{1}{40}$	0.3683E−04	0.4347E−04	0.1516E−03	0.4748E−04

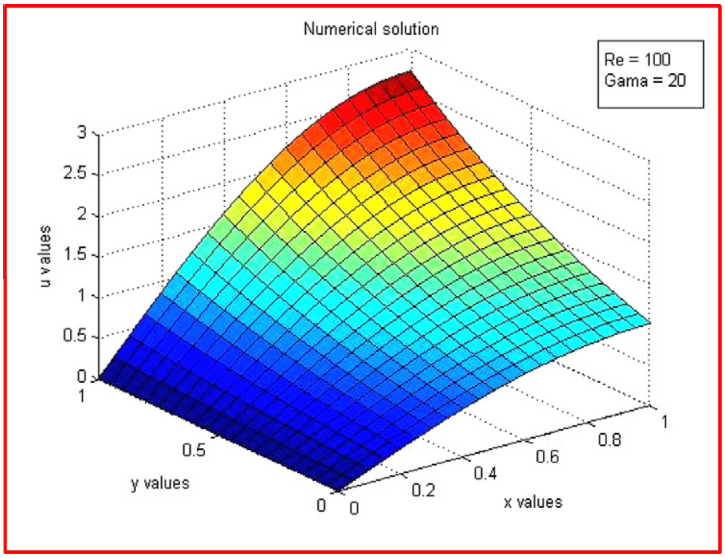

Fig. 6.1. Steady state Burger's equation $\gamma = 20$, $R_e = 100$ [Numerical Solution].

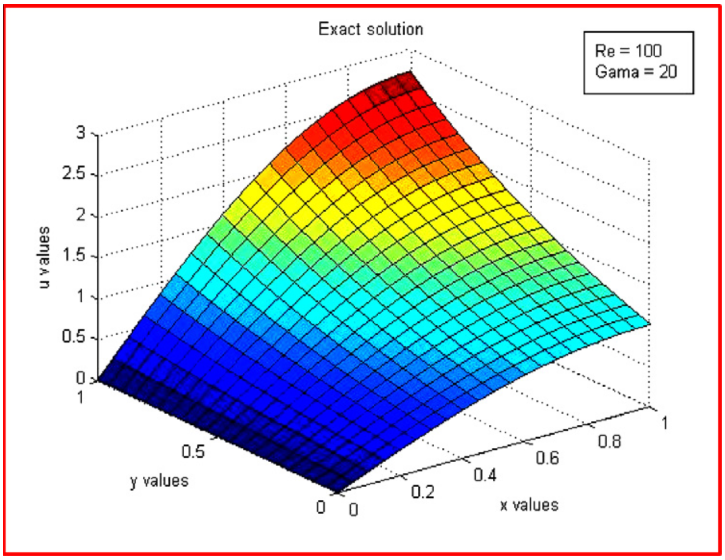

Fig. 6.2. Steady state Burger's equation $\gamma = 20$, $R_e = 100$[Exact Solution].

polar coordinates whose exact solutions are known to us. The Dirichlet boundary conditions can be obtained using the exact solutions as a test procedure. We also compare our method with the central difference scheme (44) and the methods discussed in [10] in terms of solution accuracy. In all cases, we have taken initial guess $\boldsymbol{u}(x_l, y_m) = \boldsymbol{0}$. The iterations were stopped when the absolute error tolerance became $\leqslant 10^{-10}$. All computations were carried out in double precision arithmetic using

MATLAB. Graphs depicting exact and numerical solutions for selected parameters for each of the problems discussed have been included (see Figs. 3.1–3.4).

Example 1 (*Convection–diffusion equation*). The problem is to solve (43) in the solution region $0 < x,\ y < 1$ whose exact solution is

$$u(x,y) = e^{\frac{\beta x}{2}}\frac{\sin\pi y}{\sinh\sigma}\left[2e^{\frac{\beta}{2}}\sinh\sigma x + \sinh\sigma(1-x)\right], \quad \text{where } \sigma^2 = \pi^2 + \frac{\beta^2}{4}.$$

The maximum absolute errors for u are tabulated in Tables 1.1 and 1.2. Figs. 3.1 and 3.2 demonstrate a comparison of the plots of the numerical and exact solution of $u(x,y)$ for the values $\beta = 30$ and $\gamma = (\Delta y/\Delta x^2) = 20$.

Example 2 (*Poisson's* equation in polar coordinates). The problems are to solve (41) and (42) in the solution regions $0 < r,\ \theta < 1$ and $0 < r,\ z < 1$, respectively. The exact solutions are given by $u(r,\theta) = r^2\cos\pi\theta$ and $u(r,z) = \cosh r \cosh z$, respectively. The maximum absolute errors for u are tabulated in Tables 2.1 and 2.2. A comparison of the plots of the numerical and exact solution of u for the value $\gamma = (\Delta y/\Delta x^2) = 20$ is shown in the Figs. 4.1, 4.2, 5.1 and 5.2.

Example 3 (*Steady* state Burgers' Model Equation).

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + e^x \sin\left(\frac{\pi y}{2}\right)\left[\varepsilon\left(1 - \frac{\pi^2}{4}\right) - e^x\left(\sin\left(\frac{\pi y}{2}\right) + \frac{\pi}{2}\cos\left(\frac{\pi y}{2}\right)\right)\right], \quad 0 < x, \quad y < 1, \tag{45}$$

where $R_e = \varepsilon^{-1} > 0$ is called Reynolds number. The exact solution is given by $u(x,y) = e^x \sin\left(\frac{\pi y}{2}\right)$. The maximum absolute errors for u are tabulated in Tables 3.1 and 3.2 for various values of R_e. Figs. 6.1 and 6.2 demonstrate a comparison of the plots of the numerical and exact solution of $u(x,y)$ for the values $R_e = 100$ and $\gamma = (\Delta y/\Delta x^2) = 20$.

Table 4.1
Example 4: The maximum absolute errors ($p = \frac{\Delta y}{\Delta x} = 0.8$).

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method			$O(\Delta y^2 + \Delta x^2)$-method		
	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$
$\frac{1}{16}$	0.2853E−04	0.4999E−04	0.7407E−04	0.6154E−04	0.1090E−03	0.1883E−03
$\frac{1}{32}$	0.8114E−05	0.1446E−04	0.2118E−04	0.1688E−04	0.2910E−04	0.4412E−04
$\frac{1}{64}$	0.2088E−05	0.3740E−05	0.5494E−05	0.4352E−05	0.7530E−05	0.1119E−04
$\frac{1}{128}$	0.4325E−06	0.8828E−06	0.1359E−05	0.9405E−06	0.1803E−05	0.2785E−05

Table 4.2
Example 4: The maximum absolute errors ($\gamma = \frac{\Delta y}{\Delta x^2} = 20$).

Δx	Proposed $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$-method			$O(\Delta y^4 + \Delta y^2\Delta x^2 + \Delta x^4)$-method discussed in [10]		
	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$
$\frac{1}{10}$	0.5014E−03	0.8912E−03	0.1161E−02	0.7264E−03	0.1102E−02	0.4141E−02
$\frac{1}{20}$	0.3248E−04	0.5791E−04	0.8272E−04	0.4510E−04	0.6864E−03	0.2577E−03
$\frac{1}{40}$	0.2017E−05	0.3607E−05	0.5223E−05	0.2792E−05	0.4281E−03	0.1612E−04

Table 5
Fourth order convergence: $\Delta x_1 = \frac{1}{20}, \Delta x_2 = \frac{1}{40}, \gamma = \frac{\Delta y}{\Delta x^2} = 20$.

Example	Parameters	Order of the method
01	$\beta = 10$	4.00
	$\beta = 20$	4.04
	$\beta = 30$	4.08
02	Eq. (41)	4.01
	Eq. (42)	4.00
03	$R_e = 10$	4.00
	$R_e = 100$	4.07
04	$\alpha = 5$	4.00
	$\alpha = 10$	4.00
	$\alpha = 20$	3.99

Example 4. Quasi-linear elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + (1+u^2)\frac{\partial^2 u}{\partial y^2} = \alpha u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + \exp(xy)\left[x^2 + y^2 + (\exp(xy))^2 x^2 - \alpha \exp(xy)(x+y)\right], \quad 0 < x, \quad y < 1 \tag{46}$$

The exact solution is $u(x,y) = \exp(xy)$. The maximum absolute errors for u are tabulated in Tables 4.1 and 4.2 for various values of α.

Finally, we present Table 5 which shows that our method works as a fourth order method with fixed parameter $\gamma = (\Delta y/\Delta x^2)$. The order of convergence may be obtained by using the formula

$$log\left(\frac{e_{\Delta x_1}}{e_{\Delta x_2}}\right)/log\left(\frac{\Delta x_1}{\Delta x_2}\right) \tag{47}$$

where $e_{\Delta x_1}$ and $e_{\Delta x_2}$ are maximum absolute errors for two uniform mesh widths Δx_1 and Δx_2, respectively. For computation of order of convergence of the proposed method, we have considered errors for last two values of Δx, i.e., $\Delta x_1 = \frac{1}{20}, \Delta x_2 = \frac{1}{40}$ for the above discussed elliptic partial differential equations.

7. Concluding remarks

Available numerical methods based on cubic spline approximations for the numerical solution of quasi-linear elliptic equations are of $O(\Delta y^2 + \Delta x^2)$ accurate. Although 9-point finite difference approximations of $O(\Delta y^4 + \Delta y^2\Delta x^2 + \Delta x^4)$ accurate for the solution of non-linear and quasi-linear elliptic differential equations are discussed in [10,29], but these methods require five evaluations of the function f. In this article, using the same number of grid points and three evaluations of the function f, we have derived a new stable cubic spline method of $O(\Delta y^2 + \Delta y^2\Delta x^2 + \Delta x^4)$ accuracy for the solution of quasi-linear elliptic Eq. (1). However, for a fixed parameter $\gamma = \frac{\Delta y}{\Delta x^2}$, the proposed method behaves like a fourth order method. The accuracy of the proposed method is exhibited from the computed results. Further, we have reported the numerical results for the solution of 2D nonlinear Burger's equation in Table 3.1 for $p = 0.8$. The stability of the method plays an important role for computation. In general, the stability condition for 2D nonlinear Burger's equation cannot be determined theoretically. During computation for nonlinear Burger's equation, we found that for $(\Delta x, R_e) = \left(\frac{1}{32}, 50\right), \left(\frac{1}{32}, 100\right)$ and $\left(\frac{1}{64}, 100\right)$, the $O(\Delta y^2 + \Delta x^2)$-method becomes unstable and errors overflow in these cases, whereas the proposed method is stable in these cases. In other cases, both the methods are stable. The proposed method is applicable to Poisson's equation in polar coordinates, and two dimensional Burgers' equation, which is main highlight of the proposed work.

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