Estimation of Volatility Using Open, Close, High, and Low Prices: Application to Stochastic Volatility Models and High-Frequency Data.

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Motivation

- Working with returns

$$r_{t+\Delta} \equiv Y_{t+\Delta} - Y_t = \log(S_{t+\Delta}/S_t) \sim N\left(\Delta\mu, \Delta\sigma^2\right).$$

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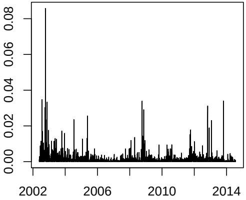

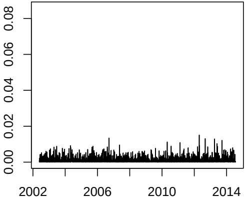

Outline

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Models for Time-Dependent Volatility

1. ARCH/GARCH:

$$\begin{aligned} r_t &= \mu + \epsilon_t & \epsilon_t &\sim N(0, \sigma_t^2), \\ \sigma_t^2 &= \omega + \alpha\sigma_{t-1}^2 + \beta\epsilon_{t-1}^2 \end{aligned}$$

2. Time-changed Brownian Motion: BM where the time of the process is scaled

$$Y_t = W_{\tau_t},\ \ t \geq 0; \quad Y_t|\tau_t \sim N(0, \tau_t)$$

3. Stochastic volatility:

$$\begin{aligned} Y_t &= \mu + Y_{t-1} + \epsilon_t^1 & \epsilon_t^1 &\sim N(0, \sigma_t^2), \\ \log(\sigma_t) &= \alpha + \phi\left[\log(\sigma_{t-1}) - \alpha\right] + \epsilon_t^2 & \epsilon_t^2 &\sim N(0, \tau^2) \end{aligned}$$

Stochastic Volatility Models

- General case

$$\begin{aligned} dY_t &= \mu(Y_t, \sigma_t)dt + \nu(Y_t, \sigma_t)dW_{t,1} \\ d\sigma_t &= \alpha(Y_t, \sigma_t)dt + \beta(Y_t, \sigma_t)dW_{t,2} \end{aligned}$$

- Continuous time OU

$$\begin{aligned} dY_t &= \mu dt + \sigma_t dW_{t,1} \\ d\log(\sigma_t) &= \underbrace{\phi(\alpha - \log(\sigma_t))dt + \omega dW_{t,2}}_{\text{mean-reverting stochastic process}} \end{aligned}$$

- Discrete time OU

$$\begin{aligned} Y_t &= \mu + Y_{t-1} + \epsilon_t^1 & \epsilon_t^1 &\sim N(0, \sigma_t^2) \\ \log(\sigma_t) &= \alpha + \phi\,[\log(\sigma_{t-1}) - \alpha] + \epsilon_t^2 & \epsilon_t^2 &\sim N(0, \tau^2) \end{aligned}$$

High-Frequency Data and Microstructure Noise

- Availability of tick-by-tick/bid-ask financial transaction data has motivated the development of models that utilize this information to better estimate asset volatility.
- A model-free estimation approach: Realized Volatility

$$RV^{(m)} = \sum_{i=1}^{m} r_i^{(m)^2}$$

It can be shown that $RV^{(m)} \xrightarrow{p} \int_t^{t+\Delta} \sigma^2(Y_s, s)ds$ as $m \to \infty$.

- Problem: microstructure noise on the microsecond level

Outline

Motivation

- GARCH/SV models have been adapted to high-frequency data using RV: subsampling and using RV to summarize intraperiod data.
- We focus on an approach which collapses high-frequency intraperiod transaction information in the form of recorded maximum and minimum prices.

Likelihood for OCHL Data

- We assume constant μ and σ over an interval $s \in [t-1, t]$ with intraperiod min/max a_t and b_t, with BM for the log-price

$$
\begin{aligned}
dY_s &= \mu ds + \sigma dW_s, \\
a_t < \; Y_s &< b_t.
\end{aligned}
$$

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- Using the Fokker-Planck equation,

$$\begin{aligned} \frac{\partial}{\partial s} q(y,s) &= -\mu \frac{\partial}{\partial y} q(y,s) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} q(y,s), \\ q(y,t-1) &= \delta(y - y_{t-1}); \quad q(a_t,s) = 0, q(b_t,s) = 0 \end{aligned}$$

$$\boxed{q(y,t) = p(m_t \geq a_t, M_t \leq b_t, Y_t = y | Y_{t-1} = y_{t-1})}$$

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- Likelihood for OCHL data

$$\boxed{-\frac{\partial^2}{\partial a_t \partial b_t} q(y,t) = p(m_t = a_t, M_t = b_t, Y_t = y | Y_{t-1} = y_{t-1})}$$

Solution by Fourier Expansion

$$q(y,t) = \exp\left(\frac{\sqrt{2\mu}}{\sigma}y\right) \times$$
$$\sum_{n=1}^{\infty} \frac{2}{L} \sin\left((y_{t-1} - a_t)\frac{n\pi}{L}\right) \exp\left(-\frac{1}{2}\sigma^2 \frac{n^2\pi^2}{L^2} t\right) \sin\left((y - a_t)\frac{n\pi}{L}\right)$$

Solution by Method of Images

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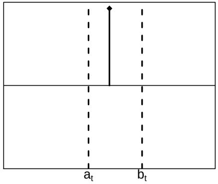

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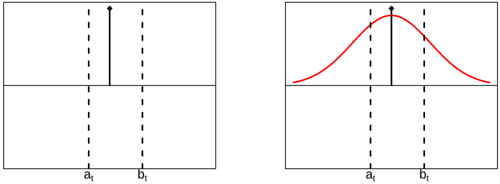

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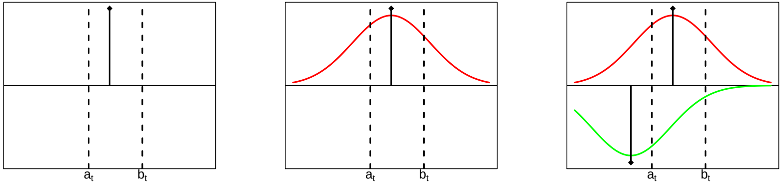

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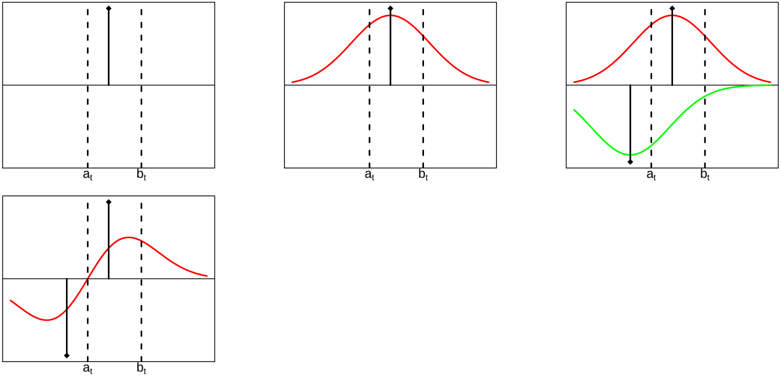

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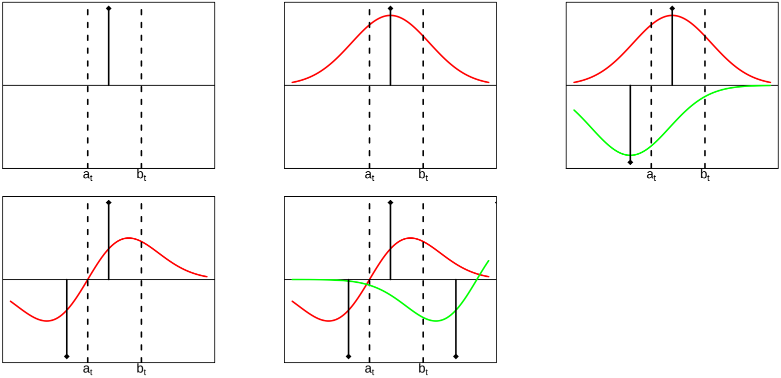

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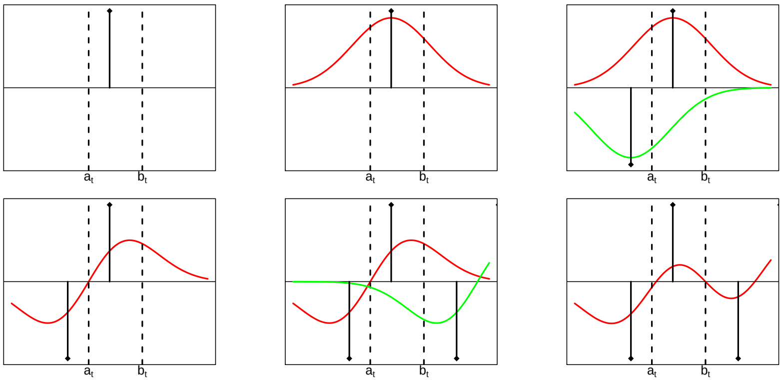

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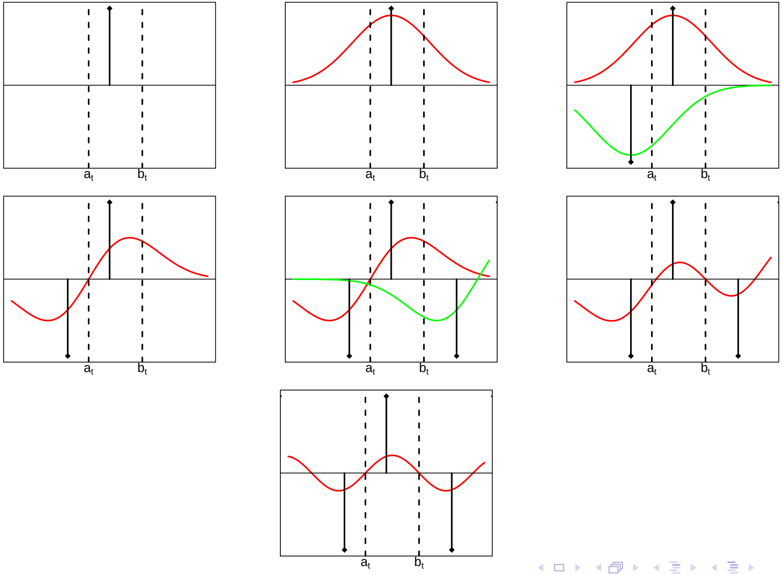


Accuracy of the Fourier Expansion Solution

- Numerical differentiation: central-difference method

$$\Delta x = \frac{1}{2^k}\frac{1}{100}(b_t - a_t).$$

- Analytic differentiation

Accuracy of the Fourier Expansion Solution

- Numerical differentiation: central-difference method

$$\Delta x = \frac{1}{2^k}\frac{1}{100}(b_t - a_t).$$

- Analytic differentiation

Relative error

	N=4	N=8	N=16	N=32	N=64
k=4	-1.290e-3	1.810e-5	1.818e-5	1.818e-5	1.818e-5
k=8	-1.308e-3	7.110e-8	7.111e-8	7.111e-8	7.111e-8
analytic	-1308e-3	-2.002e-14	0	0	0

(N = number of summands in the truncated expansion;
$\mu = 1, \sigma = 1,\ a_t = -0.0557,\ b_t = 3.1424,\ y_t = 3.019,\ y_{t-1} = 0.$)

Simulation

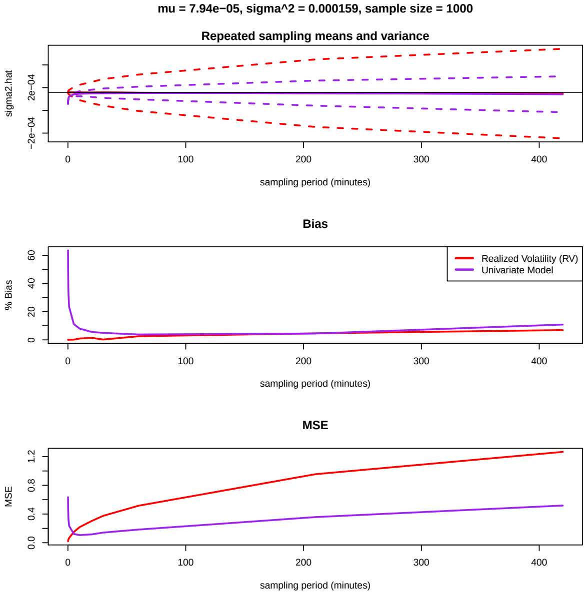
- Goal: compare OCHL estimators to RV estimators
- Forward-Euler discretization

$$Y_{k+1} = Y_k + \mu \Delta t^* + \sigma \Delta t^* \epsilon_k$$

- $\Delta t^* = 1/23400$, equivalent to sampling the returns once every second over the length of a trading day.

$$\mu = 7.936508 \cdot 10^{-5}, \qquad \sigma = 0.01259882$$

- When performing estimation, longer sampling intervals are used: 5 sec to 7 hours.
- No microstructure noise
- 1000 trading days: 1000 estimates for each sampling period Δt used.


mu = 7.94e−05, sigma^2 = 0.000159, sample size = 1000
Repeated sampling means and variance
sigma2.hat
2e−04
−2e−04
0
100
200
300
400
sampling period (minutes)
Bias
% Bias
60
40
20
0
Realized Volatility (RV)
Univariate Model
sampling period (minutes)
MSE
1.2
0.8
0.4
0.0
sampling period (minutes)

Outline

Bivariate Model

- We expect that using OCHL data for two assets can improve estimates for drift and volatility, as well as the estimate of their correlation.

$$\left.\begin{array}{rcl} dY_{1,s} & = & \mu_1 ds + \sigma_1 dW_{1,s}, \\ dY_{2,s} & = & \mu_2 ds + \sigma_2 dW_{2,s} \end{array}\right\} Cor(W_1, W_2) = \rho$$

- Through Fokker-Planck equation, with $\mathbf{y} = (y_1, y_2)$,

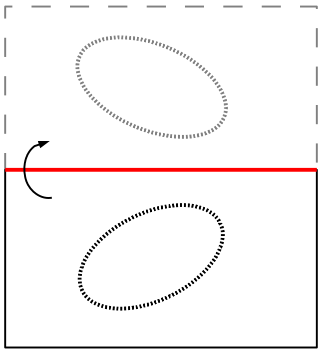
$$\frac{\partial}{\partial s} q(\mathbf{y}, s) = -\mu_1 \frac{\partial}{\partial y_1} q(\mathbf{y}, s) - \mu_1 \frac{\partial}{\partial y_2} q(\mathbf{y}, s)$$

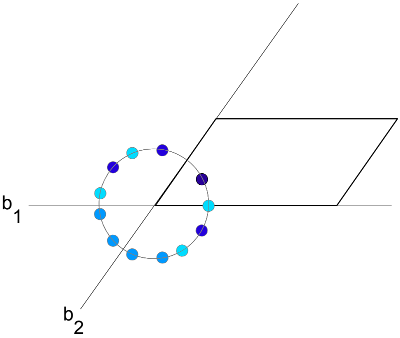
$$+\frac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial y_1^2} q(\mathbf{y}, s) + \frac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial^2 y_2} q(\mathbf{y}, s) + \rho\sigma_1\sigma_2 \frac{\partial^2}{\partial y_1 \partial y_2} q(\mathbf{y}, s)$$

$$\begin{array}{rcl} q(\mathbf{y}, t-1) & = & \delta(y_1 - y_{1,t-1})\delta(y_2 - y_{2,t-1}), \\ q(\mathbf{y}, s) & = & 0 \quad \text{for} \quad y_1 = a_{1,t}, b_{1,t} \quad \text{or} \quad y_2 = a_{2,t}, b_{2,t}. \end{array}$$

Solution through Method of Images

Method of Images does not work.





Solution through Fourier Expansion

$$
\begin{aligned}
q(\mathbf{y}, s) &= \exp(\mathbf{a}^T\mathbf{y} + bs)\times \\
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}(s) \sin\left(\frac{m\pi(y_1 - a_{1,t})}{b_{1,t} - a_{1,t}}\right) \sin\left(\frac{n\pi(y_2 - a_{2,t})}{b_{2,t} - a_{2,t}}\right) \\
&= \exp(\mathbf{a}^T\mathbf{y} + bs)\left(\mathbf{C}(s)^T\mathbf{S}\right)
\end{aligned}
$$

$$
\frac{d}{ds}\begin{pmatrix} C_{1,1}(s) \\ \vdots \\ C_{1,N}(s) \\ C_{2,1}(s) \\ \vdots \\ C_{M,N}(s) \end{pmatrix} = \underbrace{\frac{1}{2}\sigma_1^2\mathbf{A}_1 + \frac{1}{2}\sigma_2^2\mathbf{A}_2 + \rho\sigma_1\sigma_2\mathbf{B}}_{\mathbf{A}} \begin{pmatrix} C_{1,1}(s) \\ \vdots \\ C_{1,N}(s) \\ C_{2,1}(s) \\ \vdots \\ C_{M,N}(s) \end{pmatrix}
$$

Evaluating the Likelihood

$$\frac{\partial^4 q(\mathbf{y}, t)}{\partial a_{1,t} \partial b_{1,t} \partial a_{2,t} \partial b_{2,t}} =$$

$$p(m_{1,t} = a_{1,t}, M_{1,t} = b_{1,t}, m_{2,t} = a_{2,t}, M_{2,t} = b_{2,t}, \mathbf{Y}_t = \mathbf{y} | \mathbf{Y}_{t-1} = \mathbf{y}_{t-1})$$

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- Numerical differentiation

$$\frac{\partial^4 q(\mathbf{y}, t)}{\partial a_{1,t} \partial b_{1,t} \partial a_{2,t} \partial b_{2,t}} \approx \frac{\text{finite difference} + \epsilon_{\text{machine}}}{(\Delta x)^4}$$

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$$\frac{\partial^4 q(\mathbf{y},t)}{\partial a_{1,t}\partial b_{1,t}\partial a_{2,t}\partial b_{2,t}} =$$

$$p(m_{1,t}=a_{1,t}, M_{1,t}=b_{1,t}, m_{2,t}=a_{2,t}, M_{2,t}=b_{2,t}, \mathbf{Y}_t=\mathbf{y}|\mathbf{Y}_{t-1}=\mathbf{y}_{t-1})$$

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- Mixed differentiation

$$\frac{\partial^4 q(\mathbf{y},t)}{\partial a_{1,t}\partial b_{1,t}\partial a_{2,t}\partial b_{2,t}} = \exp(\mathbf{a}^T\mathbf{y}+bs)\frac{\partial^4 (\overbrace{\boxed{e^{\mathbf{A}s}\mathbf{C}(0)}}^{\mathbf{C}(s)^T}{}^T\,\mathbf{S})}{\partial a_{1,t}\partial b_{1,t}\partial a_{2,t}\partial b_{2,t}}$$

Numerical Results

- $\Delta x = \min\left\{\frac{1}{2^k}\frac{1}{100}(b_{1,t} - a_{1,t}), \frac{1}{2^k}\frac{1}{100}(b_{2,t} - a_{2,t})\right\}$
- $N = M$ in Fourier expansion
- $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

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- $N = M$ in Fourier expansion
- $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$.

Log-likelihood for $\rho = 0.3$

	N=4	N=8	N=16	N=32	N=64
k=3	-1.8300	-1.7475	-1.7601	-1.7416	-1.6371
k=6	-1.8300	-1.7475	-1.7601	-1.7421	-1.6453
k=8	-1.8564	-1.7292	-1.8217	-1.4718	$-\infty$
mixed, k=3	-1.8300	-1.7475	-1.7601	-1.7416	-1.6371
mixed, k=6	-1.8300	-1.7475	-1.7601	-1.7421	-1.6453
mixed, k=8	-1.8300	-1.7475	-1.7601	-1.7416	-1.6365

$(a_{1,t} = -0.7906,\ b_{1,t} = 0.7947,\ y_{1,t} = -0.3946,\ a_{2,t} = -0.02775,\ b_{2,t} = 1.2675,\ y_{2,t} = 1.06421)$

Numerical Results

Log-likelihood for $\rho = 0.5$

	N=4	N=8	N=16	N=32	N=64
k=3	–3.3405	-3.4384	-3.8774	-3.9335	-3.8991
k=6	-3.3409	-3.4354	-3.9105	-3.9596	-3.7720
k=8	-3.3811	-2.9068	-5.4378	$-\infty$	1.0481
mixed, k=3	-3.3405	-3.4384	-3.8776	-3.9338	-3.8981
mixed, k=6	-3.3405	-3.4384	-3.8778	-3.9337	-3.8980
mixed, k=8	-3.3405	-3.4384	-3.8778	-3.9337	-3.8980

$(b_{1,t} = 0.5524,\ a_{1,t} = -0.7481,\ y_{1,t} = -0.3282,\ b_{2,t} = 0.01645,$
$a_{2,t} = -2.7176,\ y_{2,t} = -2.1075\ )$

Numerical Results

Log-likelihood for $\rho = 0$

	N=4	N=8	N=16	N=32	N=64
k=8	-2.1731	-2.1731	-2.1731	-2.1731	-2.1731
mixed, k=8	-2.1731	-2.1731	-2.1731	-2.1731	-2.1731

$(b_{1,t} = 1.5059,\ a_{1,t} = -0.1894,\ y_{1,t} = 1.4004,\ b_{2,t} = 0.5387,\ a_{2,t} = -0.2317,\ y_{2,t} = 0.2569)$

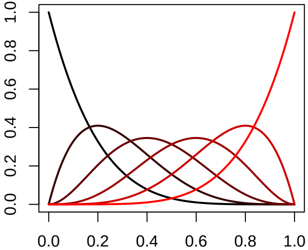
Outline

Bernstein Polynomials

- Sparser matrix

$$B_{\nu,n}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu},$$

$$\nu = 0, \ldots, n$$



$$q(\mathbf{y}, s) = \sum_{n=1}^{N} \sum_{m=1}^{M} C_{m,n}(s) B_{n,N}(y_1) B_{m,M}(y_2).$$

- Accounting for correlation

$$q(\mathbf{y}, s) = \sum_{n=1}^{N} \sum_{m=1}^{M} C_{m,n}(s) \mathcal{K}\Big(B_{n,N}(y_1), B_{m,M}(y_2) \Big).$$

- Other bases?

Bivariate Stochastic Volatility Model Using OCHL data

$$\begin{pmatrix} \log(S_{t+1,1}) \\ \log(S_{t+1,2}) \end{pmatrix} = \begin{pmatrix} \log(S_{t,1}) \\ \log(S_{t,2}) \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \mathbf{V_{t+1}} \begin{pmatrix} w_{t+1,1} \\ w_{t+1,2} \end{pmatrix}$$

$$\mathbf{V}_t = \begin{pmatrix} V_{t,11} & 0 \\ V_{t,21} & V_{t,22} \end{pmatrix}, \qquad \mathbf{\Sigma}_t = \mathbf{V}_t' \mathbf{V}_t.$$

$$\begin{pmatrix} \log(V_{t+1,11}) \\ \log(V_{t+1,22}) \\ V_{t+1,21} \end{pmatrix} = \begin{pmatrix} \mu_{11} \\ \mu_{22} \\ \mu_{21} \end{pmatrix} + \mathbf{\Psi} \left(\begin{pmatrix} \log(V_{t,11}) \\ \log(V_{t,22}) \\ V_{t,21} \end{pmatrix} - \begin{pmatrix} \mu_{11} \\ \mu_{22} \\ \mu_{21} \end{pmatrix} \right) + \boldsymbol{\nu},$$

$$\boldsymbol{\nu} \sim N_3(\mathbf{0}, \mathbf{\Sigma}_\nu)$$

Stochastic Volatility Estimation with Microstructure Noise

$$Y_{t+1} = \log(S_{t+1}) + \xi\gamma_t$$

$$\boxed{\begin{aligned} \log(S_{t+1}) &= \log(S_t) + \mu + \sigma_{t+1}\epsilon_{t+1,1} \\ \log(\sigma_{t+1}) &= \alpha + \phi[\log(\sigma_t) - \alpha] + \beta\epsilon_{t+2,2}. \end{aligned}}$$

- $\gamma_t \sim N(0,1)$, $\xi \approx D/2Q$.
- $(\epsilon_{t+1,1}, \epsilon_{t+1,2})$ follows a bivariate normal distribution with $E(\epsilon_{t,1}) = E(\epsilon_{t,2}) = 0$, $\text{Var}(\epsilon_{t,1}) = \text{Var}(\epsilon_{t,2}) = 1$ and $\text{Cor}(\epsilon_{t,1}, \epsilon_{t,2}) = \rho$.

Univariate Likelihood in the Presence of Microstructure Noise

- Our goal is to derive the likelihood for the open, close, high, and low **observed** prices within a period.
- **Observed** intraperiod maximum and minimum are not the **true** maximum and minimum taken on by the asset.
- Previous work is not directly applicable in this setting.
- Brownian motion with jumps?

Timeline for Future Work

	Fall 2014	Winter 2015	Spring 2015	Summer 2015	Fall 2015	Winter 2016
Stochastic Volatility with microstructure noise:						
Univariate Likelihood with microstructure noise:						
Bivariate Likelihood with Bernstein polynomials:						
Bivariate Stochastic Volatility model:						
Thesis writing:						

Questions

Microstructure Variance Specification

- Observe price P_t is the true price S_t plus noise

$$P_t = S_t + \xi^* \epsilon_t, \quad \xi^* = D/2$$

- The log of the observed price is approximated with a first-order Taylor expansion and with average price Q

$$\begin{aligned} \log(P_t) &= \log(S_t + \xi^* \epsilon_t) \\ &\approx \log(S_t) + \frac{\xi^*}{S_t}\epsilon_t \\ &\approx \log(S_t) + \frac{\xi^*}{Q}\epsilon_t \end{aligned}$$

- Hence, the standard deviation in the SV model is approximated with

$$\boxed{\xi = D/2Q}$$

Analytic Matrix Derivatives

- "Matrix" Cauchy Integral

$$f(H) = \frac{1}{2\pi i} \oint f(z)(\mathbf{I}z - \mathbf{H})^{-1} dz$$

- Differentiation with respect to some parameter μ

$$\frac{\partial}{\partial \mu} f(H) = \frac{1}{2\pi i} \oint f(z)(\mathbf{I}z - \mathbf{H})^{-1} \frac{\partial \mathbf{H}}{\partial \mu} (\mathbf{I}z - \mathbf{H})^{-1} dz$$

- Second-order derivative follow

$$\frac{\partial^2}{\partial \mu \partial \nu} f(H) =$$

$$\frac{1}{2\pi i} \oint f(z) \frac{\partial}{\partial \nu} \left[(\mathbf{I}z - \mathbf{H})^{-1} \frac{\partial \mathbf{H}}{\partial \mu} (\mathbf{I}z - \mathbf{H})^{-1} \right] dz$$