The Axiom of Choice Can Fail in the $<\kappa$-Mantle

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Abstract

For a given cardinal κ, the $<\kappa$-mantle is the intersection of all grounds that extend to the universe via a forcing of size $<\kappa$. We give some examples in which the κ-mantle fails to satisfy the axiom of choice, in one of these κ is Mahlo. It is known that this is impossible for κ measurable. This answers a question of Usuba.

0 Introduction

The interest of Set-Theoretic Geology is the study of the structure of grounds, that is inner models of ZFC that extend to V via forcing, and associated concepts. In an effort to get rid of generic sets, the mantle was born.

Definition 0.1. The mantle, denoted $\mathbb{M}$, is the intersection of all grounds.

This definition makes use of the fact that all grounds are uniformly definable.

Fact 0.2. *[FHR15] There is a first order $\in$-formula $\varphi(x, y)$ such that*

$$W_r = \{x|\varphi(x, r)\}$$

defines a ground for all $r \in V$ and all grounds are of this form. Moreover, if κ is a cardinal and W extends to V via a forcing of size $<\kappa$ then there is $r \in V_\kappa$ with $W = W_r$.

This allows as to quantify freely over grounds as we will frequently do. Usuba has shown in [Usu17] that $\mathbb{M}$ always is a model of ZFC. Fuchs, Hamkins and Reitz suggested in [FHR15] to study restricted forms of the mantle.

Definition 0.3. For a class Γ of forcings, the Γ-mantle $\mathbb{M}_\Gamma$ is the intersection of all grounds W that extend to V via a forcing $\mathbb{P}$ with $W \models \mathbb{P} \in \Gamma$.

Fuchs, Hamkins and Reitz demonstrated that the σ-closed mantle need not be a model of ZFC.

Fact 0.4. *[FHR15] If Γ is the class of all σ-closed forcings it is consistent that $\mathbb{M}_\Gamma \models ZF \wedge \neg AC$.*

In this note we investigate $\mathbb{M}_\Gamma$ for Γ the class of all forcings of size $<\kappa$. In this case, we denote the Γ-mantle by $\mathbb{M}_{<\kappa}$ and call it the *$<\kappa$-mantle*. The associated grounds are the *$<\kappa$-grounds*. The interest of the $<\kappa$-mantle arose in different contexts. On the one side, Usuba has shown:

Fact 0.5. *[Usu18] If κ is extendible then $\mathbb{M}_\kappa = \mathbb{M}$. In particular $\mathbb{M}_\kappa$ is a model of ZFC.*

A similar situation was found to be the case for the least iterable inner model with a strong cardinal above a Woodin cardinal for κ the unique strong cardinal in this universe, see [SS18].

The following is known:

Fact 0.6. *If κ is a strong limit then $\mathbb{M}_\kappa \models ZF$.*

A sketch of a proof can be found in [Usu18].

Remark 0.7. For any strong limit κ we have

$$\mathbb{M}_\kappa \models AC \Leftrightarrow \mathbb{M}_\kappa \text{ is a ground}$$

The implication from right to left is by definition and the one from left to right follows from Grigorieff's theorem since $\mathbb{M}_\kappa$ always contains a ground by results of Usuba in [Usu17].

Schindler has proved the following:

Theorem 0.8. *[Sch18] If κ is measurable then $\mathbb{M}_\kappa \models ZFC$.*

The big difference to Fact 0.5 is that the existence of a measurable is consistent with the failure of the bedrock axiom, that is "$\mathbb{M}$ is not a ground". Particularly, we might have $\mathbb{M}_\kappa \neq \mathbb{M}$ for κ measurable.

We also argue that "κ is measurable" cannot be replaced by "κ is Mahlo".

Theorem 0.9. *It is consistent that κ is Mahlo, but $\mathbb{M}_\kappa$ fails to satisfy the axiom of choice.*

This answers a question of Usuba, namely whether or not the κ-mantle always is a model of ZFC, see [Usu18, Question 2.7].

1 Choice May Fail in $\mathbb{M}_\kappa$

Here, we will construct a model where the $<\kappa$-mantle of a Mahlo cardinal κ does not satisfy the axiom of choice. We will start with L and assume that κ is Mahlo there. The final model will be a forcing extension of L by

$$\mathbb{P} = \prod_{\lambda \in I \cap \kappa}^{<\kappa-\text{support}} Add(\lambda, 1)$$

where I is the class of all inaccessible cardinals. Notice that $\mathbb{P}$ is a product forcing and not an iteration (in the usual sense), as we want to generate many $<\kappa$-grounds. Let G be $\mathbb{P}$-generic over L. We will show that κ is still Mahlo in $L[G]$ and that $\mathbb{M}_\kappa^{L[G]}$ does not satisfy the axiom of choice. We remark that, would we start with a model in which κ is measurable, $\mathbb{P}$ would provably force κ to not be measurable.

First, let's fix notation. For $\lambda < \kappa$, we may factor $\mathbb{P}$ as $\mathbb{P}_{\leqslant\lambda} \times \mathbb{P}_{>\lambda}$ where in each case we only take a product over all $\gamma \in I \cap \kappa$ with $\gamma \leqslant \lambda$ and $\gamma > \lambda$ respectively. Observe that $\mathbb{P}_{>\lambda}$ has $<\kappa$-support while $\mathbb{P}_{\leqslant\lambda}$ has full support. We also factor G as $G_{\leqslant\lambda} \times G_{>\lambda}$ accordingly. For $\lambda \in I \cap \kappa$ we denote the generic for $Add(\lambda, 1)^L$ induced by G as g_λ. In addition to this, for $\alpha \leqslant \kappa$ we will write I_α for the α-th inaccessible cardinal.
For $\alpha < \kappa$ let $F_\alpha : I_\alpha \to 2$ denote the function induced by g_{I_α}. It will be convenient to think of G as a $\kappa \times \kappa$-matrix F where

$$F(\alpha, \beta) = \begin{cases} F_\alpha(\beta) & \text{if } \beta < I_\alpha \\ 0 & \text{else} \end{cases}$$

Note that F is an upper left triangle matrix.

Lemma 1.1. *I is absolute between L and $L[G]$.*

Proof. First we show that all limit cardinals of L are limit cardinals in $L[G]$. It is enough to prove that all double successors δ^{++} are preserved. This is obvious for $\delta \geqslant \kappa$ as $\mathbb{P}$ has size κ. For $\delta < \kappa$, $\mathbb{P}_{>\delta}$ is $\leqslant\delta^{++}$-closed so that all cardinals $\leqslant \delta^{++}$ are preserved in $L[G_{>\delta}]$. Furthermore, $\mathbb{P}_{<\delta}$ has size at most δ^+ in $L[G_{>\delta}]$ by GCH in L. Hence δ^{++} is still a cardinal in $L[G]$.

Now we have to argue that all $\lambda \in I$ remain regular. Again, this is clear for $\lambda > \kappa$. On the other hand, assume $\delta := \mathrm{cof}(\lambda)^{L[G]} < \lambda$. As $\mathbb{P}_{>\delta}$ is δ-closed, a witnessing cofinal sequence must already be in $L[G_{\leqslant\delta}]$. But $\mathbb{P}_{\leqslant\delta}$ has size $< \lambda$ in L and thus could not have added this sequence. $\square$

In fact, $\mathbb{P}$ does not collapse any cardinals, but some more work is required to prove this. This is, however, not relevant to the argument here.

Lemma 1.2. *κ is Mahlo in $L[G]$.*

Proof. Suppose $\dot{C}$ is a $\mathbb{P}$-name for a club in κ and $p \in \mathbb{P}$. We will find $p_* \leqslant p$ and an inaccessible λ so that $p_* \Vdash \check{\lambda} \in \dot{C}$. As κ is Mahlo, we can find $X \prec H_{\kappa^+}$, $\mathbb{P}, \kappa \in X$ and $X \cap \kappa \in I \cap \kappa$, say $X \cap \kappa = \lambda$. Let $\pi : M \to H_{\kappa^+}$ be the inverse collapse map. Then $crit(\pi) = \lambda$ and $\pi(\lambda) = \kappa$. By standard Löwenheim-Skolem arguments, we can furthermore assume that ${}^{<\lambda}M \subseteq M$. Let $\bar{\mathbb{P}}$ be the preimage of $\mathbb{P}$ and $\dot{D}$ the preimage of $\dot{C}$ under π. Observe that $\bar{\mathbb{P}}_{\leqslant\gamma} = \mathbb{P}_{\leqslant\gamma}$ for all $\gamma < \lambda$. We have

$$M \models p \Vdash_{\bar{\mathbb{P}}} \dot{D} \text{ is a club in } \check{\lambda}$$

Claim 1.3. *For any $\gamma < \lambda$ and $q \in \mathbb{P}_{\leqslant\gamma}$, there is $\gamma < \eta < \lambda$ and $r \in \bar{\mathbb{P}}$ with $r \upharpoonright (\gamma+1) = q$ so that*

$$M \models r \Vdash_{\bar{\mathbb{P}}} \check{\eta} \in \dot{D}$$

Proof. Let $\langle q_i | i < \delta\rangle \in M$ be an enumeration of all conditions in $\mathbb{P}_{\leqslant\gamma}$ that are below q such that any condition appears cofinally often with $|\delta| = |\mathbb{P}_{\leqslant\gamma}|$. Note that $\bar{\mathbb{P}}_{>\gamma}$ is $\leqslant\delta$-closed, as the next forcing only appears at the next inaccessible. We construct a sequence

$$\langle r_i | i < \delta\rangle$$

of conditions in $\bar{\mathbb{P}}_{>\gamma}$ and a sequence

$$\langle \gamma_i | i < \delta\rangle$$

of ordinals. Given $i < \delta$, choose some $r_i \leqslant r_j$ for all $j < i$, $r_i \in \bar{\mathbb{P}}_{>\gamma}$ and some $\sup_{j<i}(\gamma_j \cup \gamma) < \gamma_i < \lambda$ such that

$$M \models (q_i^\frown r_i) \Vdash_{\bar{\mathbb{P}}} \check{\gamma}_i \in \dot{D}$$

if there is such a condition. Otherwise take $r_i = \mathbb{1}$ and $\gamma_i = \gamma$.
Let $\eta = \sup\{\gamma_i | i < \delta\} < \lambda$ and find some r with $r \leqslant q^\frown r_i$ for all $i < \delta$ and $r \upharpoonright (\gamma+1) = q$. We have:

$$M \models r \Vdash_{\bar{\mathbb{P}}} \check{\eta} \in \dot{D}$$

To see this, assume $r' \leqslant r$ and M believes that r' forces the opposite. We can furthermore assume that there is $\gamma \leqslant \beta < \eta$ such that

$$M \models r' \Vdash_{\bar{\mathbb{P}}} \dot{D} \cap [\check{\beta}, \check{\eta}) = \varnothing$$

and moreover that there is some $\alpha > \eta$ so that

$$M \models r' \Vdash_{\bar{\mathbb{P}}} \check{\alpha} \in \dot{D}$$

We can find some $i < \delta$ with $\beta \leqslant \gamma_i$. Since every condition in $\mathbb{P}_{\leqslant\gamma}$ appears cofinally often in the chosen enumeration, there is some $j \geqslant i$ with

$$q_j = r' \upharpoonright (\gamma+1)$$

Let us write $r' = q_j^\frown r''$. It follows from the construction that

$$M \models q_j^\frown r_j \Vdash_{\bar{\mathbb{P}}} \check{\gamma}_j \in \dot{D}$$

and $\gamma_i \leqslant \gamma_j$. But $r' = q_j^\frown r'' \leqslant q_j^\frown r_j$. This is a contradiction. $\square$

There is a decreasing sequence $\langle p_i | i < \lambda\rangle$ of conditions in $\bar{\mathbb{P}}$ and a cofinal increasing sequence $\langle \lambda_i | i < \lambda\rangle$ in λ with the following properties:

(*i*) $p_0 = p \in \mathbb{P}_{\leqslant \lambda_0}$

(*ii*) $\langle p_i | i < j \rangle \in M$ for all $j < \lambda$

(*iii*) for all $i < \lambda$, $p_{i+1} \upharpoonright \lambda_{2i} = p_i$

(*iv*) for all $i < \lambda$, $M \models p_{i+1} \Vdash_{\bar{P}} \check{\lambda}_{2i+1} \in \dot{D}$

The construction is immediate from the $<\lambda$-closure of M and the above claim. This gives rise to a condition

$$p_* = \bigcup_{i<\lambda} p_i \in \mathbb{P}$$

The proof is complete if we show $p_* \Vdash_{\mathbb{P}} \check{\lambda} \in \dot{C}$. For any $\beta < \lambda$, there is some $i < \lambda$ with $\lambda_{2i+1} \geqslant \beta$. Now we have

$$M \models p_{i+1} \Vdash_{\bar{\mathbb{P}}} \check{\lambda}_{2i+1} \in \dot{D}$$

and thus by elementarity of π:

$$H_{\kappa^+} \models p_* \leqslant p_{i+1} = \pi(p_{i+1}) \Vdash_{\mathbb{P}} \check{\lambda}_{2i+1} \in \dot{C}$$

But then the same is true in L and p_* forces λ to be a limit point of $\dot{C}$. $\square$

Next, we get an easier description of $\mathbb{M}_\kappa^{L[G]}$. Recall that whenever W is a $<\lambda$-ground of a universe V, $W \subseteq V$ satisfies the λ-approximation property. That is, for any $x \in V$ with $x \subseteq W$ so that all λ-approximations $x \cap y$ (i.e. $y \in W$ of size $< \lambda$) are in W then $x \in W$.

Lemma 1.4. $\mathbb{M}_\kappa^{L[G]} = \bigcap_{\lambda \in I \cap \kappa} L[G_{>\lambda}]$

Proof. Suppose W is a κ-ground of $L[G]$. It is enough to find $\lambda \in I \cap \kappa$ such that $L[G_{>\lambda}] \subseteq W$. Clearly, $\mathbb{P} \in L \subseteq W$. As κ is a limit of inaccessibles, we may take some $\lambda < \kappa$ inaccessible so that W is a $<\lambda$-ground. Thus $W \subseteq L[G]$ satisfies the λ-approximation property. We will show $G_{>\lambda} \in W$ (even $G_{\geqslant\lambda} \in W$). Find α with $\lambda = I_\alpha$. It is enough to show

$$F \upharpoonright (\kappa \backslash \alpha \times \kappa) \in W$$

Let $a \in W$, $a \subseteq \kappa \backslash \alpha \times \kappa$, $|a| < \lambda$. As $0^\#$ does not exist in W, there is $b \in L$, $b \subseteq \kappa \backslash \alpha \times \kappa$ of size $< \lambda$ with $a \subseteq b$. For all $\alpha \leqslant \gamma < \kappa$, the set of $\beta < I_\gamma$ with $(\gamma, \beta) \in b$ is bounded in I_γ. We may think of conditions in $\mathbb{P}$ as partial upper left triangle matrices. With this in mind, the conditions $p \in \mathbb{P}$ with "$b \subseteq \mathrm{dom}(p)$" form a dense set. Thus $F \upharpoonright b$ corresponds to a condition $p \in \mathbb{P} \subseteq W$ and hence $F \upharpoonright a = (F \upharpoonright b) \upharpoonright a \in W$. As $W \subseteq L[G]$ satisfies the λ-approximation property, we have $F \upharpoonright (\kappa \backslash \alpha \times \kappa) \in W$.

$\square$

Remark 1.5. The above argument shows that for any $\lambda \in I \cap \kappa$:

$$\mathbb{M}_\kappa^{L[G_{>\lambda}]} = \mathbb{M}_\kappa^{L[G]}$$

In fact, whenever δ is a strong limit, the $<\delta$-mantle is always absolute to any $<\delta$-ground.

We will later show that $\mathcal{P}(\kappa)^{\mathbb{M}_\kappa^{L[G]}}$ does not have a wellorder in $\mathbb{M}_\kappa^{L[G]}$.

Proposition 1.6. *The subsets of κ in $\mathbb{M}_\kappa^{L[G]}$ are exactly the fresh subsets of κ in $L[G]$, that is, the subsets $a \subseteq \kappa$ in $L[G]$ for which $\forall \lambda < \kappa \ a \cap \lambda \in L$.*

Proof. First suppose $a \subseteq \kappa$, $a \in \mathbb{M}_\kappa^{L[G]}$. If $\lambda < \kappa$ then $a \in L[G_{>\lambda}]$. As $\mathbb{P}_{>\lambda}$ is $\leqslant\lambda$-closed in L, $a \cap \lambda \in L$.
For the other direction assume $a \in L[G]$ is a fresh subset of κ and assume W is a $<\kappa$-ground of $L[G]$. There is $\lambda < \kappa$ so that $W \subseteq L[G]$ satisfies the λ-approximation property. As a is fresh, all the λ-approximations of a in W are in W. Thus $a \in W$. $\square$

We are now in good shape to complete the argument.

Theorem 1.7. *The axiom of choice fails in $\mathbb{M}_\kappa^{L[G]}$.*

Proof. It is a standard argument to show that the rows of F, namely

$$c_\beta : \kappa \to 2, \ c_\beta(\alpha) = F(\alpha, \beta)$$

for $\beta < \kappa$, generate $Add(\kappa, 1)^L$-generic filter over L. This is the reason we have chosen $\mathbb{P}$ to be $<\kappa$-supported, otherwise the above would not be true. Note that all c_β are characteristic functions of a fresh subset of κ and hence $c_\beta \in \mathbb{M}_\kappa^{L[G]}$. Of course, the sequence $\langle c_\beta | \beta < \kappa\rangle$ is not in $\mathbb{M}_\kappa^{L[G]}$, as one can compute the whole generic G from this sequence. However, we can make this sequence fuzzy to result in an element of $\mathbb{M}_\kappa^{L[G]}$. Let $\sim$ be the equivalence relation of eventual coincidence on $({}^\kappa 2)^{\mathbb{M}_\kappa^{L[G]}}$.

Claim 1.8. $\langle [c_\beta]_\sim | \beta < \kappa\rangle \in \mathbb{M}_\kappa^{L[G]}$.

Proof. By Lemma 1.4, it is enough to show that for every $\alpha < \kappa$, $L[G_{>I_\alpha}]$ contains this sequence. But $L[G_{>I_\alpha}]$ contains the sequence

$$\langle c_\beta \upharpoonright \kappa \backslash (\alpha + 1) | \beta < \kappa\rangle$$

and can compute the relevant sequence from there as $\mathbb{M}_{<\kappa}^{L[G_{>I_\alpha}]} = \mathbb{M}_{<\kappa}^{L[G]}$. $\square$

Finally, we argue that $\mathbb{M}_\kappa^{L[G]}$ does not contain a choice sequence for the fuzzy sequence. Heading towards a contradiction, let us assume that

$$\langle x_\beta | \beta < \kappa\rangle \in \mathbb{M}_\kappa^{L[G]}$$

is such a sequence. $L[G]$ knows about the sequence

$$\langle \delta_\beta | \beta < \kappa \rangle$$

where δ_β is the least δ with $x_\beta \restriction (\kappa \backslash \delta) = c_\beta \restriction (\kappa \backslash \delta)$. The set of $\lambda < \kappa$ that are closed under the operation $\beta \longmapsto \delta_\beta$ is club in κ. As κ is Mahlo in $L[G]$, there is an inaccessible $\alpha = I_\alpha < \kappa$ that is closed under $\beta \longmapsto \delta_\beta$. Now observe that

$$x_\beta(\alpha) = 1 \Leftrightarrow c_\beta(\alpha) = 1 \Leftrightarrow F_\alpha(\beta) = 1$$

holds for all $\beta < \alpha$. This is a contradiction to $F_\alpha \notin L[G_{>I_\alpha}] \supseteq \mathbb{M}_\kappa^{L[G]}$.

□

Theorem 0.9 follows.

Remark 1.9. The only critical property of L that we need to construct a model of the above form is that L has no nontrivial grounds, i.e. L satisfies the ground axiom. GCH is convenient and implies that no cardinals are collapsed, but it is not necessary. In Lemma 1.4, we made use of Jensen's Covering Theorem and needed that $0^\# \notin L$. Unsurprisingly, this can be avoided, but that leads to a longer argument.

2 The $<\omega_1$-Mantle

Up to now, we have focused on the $<\kappa$-mantle for strong limit κ. We will get similar results for the $<\omega_1$-mantle. There is some ambiguity in the definition of the $<\omega_1$-mantle. One can define it as the intersections of all grounds W so that W extends to V via a forcing so that $W \models |\mathbb{P}| < \omega_1^W$ or so that $W \models |\mathbb{P}| < \omega_1^V$. These are in general not equivalent. To make the distinction clear, we give the first version the name "Cohen mantle" and denote it by $\mathbb{M}_\mathbb{C}$. The reason for the name is, of course, that all non-trivial countable forcings are forcing-equivalent to Cohen forcing.

Lemma 2.1. *$\mathbb{M}_{\omega_1} \models$ ZF and $\mathbb{M}_\mathbb{C} \models$ ZF.*

Proof. First let us do it for $\mathbb{M}_\mathbb{C}$. Clearly, $\mathbb{M}_\mathbb{C}$ is closed under the Gödel operations. It is thus enough to show that $\mathbb{M}_\mathbb{C} \cap V_\alpha \in \mathbb{M}_\mathbb{C}$ for all $\alpha \in$ Ord. Let W be any Cohen-ground. As Cohen-forcing is homogeneous, $\mathbb{M}_\mathbb{C}^V$ is a definable class in W. Hence, $\mathbb{M}_\mathbb{C} \cap V_\alpha = \mathbb{M}_\mathbb{C} \cap V_\alpha^W \in W$. As W was arbitrary, this proves the claim.

Now onto $\mathbb{M}_{\omega_1}$. The above argument shows that all we need to do is show that $\mathbb{M}_{\omega_1}$ is a definable class in all associated grounds. So let W be such a ground. There are two cases. First, assume that $\omega_1^W = \omega_1^V$. Then W extends to V via Cohen forcing, so $\mathbb{M}_{\omega_1}$ is definable in W. Next, suppose that $\omega_1^W < \omega_1^V$. This can only happen if ω_1^V is a successor cardinal in W, say $W \models \omega_1^V = \mu^+$. In this case, W extends to V via a forcing of W-size $\leqslant \mu$ and which collapses μ to be countable. It is well known that in this situation, W extends to V via $\mathrm{Col}(\omega, \mu)$, which is homogeneous as well, so once again, $\mathbb{M}_{\omega_1}$ is a definable class in W. $\square$

Once again, choice can fail.

Theorem 2.2. *Relative to the existence of an inaccessible, it is consistent that $\mathbb{M}_{\omega_1} = \mathbb{M}_{\mathbb{C}}$ and does not have a wellorder of $\mathcal{P}(\omega_1^V)^{\mathbb{M}}_{\omega_1}$.*

In the model we will construct, ω_1 will be inaccessible in $\mathbb{M}_{\omega_1}$. For convenience, let us assume $V = L$ and let λ be an inaccessible cardinal. Let $\mathbb{P}$ be the "$<\lambda$-support version of $\mathrm{Col}(\omega, <\lambda)$", that is

$$\mathbb{P} = \prod_{\alpha<\lambda}^{<\lambda-\mathrm{supp}} \mathrm{Col}(\omega, \alpha)$$

Let us pick a $\mathbb{P}$-generic filter G over V. From now on, $\mathbb{M}_{\omega_1}$ will denote $\mathbb{M}_{\omega_1}^{V[G]}$ and $\mathbb{M}_{\mathbb{C}}$ will denote $\mathbb{M}_{\mathbb{C}}^{V[G]}$.

Lemma 2.3. *Let A be an antichain in $\mathbb{P}$ and $q \in \mathbb{P}_{\leqslant\gamma}$ for some $\gamma < \lambda$. Then there is some $p \in \mathbb{P}$ with $p \upharpoonright \gamma = q$ so that p is compatible with $<\lambda$ many elements of A.*

Corollary 2.4. *We get the following consequences:*

(*i*) $\omega_1^{V[G]} = \lambda$

(*ii*) *If $g : \omega \to \mathrm{Ord} \in V[G]$, then there is some $\alpha < \lambda$ so that $g \in V[G \upharpoonright \alpha]$.*

Proof. (*i*) Clearly $\geqslant$ holds. Assume $\dot{f}$ is a name for a cofinal increasing function $f : \omega \to \lambda$. Using the above lemma, we can inductively build a decresing sequence of conditions $\langle p_n \mid n < \omega\rangle$ and a sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ so that

(*a*) $p_n \Vdash \dot{f}(\check{n}) \leqslant \check{\alpha}_n$

(*b*) $p_{n+1} \upharpoonright \mathrm{dom}(p_n) = p_n$

To construct p_{n+1}, simply look at a maximal antichain of conditions that decide the value $\dot{f}(\check{n})$ and let p_{n+1} be the condition given by the lemma with $q = p_n$. p_{n+1} is then only compatible with $< \lambda$ many conditions in A and thus letting α_{n+1} be the supremum of the corresponding values for $\dot{f}(\check{n})$, we get (*a*). This allows us to take the fusion $p = \bigcup_{n<\omega} p_n$ of the sequence. Let $\alpha = \sup_{n<\omega} \alpha_n$. Then $p \Vdash \sup\mathrm{ran}(\dot{f}) \leqslant \check{\alpha}$. As $\alpha < \lambda$, this is a contradiction.

(*ii*) Let $\dot{g} \in V$ be a name for g. In $V[G]$, find a decreasing sequence of conditions $\langle p_n \mid n < \omega \rangle$ in G so that p_n decides the value of $\dot{g}(\check{n})$ (from the perspective of V). Let $\alpha = \sup_{n<\omega} \sup \operatorname{dom} p_n$. By (*i*), $\alpha < \lambda$. But then $V[G \upharpoonright \alpha]$ can compute the whole of g.

$\square$

Let us define an auxiliary model $N = \bigcap_{\alpha<\lambda} V[G \upharpoonright [\alpha, \lambda)]$. It is clear that $\mathbb{M}_{\omega_1}^{V[G]} \subseteq N$.

Proposition 2.5. *1.* $N \models \mathrm{ZF}$

2. $N \cap \mathcal{P}(\lambda) = \mathbb{M}_{\omega_1} \cap \mathcal{P}(\lambda) = \mathbb{M}_{\mathbb{C}} \cap \mathcal{P}(\lambda) = \{a \subseteq \lambda \mid \forall \beta < \lambda \ a \cap \beta \in V\}$

Proof. (*i*) Once again it is enough to show that N is definable in all $V[G \upharpoonright \alpha]$ for $\alpha < \lambda$. But this is clear as $N = \bigcap_{\alpha \leqslant \beta < \lambda} V[G \upharpoonright [\beta, \lambda)]$ for all $\alpha < \lambda$.

(*ii*) $\mathbb{M}_{\omega_1} \cap \mathcal{P}(\lambda) \subseteq \mathbb{M}_{\mathbb{C}} \cap \mathcal{P}(\lambda) \subseteq N \cap \mathcal{P}(\lambda)$ is trivial. If $a \in N \cap \mathcal{P}(\lambda)$ and $\beta < \lambda$ then $a \cap \beta \in V[G \upharpoonright \alpha]$ for some α by clause (*ii*) of the above corollary. As $a \in N$, $a \cap \beta \in V[G \upharpoonright [\alpha, \lambda)]$, too. This can only happen if $a \cap \beta \in V$. The final inclusion follows since if W is a ground of V which extends to V via $\mathbb{Q}$ of size $< \lambda$, then $\mathbb{Q}$ cannot add a fresh subset of λ.

$\square$

Proof. (Theorem 2.2) We will show that in $V[G]$, $\mathbb{M}_{\omega_1} = \mathbb{M}_{\mathbb{C}}$ and this model does not posess a wellorder of its version of $\mathcal{P}(\lambda)$. In fact, we will show that N does not have such a wellorder, which is enough by (*ii*) of the above proposition. Once again, let $\sim$ be the equivalence relation on functions $f : \lambda \to \lambda \in \mathbb{M}_{\omega_1}$ of eventual coincidence. For $n < \omega$ let $d_n : \lambda \to \lambda, d_n(\alpha) = \bigcup G(n, \alpha)$. As in the other argument, $\langle [d_n]_\sim \mid n < \omega \rangle \in N$. If N had a wellorder of $\mathcal{P}(\lambda)$, then there would be a selector $\langle x_n \mid n < \omega \rangle \in N$. In $V[G]$, one can define the sequence $\langle \delta_n \mid n < \omega \rangle$ by letting δ_n be the least point after which x_n and d_n coincide. As $\lambda = \omega_1$ in $V[G]$, the δ_n are bounded by some $\delta < \lambda$. But this means that $G \upharpoonright [\delta, \lambda) \in N$, a contradiction.

$\square$

3 The Successor Cardinal Case

We show that, under $V = L$, for every regular κ there is a forcing extension in which $\mathbb{M}_{\kappa^+}$ is not a model of ZFC.

Theorem 3.1. *Suppose κ is regular and $\diamondsuit_\kappa$ holds. Then after forcing with*

$$\mathbb{P} = \prod_{\alpha<\kappa^+}^{\kappa-support} \mathrm{Add}(\kappa, 1)$$

$\mathbb{M}_{\kappa^+}$ *is not a model of ZFC.*

First we will make need to analyse that forcing.

Definition 3.2. For $\kappa < \lambda$ we say that a forcing $\mathbb{Q}$ satisfies Axiom A(κ, λ), abbreviated AA(κ, λ), if there is a sequence $\langle \leqslant_\alpha |\ \alpha < \kappa \rangle$ of partial orders on $\mathbb{Q}$ with the following properties:

(*i*) $\forall \alpha + 1 < \kappa \ \ \leqslant_{\alpha+1} \subseteq \leqslant_\alpha \subseteq \leqslant_{\mathbb{Q}}$

(*ii*) for all antichains A in $\mathbb{Q}$, $\alpha < \kappa$ and $p \in \mathbb{Q}$ there is $q \leqslant_\alpha p$ so that $|\{a \in A \mid a \| q\}| < \lambda$

(*iii*) for all $\beta \leqslant \kappa$ if $\vec{p} = \langle p_\alpha \mid \alpha < \beta \rangle$ satisfies $p_{\alpha+1} \leqslant_\alpha p_\alpha$ for all $\alpha < \beta$ then there is a fusion p_β of $\vec{p}$, that is $p \leqslant_\alpha p_\alpha$ for all $\alpha < \beta$

Remark 3.3. The usual Axiom A is thus Axiom A(ω, ω_1)

Proposition 3.4. *If $\mathbb{Q}$ is AA(κ, λ) for $\kappa < \lambda$ and λ regular, then in $V^{\mathbb{Q}}$ there is no surjection from κ onto λ.*

Proof. This is a straightforward adaptation of the proof that Axiom A forcings preserve ω_1. ☐

Lemma 3.5. *If κ is regular and $\diamondsuit_\kappa$ holds then*

$$\mathbb{Q} = \prod_{\alpha<\kappa}^{\textit{full support}} \mathrm{Add}(\kappa, 1)$$

satisfies AA(κ, κ^+).

Proof. We let $p \leqslant_\alpha q$ if $p \leqslant q$ and $p \restriction \alpha = q \restriction \alpha$. It is easy to see that (*i*) and (*iii*) of Definition 3.2 hold, so let us show (*ii*). Therefore let $\alpha < \kappa$, $p \in \mathbb{Q}$ and an antichain A in $\mathbb{Q}$ be given. As $\diamondsuit_\kappa$ holds, there is a sequence $\langle d_\beta \mid \beta < \kappa$ with $d_\beta \in \mathbb{Q}_\beta$ so that for any $q \in \mathbb{Q}$ there is some β with $q \restriction \beta = d_\beta$. We will define a sequence $(p_\beta)_{\alpha \leqslant \beta \leqslant \kappa}$ inductively so that always $p_{\beta+1} \leqslant_\beta p_\beta$. We put $p_\alpha = p$. At limit stages β we let p_β be the canonical fusion of the prior $p_\gamma' s$. So assume p_β is defined. In the case that d_β and p_β are incompatible we put $p_{\beta+1} = p_\beta$. Otherwise, we may now find $p_{\beta+1} \leqslant_\beta p_\beta$ so that $d_\beta \cup p_{\beta+1} \restriction [\beta, \kappa)$ is below a condition in A. Now clearly $q := p_\kappa \leqslant_\alpha p$ and I claim that q is compatible with at most κ-many conditions in A. To see this, suppose $a \in A$ is compatible with q. We may find $\beta < \kappa$ so that $d_\beta = a \restriction \beta$. In the construction of $p_{\beta+1}$ from p_β, we must have that d_β was compatible with p_β. Therefore $d_\beta \cup p_{\beta+1} \restriction [\beta, \kappa)$ is below some condition in A, which must be a itself. This shows that for any $a \in A$ that is compatible with q, there is $\beta < \kappa$ so that $q \restriction [\beta, \kappa) \leqslant a \restriction [\beta, \kappa)$. As $\mathbb{Q}_\beta$ has size κ, it follows that there are at most κ-many such $a \in A$. ☐

Corollary 3.6. *In the case of the above lemma, $\mathbb{Q}$ preserves all cardinals.*

Proof. $\mathbb{Q}$ is $<\kappa$-closed, satisifes $\mathrm{AA}(\kappa, \kappa^+)$ and has size κ^+. $\square$

This was a warm-up before we do what we acctually care about. Note that $\mathbb{P}$ denotes the forcing from Theorem 3.1.

Lemma 3.7. *If κ is regular and $\Diamond_\kappa$ holds then $\mathbb{P}$ preserves all cardinals $\leqslant \kappa^+$. Moreover, if G is $\mathbb{P}$-generic and $g : \kappa \to \mathrm{Ord}$ is in $V[G]$ then there is $\alpha < \kappa^+$ with $g \in V[G_\alpha]$.*

For $x \subseteq \kappa^+$ we will write $p \leqslant_x q$ if $p \leqslant q$ and $p \upharpoonright x = q \upharpoonright x$. We will make use of $\leqslant_x$ only for such x of size $<\kappa$.

Proposition 3.8. *Suppose κ is regular and $\Diamond_\kappa$. If $p \in \mathbb{P}$, $A \subseteq \mathbb{P}$ is an antichain and $x \in \mathcal{P}_\kappa(\kappa^+)$ then there is $q \leqslant_x p$ such that q is compatible with at most κ-many elements of A.*

Proof. Again let $\langle d_\beta \mid \beta < \kappa\rangle$ be the "diamond sequence for $\mathbb{Q}$" that appeared in the proof of Lemma 3.5. We will need to do some additional bookkeeping. Let

$$h : \kappa \to \kappa \times \kappa$$

surjection such that if $h(\beta) = (\alpha, \gamma)$ then $\alpha \leqslant \beta$. We will construct two sequences $\langle p_\beta \mid \beta \leqslant \kappa\rangle$ and $\langle x_\beta \mid \beta < \kappa\rangle$ as well as a function $s : \kappa \times \kappa \to \kappa^+$ so that for all $\beta < \kappa$:

- (i) $p_0 = p$ and $x_0 = x$
- (ii) $s \upharpoonright \{\beta\} \times \kappa$ is a surjection onto $\mathrm{supp}(p_\beta)$
- (iii) $x_{\beta+1} = x_\beta \cup \{s(h(\beta))\}$
- (iv) $p_{\beta+1} \leqslant_{x_\beta} p_\beta$
- (v) if β is a limit then $x_\beta = \bigcup_{\gamma<\beta} x_\gamma$ and $\forall \alpha < \kappa^+ \; p_\beta(\alpha) = \bigcup_{\gamma<\beta} p_\gamma(\alpha)$

We do not care about the particular choice of s so we need only specify $p_{\beta+1}$ outside x_β at stage β of the construction. To do this, we let $e_\beta \in \mathbb{Q}$ so that $\mathrm{supp}(e_\beta) = s[h[\beta]]$ and $e_\beta(s(h(\gamma))) = d_\beta(\gamma)$, if this is well-defined. Now we let $p_{\beta+1} \leqslant_{x_\beta} p_\beta$ such that

$$e_\beta \upharpoonright x_\beta \cup p_{\beta+1} \upharpoonright (\kappa^+ \backslash x_\beta)$$

is below a condition in A, if that is possible. If one of these steps does not work, we let $p_{\beta+1} = p_\beta$.

Finally set $y = \bigcup_{\beta<\kappa} x_\beta$ and $q(\alpha) = \bigcup_{\beta<\kappa} p_\beta(\alpha)$.

Claim 3.9. $q \in \mathbb{P}$

Proof. Note that $\mathrm{supp}(q) = y$ by construction and that y has size at most κ. For any $\alpha \in y$ there is $\beta < \kappa$ with $\alpha \in x_\beta$. Thus $q(\alpha) = p_\beta(\alpha) \in \mathrm{Add}(\kappa, 1)$ so that $q \in \mathbb{P}$. □

We must show that q is compatible with at most κ-many conditions in A. So assume a is such a condition.

Claim 3.10. *There is $\beta < \kappa$ so that e_β is well-defined and $e_\beta \upharpoonright x_\beta = a \upharpoonright x_\beta$.*

Proof. We define $b \in \mathbb{Q}$ by $b(\alpha) = a(s(h(\alpha)))$. Then there is β with $b \upharpoonright \beta = d_\beta$. It is easy to see now that β is as desired. □

Thus at stage β in the construction we tried to extend p_β outside x_β so that

$$a \upharpoonright x_\beta \cup p_{\beta+1} \upharpoonright (\kappa^+ \backslash x_\beta)$$

is below some condition in A. This is possible for a, and only for a as q and a are compatible. We have shown that for any $a \in A$ that is compatible with q, there is $\beta < \kappa$ such that $q \upharpoonright (\kappa^+ \backslash x_\beta) \leqslant a \upharpoonright (\kappa^+ \backslash x_\beta)$. As there are only κ-many $r \in \mathbb{Q}$ with support contained in x_β, this implies that there are at most κ-many such a. □

Proof. (Lemma 3.7) $\mathbb{P}$ is $<\kappa$-closed so that $\mathbb{P}$ does not collapse any cardinal $\leqslant \kappa$. Let $\dot{f}$ be a $\mathbb{P}$-name for a function from κ to κ^+ and p be any condition in $\mathbb{P}$. We will do a construction similar to the one in the proof of Proposition 3.8, namely of sequences $\langle p_\alpha \mid \alpha \leqslant \kappa \rangle$, $\langle x_\alpha \mid \alpha < \kappa \rangle$. Furthermore let h be as in the proof before and again a function $s : \kappa \times \kappa \to \kappa^+$ will be defined. Additionally we will construct a sequence $\langle \epsilon_\beta \mid \beta < \kappa \rangle$. We will demand that for all $\beta < \kappa$:

- (i) $p_0 = p$, $x_0 = \varnothing$
- (ii) $s \upharpoonright \{\beta\} \times \kappa$ is a surjection onto $\mathrm{supp}(p_\beta)$
- (iii) $x_{\beta+1} = x_\beta \cup \{s(h(\beta))\}$
- (iv) $p_{\beta+1} \leqslant_{x_\beta} p_\beta$
- (v) if β is a limit then $x_\beta = \bigcup_{\gamma<\kappa} x_\gamma$ and $\forall \alpha < \kappa^+$ $p_\beta(\alpha) = \bigcup_{\gamma<\beta} p_\gamma(\alpha)$
- (vi) $p_{\beta+1} \Vdash \dot{f}(\check{\beta}) \leqslant \check{\epsilon}_\beta$

The construction is straightforward. Again we can find a fusion $q \in \mathbb{P}$ of $\langle p_\beta \mid \beta < \kappa \rangle$ along $\langle x_\beta \mid \beta < \kappa \rangle$, that is $q \leqslant_{x_\beta} p_\beta$. Put $\epsilon_* = \sup_{\beta<\kappa} \epsilon_\beta$. Then $q \leqslant p$ and $q \Vdash \sup(\mathrm{ran}\, \dot{f}) \leqslant \check{\epsilon}_*$. Thus κ^+ is preserved.

For the moreover part, find $\dot{g} \in V^{\mathbb{P}}$ with $\dot{g}^G = g$ and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{g} : \check{\kappa}^+ \to \mathrm{Ord}$ is a function". In $V[G]$ choose $p_\alpha \in G$ so that p_α decides $\dot{g}(\check{\alpha})$ (as $g(\alpha)$). As κ^+ is not collapsed the supremum of the supports of the p_α, $\alpha < \kappa$, are bounded in κ^+. This gives some α so that $g \in V[G_\alpha]$. □

Remark 3.11. If additionally GCH holds at κ^+ then $\mathbb{P}$ does not collapse any cardinals at all by a standard Δ-system argument.

Proof. (Theorem 3.1) Let G be $\mathbb{P}$-generic over L. By Lemma 3.7, all L-cardinals $\leqslant \kappa^+$ are still cardinals in $L[G]$ (in fact, all cardinals are preserved).. Let $N = \bigcap_{\alpha<\kappa^+} L[G_{[\alpha,\kappa^+)}]$. Using that N is definable in every $L[G_{[\alpha,\kappa^+)}]$, it is easy to check that N is a model of ZF. We call $A \subseteq \kappa^+$ fresh if $A \cap \alpha \in L$ for all $\alpha < \kappa^+$.

Claim 3.12. $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}} = \mathcal{P}(\kappa^+)^N = \{A \subseteq \kappa^+ \mid A \textit{ is fresh}\}^{L[G]}$

Proof. $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}} \subseteq \mathcal{P}(\kappa^+)^N$ is trivial. Suppose $A \subseteq \kappa^+, A \in N$. Given $\alpha < \kappa^+$, by Lemma 3.7, there is $\beta < \kappa^+$ so that $A \cap \alpha \in L[G_\beta]$. Thus $A \in L[G_\alpha] \cap L[G_{[\alpha,\kappa^+)} = L$. For the last inclusion assume $A \in L[G]$ is a fresh subset of κ^+ and W is any κ^+-ground of $L[G]$. It follows that $W \subseteq L[G]$ satisfies the κ^+-approximation property so that $A \in W$ as any bounded subset of A is in $L \subseteq W$. □

We will show that there is no wellorder of $\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}}$ in $\mathbb{M}_{\kappa^+}$. So assume otherwise. Let $\sim$ be the equivalence relation of eventual coincidence on ${}^{\kappa^+}2$ in N. We can realise G as a matrix where the α-th row is $\mathrm{Add}(\kappa, 1$-generic over L. Now the columns are in fact $\mathrm{Add}(\kappa^+, 1)$-generic over L. Let us write c_α for the α-th column ($\alpha < \kappa^+$) and d_β for the β-th row ($\beta < \kappa$). For any $\alpha < \kappa^+$ we have that $\langle d_\beta \restriction [\alpha, \kappa^+) \mid \beta < \kappa\rangle \in L[G_{[\alpha,\kappa^+)}]$. Thus

$$\langle [d_\beta]_\sim \mid \beta < \kappa\rangle \in N$$

and by our assumption there must be a choice function, say $\langle x_\beta \mid \beta < \kappa\rangle$, in N. In $L[G]$ we can definde the sequence $\langle \delta_\beta \mid \beta < \kappa\rangle$, where δ_β is the least point after which x_β and d_β coincide. As κ^+ is not collapsed by $\mathbb{P}$, we can strictly bound all δ_β by some $\delta_* < \kappa^+$. But then

$$\langle x_\beta(\delta_*) \mid \beta < \kappa\rangle \in N$$

is $\mathrm{Add}(\kappa, 1)$-generic over L, which contradicts that N and L have the same subsets of κ. □

Remark 3.13. In this case, one can show that the $L[G_{[\alpha,\kappa^+)}]$, $\alpha < \kappa^+$ do not form a dense "set" of κ^+-small grounds. For example, we can modify $G_{[1,\kappa^+)}$ by adding the generic sequence of the first factor onto every row mod 2, call this H. Then $L[H]$ is a κ^+-ground of $L[G]$ in which no $L[G_{[\alpha,\kappa^+)}]$ is included. It also follows that $\mathbb{M}_{\kappa^+} \subsetneq N$ as

$$\langle [d_\beta]_\sim \mid \beta < \kappa\rangle \in N \backslash \mathbb{M}_{\kappa^+}$$

It would be interesting to see a better description of $\mathbb{M}_{\kappa^+}$. For example, is

$$\mathbb{M}_{\kappa^+} = L(\mathcal{P}(\kappa^+)^{\mathbb{M}_{\kappa^+}})$$

true? Is $\mathbb{M}_{\kappa^+}$ even a model of ZF?

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