A transitive model of Z + ¬TC

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Abstract

It is wellknown that the original Zermelo set theory Z^- is not able to prove that every set is contained in a transitive set (i.e. the axiom TC of transitive containment). However the models of $\mathrm{Z}^- + \neg\mathrm{TC}$ in the literature satisfy the axiom of foundation, but not the foundation scheme for classes (in such models TC necessarily fails) and thus they are illfounded. The main purpose of these notes is to give an easy example of a transitive model of $\mathrm{Z}^- + \neg\mathrm{TC}$. This model also satisfies the theory Z^-+ foundation scheme, which we will call Z. To the best of the authors knowledge there is no proof of $\mathrm{Z} \nvdash \mathrm{TC}$ in the literature. Along the way we produce a few other pathological models of Z. Finally we show that models of $\mathrm{Z} + \mathrm{TC}$ need not have a hierarchy.

Definition 1. The theory Z^- of original Zermelo set theory consists of the following axioms:

(*i*) Extensionality

(*ii*) Pairing

(*iii*) Union

(*iv*) Power Set Axiom

(*v*) Separation Scheme

(*vi*) Infinity

The axiom of foundation (for sets) is denoted by SFound and the foundation scheme for classes by CFound. The theory Z is $\mathrm{Z}^- + \mathrm{CFound}$. We formulate the axiom of choice as "all sets of disjoint sets admit a selector" and let ZC be the theory $\mathrm{Z} + \mathrm{Choice}$. If T is a theory then $T^{\neg\mathrm{Inf}}$ is the theory where the axiom of infinity is dropped from T and its negation is added.

Definition 2. The axiom of transitive containment, denoted by TC, is the statement "every set is a subset of a transitive set".

We will consider a general method of producing models of Z and $\mathrm{Z}+\mathrm{TC}$ that avoid certain sets. Recall that $\mathrm{tc}(x) = \bigcup_{n<\omega} \bigcup^n x$ where $\bigcup^0 x = x$ and $\bigcup^{n+1} = \bigcup \bigcup^n x$. We will write $\bigcup^{\leqslant n} x$ for $\bigcup_{i=0}^n \bigcup^i x$.

Definition 3. Let X be any set.

(*i*) A set X is called avoidable if it is nonempty and for any $x, y \in X$ there is $z \in X$ with $x, y \in z$.

(*ii*) The model $M = \{a \mid X \nsubseteq \mathrm{tc}(a)\}$ is called *the model that avoids X.*

(*iii*) The model $M = \{a \mid \forall n < \omega \ X \nsubseteq \bigcup^{\leqslant n} x\}$ is called *the model that narrowly avoids X.*

Remark 4. Any nonempty X that is closed under finite unions and either taking singletons or the $+1$-operation is avoidable. This includes for example V_α and α for all limit ordinals α.

The following explains why we chose the term avoidable. Also the proof is more important than the statement itself.

Lemma 5. *Let X be avoidable.*

(*i*) *The model M that avoids X is a transitive model of* $\mathrm{Z} - \mathrm{Inf} + \mathrm{TC}$.

(*ii*) *The model N that narrowly avoids X is a transitive model of* $\mathrm{Z} - \mathrm{Inf}$.

Moreorver if $X \nsubseteq \omega$ then both M and N satisfy the axiom of infinity.

Proof. Let us only prove the slighly more subtle case (*ii*). First of all as $X \neq \varnothing$, $\varnothing \in N \neq \varnothing$. If $a \in b$ then $\bigcup^{\leqslant n} a \subseteq \bigcup^{\leqslant n+1} b$ for any $n < \omega$ and hence N is transitive, so that N is a model of extensionality and CFound. Similarly if $a \subseteq b$ then $\bigcup^{\leqslant n} a \subseteq \bigcup^{\leqslant n} b$ so that N is closed under subsets. As N is a definable class in V, N satisfies the separation scheme. Let $a, b \in N$. We check the remaining axioms:

- **Pairing** Let $n < \omega$ and find $x, y \in X$ so that $x \notin \bigcup^{\leqslant n+1} a$, $y \notin \bigcup^{\leqslant n+1} b$. Find $z \in X$ with $x, y \in z$. Then

$$\bigcup\nolimits^{\leqslant n+1}\{a, b\} = \{a, b\} \cup \bigcup\nolimits^{\leqslant n} a \cup \bigcup\nolimits^{\leqslant n} b \not\supseteq z$$

  as is straightforward to check.

- **Union** For any $n < \omega$, $\bigcup^{\leqslant n} \bigcup a = \bigcup^{\leqslant n+1} a$ and thus $\bigcup a \in N$.

- **Power Set** We show that the real powerset $\mathcal{P}(a)$ lies in N. Let $n < \omega$ and pick $x \in X$ so that $x \notin \bigcup^{\leqslant n+1} a$ and $z \in X$ with $x \in z$. We have

$$\bigcup\nolimits^{\leqslant n+1} \mathcal{P}(a) = \mathcal{P}(a) \cup \bigcup\nolimits^{\leqslant n} a \not\supseteq z$$

- **Choice** Assume $a \in N$ is a set of disjoint non-empty sets. Pick any selector c in V, we will show $c \in N$. For any $n < \omega$ we have

$$\bigcup^{\leqslant n} c \subseteq \bigcup^{\leqslant n+1} a$$

which shows $c \in N$.

Finally in both cases if $X \nsubseteq \omega$ then $\omega \in M, N$ so that they are models of the axiom of infinity. □

Remark 6. In the above lemma it is not enough to assume that X is non-empty and $\forall x \in X \exists z \in X\ x \in z$ to ensure M/N are closed under pairing. All other arguments would go through tough.

Corollary 7. $\mathrm{ZC} + \mathrm{TC}$ *does not prove* V_ω *to be a set.*

Proof. The model that avoids V_ω witnesses this. □

Definition 8. The Zermelo ordinals Ord^* consisting of α^* for any ordinal α are defined as follows:

(*i*) $0^* = 0$

(*ii*) $(\alpha + 1)^* = \{\alpha^*\}$

(*iii*) $\alpha^* = \{\beta^* \mid \beta < \alpha\}$

The ordering on Ord^* is the natural ordering induced by the order on the ordinals. It can be equivalently defined by $\alpha^* < \beta^*$ iff $\alpha^* \in \mathrm{tc}(\beta^*)$.

Corollary 9. $\mathrm{ZC}^{\neg\mathrm{Inf}}$ *does not prove all sets to be finite.*

Proof. Let M be the model that avoids ω. Then $\omega \notin M$ so that $M \models \mathrm{ZC}^{\neg\mathrm{Inf}}$. However M contains sets of arbitrary V-size: All Zermelo ordinals are in M as $2 \notin \mathrm{tc}(\alpha^*)$ for any α. Now $\omega^* \in M$ is not finite for any reasonable definition of finite. □

Corollary 10. $\mathrm{ZC} + \mathrm{TC}$ *is consistent with the sentence "the avoidable sets are exactly the cofinal subsets of* ω*".*

Proof. Let $\mathcal{X}$ be the class of all avoidable sets that are not subsets of ω. Let

$$M = \{a \mid \forall X \in \mathcal{X}\ X \nsubseteq \mathrm{tc}(a)\}$$

i.e. the "model that avoids all avoidable sets that are not subsets of ω". The proof Lemma 5 still works in this situation and shows that $M \models ZC$. If $X \in M$ is so that

$$M \models \text{"}X \text{ is avoidable''}$$

then this is true in V as well. But $X \in M$ implies $X \notin \mathcal{X}$, so that $X \subseteq \omega$. Finally observe that the avoidable subsets of ω are exactly the cofinal ones. □

The following chain of implications is folklore:

$$\mathrm{Z}^- + \mathrm{SFound} + \mathrm{TC} \Rightarrow \mathrm{Z}^- + \mathrm{CFound} = \mathrm{Z} \Rightarrow \mathrm{Z}^- + \mathrm{SFound}$$

It is a theorem of Jensen-Schröder that the second implication cannot be reversed and thus in particular, $\mathrm{Z}^- + \mathrm{SFound}$ does not prove TC. To the best of the authors knowledge, there is no proof in the literature that shows the first implication is not reversible, i.e. that Z does not prove the axiom of transitive containment. We will give a proof here.

Definition 11. We say that a set x is close to transitive if $\bigcup^{\leqslant n} x$ is transitive for some $n < \omega$. Otherwise x is far from transitive.

Remark 12. Note that the functions $x \mapsto \bigcup^n x$ are uniformly definable over any model of the theory Z^- via $y = \bigcup^n x$ iff $\exists f : n+1 \to V\ f(0) = x \wedge \forall i < n\ f(i+1) = \bigcup f(i) \wedge y = f(n)$. Hence the same is true for $x \mapsto \bigcup^{\leqslant n} x$ as well.

Note that Z^- trivially proves that all sets that are close to transitive have a transitive closure.

Proposition 13. *There is a set A that is far from transitive and whose transitive closure is avoidable.*

Proof. For a set a let $\{a\}^n$ be the result of applying the singleton operation to a n-times (with $\{a\}^0 = a$). Let $A = \{\{V_n\}^n \mid n < \omega\}$. Then $\mathrm{tc}(A) = V_\omega$ is avoidable. Moreover A is far from transitive: For $n \geqslant 2$ we have $V_n \notin \bigcup^{\leqslant n} A$. $\square$

Theorem 14. *There is a transitive model of* $\mathrm{ZC} + \neg\mathrm{TC}$*. In particular,* ZC *does not prove* TC*.*

Proof. Let A be far from transitive so that $X = \mathrm{tc}(A)$ is avoidable. Then the model N that narrowly avoids X is transitive and a model of ZC by Lemma 5. However $A \in N$ as A is far from transitive so that $X \nsubseteq \bigcup^{\leqslant n} A$ for any $n < \omega$, but $\mathrm{tc}(A) = X \notin N$. It is easy to see that Z^- proves that if a set a is contained in a some transitive set (as a subset) then $\mathrm{tc}(a)$ exists and $\mathrm{tc}(a) = \bigcup_{n<\omega} \bigcup^{\leqslant n} a$. Thus A witnesses that TC fails in N. $\square$

Definition 15. Let M be an $\in$-model. We say that M has (or admits) a weak hierarchy if there is a definable class L and a definable linear order $<$ on L so that there is a definable sequence

$$\langle H_i \mid i \in L \rangle$$

satisfying the following properties:

(*i*) $i < j \Rightarrow H_i \subseteq H_j$

(*ii*) $M = \bigcup_{i \in L} H_i$

Furthermore we say that M has (or admits) a hierarchy if it has a weak hierarchy that is indexed by ordinals, is continuous at limits and all H_i are transitive.

Usually models of Z or a fragment thereof that appear in practice all have a hierarchy since they are often themselves limit points of a hierarchy of a larger model. This includes all models of the form V_α, $J_\alpha[B]$ for limit ordinals α as well as H_κ for limit cardinals κ. Of course all models of ZFC have a hierarchy, namely the Von-Neumann hierarchy. The construction of the Von-Neumann hierarchy depends on both the power set axiom as well as on some instances of the replacement scheme. Are there still hierarchies or weak hierarchies when only one of the two is available? Let us first consider on the case where only power set is available.

Note that models of Z that admit a hierarchy necessarily satisfy some sentences that are not provable in Z, for example TC (as all points in a hierarchy are assumed to be transitive and any set is an element of some such point).

Proposition 16. *A model of* Z *admits a hierarchy iff it admits a weak hierarchy with all its elements transitive.*

Proof. Assume M is a model of Z that admits a weak hierarchy as witnessed by $L, <, \langle H_i \mid i \in L\rangle$ so that all H_i are transitive. For $\alpha \in \mathrm{Ord}^M$, let

$$H'_\alpha = \bigcup\{H_i \mid i \in L \wedge \alpha \notin H_i\}$$

Then $H'_\alpha \in M$ as it can be separated out of any H_j where $j \in L$ is so that $\alpha \in H_j$. Note that for such j, $\alpha \subseteq H_j$ as H_j is transitive. It is now straightforward to check that $\mathrm{Ord}^M, \in, \langle H'_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ is a hierarchy for M. □

Corollary 17. *A model of* Z + TC *admits a hierarchy iff it admits a weak hierarchy.*

Proof. If $\langle H_i \mid i \in L\rangle$ is the weak hierarchy then $\langle \mathrm{tc}(H_i) \mid i \in L\rangle$ is a weak hierarchy with all its elements transitive. By Proposition 16 this is enough. □

Theorem 18. *There are transitive models of* ZC + TC *that do not have a weak hierarchy.*

Proof. Consider the model M that avoids V_ω. It is enough to show that M does not admit a hierarchy. Note that M still contains all ordinals. Assume M had a hierarchy as witnessed by $\langle H_i \mid i \in \mathrm{Ord}\rangle$. Working in V, find $i_n \in L$ with $V_n \in H_{i_n}$ for any $n < \omega$ and let $i_* = \sup_{n<\omega} i_n$. Then $V_\omega \subseteq H_{i_*}$ and since V_ω is definable in M, V_ω can be separated out of H_{i_*}, a contradiction to $V_\omega \notin M$. □

Remark 19. There is another such model that has only boundedly many ordinals, namely the model that avoids α for ones favourite limit ordinal $\alpha > \omega$. The point is that this model can still define a wellorder of ordertype Ord, namely $\mathrm{ran}(f)$ with $i < j$ iff $i \in \mathrm{tc}(j)$. Note that the relevant transitive closures exist in this model. A similar argument as above shows that this model cannot admit a weak hierarchy.

Question 20. Is the class of all models of ZC that admit a (weak) hierarchy axiomatisable?

I conjecture that the answer is no in both cases.

The proper classes of models of Z that admit a hierarchy have a uniform cofinality in the following sense:

Lemma 21. *Assume M is a model of* Z *with a hierarchy and C is a proper class of M that admits a weak hierarchy in the sense that there is a definable linear order L and a $\subseteq$-increasing definable sequence $\langle K_i \mid i \in L\rangle$ such that $C \bigcup_{i \in L} K_i$. Then there is a definable increasing cofinal map $h : L \to \mathrm{Ord}^M$.*

Proof. Let $\langle H_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ be a hierarchy of M. The map is given by

$$h(i) = \min\{\alpha \in \mathrm{Ord} \mid K_i \subseteq H_\alpha\}$$

There always is an α with $K_i \subseteq H_\alpha$ since there is one with $K_i \in H_\alpha$ and since H_α is transitive, $K_i \subseteq H_\alpha$ follows. Since the K_i are $\subseteq$ increasing along L, h is increasing. Assume h were not cofinal in Ord^M. Then there were some α with $\mathrm{ran}\, h \subseteq \alpha$ so that $K_i \subseteq H_\alpha$ for all $i \in L$. But then $C \subseteq H_\alpha$ in contradiction to C being a proper class. □

The conclusion of the lemma above can be expressed as a scheme of $\in$-sentences. This scheme is not provable in ZC + TC alone. This is implicit in the proof of Theorem 18. There are also models of ZC + TC that have a bounded number of ordinals, but all Zermelo ordinals or vice versa.