An inner model with a Woodin from the Zipper Lemma

With the Zipper Lemma, we can get to the situiation where $L(V_\delta) \models$ "δ is Woodin", but we might have that choice does not hold in $L(V_\delta)$. As choice holds in V_δ, the only obstruction to choice there is the possible nonexistence of a wellorder of V_δ. There is a canonical forcing for adding such a wellorder. We show that this forcing does not destroy the Woodinness of δ. As we work in a choiceless context, Woodinness will mean here to be Woodin for all $A \subseteq V_\delta$ and not just all $A \subseteq \delta$ (i.e. there is $\kappa < \delta$ so that for all $\lambda < \delta$ there is an appropriate elementary embedding j with $j(A) \cap V_\lambda = A \cap V_\lambda$). Note that in the situation of the Zipper Lemma, this (in a choiceless context) stronger version of Woodinness is true in $L(V_\delta)$.

Theorem 1. *Assume that δ is A-Woodin for all $A \in V_{\delta+1}^{L(V_\delta)}$. Then there is a forcing extension of $L(V_\delta)$ in which δ is Woodin and choice holds.*

First we prove a lemma.

Lemma 2. *Suppose δ is $A-$Woodin for some $A \subseteq \delta$. Then there is κ $<\delta - A$-strong and $f : \kappa \to V_\kappa$ so that for any $\kappa < \lambda < \delta$ and $x \in V_\lambda$ there is a $\lambda - A$-strong embedding j with critical point κ so that $j(f)(\kappa) = x$.*

Proof. This is essentially the construction of a Laver function for a supercompact. Fix a wellorder $\prec$ of V_δ. Define $g : \delta \to V_\delta$ by induction as follows: $g(\alpha) = 0$ unless α is $\lambda - A$-strong for some $\lambda < \delta$ and there is $x \in V_\lambda$ so that no $\lambda - A$-strong embedding i with critical point α satisfies $i(g \upharpoonright \alpha)(\alpha) = x$. In that case choose λ minimal and let $g(\alpha)$ be the $\prec$-minimal $x \in V_\lambda$ with that property. Now there is a club of cardinals α so that $\operatorname{ran}(g \upharpoonright \alpha) \subseteq V_\alpha$. As δ is Woodin, we can find a κ in this club that is $<\delta - A$-strong. I claim that κ and $f = g \upharpoonright \kappa$ are as desired. Suppose this fails. Let $g(\kappa) = x$ and λ least with $x \in V_\lambda$. Let $j : V \to M$ be a $(|V_\lambda| + \omega)-A-$strong embedding with $\operatorname{crit}(j) = \kappa$. Observe that $j(f) \upharpoonright \kappa = f$ as $\operatorname{ran}(f) \subseteq V_\kappa$. It follows that in M, there is no $\lambda - j(A)-$strong embedding i with critical point κ so that $i(j(f) \upharpoonright \kappa)(\kappa) = x$ and thus the definition of $j(f)$ at stage κ is non-trivial. Lets look at $y = j(f)(\kappa)$ and μ least with $y \in V_\mu$. Clearly $\mu \leqslant \lambda$. Furthermore we get that there can be no $\mu-A-$strong embedding i at κ in V with $i(f)(\kappa) = y$, otherwise an appropriate extender witnessing this would be in M and be $\mu-j(A)-$strong there. Consider E the $(\kappa, |V_\mu|)$-extender derived from j. Let $i : V \to Ult(V, E) = N$ be the induced elementary embedding and note that i is $\mu-A-$strong. Furthermore,

$$k : N \to M, k([h, a]_E) = j(h)(a)$$

is an elementary embedding with $\operatorname{crit}(k) \geqslant \mu$ and $k \circ i = j$. It is clear that $k \upharpoonright V_\mu = \operatorname{id} \upharpoonright V_\mu$. But then $i(f)(\kappa) = k(i(f)(\kappa)) = j(f)(\kappa) = y$, contradiction. $\square$

Corollary 3. *Suppose δ is $A-$Woodin for some $A \subseteq \delta$. Then there is $\kappa < \delta$ that is $<\delta-A-$strong and a function $p : \kappa \to V_\kappa$ so that for any $\lambda < \delta$ and any $q : \lambda \to V_\lambda$ that extends p, there is a $\lambda-A-$strong embedding j at κ with $j(p) \upharpoonright \lambda = q$.*

Proof. Let $\kappa < \delta$ and $f : \kappa \to V_\kappa$ as in the lemma. Let p be the concatenation of all $f(\alpha)$ whenever $f(\alpha)$ is a sequence in V_κ. Suppose λ and q are as in the claim. Find a $\lambda-A-$strong embedding j at κ so that $j(f)(\kappa)$ is the sequence induced by $q \upharpoonright [\kappa, \lambda)$. By elementarity, $j(p) \upharpoonright \lambda = q$. □

Proof. (of the Theorem) Write $L(V_\delta) = M$. The forcing we use is

$$\mathbb{P} = \{f \mid f : \alpha \to V_\delta, \alpha < \delta\}$$

Let G be $\mathbb{P}$-generic over M. Clearly $\mathbb{P}$ adds a wellorder of V_δ and it is easy to see that $M[G] \models ZFC$. It remains to show that δ is still Woodin in $M[G]$. Let $A \subseteq \delta$ in $M[G]$. We can pick a "nice-enough" name $\dot{A}$ for A in M, that is one which is of the form

$$\{(\check{\alpha}, p) \mid \alpha < \delta \wedge p \in D_\alpha\}$$

where $D_\alpha \subseteq \mathbb{P}$ is, say, open. (Just take $D_\alpha = \{p \in \mathbb{P} \mid p \Vdash \check{\alpha} \in \dot{A}'\}$ where $\dot{A}'$ is any name for A).

Claim 4. *For any $\lambda < \delta$ there is $\gamma_0 < \delta$ so that for all $\gamma_0 \leqslant \gamma < \delta$ we have $A \cap \lambda = (\dot{A} \cap V_\gamma)^G \cap \lambda$.*

Proof. The reason that some $\alpha < \lambda$ is in A is that eventually G runs into the open set D_α, thus we can take γ_0 large enough so that V_{γ_0} contains a witness p_α for this whenever $\alpha \in A \cap \lambda$. □

Consider

$$D = \{p \in \mathbb{P} \mid \mathrm{dom}(p) \text{ is } {<}\delta - \dot{A} - \text{strong and } p \text{ is as in the corollary}\}$$

Note that in the corollary, we can make an initial segment of p anything we want. This shows that D is dense, so pick some $p \in G \cap D$ and let $\kappa = \mathrm{dom}(p)$.

Claim 5. *For any $\lambda < \delta$ there is a $\lambda-$strong embedding $j : M[G] \to N$ with $\mathrm{crit}(j) = \kappa$ and $A \cap \lambda \subseteq j(A) \cap \lambda$.*

Proof. In M, pick an extender $E \in V_\delta$ so that the induced embedding $i : M \to W$ satisfies:

1. $\mathrm{crit}(i) = \kappa$
2. i is $\gamma-\dot{A}-$strong, where $\lambda < \gamma$ is as in Claim 4

3. $i(p) \restriction \gamma = \bigcup G \restriction \gamma$

As $V_\delta^{M[G]} = V_\delta^M$, E is an extender in $M[G]$ and the induced embedding $i_G : M[G] \to N$ extends i and still satisfies the above properties. Let $H = i_G(G)$. As $p \in G$ and thus $j(p) \in H$, we have by 3. that $\bigcup G \restriction \gamma = \bigcup H \restriction \gamma$. This implies:

$$
\begin{aligned}
A \cap \lambda = \dot{A}^G \cap \lambda = (\dot{A} \cap V_\gamma)^G \cap \lambda = (i_G(\dot{A}) \cap V_\gamma)^G \cap \lambda \\
\subseteq (i_G(\dot{A}) \cap V_\gamma)^H \cap \lambda \subseteq i_G(\dot{A})^H \cap \lambda = i_G(A) \cap \lambda
\end{aligned}
$$

□

This shows that if we would do the same proof all over, not just with the name $\dot{A}$, but with a similar "nice-enough" name $\dot{C}$ for the complement of A, in the end we would get $A \cap \lambda \subseteq i_G(A) \cap \lambda$ as well as $(\delta \backslash A) \cap \lambda \subseteq i_G(\delta \backslash A) \cap \lambda$. This clearly gives $A \cap \lambda = i_G(A) \cap \lambda$. □