An ω_1-dense Ideal on ω_1 from an Almost Huge Cardinal

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Abstract

We give a detailed account of how to force an ω_1-dense ideal on ω_1 from an almost huge cardinal, a result due to Woodin. The note is mainly based on Theorem 7.60 in [For10].

1 ω_1-Dense Ideals

Definition 1. An ω_1-dense ideal on ω_1 is a countably complete normal ideal $\mathcal{I}$ on ω_1 so that $(\mathcal{P}(\omega_1)/\mathcal{I})^+$ has a dense subset of size ω_1.
Given any normal countably complete ideal $\mathcal{J}$ on ω_1 we let $Y_{Col}(\mathcal{J})$ denote the set of functions $f : \omega_1 \to H_{\omega_1}$ with the following properties:

- (i) For nonzero $\alpha < \omega_1$, $f(\alpha)$ is a filter in $\mathrm{Col}(\omega, \alpha)$
- (ii) for any $S \in \mathcal{J}^+$ there is $p \in \mathrm{Col}(\omega, \omega_1)$ with

$$S_p := \{\alpha < \omega_1 \mid p \in f(\alpha)\} \subseteq S \pmod{\mathcal{J}}$$

and any such S_p is not in $\mathcal{J}$.

Lemma 2. *The following are equivalent:*

- (i) *There is an ω_1-dense ideal on ω_1.*
- (ii) *In $V^{\mathrm{Col}(\omega,\omega_1)}$ there is a definable elementary embedding $j : V \to M$ with critical point ω_1^V and M transitive.*
- (iii) *In $V^{\mathrm{Col}(\omega,\omega_1)}$ there is a V-ultrafilter on ω_1^V that is countably complete and normal for sequences in V so that $\mathrm{Ult}(V, U)$ is wellfounded.*
- (iv) *There is a normal countably complete ideal $\mathcal{I}$ such that $\mathrm{Col}(\omega, \omega_1)$ embeds densly into $(\mathcal{P}(\omega_1)/\mathcal{I})^+$.*
- (v) *There is a normal countably complete ideal $\mathcal{I}$ on ω_1 with $Y_{Col}(\mathcal{I}) \neq \emptyset$.*

Proof. For $(i) \Rightarrow (ii)$ note that, for $\mathcal{I}$ the dense ideal, forcing with $(\mathcal{P}(\omega_1)/\mathcal{I})^+$ has a dense subset of size ω_1 and collapses ω_1, thus is forcing equivalent to $\mathrm{Col}(\omega, \omega_1)$. It is clear that there is such an embedding after forcing with $(\mathcal{P}(\omega_1)/\mathcal{I})^+$.
$(ii) \Rightarrow (iii)$ is standard.
Let's do $(iii) \Rightarrow (iv)$. Let $\dot{U}$ be a $\mathrm{Col}(\omega, \omega_1)$-name for such an ultrafilter. We show that the "hopeless" ideal

$$\mathcal{I} = \{A \subseteq \omega_1 \mid \mathbb{1} \Vdash_{\mathrm{Col}(\omega,\omega_1)} \check{A} \notin \dot{U}\}$$

works. $\mathcal{I}$ is clearly a countably complete, normal ideal on ω_1.

Claim 3. *$\mathcal{I}$ is saturated.*

Proof. Let $\vec{A} = \langle A_i \mid i < \omega_2\rangle$ be a sequence in $\mathcal{I}^+$. For any $i < \omega_2$ there is some $p_i \in \mathrm{Col}(\omega, \omega_1)$ with

$$p_i \Vdash \check{A}_i \in \dot{U}$$

There must be some $i < j < \omega_2$ with $p_i = p_j$ and thus

$$p_i = p_j \Vdash \check{A}_i \cap \check{A}_j \in \dot{U}$$

so that $\vec{A}$ is not an antichain. □

Thus $\mathcal{P}(\omega_1)/\mathcal{I}$ is a complete Boolean algebra. Define the Boolean algebra homomorphism

$$i : (\mathcal{P}(\omega_1)/\mathcal{I})^+ \to \mathrm{RO}(\mathrm{Col}(\omega, \omega_1)),\ [A]_{\mathcal{I}} \mapsto \left\|\check{A} \in \dot{U}\right\|$$

Claim 4. *i is a complete embedding.*

Proof. Let $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ be a maximal antichain of $\mathcal{I}^+$ and set

$$X = \{\alpha < \omega_1 \mid \exists\beta < \alpha\ \alpha \in A_\beta\}$$

Since I is a normal ideal containing the bounded ideal, $\omega_1 \backslash X \in I$ so that

$$\mathbb{1} \Vdash_{\mathrm{Col}(\omega,\omega_1)} \check{X} \in \dot{U}$$

Let g be $\mathrm{RO}(\mathrm{Col}(\omega, \omega_1))$-generic and $U = \dot{U}^g$. Consider the map

$$f : X \to \omega_1,\ \alpha \mapsto \beta \text{ where } \beta \text{ is least with } \alpha \in A_\beta$$

As U is V-normal, f is constant on some $Y \in U$ with value β. Then $Y \subseteq A_\beta$ so that $A_\beta \in U$, i.e. $i(A_\beta) \in g$. □

In particular, $\mathrm{ran}(i)$ is a complete Boolean subalgebra of $\mathrm{RO}(\mathrm{Col}(\omega, \omega_1))$ that collapses ω_1 and is by Lemma 15 isomorphic to $\mathrm{RO}(\mathrm{Col}(\omega, \omega_1))$. Thus there is a dense embedding

$$e : \mathrm{Col}(\omega, \omega_1) \to (\mathcal{P}(\omega_1)/\mathcal{I})^+$$

For $(iv) \Rightarrow (v)$, let $\mathcal{I}$ be such an ideal and e the given dense embedding. For the rest of the proof, we identify $\mathrm{Col}(\omega, \alpha)$ with the poset ${}^{<\omega}\alpha$ ordered by reverse inclusion. We can now inductively choose x_p for $p \in \mathrm{Col}(\omega, \omega_1)$ such that:

1. $[x_p]_{\mathcal{I}} = e(p)$
2. $x_\varnothing = \omega_1$
3. $x_p = \bigcup_{\alpha<\omega_1} x_{p\frown\alpha}$
4. if $\alpha < \beta < \omega_1$ then $x_{p\frown\alpha} \cap x_{p\frown\beta} = \varnothing$

For nonzero $\alpha < \omega_1$, we let $f(\alpha) = \{p \in \mathrm{Col}(\omega, \alpha) \mid \alpha \in x_p\}$. The properties of the x_p guarantee that $f(\alpha)$ is a filter. We see that

$$S_p = \{\alpha < \omega_1 \mid p \in f(\alpha)\} = x_p \in \mathcal{I}^+$$

and as the x_p induce a dense subset of $(\mathcal{P}(\omega_1)/\mathcal{I})^+$, for any $S \in \mathcal{I}^+$ there is a p with

$$S_p = x_p \subseteq S (\mathrm{mod}\ \mathcal{I})$$

Hence $f \in Y_{Col}(\mathcal{I})$.
$(v) \Rightarrow (i)$ is trivial.

$\square$

Proposition 5. *The following are equivalent:*

(i) There is a ω_1-dense ideal on ω_1 so that the induced generic embedding after forcing with $(\mathcal{P}(\omega_1)/\mathcal{I})^+$ restricted to the ordinals is independent of the choice of generic filter.

(ii) There is a $\mathrm{Col}(\omega, \omega_1)$-name $\dot{U}$ for a V-ultrafilter on ω_1^V that is countably complete and normal for sequences in V such that $\mathrm{Ult}(V, \dot{U})$ is forced to be wellfounded and $j_{\dot{U}^g} \restriction \mathrm{Ord}$ does not depend on the choice of generic g.

Proof. $(i) \Rightarrow (ii)$ is trivial, so we will do $(ii) \Rightarrow (i)$. As in the proof of the lemma above we let $\mathcal{I} = \{A \subseteq \omega_1 \mid \mathbb{1} \Vdash_{\mathrm{Col}(\omega,\omega_1)} \check{A} \notin \dot{U}\}$ and get that

$$i : (\mathcal{P}(\omega_1)/\mathcal{I})^+ \to \mathrm{RO}(\mathrm{Col}(\omega, \omega_1)),\ [A]_{\mathcal{I}} \mapsto \left\| \check{A} \in \dot{U} \right\|$$

is a complete embedding. Let h be $(\mathcal{P}(\omega_1)/\mathcal{I})^+$-generic and

$$U = \{x \subseteq \omega_1^V \mid [x]_{\mathcal{I}} \in h\}$$

As $\mathrm{ran}(i)$ is a complete subforcing of $\mathrm{RO}(\mathrm{Col}(\omega, \omega_1))$, we can further force over $V[h]$ to find a $\mathrm{Col}(\omega, \omega_1)$-generic extension $V[g]$ of V so that $h = g \cap \mathrm{ran}(i)$. It is now easy to see that

$$U = \dot{U}^g$$

which implies that $\mathcal{I}$ is as desired by our assumption on $\dot{U}$. □

Lemma 6. *Suppose κ_0 is κ_1-almost huge. Then there is a (long) (κ_0, λ)-extender E such that:*

(i) *j_E witnesses that κ_0 is κ_1-almost huge*

(ii) *j_E is continuous at κ_1*

(iii) $j_E(\kappa_1) = \lambda$

(iv) *λ has size κ_1*

Proof. Let $j : V \to M$ be any embedding witnessing that κ_0 is κ_1-almost huge. Let $\lambda = \sup j[\kappa_1]$ and E the derived (κ_0, λ)-extender. Clearly, j_E has critical point κ_0 and satisfies $j_E(\kappa_0) = \kappa_1$. Furthermore, $M_\lambda = (M_E)_\lambda$ as κ_1 is inaccessible. Therefore for any $\alpha < \kappa_1$, we have $j_E[\alpha] = j[\alpha] \in M_\lambda \subseteq M_E$. In addition to this, ${}^{<\kappa_1}([\lambda]^{<\omega}) \subseteq M_\lambda \subseteq M_E$ (where the first inclusion holds as λ has cofinality κ_1) and this is enough to conclude that M_E is closed under sequences of length $<\kappa_1$. Next we show that j_E is continuous at κ_1, it follows that $j_E(\kappa_1) = \lambda$. So let $\alpha < j_E(\kappa_1)$. Then there is $a \in [\lambda]^n$ for some n and β such that $\alpha = j_{a,\infty}(\beta)$, where $j_{a,\infty}$ is the factor embedding $M_{E_a} \to M_E$. Clearly, $\beta < j_a(\kappa_1)$, where j_a is the embedding $V \to M_{E_a}$.

Claim 7. *κ_1 is a fixed point of j_a.*

Proof. j_a is the ultrapower embedding given by

$$E_a = \{A \subseteq [\kappa_1]^n \mid a \in j_E(A)\}$$

but in fact, j_a is also given by the ultrapower by

$$E'_a = \{A \subseteq [\gamma]^n \mid a \in j_E(A)\}$$

where $\gamma < \kappa_1$ is large enough such that $\max a < j(\gamma) = j_E(\gamma)$. Now κ_1 is an inaccessible above γ and thus a fixed point of j_a. □

We can now see immediately that $\alpha \leqslant j_E(\beta) < j_E(\kappa_1)$ which shows that j_E is continuous at κ_1. It may not be the case that λ has size κ_1 in our situation. Note that, to compute j_E, only the bounded subsets of κ_1 are relevant as (M_E, j_E) is the direct limit of the (M_{E_a}, j_a) and only the bounded subsets of κ_1 are relevant to compute a given j_a by the computation above. Let $E' = \langle E'_a \mid a \in [\lambda]^{<\omega}\rangle$. Let θ be regular and large enough. Find an elementary substructure $X \prec H_\theta$ of size κ_1 such that $E' \in X$ and $H_{\kappa_1} \cup \{\kappa_1\} \subseteq X$ and let Y be the transitive collapse of X. If F' is the image of E', and λ' is the image of λ, it is easy to check that F' generates a (κ_0, λ') extender in the same way E' generates E. This is because Y knows all bounded subsets of κ_1. It is clear that F retains all the properties $(i) - (iii)$ with λ replaced by λ', but now additionally λ' has size κ_1. □

Lemma 8. *Assume W is an inner model of ZFC so that ω_1^V is inaccessible in W and every real of V is an element of a forcing extension $W[h]$ of W for a forcing of size $<\omega_1^V$ with $h \in V$. Then there is a forcing $\mathbb{P}$ so that if G is $\mathbb{P}$-generic there is $g \in V[G]$ so that g is $\mathrm{Col}(\omega, <\omega_1^V)$-generic over W and $\mathbb{R}^{W[g]} = \mathbb{R}^V$.*

Proof. Let $\mathbb{P}$ consists of filters f that are $\mathrm{Col}(\omega, <\alpha_f)$-generic over W for some $\alpha_f < \omega_1^V$, with $f_0 \leqslant f_1$ if $\alpha_{f_0} \geqslant \alpha_{f_1}$ and $f_1 \subseteq f_0$. Suppose G is $\mathbb{P}$-generic and let $g = \bigcup G$. Using that ω_1^V is inaccessible in W, it is easy to see that g is $\mathrm{Col}(\omega, <\omega_1^V)$-generic over W and that $\mathbb{R}^{W[g]} \subseteq \mathbb{R}^V$. For the other inclusion let $r \in \mathbb{R}^V$. I claim that

$$D = \{f \in \mathbb{P} \mid r \in W[f]\}$$

is dense in $\mathbb{P}$. So let $f \in \mathbb{P}$ be given and note that we can identify f with a real. Thus there is a ω_1^V-small forcing extension $W[h]$ of W with $h \in V$ and $f, r \in W[h]$. We have that $W \subseteq W[f] \subseteq W[h]$ and so there is a forcing $\mathbb{Q} \in W[f]$ of $W[f]-$size $<\omega_1^V$ so that $W[h] = W[f][h']$ for some h' $\mathbb{Q}$-generic over $W[f]$. By the universal property of the Levy collapse, we can absorb $\mathbb{Q}$ into a forcing of the form $\mathrm{Col}(\omega, [\alpha_f, \beta))$ for some $\beta < \omega_1^V$ large enough and find $f' \in V$ generic for this forcing over $W[f]$ so that $W[h] \subseteq W[f][f']$. This means that $f \cup f'$ is $\mathrm{Col}(\omega, <\beta)$-generic over W and thus is a condition in D below f. This implies that $r \in W[g]$. □

Theorem 9. *If ZFC+"There is an almost huge cardinal" is consistent then so is ZFC+"There is an ω_1-dense ideal on ω_1 whose generic embedding restricted to the ordinals does not depend on the generic filter"+CH is consistent.*

Proof. Suppose κ_0 is κ_1-almost huge and let E be a (κ_0, λ)-extender given by Lemma 6 and let $j : V \to M$ denote the induced elementary embedding. We let $\kappa_2 = \lambda$ to emphasize $\lambda = j^2(\kappa_0)$. Let g_0 be $\mathrm{Col}(\omega, <\kappa_0)$-generic over V and $\mathbb{R}_* = \mathbb{R}^{V[g_0]}$. Let g_1 be $\mathrm{Col}(\kappa_0, [\kappa_0, \kappa_1))$-generic over $V[g_0]$.

Note that $V(\mathbb{R}_*)[g_1]$ makes sense and satisfies choice. Further observe that $V(\mathbb{R}_*)[g_1] = V[g_1]$, so we will write the latter instead to ease notation. $V[g_1]$ will be our target model. So we will show that there is a suitable elementary embedding in $V[g_1, g]$ that allows us to apply Proposition 5. Therefore let g be $\mathrm{Col}(\omega, \kappa_0)$-generic over $V[g_1]$ and put $\mathbb{R}^* = \mathbb{R}^{V[g_1,g]}$.

Claim 10. *In $V[g_1, g]$, j lifts to an elementary embedding:*

$$j^+ : V(\mathbb{R}_*) \to M(\mathbb{R}^*)$$

Proof. Let $g_1^- = g_1 \cap \mathrm{Col}(\kappa_0, \{\kappa_0\})$. In $V(\mathbb{R}_*)[g^-] = V[g^-]$, we can apply Lemma 8 to find a forcing $\mathbb{P}$ that adds a V-generic filter h for $\mathrm{Col}(\omega, {<}\kappa_0)$ so that $\mathbb{R}^{V[h]} = \mathbb{R}_*$. Surely we can find such an h in $V[g_1, g]$. We can apply that lemma again in $V[g_1, g]$ to see that in some forcing extension there is H a V-generic filter for $\mathrm{Col}(\omega, {<}\kappa_1)$ with $h \subseteq H$ and $\mathbb{R}^{V[H]} = \mathbb{R}^*$. This gives a lift

$$i : V[h] \to M[H]$$

of j. Put:

$$j^+ = i \upharpoonright V(\mathbb{R}_*) : V(\mathbb{R}_*) \to M(\mathbb{R}^*)$$

As $i(\mathbb{R}_*) = \mathbb{R}^*$, it is clear that j^+ is elementary. It is our duty to show that j^+ is already definable in $V[g_1, g]$. Given $x \in V(\mathbb{R}_*)$, there are $\alpha \in \mathrm{Ord}$, $r \in \mathbb{R}_*$, $a \in V$ and a formula φ so that

$$x = \{y \in V(\mathbb{R}_*)_\alpha \mid V(\mathbb{R}_*)_\alpha \models \varphi(y, r, a)\}$$

and thus

$$i(x) = \{y \in M(\mathbb{R}^*)_{j(\alpha)} \mid M(\mathbb{R}^*)_{j(\alpha)} \models \varphi(y, r, j(a))\}$$

which shows that $j^+(x) = i(x)$ is definable in $V[g_1, g]$ uniformly in x. $\square$

Claim 11. *From the perspective of $V[g_1, g]$, ${}^\omega M(\mathbb{R}^*) \subseteq M(\mathbb{R}^*)$.*

Proof. If we let

$$I(\alpha, r, a, \varphi) = \{y \in M(\mathbb{R}^*)_\alpha \mid M(\mathbb{R}^*)_\alpha \models \varphi(y, r, a)\}$$

then every element of $M(\mathbb{R}^*)$ is of the form $I(\alpha, r, a, \varphi)$ for some $\alpha \in \mathrm{Ord}, r \in \mathbb{R}^*, a \in M$ and a formula φ. As a countable sequence of reals is coded by a real again and as $M(\mathbb{R}^*)$ knows all the reals, it is sufficient to prove that $M(\mathbb{R}^*)$ contains all countable sequences of ordinals in $V[g_0, g]$. Note that

$$M' = M(\mathbb{R}_*)[g_1, g] = M[g_1, g]$$

is closed under ω-sequences and that $\mathbb{R}^*$ are the reals of that model. If $\vec{\alpha}$ is any such sequence then $\vec{\alpha} \in M'$ and moreover there is $\beta < \kappa_1$ so that:

$$\vec{\alpha} \in M(\mathbb{R}_*)[g, g_1 \cap \mathrm{Col}(\kappa_0, [\kappa_0, {<}\beta))] \subseteq M(\mathbb{R}^*)$$

$\square$

Claim 12. *In* $V[g_1, g]$ *there is some* H *that is* $\mathrm{Col}(\kappa_1, [\kappa_1, \kappa_2))^{M(\mathbb{R}^*)}$*-generic over* $M(\mathbb{R}^*)$ *with* $j^+[g_1] \subseteq H$.

Proof. First we quickly build a generic H^- for

$$\mathrm{Col}(\kappa_1, \{\kappa_1\})^{M(\mathbb{R}^*)} \cong \mathrm{Add}(\kappa_1, 1)^{M(\mathbb{R}^*)}$$

with $j^+[g_1^-] \subseteq H^-$. This is possible as

- $|\mathcal{P}(\mathrm{Add}(\kappa_1, 1)) \cap M(\mathbb{R}^*)|^{V[g_1,g]} < |\kappa_2|^{V[g_1,g]} = \kappa_1$
- $\kappa_1 = \omega_1^{V[g_1,g]}$, and
- $M(\mathbb{R}^*)^\omega \subseteq M(\mathbb{R}^*)$

Note that $M(\mathbb{R}^*)[H^-] = M[H^-]$ is still closed under countable sequences. For $\kappa_1 < \alpha \leqslant \kappa_2$, let us write $\mathbb{Q}_\alpha$ for $\mathrm{Col}(\kappa_1, (\kappa_0, \alpha))$. Again, there are at most κ_1-many dense subsets of any $\mathbb{Q}_\alpha$ in $M[H^-]$. Let

$$\langle D^\alpha_\beta \mid \beta < \kappa_1 \rangle$$

be an enumeration of them. As $\sup j[\kappa_1] = j(\kappa_1)$, we can find a bookkeeping bijection $b : \kappa_1 \to \kappa_2 \times \kappa_1$ such that if $b(\gamma) = (\alpha, \beta)$ then $\alpha \leqslant j(\gamma)$. By induction, define a descending sequence

$$\langle p_\gamma \mid \gamma < \kappa_1 \rangle$$

of conditions in $\mathbb{Q}_{\kappa_2}$ such that $p_\gamma \in \mathbb{Q}_{j(\gamma)}$. Let $p_0 = \varnothing$. If p_γ is defined, then find $q \in \mathbb{Q}_\alpha$ below $p_\gamma \upharpoonright \kappa_1 \times \alpha$ that is in D^α_β where $(\alpha, \beta) = b(\gamma)$, given that $\alpha > \kappa_1$. We set

$$p_{\gamma+1} = p_\gamma \cup q \cup ((\bigcup j[g_1]) \upharpoonright \kappa_0 \times \{j(\gamma)\})$$

By the closure of $M[H^-]$, $p_{\gamma+1}$ is a condition in $M[H^-]$. If γ is a limit, we let $p_\gamma = \bigcup_{\delta<\gamma} p_\delta$. Again, $p_\gamma \in M[H^-]$.

This sequence generates a filter H^+. We will show that it is generic over $M[H^-]$. $\mathbb{Q}_{\kappa_2}$ has the κ_2-cc, so that any maximal antichain A in there is already a maximal antichain in $\mathbb{Q}_\alpha$ for some $\kappa_1 < \alpha < \kappa_2$. Hence the downwards closure of A is a dense subset there which is met by H^+ by construction. Thus H^+ meets A. This shows that $H = H^- \times H^+$ is as desired.

□

This allows us to lift j^+ to an elementary embedding

$$j^{++} : V[g_1] \to M[H]$$

Let $U = \{A \in \mathcal{P}(\kappa_0)^{V[g_1]} \mid \kappa_0 \in j^{++}(A)\}$. U induces an elementary embedding

$$i : V[g_1] \to \mathrm{Ult}(V[g_1], U) =: N$$

and a factor embedding

$$k : N \to M[H]$$

with $j^{++} = k \circ i$.

Claim 13. *(i)* $\mathrm{crit}(k) \geqslant \kappa_2$ *(if it exists)*

(ii) $i = j^{++}$ *and* $N = M[H]$.

Proof. (i) As N contains all subsets of κ_0 in $V[g_1]$, $\kappa_1 = \omega_2^{V[g_1]} \leqslant \omega_1^N$ and as $\kappa_1 = k \circ i(\kappa_0)$, $\omega_1^N \leqslant k(\omega_1^N) = \kappa_1$. Hence $\mathrm{crit}(k) \geqslant \omega_2^N$. So suppose $\alpha < \kappa_1$. We get $i(\alpha) < \omega_2^N$ and thus $i(\alpha) = k \circ i(\alpha) = j(\alpha)$. Since j is continuous at κ_1, this finally implies that $\mathrm{crit}(k) \geqslant \omega_2^N = \kappa_2$.

(ii) As j was originally given by a (κ_0, κ_2)-extender, j^{++} is still induced by its derived (κ_0, κ_2)-extender. The claim follows if we can show that i and j^{++} coincide on $\mathcal{P}(\kappa_1)^{V[g_1]}$, which is immediate by (i).

□

By Proposition 5, we get that there is a ω_1-dense ideal on ω_1 in $V[g_1]$ whose induced generic elementary embedding restricted on the ordinals (in fact restricted to V) does not depend on the choice of generic. Finally, it is clear that CH holds in $V[g_1]$. □

Remark 14. It is worth noting that in the above proof, the final model $V[g_1]$ is, even though we dropped at some point to an inner model, a forcing extension of V. This is since g_1 was chosen $V[g_0]$-generic and hence

$$V \subseteq V[g_1] \subseteq V[g_0, g_1]$$

The intermediate model theorem yields that $V[g_1]$ is indeed a forcing extension of V. Thus it is, given enough large cardinals, possible to force the existence of an ω_1-dense ideal on ω_1.

2 Addendum

In this section we will prove the following which was used in the argument of $(iii) \Rightarrow (iv)$ of Lemma 2.

Lemma 15. *Suppose $\mathcal{A}$ is a complete Boolean subalgebra of* $\mathrm{RO}(\mathrm{Col}(\omega, \lambda))$ *so that $\mathcal{A}^\times$ collapses λ to ω. Then $\mathcal{A} \cong$* $\mathrm{RO}(\mathrm{Col}(\omega, \lambda))$.

Proposition 16. *Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{A}$ is a complete Boolean subalgebra of $\mathcal{B}$ and $\mathcal{B}^\times$ has a dense subset of size λ then $\mathcal{A}^\times$ has a dense subset of size $\leqslant \lambda$.*

Proof. Let $D \subseteq \mathcal{B}^\times$ be a dense subset of size λ. Let

$$\pi : D \to \mathcal{A}$$

be given by

$$\pi(d) = \inf\{a \in \mathcal{A} \mid d \leqslant_{\mathcal{B}} a\}$$

Clearly $0 \notin \operatorname{ran}(\pi)$. Now if $a \in \mathcal{A}^\times$, there is some $d \in D$ with $d \leqslant_{\mathcal{B}} a$ by density of D. Thus $\pi(d) \leqslant_{\mathcal{A}} a$. This shows that $\operatorname{ran}(\pi)$ is dense in $\mathcal{A}^\times$. □

The following fact is wellknown:

Fact 17. *If $\mathbb{P}$ is a separative partial order of size λ which collapses λ to ω. Then there is a dense embedding*

$$\pi : \mathrm{Col}(\omega, \lambda) \to \mathbb{P}$$

Proposition 18. *If $\mathbb{P}$ is a partial order, $\mathcal{A}, \mathcal{B}$ are complete Boolean algebras so that $\mathbb{P}$ embeds densely into $\mathcal{A}^\times$ as well as $\mathcal{B}^\times$ then $\mathcal{A} \cong \mathcal{B}$.*

Proof. For $\mathcal{C} \in \{\mathcal{A}, \mathcal{B}\}$ let $\pi_{\mathcal{C}} : \mathbb{P} \to \mathcal{C}^\times$ be a dense embedding and let $D_{\mathcal{C}} = \operatorname{ran}(\pi_{\mathcal{C}})$. Note that since $D_{\mathcal{C}}$ is dense in $\mathcal{C}^\times$, any $c \in \mathcal{C}$ satisfies

$$c = \sup\{d \in D_{\mathcal{C}} \mid d \leqslant_{\mathcal{C}} c\}$$

with the convention $\sup \varnothing = 0_{\mathcal{C}}$. Put

$$\eta : D_{\mathcal{A}} \to D_{\mathcal{B}},\ \eta = \pi_{\mathcal{B}} \circ \pi_{\mathcal{A}}^{-1}$$

Let

$$\mu : \mathcal{A} \to \mathcal{B}$$

be given by

$$\mu(a) = \sup\{\eta(d) \mid d \in D_{\mathcal{A}} \wedge d \leqslant_{\mathcal{A}} a\}$$

It is now easy to check μ is an isomorphism of Boolean algebras. □

Proof. (*Of Lemma 15*) Let us write $\mathcal{B}$ for $\mathrm{RO}(\mathrm{Col}(\omega, \lambda))$. $\mathcal{B}$ has a dense subset of size λ, namely $\mathrm{Col}(\omega, \lambda)$. By Proposition 16, $\mathcal{A}^\times$ has a dense subset D of size $\leqslant \lambda$. As $\mathcal{A}^\times$ collapse λ, D must have size exactly λ. Also D is forcing equivalent to $\mathcal{A}^\times$ and thus collapses λ to ω as well. By Fact 17, there is a dense embedding

$$\pi : \mathrm{Col}(\omega, \lambda) \to D \subseteq \mathcal{A}^\times$$

But by the previous proposition, there is up to isomorphism a unique complete Boolean algebra that $\mathrm{Col}(\omega, \lambda)$ densely embeds into, thus

$$\mathcal{A} \cong \mathcal{B}$$

□

References

[For10] Matthew D. Foreman. Ideals and generic elementary embeddings. 2010.