Lecture Notes

Andreas Lietz

March 2024

These lecture notes are intended for the introductory Set Theory lecture at TU Wien in the summer semester of 2024. If you have any suggestions, remarks or find typos/errors, feel free to send me an email!

1 The Continuum Hypothesis

5.3.24

The real number line is perhaps the best studied mathematical object there is. Set Theorists are particularly interested in the subsets of $\mathbb{R}$ and the first interesting thing to try is classifying sets of reals by their size. Of course we can realize any finite size via the set $\{0, \dots, n\}$ for $n \in \mathbb{N}$, as well as the size of $\mathbb{N}, \mathbb{R}$ themselves as obviously $\mathbb{N}, \mathbb{R} \subseteq \mathbb{R}$. The statement that this is a complete classification is known as the Continuum Hypothesis.

Definition 1.1 (Continuum Hypothesis)**.** The **Continuum Hypothesis** (CH) states that every infinite $X \subseteq \mathbb{R}$ is either countable, so in bijection with $\mathbb{N}$ or has the same size as $\mathbb{R}$, so is in bijection with $\mathbb{R}$.

Whether or not the continuum hypothesis is true was one of the most important mathematical questions of the 20th century, appearing as the first of the 23 questions posed by David Hilbert at the ICM in the year 1900.

The Austrian Kurt Gödel proved in the 30s that CH is at least not contradictory. It took another 30 years for Paul Cohen to show the dual statment: The negation of CH is not contradictory either, netting him a fields medal.

Proving Gödels result will be a central part of this lecture. Let us now begin with Cantor's early attempts at settling CH. His idea was to show that simple sets of reals cannot contradict CH and then push through to more and more complex sets of reals until finally CH is proven completely. While this project cannot be fully completed it was nonetheless a very fruitful strategy. Nowadays, Set Theorists have a good understanding of how complicated counterexamples to CH must be if they exist.

Theorem 1.2 (Cantor-Bendixson)**.** *Closed sets of reals are not counterexamples to* CH*, i.e. an uncountable closed set is in bijection with* $\mathbb{R}$*.*

We will show this by proving that any closed set of reals is the union of a perfect closed set P and a countable set A. Moreover, non-empty perfect closed sets are in bijection with $\mathbb{R}$.

Definition 1.3. A set $P \subseteq \mathbb{R}$ is **perfect** if for all $x \in P$, $x \in \overline{P \setminus \{x\}}$.

We will not try to give the most efficient proof, rather we want to illustrate some Set Theoretical ideas.

We will replace $\mathbb{R}$ by the interval $[0,1]$ and represent closed sets $C \subseteq [0,1]$ by binary trees. For 0-1-sequences $s, t \in \{0,1\}^{\leq \mathbb{N}}$ write $s \leq t$ if s is an initial segment of t, i.e. if there is some $r \in \{0,1\}^{<\mathbb{N}}$ so that $t = s^\frown r$.

Definition 1.4 (Binary Trees)**.** (*i*) A binary tree is a subset $T \subseteq \{0,1\}^{<\mathbb{N}}$ of finite 0-1-sequences which is closed under initial segments, i.e. if $t \in T$ and $s \leq t$ then $s \in T$.

(*ii*) A branch through a binary tree T is a subset $b \subseteq T$ which is closed under initial segments and linearly ordered by $\leq$.

(*iii*) The set of cofinal branches through T is

$$[T] := \{b \subseteq T \mid b \text{ is an infinite branch}\}.$$

For $b \in [T]$, b^* is the unique infinite sequence in $\{0,1\}^{\mathbb{N}}$ which all points in b are an initial segment of.

(*iv*) A binary tree T **represents** the set

$$[\![T]\!] := \{x \in [0,1] \mid \exists b \in [T]\ x = (0.b^*)_2\}$$

Here, $(0.a_1a_2a_3\dots)_2 = \sum_{n=1}^{\infty} a_1 \cdot 2^{-n}$ is the evaluation of a binary representation.

Proposition 1.5. *The following are equivalent for a set $D \subseteq [0,1]$:*

(*i*) *D is closed.*

(*ii*) *There is a binary tree T representing D, that is $D = [\![T]\!]$.*

Proof. $(i) \Rightarrow (ii)$: The set

$$T_D := \{t \in \{0,1\}^{<\mathbb{N}} \mid \exists b \in \{0,1\}^{\mathbb{N}}\ (0.b)_2 \in D \wedge t \leq b\}$$

is a binary tree with $[\![T_D]\!] = D$. "$\subseteq$" is obivous, while "$\supseteq$" holds as D is closed: If $x \in [\![T_D]\!]$ the there is $b \in [T_D]$ with $x = (0.b*)_2$. Find sequences $a_n \in \{0,1\}^{\mathbb{N}}$ with $(0.a_n)_2 \in D$ and $b \restriction n \leq a_n$ where

$$b \restriction n = b_1 \dots b_n$$

for $b^* = b_1b_2\dots$. It follows that $|(0.a_n)_2 - (0.a_m)_2| \leq 2^{-n}$ for $n \leq m$ so that

$$(0.b^*)_2 = \lim_{n\to\infty} (0.a_n)_2 \in D.$$

$(ii) \Rightarrow (i)$: We show that $[\![T]\!]$ is closed for all binary trees T. Suppose that $x_n \in [\![T]\!]$ for $n \in \mathbb{N}$ and $x_n \xrightarrow{n\to\infty} x$. As $x_n \in [\![T]\!]$, there is a sequence

$$a_1^n a_2^n \dots \in \{0,1\}^{\mathbb{N}}$$

with all finite initial segments in T and $x_n = (0.a_1^n a_2^n \dots)_2$.

Claim 1.6. *There is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ so that $(a_m^{n_k})_{k\in\mathbb{N}}$ is eventually constant for all $m \in \mathbb{N}$.*

Proof. Define sequences $(n_k^l)_{k\in\mathbb{N}}$ by induction on l. Let $n_k^0 = k$ for $k \in \mathbb{N}$ and now suppose that $(n_k^l)_{k\in\mathbb{N}}$ has been defined. $(a_l^{n_k^l})_{k\in\mathbb{N}}$ is a sequence which only takes one of two values, so we can then find a subsequence $(n_k^{l+1})_{k\in\mathbb{N}}$ on which it is constant.

Finally, the diagonal sequence $n_k = n_k^k$ does the job. □

(We have basically proven here that $\{0,1\}^{\mathbb{N}}$ is compact. The reader comfortable with this fact can ignore the claim above)

Let b_m be the eventual value of $((b_m)^{n_k})_{k\in\mathbb{N}}$. Then it is easy to see that

$$[\![T]\!] \ni (0.b_1b_2\ldots)_2 = \lim_{k\to\infty}(0.a_1^{n_k}a_2^{n_k}\ldots)_2 = \lim_{n\to\infty} x_n = x.$$

□

We can also describe perfect closed sets in terms of binary trees.

Definition 1.7. Suppose T is a binary tree.

1. A node $t \in T$ **splits** if both $t^\frown 0$, $t^\frown 1$ are in T.
2. The tree T is **perfect** iff every $s \in T$ can be extended to some $s \leq t \in T$ which splits in T.

Proposition 1.8. *A closed set $D \subseteq [0,1]$ is perfect iff there is a perfect binary tree T representing D.*

Partial proof. We only show the easier direction as we have no use for the other implication anyway. Clearly $[\![\emptyset]\!] = \emptyset$ is perfect, so let T be a non-empty perfect tree and $x \in [\![T]\!]$, say $x = (0.a_1a_2\ldots)_2$ and all finite initial segments of $a_1a_2\ldots$ are in T. For each $k \in \mathbb{N}$, let a_{n_k} be the k-th splitting point along the branch b given by $a_1a_2\ldots$, which must exist as T is perfect. Further, since T is perfect, we can extend $a_1\ldots a_{n_k}^\frown(1-a_{n_k+1})$ to an infinite branch b_k, so b and b_k differ first at their n_k+1-th node. In particular,

$$|(0.b^*)_2 - (0.b_k^*)_2| \leq 2^{-k}$$

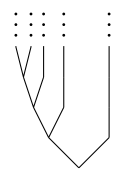
which shows $(0.b^*) = x \in \overline{[\![T]\!] \setminus \{x\}}$ □

Next we describe how we can reduce binary trees to perfect binary trees. The idea is to cut off isolated branches which do not split anymore.

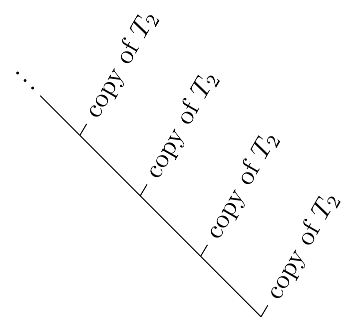
Definition 1.9. If T is a binary tree then the **derivative of** T is the binary tree

$$T' := \{t \in T \mid T \text{ splits above } t\}.$$

In some sense T' is closed to a perfect tree than T was. However T' certainly need not be perfect. Consider for example to following tree T_2:



Then T_2' is the leftmost branch of T_2 and not perfect. In fact $(T_2')' = \emptyset$. We can easily continue to produce a tree whose 3rd derivative is $\emptyset$, but not the 2nd, e.g. the tree T_3:



For a binary tree T, define inductively $T^{(0)} = T$ and $T^{(n+1)} = (T^{(n)})'$. So for every n there is a binary tree T with $T^{(n+1)} = \emptyset \neq T^{(n)}$. We set $T^\omega = \bigcap_{n<\omega} T_n$. It is still not guaranteed that T^ω is perfect. Does this mean we have to abandon ship and this construction is not helpful? No! We just have to continue this continue this construction transfinitely! To do so properly, we have to introduce ordinals. In the end we will have the following:

Lemma 1.10. *For every binary tree T, there is some countable ordinal α so that $T^{(\alpha)}$ is perfect.*

Note that a binary tree S is perfect iff $S' = S$, so the above happens only at the first α so that $T^{(\alpha+1)} = T^{(\alpha)}$.

Now, if $C \subseteq [0,1]$ is closed, let T_C be a binary tree representing C. Then let α be countable with $T_C^{(\alpha)}$ perfect. We set $P = [\![T_C^{(\alpha)}]\!]$, which is perfect, and $A = C \setminus P$. We have to show that A is countable.

Proposition 1.11. *If T is a binary tree then $[\![T]\!] \setminus [\![T']\!]$ is countable.*

Proof. We cut off at most countable many branches and each branch is responsible for the binary representation of at most one real number in $[\![T]\!]$ the branch does not split. $\square$

Hence we can write

$$A = [\![T_C]\!] \setminus [\![T_C^{(\alpha)}]\!] = \bigcup_{\beta<\alpha} [\![T_C^{(\beta)}]\!] \setminus [\![T_C^{(\beta+1)}]\!]$$

which is a countable union of countable sets and hence countable.

To complete the proof of the Cantor-Bendixson Theorem, it remains to show that non-empty perfect closed sets are large.

Lemma 1.12. *If $P \subseteq [0,1]$ is nonempty and perfect closed then there is a bijection between P and $[0,1]$.*

We make use of a theorem we promise to prove at a later stage.

Theorem 1.13 (Cantor-Schröder-Bernstein)**.** *If there are injections $X \hookrightarrow Y$ and $Y \hookrightarrow X$ then there is a bijection between X and Y.*

Proof of Lemma 1.12. Let P be non-empty perfect closed. Clearly there is an injection $P \hookrightarrow \mathbb{R}$, e.g. the inclusion, so it remains to find an injection $[0,1] \hookrightarrow P$. Let T be a perfect tree representing P. We may arrange that every $x \in P$ is uniquely represented by a branch through T in the sense that if $b, c \in [T]$ are different then $(0.b^*)_2 \neq (0.c^*)_2$, the details are left to the reader. We first define an embedding $j\colon \{0,1\}^{<\mathbb{N}} \to T$ of the full binary tree into P by induction. We make sure that all nodes in $\mathrm{ran}(j)$ are splitting nodes of T. Map the empty sequence to the (unique) shortest splitting node of T (this exists as P is non-empty, so T is non-empty). Next, if $j(s)$ is defined, for $i = 0, 1$ let $j(s^\frown i)$ be the next splitting node of T above $j(s)^\frown i$. As j respects the initial segment relation $\leq$, j lifts to a map on the cofinal branches

$$j^+\colon [\{0,1\}^{\mathbb{N}}] \to [T]$$

via $j^+(b) = j[b]$, the pointwise image of b under j. As j is injective, so is j^+.

Putting everything together, we get an injection

$$[0,1] \hookrightarrow \{0,1\}^{\mathbb{N}} = [\{0,1\}^{<\mathbb{N}}] \xhookrightarrow{j^+} [T] \hookrightarrow [\![T]\!] = P$$

where the first arrow is choosing a binary representation and the last map is $b \mapsto (0.b^*)_2$. $\square$

Tree constructions as above are immensely useful in Set Theory. When working with real numbers, the non-uniqueness of binary representation is sometimes somewhat annoying (as it is above as well). For that reason, the interval $[0,1]$ is usually replaced by the infinite binary sequences $\{0,1\}^{\mathbb{N}}$ and $\mathbb{R}$ is replaced by $\mathbb{N}^{\mathbb{N}}$. While the replacements are not homeomorphic to the originals, the differences are minor and can be neglected in almost all cases of interest. 6.3.24

2 Zermelo-Fraenkel Set Theory