0 Preserving Suslin Trees with Finite Support

Definition 1. A sequence $\langle \mathbb{B}_\alpha \mid \alpha < \delta \rangle$ of cBa's is called continuous if $\mathbb{B}_\alpha <_{\text{reg}} \mathbb{B}_\beta$ for all $\alpha < \beta < \delta$ and for all limit $\alpha < \delta$, $\mathbb{B}_\alpha = \text{dir lim}_{\beta<\alpha} \mathbb{B}_\beta$, that is $\mathbb{B}_\alpha$ is the completion of $\bigcup_{\beta<\alpha} \mathbb{B}_\beta$ or equivalently that union is dense in $\mathbb{B}_\alpha$.

Lemma 2. *if $\langle \mathbb{B}_\alpha \mid \alpha < \gamma \rangle$ is a continuous chain of cBa's and κ is regular uncountable so that every $\mathbb{B}_\alpha$ has the κ-cc then the direct limit $\mathbb{B}$ of this sequence is κ-cc.*

Proof. We proof this by induction on δ. If δ is a successor this is trivial. If δ is a limit, this is trivial for $\text{cof}(\delta) \neq \kappa$, so assume $\text{cof}(\delta) = \kappa$. Let $\{b_\alpha \mid \alpha < \kappa\}$ be a sequence in $\mathbb{B}^\times$. Let $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ be a normal sequence of limit ordinals cofinal in κ. For $\alpha < \kappa$ let p_α be the projection of $\mathbb{B}$ onto $\mathbb{B}_\alpha$, that is

$$p_\alpha(b) = \inf\{c \in \mathbb{B}_\alpha \mid b \leqslant c\}$$

For $\alpha < \delta$ limit let

$$\mathbb{B}_{<\alpha} = \bigcup_{\beta<\alpha} \mathbb{B}_\beta$$

It is easy to show the following:

Claim 3. *If $\alpha < \kappa$ and $c \leqslant p_\alpha(b)$, $c \in \mathbb{B}_\alpha$ and $c, b \neq 0$, then $c \wedge b \neq 0$.*

$\square$

Now for any $\alpha < \delta$ choose $c_\alpha \in \mathbb{B}_{<\gamma_\alpha}$ so that

$$c_\alpha \leqslant p_{\gamma_\alpha}(b_\alpha)$$

and find $\eta_\alpha < \gamma_\alpha$ so that $c_\alpha \in \mathbb{B}_{\eta_\alpha}$. There is a stationary $S \subseteq \kappa$ and some $\eta < \kappa$ so that $\eta = \eta_\alpha$ for α in S. Observe that

$$\{c_\alpha \mid \alpha \in S\}$$

cannot be an antichain as $\mathbb{B}_\eta$ has the $\kappa-$cc. Thus there are $\alpha < \beta$ both in S so that $c_\alpha \parallel c_\beta$, witnessed by $c \in \mathbb{B}_\eta$. I claim that b_α and b_β are compatible. Let $a = c \wedge b_\alpha \wedge b_\beta$. We show that $a \neq 0$. By the claim, $c \wedge b_\alpha \neq 0$. Furthermore $c \wedge b_\alpha \in \mathbb{B}_{\gamma_\alpha} \subseteq \mathbb{B}_{\gamma_\beta}$ and

$$c \wedge b_\alpha \leqslant c \leqslant p_{\gamma_\beta}(b_\beta)$$

and by the claim again, $a = c \wedge b_\alpha \wedge b_\beta \neq 0$.

$\square$

Definition 4. We call a sequence $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ of p.o.'s continuous iff $\mathbb{P}_\alpha <_{\text{reg}} \mathbb{P}_\beta$ for $\alpha < \beta < \delta$ and for limit $\alpha < \delta$, $\mathbb{P}_\alpha = \bigcup_{\beta<\alpha} \mathbb{P}_\beta$ with the natural ordering.

Corollary 5. *If $\langle \mathbb{P}_\alpha \mid \alpha < \delta \rangle$ is a continuous sequence of separative p.o.'s which all satisfy the κ-cc for regular uncountable κ, then the direct limit $\mathbb{P}$, namely $\bigcup_{\alpha<\delta} \mathbb{P}_\alpha$, satisfies the κ-cc.*

Proof. For $\alpha < \delta$ let $\mathbb{B}_\alpha \cong \mathrm{RO}(\mathbb{P}_\alpha)$ so that $\mathbb{B}_\alpha$ is a subalgebra for $\mathbb{B}_\beta$ if $\alpha < \beta < \delta$. It is easy to check that $\langle \mathbb{B}_\alpha \mid \alpha < \delta \rangle$ is a continuous sequence of cBa's and that each $\mathbb{B}_\alpha$ has the κ-cc. Thus $\mathbb{B} = \mathrm{RO}(\mathbb{P})$ has the κ-cc and so $\mathbb{P}$ does, too. □

Corollary 6. *If T is a Suslin tree, then finite support iterations preserve the property ccc+"preserving T". That is if $\mathbb{Q} = \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ is a finite support iteration such that $\mathbb{Q}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is ccc and preserves $\check{T}$" then $\mathbb{Q} = \mathbb{Q}_\delta$ is ccc and preserves T, too.*

Proof. By induction on δ. This is trivial for δ successor, so assume otherwise. Clearly, $\mathbb{Q}$ is ccc so that $\mathbb{Q}$ preserves T if and only if $\mathbb{Q} \times T$ is ccc. Now

$$\mathbb{Q} \times T = \mathrm{dir\ lim}_{\alpha<\delta}\, \mathbb{Q}_\alpha \times T$$

and $\langle \mathbb{Q}_\alpha \mid \alpha < \delta \rangle$ is a continuous sequence of p.o.'s (simply by finite support). As each $\mathbb{Q}_\alpha$ is ccc and preserves T by induction, $\mathbb{Q}_\alpha \times T$ is ccc for all $\alpha < \delta$. Hence $\mathbb{Q} \times T$ is ccc as well. □