Dynamics

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Introduction

This set of lecture notes is based on lectures given by Andrei Gruzinov at NYU in the Spring of 2021, for the class PHYS-UA 120. Additionally, the books Mechanics by Landau and Lifshitz, Classical Mechanics by John R. Taylor, and The Variational Principles of Mechanics by Cornelius Lanczos are used in order to improve the notes.

Remark. In these notes, we will use the so-called "postfix" notaition for integrals:

$$\int dx f(x).$$

In theoretical physics, since integrands can be very long and complex, this notation has the benefit of clearly indicating what the variable is. The important thing is to note that these conventions have no bearing on the underlying mathematics. For instance, the physicist Juan Maldacena sometimes writes integrals like so:

$$\int f(x)$$

omitting the dx.

Once we get the physicists intuition in these mathematical expressions, we will become more comfortable using these expressions.

Consider the simple harmonic oscillator, governed by the force equation $F = -kx$ and the differential equation

$$\frac{d^2x}{dx^2} + \frac{k}{m}x = 0.$$

This begs the question: why is the force linear (to a very good approximation)? This is because we know that at $x = 0$, $F = 0$. Then we know that if we plot x versus F, then we know that the equation crosses the origin. Assuming that the function is C^2, we see that there is a linear approximation to the function near the origin. Indeed, Hooke's law is only valid for small scales.

However, we can recast this problem as a minimization problem. The quantity we are minimizing is known as *action*.

1 Calculus of Variations

1.1 Generalized Coordinates

In physics, one of the fundamental objects we study is a particle. In general, this is any object whoes dimensions can be ignored when we describe its motion. For instance, we are allowed to approximate the Earth with a point when attempting to describe its motion, since interplanetary distances are much more vast than the size of the Earth. In short, particles are things to which we can assign coordinates.

The position of a particle in space is defined by a vector $\mathbf{r}$ from the origin. In Cartesian coordinates, its comonents are x, y, z given by its position along the three axes. The *velocity* and *acceleration* are given by $\mathbf{v}$ and $\mathbf{a}$, respectively, where

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$$
$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}.$$

It's customary to put a dot over a quantity to denote its time derivative.

Given N particles in 3-D space, we need $3N$ coordinates to describe it. Moreover, the number of independent coordinates needed to uniquely describe the position of a system is called the *degrees of freedom*. For instance, a spring only needs one coordinate to describe the system, which is its length x; a ball thrown in a perfect vacuum needs two; and a bug flying in a room needs three. Notice how all of these objects have been approximated by points as well.

These degrees of freedom don't need to be the Cartesian coordinates; for instance, to describe a fly in a room, we can just as well use spherical coordinates ρ, θ, and ϕ.

Definition 1.1.1 (Generalized Coordinates and Velocities)**.** For a system with n degrees of freedom, any n quantities $q_1, \ldots, q_n$ which completely define a system with n degrees of freedom are known as the *generalized coordinates* of a system. Their time derivatives $\dot{q}_1, \ldots, \dot{q}_n$ are known as the *generalized velocities*.

Notation. To denote the set of all coordinates, we sometimes use $\mathbf{q}$. To denote the set of the velocities, we use $\dot{\mathbf{q}}$.

It's generally not enough to state the generalized coordinates of a system. More often we have to know the velocities of each particle. Once we know that, any further evolution of the state of a deterministic system can in principle be calculated. The relations between accelerations, velocities, and coordinates are called the *equations of motion*.

1.2 Functions and Functionals

Recall the definition of a function in one real variable. It is a mapping f, where

$$f : \mathbb{R} \to \mathbb{R}$$

such that if $x = y$, then $f(x) = f(y)$. Alternatively, it can be thought of as passing the vertical line test.

However, a functional can be thought of as a function that takes functions as input: On the other hand, a functional is a mapping from a set of functions to the set of real numbers:

Definition 1.2.1 (Functional)**.** Let V be a vector space over a field K. Then a *functional* is a mapping F such that

$$F : V \to K.$$

That vector space may be vectors, or it could be the Banach space of Lebesgue-integrable functions L^1. For our purposes, it will be the vector space of continuous functions:

$$F[f] : \{\text{the set of continuous functions on } \mathbb{R}\} \to \mathbb{R}.$$

For instance, an example of a functional F corresponds to "evaluate f at 0". That is, given f, $F[f] = f(0)$. A more interesting example might be

$$F[f(x)] = \int_0^1 dx \left(\frac{1}{2}f'^2 - \frac{1}{2}f^2\right)$$

whose significance will be revealed later.

1.3 Stationary Values of Functionals

Akin to minimization and maximization problems in calculus, we can compute stationary values of functionals. Given a function $U(x, y, z)$, we know that the function is locally at a minimum or maximum (or saddle point!) when the following is satisfied:

$$\nabla U(x, y, z) = 0.$$

For a potential, this means that the component of the force in the x, y, and z directions are all 0. We wish to formulate some similar conditions for functionals. This process of trying to provide optima for functionals is known as the Calculus of Variations. Solving these variational problems is in some sense finding the solution of a differential equation, as we will see shortly.

Most often, a functional is written as an intgral of the form

$$F = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\ dt$$

where $\mathcal{L}$ is a function known as the *Lagrangian*. Supposing there is a minimum, let $q(t)$ be the function for which F is at a minimum. Then if we perturb our function q,

$$q(t) \mapsto q(t) + \delta q(t)$$

then F increases. Note that since q must satisfy certain boundary conditions, we get that $\delta q(t_1) = \delta q(t_2) = 0$. Then the change in F is

$$\delta F = \int_{t_1}^{t_2} \mathcal{L}(q + \delta q, \dot{q} + \delta\dot{q}, t)\ dt - \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\ dt$$

If we expand the terms out and compute the difference, we get

$$\delta F = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}\right) dt.$$

[WHY]? Next, integrating by parts gives us

$$\delta F = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right)\delta q$$

where the first term vanishes, since from our boundary conditions $\delta q(t_1) = \delta q(t_2) = 0$. Therefore, the requirement that F is stationary is the same as saying that $\delta F = 0$ for all variations δq. This holds if and only if we have that the differential terms in the middle are 0.

Theorem 1.3.1 (Euler-Lagrange Equation)**.** Given a Lagrangian $\mathcal{L}(q, \dot{q}, t)$ and a functional F of the form

$$F = \int dt\ \mathcal{L}(q, \dot{q}, t)$$

is stationary if the *Euler-Lagrange equation* is satisfied:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right). \tag{1.3.1}$$

Calculus of Variations (CoV for short) helps us greatly simplify mechanical systems. For instance, consider the trajectory of a spring:

[DRAW IMAGE]

Amaxingly, we can state that the actual trajectory of the oscillator minimizes the integral

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right).$$

Every C^2 function is locally approximable by a line. The derivative can be interpreted as

$$\delta y = \alpha \delta x$$

where if $\alpha > 0$, a positive change in x increases f, and if $\alpha < 0$, a negative change in x increases f. The only way we can be sure that there is (locally) no more optimal move is if $\alpha = 0$.

Now consider the functional

$$S[f(x)] = \int_0^1 dx f(x);\ f(0) = f(1) = 0.$$

If we want to maximise or minimize the functional, then we get $-\infty$ or $+\infty$, since we can arbitrarily make the function enclsoe more and more area.

Moreover, we want to ensure that close to the boundaries.

By definition, we see that

$$S[f(x) + \delta f(x)] = \int_0^1 (f(x) + \delta f(x))$$

where $\delta f = f(x - dx) - f(x)$.

$$\delta S = S[f + \delta f] - S[f]$$

Now we subtract thw two integrals:

$$\delta S = \int_0^1 dx \delta f(x)$$

Thus the variation of the functional with variation of the function does not vanish to first order. Therefore, this function has neither a maximum or a minimum.

This second example will be nontrivial. Let us define a functional

$$S = \int_0^1 dx f'^2;\ f(0) = 0;\ f(1) = 1.$$

We can try the function $f(x) = x$. It has a derivative of 1, and us $S = 1$. If we take $f(x) = x^2$, then we get $S = 4/3$. If we try $\sqrt{x}$, we indeed get a larger area; $S = +\infty$ to be exact. Thus, using our variational principles,

$$S[f+\delta f]-S[f]=\int_0^1 dx(f'+\delta f')^2=\int_0^1 dx(f'^2+3f'\delta f'+\delta f'^2)$$

so subtracting, we get

$$\delta S=\int_0^1 dx(2f'\delta f'+\delta f'^2)$$

Just as locally, every well-behaved function is a line., so too is every functional locally a linear functional. We then see that, dropping the quadratic term (since it is small),

$$\delta S=-2\int_0^1 dx f''\delta f$$

where we have integrated by parts.

Therefore, we must know that f'' must be 0 everywhere, since we will be able to increase or decrease our functional. Therefore $f''=0$ everywhere. We treat f'' as our coefficient of proportionality. With the boundary conditions, the only function that achieves this is $f(x)=x$.

Now are some nice results we can get from calculus of variations.

1.4 Geodesics

1.4.1 The Geodesic Problem

Suppose we are given two points on a plane, (x_1,y_1) and (x_2,y_2), and we want to find the shortest path between the two. The problem can be made rigorous through the Calculus of Variations. Consider a differentiable function $y(x)$, which determines a possible path between the two points. Its length L can be calculated by

$$L[y]=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

In essence, we are trying to find a function such that the the quantity L is at a stationary point, or that changing our function y by even a little bit will give us a larger value for L. This is akin to a function being minimized for any change in x.

In general, given an arbitrary length function, we can take the integral and

We can express the length of the line given by $y=f(x)$, then the length fo the line is

$$L[f]=\int_0^1 dx\sqrt{1+f'(x)^2}.$$

Length, therefore, is a functional of the shape. Therefore, a small variation of L given by a small variation of f is given by

$$\frac{1}{2\sqrt{1+f'^2}}2f'\delta f'$$

Therefore we get

$$\left(\frac{f'}{\sqrt{1+f'^2}}\right)'=0$$

so the geodesic in Euclidean space is a line.

If we want to minimize distance on a sphere, say the Earth, we can solve this problem analogously. Put one point on the great circle as the origin, and the other one at an angle ϕ; thus a deviatoin of the length is

$$dS^2=R^2(d\theta^2+\sin^2\theta d\phi^2).$$

Thus, since length is a functional, we see

$$\mathcal{L}[\phi(\theta)] = R\int d\theta(1+\sin^2\theta\phi'^2)^{1/2}.$$

1.4.2 Hyperbolic Space

Lastly, in hyperbolic space, we can consider the Lobachevsky plane. The geodesics of this plane can be calculated by minimizing the length functional characterized by

$$L[y(x)] = \int d\ell = \int dx\frac{\sqrt{1+y'^2}}{x}$$

This gives us the Euler-Lagrange equation

$$\begin{aligned}\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'} &\Rightarrow 0 = \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+y'^2}}\right)\\ &\Rightarrow \frac{y'}{x\sqrt{1+y'^2}} = C\\ &\Rightarrow y' = \frac{Cx}{\sqrt{1-C^2x^2}}\\ &\Rightarrow y = \sqrt{\frac{1}{C^2}-x^2}+y_0\end{aligned}$$

Where y_0 is a constant. However, we note that this equation can be rearranged to be

$$x^2+(y-y_0)^2 = \frac{1}{C^2};\ x>0,\ C\neq 0.$$

This represents a semicircle on the y-axis. We can consider C to be the radius of curvature.

1.5 The Isoperimetric Problem

We know intuitively that the maximal possible area given a fixed perimeter is a circle. We can think of our solution as a curve in $\mathbb{R}^3$ given by the parametrization

$$\mathbf{c}(t) = (x(t), y(t), 0)$$

subject to the boundary condition $x(0) = x(1)$, $y(0) = y(1)$. The differential length is simply

$$ds^2 = dx^2 + dy^2$$

which imposes the condition

$$L = \int dt\sqrt{\dot{x}^2+\dot{y}^2}.$$

The differential area is given by half of the parallelogram determined by $\mathbf{c}$ and $\mathbf{c}'$, or

$$dA = \frac{1}{2}\mathbf{c}\times\mathbf{c'}dt = (x\dot{y}-y\dot{x})dt$$

and we look for the extrema of

$$\mathcal{A}[\mathbf{c}(t)] = \int dA = \int dt\ x\dot{y}-y\dot{x}.$$

This gives rise to the Euler-Lagrange equations

$$\frac{\partial \mathcal{A}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{A}}{\partial \dot{x}}$$
$$\frac{\partial \mathcal{A}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{A}}{\partial \dot{y}}.$$

Explicitly, these are

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(\frac{1}{2}y - \lambda\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0$$
$$\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x - \lambda\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0$$

If we take t to be an arclength parameter, we get the simplified equations

$$\dot{y} + \lambda\ddot{x} = 0$$
$$-\dot{x} + \lambda\ddot{y} = 0$$

The solutions to this differential equation are

$$x = x_0 + \lambda\cos\frac{t-t_0}{\lambda}$$
$$y = y_0 + \lambda\cos\frac{t-t_0}{\lambda}$$

and these are the perimetric equations for a circle. Therefore given a circumference of 100m, we can find the maximal area enclosed:

1.6 Galilean Groups and Invariance

In this section, we investigate calssical notions of relativity and inertial frames. Note that what follows is a good approximation as long as the velocities involved are small (i.e., non-relativistic).

We can consider an n-dimensional affine space A^n, which differs from the vector space $\mathbb{R}^n$ in that it does not have a fixed origin. The elements in an affine space are called points, and

In particular, A^n is a set with no identity, and $\mathbb{R}^n$ is a group that acts on the set A^n as the so-called *group of parallel displacements*. Essentially, with this action, each vector in $\mathbb{R}^4$ can be considered a way to translate the system:

$$a \mapsto a + \mathbf{r};\ a \in A^n,\ \mathbf{r} \in \mathbb{R}^n$$

where the resultant is an element of A^n. In Galileo's picture, spacetime has the following structure:

1. The universe is a four-dimensional affine space A^4. The points in A^4 are known as *events* or world-points.

2. Time is a linear mapping $t : \mathbb{R}^4 \to \mathbb{R}$ frim the vector space of parallel displacements to the real time axis.

3.

2 The Principle of Least Action

Whenever we find stationary values, we implicitly are also solving an optimization problem. If we consider a particle, we assume that there is some optimization principle behind it. This is the key shift in reformulating Newton's equations;

The "Principle of Least Action" is a bit of a misnomer; rather than a global minimum, we find a function for which the value of the action integral is stationary.

Theorem 2.0.1 (Principle of Least Action)**.** The true trajectory $q(t)$ of a particle is given by making the following action integral stationary:

$$\mathcal{S}[q(t)] = \int_{t_1}^{t_2} dt\ \mathcal{L}(q, \dot{q}, t).$$

The advantages to the Principle of Least action are many; for instance, the stationary value is invariant with respect to coordinate transformations, even though the Lagrangian may change.

2.1 Motion in One Dimension

The question is, what $\mathcal{L}$ should we consider such that we get out Newton's equations? The quantities related to the equations of motion which depend on q and $\dot{q}$ are, respectively, the potential energy U and kinetic energy U.

Theorem 2.1.1. For a particle in 1 dimension, the function $\mathcal{L}$ which will give us Newton's equations when we solve the Euler-Lagrange equation is

$$\mathcal{L}(x, \dot{x}, t) = T - U = \frac{1}{2}m\dot{x}^2 - U(x). \tag{2.1.1}$$

Proof. Plugging in the function $\mathcal{L}(x, \dot{x}, t) = T - U$ into the Euler-Lagrange equation, we get

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \iff -\frac{dU}{dx} = m\ddot{x}$$

However, recalling that $F = -\partial U/\partial x$, we get

$$F = m\ddot{x}.$$

□

For the same oscillating system, we no longer look at force. Instead, we look at a quantity called *action*, which is a function of the trajectory:

$$\mathcal{S}[x(t)] = \int_{t_1}^{t^2} dt\mathcal{L}(x, \dot{x}).$$

Where $\mathcal{L}$ is the *Lagrangian* of the system defined by

$$\mathcal{L}(x, \dot{x}) = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

In spacetime, recall that our oscillator moves along its worldline from an initial point to an ending point. We associate action with not only the actual trajectory, but with every single possible one.

"Action is just a length in a 4 dimensional spacetime, and thus length is the most natuarl object." ??????

The system moves to minimize action for initial and final positions $x(t_i)$ and $x(t_f)$. Thus we obtain the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'}$$

Thus we obtain

$$kx = \frac{d}{dt}(-m\dot{x})$$

Which gives us the familiar equation

$$\ddot{x} + \frac{k}{m}x = 0.$$

2.1.1 Newton or Hamilton?

Which is the truth, which one is better? From the point of view from the equation, these two formulations are equivalent. In Newton's picture,

no friction in hamilton? In hamilton;s picture, energy is always conserved

A chicken is not a bird; a logarithm is not infinity. Landau

Suppose you travel with acceleration g with a spaceship.

2.2 Generalized Coordinates

Consider our harmonic oscillator,

$$\ddot{x} + \omega^2 x = 0.$$

However, we replace x by the expression

$$x = a(u + u^3),$$

where a has dimensions of meters, and u is dimensionless. By the chain rule, we can find $\dot{x}$:

$$\dot{x} = a(1 + 3u^2)\dot{u}$$
$$\ddot{x} = a\left((1 + 3u^2)\ddot{u} + x\right)$$

Only in Cartesian Coordinates are Newton's laws dependent only on position. In generalized coordinates,

$$\delta\mathcal{S} = \int dt\,(\mathcal{L}(q + \delta q, \dot{q} + \delta\dot{q}) - \mathcal{L}(q, \dot{q}))$$

The question of calculating the quantity in the parenthesis is the same as the problem

$$f(x + \varepsilon_1, y + \varepsilon_2) - f(x, y).$$

Consider the surface $z = f(x, y)$. Then for this surface, we can consider it locally to be represented by a tangent plane. Then we can see that

$$f(x + \varepsilon_1, y + \varepsilon_2) - f(x, y) = \frac{\partial f}{\partial x}\varepsilon_1 + \frac{\partial f}{\partial y}\varepsilon_2.$$

Therefore, our equation can be written as

$$\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}$$

Therefore with respect to η, we have $\delta\eta$ is

$$\delta\eta =$$

2.2.1 How to Construct a Lagrangian

We start with the equation

$$L = T - U$$

where

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

and where

$$U = U(\mathbf{r}_1, \ldots, \mathbf{r}_n).$$

The reason we are interested in forces which are gradients of potentials is because those are precisely the forces which give us a conservation law; this is elucidated by the fact that for any function ∇f, we get

$$\oint \nabla f \cdot r = 0.$$

3 Symmetry and Conservation

3.1 Symmetry

Consider a perfectly smooth sphere, with no patterns or any irregularities whatsoever on the surface. Then, in principle, it would be impossible to distinguish the end result of two different rotations from one another. By contrast, if you have a cube, there are only certain rotations that will yield indistinguishable states. The sphere, then, has continuous symmetry, whereas the cube has discrete symmetry[1]

In a nutshell, symmetry is when you do something to your system and nothing changes. This can be a powerful tool; the reason why the field lines of a point charge go radially outward is because if we rotate the (spherical) particle, in theory we should not be able to distinguish it, since the source of the field is identical.

There should be a small distinction about the symmetry of things, and the symmetry of mechanical systems; a system can posess symmetry even though its concrete configuration is not invariant under the corresponding transformation, as we will see.

Formally, we define a continuous symmetry of the Lagrangian:

Definition 3.1.1 (Continuous Symmetry)**.** Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of maps such that for a generalized coordinate q_i, $\sigma : q_i(t) \mapsto Q_i(s,t)$, such that $Q_i(0,t) = q(t)$. Then this transformation is called a *continuous symmetry* of the Lagrangian $\mathcal{L}$ if

$$\frac{\partial}{\partial s}\mathcal{L}(Q_i(s,t), \dot{Q}_i(s,t), t) = 0.$$

3.2 Noether's Theorem

The derivations of the conserved quantities do not betray the deep principles that unites symmetry and conservation. It's instructive to think of a physicsist in an inert rocket ship, surrounded by empty, homogeneous space. The experimental setup consists of a spring-loaded ball which shoots out and hits a button. Then we note the three following trivialities:

(1) This experiment can be performed at any time, and will behave in the same manner.

(2) The experiment can be shifted either to the left or right without having an impact on the ensuing experiment.

(3) The device can be oriented in any direction, and this will not have an effect on our experiment.

Note that these can be thought of as transformations of time, position, and angle respectively. These three seemingly obvious statements yield profound results. As a matter of fact, we have the following implications:

(1) Homogeneity of Space $\Rightarrow$ Conservation of Momentum

(2) Homogeneity of Time $\Rightarrow$ Conservation of Energy

(3) Isotropy of Space $\Rightarrow$ Conservation of Angular Momentum

One might wonder if these continuous symmetries always imply conservation laws. Indeed, this is precisely what Noether demonstrated. The theorem can be stated thusly:

[1] The cube actually has 48 isometry elements, forming the group $O_h \cong S_4 \times C_2$.

Theorem 3.2.1 (Noether)**.** If a transformation σ is a continuous symmetry of the system, and $\sigma : q_i(t) \to Q(s,t)$, then the quantity

$$\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial Q_i}{\partial s}$$

is conserved. In particular,

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}} = \text{const.}$$

Proof. We have that

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{\partial \mathcal{L}}{\partial Q_i} \frac{\partial Q_i}{\partial s} + \frac{\partial \mathcal{L}}{\partial \dot{Q}_i} \frac{\partial \dot{Q}_i}{\partial s}$$

so we obtain

$$\begin{aligned} 0 = \left[\frac{\partial \mathcal{L}}{\partial s}\right]_{s=0} &= \left[\frac{\partial \mathcal{L}}{\partial q_i} \frac{\partial Q_i}{\partial s}\right]_{s=0} + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{Q}_i}{\partial s}\right]_{s=0} \\ &= \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \frac{\partial Q_i}{\partial s}\right]_{s=0} + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial \dot{Q}_i}{\partial s}\right]_{s=0} \\ &= \frac{d}{dt}\left(\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial Q_i}{\partial s}\right]_{s=0}\right) \end{aligned}$$

Therefore, the quantity

$$\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{\partial Q_i}{\partial s}$$

evaluated at $s = 0$, is conserved for all time. The particular case is just an application of the Euler-Lagrange equation:

$$0 = \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

from which we get that the quantity $\partial \mathcal{L}/\partial \dot{q}$ is conserved. □

We will set out to prove these implications.

3.2.1 Homogeneity of Space

We can write the Lagrangian for our system of particles. If we perturb our positions by a constnant factor, then since space is homogeneous, we will expect that the resultant Lagrangian does not depend on our perturbation (i.e., translation is a continuous symmetry of the Lagrangian). Every vector $\mathbf{r} \mapsto \mathbf{r} + \boldsymbol{\eta}$. Then our Lagrangian in terms of our coordinates is

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = \sum_i \frac{1}{2} m \dot{\mathbf{r}}_i^2 - U\left(\mathbf{r}_i - \mathbf{r}_j\right).$$

However, we see that

$$L(\mathbf{r}_i + \boldsymbol{\eta}, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t).$$

3.2.2 Homogeneity of Time

3.2.3 Isotropy of Space

In mathematical terms, perturbing our time $t \mapsto t + s$, then we

3.3 Conservation in Newton's Picture

Conservation laws, and in particualr energy, rule the world. We have two other conservation laws, which are conservation of momentum and angular momentum. Like there is no space and time (only spacetime), we have that same equivalence between momentum and energy. Moreover, the fundamental constituents of our universe—elementary particles—are endowed with a certain intrinsic angular momentum called *spin*. Therefore, special attention ought to be paid to them.

For a system, we can index the force on each particle by

$$m_i \mathbf{a}_1 = \mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i}$$

Then, as you should recall from a first course in mechanics, we have the following laws:

Theorem 3.3.1. Suppose that no external forces act on a mechanical system. Then the following are conserved:

(1) $E = \sum_i \frac{1}{2} m_i v_i^2 + U$

(2) $\mathbf{P} = \sum_i m \mathbf{v}_i := \sum_i \mathbf{p}_i$

(3) $\mathbf{J} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$

Proof.

(1) Over time, positions and velocities may change, but we must prove that Energy does not depend on time, or that $\partial_t E = 0$. Note that $v_i^2 = \mathbf{v_i} \cdot \mathbf{v}_i$. We can differentiate these with respect to time, and see that they are conserved over time: From the chain rule,

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \sum m_i \mathbf{v}_i \cdot \mathbf{a}_i + \frac{\partial U}{\partial t} \\
&= \sum m_i \mathbf{v}_i \cdot \mathbf{a}_i + \sum_i \frac{\partial U}{\partial \mathbf{r}_i} \cdot \mathbf{v}_i \\
&= \sum_i \mathbf{v}_i \cdot \left(m_i \mathbf{a}_i + \frac{\partial U}{\partial \mathbf{r}_i} \right) \\
&= 0.
\end{aligned}$$

(2) We get that

$$\begin{aligned}
\mathbf{P} = \sum_i m_i \mathbf{a}_i &= \sum_i \mathbf{F}_i \\
&= \sum_{ij} \mathbf{F}_{ij}, \\
&= \frac{1}{2} \sum_{ij} \mathbf{F}_{ij} + \mathbf{F}_{ji} \\
&= 0
\end{aligned}$$

by Newton's third law and since we can break up the forces into the bodies on which they act.

□

F If there is time-translation symmetry of the Lagrangian, then

$$\partial t\mathcal{L} = 0 \Rightarrow E := \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - L = \text{const.}$$

The fact that the Lagrangian's derivative with respect to time is 0 implies that there is a symmetry principle underlying it. This continuous symmetry implies the conservation law.

With the example of a harmonic oscillator,

$$\mathcal{L} = \mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

If $L = T - U$, and $E = T + U = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - L = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - (T - U) = \left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - T\right) + U$, then this implies that iff

$$\dot{q}\frac{\partial T}{\partial \dot{q}} = 2T.$$

This has to do that T is a quadratic expression in $\dot{q}$.

Swings argument:

When you are moving up and down on a swing, you are effectively changing the length of your pendulum. Now there is a spring constant which is variable in time. By Noether's theoerm, energy is not conserved.

3.4 Why Noether?

Recall how we started discussing conservation laws with the example of a measurement device and a ball. The experiment was invariant with respect to time, position, and angle. For a Lagrangian $\mathcal{L}(q, \dot{q}, t)$, we have

$$\frac{\partial \mathcal{L}}{\partial q_1} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = const$$

so the generalized momentum is constant.

Symmetry is when you do something to a system, but nothing changes -feynman. But if you can do something to a system where nothing changes, then perhaps you can change your coordinates such that the Lagrangian doesn't change. This is in fact true.

Consider a 2-dimenional swinging mass with changing length, $r = r(t)$. Will we have a conservation law? The answer is yes; there is no time translation invariance, but there is angular translation invariance. If we coordinatize our system with polar coordinates, then our lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Notice that

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \text{const.}$$

Thus this shows that angular momentum is conserved:

$$mr^2\dot{\theta} = mrv_\theta.$$

Having a symmetry means that a Lagrangian does not depend on a certain variable.

3.5 Center of Mass Frame

This center of mass (CM) frame is a consequence of conservation of momentum. That is, if no external forces are acting on the system, we are free to translate our system wherever, and thus have conservation of momentum:

$$\begin{aligned}\sum \mathbf{p}_i &= \text{ const.} \\ &= M\mathbf{V}_0\end{aligned}$$

where $M = \sum m_i$. This is exactly what you need for the concept of a center of mass. This is a concept hugely important in physics. Now we can integrate this with respect to time,

$$\sum_i m_i \mathbf{r}_i = M\mathbf{V}_0 t + M\mathbf{r}_0.$$

Recalling Galilean invariance, we can view our system as $\mathbf{r}_i \mapsto \mathbf{r}_i + \mathbf{V}_0 t + \mathbf{r}_0$. Therefore, we get

$$\sum_i m_i \mathbf{r}_i = 0.$$

Due to momentum conservation, we can always find a reference frame such that the center of mass is at the origin, for all time.

3.6 Virial Theorem

Before we state the result of the theorem, we must define The theorem can be stated as follows:

Definition 3.6.1 (Homogeneous function)**.** Let $U : V \to W$, where V, W are vector spaces over a field K. Then U is said to be homogeneous of degree k if

$$U(\alpha \mathbf{v}) = \alpha^k U(\mathbf{v})$$

where $k \in \mathbb{Z}$. Moreover,

$$\mathbf{r} \cdot \frac{\partial U}{\partial \mathbf{r}} = kU$$

The following theorem concerns the average quantities of mechanical systems, and can be very useful; if we have a mechanical system that proceeds for a long time, it may make sense to obtain the averages of certain quantities, in order to set limits.

Theorem 3.6.1 (Virial Theorem)**.** Assume that the potential energy U is a homogeneous function of degree k, which means

$$\mathbf{r} \cdot \frac{\partial U}{\partial \mathbf{r}} = kU.$$

for instance, $U = \frac{1}{r} = \mathbf{r}\frac{\partial U}{\partial \mathbf{r}}$. Then

$$\begin{aligned}\langle U \rangle &= \frac{2}{k+2} E \\ \langle T \rangle &= \frac{k}{k+2} E\end{aligned} \tag{3.6.1}$$

Proof. FILL IN □

In particular, when $k = 2$, we have that $\langle T \rangle = \langle U \rangle$. This corresponds to the particle being in a quadratic potential, which is true for small oscillations. If $k = -1$, then $\langle T \rangle = -E$, and $\langle U \rangle = 2E$.

Consider $f(r)$. Then $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ and $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$\frac{\partial f}{\partial x} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

Therefore the gradient is

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial r} \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right) \\ &= f' \nabla r(x, y, z). \end{aligned}$$

4 1D Motion

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Although 1-dimensional motion is one of the simplest physical systems, it has broad applicability in many areas of physics. For instance, in more advanced physical theories, the concept of a particle is replaced by the notion of a field, and particles result as oscillations of these fields.

Indeed, many complex physical systems can be approximated by one-dimensional motion, further underscoring the need to flesh out these concepts in full.

The most general form of the Lagrangian for a system with one degree of freedom q is

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2}a(q)\dot{q}^2 + U(q). \tag{4.0.1}$$

4.1 Solvable and Unsolvable 1D Motion

A 1-D system is a system that has one degree of freedom, or a Lagrangian that is of the form $\mathcal{L}(q, \dot{q}, t)$. We can think of the angle of a mathematical pendulum or the displacement of a spring.

In general, ordiary and partial differential equations are unsolvable in the analytical sense. For instance, given a Lagrangian with one degree of freedom, the resultant Euler-Lagrange equation will not be analytically solvable.

However, if energy is conserved (particularly, $\partial_t \mathcal{L} = 0$),

$$E := \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - L$$

Thus we can express $\dot{q}$ as a function of q (known as "solving the equation with respect to $\dot{q}$"):

$$\dot{q} = f(q)$$

which is a first-order ODE, and is in fact much simpler to solve using the method of quadratures. What this means is given an equation like the one above, its exact solution may be expressed in the form of an integral:

$$\begin{aligned} \frac{dq}{dt} = f(q) \Rightarrow dt &= \frac{dq}{f(q)} \\ \Rightarrow t &= \int \frac{dq}{f(q)} \end{aligned}$$

from which we can derive an equation for t in terms of q, which can then be solved to get the equation of motion $q(t)$. However, we must understand the caveat that the difficulty of solving an ODE is converted to the difficulty of solving an integral, which in general is also very difficult.

For the example of an oscillator, we get that the energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

and so we relate $\dot{x} = dx/dt$ to x:

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx^2\right)}.$$

Solving for time, we see that

$$t = \int \frac{dx}{\frac{2}{m}\sqrt{E - \frac{1}{2}kx^2}}$$

which is a tricky (but solvable!) integral; recognizing that $E = \frac{1}{2}kx_0^2$, we get that

$$t = \frac{1}{\omega}\int \frac{dx}{\sqrt{x_0^2 - x^2}}.$$

We can either recognize that this is the integral of $\sin^{-1}$. From this, we get the equation of motion.

We can now focus on the general problem of a general potential.

4.2 Particle in a Potential

Consider the Lagrangian for a particle in a general potential.

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$$

which is a form of the Lagrangian which is ubiquitous in physics, especially in quantum physics. When energy is conserved, we can integrate it and get

$$E = \frac{1}{2}m\dot{x}^2 + U(x)$$

$$\frac{dx}{dt} = v = \dot{x} = \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

Thus we can integrate dx/v, since $\int dx/v = \int dx/(dx/dt) = \int dt$. Therefore, the time of the period, T, can be written as

$$T = \sqrt{2m}\int \frac{dx}{\sqrt{(E - U(x))}}.$$

If we consider our system to be

$$T = \sqrt{2m}\int \frac{dx}{\sqrt{\left(E - \frac{1}{2}kx^2\right)}}$$

and since $E = \frac{1}{2}kx_0^2$,

$$= \frac{1}{\omega}\int \frac{dx}{\sqrt{x_0^2 - x^2}}.$$

Understanding time-independent one-dimensional systems is an indispensible part of physics. For a time independent system (i.e., conserved energy), we have that for a quadratic potential, our energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

If our energy is finite, we can mark the intersection of the potential with the line $U = E$, which occurs at $-x_0, x_0$. Since $|x| \leq x_0$, this implies that $E \geq \frac{1}{2}kx^2$. This means that our x_0 is given by

$$x_0 = \sqrt{\frac{2E}{k}}.$$

We can then interpret our oscillating system in terms of restrictions of our system. For an ideal spring, if the energy of the system is finite, then the motion will correspondingly be finite. The points at which

$$U(x) = E$$

are called *turning points*. The velocity will be correspondingly 0. If our potential is not parabolic, for instance, it may be possible for the particle to overcome the function. In these cases, the motion is infinite. A useful analogy is a ball rolling on a hill. There may be a local dip in the hill, but its possible that given enough energy, the ball may exit the dip and keep going forever.

4.3 Estimating Periods of Oscillations

3/1 Oftentimes, the integral form of the period is impossible to integrate analytically. Therfore, in this section, we will explore techniques to give rough estimates of the period.

One of the simplest techniques to estimate the period of a system is to try analysis of dimensions. This involves combining the given constants in a special way so as to give a quantity that has units of time. For isntance, if we are given a system whose energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

we can try and construct an expression in terms of x_0, k, m in order to arrive at something with units of time. Recognizing that $[m] = \text{kg}$, and $[k] = \text{kg/s}^2$, a reasonable guess might be that

$$T \propto \sqrt{\frac{m}{k}}$$

which is indeed true; $T = 2\pi\omega$. Such a statement proves useful in finding the asymptotic behavior of a system. Oftentimes, the combination of known constants should be the simplest possible in doing such estimates, or at least the one that requires the fewest assumptions. The method is not perfect, since it is impossible to account for dimensionless constants. Luckily, most dimensionless constants are of order unity in physics.

Next, we can do order of magnitude estimates. We can say that $x \sim x_0$ is because when $x \to x_0$, $v \to 0$, so we spend more time near the top than at the bottom. Crudely, $x \sim x_0$, or x is on the order of x_0, then our acceleration is on the order of x_0/T^2, where T is the period. Then, since our force for a harmonic oscillator is linear, on the order of kx_0, then

$$F = ma = m\frac{x_0}{T^2} \sim kx_0 \Rightarrow T \sim \sqrt{\frac{m}{k}}.$$

Lastly, we can use the available formula to try and estimate the period. If our exact equation for the period is

$$T = \sqrt{2m}\int_{-x_0}^{x_0} \frac{dx}{\sqrt{x_0^2 - x^2}},$$

we can try to think about how this depends on x_0. Let $x_0 u = x$. Our integral is then

$$\begin{aligned} T &= \sqrt{2m}\int_{-1}^{1} \frac{x_0 du}{x_0\sqrt{1-u^2}} \\ &= \sqrt{2m}\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} \end{aligned}$$

which has the advantage of giving an exact formula for any potential dimensionless constants.

4.4 Bounded and Unbounded Motion

Our goal is to further study general problems of 1-dimensional motion. Before the big bang, there was an inflationary period. The potential is modeled by a scalar field moving in a potential.

A mathematical pendulum is a rigid massless rod with an attached mass, but instead of small oscillations, we want to consider all possible forms of motion. Our potential energy is, for a gravitational field,

$$U = -mg\cos\theta$$

and its Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

which has multiple points of equilibria, stable and unstable. The equation of motion is given by the differential equation

$$\ddot{\theta} = -\frac{g}{R}\sin\theta.$$

In particular, if our energy is below mg, then we are in a stable equilibrium, and the motion is bounded. However, for energies greater than mg, the motion of the system can go past unstable equilibria, and the motion unbounded, as the pendulum swings in one direction forever.

3/3 We can draw the following conclusions:

(1) $-mgR < E < -mgR$ is when the motion is bounded.

(2) $E > mgR$ is when the motion is unbounded.

(3) $E = -mgR$ is a stable equilibrium.

(4) $E = mgR$ is an unstable equilibrium.

The last two cases seems anomalous, since one may object that we will never encounter a pendulum which has exactly zero motion in the real world. However, for an exact solution, we can extend that to nearby motions.

In case (3), if we perturb our oscillator by a small angle $\delta\theta$, our system will remain near the bottom. However, if we perturb our oscillator by $\theta = \pi - \delta\theta$, then the system falls and we achieve large oscillations. Understanding how these unstable equilibria evolve is indispensible in understanding other phenomena.

For instance, we all have heavy elements in our bodies, and we ultimately come from stars. Stars come from initially homogeneous initial density, with small quantum fluctuations. Where the densities were larger, the gravitational attraction kept accumulating, and eventually result in the inhomogeneities in our universe today.

One may ask how it is possible to even end up at an unstable equilibrium, besides delicately balancing a pendulum at the top of its rotation. The answer is that the profile of the potential energy may evolve over time, and points which were once stable equilibria may become unstable.

Without a graph, the following is a useful theorem in classifying unstable equilibria:

Theorem 4.4.1 (Chetaev Instability Theorem)**.** For the system $\dot{\mathbf{x}} = X(\mathbf{x})$, with an equilibrium point at the origin, and a C^∞ function $U(\mathbf{x})$ such that

(1) The origin is a boundary point of the set $G = \{\mathbf{x} : U(x) > 0\}$

(2) There is a neighborhood $\mathcal{D}$ of the origin such that

$$U'(\mathbf{x}) > 0$$

for all $\mathbf{x} \in G \cap \mathcal{D}$,

then the origin is considered to be a point of *unstable equilibrium.*

Back to our original example of the pendulum, we can attempt to calculate what happens at the unstable equilibria. Let $\theta = \pi - x$, where $|x| \ll 1$. Then for our differential equation, this becomes

$$-\ddot{x} = -\omega^2 \sin x \;\Rightarrow \ddot{x} \approx \omega^2 x$$

which gives us the approximate acceleration for small x. The soltuion is immediate:

$$x = Ae^{\omega t} + Be^{-\omega t}$$

which means that

$$x_0 = A + B \tag{4.4.1}$$
$$v_0 = \omega(A - B) \tag{4.4.2}$$

which gives us

$$x = \frac{1}{2}x_0\left(e^{\omega t} + e^{-\omega t}\right)$$

which tells us that the growth of the initially small displacement is initially exponential. If we want to calculate the time it takes for the pendulum to fall down, we can approximate it by the equation

$$\frac{1}{2}x_0 e^{\omega t} = x_f$$

since for x_f of order unity, the pendulum is already falling very rapidly. Therefore,

$$T = \frac{1}{\omega}\ln\left(\frac{x_0}{x_f}\right)$$

For instance, if we consider a balanced 10 cm long pencil, initially deviating 3 millimeters from equilibrium, and $x_f = 3cm$, then our $1/\omega$ is

$$\sqrt{\frac{R}{g}} = 0.1 \text{ seconds}$$

and our logarithmic factor is

$$\ln(100) = 2\ln(10) \approx 4.6$$

so our final answer is 0.46 seconds, which is reasonable. Moreover, our approximation is surprisingly robust, since if we change the initial displacement to 1 millimeter (a factor of 3 lower), then our ending time is around 0.56 seconds, which is a factor of around 1.2 greater. What the logarithm also tells us is that within our approximation, it is hard to break the order of magnitude of $1/\omega$ for our final answer in an experimental setting, since that would require immense precision. To get a pencil to fall in 10 seconds, we would need for

$$x_i \sim x_f e^{-100} \sim 1 \times 10^{-40} \text{ cm}$$

which is smaller than the width of a proton, smaller than the width of an electron, even smaller than a planck length! Simply put, it doesn't even make physical sense. And practically speaking, its also impossible, since thermal fluctuations and molecule collisions make this endeavor fruitless.

A possible way to cheat is to vibrate the equilibrium point sufficiently fast, and it turns out that the unstable equilibrium will become stable.

4.5 Two Bodies in One Dimension

So far we have talked about a particle in an external potential. Now we can shift our focus to particles within a closed system. In this scenario, we indeed have momentum conservation. Then we must consider the other body which creates a potential. For instance, we can consider a spring with two masses m_1 and m_2 on either side. Our Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}\kappa(x_1 - x_2)^2$$

which is a system with two degrees of freedom. However, when we introduce the constraint that momentum is conserved, then this system has only one degree of freedom. This is because there exists a reference frame such that the center of mass of this object is 0. Thus,

$$m_1 x_1 + m_2 x_2 = 0.$$

Therefore, it makes sense to coordinatize our system in terms of

$$x = x_2 - x_1$$

which give us

$$x_1 = -\frac{m_2}{m_1 + m2} x$$
$$x_2 = \frac{m_1}{m_1 + m_2} x.$$

And now we can see a powerful result of Lagrangian mechanics: the system minimizes action independently of the coordinates we choose. Thus,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2} m_1 \left(-\frac{m_2}{m_1 + m2} \dot{x} \right)^2 \\ &+ \frac{1}{2} m_2 \left(\frac{m_1}{m_1 + m_2} \dot{x} \right)^2 \\ &- \frac{1}{2} \kappa x^2.\end{aligned}$$

we can simplify the expression by noting that $m_1 m_2^2 + m_2 m_1^2 = m_1 m_2 (m_1 + m_2)$, so we can cancel the squared $(m_1 + m_2)$ term in the denominator. Therefore,

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \kappa x^2$$
$$m \equiv \frac{m_1 m_2}{m_1 + m_2}$$

where the term m is called the *reduced mass*.

This concept is very useful throughout physics. For instance, in quantum mechanics, a proton and electron are in orbit, with a characteristic energy. The lowest energy of hydrogen will have an energy of 13.6 eV. If we replace the proton with a positron, they circle around the center of mass. The energy is calculated in terms of charge, mass, and Planck's constant. The moment the proton is replaced by the positron, we only need to replace the electron mass with the reduced mass, which is $m_e/2$ in this instance. If we want to calculate the energy of this "positronium," then we take the energy of hydrogen and replace the electron mass with the electron mass over 2. Here, the individual masses do not matter, only this reduced mass. This is because the proton is much heavier than the electron, so the reduced mass in the hydrogen atom is approximately the mass of an electron. This means that the center of mass frame is almost coincident with the center of mass of the proton itself.

The understanding that with concervation of momentum, the two-body problem is really a one-body problem, is very important. We will see very soon that the general two-body problem in a potential field can, under conservation of momentum, be reduced to one body in a central field.

If we have three bodies, we can use this trick to reduce it to a two-body problem in a central field; however, this is unsolvable in general and becomes chaotic. If the system is set up in an equilateral triangle, with very small initial velocities, the end result is that two of them form a tight binary system and the third is ejected off to infinity. This is because our system is virialized; since the binary has very low energy, the remainder is transferred into the third star, which is thus ejected off into infinity.

Our own solar system is chaotic, and indeed evolves over time—albeit, very slowly. Initially, out

[ARNOLD EXCERPT ABOUT MOON?]

5 Kepler's Problem 3/8

It took Newton his entire *Principia* to solve Kepler's problem. With the techniques we have developed so far, the complexity can greatly be reduced.

5.1 General 2-body problem

Suppose we have two bodies with potential $U(|\mathbf{r}_1 - \mathbf{r}_2|)$. Since each particle has three components for the position, this equation so far has six degrees of freedom. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) - U\left(|\mathbf{r}_1 - \mathbf{r}_2\right)$$

If we consider the center of mass frame, we can change the Lagrangian to be

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(r);\ m := \frac{m_1 m_2}{m_1 + m_2}.$$

This is the motion of one particle in a central potential. The degrees of freedom have been reduced from six to three.

Automatically, we must look for symmetries in order to simplify the problem. Clearly, angle and time don't enter into the equation, so energy and angular momentum are conserved by Noether's theorem. Writing down the conservecd quantities of our system,

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{r}}^2 + U(r) \\ \mathbf{J} &= m\mathbf{r} \times \dot{\mathbf{r}}. \end{aligned}$$

The insight here is that since we have a conserved vector (in both direction and magnitude), then we have that $\mathbf{r}, \dot{\mathbf{r}}$ define a plane for all time, and $\mathbf{J}$ is normal to it. Thus, we can deduce that all the motion occurs in a single plane, simplifying our system from three degrees of freedom to two. Then we can coordinatize the problem in polar coordinates to further simplify the problem. The energy can be re-written as

$$\begin{aligned} E &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \\ J &= mr^2\phi. \end{aligned}$$

Where J is no longer a vector since we know its direction. Then, we have that

$$\dot{\phi} = \frac{J}{mr^2}.$$

Substituting it into our energy equation,

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{r}}^2 + U_{\text{eff}}(r) \\ U_{\text{eff}} &:= U(r) + \frac{J^2}{2mr^2} \end{aligned}$$

where U_{eff} is included in our effective potential. The term with J is known as the centrifugal energy. Then our energy equation is just regular conservation of motion in one dimension, which is an enormous simplification from what we started with. Given energy and angular momentum, we can solve the energy equation in quadrature. Once we get r as a function of time, we can solve for ϕ.

If we consider our potential $U(r) = \frac{1}{2}kr^2$, then we can plot the energy:

Now the nontrivial part of the problem. We know that our motion is periodic, with period T_r. This is the time it takes for r to go to minimum to maximum and back to minimum. Now for ϕ, we have that ϕ is also periodic, with period T_ϕ, which won't agree with the period of r. Thus the orbit may not necessarily be closed. Only Hooke and Newton give us closed orbits, and closure of the orbit is a particular characteristic of potentials like $1/r$. Even if our potential was slightly different, say $1/r^{1.0001}$, then our ellipse will not close and thus it will eventually precess.

Theorem 5.1.1. For a potential of the form

$$U_{\text{eff}} = \frac{1}{2}kx^2 + \frac{J^2}{2mr^2}$$

then the orbit does not precess, or $T_r = nT_\phi$ where $n \in \mathbb{N}$.

Proof. If we set up our newton's equation, we see that in 3-D space,

$$m\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} = -k\mathbf{r}$$

Then the solution, after solving in x, y, and z, is

$$\mathbf{r} = (a\cos\omega t, b\sin\omega t, s).$$

Which is an ellipse and is closed. □

5.2 Kepler's Laws

Read Feynman book, Vladimir Arnold book. According to Arnold, Principia was written to show an ellipse, and the other two laws are trivialities.

Theorem 5.2.1 (Kepler's Laws)**.**

1. The orbit of two bodies is elliptical.
2. The oritented area swept out by the orbit can be calculated by

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}\oint \mathbf{r} \times d\mathbf{r} \\ &= \frac{1}{2}\oint dt\ \mathbf{r} \times \dot{\mathbf{r}} \\ &= \oint dt\ \dot{\mathbf{A}}\end{aligned}$$

   The rate of change of the area is given by $\frac{1}{2}\mathbf{r} \times \dot{\mathbf{r}}$. Thus,

$$\begin{aligned}\ddot{\mathbf{A}} &= \frac{1}{2}\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \frac{1}{2}\mathbf{r} \times \ddot{\mathbf{r}} \\ &= 0 \\ &\iff \mathbf{J} = \text{const.}\end{aligned}$$

3. The period is

$$T \propto a^{3/2}.$$

5.3 The Ellipse

"An ellipse is a circle inscribed into a square with unequal sides." Russian mathematical joke.

What this really means is that an ellipse is a circle that has been squeezed. Mathematically, its equation is

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1.$$

An alternative definition, and perhaps more useful one, is that the lengths of the vectors originating from the foci to the edge of the ellipse is constant.

Lastly, we can define it in terms of its eccentricity. In polar coordinates, an ellipse can be written as

$$r=\frac{p}{1+e\cos\phi}$$

Where converting back into cartesian coordinates,

$$\begin{aligned} r+ex&=p\\ r&=p-ex\\ x^2+y^2&=p^2-2epx+e^2x^2\\ (1-e^2)x^2+y^2&=p^2-2epx. \end{aligned}$$

Suppose you know a solution to some equations of motion is an ellipse. Can we recover the parameters of this ellipse from our integrals of motion?

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We can write the effective potential as

$$U_{\text{eff}}:=-\frac{GMm}{r}+\frac{J^2}{2mr^2}$$

Often times, the motion of systems reduces to motion in one dimension. At large distances, we have a nice coulomb potential; at small distances, it is dominated by the right hand term. Therefore, the effective potential looks like [IMAGE]

Just like we did our mathematical pendulum, there are four different cases; the absolute bottom of the potential, negative energy, zero energy, and positive energy.

We can have 1-D motion where the particle stays at the bottom of the potential. These correspond to circular, elliptical, parabolic, and hyperbolic orbits.

Deriving ellipse: We have

$$\begin{aligned} E&=\frac{1}{2}m\dot{r}^2-\frac{GMm}{r}+\frac{J^2}{2mr^2}\\ J&=mr^2\dot{\phi}. \end{aligned}$$

We want $dr/d\phi$; we dont care how fast it moves, we are only interested in the trajectory.Thus

$$\dot{r}=\frac{dr}{dt}=\frac{dr}{d\phi}\frac{d\phi}{dt}=\frac{dr}{d\phi}\frac{J}{mr^2}.$$

Then the energy conservation law becomes a differential equation for the trajectory. Then

$$\frac{1}{2}$$

FILL IN

$$E=\frac{J^2}{2mr^4}\left(\frac{dr}{d\phi}\right)^2-\frac{GMm}{r}+\frac{J^2}{2mr^2}$$

which is an equation we can solve in quadrature. We will use a trick to avoid doing any integration. This equation is a linear harmonic oscilaltor in disguise. Let $u = 1/r$. Then

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}.$$

then

$$E = \frac{J^2}{2m}\left(\frac{du}{d\phi}\right)^2 + \frac{J^2}{2m}u^2 - GMmu.$$

If we treat the latter two terms as our potential energy, we have that it is a parabola displaced from the origin. Multiplying both sides by $2m/J^2$, then

$$E = \left(\frac{du}{d\phi}\right)^2 + (u - u_1)^2 = u_2^2$$

and so

$$u_1 = \frac{GMm^2}{J^2}$$

and

$$u_2^2 = u_1^2 + \frac{2mE}{J^2}$$

This is our standard harmonic oscillator, so the solution can be written as with a sinusoidal solution

$$u = u_1 + u_2\cos\phi.$$

This is immediate if $u \mapsto u_1 + x$ and $\phi \mapsto t$, and $u_2^2 \mapsto x_0^2$; then this is a linear harmonic oscillator with amplitude x_0, whose solution is known. This means that

$$\begin{aligned} r = \frac{1}{u_1 + u_2\cos\phi} &= \frac{1/u_1}{1 + (u_2/u_1)\cos\phi} \\ &= \frac{p}{1 + e\cos\phi} \end{aligned}$$

where

$$\begin{aligned} p &= \frac{1}{u_1} = \frac{J^2}{GMm^2} \\ e &= \frac{u_2}{u_1} = \sqrt{1 + \frac{2mEJ^2}{G^2M^2m^4}} = \sqrt{1 + \frac{2J^2E}{G^2M^2m^3}} \end{aligned}$$

which is the canonical polar representation of the ellipse.

Argument of reasoning by extreme solutions: standard trick in physics.

Russian phrase: OF course Alexander was a great hero: but why should you break chairs?

For a standard circle, we should get zero eccentricity:

$$\begin{aligned} \frac{GM}{a^2} &= \frac{v^2}{a} \\ E = -\frac{GMm}{a} + \frac{1}{2}mv^2 &= -\frac{GMm}{2a} \\ J &= mav \end{aligned}$$

And lastly,

$$p = \frac{m^2a^2}{GMm^2} \cdot \frac{GM}{a} = a$$

as desired.

Notice that $E < 0, e < 1$ and $E > 0, e > 1$ is an ellipse.

If we want to calculate the period, we get that for a circle,

$$T = \frac{2\pi a}{v} = 2\pi \left(\frac{a^3}{GM}\right)^{1/2}$$

We want r as a function of time. Here, we go back to the statement that 1-D motion with conservation of energy can always be solved in quadratures.

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{J^2}{2mr^2}$$

whence

$$\dot{r} = \sqrt{\frac{2}{m}\left(E + \frac{GMm}{r} - \frac{J^2}{2mr^2}\right)}$$

Or $dr/dt = f(r)$. This is

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{(E + \frac{GMm}{r} - \frac{J^2}{2mr^2})}}$$

Schematically, this integral is like

$$\int \frac{dx}{\sqrt{-1 + \frac{c}{x} - \frac{1}{x^2}}}$$

where E is negative for an ellipse.

$$\int \frac{xdx}{\sqrt{-x^2 + cx - 1}}$$

where the constant c is needed in order to prevent the square root from being negative. At this point, this is a standard procedure; the denominator is like a parabola:

$$\int \frac{xdx}{\sqrt{c - (x-2)^2}}$$

Changing variables like $x \mapsto x + 2$,

$$\int \frac{(x+2)dx}{\sqrt{3 - x^2}}$$

Change $x \mapsto \sqrt{3}x$ abd then $x = \cos\xi$,

$$\int \frac{(2 + \sqrt{3}\sin\phi)d\phi}{f} = 2\phi - \sqrt{3}\cos\phi$$

Implicit formula:

$$t = \xi - e\sin\xi, \; r = 1 - e\cos\xi.$$

6 Collisions

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Collisions are related to the largest physics experiment ever devised; the Large Hadron Collider (LHC) in Geneva, Switzerland. It was built to accelerate protons to high velocities, and smash them to see what comes out. Another collision experiment that resulted in a discovery is Rutherford's experiment, which established the structure of an atom, as having a nucleus and a cloud of electrons. He took a thin gold foil, and shot alpha particles at it. The fact that some of the particles were scattered at large angles allowed them to determine the structure. Lastly, Boltzmann's proof of entropy growth was determined by collisions.

The most natural examples of collisions are particle decays, given by

$$\pi^0 \to \gamma + \gamma$$
$$n \to p + e^- + \overline{\nu}_0.$$

There are particle decays which are non-relativistic. Thus,

$$^{238}\mathrm{U} \to {}^{234}\mathrm{Th} + \alpha$$

Moreover, the laws of conservation are only valid for relatively low temperatures. For instance, we can model the collision of Helium atoms by the classical formulas of elastic collisions. This is because despite the whizzing electrons, the electrons only occur at specific energy levels. The temperatures needed to actually start changing this structure is on the order of 40,000 Kelvin. For protons, the LHC probes temperatures of 10 TeV, or 10^{17}K.

6.1 Elastic Collisions

In the simplest formulation, we can think of collisions of billiard balls, and we want to determine what results after they collide. We will consider them as point-masses and ignore their rotation. For two particles, we can naively apply conservation laws to state that for two particles with masses m_1, m_2, and corresponding velocitis $\mathbf{v}_1, \mathbf{v}_2$, their resultant momenta and energy will be given by

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

However, note that we must wait a sufficiently long time for the second equation to be true. In reality, we assume that the energy is given by

$$E = \frac{1}{2}m_1v_1^2(t) + \frac{1}{2}m_2v_2^2(t) + U\left(|\mathbf{r}_1(t) - \mathbf{r}_2(t)|\right)$$

where $U \to 0$ as $t \to \pm\infty$.

In the above form, the consequences of these equations are not obvious; if we look at them from the right perspective, we may get some elucidation. The equation is the simplest in the CM frame:

$$\mathbf{v}_1' = \mathbf{V} + |\mathbf{v_1} - \mathbf{V}|\,\hat{n}_0$$
$$\mathbf{v}_2' = \mathbf{V} + |\mathbf{v_2} - \mathbf{V}|\,\hat{n}_0$$
$$\mathbf{V} := \frac{m_1\mathbf{v_1} + m_2\mathbf{v}_2}{m_1 + m_2}$$

and $\hat{n}_0$ is an arbitrary vector. In the CM frame, the total momentum is 0. We can dentoe $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$. Then the velocities are expressed in the CM frame as

$$\mathbf{v}_1 = \frac{m_2}{m_1 + m_2}\mathbf{v}$$
$$\mathbf{v}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{v}$$

Therefore, our kinetic energy is $\frac{1}{2}M_r v^2$. In the CM frame, our momentum is always 0; then we can express the kinetic energy in terms of the reduced mass and the velocity difference. Then we can conclude that since this stays constant, our velocity vector can only rotate in space. Thus in the center of mass frame, we see that the only thing that happens to the velocities are rotations, since $\mathbf{V} = 0$.

Depending on the configuration of our system, this vector $\hat{n}_0$ will change. The apparatus for calculating it is called "cross-sections."

As a thought experiment, we can consider measuring velocities of individual nitrogen molecules in the air; plotting their velocities, we will get a Gaussian. If we take a pure sample of Nitrogen, and orient each of them with the same initial velocities but diferent orientations, then the distribution will initially be uniform. Over time, the collisions will result in a Gaussian distribution of the velocities.

Initially, it is unclear how elastic collisions will increase the velocities of some particles. For two particles with equal mass, $\mathbf{V} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$. If two incoming particles collide at a right angle, one will fly off with $\sqrt{2}v$ and the other will remain constant.

6.2 Cross-Sections

6.2.1 Rutherford's Experiment: An Example

The idea of the cross-section is to formally understand the probabilities for different vectors $\mathbf{n}_0$. Definitions here will be very important, and often good definitions can help simplify problems.

If we consider a sphere of radius R, then we can imagine sending a particle with some velocity to the spgere. By the law that the angle of incidence will be the angle of reflection, then the angle at which the particle is scattered depends on the area of the sphere as seen from the particle's perspective. Another definition is that it is the region a particle must go through in order to be scattered. Thus, the *total cross-section* of a sphere is just

$$\sigma = \pi R^2.$$

3/17 This will be useful in Rutherford's experiment, where the size of the nucleus is $\sim 10^{-12}$cm, and we can treat it as a hard sphere; its cross-section is therefore $\sim 3 \times 10^{-24}$ cm^2. Moreover, the number density of gold atoms per cubic centimeter can be estimated by

$$n = \frac{\rho}{n_p \times m_p} \approx \frac{20\text{g/cm}^3}{200 \times 1.7 \times 10^{-24}\text{g}} \approx 6 \times 10^{22}\ \text{cm}^{-3}$$

where ρ is the mass density of gold, n_p is the atomic weight of gold, and m_p is the mass of a proton or neutron. Therefore, suppose the thickness of a gold foil is d. In order for an alpha particle to be scattered, it must pass through a cylinder of radius R, or the radius of a single gold atom. The expected number of scatterings is therefore given by

$$\sigma dn$$

which, if it is less than 1, then gives the probability that the alpha particle will be scattered. Thus we get that for a sheet of thickness 1 mm, we get a probability of $\sim 0.2\%$.

In order to characterize the interaction of the scattered particles (e.g., coulomb interaction), we can consider the scattering angle χ as a function of b, where b is the impact parameter (the perpendicular distance from our particle to the center of the gold nucleus). We can consider the probabilities of particles scattered at angles χ and $\chi + d\chi$, which will give us a probability density function. That will determine how many particles are observed at the end of the experiment.

We also have the notion of a differential cross-section, which tells us which fraction of scattered particles scatter into which angles:

$$\frac{d\sigma}{d\chi} = 2\pi b \frac{db}{d\chi}.$$

which results in the probability density function

$$\frac{dp}{d\chi} = nd\frac{d\sigma}{d\chi}$$

It is not difficult to calculate the differential cross-section for a hard sphere, as the angle of reflection is the same as the angle of incidence. What Rutherford showed was that that the alpha particles were not scattered according to a hard sphere. Rather, the probability distribution function that resulted was in agreement with a Coulomb repulsion, not a hard sphere.

6.3 Small-Angle Scattering

In order to understand scattering, we must know the process by which a particle is scattered. If an alpha particle is sent at a gold atom with a relatively large impact parameter, then due to the Coulomb repulsion force, the resulting scattering angle will be small. In the limit of small angles, the estimate of small angles can be used as a rough estimate for large angle scattering.

Two Anecdotes: Famous professor Max Tegmark asked Andrei Gruzinov how to estimate the scattering of two dipole interactions. He wanted to know how a

Another anecdote is how would some amino acid scatter a polarized muon.

Now the actual calculation. Assume the impact parameter b is large with respect to the radius of the nucleus, and the particle is travelling at a velocity v. In the first approximation, it goes straight. The particle experiences the force

$$F \sim \frac{2ze^2}{b^2}$$
$$a \sim F/m$$

where z is the charge of the gold nucleus, 2 is the charge of the alpha particle, and e is the electron charge. The time of collision is

$$\tau \sim \frac{b}{v}.$$

Therefore, the perpendicular velocity imparted onto the alpha particle is $v_\perp = a\tau$, and so the resulting angle is

$$\chi \sim \frac{v_\perp}{v} \sim \frac{2ze^2}{mv^2}\frac{1}{b} \sim \frac{U}{T}$$

where U is the potential energy and T is the kinetic energy of the interaction. For a defleciton angle of order ~ 1, we must have an impact parameter of order

$$b \sim \frac{2ze^2}{m_\alpha v^2} = 2z\frac{e^2}{m_e c^2}\frac{m_e c^2}{m_\alpha v^2}$$

then the second term is known as the classical electron radius r_e. This is because r_e is the radius such that the electrostatic energy of the electron is fully accounted for by the Einsteinian mass-energy relation. Thus we get approximately

$$160 \times 3 \times 10^{-13}\text{ cm} \times \frac{0.5\text{ MeV}}{4\text{ MeV}} = 5 \times 10^{-12}\text{ cm}.$$

This impact parameter will result in scattering angles of order unity. Therefore, the cross section is $p \sim n\sigma d \sim 1$ for $d = 0.1$ cm.

6.4 Rutherford's Formula

3/22 We already solved the corresponding Kepler problem for potentials like $1/r$. From Kepler, we calculated it was an ellipse, with the formulae

$$r = \frac{p}{1 + e\cos\phi}$$
$$p = \frac{J^2}{GMm^2}$$
$$e^2 = 1 + \frac{2J^2E}{G^2M^2m^3}$$

Thus, we wish to calculate the cross-section to scatter by more than a given angle χ, or $\sigma(\chi)$. When we have a nucleus, we assume it is much heavier than the alpha particle. Thus, we can write the potential for a Coulomb interaction:

$$U = \frac{q_1}{q_2}r$$

where $q_1 = z_1 e$ and $q_2 = z_2 e$ where $z_1, z_2 \in \mathbb{Z}$. Then, replacing the potentials in our equations for the ellipse, we get

$$r = \frac{p}{e\cos\phi - 1}$$
$$p = \frac{J^2}{z_1 z_2 e^2 m^2}$$
$$e^2 = 1 + \frac{2J^2E}{(z_1 z_2 e^2)^2 m^3}$$

where we change the sign of p and r. This corresponds to a hyperbola, so our particle travels where $e\cos\phi > 0$. Thus when $\cos\phi = 1/e$, our particle goes to infinity. Thus, we must calculate $\sigma(\chi) = \pi b(\chi)^2$, where b is the impact parameter needed to scatter by angles more than χ. Thus $\chi = \pi - 2\phi$, so $\cos\phi = \sin(\chi/2)$. Therefore, we have that

$$e = \frac{1}{\sin\left(\frac{\chi}{2}\right)}.$$

We can calculate the angular momentum J as a function of the impact parameter and the magnitude of the velocity at infinity (i.e., the velocity where the alpha particle is on a straight path). Thus,

$$J = mbv_\infty,$$

and our energy is

$$E = \frac{1}{2}mv_\infty^2.$$

Therefore, plugging these in,

$$e^2 = 1 + \frac{2m^2b^2v_\infty^2 \frac{1}{2}mv_\infty^2}{(z_1 z_2 e^2)^2 m}.$$

Therefore,

$$e^2 = \frac{1}{\sin^2\left(\frac{\chi}{2}\right)} = 1 + \left(\frac{mv_\infty}{z_1 z_2 e^2}\right)^2 b^2$$

or, solving for b^2,

$$b^2 = \left(\frac{z_1 z_2 e^2}{mv_\infty}\right)\left(\frac{1}{\sin^2\left(\frac{\chi}{2}\right)} - 1\right)$$

Which gives us Rutherford's formula:

$$\sigma(\chi) = \left(\frac{z_1 z_2 e^2}{m v_\infty}\right)^2 \cot^2\left(\frac{\chi}{2}\right). \tag{6.4.1}$$

6.5 General Case & Capture in Bottomless Pits

Its useful to calculate scattering for a general potential in quadrature. For an arbitrary central field, we found that the problem is a 1-body problem when we consider the reduced mass and take into account conservation of angular momentum.

Suppose we have a potential which overwhelms potentials $\sim 1/r$ ar small distances, e.g. $\sim -\alpha/r^3$, $\alpha > 0$. With this potential, this potential is capable of capturing the particle and making it go through 0. The effective potential of such a system is

$$U_{\text{eff}} = -\frac{\alpha}{r^3} + \frac{J^2}{2mr^2}$$

. [DRAW PIC?] This is a phenomenon known as "capture," which is important to understanding black holes. Sometimes, black holes can be simply modeled as modified Newtonian potentials.

7 Small Oscillations

After we have developed theory, we have applied these to 1-dimensional motion. Now we will consider motion in more than one dimension, but only around equilibrium. It seems paradoxical to study these simple problems, which are constrained verisons of more general ones. However, the systems we can solve are in fact very applicable. This was a fact noted by Eugene Wigner in his essay titled "The Unreasonable Effectiveness of Mathematics."

7.1 Linear Harmonic Oscillator

As a brief reminder, the Lagrangian for a linear harmonic oscillator is like

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

and its solutions of the form s

$$\begin{aligned} x &= A\cos\omega t + B\sin\omega t \\ &= \Re\left(ae^{-i\omega t}\right) \\ &= \frac{1}{2}\left(ae^{-i\omega t} + a^* e^{i\omega t}\right) \end{aligned}$$

where $\omega^2 := k/m$. The Cauchy problem for the ODE involved two arbitrary constants, so we have them in our solutions.

Using complex expinentials is one of the best ways to express our solution. Considering the damped oscillator, our differential equation is

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0.$$

The way we solve this equation is by our *ansatz*, or "best guess." Trying the solution $x = e^{\lambda t}$, we get the equation for λ is

$$\lambda^{@} + \gamma\lambda + \omega^2 = 0$$

, and so

$$\lambda = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2}.$$

We ought to consider various limiting cases of this equation. For $\gamma, \omega \in \mathbb{R}^+$,

1. (No damping: $\gamma = 0$). In this undamped system, we have that $\lambda = \pm i\omega$. This gives a solution of $\Re(e^{-i\omega t})$.
2. 3/29!!!!!
3. (Light damping: $\gamma \ll \omega$) In this case, we can Taylor expand $i\omega\sqrt{1 - (\gamma^2/4\omega^2)}$ and see that we can remove the γ in the radical. Thus,

$$\begin{aligned} \lambda &\approx -i\omega - \frac{1}{2}\gamma \\ e^{\lambda t} &\approx e^{-\gamma t/2}e^{-i\omega t} \\ \Re(e^{\lambda t}) &\approx e^{-\gamma t/2}\cos\omega t \end{aligned}$$

   so the oscillator tends to equillibrium as $t \to \infty$ because of the decaying coefficient.
4. (Critical damping:)

5. (Heavy Damping: $\gamma \gg \omega$ or $\omega = 0$). This gives us

$$\lambda_1 = -\gamma$$
$$\lambda_2 = 0$$

The significance of the first solution is that the system does not oscillate, but rather it gradually goes towards equillibrium. Thus,

$$x(t) = \frac{v_0}{\gamma} e^{-\gamma t} + x_0$$

The second solution is that if we let $\omega = 0$, then this is akin to not having a restoring force, so the oscillator stays put exactly where it is.

7.2 Forced Oscillations and Resonance

For an oscillator that is "driven" by some force, we mean that the force equation depends on time. That is,

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = F(t) \tag{7.2.1}$$

However, any sufficiently nice function can be decomposed into pure Fourier harmonics, meaning we can turn the above equation into

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = \sum_{\omega} A_\omega \cos(\omega t + \phi_\omega)$$

where the right hand side is written as a sum. By the fact that our differential equation is linear, if f and g are solutions, then $\alpha f + \beta g$ is also a solution (note that $\alpha, \beta \in \mathbb{C}$). This fact is known as the *principle of superposition*.

Therefore, we want to solve the equation for forces of the form

$$F(t) = Fe^{-i\omega_0 t}.$$

If we can solve this problem, then we will, by the principle of superposition, be able to solve the equation for any driving force. For the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = Fe^{-i\omega t}$$

$$x(t) = Ce^{-i\omega t}.$$

Then we get an expresion for A:

$$C = \frac{F}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{7.2.2}$$

Then, we can take the real part of this expression. Note that $C = Ae^{i\alpha}$ for some real number A and some angle α. This gives us

$$x(t) = A\cos(\omega t - \alpha).$$

where

$$A = \frac{|F|}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \tag{7.2.3}$$

What this tells us is that if the driving force is sinusoidal, then the resulting motion is also sinusoidal with the same frequency but a different phase.

[PLOT A/—F—]??????? In the special case $\omega_0 = \omega$, if we drive the system at its natural frequency, the oscillation amplitude spikes. This phenomenon is known as *resonance*. For instance,

if you are turning on the faucet in an old apartment, then the pipes will start making a noise if you get the water pressure just right. Moreover, this is why soldiers must break step when they cross a bridge. The impact of their synchronized boots will cause the bridge to sway, and in some cases collapse.

Let us consdier two cases, both when $\gamma \ll \omega$. First, if $\omega = 2\omega_0$, then

$$\frac{A}{|F|} = \frac{1}{\sqrt{3\omega_0^4 + \gamma^2 4\omega_0^2}} \approx \frac{1}{\omega^2\sqrt{3}}.$$

On the other hand, if $\omega = \omega_0$, then

$$\frac{A}{|F|} = \frac{1}{\gamma\omega} \gg \frac{1}{\omega^2}.$$

7.2.1 Breaking Case: Undamped Resonance

Lastly, what happens if $\gamma = 0$ but $\omega = \omega_0$? Our solution goes off to infinity. Can we physically interpret this, and pinpoint why our equations break down? If this is true, our equation is

$$\ddot{x} + \omega^2 x = Fe^{-i\omega t}$$

which cannot be solved by our ansatz, since on the left hand side we will get a 0.

In order to solve this, we can consider the equation

$$\ddot{x} + \varepsilon\dot{x} + x = e^{it}.$$

We can have our ansatz of $x(t) = Ce^{it}$. We get that the solution for C for the equation

$$\ddot{x} + \varepsilon\dot{x} + x = \cos t$$

is

$$C = -\frac{i}{\varepsilon}$$

after plugging in our ansatz. Therefore, we can take the real part and we get

$$x(t) = \frac{\sin(t)}{\varepsilon}.$$

We can ignore the homogeneous solution (i.e., the solution for undriven oscillation) since in the limit for very large t, our initial conditions do not matter. This is because with damping, the homogeneous solution has a $e^{-\gamma t}$ term in it, so this is ignorable.

Next, we can consider the closely-related equation

$$\ddot{x} + x = \cos t.$$

We can begin with the ansatz $x(t) = u(t)e^{it}$. Then we have

$$\dot{x} = u'(t)e^{it} + iu(t)e^{it}$$
$$\ddot{x} = e^{it}\left(\ddot{u} + 2i\dot{u} - u\right)$$

Plugging this into the differential equation, we get

$$\ddot{u} + 2i\dot{u} = 1,$$

which means that $u = -\frac{i}{2}t$. Then we can take the real part, since our driving force was cos, so

$$x = \frac{1}{2}t\sin(t)$$

where we ignore the homogeneous solution.

Amazingly, what this demonstrates is that when $\omega_0 = \omega$, and $\gamma = 0$, the amplitude of our soulution is increasing! Thus, as $t \to \infty$ (or, as the prior calculation demonstrated, $\gamma \to 0$), our amplitude becomes infinite. If this carries on, our physical system will likely break.

7.3 Many Degrees of Freedom

The real power of the method of small oscillations is when it is applied to systems with many degrees of freedom. Consider a completely general system, with multiple degrees of freedom. Then

$$\begin{aligned}\mathcal{L} &= T - U \\ T &= \frac{1}{2}\sum_{i,k=1}^{N} a_{ik}(q)\dot{q}_i\dot{q}_k \\ u &= U(q).\end{aligned}$$

What does it mean to apply the method of small oscillations to such a system? It means that we want to describe what the system is doing when it is close to equiibrium. The equilibria of these systems occur when

$$\frac{d}{dt}\left(\sum_k a_{ik}(q)\dot{q}(k)\right) = -\frac{\partial U}{\partial q_i} = 0.$$

This is an extremal point of our function. If this condition is satisfied, then we are either at a minimum, a maximum, or a saddle point. In Russian, this is calleda "pereval."

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Considering a potential like $U = U(q)$, then we can consider small oscillations around the equilibrium. We have argued that small oscillations means that $U(Q)$ is a stationary point, such that all partial derivatives

$$\frac{\partial U}{\partial q_i}(Q) = 0.$$

The only meaning is that we consider motion near the equilibrium point, $q = Q + x$ where x is a vector, which is a small deviation. This is the essence of small oscillations: near equilibria, we can solve for the motion. The simplification procedure is as follows. We can expand our expressions to leading order, which in this case means quadratically.

The philosophy is that if f is an analytic function on the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ can be expanded like

$$f(x_0 + \varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n_1}\varepsilon^n.$$

To leading order, this is equal to $f(x_0)$. To get next to leading orer, we get $f(x_0) + f'(x_0)\varepsilon$, which is an even better approximation. We can keep going as long as is convenient, though we usually stop after quadratic order:

$$f(x_0) + f'(x_0)\varepsilon + \frac{1}{2}f''(x_0)\varepsilon^2.$$

After a few steps, for ε sufficiently small, the terms $\varepsilon^n/n!$ is a very small quantity, so we can very rapidly converge to the actual function in just a few steps, for small ε. Then

$$U(q) \approx U(Q) + U'(Q)x + \frac{1}{2}U''(Q)x^2.$$

We expand to quadratic order, since for regular oscillators, a quadratic approximation will give us meaningful behavior (for instance, you cannot oscillate on the slope of a line). Then

$$U(q) \simeq \frac{1}{2}\sum b_{ij}x_i x_j$$

where

$$b_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j},$$

since our constant term $U(Q)$ can be set to 0, and we are at an equilibrium where the first partial derivatives are 0.

Now that we have a Taylor approximation of U, we have that

$$T = \frac{1}{2}\sum a_{ij}(Q+x)\dot{x}_i\dot{x}_j \simeq \frac{1}{2}\sum a_{ij}\dot{x}_i\dot{x}_j$$

where we have expanded to 0th order. Recall that in one dimension, we had

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\kappa x^2$$

so in multiple dimensions, we get

$$\mathcal{L} = \frac{1}{2}\sum a_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum b_{ij}x_i x_j.$$

Our N resultant equations of motion are, from Euler-Lagrange,

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\sum_j b_{ij}x_j.$$

The coefficient $1/2$ disappears since by Clairaut's theorem,

$$b_{ik} = \frac{\partial^2 U}{\partial q_i \partial q_k} = b_{ki}.$$

Our other part of the equation is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \sum a_{ij}\ddot{x}_j$$

so from Euler-Lagrange,

$$\sum a_{ij}\ddot{x}_j = -\sum_j b_{ij}x_j.$$

These can be neatly summarized in the following matrix form:

$$M\ddot{\mathbf{x}} = -K\mathbf{x}.$$

where M is a matrix corresponding to the generalized mass a_{ik} and K is a matrix corresponding to the generalized spring constants b_i. In order to solve these equations, we can use our ansatz for undamped oscillations due to the linearity of our system. For each i,

$$x_i = \lambda^2 u_i e^{\lambda t}$$
$$\ddot{x}_i = \lambda^2 u_i e^{\lambda t}$$

and so

$$\lambda^2 \sum a_{ik}u_k = -\sum b_{ik}u_k,\ i = 1, \ldots, N.$$

The coefficient λ^2 is called an Eigenfrequency of the sytem, and the vector $\mathbf{u} = (u_1, \ldots, u_N)$ is the eigenvector to this matrix equation. This system has one trivial solution, which is $\mathbf{u} = 0$. These correspond to the initial amplitudes of our equations of motion.

For a given Eigenfrequency λ, there will be a linear subspace of solutions to the equation. These equations are ubiquitous; in quantum mechanics, they are used in finding the energy levels of a system.

7.3.1 The Coupled Oscillator

Consider a setup with two masses, m_1 and m_2, attached by a spring with constant κ, and both are attached to a wall with constants κ_1 and κ_2. Our Lagranian for a coupled oscillator is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \\ &- \frac{1}{2}\kappa_1 x_1^2 + \frac{1}{2}\kappa_2 x_2^2 \\ &+ \frac{1}{2}\kappa(x_2 - x_1)^2.\end{aligned} \tag{7.3.1}$$

Supposing $\kappa_1 = \kappa = \kappa_2$, and $m_1 = m_2$, our potential energy is

$$U = \kappa(x_1^2 + x_2^2 - x_1x_2).$$

We have the equations of motion:

$$\begin{aligned}m\ddot{x}_1 &= -\kappa(x_1 - x_2) \\ m\ddot{x}_2 &= -\kappa(x_2 - x_1)\end{aligned}$$

which we can attempt to solve with the ansatzen

$$\begin{aligned}x_1 &= u_1 e^{-i\omega t} \\ x_2 &= u_2 e^{-i\omega t}.\end{aligned}$$

Let

$$\Omega^2 = \frac{\kappa}{m}.$$

Then our eigenvalue equation becomes

$$\begin{aligned}\omega^2 u_1 &= \Omega^2(2u_1 - u_2) \\ \omega^2 u_2 &= \Omega^2(2u_2 - u_1)\end{aligned}$$

Which is

$$\begin{aligned}(\omega^2 - 2\Omega^2)u_1 + \Omega^2 u_2 &= 0 \\ \Omega^2 u_1 - (\omega^2 - 2\Omega^2)u_2 &= 0.\end{aligned}$$

We can take the determinant of this equation to compute the characteristic equation of the system:

$$\omega^2 - 2\Omega^2 = \pm\Omega^2 \;\Rightarrow \omega^2 = 3\Omega^2, \Omega^2$$

The first Eigenfrequency corresponds to contrary motion of the two masses. This is a higher frequency since all three springs exert force in the same direction. The second corresponds to motion in the same direction; in this case, the middle spring is unstretched so they oscillate like they normally would.

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We were solving a coupled oscillator with two masses, connected by a spring constant κ and governed by the differential equations

$$\ddot{x}_1 = -2\Omega^2 x_1 + \Omega^2 x_2$$
$$\ddot{x}_2 = -2\Omega^2 x_2 + \Omega^2 x_1$$

The ansatz here is $x_1, x_2 \propto e^{-i\omega t}$. Solving this system using the ansatz, we get

$$\omega_1^2 = \Omega^2$$
$$\omega_2^2 = 3\Omega^2.$$

If we have the initial condition $\dot{x}_1 = \varepsilon$, we can attempt to solve this set of differential equations.

This system has two solutions. However, how might we factor in initial conditions? For instance, one of the masses may have a nonzero velocity. For a mediocre mathematician, the answer to this question will be complicated. For a physicist, the answer is obvious; yes, because you can do it! Insofar that our world is mechanical (we can build it) and deterministic, the solution will be unique.

In order to find the solution, we can combine solutions. Therefore, the general solution to a coupled oscillator is:

$$x_1(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t + A_2 \cos\omega_2 t + B_2 \sin\omega_2 t$$
$$x_2(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t - A_2 \cos\omega_2 t - B_2 \sin\omega_2 t$$

where due to the fact that one of the Eigenfrequencies corresponds to contrary motion, our second part of the solution for x_2 incurs a minus sign. It is still unclear that there is a unique combination of A_1, B_1, A_2, B_2 which satisfy the initial conditions. As we mentioned before, there is a unique solution.

7.5 Parametric Resonance

If we have a well-tempered piano, here's an experiment you might try. If you press a C, and then a C an octave above, you can let go of the first one and the string of the first C will resonate at a small amplitude. However, if this is done with a discordant note, such as a B, then the same effect does not occur. This demonstrates the phenomenon of resonance.

The method by which children swing on swings is by parametric resonance. We can model this situation where we have a pendulum, except the massless string can be tugged in order to make it shorter. By changing the parametrs of the system, we get that pulling the string with frequency $\omega_0 = 2\omega$ is also a resonance, and we can increase the amplitude of our system, making our system unstable.

Another example of parametric resonance is before the big bang. There was a period called inflation, where a scalar field was oscillating around a local minimum of potential, which caused oscillations in other fields. By parametric resonance, these fields got transformed to other particles. These parametric resonances are a very important mechanics problem. It is a piece of highly nontrivial mathematical physics.

Formally , we are solving the problem:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 = -\frac{1}{2}\kappa x^2$$
$$m = m(t)$$
$$\kappa = \kappa(t)$$

The mass and κ vary with time. Hence the term "parametric resonance." For the example fo the swing, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\ell^2\dot{\phi}^2 - \frac{1}{2}mg\ell\phi^2$$

where $\ell = \ell(t)$ is the distance from the point where the ropes are fastened to the center of mass. Then our effective mass is $\frac{1}{2}m\ell^2$ and our effective κ is $\frac{1}{2}mg\ell\phi^2$.

Therefore, for a generalized mass and generalized κ,

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x}$$
$$\frac{d}{dt}\left(m\dot{x}\right) = \kappa x$$

Since our mass is time-dependent, we cannot pull out m. This general problem of a time-variable oscillator reduces to the problem where only κ depends on time, which is a highly nontrivial result, which we will now demonstrate. From above,

$$m\frac{d}{dt}\left(m\frac{d}{dt}x\right) = m\kappa x$$
$$\frac{d^2}{d\tau^2}x = \kappa x$$

Consider $m(t)\frac{df}{dt}$. We want to find $\tau(t)$ such that

$$m(t)\frac{df}{dt} = \frac{df}{d\tau}$$

which means that

$$\frac{m}{dt} = \frac{1}{d\tau}$$
$$d\tau = \frac{1}{m}dt$$
$$\tau = \int_0^1 dy\frac{1}{m(y)}$$

The interesting fact is that we can always find such a τ. The term here is we are changing our independent variable, constructed to satisfy the relation

$$m(t) = \frac{1}{\tau'(t)}.$$

Our equation of motion reduces to

$$\frac{d^2x}{d\tau^2} + m\kappa x = 0$$

which is isomorphic to the original problem. Therefore, there is no loss of generality in considering our system to be of the form

$$\ddot{x} = -\omega^2(t)x.$$

If we have

$$\ddot{x} = -\omega^2(1 + \varepsilon f(t))x.$$

This is akin to slightly changing the length of the rope in our pendulum, or changing our center of mass on a swing. According to Noether, the reason for energy conservation is because of time

translation symmetry. Notice now that since ω depends on time, our Lagrangian does indeed depend on time and so energy does not depend on time:

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x.$$

If we are clever, we will be able to find a smart configuration where we can increase our amplitude. We set our unperturbed osciolation energy to be

$$E := \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2.$$

Taking the time derivative,

$$\dot{E} = \dot{x}(\ddot{x} + \omega^2 x) = \dot{x}(-\omega^2 \varepsilon f(t)x).$$

Thus, the rate of change of our energy is

$$\dot{E} = -\varepsilon f(t)\omega^2 x\dot{x}.$$

If ε is small, then in the long-term trend of this quantity E_0, we can calculate this to 0th order in ε. Thus we are very close to free oscillation. Then $x = \cos\omega t$, and $\dot{x} = \sin\omega t$.

$$\dot{E}_0 = -\varepsilon f(t)\left(-\frac{1}{2}\sin 2\omega t\right).$$

then if we make $f(t) = \sin 2\omega t$, then our E_0 term is always increasing since $\dot{E}_0$ is always positive.

We can try and derive this another way. We can consider our actual time-dependent energy, which is

$$E \equiv \dot{x}\frac{\partial \mathcal{L}}{\partial x} - \mathcal{L}$$

and so

$$\dot{E} = \dot{x}\ddot{x} + \omega^2 x\dot{x} + \omega\dot{\omega}x^2.$$

Because of our differential equation $\ddot{x} = -\omega^2 x$, we know that the first two terms are 0. Therefore,

$$\dot{E} = \omega\dot{\omega}x^2$$

where $x = A\cos\omega_0 t$. If our frequency is $\omega \approx \omega_0$, then our equation for x is close to our formula for linear oscillation. We can use the half angle formula to get

$$x^2 = \frac{A^2}{2}\left(1 + \cos 2\omega_0 t\right).$$

Moreover, to leading order, $\omega\dot{\omega} \approx \omega_0\dot{\omega}$. Therefore, our energy is, to leading order,

$$\dot{E}\omega_0\dot{\omega}\frac{1}{2}A^2\left(1 + \cos 2\omega_0 t\right).$$

Our energy is, therefore,

$$E = E_0 + \frac{A^2}{2}\int_0^t ds(\dot{\omega} + \dot{\omega}\cos 2\omega_0 s).$$

Since $\omega \approx \omega_0$, we know that $\dot{\omega}$ is very small, so it can be ignored. in the sum. In the second term, if $\dot{\omega}$ is a function of the frequency, then the second term can have an average value which is not small. For instance, if $\dot{\omega} \sim \varepsilon\cos(2\omega_0 t)$, then the integral of the second term becomes

$$\frac{A^2}{4}\varepsilon t + \frac{A^2\varepsilon\sin(4\omega_0 t)}{16\omega_0}$$

However, since t can be arbitrarily large, we can add an arbitrary amount of energy to the system just by parametric resonance. This is known as secular growth of energy. Moreover, the fact that you have to tuck your legs in twice per swing is an example of varying parameters with twice the frequency of natural oscillation.

The way a child can intuitively perform this parametric resonance to swing higher is by feedback. This $\omega\dot{\omega}$ is really $\dot{\ell}$, so when x^2 is large, $\dot{\ell}$ should be large, and if x^2 is small, then $\dot{\ell}$ should be negative. At the bottom of the swing.

MAYBE I HAVE IT BACKWARDS? Particle decays in high energy physics can be thought of as parametric resonance.

8 Motion of Rigid Bodies

A rigid body is something that cannot be faithfully represented as a point particle, yet we also do not need to consider the properties of individual particles in the system.

We study problems in mechanics only when they are useful for other areas of physics, since mechanics is more or less a finished science. In this respect, rigid body motion is unusual since it does not really come up outside of mechanics. However, this does not undermine its importance; if we have a rotating body, it has intrinsic angular momentum. It turns out that the intrinsic angular momentum of a particle, called spin, is fundamental in the study of elementary particles. By some miracle, they are half-integer multiples of $\hbar$. For some reason, this spin determines whether or not the particles are bosons or fermions. If all electrons were in the same state, no chemistry would be poissible. All matter is made of particles with spin 1/2, all interactions are particles with spin 1, gravity is mediated by spin 2, and the Higgs boson particle has spin 0. There are no fundamental fields with spin 3/2, since they are non-renormalizable. However, all are possible spins of nuclei.

Another motivation is that the fine structure of levels in hydrogen are caused by the so-called spin-orbit interactions.

The motion of rigid bodies is useful in studying rotating bodies such as stars. Milankovitch cycles are caused due to the spin of the Earth on its axis.

8.1 Number of Degrees of Freedom

In Lagrangian mechanics, we have a universal recipe to deal with different systems: given action, out comes everything else. However, before calculating kinetic energy, we must first count the number of degrees of freedom. Recalling the definition, we need the numbers to fully characterize our system.

We can translate our rigid object in 3 dimensions, but we can also orient it around three intrinsic axes. Therefore, our Lagrangian with 6 degrees of freedom will be a horrible mess indeed.

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8.2 Lagrangian of a Potato

The Lagrangian of a potato in a gravitational field is a system with 6 degrees of freedom. It is given by

$$\begin{aligned}\mathcal{L}(X,Y,X,\theta,\phi,\psi,\dot{X},\dot{Y},\dot{Z},\dot{\theta},\dot{\phi},\dot{\psi}) = &\frac{1}{2}M(\dot{X}^2+\dot{Y}^2\dot{Z}^2)\\ &- MgZ\\ &+\frac{1}{2}I_1(\sin\theta\sin\psi\dot{\phi}+\cos\psi\dot{\theta})^2\\ &+\frac{1}{2}I_2\left(\sin\theta\cos\psi\dot{\phi}-\sin\psi\dot{\theta}\right)^2\\ &+\frac{1}{2}I_3\left(\cos\theta\dot{\phi}+\dot{\psi}\right)^2\end{aligned}$$

Moreover, the center of mass of the potato is decoupled from the orientation of the potato; then we can use only the first part in order to describe its motion. About three axes, this potato has three angles to describe how it is tilted around a particular axis.

The reason we are talking about potatoes is to underscore the fact that the full motion of an asymmetrical, 3-Dimensional object may inded be extremely complicated. In these instances, we may have to deviate from the standard recipe of finding a Lagrangian and calculating the equations of motion from it. The above equation can be solved in quadratures, but the solutions will involve elliptic integrals and theta funcitions.

However, these difficulties are purely mathematical; ideologically, we may be on the right track. For instanace, if we were in 2 dimensions, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) - Mgy + \frac{1}{2}I\dot{\phi}^2$$

so its motion is fully characterized by its mass and moment of inertia, with an additional degree of freedom ϕ which specifies orientation.

Rotations in three dimensions are much more complicated in part due to the fact that the structure of the group is nonabelian and complicated.

8.3 Angular Velocity Without Angles

To solve 3-dimensional motion would mean to give the equations of motion and give the coordinate of a particle as a function of time. The rotational part of the motion in 3-dimensional motion, it is

$$\phi(t) = \Omega_0 t + \phi_0.$$

To do this in 3 dimensions is a terribly complicated affair. We want to avoid using Euler angles; for solid objects, we can introduce velocity without specifying which coordinate it represents. Therefore, all rotational motion is rotational motion around some unspecified axis,

Theorem 8.3.1. Let $\mathbf{v}$ be a velocity of some point in an object. Then

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}. \tag{8.3.1}$$

where $\mathbf{V}$ is the velocity of some arbitrary point O in our object, $\mathbf{r}$ is the distance between them.

We get to choose our coordinates, so we let $\Omega = (0, 0, \Omega)$ and then the cross product between that and the radius vector is

$$\boldsymbol{\Omega} \times \mathbf{r} = (-\Omega y, \Omega x, 0).$$

This makes sense, since the velocity of rotation is proportional to the distance along an axis, and perpendicular to the radius vector.

Take any potato with a point O that is not moving. There will always instantaneously be one axis of rotation about which it rotates.

8.4 Tensor of Inertia

Notation. Suppose we want to calculate the momentum of a solid body. Then

$$\mathbf{P} = \int dm\mathbf{v} = \int d^3r\ \rho(r)\mathbf{v}(\mathbf{r})$$

Then the kinetic energy of a moving solid is as follows:

$$\begin{aligned} T &= \frac{1}{2}\sum mv^2 = \frac{1}{2}\sum m(V + \Omega \times r)^2 \\ &= \frac{1}{2}\sum m\left[\mathbf{v}^2 + 2\mathbf{v}\cdot(\boldsymbol{\Omega}\times\mathbf{r}) + (\boldsymbol{\Omega}\times\mathbf{r})^2\right] \end{aligned}$$

Sinc we can take O to be any arbitrary point, we can choose the center of mass frame. Our potential energy is

$$\begin{aligned} \Pi &= \sum mgz \\ &= MgZ_{cm} \end{aligned}$$

Then we have that, in the CM frame, $\sum m\mathbf{r} = 0$. Therefore, our $2\mathbf{v} \cdot (\mathbf{\Omega} \times \mathbf{r})$ term vanishes. As a result, our kinetic energy nicely splits into two parts:

$$\frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\sum m\left(\mathbf{\Omega} \times \mathbf{r}\right)^2$$

If we chose to cosnider any other point, we would have a very complicated system. Now, we are at liberty to focus on the second term. Then

$$\begin{aligned} T &= \frac{1}{2}\sum m\left(\mathbf{\Omega} \times \mathbf{r}\right)^2 \\ &= \frac{1}{2}\sum_{i,k=1}^{3} I_{ik}\Omega_i\Omega_k \\ &= \frac{1}{2}landau \end{aligned}$$

where we derive the second from $|\Omega \times r|^2 A^2 B^2 - (\mathbf{\Omega} \cdot \mathbf{r})^2$.

$$\begin{aligned} \frac{1}{2}\sum m\left(\mathbf{\Omega} \times \mathbf{r}\right)^2 &= \frac{1}{2}\sum m\left(\mathbf{\Omega}^2\mathbf{r}^2 - \left(\mathbf{\Omega} \cdot \mathbf{r}\right)^2\right)^2 \\ &= \frac{1}{2}\sum m\left[\mathbf{r}^2\sum_{i,k=1}^{3}\delta_{ik}\Omega_i\Omega_k - \left(\sum_{i=1}^{3}\Omega_i x_i\right)\left(\sum_{k=1}^{3}\Omega_k x_k\right)\right] \\ &= \frac{1}{2}\sum_{i,k=1}^{3}\Omega_i\Omega_k\left[\sum m\left(\mathbf{r}^2\delta_{ik} - x_i x_k\right)\right] \end{aligned}$$

Then we let

$$I_{ik} := \sum m(\mathbf{r}^2\delta_{ik} - x_i x_k).$$

Which is, in matrix form,

$$\begin{bmatrix} \sum m(x^2+y^2) & -\sum mxy & -\sum xz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(y^2+z^2) \end{bmatrix}$$

The components of this matrix are called moments of inertia. The diagonals are the moments along the three different axes.

There is an isomorphism between ellipsoids in 3 dimensions and symemtric, 3-dimensional matrices. This means that if we choose our coordinates smartly, our inertia tensor is a diagonal matrix. Then we get that the diagonals are principle moments of inertia.

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Last time, we calculated the kinetic energty of a rigid body in terms of its instantaneous velocity. Moreover, we want to know given some force, what the instantaneous acceleration be, for an ideal cylinder, say. To solve this, our expression of kinetic energy will suffice.

Just using the equation for kinetic energy, can we tell when a pencil slips as it drops? Or when a cylinder rotates around its tip?

One interesting problem is that given a rigid body leaning against a wall, and coefficients of friction, it is impossible to predict the weight as read by a scale on the floor.

Consider a cylinder being pulled along the ground with a given force $\mathbf{F}$. We do know how to calculate its kinetic energy:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2$$

And we also have that

$$I = \sum \delta m \cdot r^2 = M\langle r^2 \rangle$$

But the average value of r is given by an integral in the continuous case:

$$\langle r^2 \rangle = \frac{\int_0^R r dr r^2}{\int_0^R r dr} = \frac{1}{2}$$

And so we get $m\langle r^2 \rangle = \frac{1}{2}mR^2$. Then our expression for the kinetic energt is

$$T = \frac{3}{4}mV^2$$

Then we see that power is given by

$$Fv = \frac{dT}{dt} = \frac{3}{2}mva$$

so

$$\frac{2}{3}\frac{F}{m} = a$$

so this means that part of the work goes into actually rotating the cylinder.

We can try giving an upper bound to the average height:

$$0 \leq \langle r^2 \rangle \leq R^2$$

8.5 Angular Momentom of a Rigid Body

We can start with the familiar equation for the kinetic energy:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}\sum_{j,k=1}^{3} I_{jk}\Omega_j\Omega_k$$

And the angular momentum:

$$\mathbf{J} = \mathbf{\Omega} + \mathbf{J}_{\text{cm}}$$

where

$$\mathbf{J}_{\text{cm}} = m\mathbf{R}_{\text{cm}} \times \mathbf{v}$$

and

$$J_i = \sum_{k=1}^{3} I_{ik}\Omega_k$$

We have introduced the moment of inertia in order to calculate kineic energy of a rigid body, which is the energy of the center of mass and also rotational kinetic energy, which is given in terms of the tensor of inertia. It turns out that this tensor of inertia allows us to calculate angular momentum; The center of mass is easy to calculate but rotational angular momentum can be given by the tensor of inertia.

The intrinsic angular momentum is given by the inertia tensor and angular velocity.

In order to get the kinetic energy, we needed the concept of the inertia tensor. It turns out that this tensor allows us to calculate the angular momentum of the rigid body. The center of mass has angular momentum, which is given by the familiar equation, whereas the ith component of the intrinsic angular momentum J_i is given in terms of the inertia tensor and rotational velocity. Note that this is calculated around the center of mass.

The reason for the existence of any tensor is to convert vectors into other vectors. Moreover, we have the following useful identity:

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

useful all the time in physics.

To summarize, we can calculate the energy of rotation, and also the angular momentum of a rotation.

$$T = \frac{1}{2} \sum_{i,k=1}^{3} I_{ik} \Omega_i \Omega_k$$

$$J_i = \sum_{k=1}^{3} I_{ik} \Omega_k$$

A smart choice in axis gives us a diagonal angular momentum. If we use this coordinate system, we get that

$$T = \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

$$J = (I_1 \Omega_1, I_2 \Omega_2, I_3 \Omega_3)$$

Now that we can calculate angular momentum, we can calculate many complex systems.

Book flipping result: one of the axes has an intermediate moment of inertia. Stable rotation about largest, about smallest, but unstable about intermediate moment of inertia.

The rotation of a cylinder is very regular, so it can be modelled like a spinning top. It precesses, and it also rotates, with fixed frequencies.

8.6 Precession of the Symmetrical Top

A *symmetrical top* is an object such that $I_1 = I_2 \neq I_3$. For a bottle, or a long cylinder, $I_1 = I_2 > I_3$. We want to show that this motion can be fully understood from conservation laws. In particular, we know that $\mathbf{J}$ and E are cosntants.

Since $\mathbf{J}$ is conserved, its direction is also conserved. We can take the third axis of the cylinder to be the plane of the page, x_3. The other two axes we are free to rotate. We can make x_1 co-planar with x_3, and it becomes clear that

$$\mathbf{J} = (I_1 \Omega_1, I_2 \Omega_2, I_3 \Omega_3).$$

We claim that Ω_2 vanishes; this is because we know that J is co-planar with x_3 and x_1, Therfore, we see that it cannot have a component along the second axis, so $\Omega_2 = 0$, and so $\mathbf{\Omega}$ is in the same plane as $\mathbf{J}$ and x_3.

$$\mathbf{J} = (I_1 \Omega_1, 0, I_3 \Omega_3)$$

We know that J^2 must be constant, so

$$J^2 = I_1^2 \Omega_1^2 + I_3^2 \Omega_3^2 = \text{const.}$$

but also

$$E = \frac{1}{2} I_1 \Omega_1^2 + I_3 \Omega_3^2 = \text{const.}$$

so we conclude that Ω_1, Ω_3 are constants since $I_1 \neq I_3$. This means that the angle between $\mathbf{J}$ and $\mathbf{x}_3$ is constant. This angle is known as the *angle of precession.*

It is possible to calculate the frequency with which this precession occurs. The vector **Omega** can be composed into a vector along $\mathbf{J}$, denoted $\mathbf{\Omega}_{pr}$ and a vector along x_3, denoted $\widetilde{\mathbf{\Omega}}$. Note that x_3 and $\mathbf{J}$ are not necessarily orthogonal. Therefore,

$$\mathbf{\Omega} = \Omega_{pr}\mathbf{J} + \widetilde{\Omega}x_3$$

Simply put, it is the rotation about the axis $\mathbf{J}$ but also the angular velocity around the third axis. One might be tempted to simply subtract Ω_3 from the expression, but then the remaining term will be along x_1, since they are orthogonal in the plane. It must be emphasized that $\Omega_3 \neq \widetilde{\Omega}$. Therefore, we take the dot product of the above:

$$\begin{aligned} x_1 \cdot \Omega_1 &= \Omega_{pr} \sin\theta \\ J_1 = I_1\Omega_1 &= J\sin\theta \end{aligned}$$

and so

$$\Omega_{pr} = \frac{J}{I_1}.$$

Since we express this in terms of conserved quantities, we say this is a result since conserved quantities define our motion.

If we throw a cylinder into the air, its precession rate is given by its total angular momentum. In particular, if our angular momentum is closely oriented along x_3, then

$$J = \Omega_{\text{rot}} I_3$$

implying that

$$\Omega_{\text{pr}} \approx \frac{I_3}{I_1}\Omega_{\text{rot}}$$

so we conclude that it is rotating faster than it is precesssing.

8.7 Equations of Motion for a Rigid Body

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It makes sense to write the equations of motions as the rate of change of momentum and angular momentum. Since our system has 6 degrees of freedom, we have 6 equations of motion from angular and linear momentum. Then

$$\begin{aligned} \dot{\mathbf{P}} &= \sum \mathbf{F}_i \\ \boldsymbol{\tau} &= \sum \mathbf{r}_i \times \mathbf{F}_i \end{aligned}$$

Where we know that all a rigid body can do is move the center of mass, and rotate with respect to the center of mass.

$$\begin{aligned} \mathbf{P} &= \sum m\mathbf{V} \\ &= \sum m\left(\mathbf{V}_{cm} + \mathbf{\Omega} \times \mathbf{r}\right) \end{aligned}$$

8.8 Statics of Rigid Bodies

For equations involving bodies in a static equilibrium, then the equations we discussed are quite useful. In particular,

$$\mathbf{F} = 0$$
$$\boldsymbol{\tau} = 0.$$

Moreover, there is an easy statement that if the sum of all forces vanishes, then it does not matter which reference point we choose for our torque, since

$$\boldsymbol{\tau} = \sum r \times f$$

where if we replace r by $a + r$, then

$$\boldsymbol{\tau}' = \sum (a + r) \times f = \sum a \times f + \sum r \times f = \boldsymbol{\tau}$$

since the first term vanishes.

Obviously, our equation for $\mathbf{F}$ is only valid for the center of mass. But sometimes, it is useful to chose a different reference frame

8.9 Euler Equations

Ignoring gravity, and neglecting air resistence, we have that

$$\dot{\mathbf{J}} = \boldsymbol{\tau} = 0$$

which is enough to describe the motion of the top. If we coordinatize from our object, we want to calculate $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in terms of time. It turns out that this is not possible in terms of elementary functions. However, if we relax our ambitions, we want to present equations which are calculable in principle (e.g., by a computer). They are, s

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= -\Omega_2 \mathbf{x}_3 + \Omega_3 \mathbf{x}_2 \\
&\vdots \\
I_1 \dot{\boldsymbol{\Omega}}_1 &= (I_2 - I_3)\Omega_2 \Omega_3 \\
I_3 \dot{\Omega}_2 &= (I_3 - I_1)\Omega_3 \Omega_1 \\
I_3 \dot{\Omega}_3 &= (I_1 - I - 2)\Omega_1 \Omega_2.
\end{aligned}$$

One may object to these. Aren't $\mathbf{x}_j$ vectors, which have 2 angular components each? Then, we will have 9 initial conditions, but only 3 degrees of freedom for angles. In reality, x_j are related by fixed relations, so we will indeed have 6.

We can use intuition to analyse the first equation. The vectors $\mathbf{x}_1, \mathbf{x}_2$ form a plane. If we have Ω_3 rotation, then at a different instance, we have the orthonormal vectors, just rotated by $\Omega_3 \Delta t \mathbf{x}_2$. We can also consider the frame where $\mathbf{x}_1, \mathbf{x}_3$ are in the same plane, and see that we get the other term.

A simpler way to derive it is thus: we know that velocity at any point in the rigid body is $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$. Then,

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \boldsymbol{\Omega} \times \mathbf{x_1} \\
&= (\Omega_1 \mathbf{x}_1 + \Omega_2 \mathbf{x}_2 + \Omega_3 \mathbf{x}_3) \times \mathbf{x}_1 \\
&= -\Omega_2 \mathbf{x}_3 + \Omega_3 \mathbf{x}_2.
\end{aligned}$$

The second one essentially says that $\dot{\mathbf{J}} = \mathbf{t} = 0$. We have

$$\begin{aligned}\mathbf{J} &= J_1\mathbf{x}_1 + J_2\mathbf{x}_2 + J_3\mathbf{x}_3 \\ &= I_1\Omega_1\mathbf{x}_1 + I_2\Omega_2\mathbf{x}_2 + I_3\Omega_3\mathbf{x}_3\end{aligned}$$

Taking the derivative with respect to time,

$$\begin{aligned}\dot{\mathbf{J}} &= I_1\dot{\Omega}_1\mathbf{x}_1 + I_1\Omega_1\dot{\mathbf{x}} + \dots \\ &= I_1\dot{\Omega}_1\mathbf{x}_1 + I_1\Omega_1(-\Omega_2\mathbf{x}_3 + \Omega_3\mathbf{x}_2) + \cdots + I_3\dot{\Omega}_3\mathbf{x}_3\end{aligned}$$

As $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ change their values, then Ω_k should do so accordingly in order to ensure that the total anglular momentum remains constant. There is nothing new here, since we have really only used the fact that the total angular momentum is constant. If there were external torques, then the equations become

$$\begin{aligned}I_1\dot{\mathbf{\Omega}}_1 &= (I_2 - I_3)\Omega_2\Omega_3 + \tau_1 \\ I_3\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 + \tau_2 \\ I_3\dot{\Omega}_3 &= (I_1 - I - 2)\Omega_1\Omega_2 + \tau_3.\end{aligned}$$

We have just derived the equations which give us the projections of the angular velocity onto the principal axis of inertia of a potato, neglecting air resistence. A potato flying through the air should conserve its rotational energy, since there are no torques. In particular, we have that the kinetic energy of the center of mass, the energy due to rotation, and the potential energy must be conserved. Moreover, since we can write a separate Lagrangian for the center of mass in which energy also stays constant, then we must conclude that rotational energy must itself be constant.

We can derive this statementmathematically.

$$\begin{aligned}E &= \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \\ \dot{E} &= I_1\Omega_1\dot{\Omega}_1 + I_2\Omega_2\dot{\Omega}_2 + I_3\Omega_3\dot{\Omega}_3 \\ &= \Omega_1(I_2 - I - 3)\Omega_2\Omega_3 + \dots \\ &= \Omega_1\Omega_2\Omega_3\left[(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2)\right] \\ &= 0.\end{aligned}$$

Since the Euler equations do not contain $\mathbf{x}_k$ in them, it may be tempting to just solve for the rotation. However, it turns out that this is a nonlinear differential equation, and requires elliptic integrals.

8.10 The Intermediate Axis Theorem

Recall that if you flip a book around the axis with the largest and smallest moments of inertia, then the rotation is stable, in that it is a rotation with some precession. However, if we attempt to do it around the axis with the intermediate moment of inertia, the rotation is very complicated and unstable.

If we are only studying the Euler equations, then we have our recipe of small oscillations to find behavior near equillibria. By definition, an equilibrium is

$$\Omega_1, \Omega_2, \Omega_3 = \text{const.}$$

since there are no external forces. This implies that for our Euler equations, the right hand side is simply 0. Without loss of generality, assume that $I_1 > I_2 > I_3$. Then this immediately forces us to conclude that 2 of the 3 axes must have 0 rotation.

$$\Omega_j = \Omega_k = 0,\ j \neq k.$$

Therefore, when we rotate around the principal axes, then there are 3 equillibria. However, two of these are stable equilibria, and one of them is unstable. For instance, assume that $\Omega_1 = \Omega_2 = 0$. We can consider small displacements, when

$$0 < \Omega_1, \Omega_2 \ll \Omega_3.$$

We can write our equations to linear order. Note that since in the third Euler equation, we have the term $\Omega_1\Omega_2$, this implies that Ω_3 is a constant to linear order. For the others,

$$\dot{\Omega}_1 = \left(\frac{I_2 - I_3}{I_1}\Omega_3\right)\Omega_2$$
$$\dot{\Omega}_2 = \left(\frac{I_3 - I_1}{I_2}\Omega_3\right)\Omega_1$$

where, if we differentiate them, it should be apparent that these are the equations for a linear oscillator:

$$\ddot{\Omega}_1 = \left(\frac{I_2 - I_3}{I_1}\Omega_3\right) \times \left(\frac{I_3 - I_1}{I_2}\Omega_3\right)\Omega_1$$

with a frequency

$$\omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\Omega_3^2.$$

This equation has interesting implications. If I_3 is the smallest or the largest, then our ω is real and we have a nice harmonic oscillator. However, if I_3 is between I_1 and I_2, then our ω is imaginary and we have a complicated resulting motion.

Remarkably, we have that energy and angular momentum are conserved:

$$E = \frac{1}{2}\left(\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3}\right)$$
$$J^2 = J_1^2 + J_2^2 + J_3^2.$$

As our object rotates around the air, the first equation means our components J_k are on an ellipsoid, and the second equation means our component are on a sphere. Therefore, they must lie on the intersection of an ellipsoid and a sphere. If we make the radius of our sphere to be the same as the major axis of the ellipse, then we can have this intersection be just the two endpoints. If we relax this condition a little bit, the intersection is just a small circle on either end. The same goes for the minor axis.

However, if we make the radius of the sphere equal to the intermediate axis, the intersection becomes two intersecting curves. This is a saddle point, and accounts for the resulting unstable motion.

8.10.1 Why do spinning tops not fall?

If you try to balance a top that is not spinning, it falls down very quickly. When we give it a spin, it stays up for considerably longer. We will deduce this intuitively, as well as provide the mathematics, but the latter will not be very intuitive.

For all practical purposes, we can consider a top to be a rigid rod which can move without friction, and can spin about its own axis without frinction. It is in a gravitational field.

If it is not spinning, why does the top fall? This is due to the fact that if we choose the point of contact with the ground as the point of rotation, then there is a nonzero torque from the center of mass, given by gravity. Therefore, the top falls. Even if there was a reaction force, it would have to pass through the point of contact, so it provides zero torque.

For a spinning top, let us write equations:

$$\begin{aligned}\dot{\mathbf{x}}_1 &= -\Omega_2\mathbf{x}_3 + \Omega_3\mathbf{x}_2\\ &\dots\\ I_1\dot{\mathbf{\Omega}}_1 &= (I_2 - I_3)\Omega_2\Omega_3 + \tau_1\\ I_3\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 + \tau_2\\ I_3\dot{\Omega}_3 &= (I_1 - I - 2)\Omega_1\Omega_2 + \tau_3.\end{aligned}$$

We can find the torque, which is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = mgR\hat{z} \times \mathbf{x}_3$$

However, what is worth noting is that since the torque is a cross product with the third axis, so there is no component of the torque in the $\mathbf{x}_3$ direction. This means that $\dot{J}_3$ is 0, which implies Ω_3 is constant. This shouldn't be surprising, since from Noether's theorem, rotations about the third axis do not change the configuration of our system, so we must have a symmetry and therefore conserve angular momentum on the third axis.

Another symmetry is energy conservation. Therefore, it must also be true that the energy, given by

$$E = \frac{1}{2}I_1(\Omega_1^2 + \Omega_2^2) + I_3\Omega_3^2 mgR\hat{z} \cdot \mathbf{x}_3 = \text{const.}$$

must be constant. Note that $U = mgh$, but h is just the projection of the force onto $\mathbf{x}_3$. Using the equations of motion, we see that this must be conserved.

Moreover, we can conclude that

$$J_z = I_1\Omega_1\mathbf{x}_1 \cdot \hat{z} + I_2\Omega_2\mathbf{x}_2 \cdot \hat{z} + I_3\Omega_3\mathbf{x}_3 \cdot \hat{z} = \text{const.}$$

since we can rotate our system around the vertical axis, and nothing happens. Therefore, it is a symmetry of the system and hence must be conserved. Even if the top falls, its resulting angular momentum has no component in the z direction.

Using conservation laws, we can now show why quickly spinning tops are stable. Foremost, let us assume our speed is large. With respect to what, you may ask? By that we mean that the kinetic energy due to rotation is much larger than its potential energy, or

$$I_3\Omega_3^2 \ll mgR.$$

Suppose we are originally completely vertical, and so the potential energy term in our energy equation is $mgR\hat{z} \cdot \mathbf{x}_3 \approx mgR$. If if falls on the ground, we acquire some Ω_1 and Ω_2 rotation. If energy is equal to mgR, then Ω_1 and Ω_2 must remain much smaller than Ω_3, otherwise we will violate energy conservation, since then our total energy will be larger.

However, since J_z is constant, then the largest amount of energy that can be imparted into Ω_1 and Ω_2 is mgR. This means that most of the spin is along Ω_3. However, if our J_z is mostly constant, then this means that most of the rotation is the projection of $\Omega_3\mathbf{x}_3$ onto the z axis.

We can also explain this phenomenon in terms of forces. Imagine that the disk portion of the top is really just spokes with masses on them. Then [VSauce has a better explanation.]

9 The Canonical Formalism

We take canonical to mean standard. In this section, we will discuss the Hamiltonian formulation of classical mechanics, which is directly used in understanding quantum mechanics. We will go over the Hamilton-Jacobi equations, which will be used in understanding the wave-particle duality. Moreover, there are only fields, and quanta of these fields are particles.

Particles are described by ordinary differential equations (think of the oscillator), and waves are described by partial differential equations:

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2}.$$

Sometimes, these partial differential equations can behave like ODEs. The Hamilton-Jacobi equations are a nice illustration.

Lastly, we will understand Liouville's theorem, which is very useful in studying statistical mechanics, as well as in physics in general.

9.1 Hamilton's Equations

Suppose you have a lagrangian system,

$$\mathcal{L} = \mathcal{L}(q, \dot{q}).$$

Then, this system can also be described by the following equations:

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

where

$$H = p\dot{q} - \mathcal{L}$$
$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}.$$

However, you may notice that since p is the momentum, our Hamiltonian is just our energy. Why bother writing the Hamiltonian if we already have energy? This is because energy is reserved for the expression of q and $\dot{q}$. We can illustrate this for an oscillator.

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\kappa q^2 \\
m\ddot{q} &= -\kappa q \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{q}} \\
E &= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}\kappa q^2 \\
H &= \frac{p^2}{2m} + \frac{1}{2}\kappa q^2 \\
\dot{q} &= \frac{p}{m} \qquad\qquad \dot{p} = -\kappa q
\end{aligned}$$

Then, plugging $p = m\dot{q}$ into the last equation, we reproduce the original equation of motion. Hamiltonian is thus a way to convert second order differential equations into a pair of first-order ones. However, this is far from the only use of this formalism.

Suppose we had no idea how to solve

$$\ddot{x} = -x.$$

Then we could write

$$\begin{cases} \dot{=} v \\ \dot{v} = -x. \end{cases}$$

This first order system allows for a numerical solution. We can prove the equivalence between the two:

Proof. Since our Lagrangian is a function of q and $\dot{q}$, the state of our system is fully defined by two parameters, so we have a 2-manifold of states. Consider a general variation of the state, where we consider displacements of $q + dq$ and $\dot{q} + d\dot{q}$. Then

$$dH = dp\dot{q} + pd\dot{q} - \frac{\partial \mathcal{L}}{\partial q}dq - \frac{\partial \mathcal{L}}{\partial \dot{q}}d\dot{q}$$

Howver, by the way we defined p,

$$dH = dp\dot{q} - \frac{\partial \mathcal{L}}{\partial q}dq.$$

This means that

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\frac{\partial H}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}.$$

Since

$$\dot{p} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{dq}$$

from the Euler-Lagrange equations. We thus result in the equations

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

□

We could have shown a Legrendre transformation. Many energetic quantities in thermodynamics are constructed in a way similar to this, where we combine other thermodynamic quantities by replacing the independent variables. For instance, you can work with pressure and temperature, or pressure and volume, etc.

A nice property of the Hamilton equations is that a piece of Noether's theorem is immediately obvious. Suppose our Hamiltonian does not depend on time; then it is an immediate consequence of the equations that the Hamiltonian will be conserved. For a general time-dependent Hamiltonian,

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$$

as time moves on, and our particle moves on the $p-q$ manifold, then our H becomes a function of time:

$$H(t, q(t), p(t)).$$

For instance, we can see

$$H(t, q, p) = \frac{p^2}{2m} + \frac{1}{2}\kappa(t)q^2.$$

Given this, from Hamilton's equations we can predict how p, q will depend on time. Once they depend on time, our H becomes a function of time. Once we have that, we can calculate the rate of change of the Hamiltonian with respect to time, given by what we had above. Therefore, throwing in our equations,

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial t} = 0$$

if we have that the Hamiltonian does not explicitly depend on time.

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9.2 How to convert Lagrange to Hamilton

In this section, we will develop tools to deduce Hamiltonians for simple mechanical systems. Given a Lagrangian $\mathcal{L}(q, \dot{q})$, we can calculate momentum by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}.$$

This quantity is constant if $\partial_q \mathcal{L} = 0$. There is also a quantity called energy,

$$E = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$$

which is constant if $\partial_t \mathcal{L} = 0$. Thus, the Hamiltonian is nothing more than

$$H(q, p) = E(q, \dot{q}(q, p)).$$

However, from the first expression, we can calculate $\dot{q}$ in terms of p and q. Then by some miracle, we can use this H to write the second order differential equations into two first-order ones.

Now, we will calculate the Hamiltonian for generalized coordinates. Our Lagrangian for a system with n degrees of freedom, and in completely generalized coordinates, is

$$\mathcal{L} = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(q)\dot{q}_j\dot{q}_k - U(q)$$

The claim is that the Hamiltonian can be expressed in the form

$$H = \frac{1}{2}\sum_{j,k=1}^{n} b_{jk}(q)p_j p_k + U(q)$$

$$\sum_{i=1}^{n} b_{jk}a_{ki} = \delta ij.$$

In the simplest case where $a_{jk} = m$, then we get the familiar formula $\frac{1}{2}mv^2$.

Our momentum p_i is

$$\begin{aligned} p_i &\equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \\ &= \sum_{k=1}^{n} a_{ik}(q)\dot{q}_k \end{aligned}$$

where we get rid of the $1/2$ in front since we have a term with q_i^2 in it, and the n terms with $\dot{q}_i\dot{q}_k$ in them are combined, since $a_{ij} = a_{ji}$. Next, we can consider energy,

$$\begin{aligned} E &= \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \\ &= \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \\ &= \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L} \\ &= \sum_{i=1}^{n} \dot{q}_i \sum_{k=1}^{n} a_{ik}(q)\dot{q}_k \\ &= \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}\dot{q}_i\dot{q}_k + U(q) \end{aligned}$$

Now we want to express our summands in terms of p. From linear algebra, if our set of variables p_i are linearly related to the $\dot{q}_k$, as given by

$$p_i == \sum_{k=1}^{n} a_{ik}(q)\dot{q}_k,$$

then the inverse matrix B gives us the q's in terms of the p's. The above then becomes

$$E = \frac{1}{2}\sum_{i,k}^{n} a_{ik}\left(\sum_{j=1}^{n} b_{ij}p_j\right)\left(\sum_{m=1}^{n} b_{km}p_m\right)$$

We then can isolate the i index, and get

$$\sum_{i=1}^{n} a_{ki}b_{ij} = \delta_{kj}$$

since we are just inverting a matrix, so we get 1 if $j = k$ and 0 otherwise. Then, putting this into the above equation, so we can sum over p_k and get

$$E = \frac{1}{2}\sum_{i,k} p_k\left(\sum_{m=1}^{n} b_{kn}p_n\right)$$

Repeat this if you want to do theoretical physics in the future.

Just like you can jump from a Lagrangian to a Hamiltonian, the reverse procedure is also similar. We will not go over this here.

9.3 Phase Space and Phase Flow

The benefit of the Hamiltonian formalism is that it is a very useful shift in perspective. Consider our simple harmonic oscillator. The Lagrangian and Hamiltonian are

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{\kappa}{2}x^2$$
$$H = \frac{p^2}{2m} + \frac{\kappa}{2}x^2$$

And the corresponding equations of motion are

$$\begin{cases} \dot{x} &= \frac{p}{m} \\ \dot{p} &= -kappax \end{cases}$$

The achievement here is to map the motion of the oscillator into the so-called *phase space*, where we have coordinates (x, p) which dentote position and momentum. If we let the motion continue, we get an ellipse in space. It turns out that thinking of motion in terms of this phase space is very important.

We oftentimes use the phase-space profile to predict the motion, not use motion to draw the picture. This is especially true for more complicated systems.

For instance, we can select a point (x_0, p_0), where $x_0, p_0 > 0$, which corresponds to stretching the spring and giving it an initial velocity. Then we know that our x increases, since it changes like p_0/m, and p decreases, since it changes like $-\kappa x_0$. Afterwards, we now find ourselves at a new point, (x_1, p_1). We can now do this process again, and so at each point in phase space we only need to use our position in phase space to determine where to go next. Thus, we can think of the pair of equation as giving us velocity of our phase space point.

Moreover, phase space makes many things clearer. For instance, $\partial_t H = 0$ implies that the phase space orbits do not intersect. This is becasue if we have time-independence, then each unique point has a unique phase velocity, so intersections are impossible. Thus, we have a closed form in phase space, which corresponds to a level set of our energy function, or our Hamiltonian. Thus we see that closed trajectories correspond to conserved energies.

We can further illustrate the importance of the phase space picture by an example. Suppose there are two vilages, A, B both represented by points on the Euclidean plane. Moreover, there are two people on two different roads, holding onto a 3-meter rope. Suppose they can walk from A to B on the different roads, but still holding onto the rope. Now assume that two peasants are driving carriages, each 1.5 meters wide, going in oppposite direction on the two different roads. We are given no information about their speeds. Can the two peasants drive past each other without touching?

Intuitively we expect this situation to be like teh two people holding the rope. We will now demonstrate the solution by demonstrating that a good selection of phase space can make these problems easier. We can coordinatize the position of the two particles by coordiantizing the length of the road crossed. x will be the proportion person 1 crosses road 1, and y will be the fraction travelled by person 2 on road 2. Thus, we see this represented in phase space by a curve from $(0, 0)$ to $(1, 1)$. The positions of the peasants start at $(0, 1)$ and end at $(1, 0)$. By continuity, these curves must intersect at some point.

Phase space is a geometric object, which is used to coordinatize the state of the system.

Another, more serious example, is in dark matter. We have

$$m_{\text{dm}} \gtrsim 1 \text{ KeV} \approx \frac{1}{500} \times 10^{-27} g \tag{9.3.1}$$

if dark matter is a fermion. Most think dark matter is some form of an elementary particle, though as of the date this was written, dark matter has not been found. Dark matter is everywhere; in its

orbit around the galaxy, this dark matter should in principle be passing by us. We believe equation (9.3.1), since it follows from the density in phase space. When you learn quantum mechanics, you will learn that electrons cannot be more dense than a certain amont. In particular, each electron must occupy

$$(2\pi\hbar)^3$$

volume in phase space.

The dark matter particles swim in a collective field of graviational attraction, and they will move about a common center of mass. In the phase space, we have a 6-dimensional phase space, and we can freeze the phase space in some moment in time, and they are enclosed in a finite volume. Thus, a snapshot of the galaxy will be dark matter particles in this high-dimensional phase space. The result from quantum mechanics is that they must also have $(2\pi\hbar)^3$ volume, We know that they are moving with a certain velocity, and in order to be gravitationally bound, reqires a given amount of mass. The point is that this constraint that the phase space density of dark matter particles should not exceed this quantum bound, then this gives us the lower bound on the mass of the dark matter particles.

These days, we do not think that the mass of dark matter is too high, since then it woud interact with our detectors. Again, it cannot be too low, since it would be easily produced in accelerators.

Professor Gruzinov wrote in a paper [PAPER] the following estimate of the mass of a Dark matter particle,

$$m_{\text{dm}} \gtrsim 10^{-22} \text{ eV} \approx \tag{9.3.2}$$

if dark matter is bosonic, since bosons do not mind being arbitrarily dense in phase space. This is known as Fuzzy Dark Matter (FDM).

A final illustration of the importance of phase space is order and chaos. There is the concept of dynamical chaos, and there are manifestos of chaos and catastrophe, saying that real-life systems are not exactly solvable, and we must study these chaos with computers. This is mostly nonsense. In the solar system, there is chaos long-term, but to a good approximation our solar system is ordered. There is then a balance between order and chaos in our observations. We started studying chaos after some people drew funny pictures on their computers. People like Poincaré knew about chaos at the end of the 19th century, long before the advent of computers. It can be understood in terms of phase space very easily.

1 degree of freedom means we have a closed trajectory in 2-dimensional phase space. What the point does in phase space is uniquely predicted by its original position. Thus, closure means that we have periodic behavior. For more than 2 degrees of freedom, we have a 4-dimensional phase space. With conservation of energy, we get a 3-manifold of constant energy, $H(p_1, p_2, q_1, q_2) = E$. There is a lot of room in 3-dimensional space, such that the result of the motion can be very sensitively dependent on the initial position in phase space. In general, multi-dimensional mechanical systems will have some chaotic behaviors, since the trajectories have no reason to display closed, periodic motion.

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9.4 Liouville's Theorem

Definition 9.4.1. The *phase flow* is the one-parameter group of transformations of phase space

$$g' : (\mathbf{p}(0), \mathbf{q}(0)) \mapsto (\mathbf{p}(t), \mathbf{q}(t)) \tag{9.4.1}$$

where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions to Hamilton's equations.

Theorem 9.4.1. Hamiltonian Phase Flow preserves volume is incompressoble.

We can take a simple oscillator, with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\kappa}{2}x^2$$

The resulting phase flow will be around the center, without loss of generality. We can consider infinitely many oscillators with various initial positions. We can calculate the velocity in phase space,

$$(\dot{x}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\right) = \left(\frac{p}{m}, -\kappa x\right)$$

where, ignoring units, we can relabel this to just be $\mathbf{v} = (y, -x)$. This vector field is pure rotation, since if we multiply $\mathbf{v}$ by a small time Δt, then the result is like ϕ rotation. Therefore, our original set X of initial positions and velocities is mapped to the same volume. This is simple, however, since the shape of the initial set X is preserved.

Now, we can consider a mathematical pendulum. Up to dimensions, we get that

$$H = \frac{p^2}{2} - \cos(x)$$

In this instance, we have that the period depends on the initial amplitude; therefore, the initial shape of the volume may be deformed.

Liouville's theorem is the foundation of statistical mechanics. If we go outside, and we have a lens, and we focus the sun, we can burn our eyes. If we look away from the sun and to a place with uniform brightness, then we cannot lens this uniform distribution. With respect to the observer, it changes the spatial extent of the sun without changing its brightness. We cannot, for the same reason, lens cosmic rays from the magnetic fields of galaxies. Therefore, we can only lens anisotropies of cosmic rays.

This is a particular example of why Liouville's theorem forms the basis of statistical mechanics and kinetic theory. We can think of molecules in a room which collide every now and then. There is a Maxwell distribution for molecules in thermal velocity. Then

$$f(\mathbf{v}) \propto e^{-\frac{mv^2}{2kT}}. \tag{9.4.2}$$

This can be shown from Liouville's theorem, plus some extra assumptions. As a consequence of these collisions, this probability distribution occurs. Take any particle and measure its velocity, with the mean energy equal to temperature. Between collision, it is simple motion in a gravitational field. We thus have 6-dimensional phase space for a single particle. For a cloud of particles, they will also form trajectories in the same phase space. Thus, since they preserve phase space volume. We can introduce the distribution function which is the density in phase space, or the number of molecules per unit volume per unit velocity. Thus, the figure of merit for our calculation is the number of particles per unit of phase space.

When the particles collide, this can kick some of the particles into different positions in phase space. Between collision, phase space and the distribution function is preserved:

$$f = \text{const.}$$
$$E = \text{const.}$$

In principle, only energy is conserved. Therefore, it makes sense that our distribution function is a function of energy only. If we have a combined system with two parts, then the energy $E = E_1 + E_2$, and the distribution function can be written as

$$f = f_1(E_1) \cdot f_2(E_2)$$

The only function that has this property

$$f(x+y) = f(x)f(y)$$

is the exponential. Therefore, it follows that we have

$$f \sim ae^{-bx}.$$

We can now go on to prove the theorem due to Liouville. Consider an infinitesimal change in the phase space under infinitesimal changes in time. Then

$$(x,p) \mapsto (x + \dot{x}dt, p + \dot{p}dt) = \left(x + \frac{p}{m}dt, p - \frac{dU}{dx}dt\right)$$

since our flow is Hamiltonian. We can show that if we transform a point like so, we need to find the Jacobian:

$$(x,y) \mapsto (u,v)$$

Where

$$A_{uv} = A_{xy}\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

Therfore, our Jacobian is

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial x \partial p}dt & \frac{\partial^2 H}{\partial p^2}dt \\ -\frac{\partial^2 H}{\partial x^2}dt & 1 - \frac{\partial^2 H}{\partial p \partial x}dt \end{vmatrix} \\ &= \left(1 + \frac{\partial^2 H}{\partial x \partial p}dt\right)\left(1 - \frac{\partial^2 H}{\partial p \partial x}dt\right) + \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial p^2}dt^2\end{aligned}$$

Therefore, to first order in dt, we have that this is equal to

$$1 + dt\left(\frac{\partial^2 H}{\partial x \partial p} - \frac{\partial^2 H}{\partial p \partial x}dt\right) = 1$$

from Clairaut's theorem.

We can derive where a Jacobian comes from: if we have an infinitesimal square dy by dx, then transorming our coordinates $(x,y) \mapsto (u(x,y), v(x,y))$, then phase flow is a mapping from the plane onto the plane. Then locally, every transformation is locally a linear function:

$$\left(u_0 + \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y, v_0 + \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y\right)$$

WLOG, we can let $u_0 = v_0 = 0$. Therefore, we have

$$\left(\frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y, \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y\right)$$

If we look at the vectors dx, dy under this transformation, we get that $(0,0) \mapsto (0,0)$, and then

$$\begin{aligned}(dx, 0) &\mapsto \left(\frac{\partial u}{\partial x}dx, \frac{\partial v}{\partial x}dx\right) \\ (0, dy) &\mapsto \left(\frac{\partial u}{\partial y}dy, \frac{\partial v}{\partial y}dy\right)\end{aligned}$$

and so the area of this parallelogram is given by the cross product.

We should also understand it in a different way, namely via incompressibility. Therefore, the motion in phase space can be represented by getting the velocity of phase space:

$$\mathbf{v}_p = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x}\right)$$
$$\nabla \cdot \mathbf{v}_p = 0$$

where this latter condition implies incompressibility.

In water, we cannot have that everywhere, our velocity is all pointed indwards. Thus, certain types of velocities should not occur, namely sinks or sources.

9.5 Poincaré Recurrence Theorem

The preservation of Phase space volume is the following result due to Poincaré. For an arbitrarily small neighborhood around in phase space, for almost every point in phase spcae, eventually, the trajectory will enter the neighborhood. The number of molecules in the room is possibly 10^{25}. The phase space is enormous. As the system is moving, it will return exactly to its original positin.

How is it possible to reconcile this with entropy growing? This led to Boltzmann's suicide. This would require extremely long timescales to occur.

Theorem 9.5.1.

Aside: importance of notation. Implicit in the statement

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial t}$$

The reason we have a partial derivative equal to a regular one is that we implicitly treat $H = H(x(t), p(t), t)$ on the left hand side. On the right, we treat $H = H(x, p, t)$ as multivariable. Thus, if our hamiltonian does not depend on time, then energy is conserved. Thus we really mean

$$\frac{d}{dt}H(x(t), p(t), t) = \frac{\partial H}{\partial t}\left(x(t), p(t), t\right).$$

On the right, H is a function of 3 variables, and its partial derivative is also a function of 3 variables. Then we plug in the values of x, p, t at a certain one time t. On the left, we take an oscillator moving in phase space. For this particular oscillator, and for this particular time, we have that a certain value of our Hamiltonian. Thus, out of a 3-variable function, we create a 1-variable function of time, and then differentiate that.

Consider an oscilaltor with time dependent spring constant:

$$H = \frac{p^2}{2m} + \frac{1}{2}\kappa(t)x^2$$

there will be some motion, but its H will change with time. Therefore, we can calculate dH/dt. Our right hand side is, given these variables, we take the partial derivative with respect to time:

$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{d\kappa}{dt}x^2.$$

Now we take this function of three variables (in this instance, p has been dropped, but this needn't be the case). Now we plug in $x(t), p(t), t$ to get

$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{d\kappa}{dt}|_t x(t)^2.$$

end aside.

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9.6 Adiabatic Invariance

These invariants turn out to be very importnat in physics.

Suppose we have a 1-dimensional Hamiltonian system:

$$H(x,p)$$

then our phase space looks like a closed curve which is a level set of the function H. if H depends on time, then there may be some projectoring phase space, and so our curve can intersect itself. If ∂_t is small, then the evolution of the phase diagram evolves. In one period, it is almost closed. Then we claim that

$$I = \frac{1}{2\pi}\int_{H(x,p,t)<E(t)} dp\ dx = \text{const.}$$

For a simple harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{\kappa}{2}x^2$$

If κ is constant, then we have an ellipse in our phase space. Moreover, this is an ellipse such that $H = E$. Now, let us make $\kappa = 1/m$ in appropriate units. Then the phase space portrait will be circles. [fill in]

Now comes the miracle; no matter what our system is, if we gradually change κ_1 to κ_2, then the lines will gradually move to isolines in order to keep the same area.

Now a case when adiabatic invariance is simpler.

Consider a ping-pong ball bouncing under a paddle, ignoring gravity. Here, the phase space portrait is closed, and so our portrait looks like a rectangle. The area enclosed by this trajectory, which is proportional to the adiabatic invariant, is $a \cdot mv$. In particular, we say that $av = \text{const.}$. If we slowly move the wall closer, in order to keep the area, we have to grow our velocity by a factor of two.

In this system, it is obvous that we require gradual motion: if the pingpong ball is mid flight, then we have done nothing to its velocity.

The amount of collisions is tv/a. Then u changes and $va = \text{const.}$ in the limit as $u \to 0$. To order of magnitude, $I \sim px$. However, characteristic momentum can be written as mx/t so our invariant is of order mx^2/t. This is of order

$$tm\frac{x^2}{t^2} \sim \frac{E}{\omega}$$

Now consider two mirrors which have metal plates, and an electromagnetic wave between the two. That is, consider the longest standing electromagnetic wave across two mirrors. It has a vertical electric field as a function of space x.

As we gradually move our mirror to the right, our frequency is $\omega = c \cdot \frac{2\pi}{\lambda} = \frac{c\pi}{a}$. We reduce the frequency when we gradually move the mirror. This setup is isomorphic to an oscillator, so our energy must change accordingly:

$$\frac{E}{\omega} = \text{const.}$$

However, since $E = n\hbar\omega$, where $n \in \mathbb{N}$, then

$$n = \text{const.}$$

So the number of photons is constant, but the photons are doing positive work on the

Solzhenyitsin

10 Chaos and Catastrophe

10.1 The Chaos Manifseto

The claim to fame of chaos theory was a repudiation of the simple solvable system were not representative of the true world.

By now, dust has settled. Devoting a small section to it is appropriate; Einstein was closer to the truth, when he said we can reach exquisite clarity at the expense of generality. Luckily, the world is so constructed that its simplicity is ubiquitous.

10.1.1 3-body problem

Historically, these problems were not solved by computers, but by mathematicians like Poincaré. Using our Hamiltonian approach, we can easily solve these problems numerically.

Randomly select positions and velocities, small, and then draw the x and y coordinates after each 1000 time-steps.

For a gravitational 2-body problem Then we see :

Where they orbit the common center of mass in elliptical trajectories. This is order, calculated mathematically by Newton.

Now, if we draw 3 bodies, we get a mess. Eventually, most of them fly off to infinity.

[KEEP EXPLAINING]

And the escape velocity is given, and almost surely, terminates by ejecting one star and the other two are binaries.

This is a well-known problem in astrophysics. If you take a large collection of stars, you might use thermodynamics, since they are like a collection of particles. However, at small distances, there is a huge negative gravitational energy; this must translate to positive kinetic energy.

Sure, there is chaos. But for us to understand how this chaos works, we have to do it in terms of the systems we do understand. For the three bodies, we can find a center of mass system, but this is a 12-dimensional phase space, barring some simplifications we can make. Given a large enough phase space, even simple setups can lead to difficult effects.

Normalize everything to 1. Then run it for time is 100. In the case of the two ellipses, as we decrease the accuracy, we ask what the x coordinate of velocity of planet 1. In the case of the two-body problem,

$$10^{-4} = -0.1078$$
$$5 \times 10^5$$

So the smaller we take our time step, the better it should approximate reality. Therefore we have predicted the final position after a time 100.

These numbers were always changing. These results don't look anything like determinism. They look like flipping a coin. So what's going on? This means that our system is extremely sensitive to initial conditions. This is so-called "deterministic chaos."

[A LOT MISSED]

10.2 Mathematical Billiards, Lyapunov Exponent

Feynman: You have not understood anything, until you can give a number.

This is exaggerated, but still has a lot of truth to it. We want to be able to characterize chaos by numbers. This is characterized by the Lyapunov exponent.

If we have a perfectly reflecting wall, adn perfectly reflectingcircle. A point particle can bounce off these walls. Is bounces are given by angle of incidence is equal to angle of reflection. Change

the initial y component by 10^{-8}. By the end of the simulation, the two paths are not remotely coincident. Thus, tiny deviations lead to huge consequences.

Can we characterize this process of deviations? Let $\delta(t)$ be the distance between the two trajectories. Thus, on average,

$$\delta(t) = \delta(0)e^{\lambda t}$$

We understand that the growth of this must be exponential, due to the enormous differences after certain periods of time. This parameter λ is known as the Lyapunov exponent. It characterizes the exponential divergence of these trajectories.

Let us calculate the Lyapunov exponent for a trajectory. Let us concentrate on one part of our setup. A possible trajectory is going back and forth aganist the wall. Suppose we have a tiny shift, without changing the velocity. Suddenly, it exscapes. IF we have distance δ_0 in x coordinates, then we have distance $\theta = 2\delta_0/R$. Then

$$\overline{\delta} = \delta + \theta R = 3\delta$$

Thus, in time $t = 2R/v$, then

$$\lambda = \frac{h3v}{2R}$$

.

10.3 Weather

Arnold says the following: He estimates the duration of a weather forecast:

Surely, we cannot say whether or not it will rain from a year today, but we can forecast tomorrow, so with what rate does the divergence occur?

There are winds going around the earth, so $t_L \sim R/V_t$ where v_t is the velocity of "trade winds," which is on the order of 100 km/h winds and R is the radius of the earth. Thus

$$t_L \sim \frac{6400\text{km}}{100\text{km/h}} \sim 3 \text{ days.}$$

Therefore, we multiply our error by 2 in 3 days. Therefore, we can predict stuff a week in advance if we are accurate enough.

Disturbing practical implications- may. Logistic model.

This logistic map demonstrates erratic behavior. Deterministic systems, if they are complicated enough, will naturally demonstrate chaotic behavior due to their enormous phase space. Thus the Lyapunov exponent is the only way to characterize this which tells us how fast trajectories diverge.

Real vogue of the day back in the 70s , all the seminars about chaos. Crazy claims, by studying chaotic behavior, heartbeat rate, predict heart diseases.

Dyson stating they never work on fashionable things.

Learn physics and mathematics, you will use programming in due time. He is using C, uses FORTRAN at work. Uses python to draw pictures.

Numerical Recipes book. Algorithms

FORTRAN still the fastest due to good compilers. his friend uses C