Faculty for Informatics

Technical University of Munich

Natural Language Processing IN2361

PD Dr. Georg Groh

Social Computing Research Group

Chapter A
Hidden Markov Models

- content is based on [1]
- certain elements (e.g. equations or tables) were taken over or taken over in a modified form from [1]
- citations of [1] or from [1] are omitted for legibility
- errors are fully in the responsibility of Georg Groh
- BIG thanks to Dan and James for a great book!

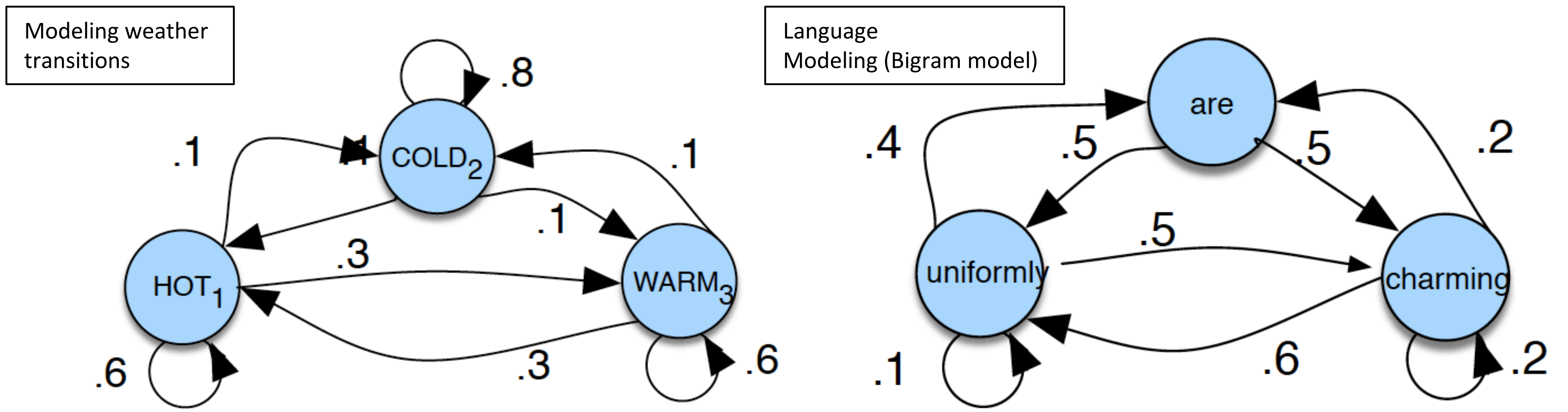
Hidden Markov Models

- HMM: probabilistic sequence classifier: input: sequence of observations; output: probability distribution over sequence of labels

- still used a lot in speech recognition and text-based NLP

- often “traditional” ML methods still state of the art, especially when data is scarce (if you do not have enough gun-powder, trying to shoot at stuff with a big cannon (complicated NNs) will just produce a mere “poop”

(First Order) Markov Chains



(a) (b)

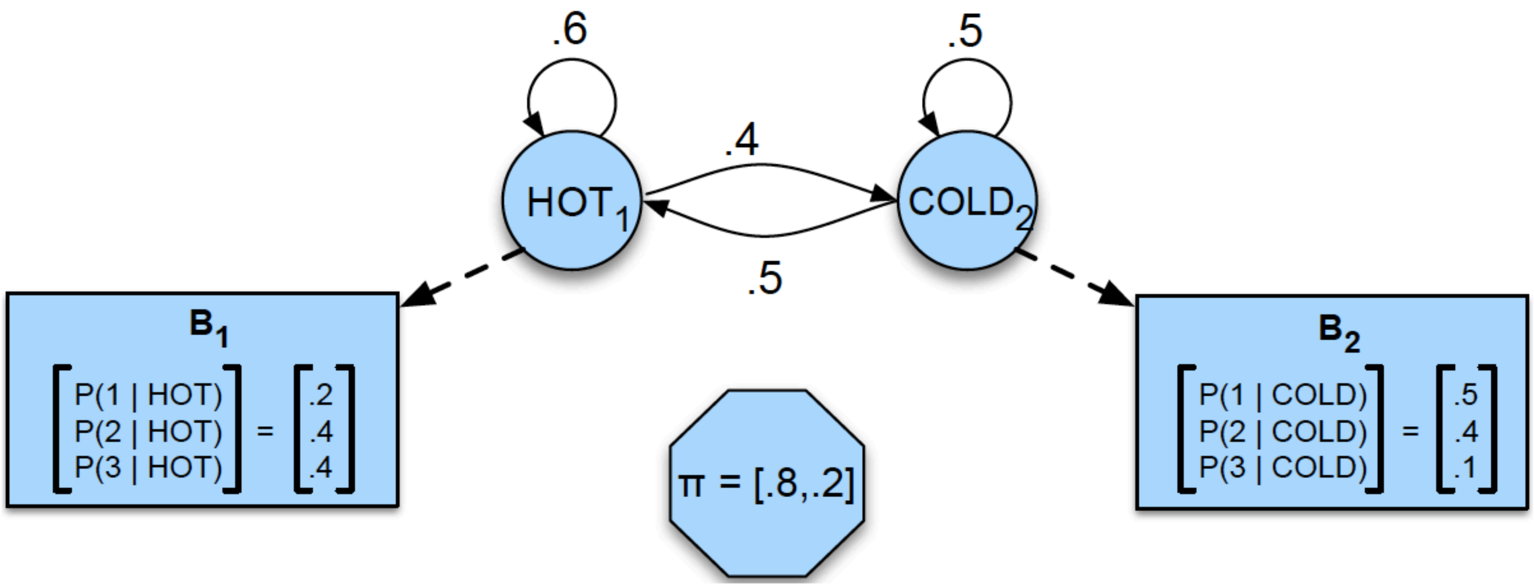
(first order)	**Markov Assumption:** $P(q_i = a \mid q_1 \ldots q_{i-1}) = P(q_i = a \mid q_{i-1})$
$Q = q_1 q_2 \ldots q_N$	a set of N **states**
$A = a_{11} a_{12} \ldots a_{n1} \ldots a_{nn}$	a **transition probability matrix** A, each a_{ij} representing the probability of moving from state i to state j, s.t. $\sum_{j=1}^{n} a_{ij} = 1 \quad \forall i$
$\pi = \pi_1, \pi_2, \ldots, \pi_N$	an **initial probability distribution** over states. π_i is the probability that the Markov chain will start in state i. Some states j may have $\pi_j = 0$, meaning that they cannot be initial states. Also, $\sum_{i=1}^{n} \pi_i = 1$

Hidden Markov Models

- If we know transition matrix: it's easy to compute probability of a given observed sequence: $P(q_t, q_{t+1}, \dots, q_{t+m}) = P(q_t)\, a_{q_t\, q_{t+1}} a_{q_{t+1}\, q_{t+2}}, \dots, a_{q_{t+m-1} q_{t+m}}$

- often: states are not observed ("hidden", "causal", latent variables). example: part of speech (POS) tags → HMM

- HMM example: observe: # of ice creams eaten; hidden states: cold weather, hot weather:



Hidden Markov Models

Markov Assumption: $P(q_i|q_1...q_{i-1}) = P(q_i|q_{i-1})$

Output Independence: $P(o_i|q_1 \ldots q_i, \ldots, q_T, o_1, \ldots, o_i, \ldots, o_T) = P(o_i|q_i)$

$Q = \{q_1 q_2 \ldots q_N\}$	a set of N **states**
$A = a_{11} \ldots a_{ij} \ldots a_{NN}$	a **transition probability matrix** A, each a_{ij} representing the probability of moving from state i to state j, s.t. $\sum_{j=1}^{N} a_{ij} = 1 \quad \forall i$

$a_{ij} = P(q_t = j | q_{t-1} = i)$

$O = (o_1 o_2 \ldots o_T) = o_{1:T}$	a sequence of T **observations**, each one drawn from a vocabulary $V = v_1, v_2, \ldots, v_V$
$B = b_i(o_t)$	a sequence of **observation likelihoods**, also called **emission probabilities**, each expressing the probability of an observation o_t being generated from a state i

$b_i(o_t) = P(o_t | q_t = i)$

Markov Assumption: $P(q_i|q_1 \ldots q_{i-1}) = P(q_i|q_{i-1})$

Output Independence: $P(o_i|q_1 \ldots q_i, \ldots, q_T, o_1, \ldots, o_i, \ldots$

$Q = \{$

$A = a$

$O = ($

remark:
the wording in Jurafsky and the literature on HMMs is sometimes a bit blurry in distinguishing whether something is a likelihood and whether something is a probability.
correct: $P(x|\theta)$ is a "probability of x given θ" and a "likelihood of θ" (given x)
so here $b_i(o_t) = P(o_t|q_t = i)$ should be called a "likelihood of the state" or a "probability of the observation" given the state.

$B = b_i(o_t)$	a sequence of **observation likelihoods**, also called **emission probabilities**, each expressing the probability of an observation o_t being generated from a state i

$b_i(o_t) = P(o_t|q_t = i)$

Fundamental HMM Problems

- **Problem 1 (Likelihood):** Given an HMM $\lambda = (A, B)$ and an observation sequence O, determine the likelihood $P(O|\lambda)$.
  - other formulation: compute $P(o_{1:T}|\lambda)$
  - solution: Forward algorithm

- **Problem 2 (Decoding):** Given an observation sequence O and an HMM $\lambda = (A, B)$, discover the best hidden state sequence Q.
  - other formulation: compute $argmax_{q_{1:T}} P(q_{1:T}|o_{1:T}, \lambda)$
  - solution: Viterbi algorithm

- **Problem 3 (Learning):** Given an observation sequence O and the set of states in the HMM, learn the HMM parameters A and B.
  - other formulation: compute $argmax_{A,B} P(o_{1:T}|\lambda(A, B, Q))$
  - solution: Forward-Backward (Baum Welch) algorithm (an instance of Expectation Maximization (EM))

- **Other problems**:
  - filtering / prediction: $P(q_{T+k}|o_{1:T}, \lambda)$ (solution: forward algorithm variant);
  - smoothing: $P(q_{T-k}|o_{1:T}, \lambda)$ (solution: backward algorithm)

Fundamental HMM Problems

- **Problem 1 (Likelihood):** Given an HMM $\lambda = (A, B)$ and an observation sequence O, determine the likelihood $P(O|\lambda)$.

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the wording in Jurafsky and the literature on HMMs is sometimes a bit blurry in distinguishing whether something is a likelihood and whether something is a probability.
correct: $P(x|\theta)$ is a "probability of x given θ" and a "likelihood of θ" (given x)
so here $b_i(o_t) = P(o_t|q_t = i)$ should be called a "likelihood of the state" or a "probability of the observation" given the state.

n: compute $P(o_{1:T}|\lambda)$

d algorithm

Give

$(A, B$

n: co

algor

remark:
do not mix up $Q = \{q_1, q_2, \dots, q_N\}$ (the set of states of/in the HMM, (which may better haven been denoted as $Q = \{q^{(1)}, q^{(2)}, \dots, q^{(N)}\}$)) which is meant below, and $Q = q_{1:t} = (q_1, q_2, \dots, q_t)$ (a sequence of states corresponding to a sequence of observations $O = o_{1:t} = (o_1, o_2, \dots, o_t)$ at times $t = 1, t = 2, \dots t = t$

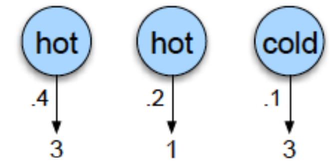
- **Problem 3 (Learning):** Given an observation sequence O and the set of states in the HMM, learn the HMM parameters A and B.
  - other formulation: compute $argmax_{A,B}\, P(o_{1:T}|\lambda(A, B, Q))$
  - solution: Forward-Backward / Baum Welch algorithm (an instance of Expectation Maximization (EM))

- **Other problems**:
  - filtering / prediction: $P(q_{T+k}|o_{1:T}, \lambda)$ (solution: forward algorithm variant);
  - smoothing: $P(q_{T-k}|o_{1:T}, \lambda)$ (solution: backward algorithm)

The Forward Algorithm

- if we knew $O = o_{1:T}$ AND $Q = q_{1:T}$, we could simply compute

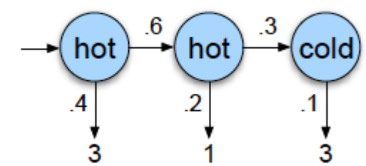
$$P(O|Q) = \prod_{i=1}^{T} P(o_i|q_i)$$



$$P(3\ 1\ 3|\text{hot hot cold}) = P(3|\text{hot}) \times P(1|\text{hot}) \times P(3|\text{cold})$$

- we do however not know $Q = q_{1:T}$ → use the joint

$$P(O,Q) = P(O|Q) \times P(Q) = \prod_{i=1}^{T} P(o_i|q_i) \times \prod_{i=1}^{T} P(q_i|q_{i-1})$$



$$\begin{aligned} P(3\ 1\ 3, \text{hot hot cold}) &= P(\text{hot}|\text{start}) \times P(\text{hot}|\text{hot}) \times P(\text{cold}|\text{hot}) \\ &\times P(3|\text{hot}) \times P(1|\text{hot}) \times P(3|\text{cold}) \end{aligned}$$

and sum out Q:

$$P(O) = \sum_{Q} P(O,Q) = \sum_{Q} P(O|Q)P(Q)$$

$$\begin{aligned} P(3\ 1\ 3) = {} & P(3\ 1\ 3, \text{cold cold cold}) + \\ & P(3\ 1\ 3, \text{cold cold hot}) + P(3\ 1\ 3, \text{hot hot cold}) + \ldots \end{aligned}$$

The Forward Algorithm

- Caveat: for a set of states with N states, there are N^T such combinations of state sequences in this example → previous suggestion is inefficient

- → use dynamic programming (store intermediate results) in recursively formulated Forward algorithm → takes only $O(N^2 T)$ time

$$\boxed{\alpha_t(j) = P(o_1, o_2 \dots o_t, q_t = j|\lambda)}$$

1. Initialization:

$$\alpha_1(j) = \pi_j b_j(o_1) \quad 1 \le j \le N$$

2. Recursion:

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

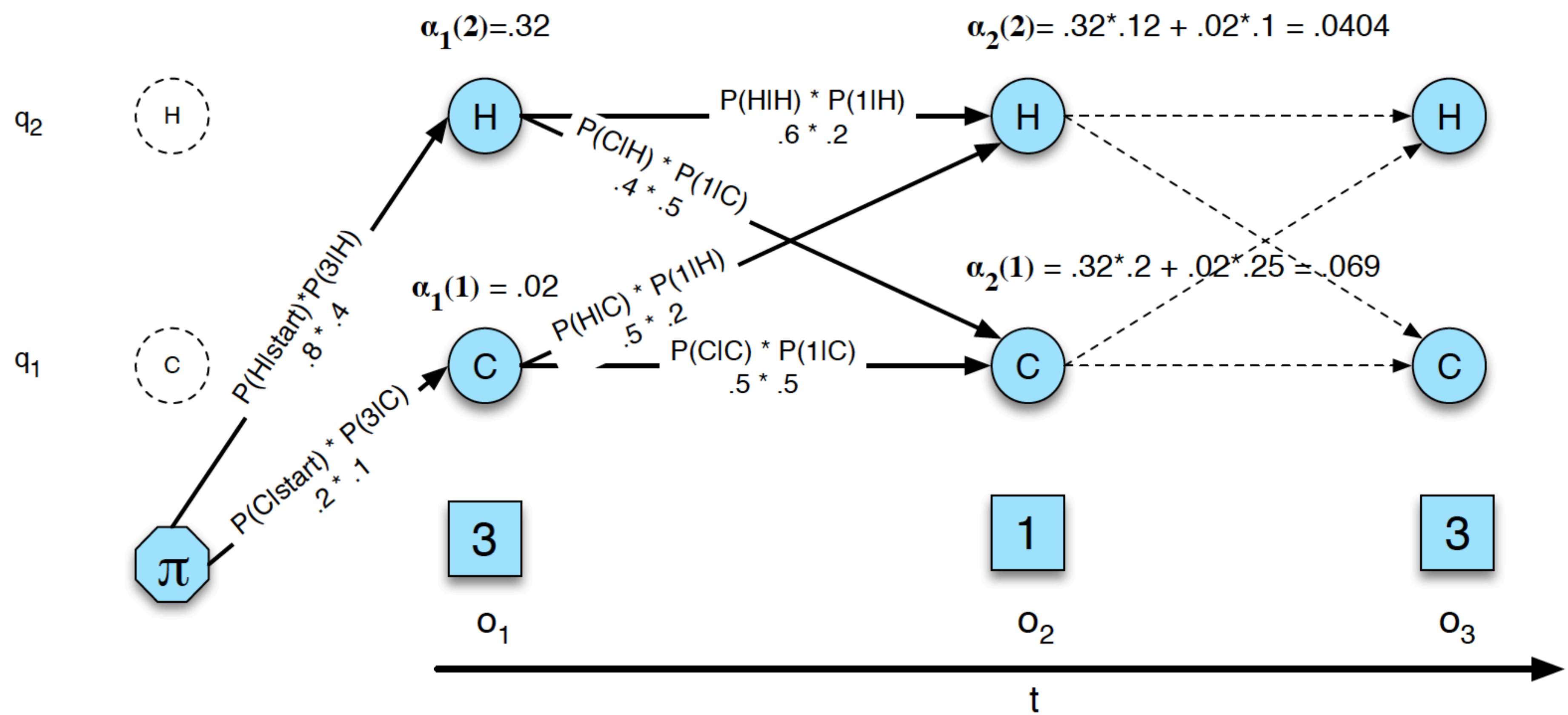
3. Termination:

$$P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

The Forward Trellis

$$\alpha_t(j) = P(o_1, o_2 \ldots o_t, q_t = j|\lambda)$$
$$= \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t)$$

$\alpha_{t-1}(i)$	the **previous forward path probability** from the previous time step
a_{ij}	the **transition probability** from previous state q_i to current state q_j
$b_j(o_t)$	the **state observation likelihood** of the observation symbol o_t given the current state j



1. Initialization:

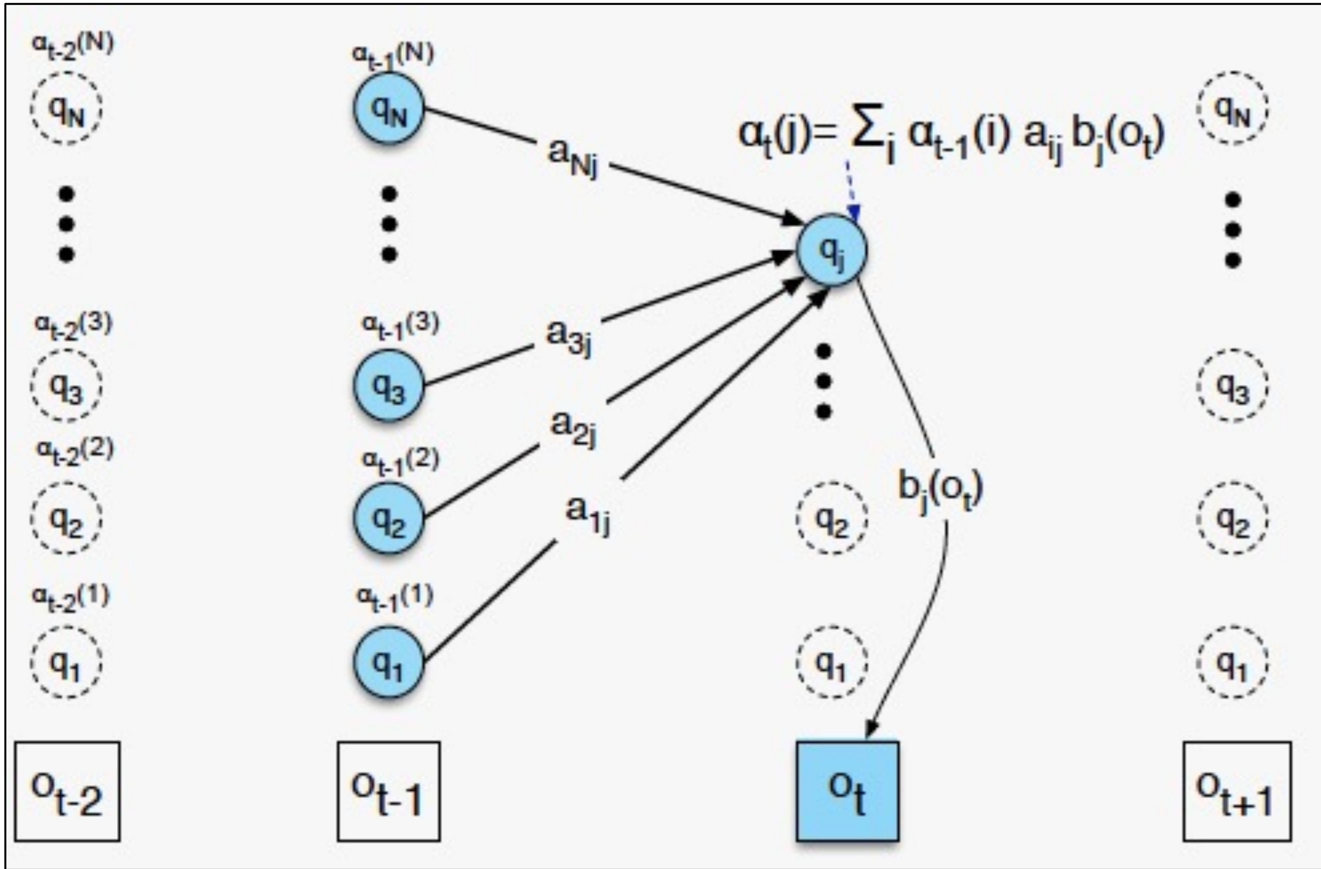
$$\alpha_1(j) \;=\; \pi_j b_j(o_1) \;\; 1 \le j \le N$$

2. Recursion:

$$\alpha_t(j) = \sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

3. Termination:

$$P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$$



The Forward Algorithm

proof of recursion:

$$\alpha_t(j) = P(q_t = j, o_{1:t}|\lambda)$$ (definition)

$$= P(q_t = j, o_t, o_{1:t-1}|\lambda)$$ (defintion of $o_{1:t}$ notation)

$$= P(o_t|q_t = j, o_{1:t-1}, \lambda)P(q_t = j, o_{1:t-1}|\lambda)$$ (simple decomposition of joint)

$$= P(o_t|q_t = j, \lambda)\sum_{i=1}^{N} P(q_t = j, q_{t-1} = i, o_{1:t-1}|\lambda)$$ (first term: definition of HMM: observations only depend on state; second term: summing out variables)

$$= P(o_t|q_t = j, |\lambda)\sum_{i=1}^{N} P(q_t = j|q_{t-1} = i, o_{1:t-1}, \lambda)P(q_{t-1} = i, o_{1:t-1}|\lambda)$$ (simple decomposition of joint)

$$= P(o_t|q_t = j, |\lambda)\sum_{i=1}^{N} P(q_t = j|q_{t-1} = i, \lambda)P(q_{t-1} = i, o_{1:t-1}|\lambda)$$ (definition of HMM: state only depends on prev. state)

$$= b_j(o_t)\sum_{i=1}^{N} a_{ij}\alpha_{t-1}(i)$$ (definition / notations)

Using Forward Algorithm For Filtering

i.e. compute

$$
\begin{aligned}
\tilde{\alpha}_t(j) &= P(q_t = j | o_{1:t} \lambda) \\
&= P(q_t = j | o_t, o_{1:t-1} \lambda) \\
&\propto P(o_t | q_t = j, o_{1:t-1}, \lambda) P(q_t = j | o_{1:t-1}, \lambda) \\
&= P(o_t | q_t = j, \lambda) \sum_{i=1}^{N} P(q_t = j, q_{t-1} = i | o_{1:t-1}, \lambda) \\
&= P(o_t | q_t = j, | \lambda) \sum_{i=1}^{N} P(q_t = j | q_{t-1} = i, o_{1:t-1}, \lambda) P(q_{t-1} = i | o_{1:t-1}, \lambda) \\
&= P(o_t | q_t = j, | \lambda) \sum_{i=1}^{N} P(q_t = j | q_{t-1} = i, | \lambda) P(q_{t-1} = i | o_{1:t-1}, \lambda) \\
&= b_j(o_t) \sum_{i=1}^{N} a_{ij} \tilde{\alpha}_{t-1}(i)
\end{aligned}
$$

(application of Bayes rule → we require normalization after each recursion)

Using Forward Algorithm For Prediction

i.e. compute

- first use k recursive steps of

$$\begin{aligned}
P(q_{t+k} = j|o_{1:t}\lambda) &= \sum_{m=1}^{N} P(q_{t+k} = j, q_{t+k-1} = m|o_{1:t}, \lambda) \\
&= \sum_{m=1}^{N} P(q_{t+k} = j|q_{t+k-1} = m, o_{1:t}, \lambda) P(q_{t+k-1} = m|o_{1:t}, \lambda) \\
&= \sum_{m=1}^{N} P(q_{t+k} = j|q_{t+k-1} = m, \lambda) P(q_{t+k-1} = m|o_{1:t}, \lambda) \\
&= \sum_{m=1}^{N} a_{mj} P(q_{t+k-1} = m|o_{1:t}, \lambda)
\end{aligned}$$

to reduce to filtering problem, then proceed with filtering.

Naturally these k steps depend only on Markov chain (the a_{mj}). If k is large: we will get the probability of state j in the stationary distribution of the Markov chain

Decoding: The Viterbi Algorithm

- Decoding: Given HMM $\lambda(A, B)$ and an observation sequence $O = o_{1:T}$, find the most probable state sequence $Q = q_{1:T}$

- Again: blindly „trying out“ all possible sequences

$$Q_{max} = argmax_Q P(Q|O) = argmax_Q P(O|Q)P(Q)$$

is prohibitively expensive, because an exponential number of possible state sequences exist → again: recursive, dynamic programming approach: Viterbi algorithm

Decoding: The Viterbi Algorithm

$$\boxed{v_t(j) = \max_{q_{1:t-1}} P(q_{1:t-1}, q_t = j, o_{1:t}|\lambda)}$$

1. **Initialization:**

$$\begin{aligned} v_1(j) &= \pi_j b_j(o_1) & 1 \le j \le N \\ bt_1(j) &= 0 & 1 \le j \le N \end{aligned}$$

2. **Recursion**

$$\begin{aligned} v_t(j) &= \max_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \le j \le N, 1 < t \le T \\ bt_t(j) &= \operatorname{argmax}_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \le j \le N, 1 < t \le T \end{aligned}$$

3. **Termination:**

$$\text{The best score:} \quad P* = \max_{i=1}^{N} v_T(i)$$

$$\text{The start of backtrace:} \quad q_T* = \operatorname{argmax}_{i=1}^{N} v_T(i)$$

simple step by step backtrace $bt_t(j)$ (not to be confused with the backward message for smoothing) determines the most probable state in time-step t if most probable state in time-step t+1 was $q_{t+1} = j$ →
desired result: $q_{1:T} = \mathrm{bt}_1(\mathrm{bt}_2(\ldots \mathrm{bt}_{T-1}(q_T *) \ldots)), \quad \mathrm{bt}_2(\mathrm{bt}_3(\ldots \mathrm{bt}_{T-1}(q_T *) \ldots)), \ldots, \mathrm{bt}_{T-1}(q_T *)$

Decoding: The Viterbi Algorithm

both symbols are the same (both are "nu") (just different fonts (Latex / Jurafksy mixed)

$$v_t(j) = \max_{q_{1:t-1}} P(q_{1:t-1}, q_t = j, o_{1:t} | \lambda)$$

$$\begin{aligned} v_1(j) &= \pi_j b_j(o_1) \qquad 1 \leq j \leq N \\ bt_1(j) &= 0 \qquad\qquad\quad 1 \leq j \leq N \end{aligned}$$

2. **Recursion**

$$\begin{aligned} v_t(j) &= \max_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \leq j \leq N, 1 < t \leq T \\ bt_t(j) &= \operatorname*{argmax}_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \leq j \leq N, 1 < t \leq T \end{aligned}$$

3. **Termination:**

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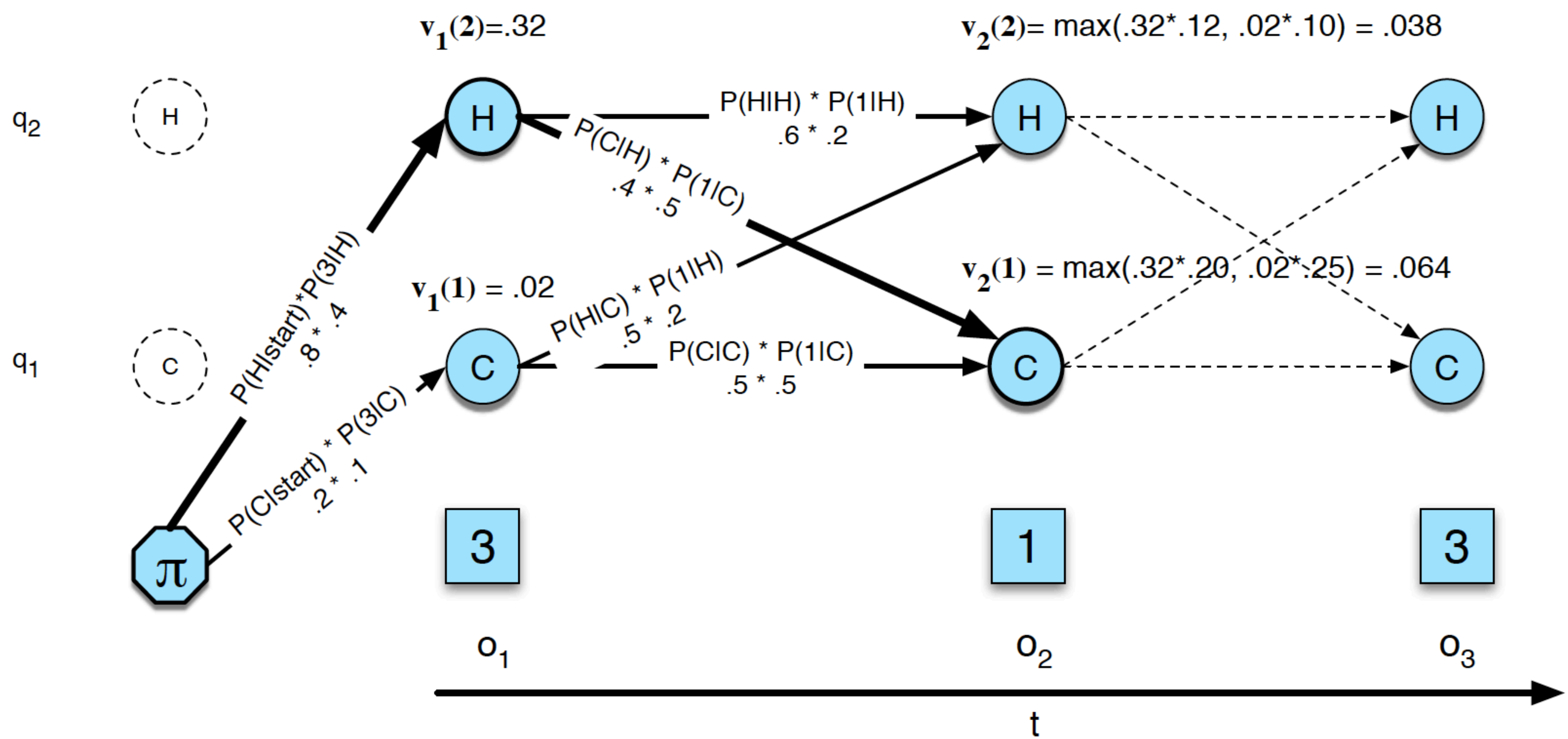
The Viterbi Trellis

$$v_t(j) = \max_{q_{1:t-1}} P(q_{1:t-1}, q_t = j, o_{1:t}|\lambda)$$

$$= \max_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t)$$

$v_{t-1}(i)$	the **previous Viterbi path probability** from the previous time step
a_{ij}	the **transition probability** from previous state q_i to current state q_j
$b_j(o_t)$	the **state observation likelihood** of the observation symbol o_t given the current state j



1. **Initialization:**

$$v_1(j) = \pi_j b_j(o_1) \qquad 1 \le j \le N$$
$$bt_1(j) = 0 \qquad 1 \le j \le N$$

2. **Recursion**

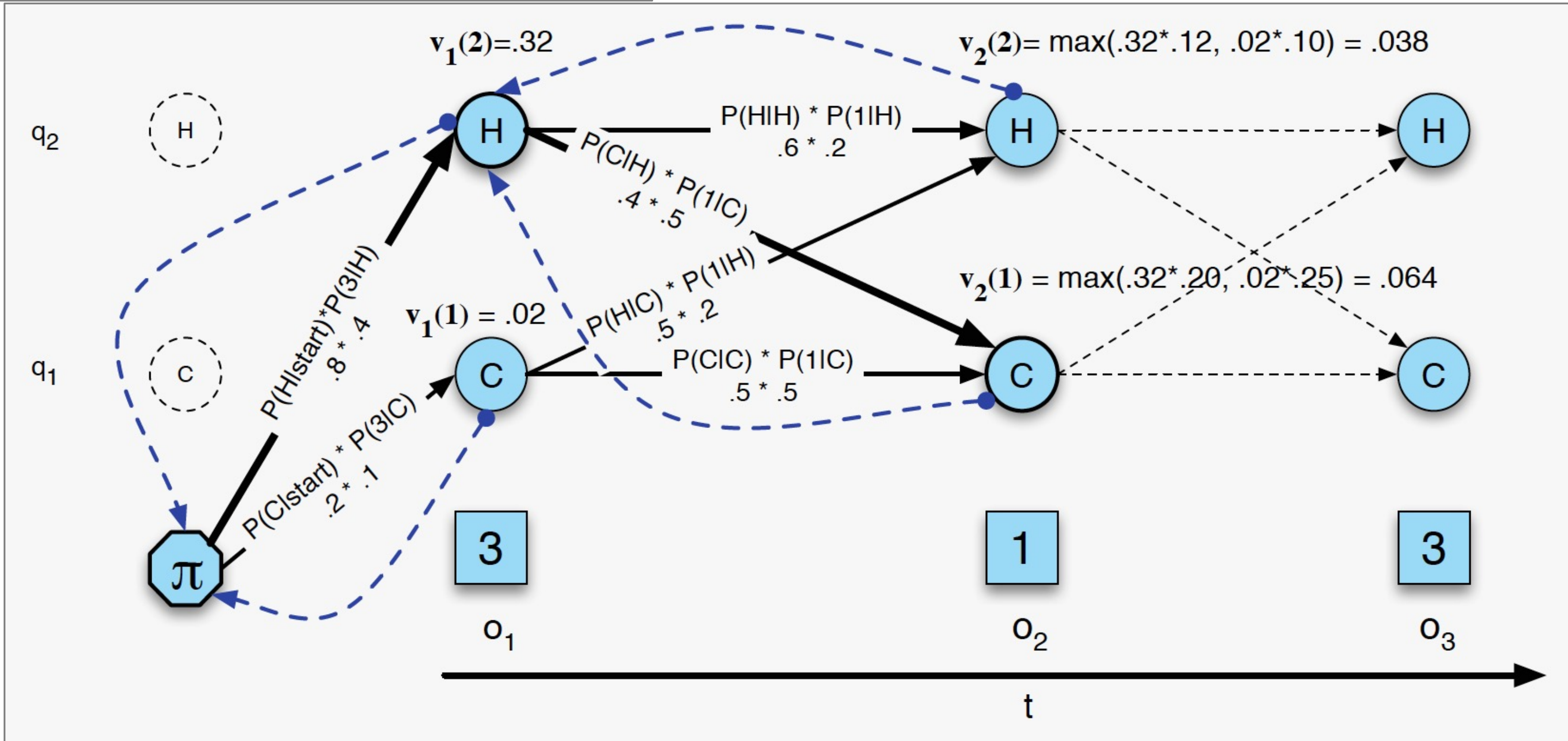
$$v_t(j) = \max_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$
$$bt_t(j) = \operatorname*{argmax}_{i=1}^{N} v_{t-1}(i)\, a_{ij}\, b_j(o_t); \quad 1 \le j \le N, 1 < t \le T$$

3. **Termination:**

The best score: $P* = \max_{i=1}^{N} v_T(i)$

The start of backtrace: $q_T* = \operatorname*{argmax}_{i=1}^{N} v_T(i)$

simple step by step backtrace $bt_t(j)$
determines the most probable state in time-step t if most probable state in time-step t+1 was $q_{t+1} = j$
→
desired result:
$q_{1:T} = \text{bt}_1(\text{bt}_2(\ldots \text{bt}_{T-1}(q_T *) \ldots))$,
$\text{bt}_2(\text{bt}_3(\ldots \text{bt}_{T-1}(q_T *) \ldots))$, …. , $\text{bt}_{T-1}(q_T *)$



Decoding: The Viterbi Algorithm

proof of recursion:

one could as well have started from $P(q_{1:t-1}, q_t = j | o_{1:t}\lambda)$ since $o_{1:T}$ is observed and thus $P(o_{1:t}|\lambda)$ is a constant.

$$v_t(j) = \max_{q_{1:t-1}} P(q_{1:t-1}, q_t = j, o_{1:t}|\lambda)$$

$$= \max_{q_{1:t-1}} P(q_{1:t-1}, q_t = j, o_t, o_{1:t-1}|\lambda)$$

$$= \max_{q_{1:t-1}} P(o_t|q_t = j, q_{1:t-1}, o_{1:t-1}, \lambda) P(q_t = j, q_{1:t-1}, o_{1:t-1}|\lambda)$$

$$= \max_{q_{1:t-1}} P(o_t|q_t = j, \lambda) P(q_t = j, q_{1:t-1}, o_{1:t-1}|\lambda)$$

(Definition of HMM (Markov Assumption of observation model): observations only depend on the state in the respective time step)

$$= \max_{q_{1:t-2}} \max_{i=1}^{N} P(o_t|q_t = j, \lambda) P(q_t = j, q_{t-1} = i, q_{1:t-2}, o_{1:t-1}|\lambda)$$

making max explicit for t-1 state

$$= P(o_t|q_t = j, |\lambda) \max_{i=1}^{N} P(q_t = j|q_{t-1} = i, q_{1:t-2}, o_{1:t-1}, \lambda) \max_{q_{1:t-2}} P(q_{1:t-2}, q_{t-1} = i, o_{1:t-1}|\lambda)$$

$$= P(o_t|q_t = j, |\lambda) \max_{i=1}^{N} P(q_t = j|q_{t-1} = i, \lambda) \max_{q_{1:t-2}} P(q_{1:t-2}, q_{t-1} = i, o_{1:t-1}|\lambda)$$

Definition of HMM (Markov Assumption for state transition model)

$$= b_j(o_t) \max_{i=1}^{N} a_{ij} v_{t-1}(i)$$

(definition / notations)

→ essentially the same as for Forward, just replace sum with max

Forward-Backward (Baum Welch) Algorithm

- Goal: given structure of HMM (set of states) and an observation sequence $O = o_{1:T}$, learn the state transition model A and the observation (emission) model B (essentially via MLE with EM-algorithm)

- for a simple Markov model we know the state sequence, so the MLE for A is given by counting:

$$a_{ij} = \frac{C(i \to j)}{\sum_{q \in Q} C(i \to q)}$$

but for an HMM, the states are hidden variables.

- Idea: combine Forward algorithm (recursively passing a forward message $\alpha_t(j)$ "→") with Backward algorithm (recursively passing a backward message $\beta_t(i)$ "←") and iterate (EM style)

Backward Part

$\beta_t(i) = P(o_{t+1}, o_{t+2} \ldots o_T | q_t = i, \lambda)$ is the probability for future observations, assuming to be now in state i

(compare forward message $\alpha_t(j) = P(o_1, o_2 \ldots o_t, q_t = j | \lambda)$)

1. **Initialization:**

$$\beta_T(i) \; = \; 1, \quad 1 \leq i \leq N$$

2. **Recursion**

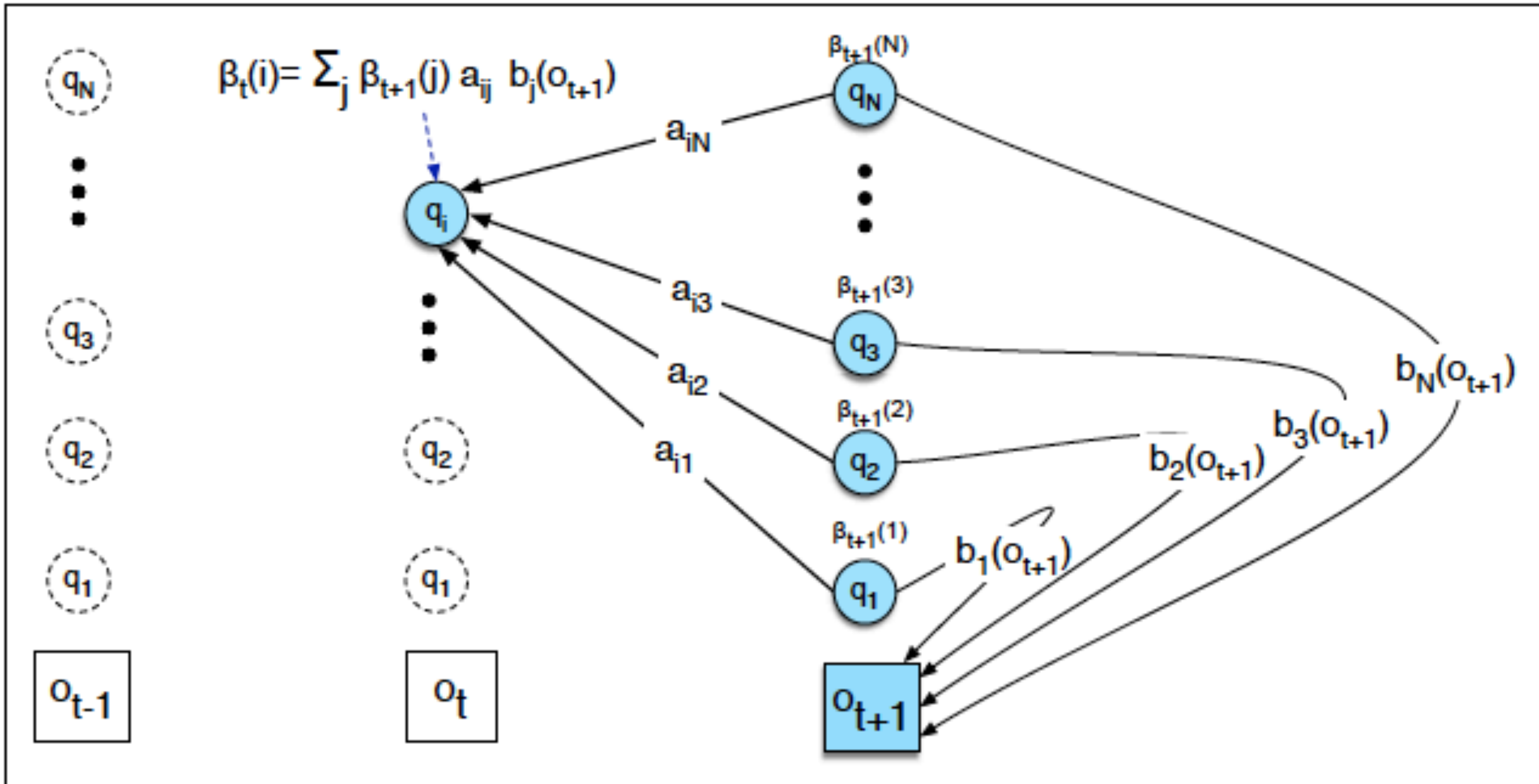
$$\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j), \quad 1 \leq i \leq N, 1 \leq t < T$$

3. **Termination:**

$$P(O|\lambda) = \sum_{j=1}^{N} \pi_j\, b_j(o_1)\, \beta_1(j)$$

$$\beta_t(i) = P(o_{t+1}, o_{t+2} \ldots o_T | q_t = i, \lambda)$$

$$\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j)$$



Backward Part

proof of recursion

$$\begin{aligned}
\beta_t(i) &= P(o_{t+1:T}|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1:T}, q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1:T}|q_{t+1} = j, q_t = i, \lambda)P(q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1:T}|q_{t+1} = j, \lambda)P(q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1}, o_{t+2:T}|q_{t+1} = j, \lambda)P(q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1}|o_{t+2:T}, q_{t+1} = j, \lambda)P(o_{t+2:T}|q_{t+1} = j, \lambda)P(q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} P(o_{t+1}|q_{t+1} = j, \lambda)P(o_{t+2:T}|q_{t+1} = j, \lambda)P(q_{t+1} = j|q_t = i, \lambda) \\
&= \sum_{j=1}^{N} b_j(o_{t+1})\beta_{t+1}(j)a_{ij}
\end{aligned}$$

Forward – Backward Algorithm

goal: iteratively improve estimates for a_{ij} and $b_i(o_t)$

1. estimate for a_{ij} :

$$\hat{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

Define:

$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j | O, \lambda)$$

$$= \frac{P(q_t = i, q_{t+1} = j, O|\lambda)}{P(O|\lambda)}$$

$$= \frac{\alpha_t(i)\, a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\sum_{j=1}^{N} \alpha_t(j) \beta_t(j)}$$

since

$$P(O|\lambda) = \sum_{j=1}^{N} \alpha_t(j) \beta_t(j)$$

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \sum_{k=1}^{N} \xi_t(i,k)}$$

Forward – Backward Algorithm

goal: iteratively improve estimates for a_{ij} and $b_i(o_t)$

2. estimate for $b_i(o_t)$

$$\hat{b}_j(v_k) = \frac{\text{expected number of times in state } j \text{ and observing symbol } v_k}{\text{expected number of times in state } j}$$

Define:

$$\gamma_t(j) = P(q_t = j|O,\lambda)$$

$$= \frac{P(q_t = j, O|\lambda)}{P(O|\lambda)}$$

$$= \frac{\alpha_t(j)\beta_t(j)}{P(O|\lambda)}$$

$$= \frac{\alpha_t(j)\beta_t(j)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)}$$

$$\hat{b}_j(v_k) = \frac{\sum_{t=1 \, s.t. O_t = v_k}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)}$$

Forward – Backward Algorithm

function FORWARD-BACKWARD(*observations* of len T, *output vocabulary* V, *hidden state set* Q) **returns** $HMM=(A,B)$

initialize A and B
iterate until convergence
E-step

$$\gamma_t(j) = \frac{\alpha_t(j)\beta_t(j)}{\sum_{j=1}^{N}\alpha_t(j)\beta_t(j)} \quad \forall\, t \text{ and } j$$

$$\xi_t(i,j) = \frac{\alpha_t(i)\, a_{ij} b_j(o_{t+1})\beta_{t+1}(j)}{\sum_{j=1}^{N}\alpha_t(j)\beta_t(j)} \quad \forall\, t,\ i, \text{ and } j$$

M-step

$$\hat{a}_{ij} = \frac{\sum_{t=1}^{T-1}\xi_t(i,j)}{\sum_{t=1}^{T-1}\sum_{k=1}^{N}\xi_t(i,k)}$$

$$\hat{b}_j(v_k) = \frac{\sum_{t=1 s.t.\ O_t=v_k}^{T}\gamma_t(j)}{\sum_{t=1}^{T}\gamma_t(j)}$$

return A, B

Forward – Backward Algorithm

proof of recursion for ξ

$$\widehat{\xi}_t(i,j) = P(q_t = i, q_{t+1} = j, o_{1:T}|\lambda) \qquad \text{definition}$$

straight-forward chain decom-position of joint

$$
\begin{aligned}
&= P(q_t = i, q_{t+1} = j, o_{1:t}, o_{t+1}, o_{t+2:T}|\lambda) \\
&= P(o_{t+1}|q_t = i, q_{t+1} = j, o_{1:t}, o_{t+2:T}, \lambda) P(q_t = i, q_{t+1} = j, o_{1:t}, o_{t+2:T}|\lambda) \\
&= P(o_{t+1}|q_t = i, q_{t+1} = j, o_{1:t}, o_{t+2:T}, \lambda) P(o_{t+2:T}|q_t = i, q_{t+1} = j, o_{1:t}, \lambda) P(q_t = i, q_{t+1} = j, o_{1:t}|\lambda) \\
&= P(o_{t+1}|q_t = i, q_{t+1} = j, o_{1:t}, o_{t+2:T}, \lambda) P(o_{t+2:T}|q_t = i, q_{t+1} = j, o_{1:t}, \lambda) P(q_{t+1} = j|q_t = i, o_{1:t}, \lambda) \\
&\qquad P(q_t = i, o_{1:t}|\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&= P(o_{t+1}|q_{t+1} = j, \lambda) P(o_{t+2:T}|q_{t+1} = j, \lambda) P(q_{t+1} = j|q_t = i, \lambda) P(q_t = i, o_{1:t}|\lambda) \\
&= b_j(o_{t+1}) \beta_{t+1}(j) a_{ij} \alpha_t(i) \qquad \text{definitions}
\end{aligned}
$$

defintion of HMM:

(1) Markov assumption for observation model

(2) → observations are conditionally independent of states further in the past, given states in the past temporally closer to the observation time AND conditionally independent of observations of the past given states in the past temporally closer to the observation time

(3) Markov assumption for transition model

Forward – Backward Algorithm

proof of recursion for γ

$$
\begin{aligned}
\gamma_t(j) &= P(q_t = j, o_{1:T}|\lambda) && \text{definition} \\
&= P(q_t = j, o_{1:t}, o_{t+1:T}|\lambda) \\
&= P(o_{t+1:T}|q_t = j, o_{1:t}, \lambda)P(q_t = j, o_{1:t}|\lambda) \\
&= P(o_{t+1:T}|q_t = j, \lambda)P(q_t = j, o_{1:t}|\lambda) \\
&= \beta_t(j)\alpha_t(j) && \text{definitions}
\end{aligned}
$$

defintion of HMM: conditionally independent of observations of the past
given states in the past temporally closer or as close to the observation time

Forward – Backward Algorithm

proof of $P(O|\lambda) = \sum_{j=1}^{N} \alpha_t(j)\beta_t(j)$

$P(O|\lambda) =$

$$= \sum_{j=1}^{N} P(o_{1:T}, q_T = j|\lambda) = \sum_{j=1}^{N} \alpha_T(j)$$

via T forward-messages

$P(O|\lambda) =$

$$= \sum_{j=1}^{N} P(o_{1:T}, q_1 = j|\lambda)$$

$$= \sum_{j=1}^{N} P(q_1 = j|\lambda)P(o_1, o_{2:T}|q_1 = j, \lambda)$$

$$= \sum_{j=1}^{N} P(q_1 = j|\lambda)P(o_1|o_{2:T}, q_1 = j, \lambda)P(o_{2:T}|q_1 = j, \lambda)$$

$$= \sum_{j=1}^{N} P(q_1 = j|\lambda)P(o_1|q_1 = j, \lambda)P(o_{2:T}|q_1 = j, \lambda)$$

$$= \sum_{j=1}^{N} \pi_j b_j(o_1)\beta_1(j)$$

via T-1 backward-messages

Forward – Backward Algorithm

proof of $P(O|\lambda) = \sum_{j=1}^{N} \alpha_t(j)\beta_t(j)$

$P(O|\lambda)$ =

$$= \sum_{j=1}^{N} P(o_{1:T}, q_t = j|\lambda)$$

$$= \sum_{j=1}^{N} P(o_{1:t}, o_{t+1:T}, q_t = j|\lambda)$$

$$= \sum_{j=1}^{N} P(o_{1:t}, o_{t+1:T}|q_t = j, \lambda)P(q_t = j|\lambda)$$

$$= \sum_{j=1}^{N} P(o_{1:t}|q_t = j, \lambda)P(o_{t+1:T}|q_t = j, \lambda)P(q_t = j|\lambda)$$

$$= \sum_{j=1}^{N} P(o_{1:t}, q_t = j|\lambda)P(o_{t+1:T}|q_t = j, \lambda)$$

$$= \sum_{j=1}^{N} \alpha_t(j)\beta_t(j)$$

via t forward-messages
and T-(t-1) backward messages

Bibliography

(1) Dan Jurafsky and James Martin: Speech and Language Processing (3rd ed. draft, version Oct 2019); Online: https://web.stanford.edu/~jurafsky/slp3/ (URL, Oct 2019) (this slide-set is especially related to chapter Appendix A)

(2) S. Russell, P. Norvig: Artificial Intelligence – A Modern Approach, 3rd edition, Pearson 2010, section 15.2

Recommendations for Studying

- minimal approach:

  work with the slides and understand their contents! Think beyond instead of merely memorizing the contents

- standard approach:

  minimal approach + read the corresponding pages in Jurafsky [1]

- interested students

  == standard approach