Machine Learning for Graphs and Sequential Data

Sequential Data – Autoregressive Models

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Data Analytics and
Machine Learning

Roadmap

- Chapter: Temporal Data / Sequential Data
  1. **Autoregressive Models**
     - **Motivation & Definitions**
     - Parameter Learning
  2. Markov Chains
  3. Hidden Markov Models
  4. Neural Network Approaches
  5. Temporal Point Processes

Motivation

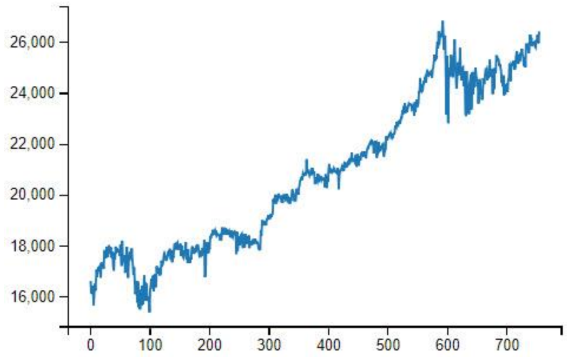
- Autoregressive (AR) models for sequences of observations $X_1, X_2, \dots, X_T$.
  - The index t can correspond to time, location, etc.
  - For now, we focus on the case of **continuous observations** occurring at **discrete time-steps**

Observations are not independent → non-i.i.d. data

- Example: Time-series forecasting
  - X_t = measurement of a sensor at time-step t
  - Applications in weather forecasting,

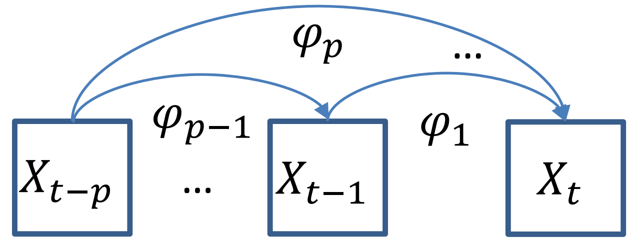
    e.g., X_t= temperature on t-th day
  - Applications in the field of economics,

    e.g., X_t= stock market quotations on t-th day



AR model - Definition

10 11 10.5 10.9



- Definition: An **autoregressive model** AR(p) of order p is defined as:

$$X_t = c + \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t$$

where $\varphi_1, \dots, \varphi_p$ are the parameters, c is a constant and $\varepsilon_t \sim N(0, \sigma)$ is a **white noise**. The variable X_{t-i} is the **lagged value** at time i.

- Intuitively, we perform a regression where the lags $([X_{t-1}, \dots, X_{t-p}])_t$ are the inputs and are $(X_t)_t$ the outputs.

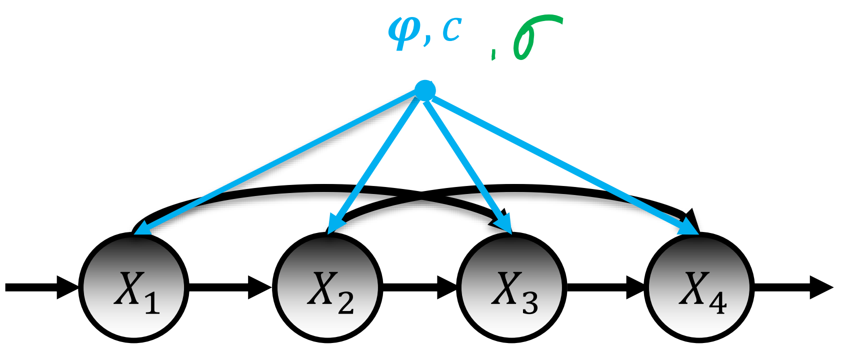
- Remark that a modification (or shock) on X_t will have a repercussion far into the future. The variables $(X_t)_t$ are not independent.

AR model – Graphical Model

- We can rewrite the AR model:

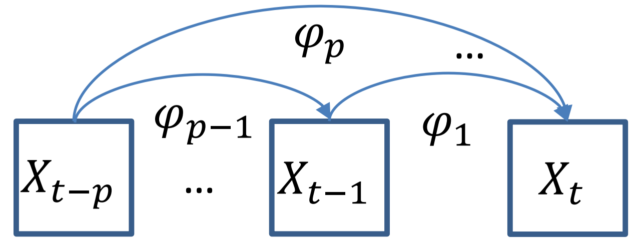
$$P(X_t | X_{t-1}, \dots X_{t-p}) \sim N(c + \sum_{i=1}^{p} \varphi_i X_{t-i}, \sigma)$$

- The graphical model representation of the AR model is:
  - The parameters $\boldsymbol{\varphi}, c$ are shared through time



- AR model can be viewed as a probabilistic model for continuous observations

AR model - Definition



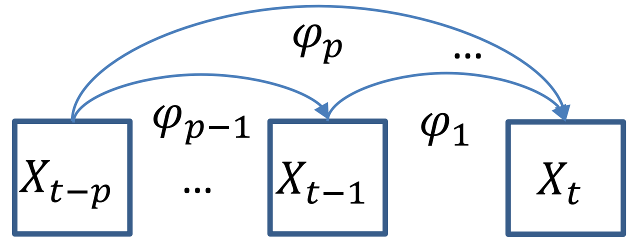
- The **mean function** of an AR model is $\mu(t) = E[X_t]$.
  By default, it depends on t.

- The **autocovariance** $\gamma(t,i) = Cov(X_t, X_{t-i})$.
  By default, it depends on t and i.

- The autocovariance function can be normalized to give the **Pearson autocorrelation function** $\rho(t,i) = \frac{Cov(X_t, X_{t-i})}{\sqrt{Var(X_t)}\sqrt{Var(X_{t-i})}}$. It lies in $[-1, 1]$.

- The autocorrelation and autocovariance are indicators of the dependence of the variable X_t with respect to the past variables X_{t-i}

AR model - Stationarity



- Definition: A process is said stationary if
  1. $E[X_t] = E[X_{t-i}] = \mu, \forall t, \forall i$
  2. $Cov(X_t, X_{t-i}) = \gamma_i, \forall t, \forall i$
  3. $E[|X_t|^2] < \infty, \forall t$

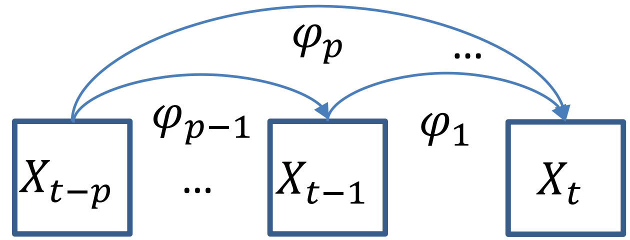
1 3 2 3.1 1.5

$i=1$

  – The mean function $E[X_t]$ is constant.
  – The autocovariance $Cov(X_t, X_{t-i})$ only depends on the lagged value at time i. It does not depend on t.
    Remark we have $\gamma_i = Cov(X_t, X_{t-i}) = Cov(X_{t-i}, X_t) = \gamma_{-i}$
  – For stationary processes, it is possible to estimate mean and autocovariance by averaging measures over time.

1 | 3
3 | 2
2 | 3.1
3.1 | 1.5

AR model - Stationarity



- Moments of a stationary AR(p):
  - $E[X_t] = \mu = \frac{c}{1-\sum_{i=1}^{p}\varphi_i}, \forall\, t$
  - $Var(X_t) = \gamma_0 = \sum_{j=1}^{p}\varphi_j\gamma_{-j} + \sigma^2, \forall\, t$
  - $Cov(X_t, X_{t-i}) = \gamma_i = \sum_{j=1}^{p}\varphi_j\gamma_{i-j}, \forall\, t, \forall\, i \in [1, p]$
  - $\rho_i = \frac{\gamma_i}{\gamma_0}$

$X_t = c + \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t$

$E[X_t] = E[\ \text{—}\ " \ \text{—}\ ]$

$\mu = c + \sum_{i=1}^{n} \varphi_i \mu + 0$

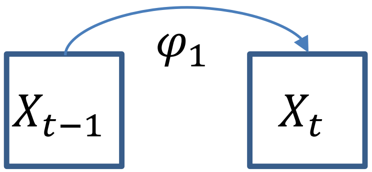
$\Leftrightarrow \mu = \dots$

AR model - Stationarity

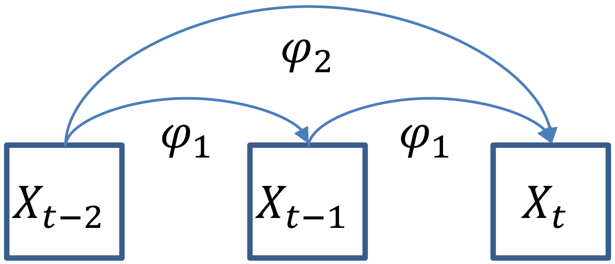
- An AR(p) process is stationary iff the roots of the characteristic polynomial $\Phi(L) = 1 - \sum_{i=1}^{p} \varphi_i L^i$ lie outside the unit circle.

WITHOUT PROOF

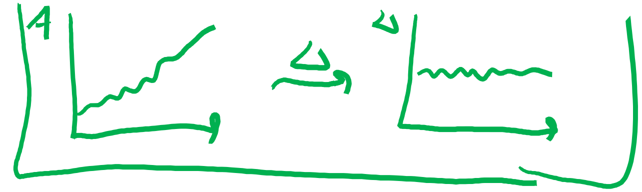
- Examples:
  - AR(1): $X_t = c + \varphi_1 X_{t-1} + \varepsilon_t$ is stationary if
    - $|\varphi_1| < 1$



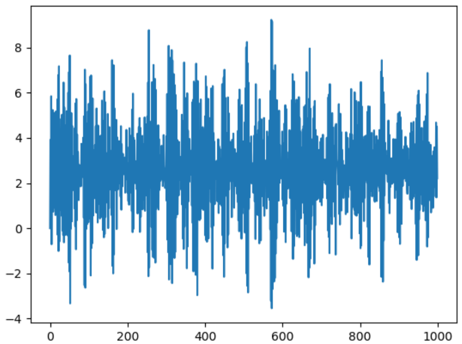
  - AR(2): $X_t = c + \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + \varepsilon_t$ is stationary if
    - $\varphi_1 + \varphi_2 < 1$
    - $\varphi_1 - \varphi_1 < 1$
    - $|\varphi_2| < 1$

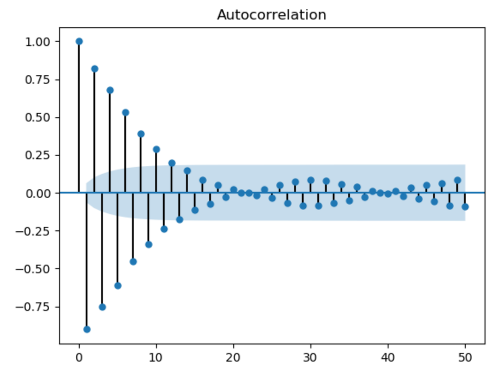


AR model - Stationarity

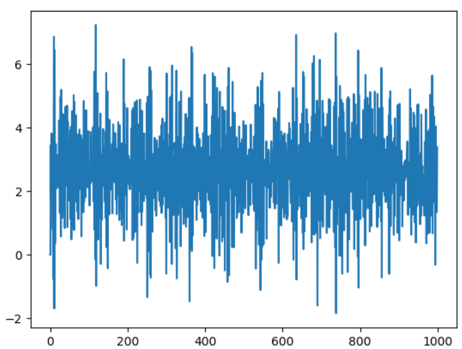


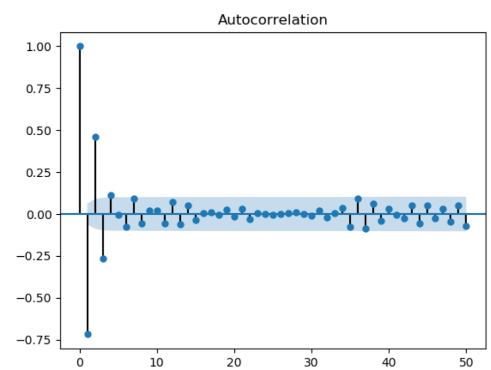
- Examples of stationary time series:
  - AR(1): $X_t = 4 - 0.9 * X_{t-1} + \varepsilon_t$





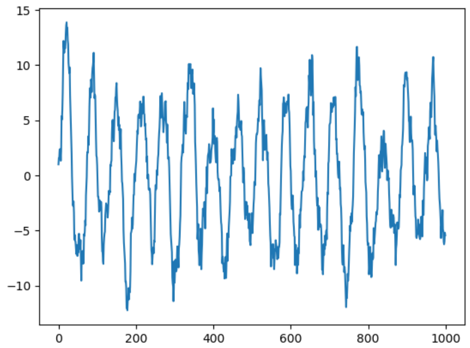
  - AR(2): $X_t = 4 - 0.8 * X_{t-1} - 0.1 * X_{t-2} + \varepsilon_t$

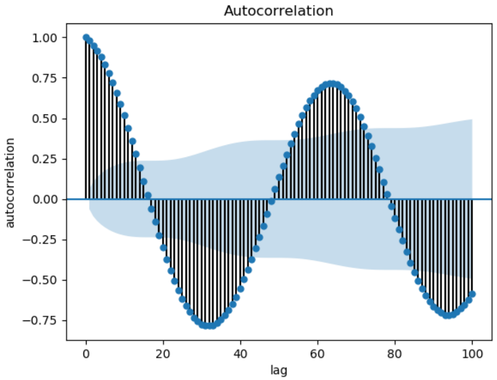




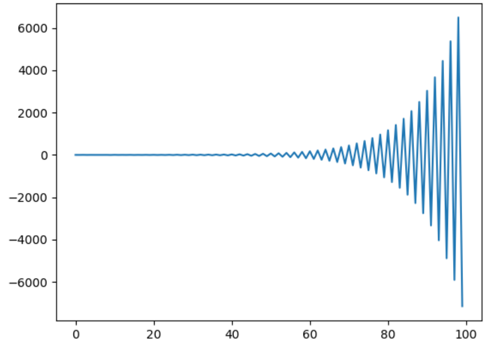
AR model - Stationarity

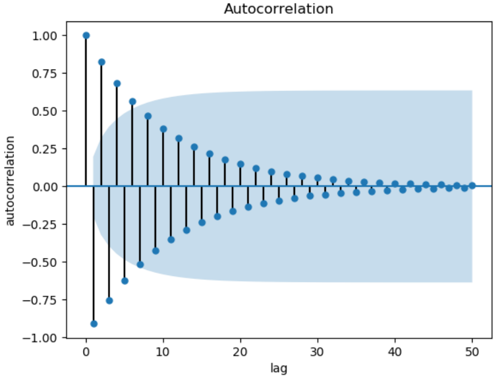
- Examples of non-stationary time series:
  - $X_t = \sin(t/10) - 0.8 \; * X_{t-1} + \varepsilon_t$





  - $X_t = -1.1 * \; X_{t-1} + \varepsilon_t$





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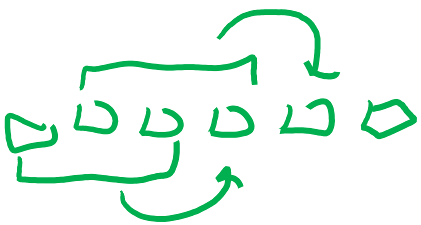
AR model – Parameter Learning (1)

- The parameters can be learned with classic least squares regression:

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{p-1} \\ \varphi_p \end{bmatrix} = (X^T X)^{-1} X^T y$$

where $X = \begin{bmatrix} X_{p-1} & \cdots & X_0 \\ X_p & \cdots & X_1 \\ \vdots & \cdots & \vdots \end{bmatrix}$

and $y = \begin{bmatrix} X_p \\ X_{p+1} \\ \vdots \end{bmatrix}$



AR model – Parameter Learning (2)

$X_t := c + \sum_{i=1}^{n} \varphi_i X_{t-i} + \varepsilon$

- The parameters can be learned by using the Yule Walker equations:

$$\gamma_0 = \sum_{j=1}^{p} \varphi_j \gamma_{-j} + \sigma^2$$
$$\gamma_1 = \sum_{j=1}^{p} \varphi_j \gamma_{1-j}$$
$$\gamma_2 = \sum_{j=1}^{p} \varphi_j \gamma_{2-j}$$
$$\dots$$
$$\gamma_p = \sum_{j=1}^{p} \varphi_j \gamma_{p-j}$$

1. Estimate the moments $\gamma_0, \gamma_1, \dots, \gamma_p$
2. Inverse Yule-Walker matrix to estimate $\varphi_1, \dots, \varphi_p$
3. Use γ_0 equation to estimate σ

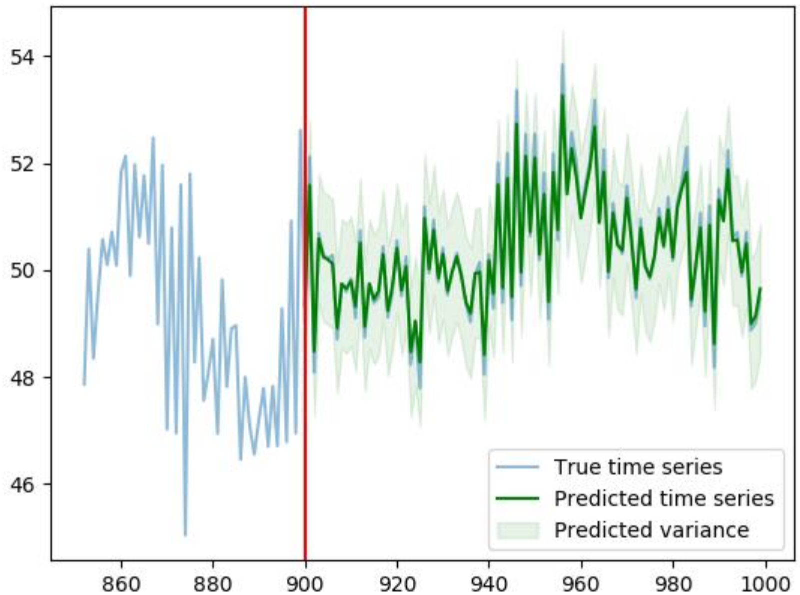
Yule-Walker matrix

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{p-1} \\ \gamma_p \end{bmatrix} = \boxed{\begin{bmatrix} \gamma_0 & \gamma_{-1} & \cdots & \gamma_{2-p} & \gamma_{1-p} \\ \gamma_1 & & & & \gamma_{2-p} \\ \vdots & & \ddots & & \vdots \\ \gamma_{p-2} & & & & \gamma_{-1} \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{p-1} \\ \varphi_p \end{bmatrix}$$

AR model – Parameter Learning

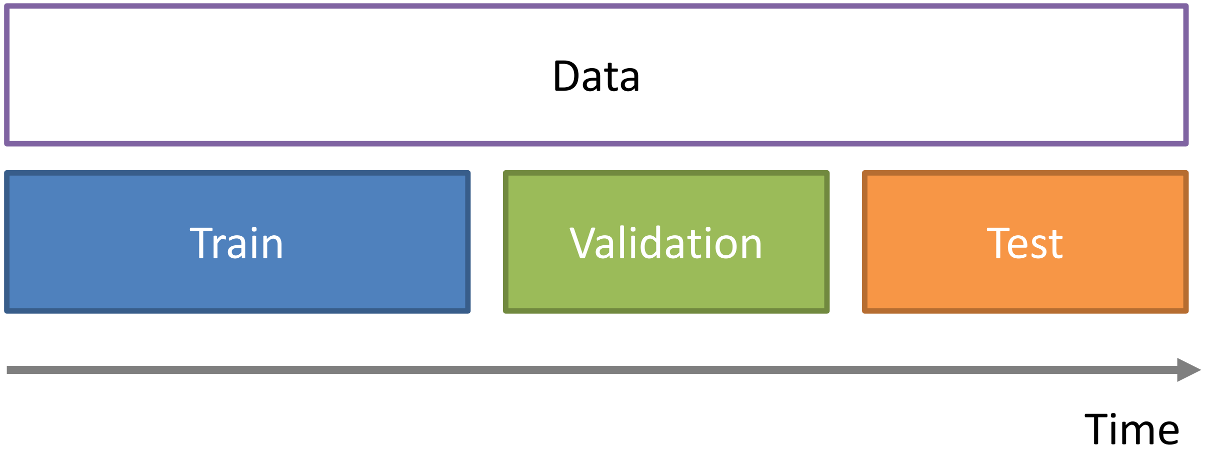
- Example: $X_t = 5 + 0.8 * X_{t-1} + 0.1 * X_{t-2} + N(0,1)$
  1. We learn parameters and the variance on the 900 first samples
     - Estimated parameters: $\varphi_1 = 0.82, \varphi_2 = 0.07, \sigma = 1.21$
  2. We predict on the last 100 samples

$X_{901} = 4.9 + 0.82 \cdot X_{900} + 0.07 X_{899} + N(0, 1.21)$



General Remark: Data split

- An important part of training is model selection
  - Usually we split data into train, validation and test set
- With time series and sequential data these sets should be split in such a way to keep the temporal ordering
- A model should be tested only on the data from the future



Questions – AR

1. What is the mean $E[X_t]$ of the following processes:
   a) $X_t = \sin(t/10) + \varepsilon_t$ where $\varepsilon_t \sim N(0, \sigma)$
   b) $X_t = 4 - 0.8 * X_{t-1} - 0.1 * X_{t-2} + \varepsilon_t$ where $\varepsilon_t \sim N(0, \sigma)$
   c) $X_t = 4 + X_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim N(0, \sigma)$ and $X_0 \sim N(0, \sigma)$

2. Does Yule Walker parameter learning assume a stationary process? Why ?

Reading Material

- [1] Stationary Models lecture, Matthieu Stigler: http://matthieustigler.github.io/Lectures/Lect2ARMA.pdf
- [2] Time Series lecture, Rauli Susmel: https://www.bauer.uh.edu/rsusmel/phd/ec2-3.pdf
- [3] Introduction on AR Model lecture, Rob Reider: https://s3.amazonaws.com/assets.datacamp.com/production/course_4267/slides/chapter3.pdf