Machine Learning for Graphs and Sequential Data Exercise Sheet 01

Normalizing Flows

Problem 1:

(a) We consider the following transformations:

- $f(\boldsymbol{z}) = \begin{bmatrix} 10z_1 + 1 \\ \cos(z_1)z_2 \\ \sin(z_1 z_2) \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R} \times \mathbb{R} \times [-1, 1]$.
- $f(\boldsymbol{z}) = \begin{bmatrix} z_1^3 \\ e^{z_1} z_2^5 \\ e^{-z_1 - z_2} z_3^7 \end{bmatrix}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Are these transformations invertible ?

(b) We consider the transformation $f(\boldsymbol{z}) = \begin{bmatrix} \sin(z_1) \\ \cos(z_2) \end{bmatrix}$ from $[a, b] \times [c, d]$ to $[-1, 1]^2$. Under what conditions on a, b, c, d is this tranformation invertible ?

(c) We consider the tranformation $f(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, where $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{b} \in \mathbb{R}^2$. Under what conditions on $\boldsymbol{A}$ and $\boldsymbol{b}$ is this tranformation invertible ?

Problem 2: We consider the following forward tranformation $f(\boldsymbol{z}) = \begin{bmatrix} z_1 \\ e^{z_1} z_2 \\ |1 + z_2| z_3 + \sin(z_1) \end{bmatrix} = \boldsymbol{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$. We assume a uniform base distribution $p_1(\boldsymbol{z}) = U([0, 2]^3)$. Evaluate the density $p_2(\boldsymbol{x})$ at the two points $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 2e \\ 3 + \sin(1) \end{bmatrix}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 2 \\ e^2 \\ 6 + \sin(2) \end{bmatrix}$.

Problem 3: We consider the following forward transformation $x = f(z) = \sum_{k=1}^{K} \sigma(kz)$ from $\mathbb{R}$ to $]0, K[$ with $\sigma(z) = \frac{1}{1+e^{-z}}$. We assume a Gaussian base distribution $p_1(z) = \mathcal{N}(0, 1)$. We sampled one point from the base distribution $z^{(1)} = 0$. Compute the corresponding sample $x^{(1)}$ from the transformed distribution and evaluate its density $p_2(x^{(1)})$.

Problem 4: We consider the forward transformation $x = f(z) = az + b$ from $\mathbb{R}$ to $\mathbb{R}$ where $a, b \in \mathbb{R}$ are learnable parameters. We assume a Gaussian base distribution $p_1(z) = \mathcal{N}(0, 1)$. We observed three points $x^{(1)} = 0, x^{(2)} = 1, x^{(3)} = 2$. Compute the maximum likelihood estimate of the parameters a, b .