Machine Learning for Graphs and Sequential Data Exercise Sheet 09

Temporal Point Processes

Problem 1: Consider a temporal point process, where all the inter-event times $\tau_i = t_i - t_{i-1}$ are sampled i.i.d. from the distribution with the survival function

$$S(\tau) = \exp\Big(-(e^{b\tau} - 1)\Big)$$

with a parameter $b > 0$.

a) Write down the closed-form expression for the conditional intensity function $\lambda^*(t)$ of this TPP. Simplify as far as you can.

First, let's find the intensity function for the inter-event time distribution. We denote this intensity as $h(\tau)$ and use the definition

$$h(\tau) = \frac{p(\tau)}{S(\tau)}$$

That means, we first need to find $p(\tau)$. We recall from the lecture the following two facts

$$F(\tau) = 1 - S(\tau) \qquad\qquad p(\tau) = \frac{d}{d\tau}F(\tau)$$

By combining them, we can conclude that

$$\begin{aligned} p(\tau) &= -\frac{d}{d\tau}S(\tau) \\ &= -\frac{d}{d\tau}\exp\Big(-(e^{b\tau} - 1)\Big) \\ &= b\exp(b\tau)\exp\Big(-(e^{b\tau} - 1)\Big) \end{aligned}$$

Now, we use the definition of intensity

$$\begin{aligned} h(\tau) &= \frac{p(\tau)}{S(\tau)} \\ &= \frac{b\exp(b\tau)\exp\big(-(e^{b\tau} - 1)\big)}{\exp\left(-(e^{b\tau} - 1)\right)} \\ &= b\exp(b\tau) \end{aligned}$$

We can now define the overall conditional intensity $\lambda^*(t)$ of the entire TPP as

$$\lambda^*(t) = b\exp(b(t - t_{i-1}))$$

where t_{i-1} is the last event that happened before t.

b) Write down the closed-form expression for the log-likelihood of a sequence $\{t_1, ..., t_N\}$ generated from this TPP on the interval $[0, T]$. Simplify as far as you can.

Let's start by writing the general expression for the likelihood of a TPP

$$\begin{aligned} p(\{t_1, ..., t_N\}) &= \left(\prod_{i=1}^{N} p^*(t_i) \right) S^*(T) \\ &= \left(\prod_{i=1}^{N} \lambda^*(t_i) S^*(t_i) \right) S^*(T) \end{aligned}$$

Now apply the logarithm

$$\log p(\{t_1, ..., t_N\}) = \left(\sum_{i=1}^{N} \log \lambda^*(t_i) + \log S^*(t_i) \right) + \log S^*(T)$$

We can compute the log-intensity and log-survival as

$$\log \lambda^*(t_i) = \log b + b(t_i - t_{i-1}) \qquad\qquad \log S^*(t_i) = 1 - e^{b(t_i - t_{i-1})}$$

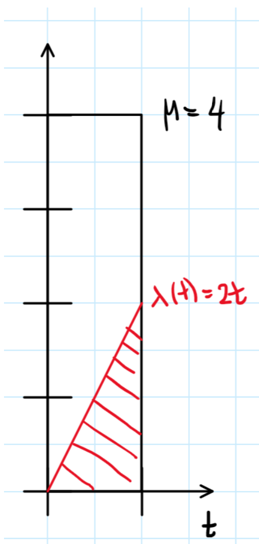
Putting everything together we obtain

$$\begin{aligned} p(\{t_1, ..., t_N\}) &= \left(\sum_{i=1}^{N} \log b + b(t_i - t_{i-1}) + 1 - e^{b(t_i - t_{i-1})} \right) + 1 - e^{b(T - t_N)} \\ &= N \log b + N + 1 + b t_N - \left(\sum_{i=1}^{N+1} e^{b(t_i - t_{i-1})} \right) \end{aligned}$$

where we denoted $t_0 = 0$ and $t_{N+1} = T$.

Problem 2: Consider an inhomogeneous Poisson process (IPP) on $[0, 1]$ with the intensity function $\lambda(t) = 2t$. We simulate a sample from this IPP using thinning. For this, we first simulate a *homogeneous* Poisson process (HPP) with intensity $\mu = 4$ and apply the thinning procedure described in the lecture. What is the expected number of events from the HPP that will be rejected when using this procedure?

Let's plot the the intensity function $\lambda(t)$ of the IPP and the intensity of the HPP μ



Recall from the lecture, that the expected number of events generated from the IPP is equal to the area under $\lambda(t)$ (red shaded area in the figure). We can easily compute that it's equal to $\int_0^1 2t dt = 1$.

Using the same reasoning, we can conclude that the expected number of events generated from the HPP is equal to 4, since HPP can be seen as an IPP with constant intensity.

By combining these two facts we conclude that on average 3 events will be discarded (which happens to correspond to the area of the unshaded region).

Problem 3: Consider an inhomogeneous Poisson process on $[0, 4]$ with the intensity function $\lambda(t) = \beta t$, where $\beta > 0$ is a parameter that has to be estimated. You have observed a single sequence $\{1, 2.1, 3.3, 3.8\}$ generated from this IPP. What is the maximum likelihood estimate of the parameter β?

We need to solve the following optimization problem

$$\max_{\beta} \log p(\{t_1, ..., t_N\}|\beta)$$

where the log-likelihood is computed as

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\begin{aligned}
\log p(\{t_1, ..., t_N\}|\beta) &= \log\left[\left(\prod_{i=1}^{N}\lambda(t_i)\right)\exp\left(-\int_0^T \lambda(u)du\right)\right] \\
&= \sum_{i=1}^{N}\log\lambda(t_i) - \int_0^T \lambda(u)du \\
&= \sum_{i=1}^{N}\log(\beta t_i) - \int_0^T \beta u du \\
&= N\log\beta - \frac{T^2}{2}\beta + \underbrace{\sum_{i=1}^{N}\log(t_i)}_{\text{constant in } \beta}
\end{aligned}
$$

Since this is a sum of concave functions, we can compute the derivative, set it to zero and solve for β in order to find the maximum

$$
\begin{aligned}
\frac{d}{d\beta}\log p(\{t_1, ..., t_N\}|\beta) &= \frac{N}{\beta} - \frac{T^2}{2} \stackrel{!}{=} 0 \\
\frac{T^2}{2}\beta &\stackrel{!}{=} N \\
\beta &= \frac{2N}{T^2} \\
\beta &= \frac{1}{2}
\end{aligned}
$$