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Research Statement

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My research area is *computability theory*, with a focus on foundational questions from *mathematical logic* especially from the point of view of *reverse mathematics*. Reverse mathematics is a research program started in the seventies by Friedman and Simpson ([Fri75], [Sim09]) which aims to calibrate the strength of principles of ordinary mathematics by showing them to be equivalent to one of some special *subsystems of second order arithmetic*. It has been one of the most active research areas in mathematical logic in the recent decades. The most important systems of reverse mathematics are RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0 in order of axiomatic strength.

I am also interested in *Ramsey theory*, in particular in the study of Ramsey like statements. These are typically of the form: fixed $k \in \mathbb{N}$ and objects $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for each coloring of the copies of $\mathcal{C}$ in $\mathcal{B}$ with k colors, there exists a copy of $\mathcal{A}$ in $\mathcal{B}$ such that all of its copies of $\mathcal{C}$ have the same color. This is usually summarized with the arrow notation $\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k$. My research focuses on Ramsey like statements from the combinatorial point of view, even though they would be interesting also from the reverse mathematical perspective.

Furthermore, I am recently growing interest in topics of *proof theory* as *proof theoretic ordinals* of subsystems of second order arithmetic ([Gen67], [Rat23]) and *fast-growing hierarchies of functions* ([APW24]).

Reverse mathematics

I am mostly interested in the study of combinatorial principles from the perspective of reverse mathematics, in particular from *order theory* and *graph theory*.
In order theory, it is possible to define a number of parameters to describe a partial order ([Har05], [Fis85]). For instance the *height* measures the maximum cardinality of a chain while the *width* measures the maximum cardinality of an antichain. Among these parameters one of the most interesting is the *dimension of a poset*.

Definition 1. If $\mathsf{X} = (X, <)$ is a poset, a set $(X, \lhd_i)_{i \in I}$ of linear orders is said to realize X if $\bigcap_{i \in I} \lhd_i = <$. The dimension of X is the least cardinality of a realization.

Together with Alberto Marcone (my supervisor) and Marta Fiori Carones, we mostly studied theorems that show how the dimension behaves if we remove from the poset respectively a point, a chain or a finite number of chains.

- $\mathsf{DB_p}$: for each poset $(X, <)$ and each $x_0 \in X$,

$$\dim(X, <) \leq \dim(X \setminus \{x_0\}, <) + 1.$$

- $\mathsf{DBc_n}$: for each poset $(X, <)$ and each pairwise incomparable and disjoint chains $C_i \subseteq X$ for $i < n$ (let $C = \bigcup_{i<n} C_i$),

$$\dim(X, <) \leq \dim(X \setminus C), <) + \max\{2, n\}.$$

○ $\mathsf{WDBc_n}$: for each poset $(X,<)$ and each family of chains $C_i \subseteq X$ for $i<n$ (let $C = \bigcup_{i<n} C_i$),

$$\dim(X,<) \leq \dim(X \setminus C, <) + 2n.$$

The principles $\mathsf{DBc_n}$ and $\mathsf{WDBc_n}$ turned out to be equivalent to $\mathsf{WKL_0}$. On the other hand $\mathsf{DB_p}$ has a very interesting: it is provable from the disjunction $\mathsf{WKL_0} \vee \mathsf{I}\Sigma^0_2$ (here $\mathsf{I}\Sigma^0_2$ is the strengthening of $\mathsf{RCA_0}$ with the induction scheme for Σ^0_2 formulas). Principles equivalent to a disjunction of systems are very rare in reverse mathematics: $\mathsf{DB_p}$ could be one of them which makes this statement interesting for further research.

I am also studying topics from graph theory in reverse mathematics perspective together with Damir Dzhafarov and Reed Solomon. A relational structure $\mathcal{M}$ is *indivisible* if for every finite coloring $c : M \to k$ of the domain of $\mathcal{M}$, there is a monochromatic subset H of M such that the induced substructure on H is isomorphic to $\mathcal{M}$. We focused on the stronger property that requires the full set of some color to yield an isomorphic substructure. Such property is called *pigeonhole property* or *strong indivisibility* ([BD99]).

Definition 2. A relational structure $\mathcal{M}$ is *strongly indivisible* if for every partition of the domain $M = X_0 \sqcup X_1$ the induced substructure on X_0 or X_1 is isomorphic to $\mathcal{M}$.

Strongly indivisible graphs have been completely classified: they are exactly the *infinite complete graph*, the *infinite totally disconnected graph* and the *random graph*. We studied the reverse mathematics of the classification results and even though we obtained some partial results ([DSV24]), the analysis is not complete and there is space for further research. Moreover, strong indivisibility has been studied for other relational structures (e.g. orders and tournaments, [BCD00]). Even for these additional structures, strong indivisibility has been completely characterized and the classification results are interesting for the reverse mathematical investigation.

Ramsey theory

Strong indivisibility is one example of the many Ramsey like statements that have been studied by mathematicians. One of the most influential has been the seminal result of Paris and Harrington of 1977 ([PH77]). This has been claimed to be the first "natural" statement independent from *Peano arithmetic*.

Statement .1. *Let $n,k,m \in \mathbb{N}$ be such that $m \geq n$. There exists $N \in \mathbb{N}$ such that for each $s \subseteq \mathbb{N}$ of cardinality N and each coloring of the n-size subsets of s in k colors, there exists a set t which is homogeneous for the coloring and such that $|t| \geq \max(m, \min t)$.*

The homogeneous set is said to be large using a criterion different from the cardinality. After this, *largeness notions* for finite sets associated to countable ordinals have been defined and have been used, among other things, to study Ramsey like statements. Many researchers ([BK99], [KoY20], [KPW07]) answered to questions of this kind: given $n,k \in \mathbb{N}$ and a countable ordinal α, which countable ordinal β satisfies $\beta \to (\alpha)^n_k$? In other words, how big should β be, to be sure that each coloring of the n-tuples of a β-large set in k colors has an α-large homogeneous set? The natural question that is driving my research is what happens if instead of coloring the n-tuples for some $n \in \mathbb{N}$, we color the γ-large sets for some countable ordinal γ. Together with Alberto Marcone and Antonio Montalbán, I am trying to find a way to compute the ordinal β for any given pair of ordinals α and γ below a certain threshold. Even though we do not get sharp Ramsey ordinals for a fixed number k of colors (which is not surprising, as sharp Ramsey numbers are known only in very few finite cases) we can compute the limit of these ordinals β as k goes to infinite. This gives a non trivial upper bound for fixed number of colors and sets a much more general framework than the one that has been used until now.

I have recently started to study Ramsey like statements from the reverse mathematics perspective too. Carlucci and Zdanowski ([CZ14]) prove that Ramsey theorem for colorings of ω-large sets is equivalent to ACA_0^+, a subsystem of second order arithmetic which consists of RCA_0 strengthened with the assertion that for each set its ω-th Turing jump exists. This suggests a connection between Ramsey theorem for α-large sets and the existence of the α-th Turing jump. Marcone and Montalbán ([MM11]) studied the axiomatic systems of reverse mathematics corresponding to transfinite jumps and showed them to be equivalent to some well ordering principles involving Veblen functions. I plan to show that each of this systems is equivalent to countably many different Ramsey like statements, according to the largeness of the sets colored. Recently, other Ramsey like statements (*free set*, *thin set* and *rainbow Ramsey* [CGHP24]) have been generalized to colorings of the ω-large sets and studied in reverse mathematics. A further generalization to colorings of α-large sets for $\alpha > \omega$ could be a new interesting research topic.

My research interests embrace classification problems in mathematical logic including (but not limited to) reverse mathematics and Weihrauch reducibility. My main focus until now has been on combinatorial principles, especially from order theory, graph theory and Ramsey theory. However I am also interested in problems coming from different areas of mathematics like algebra and geometry. Finally, I am fascinated by problems coming from other branches of mathematical logic like proof theory and more specifically ordinal analysis.

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