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Research Statement

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My research area lies in *computability theory*, with a focus on foundational questions in *mathematical logic*, especially through the lens of *reverse mathematics*. Reverse mathematics, initiated in the 1970s by Friedman and Simpson [Fri75, Sim09], aims to measure the exact logical strength of mathematical theorems by determining which axioms are necessary and sufficient to prove them. The striking observation behind the program is that many classical results, particularly those formulated without explicit set theoretic notions, turn out to be equivalent to one of a few canonical *subsystems of second-order arithmetic*. These so-called "Big Five"systems, ordered by increasing logical strength, are

$$\mathsf{RCA}_0,\ \mathsf{WKL}_0,\ \mathsf{ACA}_0,\ \mathsf{ATR}_0,\ \Pi^1_1\text{-}\mathsf{CA}_0.$$

This framework has become one of the most active areas in mathematical logic, offering a unified way to understand the computational and conceptual content of ordinary mathematics.

I am also interested in *Ramsey theory*, in particular in the study of *Ramsey like statements* based on general *largeness notions*. These notions can be defined either through systems of fundamental sequences (as in ordinal analysis) or via combinatorial structures such as blocks and barriers, the two perspectives being deeply connected, yet technically distinct. Ramsey like statements are typically of the form: given a fixed number of colors $k \in \mathbb{N}$ and structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, every coloring of the copies of $\mathcal{C}$ in $\mathcal{B}$ admits a substructure $\mathcal{A}' \subseteq \mathcal{B}$ isomorphic to $\mathcal{A}$ such that all copies of $\mathcal{C}$ within $\mathcal{A}'$ receive the same color. This is usually expressed in arrow notation as

$$\mathcal{B} \to (\mathcal{A})^{\mathcal{C}}_k.$$

My research investigates such principles from a combinatorial point of view, while also exploring their classification in the context of reverse mathematics and computability theory.

Recently, I have also developed a growing interest in topics from *proof theory*, especially the study of *proof theoretic ordinals* of subsystems of second order arithmetic [Gen67, Rat23] and *fast growing hierarchies of functions* [APW24]. These lines of work connect naturally with my current research, as they provide quantitative and structural tools for measuring the strength of combinatorial and logical principles.

In future work, I aim to deepen the connection between Ramsey theory and proof theory by analyzing the proof theoretic and computational strength of new Ramsey like principles defined via generalized largeness notions. A long term goal is to classify these principles within the reverse mathematics hierarchy and to identify the precise subsystems corresponding to their combinatorial behavior. Finally, I am also interested in classifying these results with finer tools such as Weihrauch reducibility.

Reverse mathematics

I am primarily interested in the study of combinatorial principles from the perspective of *reverse mathematics*, with a particular emphasis on *order theory* and *graph theory*.

In order theory, several parameters can be defined to describe the structure of a partially ordered set [Har05, Fis85]. For example, the *height* of a poset measures the maximal cardinality of a chain, while the *width* measures the maximal cardinality of an antichain. Among these, one of the most intriguing parameters is the *dimension* of a poset.

Definition 1. If $\mathsf{P} = (P, \preceq)$ is a poset, a family $(P, \trianglelefteq_i)_{i \in I}$ of linear orders *realizes* P if $\bigcap_{i \in I} \trianglelefteq_i \, = \, \preceq$. The *dimension* of P is the least cardinality of such a realization.

Together with Alberto Marcone (my PhD supervisor) and Marta Fiori Carones, I have studied theorems describing how the dimension of a poset behaves when removing a point, a chain, or finitely many chains. The following principles capture these relationships:

- $\mathsf{DB_p}$: For each poset $(P, \preceq)$ and each $x_0 \in P$,
$$\dim(P, \preceq) \leq \dim(P \setminus \{x_0\}, \preceq) + 1.$$
- $\mathsf{DBi_n}$: For each poset $(P, \preceq)$ and each finite family of pairwise incomparable chains $C_i \subseteq P$ for $i < n$,
$$\dim(P, \preceq) \leq \dim(P \setminus \textstyle\bigcup_{i<n} C_i, \preceq) + \max\{2, n\}.$$
- $\mathsf{DBc_n}$: For each poset $(P, \preceq)$ and each finite family of chains $C_i \subseteq P$ for $i < n$,
$$\dim(P, \preceq) \leq \dim(P \setminus \textstyle\bigcup_{i<n} C_i, \preceq) + 2n.$$

The principles $\mathsf{DBi_n}$ and $\mathsf{DBc_n}$ are equivalent to WKL_0. The case of $\mathsf{DB_p}$ is particularly interesting: it can be proved from the disjunction $\mathsf{WKL}_0 \vee \mathsf{I}\Sigma^0_2$, where $\mathsf{I}\Sigma^0_2$ is the strengthening of RCA_0 with the induction scheme for Σ^0_2 formulas. Principles equivalent to disjunctions of subsystems are extremely rare in reverse mathematics, and $\mathsf{DB_p}$ appears to be a promising example of such a phenomenon, making it a compelling direction for further research.

I am also investigating problems in *graph theory* from a reverse mathematical perspective, in collaboration with Damir Dzhafarov and Reed Solomon. A relational structure $\mathcal{M}$ is called *indivisible* if for every finite coloring $c\colon M \to k$ of its domain, there exists a monochromatic subset $H \subseteq M$ such that the induced substructure on H is isomorphic to $\mathcal{M}$. We focused on a stronger property, requiring that one of the color classes itself induces a copy of $\mathcal{M}$. This property, known as the *pigeonhole property* or *strong indivisibility* [BD99], can be formalized as follows.

Definition 2. A relational structure $\mathcal{M}$ is *strongly indivisible* if for every partition of its domain $M = X_0 \sqcup X_1$, the induced substructure on X_0 or X_1 is isomorphic to $\mathcal{M}$.

Strongly indivisible graphs have been completely classified: they are precisely the *infinite complete graph*, the *infinite totally disconnected graph*, and the *random graph*. We have investigated the reverse mathematics of these classification results. Although we have obtained partial results [DSV24], the analysis remains incomplete, leaving room for further exploration. Strong indivisibility has also been studied for other relational structures, such as *orders* and *tournaments* [BCD00], where analogous classification theorems are known and promising for reverse mathematical analysis.

Finally, another project I am pursuing with Jun Le Goh concerns the study of *ultrahomogeneous relational structures*. In reverse mathematics, a structure is *ultrahomogeneous* if every isomorphism between two of its finite substructures extends to an automorphism. We are examining classification theorems for ultrahomogeneous graphs [LW80] and partial orders [Sch79]. Interestingly, while the classification of ultrahomogeneous partial orders can be proved in RCA_0, the corresponding result for graphs (at least in its classical proof) requires that ω^ω is well ordered, a statement not provable in RCA_0. This makes the classification of ultrahomogeneous graphs a rare and potentially very interesting principle from the standpoint of reverse mathematics.

Ramsey theory

Strong indivisibility is one example of the many Ramsey like statements that have been studied by mathematicians. One of the most influential is the seminal result of Paris and Harrington (1977) [PH77], often regarded as the first "natural"statement independent of *Peano arithmetic*.

Statement 1. *Let $n, k, m \in \mathbb{N}$ be such that $m \geq n$. There exists $N \in \mathbb{N}$ such that for each $s \subseteq \mathbb{N}$ of cardinality N and each coloring of the n-element subsets of s with k colors, there exists a set t which is homogeneous for the coloring and satisfies $|t| \geq \max(m, \min t)$.*

Here, the homogeneous set is required to be "large"according to a criterion more refined than mere cardinality. Following this result, various *largeness notions* for finite sets of natural numbers associated to countable ordinals have been introduced and applied to the study of Ramsey like statements. Many researchers [BK99, KY20, KPW07] have addressed questions of the following kind: given $n, k \in \mathbb{N}$ and a countable ordinal α, which countable ordinal β satisfies $\beta \to (\alpha)^n_k$? In other words, how large must β be to guarantee that every coloring of the n-tuples of a β-large set in k colors contains an α-large homogeneous subset?
The natural question driving my research is what happens when, instead of coloring n-tuples for some $n \in \mathbb{N}$, one colors the γ-large sets for a countable ordinal γ. Together with Alberto Marcone and Antonio Montalbán, I am working on methods to compute the ordinal β corresponding to each pair of ordinals α and γ below a certain threshold. While we do not obtain sharp Ramsey ordinals for a fixed number k of colors (which is not surprising, as sharp Ramsey numbers are known in very few finite cases), we can compute the limit of these ordinals β as k tends to infinity. This provides nontrivial upper bounds for fixed numbers of colors and establishes a significantly broader framework for studying these combinatorial principles.

I have also begun to study Ramsey like statements from the perspective of *reverse mathematics*. Carlucci and Zdanowski [CZ14] proved that Ramseys theorem for colorings of ω-large sets is equivalent to ACA_0^+, the subsystem of second-order arithmetic obtained by strengthening RCA_0 with the assertion that for each set, its ω-th Turing jump exists. This suggests a close connection between Ramseys theorem for α-large sets and the existence of the α-th Turing jump. Marcone and Montalbán [MM11] analyzed the axiomatic systems of reverse mathematics corresponding to transfinite jumps (below Γ_0, the first fixed point of the binary Veblen function which is also the proof theoretic ordinal of ATR_0) and showed them to be equivalent to certain well ordering principles involving Veblen functions. My goal is to demonstrate that each of these systems is equivalent to countably many distinct Ramsey like statements, depending on the largeness notion used for the colored sets. A related direction of investigation concerns the computation of the proof theoretic ordinals of these and other intermediate subsystems between ACA_0 and ATR_0 and to try to study the classification of Ramsey like statements involving largeness notions beyond Γ_0.
Recently, several other Ramsey like statements such as the *free set*, *thin set*, and *rainbow Ramsey* theorems have been generalized to colorings of ω-large sets and studied in reverse mathematics [CGHP24]. A further generalization to colorings of α-large sets for $\alpha > \omega$ could lead to new and significant developments. Finally, I am interested in pursuing a finer classification of these Ramsey like principles through *Weihrauch reducibility*, which provides a uniform computational framework for comparing mathematical theorems.

My research interests broadly encompass classification problems in mathematical logic, particularly within reverse mathematics and Weihrauch reducibility. So far, my work has mainly focused on combinatorial principles arising in order theory, graph theory, and Ramsey theory. However, I am also keen to explore problems that emerge from other mathematical areas such as algebra and geometry, where logical methods may yield new insights. In the future, I aim to deepen the connections between combinatorial principles, proof theory, and computability by studying proof theoretic ordinals of subsystems of second order arithmetic. This line of research not only contributes to our understanding of the logical foundations of mathematics but also highlights the pervasive role of combinatorial reasoning across different branches of the discipline.

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