Generalized Recovery

Option-based P-density with time-separable utility function*

Andreas Foitzik

Karlsruhe Institute of Technology

Seminar

Applied Risk and Asset management

March 28, 2018

Abstract

This documentation characterizes a method to recover physical probabilities, marginal utilities and the discount rate from risk-neutral probabilities. We focus on large-state-space scenarios, with more states than time periods. So far, recovery is not possible without further assumptions. To provide a solution, we introduce an assumption to reduce the number of unknown parameters for the pricing kernel. Based on dimensionality reduction, recovery is possible in real-world large-state-space scenarios.

*Thanks to my supervisor Simon Walther for the support

Contents

List of abbreviations

S States

T Time periods

N Number of pricing kernel Parameters

B Design Matrix

OLS Ordinary Least Square

P Physical-Probability Distribution

Q Risk-Neutral-Probability Distribution

VIX Volatility Index

SDF Stochastic Discount Factor

1 Introduction

The Generalized Recovery method is an option based P-density with a time separable utility function. In section 2 we characterize the Generalized Recovery method and in which scenarios we are able to recover physical probabilities, marginal utilities and the discount rate. We show that recovery is possible in $S < T$ and $S = T$ scenarios, but not in real world large-state-space scenarios. For large-state-space scenarios $S > T$ we introduce an extended time separable utility function, the General Utility with N-Parameters.

In section 3 we introduce our database. We use EuroStoxx 50 option based data from 2002-01-02 to 2015-06-30 with maturities of 7, 30, 60, 91, 182, 365 days. Our database consists of risk-neutral probabilities, risk-free rates and physical probabilities. The risk-neutral probabilities are characterized by Arrow-Debreu Securities. We use physical probabilities, to test the predictive power of our recovered results.

To finally recover physical probabilities, marginal utilities and the discount rate, we introduce our Recovery Methodology in section 4. The Recovery Methodology consider seven consecutive steps. First, we define our considered state space. Second, we assign each considered state an Arrow-Debreu-Price. Third, we reduce the number of unknowns by partitioning all states in multiple state-spaces. At fourth, we linearise the connection between Arrow-Debreu-Prices and physical probabilities with a Design-Matrix. Fifth, while we determine the marginal utilities and the discount rate, we recover our pricing kernel in the sixth step. At seventh, we finally recover our physical probabilities.

In section 5 we compare our recovered results with the realized values. Therefore, we test whether an higher ex ante expected return is related with a higher ex post realized return. Based on Ordinary Least Square (OLS)-Method we regress the ex post realized return on the ex ante recovered expected return. We try to find a significant positive linear relationship between the recovered physical probabilities and the realized physical probabilities.

Finally we sum up and evaluate our approach in section 6. We look at the advantages and disadvantages and to further contributions.

2 Method

The Generalized Recovery method offers an approach to recover physical probabilities, marginal utilities and the discount rate from risk-neutral probabilities. The objective of the recovery theorem is defined as connection between risk-neutral probabilities and physical probabilities (1). $\pi^{i,j}$ determines the state price of an Arrow-Debreu Security as a risk-neutral probability. Each Arrow-Debreu Security pays 1 if state j occurs at time t. $p^{i,j}$ represents the physical transition probability of transitioning from state i to j. The connection between $\pi^{i,j}$ and $p^{i,j}$ is expressed by the pricing kernel $m^{i,j}$. (Jensen u. a., 2016, P. 7)

$$\pi^{i,j} = p^{i,j} m^{i,j} \tag{1}$$

The pricing kernel in equation (5) is defined as product of the discount rate δ and the marginal utility. The marginal utility is defined as ratio between the utility of a potential next state u^j and the utility of the current state u^i. (Jensen u. a., 2016, P. 8)

$$m^{i,j} = \delta \frac{u^j}{u^i} \tag{2}$$

Given only Arrow-Debreu Prices $\pi^{i,j}$ for each state, it is crucial to recover the pricing kernel first. Afterwards we use the recovered pricing kernel to get physical probabilities (3).

$$\frac{\pi^{i,j}}{m^{i,j}} = p^{i,j} \tag{3}$$

The Generalized Recovery method consists of three different scenarios. The first scenario contains less states than future time periods S < T. The second scenario contains the same amount of states and future time periods S = T. In the third scenario we consider a real world scenario with more states than time periods S > T. It is called a real world scenario, because even with a standard multinomial tree we are getting two new states for each new time period. Since we want to recover not simply the sign of the physical transition probability we need to consider far more states. (Jensen u. a., 2016, P.3)

To visualize the Generalized Recovery approach for those three scenarios we regard system (4). This approach solves for S < T scenarios as well. In this system we consider two possibles states $i, j \in \{1, 2\}$ and two future time horizons. As a result we get two unknown physical transition probabilities $p^{1,1}_{t,t+1}, p^{1,1}_{t,t+2}$ and four unknown marginal utilities $m^{i,j}$. This means 6 unknowns and just 4 equations. To solve this system we need to reduce the number of unknowns to at least 4. (Jensen u. a., 2016, P. 10)

$$\begin{aligned}
\pi^{1,1}_{t,t+1} &= p^{1,1}_{t,t+1} m^{1,1}_{t,t+1} \\
\pi^{1,2}_{t,t+1} &= (1 - p^{1,1}_{t,t+1}) m^{1,2}_{t,t+1} \\
\pi^{1,1}_{t,t+2} &= p^{1,1}_{t,t+2} m^{1,1}_{t,t+2} \\
\pi^{1,2}_{t,t+2} &= (1 - p^{1,1}_{t,t+2}) m^{1,2}_{t,t+2}
\end{aligned} \tag{4}$$

To do so, we introduce the time separable utility assumption.

Assumption 1 - Time separable utility *Defines, that the marginal utility u^j for each state j is independent from time τ and consistent for all time periods. As a result, for all future time periods τ the pricing kernel can be written as*

$$m^{i,j}_{t,t+\tau} = \delta^{\tau} \frac{u^j}{u^i} \tag{5}$$

To simplify the notation of (4) we introduce the vector h (6) for utility ratios.

$$h = (1, \frac{u^2}{u^1}, ... \frac{u^S}{u^1})' \equiv (1, h_2, ..., h_s)' \tag{6}$$

(Jensen u. a., 2016, P. 9)

Simplification (6) leads to the following system (7).

$$\begin{aligned}
\pi_{t,t+1}^{1,1} &= p_{t,t+1}^{1,1}\delta^1 1\\
\pi_{t,t+1}^{1,2} &= (1-p_{t,t+1}^{1,1})\delta^1 h_2\\
\pi_{t,t+2}^{1,1} &= p_{t,t+1}^{1,1}\delta^1 1\\
\pi_{t,t+2}^{1,2} &= (1-p_{t,t+1}^{1,1})\delta^1 h_2
\end{aligned} \tag{7}$$

As a result, we get 4 equations and 4 unknowns to solve this system. This system is extendable to each similar scenario, as well as to scenarios where we have less states than time periods. (Jensen u. a., 2016, P.11)

Nevertheless our goal is to recover large-state-space scenarios $S > T$, which represent real world scenarios. In general a system contains the following unknowns:

- ST equations
  - S equations at each time period T
  - one equation for each Arrow-Debreu Price
- 1 unknown discount rate delta
- $S-1$ marginal utilities, because utility for current state is 1
- $S-1$ unknown probabilities for each future time period

As a result, we have ST equations with $1+S-1+(S-1)T = ST+S-T$ unknowns. Based on this simple counting argument, recovery is only possible when $S \leq T$. To solve real world scenarios with $S > T$, we need the additional General Utility with N-Parameters assumption. This assumption on the pricing kernel allows us to reduce the amount of S unknown parameters to N. We define the lower dimensional pricing kernel by $N+1 < T$ parameters. N presents the amount of unknown state-spaces and 1 the unknown discount factor. As long as the sum of those two is less then T, recovery is possible for large-state-space scenarios. (Jensen u. a., 2016, P.3)

Assumption 1* - General Utility with N-Parameters *Extension of Assumption 1 - Time separable utility, to reduce the number of S unknown states to N unknown* *state-spaces. The pricing kernel at time τ in state s (given the initial state 1 at time 0) can be written as*

$$m_{1,s}^{0,\tau} = \delta^{\tau} h_s(\Theta) \tag{8}$$

where $\delta \in (0, 1]$ (Jensen u. a., 2016, P. 23)

As a result, we rewrite the pricing kernel (8) as a function of N parameters $\theta_1, ..., \theta_N$. While we reduce the dimensionality, we restrict the number of $S + 1$ unknowns to $N + 1 < T$. As long as our chosen parameter N is smaller than $T - 1$, we find reasonable solutions. This whole set-up is called Large-State-Space Framework. (Jensen u. a., 2016, P. 4)

3 Data

As database we use EuroStoxx 50 option based data. The time period ranges from 2002-01-02 to 2015-06-30. The maturities are 7, 30, 60,91,182, 365 days. Each maturity represents one future time horizon for which we derive a physical probability forecast. To do so, we individually recover the physical probability distribution for each date and for each maturity.

3.1 Risk-Neutral Probabilities

To use the Generalized Recovery method, we need at least the Arrow-Debreu Prices under the risk-neutral density, called Q-Data. This density is characterized by its Q-Moments. The Q-Moments consists of an expected value, variance, skewness and kurtosis for each date.

3.2 Risk-free rate

For each date and each maturity we need the annualized risk-free rate, to linearise the discount rate.

3.3 Physical Probabilities

The physical probability distribution is also called P-density. To test the recovered P-moments, we compare it with the realized P-moments. Based on given EuroStoxx 50 5-min intervals option data, we calculate the realized returns, variance, skewness and kurtosis. Afterwards we compare our expected P-moments with the realized P-moments. (Amaya u. a., 2015)

4 Recovery Methodology

The Recovery Methodology describes the procedure from Arrow-Debreu Prices towards the recovered physical probabilities. This connection is expressed by equation (9). Thereby Π represents all Arrow-Debreu Prices. D represents a diagonal matrix of discount-factors. P represents the physical probabilities and H represents the vector of marginal utilities. To recover the P-density we reshape equation (9). The inverse D^{-1} is defined as $\frac{1}{D}$ and the inverse H^{-1} is defined as $\frac{1}{H}$. (Jensen u. a., 2016, P.12f)

$$\begin{aligned} \Pi &= DPH \\ P &= D^{-1}\Pi H^{-1} \end{aligned} \tag{9}$$

The recovery procedure is based on seven consecutive steps. First, we define a state space with all states to consider. Second, for each maturity we assign each considered state an Arrow-Debreu-Price as risk-neutral probability. Third, we make use of Assumption 1* - General Utilitiy with N-Paramenters. Thereby, we define the dimensionality of the pricing kernel with N parameters θ. This leads to the necessary reduction of S unknown states, towards N unknown paramenters θ. At fourth, we define a Design-Matrix to linearise the pricing kernel. Fifth, we determine the N parameters of the pricing kernel θ and the discount factor δ. At sixth, we calculate the pricing kernel, as connection between the Q-Density and P-Density. At seventh, we finally get the P-Density for each date.

4.1 State Space

In our scenario, one state represents the potential return (10) of an option. The return is calculated by the implied strike of an option divided by the underlying forward price. If a state is equal to one, it presents no change according its previous state. A state smaller than one defines a decrease, which is equal to a loss. The other way around defines an increase, profit. We try to recover the transition probabilities between those states.

$$state := return = implStrike/underlyingforwardprice \tag{10}$$

As a result, only a finite number of states $1, ..., S$ exists. Because we are interested in a certain range of transition probabilities, we define a state space. This state space is restricted by a lower and a upper bound. These bounds are determined by the equation (11). To calculate these bounds we use the current VIX_t value at time t. As VIX_t value we annualise the daily Q-Variance. The VIX_t value should be as high as possible, that we consider as many states as possible. Therefore we use the VIX_t value of the option with maturity of 365 day. (Jensen u. a., 2016, P.31f)

$$\begin{aligned} \text{lower bound } &= S_t - 2.5 * S_t * VIX_t \\ \text{upper bound } &= S_t + 4 * S_t * VIX_t \end{aligned} \tag{11}$$

We calculate these bounds individually for each date and for each maturity. For each date we get approximately 100 states. To find a reasonable solution therefore, we need to make use of the Large-State-Space-Framework.

4.2 Q-Matrix - Arrow-Debreu Prices

For each state within the state space we assign its Arrow-Debreu Price, as in figure 1. Therefore we define the Q-Matrix with T rows and S columns, TxS. By the assignment of Arrow-Debreu Prices it is important to provide sufficient for all states. Which means, we want to avoid too many zeros at the right or left tail of the Q-Density. Additionally we want to avoid missing values in the center of the matrix. To do so we interpolate between missing values. The lower and upper bound of the state space is nicely shown in figure 1, as well as zeros at the right tail.

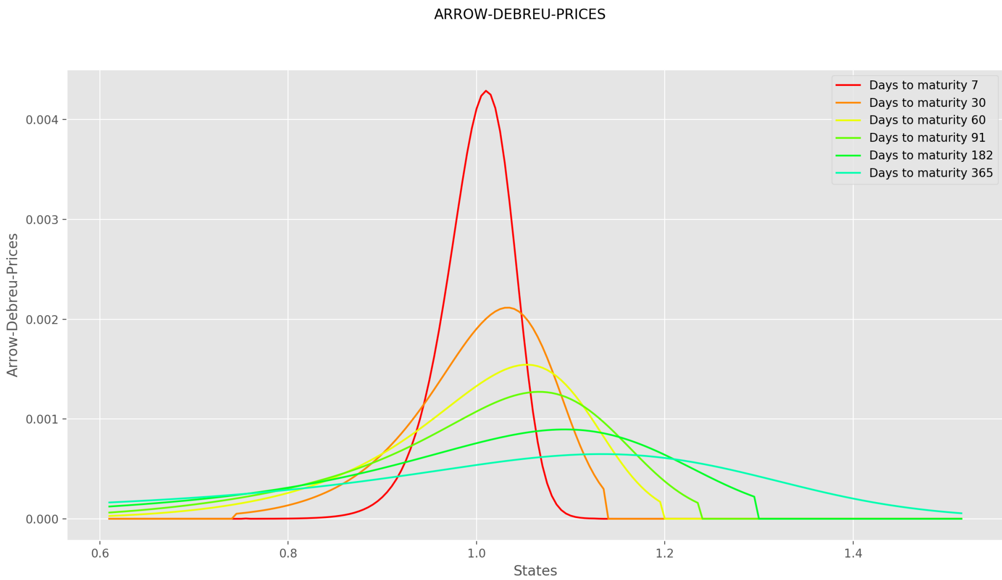

Figure 1: Arrow-Debreu-Prices for each maturity

4.3 Multiple State Spaces

We consider a real-world scenario with more states than time periods. To find a suitable solution, we make use of our extended General Utility with N-Parameters assumption. Thereby we reduce the number of S unknown states to N unknown parameters of the pricing kernel. To apply this assumption to our approach, we need to partition our state space into $N-1$ regions, as shown in system (12). $N-1$ because we add an initial slope afterwards. One of the main question remains unanswered, how to choose a proper N for the approach. So far, it is just a trial and error approach, which N fits best for each individual set-up. The $N-1$ regions are defined as follows:

$$\begin{aligned}
&\text{lowest region} := \text{from } (1-2.5*VIX_t)*S_t \text{ to } (1-2*VIX_t)*S_t \\
&\text{N-3 regions with equal size} := \text{from } (1-2*VIX_t)*S_t \text{ to } (1+2*VIX_t)*S_t \\
&\text{highest region} := \text{from } (1+2*VIX_t)*S_t \text{ to } (1+4*VIX_t)*S_t
\end{aligned} \tag{12}$$

As a result, we get a lowest region and a highest region. In between those regions, we consider $N-3$ regions of equal size. We chose N to be equal to 4, where Θ is an 4-dimensional column vector with $\theta_1, ..., \theta_4 \geq 0$. (Jensen u. a., 2016, P.32)

4.4 Design-Matrix

Design-Matrix B expresses a piecewise linear structure on the lower-dimensional inverse pricing kernel (13). This piecewise linear structure of the SxT Design-Matrix is exposed by the unknown parameter Θ. The product of $B\Theta$ is the inverse pricing kernel H^{-1}. (Jensen u. a., 2016, P.32)

$$H^{-1}e = B\Theta \tag{13}$$

To be able to express the piecewise linear structure we define B as in (14). $n_1, ..., n_4$ represents the amount of states in each of four state spaces. We derive the linear structure for each state space separately. To do so, we assign each first state the value $\frac{1}{n}$ and each last state $\frac{n}{n}$.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{n_2} & 0 & 0 \\ 1 & \frac{2}{n_2} & 0 & 0 \\ 1 & ... & 0 & 0 \\ 1 & \frac{n_2}{n_2} & 0 & 0 \\ 1 & 1 & \frac{1}{n_3} & 0 \\ 1 & 1 & \frac{2}{n_3} & 0 \\ 1 & 1 & ... & 0 \\ 1 & 1 & \frac{n_3-1}{n_3} & 0 \\ 1 & 1 & \frac{n_3}{n_3} & 0 \\ 1 & 1 & 1 & \frac{1}{n_4} \\ 1 & 1 & 1 & ... \\ 1 & 1 & 1 & \frac{n_4}{n_4} \end{bmatrix} \tag{14}$$

The first column of B determines the initial level θ_1 of the inverse pricing kernel and is therefore constant. θ_2 determines the initial slope of the inverse pricing kernel. θ_3 is the slope of the next state space and θ_4 is the last slope of the inverse pricing kernel. The inverse pricing kernel first increases and is then flat. Based on this definition we achieve a piecewise linear structure as in figure 2. As a result, the inverse pricing kernel is monotonically increasing. Equivalently, the pricing kernel is monotonically decreasing. (Jensen u. a., 2016, P.32)

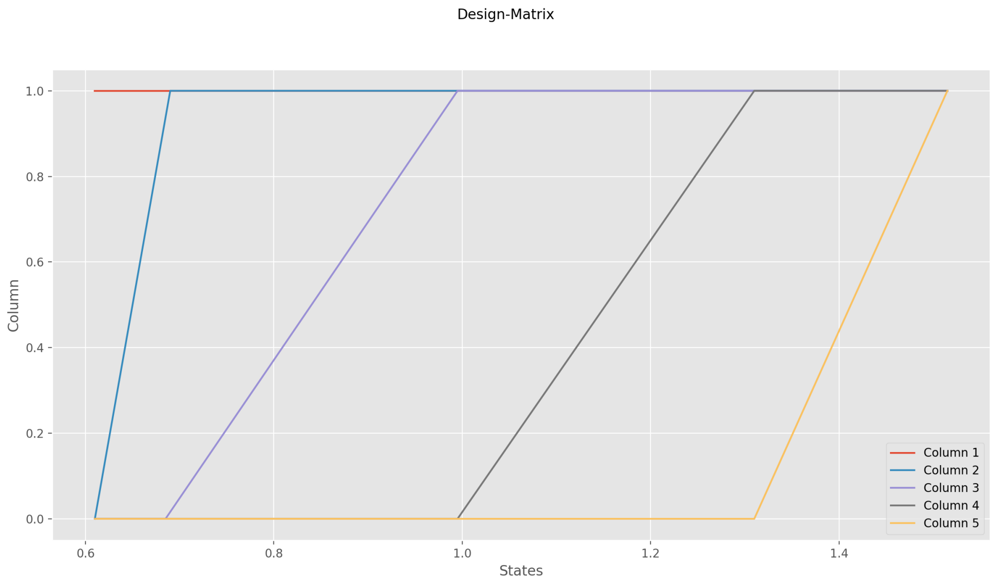

Figure 2: Piecewise linear structure of Design-Matrix B

4.5 Recover Theta and Delta

To determine the connection between Arrow-Debreu Prices and physical probabilities, it is mandatory to determine the pricing kernel. Before we obtain the pricing kernel as in (13), we need to execute two steps. First we approximate the discount rate (15) and second we solve the minimization objective (16).

The recovery problem is almost linear, except for the powers of the discount rate δ. To provide a closed-form linear solution, we linearise the discounting of δ^τ around a point δ_0 with system (15). Thereby, a and b are vectors for each date with entries for each maturity τ. (Jensen u. a., 2016, P.21)

$$\begin{aligned} a_\tau &= -(\tau - 1)\delta_0^\tau \\ b_\tau &= \tau\delta_0^{\tau-1} \\ \delta^\tau &= a_\tau + b_\tau\delta \end{aligned} \tag{15}$$

As second step, we minimize the norm of the objective to approximate Θ and δ (16). As Θ seeds we use 0.1 and as δ seed we use the average of δ^τ. Depending on the values it is based on a trial and error approach to find proper seeds.

$$\begin{aligned} \min_{\forall \theta, \delta} norm(\Pi B\theta - (a + b\delta)) \\ s.t. \theta > 0 \\ \delta \in (0, 1] \end{aligned} \tag{16}$$

As a result, we get the discount rate δ as a value between zero and one and a vector θ with $\theta_1, \theta_2, \theta_3, \theta_4$ entries. (Jensen u. a., 2016, P.33)

4.6 Pricing Kernel

Based on the Design-Matrix B and the approximated column-vector θ we determine our inverse pricing kernel as in equation (13). For instance we obtain the inverse pricing kernel in figure 3 at date 2002-10-14.

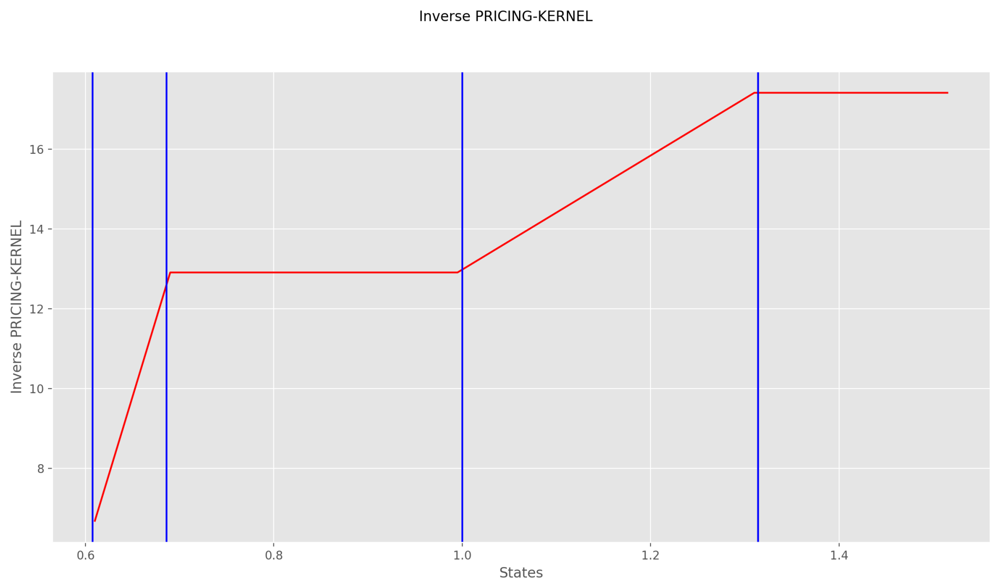

Figure 3: Inverse Pricing Kernel

System (17) shows the relation between the inverse pricing kernel and the pricing kernel. It is simply defined as the fraction of $\frac{1}{H_{init}^{-1}}$.

$$H_{init} = \frac{1}{H_{init}^{-1}} \tag{17}$$

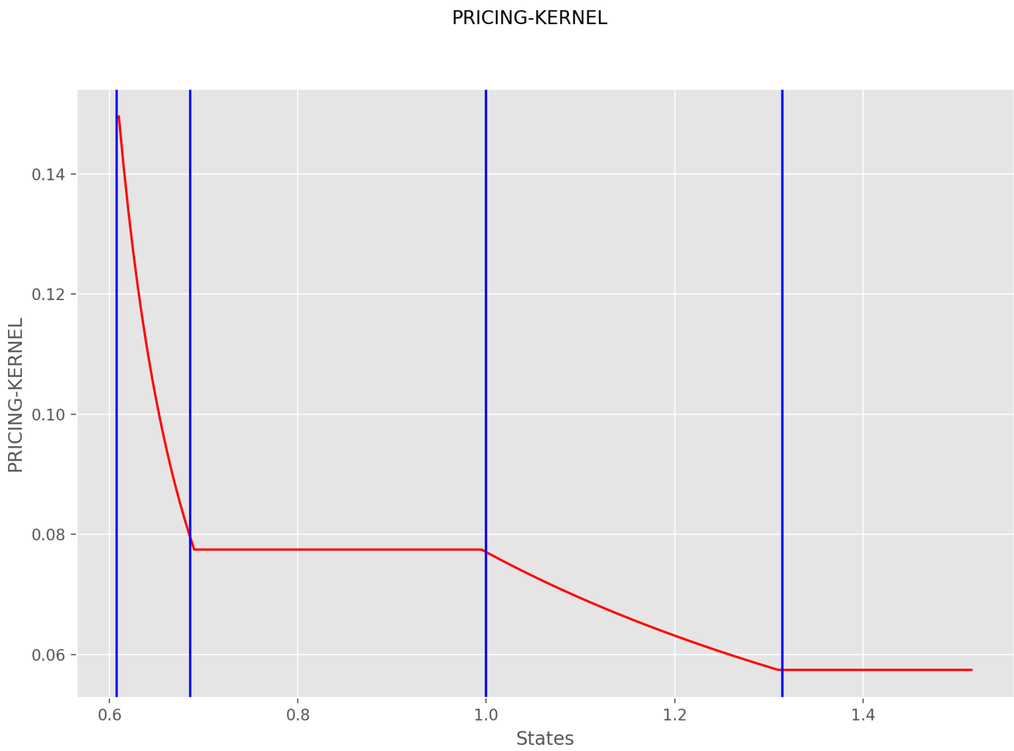

Figure 4: Pricing Kernel

We have finally recovered the pricing kernel, based on marginal utilities and the discount rate. Now we are able to recover the physical probability distribution.

4.7 Recover P-Density

We previously defined the physical probability distribution as in equation (9). We reshape this equation towards (18). Thereby, D^{-1} defines the inverse diagonal-matrix of discount factors and $diag(B\theta)$ the diagonal-matrix as inverse pricing kernel.

$$P_{init} = D^{-1}\Pi diag(B\theta) \tag{18}$$

As a result, we obtain the initial physical probability distribution. Because $\theta_1, \theta_2, \theta_3, \theta_4$ and the discount rate δ are just approximations. Therefore, we normalize P to get row sums of one (19). (Jensen u. a., 2016, P.33)

$$P_{norm} = \frac{P_{init}}{H_{init}} \tag{19}$$

Based on P_{norm} we calculate the expected cross-return, variance, skewness and kurtosis, as shown in figure 5 for the date 2002-10-14.

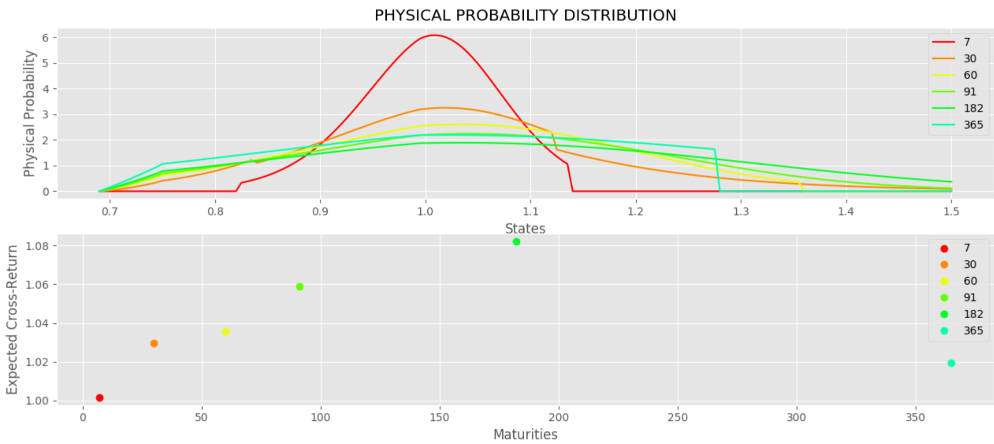

Figure 5: Recovered Physical Probability Distribution

Based on equation (20) we obtain the normalized pricing kernel of figure 6 for each maturity.

$$H_{norm} = \frac{Q}{P_{norm}} \tag{20}$$

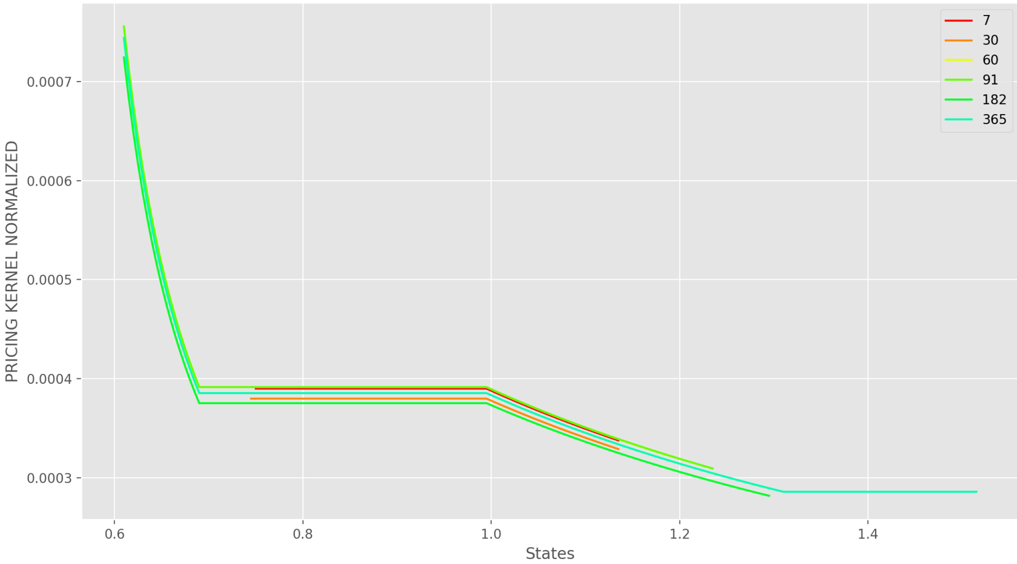

Figure 6: Normalized Pricing Kernels

5 Results

Based on the Generalized Recovery methodology we obtain a P-density for each date and each maturity. To gain a further benefit, we test the predictive power of our recovered P-density. Therefore we try to find a significantly positive relationship between the moments of the recovered and realized P-density.

5.1 Recovered P-Density

For each recovered P-Density we calculate the expected value, volatility, skewness and kurtosis. To compare those values we visualize the annualized values. We receive higher values for short maturities and lower values for longer maturities. We notice the proper relations between all maturities.

5.1.1 Recovered Expected-Returns

As a result, we receive higher annualized expected cross-returns for short maturities and lower for long maturities.

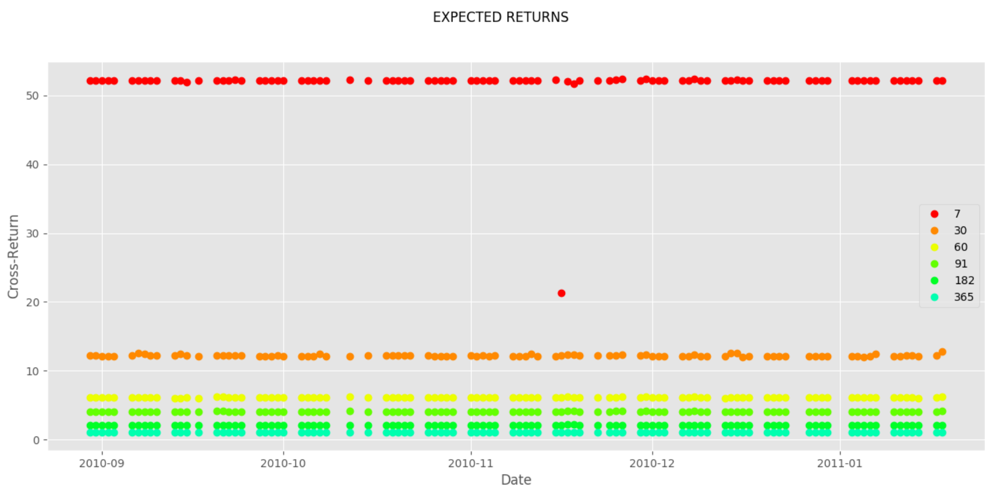

Figure 7: Recovered Returns

5.1.2 Recovered Variance

The annualized recovered variance is higher for short maturities and lower for longer maturities.

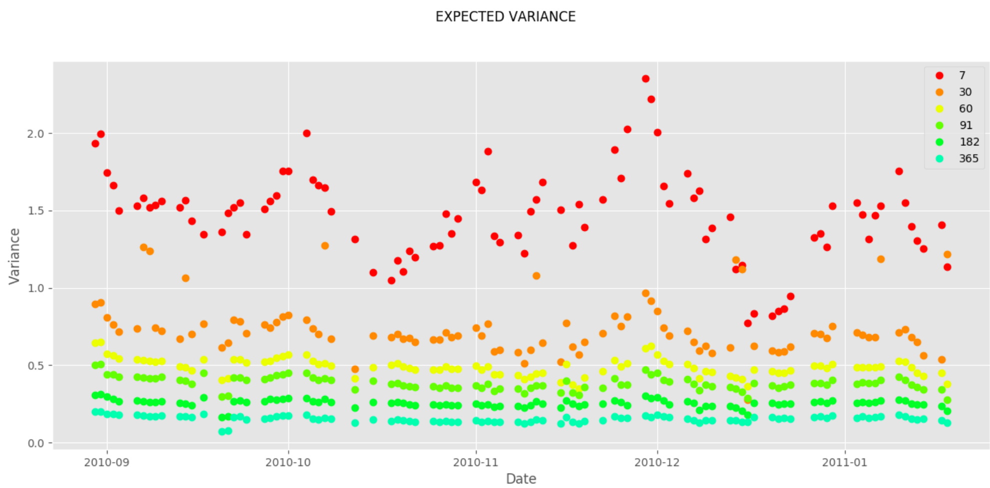

Figure 8: Recovered Variance

5.1.3 Recovered Skewness

Skewness is also very high for short maturities and decreases for longer maturities.

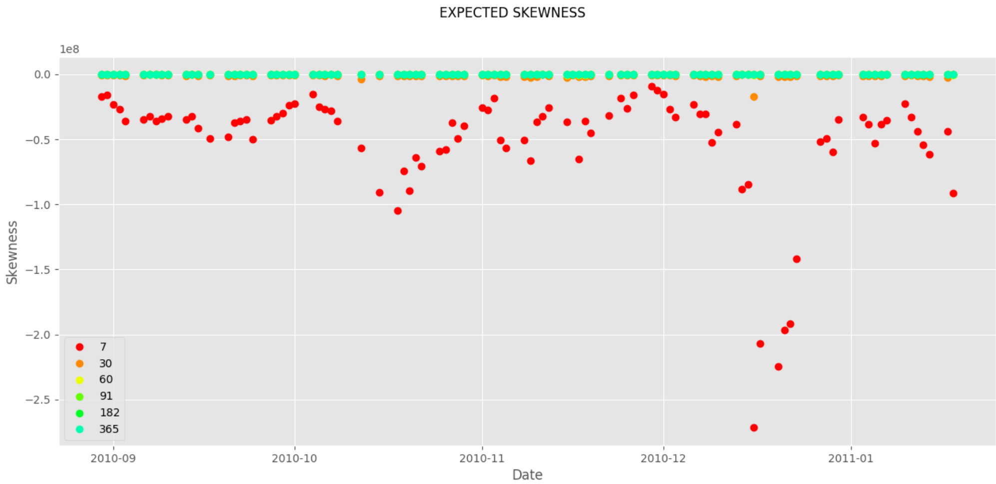

Figure 9: Recovered Skewness

5.1.4 Recovered Kurtosis

The values of our recovered excess kurtosis are especially for the maturity of seven days very high. As the maturity increases the kurtosis decreases.

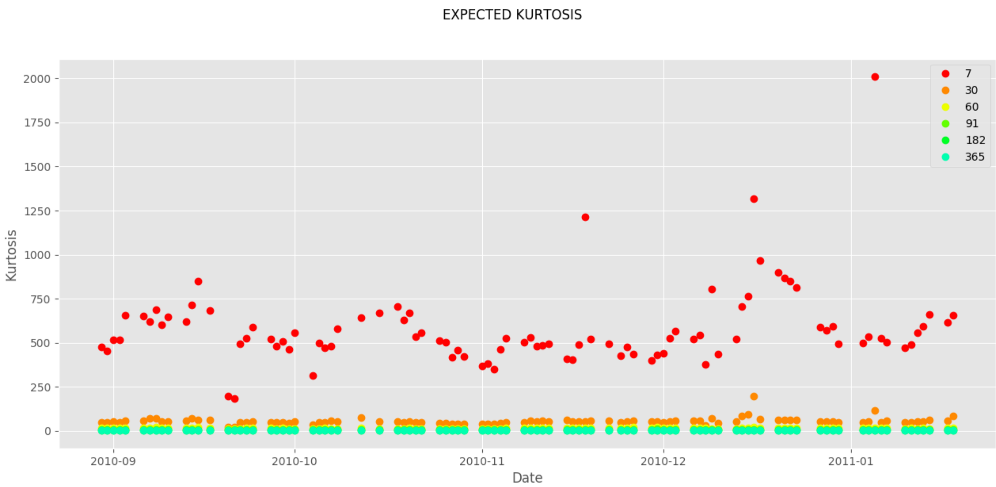

Figure 10: Recovered Kurtosis

5.2 Linear Relationship

To gain from Generalized Recovery we try to determine a linear relationship between the recovered and realized results (21). This linear relationship should provide rise to reasonable expected returns, which means a time-varying risk premia. To determine this linear relationship we test, whether recovered moments of the P-density have any predictive power. In general, whether a higher recovered value is equal to a higher realized value. Therefore we use the Ordinary Least-Square Method-OLS. (Jensen u. a., 2016, P.36)

$$realized[x_{t+\tau}] = \beta_{OLS,0} + \beta_{OLS,1} * recovered[x_{t+\tau}] + \epsilon_{t+\tau} \tag{21}$$

To determine a significantly positive linear relationship, we try to obtain a positive $\beta_{OLS,1} > 0$. In an expandable manner, the t-statistic value of $\beta_{OLS,1}$ should also be significantly larger than five.

5.2.1 OLS-Returns

With the linear regression (22) we test, whether a higher ex ante expected return is associated with a higher ex post realized return. (Jensen u. a., 2016, P.36)

$$Realized[r_{t+\tau}] = \beta_{OLS,0} + \beta_{OLS,1} * E[r_{t+\tau}] + \epsilon_{t+\tau} \tag{22}$$

As a result, we obtain table 1. We see, there is no prove of a significantly positive relationship. The relationship is slightly negative and the significance of β values increase with their maturity.

Maturity in days	$\beta_{OLS,0}$	$t - stat_{OLS,0}$	$\beta_{OLS,0}$	$t - stat_{OLS,1}$
7	1.022453	73.449	-0.022486	-1.617
30	1.089284	37.198	-0.087236	-2.998
60	3.623971	28.758	-2.579467	-20.814
91	1.944364	26.560	-0.919412	-12.880
185	4.461431	35.730	-3.293990	-27.703
365	4.547679	33.959	-3.303818	-26.410

Table 1: OLS-Estimates for return prediction

5.2.2 OLS-Variance

With the linear regression (23) we test, whether a higher ex ante variance is associated with a higher ex post realized variance.

$$Var_{real}[var_{t+\tau}] = \beta_{OLS,0} + \beta_{OLS,1} * Var_{rec}[r_{t+\tau}] + \epsilon_{t+\tau} \tag{23}$$

Table 2 represents the linear regression results. There is as well no prove of a significantly positive relationship. Instead, we notice a negative relationship. The significance of β values increase with their maturity.

Maturity in days	$\beta_{OLS,0}$	$t - stat_{OLS,0}$	$\beta_{OLS,0}$	$t - stat_{OLS,1}$
7	5.9300	377.96	-0.285106	-0.6823
30	22.604	341.73	-1.426730	-1.4353
60	42.862	368.08	-3.881952	-2.8400
91	64.760	732.53	-4.649809	-5.100
185	128.20	1079.74	-4.522674	-4.6628
365	256.48	2566.80	-8.505162	-11.9822

Table 2: OLS-Estimates for variance prediction

5.2.3 OLS-Skewness

With equation (24) we try to prove, that a higher ex ante skewness is associated with a higher ex post realized skewness.

$$Skewness_{real}[skewness_{t+\tau}] = \beta_{OLS,0} + \beta_{OLS,1} * Skewness_{rec}[r_{t+\tau}] + \epsilon_{t+\tau} \tag{24}$$

In table 3 we can determine a partly significant positive relationship. Which gives us a rise towards, that a higher ex ante skewness is associated with a higher ex post realized skewness. Especially interesting is, that the significance decreases and increases again.

5.2.4 OLS-Kurtosis

We test (25), whether a higher ex ante kurtosis is associated with a higher ex post realized kurtosis.

Maturity in days	$\beta_{OLS,0}$	$t-stat_{OLS,0}$	$\beta_{OLS,0}$	$t-stat_{OLS,1}$
7	1.000285	135424	4.97e-11	15.536
30	1.000285	127178	1.64e-11	10.290
60	1,000260	144861	1.15e-11	1.762
91	1.000258	201687	3.98e-11	0.396
185	1.000262	225010	4.74e-11	0.867
365	1.000307	238673	2.07e-11	24.10

Table 3: OLS-Estimates for skewness prediction

$$Kurtosis_{real}[kurtosis_{t+\tau}] = \beta_{OLS,0} + \beta_{OLS,1} * Kurtosis_{rec}[r_{t+\tau}] + \epsilon_{t+\tau} \tag{25}$$

Table 4 shows positive and negative relationship. But neither of them is significant.

Maturity in days	$\beta_{OLS,0}$	$t-stat_{OLS,0}$	$\beta_{OLS,0}$	$t-stat_{OLS,1}$
7	0.412	515.533	-0.000024	-0.228
30	0.210	577.87	0.000315	2.989
60	0.153	428.807	0.000081	0.4813
91	0.124	2387.37	0.000036	1.752
185	0.088	1522.549	-0.000040	-1.372
365	0.062	5452.001	-0.000014	-3.526

Table 4: OLS-Estimates for kurtosis prediction

6 Conclusion

As a conclusion, we value the Generalized Recovery method as very promising. One of the main advantages is, that we make no assumption on the underlying physical probability distribution. Instead, we rely only on the Generalized Utility with N-Parameters. The method also provides a closed form linear solution through the linearisation of the discount rate. Based on the fact that we chose the size of parameter N by ourself, it provides quite a flexibility. To do so, it also requires a good intuition of the underlying data.

The results presents evidence for a linear relationship. We show, that the recovered probabilities contain information about future P-density moments. Based on previously used data, we recommend to use more than just six time periods. In a previous scenario we used twelve time periods and received a continuously increasing shape of the piecewise linear pricing kernel as well as more reasonable results.

7 References

[Amaya u. a. 2015] Amaya, Diego ; Christoffersen, Peter ; Jacobs, Kris ; Vasquez, Aurelio: Does realized skewness predict the cross-section of equity returns? In: *Journal of Financial Economics 118* (2015), S. 135 – 167

[Jensen u. a. 2016] Jensen, Christian S. ; Lando, David ; Pedersen, Lasse H.: *Generalized Recovery.* 2016