1 The system

The vessel is modelled by

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \tag{1}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{2}$$

where

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad \boldsymbol{M}_A = \begin{bmatrix} -X_{\dot{u}} & 0 & 0 \\ 0 & -Y_{\dot{v}} & -Y_{\dot{r}} \\ 0 & -N_{\dot{v}} & -N_{\dot{r}} \end{bmatrix} \tag{3}$$

$$\boldsymbol{C}_{RB} = \begin{bmatrix} 0 & 0 & -m(x_g r + v) \\ 0 & 0 & mu \\ m(x_g r + v) & -mu & 0 \end{bmatrix} \quad \boldsymbol{C}_A = \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + \frac{1}{2}(N_{\dot{v}} Y_{\dot{r}})r \\ 0 & 0 & -X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r + \frac{1}{2}(N_{\dot{v}} Y_{\dot{r}})r & X_{\dot{u}} u_r & 0 \end{bmatrix} \tag{4}$$

$$\boldsymbol{D}_L = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix} \quad \boldsymbol{D}_{NL} = \begin{bmatrix} -X_{|u|u}|u| - X_{uuu}u^2 & 0 & 0 \\ 0 & -Y_{|v|v}|v_r| - Y_{|r|v}|r| & -Y_{|v|r}|v_r| - Y_{|r|r}|r| \\ 0 & -N_{|v|v}|v_r| - N_{|r|v}|r| & -N_{|v|r}|v_r| - N_{|r|r}|r| \end{bmatrix} \tag{5}$$

Using property 8.1 of [Fossen] we can define the dynamics using only the water relative velocity $\boldsymbol{\nu}_r$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = \boldsymbol{\tau} \tag{6}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{7}$$

Where $\boldsymbol{\nu}_c$ is the velocity of the water current in global coordinates. And

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A \tag{8}$$

$$\boldsymbol{C}(\boldsymbol{\nu}_r) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) + \boldsymbol{C}_A(\boldsymbol{\nu}_r) \tag{9}$$

Additionally $\boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{0}$ as there exists no restoring forces in surge, sway and yaw. Isolating the acceleration $\dot{\boldsymbol{\nu}}_r$ in (6) we have

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r) \tag{10}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{11}$$

Assuming that $\boldsymbol{M}$ and $\boldsymbol{C}$ as well as the linear damping coefficients X_u and N_r are known. The system equation can be split into a known part and a unknown or uncertain part. This can be written as

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{d}\boldsymbol{\nu}_r + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{12}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{13}$$

Where

$$\boldsymbol{d} = \begin{bmatrix} X_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_r \end{bmatrix} \tag{14}$$

And the uncertain part is described by

$$\Phi(\boldsymbol{\nu}_r)^T = \begin{bmatrix} |u|u & 0 & 0 \\ u^3 & 0 & 0 \\ 0 & v & 0 \\ 0 & r & 0 \\ 0 & |v|v & 0 \\ 0 & |r|v & 0 \\ 0 & |v|r & 0 \\ 0 & |r|r & 0 \\ 0 & 0 & v \\ 0 & 0 & |v|v \\ 0 & 0 & |r|v \\ 0 & 0 & |v|r \\ 0 & 0 & |r|r \end{bmatrix} \qquad \boldsymbol{\vartheta} = \begin{bmatrix} X_{|u|u} \\ X_{uuu} \\ Y_v \\ Y_r \\ Y_{|v|v} \\ Y_{|r|v} \\ Y_{|v|r} \\ Y_{|r|r} \\ N_v \\ N_{|v|v} \\ N_{|r|v} \\ N_{|v|r} \\ N_{|r|r} \end{bmatrix} \tag{15}$$

Lastly the known part is collected in a function $\boldsymbol{f}(\boldsymbol{\nu}_r)$.

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(-\boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{d}\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{16}$$

$$= \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{17}$$

$$\tag{18}$$

The system is now in a form where the known part $\boldsymbol{f}(\boldsymbol{\nu}_r)$, the uncertain part $\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}$ and the input $\boldsymbol{\tau}$ are clear to distinguish from one another.

2 Adaptive Velocity Control

2.1 Control Design

The control objective is to minimize $\tilde{\nu}(t) \triangleq \nu(t) - r_\nu(t)$. That is to minimize the difference (error) between the velocity of the vessel $\nu(t)$ and the reference velocity $r_\nu(t)$. A new variable $\boldsymbol{z}$ is introduced to describe this difference.

$$\boldsymbol{z} \triangleq \nu - r_\nu \tag{19}$$

Choosing the Lyapunov function

$$V \triangleq \frac{1}{2}\boldsymbol{z}^T\boldsymbol{z} + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}} \tag{20}$$

Which has the derivative

$$\dot{V} = \boldsymbol{z}^T\dot{\boldsymbol{z}} + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\vartheta}}} \tag{21}$$

$$= \boldsymbol{z}^T\left(\dot{\boldsymbol{\nu}}_r - \dot{r_\nu}\right) + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{22}$$

$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}\right) - \dot{r_\nu}\right) + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{23}$$

$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right) - \dot{r_\nu}\right) + \boldsymbol{z}^T\boldsymbol{M}^{-1}\Phi(\boldsymbol{\nu}_r)\tilde{\boldsymbol{\vartheta}} + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{24}$$

$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right) - \dot{r_\nu}\right) + \tilde{\boldsymbol{\vartheta}}^T\left(\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} + \Phi(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-T}\boldsymbol{z}\right) \tag{25}$$

$$\tag{26}$$

It is seen that choosing the control law

$$\boldsymbol{\tau} = \boldsymbol{M}\left(-\boldsymbol{K}\boldsymbol{z} - \boldsymbol{f}(\boldsymbol{\nu}_r) + \dot{r_\nu}\right) - \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} \tag{27}$$

And the adaptation law

$$\dot{\hat{\boldsymbol{\vartheta}}} = -\boldsymbol{\Gamma}\Phi(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-T}\boldsymbol{z} \tag{28}$$

Will makes the Lyapunov function derivative become

$$\dot{V} = -\boldsymbol{z}^T\boldsymbol{K}\boldsymbol{z} \tag{29}$$

Which is negative semi-definite if $\boldsymbol{K}$ is positive definite.

Barbalat's lemma

3 Adaptive Position Control

3.1 The system

As in the previous section the system is

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} \tag{30}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{31}$$

where the known part $\boldsymbol{f}(\boldsymbol{\nu}_r)$, the uncertain part $\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}$ and the input $\boldsymbol{\tau}$ of the system is separated. Considering the water currents as a unknown quantity

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} \tag{32}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1\boldsymbol{\vartheta} \tag{33}$$

3.2 Control objective

The control objective is to minimize $\tilde{\eta}(t) \triangleq \eta(t) - r_\eta(t)$. That is to minimize the difference (error) between the position of the vessel $\eta(t)$ and the reference position $r_\eta(t)$. A new variable $\boldsymbol{z}$ is introduced to describe this difference.

$$\boldsymbol{z}_1 \triangleq \eta - r_\eta \tag{34}$$

$$\dot{\boldsymbol{z}}_1 = \dot{\boldsymbol{\eta}} - \dot{r}_\eta = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1\boldsymbol{\vartheta} - \dot{r}_\eta \tag{35}$$

Since $\dot{\boldsymbol{z}}_1$ does not contain the input $\boldsymbol{\tau}$ it is necessary to consider the velocity $\boldsymbol{\nu}_r$ as a virtual input α. This is done in order that backstep though the system equations until we find the real input $\boldsymbol{\tau}$. For now $\boldsymbol{\nu}_r$ is considered a virtual input, that is, a state the can be controlled to act as the input of a subsystem.

3.3 Step 1

Choosing the Lyapunov function

$$V \triangleq \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}} \tag{36}$$

Which has the derivative

$$
\begin{aligned}
\dot{V} &= \boldsymbol{z}_1^T \dot{\boldsymbol{z}}_1 + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} && (37)\\
&= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \boldsymbol{\vartheta} - \dot{r}_\eta\right) + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} && (38)\\
&= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta\right) + \boldsymbol{z}_1^T \Phi_1 \tilde{\boldsymbol{\vartheta}} + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} && (39)\\
&= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta\right) + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}}\right) && (40)\\
& && (41)
\end{aligned}
$$

It is seen that choosing the control law for the virtual input α

$$
\boldsymbol{\alpha} = \boldsymbol{R}(\psi)^T - \Phi_1 \hat{\boldsymbol{\vartheta}} + \dot{r}_\eta - \boldsymbol{K}_1 \boldsymbol{z}_1 \tag{42}
$$

And the adaptation law

$$
\dot{\hat{\boldsymbol{\vartheta}}} = -\boldsymbol{\Gamma} \Phi_1^T \boldsymbol{z}_1 = -\boldsymbol{\Gamma} \tau(\boldsymbol{z}_1) \tag{43}
$$

where $\tau(\boldsymbol{z}_1) = \Phi_1^T \boldsymbol{z}_1$ is a tuning function, that must not to be confused with the input $\boldsymbol{\tau}$.

3.4 Derivative of a rotation matrix

Consider the rotation matrix $R(\psi)$

$$
\boldsymbol{R}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{44}
$$

And its derivative

$$
\dot{\boldsymbol{R}}(\psi) = \frac{d}{d\psi} \boldsymbol{R}(\psi) \dot{\psi} = \frac{d}{d\psi} \boldsymbol{R}(\psi) r \tag{45}
$$

$$
\dot{\boldsymbol{R}}(\psi) = \begin{bmatrix} -\sin(\psi)r & -\cos(\psi)r & 0 \\ \cos(\psi)r & -\sin(\psi)r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{46}
$$

Which can be written as

$$
\dot{\boldsymbol{R}}(\psi) = \boldsymbol{S}(r) \boldsymbol{R}(\psi) \tag{47}
$$

With

$$
\boldsymbol{S}(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}
$$

This is easily verified by doing the matrix multiplication.

3.5 Step 2

Now we must ensure that $\boldsymbol{\nu}_r$ takes the value α. Thus the control objective of the second step is

$$
\begin{aligned}
\boldsymbol{z}_2 &\triangleq \boldsymbol{\nu}_r - \boldsymbol{\alpha} && (49)\\
\dot{\boldsymbol{z}}_2 &= \dot{\boldsymbol{\nu}}_r - \dot{\boldsymbol{\alpha}} && (50)
\end{aligned}
$$

$$
\dot{\boldsymbol{\alpha}} = \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T - \Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1 \tag{51}
$$

Augmenting the Lyapunov function with a term for $\boldsymbol{z}_2$ we have

$$V_2 \triangleq V_1 + \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 \tag{52}$$

Which has the derivative

$$\dot{V}_1 = \boldsymbol{z}_1^T\left(\boldsymbol{R}(\psi)(\boldsymbol{z_2}+\boldsymbol{\alpha}) + \Phi_1\hat{\boldsymbol{\vartheta}} - \dot{r}_\eta\right) + \tilde{\boldsymbol{\vartheta}}^T\left(\Phi_1^T\boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{53}$$

$$\dot{V}_1 = \boldsymbol{z}_1^T\boldsymbol{R}(\psi)\boldsymbol{z_2} - \boldsymbol{z}_1^T\boldsymbol{K}_1\boldsymbol{z}_1 + + \tilde{\boldsymbol{\vartheta}}^T\left(\Phi_1^T\boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{54}$$

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \boldsymbol{z}_2^T\dot{\boldsymbol{z}}_2 && (55)\\
&= \dot{V}_1 + \boldsymbol{z}_2^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} - \boldsymbol{R}(\psi)^T\boldsymbol{S}(r)^T + \Phi_1\dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta + \boldsymbol{K}_1\dot{\boldsymbol{z}}_1\right) && (56)\\
&= \dot{V}_1 + \boldsymbol{z}_2^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} - \boldsymbol{R}(\psi)^T\boldsymbol{S}(r)^T + \Phi_1\dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta + \boldsymbol{K}_1\dot{\boldsymbol{z}}_1\right) + \boldsymbol{z}_2^T\boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\tilde{\boldsymbol{\vartheta}} && (57)\\
& && (58)
\end{aligned}$$

Choosing the control law

$$\boldsymbol{\tau} = -\Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} + M\left(-\boldsymbol{K}_2\boldsymbol{z}_2 - \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{R}(\psi)^T\boldsymbol{S}(r)^T - \Phi_1\dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1\dot{\boldsymbol{z}}_1 - \boldsymbol{z}_1^T\boldsymbol{R}(\psi)\boldsymbol{z_2}\right) \tag{59}$$

$$\dot{V} = -\boldsymbol{z}_1^T\boldsymbol{K}_1\boldsymbol{z}_1 - \boldsymbol{z}_2^T\boldsymbol{K}_2\boldsymbol{z}_2 + \tilde{\boldsymbol{\vartheta}}^T\left(\Phi_1^T\boldsymbol{z}_1 + \Phi_2(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-1}\boldsymbol{z}_2 + \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{60}$$

The adaptation law

$$\dot{\hat{\boldsymbol{\vartheta}}} = -\boldsymbol{\Gamma}\left(\Phi_1^T\boldsymbol{z}_1 + \Phi_2(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-1}\boldsymbol{z}_2\right) \tag{61}$$