

1 The system

The system is describe by

$$\begin{aligned}\dot{\phi} &= r \\ \dot{r} &= (r_{1x} + r_{2x}) \sin(\xi)(T_{nn}n^2 + T_{nv}V_A n) \\ \dot{\xi} &= \tau_\xi\end{aligned}$$

Assuming constant propeller speed

$$\begin{aligned}\dot{\phi} &= r \\ \dot{r} &= k_r \sin(\xi)(k_{nn} + k_{nv}V_A) \\ \dot{\xi} &= \tau_\xi\end{aligned}$$

Rewriting this

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1^T \vartheta \\ \dot{x}_2 &= k \sin(x_3) + \phi_2^T \vartheta \\ \dot{x}_3 &= u\end{aligned}$$

where $k = k_r + k_{nn}$. The control objective can then by formulated as a reference tracking of x_1 .

$$\begin{aligned}z_1 &\triangleq x_1 - r \\ z_2 &\triangleq x_2 - \alpha_1(x_1, \hat{\vartheta}) \\ z_3 &\triangleq x_3 - \alpha_2(x_1, x_2, \hat{\vartheta})\end{aligned}$$

1.1 Step 1

The dynamics of z_1 is

$$z_1 \triangleq x_1 - r \tag{1}$$

$$\dot{z}_1 = \dot{x}_1 - \dot{r} \tag{2}$$

$$= x_2 + \phi_1^T \vartheta - \dot{r} \tag{3}$$

$$= z_2 + \alpha_1(x_1, \hat{\vartheta}) + \phi_1^T \vartheta - \dot{r} \tag{4}$$

Considering x_2 as a virtual input we can find a stabilizing function for z_1

$$\alpha_1 = -k_1 z_1 - \phi_1^T \hat{\vartheta} + \dot{r} \tag{5}$$

If x_2 takes the value of α_1 we have

$$\dot{z}_1 = x_2 + \phi_1^T \vartheta - \dot{r} = -k_1 z_1 - \phi_1^T \hat{\vartheta} + \dot{r} + \phi_1^T \vartheta - \dot{r} = -k_1 z_1 - \phi_1^T \tilde{\vartheta} \tag{6}$$

As $\hat{\vartheta} = \vartheta + \tilde{\vartheta}$. We see that if the estimate $\hat{\vartheta}$ is perfect ($\tilde{\vartheta} = 0$), then z_1 will be GES for $k_1 > 0$. In practice we will of course never have a perfect estimate.

Choosing the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{\vartheta}^2 \quad (7)$$

$$\dot{V}_1 = z_1\dot{z}_1 + \frac{1}{\gamma}\tilde{\vartheta}\dot{\tilde{\vartheta}} \quad (8)$$

$$= z_1(z_2 + \alpha_1(x_1, \hat{\vartheta}) + \phi_1^T \vartheta - \dot{r}) + \frac{1}{\gamma}\tilde{\vartheta}\dot{\tilde{\vartheta}} \quad (9)$$

$$= z_1(z_2 - k_1 z_1 - \phi_1^T \hat{\vartheta} + \dot{r} + \phi_1^T \vartheta - \dot{r}) + \frac{1}{\gamma}\tilde{\vartheta}\dot{\tilde{\vartheta}} \quad (10)$$

$$= z_1 z_2 - k_1 z_1^2 - z_1 \phi_1^T \tilde{\vartheta} + \frac{1}{\gamma}\tilde{\vartheta}\dot{\tilde{\vartheta}} \quad (11)$$

$$= z_1 z_2 - k_1 z_1^2 + \tilde{\vartheta} \left(-z_1 \phi_1^T + \frac{1}{\gamma}\dot{\tilde{\vartheta}} \right) \quad (12)$$

Where the property $\dot{\tilde{\vartheta}} = \dot{\hat{\vartheta}}$ was used. This is due to ϑ being constant or slowly varying. We see that choosing the adaptation law

$$\dot{\hat{\vartheta}} = \gamma z_1 \phi_1^T \quad (13)$$

$$= \gamma(x_1 - r) \phi_1^T \quad (14)$$

$$= \gamma \tau_1(x_1) \quad (15)$$

Will make the last terms vanish.

1.2 Step 2

The dynamics of z_2 is

$$z_2 \triangleq x_2 - \alpha_1(x_1, \hat{\vartheta}) \quad (16)$$

$$\dot{z}_2 = \dot{x}_2 - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial \hat{\vartheta}} \dot{\hat{\vartheta}} \quad (17)$$

$$= kv + \phi_2^T \vartheta + k_1 \dot{x}_1 + \phi_1^T \dot{\hat{\vartheta}} \quad (18)$$

$$v = \frac{-\phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}} - k_2 z_2}{k} \quad (19)$$

$$\dot{z}_2 = -k_2 z_2 - \phi_2^T \tilde{\vartheta} \quad (20)$$

$$\dot{z}_2 = kv + \phi_2^T \vartheta + k_1 \dot{x}_1 + \phi_1^T \dot{\hat{\vartheta}} \quad (21)$$

$$= -\phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}} - k_2 z_2 + \phi_2^T \vartheta + k_1 \dot{x}_1 + \phi_1^T \dot{\hat{\vartheta}} \quad (22)$$

$$= -\phi_2^T \hat{\vartheta} - \phi_1^T \dot{\hat{\vartheta}} - k_2 z_2 + \phi_2^T \vartheta + \phi_1^T \dot{\hat{\vartheta}} \quad (23)$$

$$(24)$$

Choosing the Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2 \quad (25)$$

$$\dot{V}_2 = z_1 z_2 - k_1 z_1^2 + \tilde{\vartheta} \left(-\tau_1(x_1) + \frac{1}{\gamma} \dot{\hat{\vartheta}} \right) + z_2 \left(kv + \phi_2^T \vartheta + k_1 \dot{x}_1 + \phi_1^T \dot{\hat{\vartheta}} \right) \quad (26)$$

$$v = \frac{-k_2 z_2 - z_1 - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}}}{k} \quad (27)$$

$$\dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 + \tilde{\vartheta} \left(-\tau_1(x_1) + \frac{1}{\gamma} \dot{\hat{\vartheta}} - z_2 \phi_2^T \right) \quad (28)$$

We see that chosing the adaptation law

$$\dot{\hat{\vartheta}} = \gamma(z_1 \phi_1^T + z_2 \phi_2^T) \quad (29)$$

$$(30)$$

Will make the last terms vanish.

$$v = \frac{-k_2(x_2 - (-k_1 z_1 - \phi_1^T \hat{\vartheta} + \dot{r})) - (x_1 - r) - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}}}{k} \quad (31)$$

$$v = \frac{-k_2 x_2 - k_2 k_1 z_1 - k_2 \phi_1^T \hat{\vartheta} + k_2 \dot{r} - x_1 + r - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}}}{k} \quad (32)$$

$$v = \frac{-k_2 x_2 - k_2 k_1 x_1 + k_2 k_1 r - k_2 \phi_1^T \hat{\vartheta} + k_2 \dot{r} - x_1 + r - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}}}{k} \quad (33)$$

$$v = \frac{-k_2 x_2 + (r - x_1)(k_2 k_1 + 1) + k_2 \dot{r} - k_1 \dot{x}_1 - \phi_1^T (k_2 \hat{\vartheta} + \dot{\hat{\vartheta}}) - \phi_2^T \hat{\vartheta}}{k} \quad (34)$$

$$-1 \leq v \leq 1 \quad (35)$$

$$-1 \leq \frac{-k_2 z_2 - z_1 - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}}}{k} \leq 1 \quad (36)$$

$$-k \leq -k_2 z_2 - z_1 - \phi_2^T \hat{\vartheta} - k_1 \dot{x}_1 - \phi_1^T \dot{\hat{\vartheta}} \leq k \quad (37)$$