1 The system

The vessel is modelled by

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_{A}\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_{A}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \tag{1}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{2}$$

where

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad \boldsymbol{M}_{A} = \begin{bmatrix} -X_{\dot{u}} & 0 & 0 \\ 0 & -Y_{\dot{v}} & -Y_{\dot{r}} \\ 0 & -N_{\dot{v}} & -N_{\dot{r}} \end{bmatrix} \tag{3}$$

$$\boldsymbol{C}_{RB} = \begin{bmatrix} 0 & 0 & -m(x_g r + v) \\ 0 & 0 & mu \\ m(x_g r + v) & -mu & 0 \end{bmatrix} \quad \boldsymbol{C}_{A} = \begin{bmatrix} 0 & 0 & Y_{\dot{v}}v_r + \frac{1}{2}(N_{\dot{v}}Y_{\dot{r}})r \\ 0 & 0 & -X_{\dot{u}}u_r \\ -Y_{\dot{v}}v_r + \frac{1}{2}(N_{\dot{v}}Y_{\dot{r}})r & X_{\dot{u}}u_r & 0 \end{bmatrix} \tag{4}$$

$$\boldsymbol{D}_{L} = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix} \quad \boldsymbol{D}_{NL} = \begin{bmatrix} -X_{|u|u}|u| - X_{uuu}u^2 & 0 & 0 \\ 0 & -Y_{|v|v}|v_r| - Y_{|r|v}|r| & -Y_{|v|r}|v_r| - Y_{|r|r}|r| \\ 0 & -N_{|v|v}|v_r| - N_{|r|v}|r| & -N_{|v|r}|v_r| - N_{|r|r}|r| \end{bmatrix} \tag{5}$$

Using property 8.1 of [Fossen] we can define the dynamics using only the water relative velocity $\boldsymbol{\nu}_r$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = \boldsymbol{\tau} \tag{6}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{7}$$

Where $\boldsymbol{\nu}_c$ is the velocity of the water current in global coordinates. And

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_{A} \tag{8}$$

$$\boldsymbol{C}(\boldsymbol{\nu}_r) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) + \boldsymbol{C}_{A}(\boldsymbol{\nu}_r) \tag{9}$$

Additionally $\boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{0}$ as there exists no restoring forces in surge, sway and yaw. Isolating the acceleration $\dot{\boldsymbol{\nu}}_r$ in (6) we have

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r) \tag{10}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{11}$$

Assuming that $\boldsymbol{M}$ and $\boldsymbol{C}$ as well as the linear damping coefficients X_u and N_r are known.

$$\boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = -\boldsymbol{d}\boldsymbol{\nu}_r - \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} \tag{12}$$

The system equation can then be split into a known part and a unknown or uncertain part. This can be written as

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{d}\boldsymbol{\nu}_r + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{13}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{14}$$

Where

$$\boldsymbol{d} = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -N_r \end{bmatrix} \tag{15}$$

And the uncertain part is described by

$$\Phi(\boldsymbol{\nu}_r)^T = \begin{bmatrix} |u|u & 0 & 0 \\ u^3 & 0 & 0 \\ 0 & v & 0 \\ 0 & r & 0 \\ 0 & |v|v & 0 \\ 0 & |r|v & 0 \\ 0 & |v|r & 0 \\ 0 & |r|r & 0 \\ 0 & 0 & v \\ 0 & 0 & |v|v \\ 0 & 0 & |r|v \\ 0 & 0 & |v|r \\ 0 & 0 & |r|r \end{bmatrix} \qquad \boldsymbol{\vartheta} = \begin{bmatrix} X_{|u|u} \\ X_{uuu} \\ Y_v \\ Y_r \\ Y_{|v|v} \\ Y_{|r|v} \\ Y_{|v|r} \\ Y_{|r|r} \\ N_v \\ N_{|v|v} \\ N_{|r|v} \\ N_{|v|r} \\ N_{|r|r} \end{bmatrix} \tag{16}$$

Lastly the known part is collected in a function $\boldsymbol{f}(\boldsymbol{\nu}_r)$.

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}(\boldsymbol{d}\boldsymbol{\nu}_r - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{17}$$
$$= \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}) \tag{18}$$
$$\tag{19}$$

The system is now in a form where the known part $\boldsymbol{f}(\boldsymbol{\nu}_r)$, the uncertain part $\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}$ and the input $\boldsymbol{\tau}$ are clear to distinguish from one another.

2 Adaptive Velocity Control

2.1 Control Design

The control objective is to minimize $\tilde{\nu}(t) \triangleq \nu(t) - r_\nu(t)$. That is to minimize the difference (error) between the velocity of the vessel $\nu(t)$ and the reference velocity $r_\nu(t)$. A new variable $\boldsymbol{z}$ is introduced to describe this difference.

$$\boldsymbol{z} \triangleq \nu - r_\nu \tag{20}$$

Choosing the Lyapunov function

$$V \triangleq \frac{1}{2}\boldsymbol{z}^T\boldsymbol{z} + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}} \tag{21}$$

Which has the derivative

$$\dot{V} = \boldsymbol{z}^T\dot{\boldsymbol{z}} + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\vartheta}}} \tag{22}$$
$$= \boldsymbol{z}^T\left(\dot{\boldsymbol{\nu}}_r - \dot{r_\nu}\right) - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{23}$$
$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}\right) - \dot{r_\nu}\right) - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{24}$$
$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right) - \dot{r_\nu}\right) + \boldsymbol{z}^T\boldsymbol{M}^{-1}\Phi(\boldsymbol{\nu}_r)\tilde{\boldsymbol{\vartheta}} - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} \tag{25}$$
$$= \boldsymbol{z}^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right) - \dot{r_\nu}\right) + \tilde{\boldsymbol{\vartheta}}^T\left(\Phi(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-T}\boldsymbol{z} - \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{26}$$
$$\tag{27}$$

It is seen that choosing the control law

$$\boldsymbol{\tau} = \boldsymbol{M}\left(-\boldsymbol{K}\boldsymbol{z} - \boldsymbol{f}(\boldsymbol{\nu}_r) + \dot{r_\nu}\right) - \Phi(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} \tag{28}$$

And the adaptation law

$$\dot{\hat{\boldsymbol{\vartheta}}} = \boldsymbol{\Gamma}\Phi(\boldsymbol{\nu}_r)^T \boldsymbol{M}^{-T}\boldsymbol{z} \tag{29}$$

Will makes the Lyapunov function derivative become

$$\dot{V} = -\boldsymbol{z}^T\boldsymbol{K}\boldsymbol{z} \tag{30}$$

Which is negative semi-definite if $\boldsymbol{K}$ is positive definite.

Barbalat's lemma

3 Adaptive Position Control

3.1 The system

As in the previous section the system is

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} \tag{31}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{32}$$

where the known part $\boldsymbol{f}(\boldsymbol{\nu}_r)$, the uncertain part $\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}$ and the input $\boldsymbol{\tau}$ of the system is separated. Considering the water currents as a unknown quantity

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} \tag{33}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1\boldsymbol{\vartheta} \tag{34}$$

3.2 Control objective

The control objective is to minimize $\tilde{\eta}(t) \triangleq \eta(t) - r_\eta(t)$. That is to minimize the difference (error) between the position of the vessel $\eta(t)$ and the reference position $r_\eta(t)$. A new variable $\boldsymbol{z}$ is introduced to describe this difference.

$$\boldsymbol{z}_1 \triangleq \eta - r_\eta \tag{35}$$

$$\dot{\boldsymbol{z}}_1 = \dot{\boldsymbol{\eta}} - \dot{r}_\eta = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1\boldsymbol{\vartheta} - \dot{r}_\eta \tag{36}$$

Since $\dot{\boldsymbol{z}}_1$ does not contain the input $\boldsymbol{\tau}$ it is necessary to consider the velocity $\boldsymbol{\nu}_r$ as a virtual input α. This is done in order that backstep though the system equations until we find the real input $\boldsymbol{\tau}$. For now $\boldsymbol{\nu}_r$ is considered a virtual input, that is, a state the can be controlled to act as the input of a subsystem.

3.3 Step 1

Choosing the Lyapunov function

$$V \triangleq \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}} \tag{37}$$

Which has the derivative

$$\dot{V} = \boldsymbol{z}_1^T \dot{\boldsymbol{z}}_1 + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\tilde{\boldsymbol{\vartheta}}} \tag{38}$$

$$= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \boldsymbol{\vartheta} - \dot{r}_\eta \right) + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \tag{39}$$

$$= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta \right) + \boldsymbol{z}_1^T \Phi_1 \tilde{\boldsymbol{\vartheta}} + \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \tag{40}$$

$$= \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)\alpha + \Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta \right) + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \right) \tag{41}$$

It is seen that choosing the control law for the virtual input α

$$\boldsymbol{\alpha} = \boldsymbol{R}(\psi)^T \left(-\Phi_1 \hat{\boldsymbol{\vartheta}} + \dot{r_\eta} - \boldsymbol{K}_1 \boldsymbol{z}_1 \right) \tag{42}$$

$$= \boldsymbol{R}(\psi)^T \boldsymbol{f}_\alpha \tag{43}$$

Inserting $\boldsymbol{\alpha}$ in (41) we get

$$\dot{V}_1 = -\boldsymbol{z}_1^T \boldsymbol{K}_1 \boldsymbol{z}_1 + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \right) \tag{44}$$

And the adaptation law

$$\dot{\hat{\boldsymbol{\vartheta}}} = -\boldsymbol{\Gamma} \Phi_1^T \boldsymbol{z}_1 = -\boldsymbol{\Gamma} \tau(\boldsymbol{z}_1) \tag{45}$$

where $\tau(\boldsymbol{z}_1) = \Phi_1^T \boldsymbol{z}_1$ is a tuning function, that must not to be confused with the input $\boldsymbol{\tau}$.

3.4 Derivative of a rotation matrix

Consider the rotation matrix $R(\psi)$

$$\boldsymbol{R}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

And its derivative

$$\dot{\boldsymbol{R}}(\psi) = \frac{d}{d\psi} \boldsymbol{R}(\psi) \dot{\psi} = \frac{d}{d\psi} \boldsymbol{R}(\psi) r \tag{47}$$

$$\dot{\boldsymbol{R}}(\psi) = \begin{bmatrix} -\sin(\psi) r & -\cos(\psi) r & 0 \\ \cos(\psi) r & -\sin(\psi) r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

Which can be written as

$$\dot{\boldsymbol{R}}(\psi) = \boldsymbol{S}(r) \boldsymbol{R}(\psi) \tag{49}$$

With

$$\boldsymbol{S}(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{50}$$

This is easily verified by doing the matrix multiplication.

3.5 Step 2

Now we must ensure that $\boldsymbol{\nu}_r$ takes the value α. Thus the control objective of the second step is

$$\boldsymbol{z}_2 \triangleq \boldsymbol{\nu}_r - \boldsymbol{\alpha} \tag{51}$$

$$\dot{\boldsymbol{z}}_2 = \dot{\boldsymbol{\nu}}_r - \dot{\boldsymbol{\alpha}} \tag{52}$$

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T - \Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1 \tag{53}$$

$$\begin{aligned}
\dot{\boldsymbol{\alpha}} &= \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T \boldsymbol{f}_\alpha + \boldsymbol{R}(\psi)^T \dot{\boldsymbol{f}}_\alpha & (54)\\
&= \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T \boldsymbol{f}_\alpha + \boldsymbol{R}(\psi)^T \left(-\Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1\right) & (55)\\
&= \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T \boldsymbol{f}_\alpha + \boldsymbol{R}(\psi)^T \left(-\Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 (\boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1 \boldsymbol{\vartheta} - \dot{r}_\eta)\right) & (56)\\
&= \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T \boldsymbol{f}_\alpha + \boldsymbol{R}(\psi)^T \left(-\Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 \Phi_1 \boldsymbol{\vartheta} + \boldsymbol{K}_1 \dot{r}_\eta\right) + \boldsymbol{K}_1 \boldsymbol{\nu}_r & (57)
\end{aligned}$$

Augmenting the Lyapunov function with a term for $\boldsymbol{z}_2$ we have

$$V_2 \triangleq V_1 + \frac{1}{2} \boldsymbol{z}_2^T \boldsymbol{z}_2 \tag{58}$$

Which has the derivative

$$\dot{V}_1 = \boldsymbol{z}_1^T \left(\boldsymbol{R}(\psi)(\boldsymbol{z_2} + \boldsymbol{\alpha}) + \Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta\right) + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{59}$$

$$\dot{V}_1 = \boldsymbol{z}_1^T \boldsymbol{R}(\psi) \boldsymbol{z_2} - \boldsymbol{z}_1^T \boldsymbol{K}_1 \boldsymbol{z}_1 + + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{60}$$

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \boldsymbol{z}_2^T \dot{\boldsymbol{z}}_2 & (61)\\
&= \dot{V}_1 + \boldsymbol{z}_2^T \left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta} - \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T + \Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta + \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1\right) & (62)\\
&= \dot{V}_1 + \boldsymbol{z}_2^T \left(\boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{M}^{-1}\Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} - \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T + \Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta + \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1\right) + \boldsymbol{z}_2^T \boldsymbol{M}^{-1} \Phi_2(\boldsymbol{\nu}_r) \tilde{\boldsymbol{\vartheta}} & (63)\\
& & (64)
\end{aligned}$$

Choosing the control law

$$\boldsymbol{\tau} = -\Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} + M \left(-\boldsymbol{K}_2 \boldsymbol{z}_2 - \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T - \Phi_1 \dot{\hat{\boldsymbol{\vartheta}}} + \ddot{r}_\eta - \boldsymbol{K}_1 \dot{\boldsymbol{z}}_1 - \boldsymbol{z}_1^T \boldsymbol{R}(\psi) \boldsymbol{z_2}\right) \tag{65}$$

$$\dot{V} = -\boldsymbol{z}_1^T \boldsymbol{K}_1 \boldsymbol{z}_1 - \boldsymbol{z}_2^T \boldsymbol{K}_2 \boldsymbol{z}_2 + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{z}_1 + \Phi_2(\boldsymbol{\nu}_r)^T \boldsymbol{M}^{-1} \boldsymbol{z}_2 + \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{66}$$

The adaptation law

$$\dot{\hat{\boldsymbol{\vartheta}}} = -\boldsymbol{\Gamma} \left(\Phi_1^T \boldsymbol{z}_1 + \Phi_2(\boldsymbol{\nu}_r)^T \boldsymbol{M}^{-1} \boldsymbol{z}_2\right) \tag{67}$$

4 Adaptive Position Control 2

4.1 The System

As in the previous section the system is

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}\right) \tag{68}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c \tag{69}$$

where the known part $\boldsymbol{f}(\boldsymbol{\nu}_r)$, the uncertain part $\Phi(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}$ and the input $\boldsymbol{\tau}$ of the system is separated. Considering the water currents as a unknown quantity

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r)\boldsymbol{\vartheta}\right) \tag{70}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}_r + \Phi_1\boldsymbol{\vartheta} \tag{71}$$

4.2 Control objective

The control objective is to minimize $\tilde{\eta}(t) \triangleq \eta(t) - r_\eta(t)$. That is to minimize the difference (error) between the position of the vessel $\eta(t)$ and the reference position $r_\eta(t)$. A new variable $\boldsymbol{z}$ is introduced to describe this difference.

$$\boldsymbol{z}_1 \triangleq \boldsymbol{R}(\psi)^T(\eta - r_\eta) \tag{72}$$

$$\boldsymbol{z}_2 \triangleq \boldsymbol{\nu}_r - \boldsymbol{\alpha} \tag{73}$$

4.3 Step 1

Choosing the Lyapunov function

$$V_1 \triangleq \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2}\tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\vartheta}} \tag{74}$$

Which has the derivative

$$\dot{V}_1 = \boldsymbol{z}_1^T\dot{\boldsymbol{z}}_1 \tag{75}$$

We first have to find the derivative of $\boldsymbol{z}_1$

$$\begin{aligned}
\dot{\boldsymbol{z}}_1 &= \boldsymbol{R}(\psi)^T\boldsymbol{S}(r)^T(\eta - r_\eta) + \boldsymbol{R}(\psi)^T(\dot{\boldsymbol{\eta}} - \dot{r}_\eta) && (76)\\
&= \boldsymbol{S}(r)^T\boldsymbol{z}_1 + \boldsymbol{R}(\psi)^T(\dot{\boldsymbol{\eta}} - \dot{r}_\eta) && (77)\\
&= \boldsymbol{S}(r)^T\boldsymbol{z}_1 + \boldsymbol{\nu}_r + \boldsymbol{R}(\psi)^T(\Phi_1\boldsymbol{\vartheta} - \dot{r}_\eta) && (78)
\end{aligned}$$

where the property $\boldsymbol{R}(\psi)^T\boldsymbol{S}(r)^T = \boldsymbol{S}(r)^T\boldsymbol{R}(\psi)^T$ is used. Now we can find $\dot{V}_1$ and isolate the terms that depend on $\tilde{\boldsymbol{\vartheta}}$

$$\begin{aligned}
\dot{V}_1 &= \boldsymbol{z}_1^T\dot{\boldsymbol{z}}_1 + \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\vartheta}}} && (79)\\
&= \boldsymbol{z}_1^T\left(\boldsymbol{S}(r)^T\boldsymbol{z}_1 + \boldsymbol{\nu}_r + \boldsymbol{R}(\psi)^T(\Phi_1\boldsymbol{\vartheta} - \dot{r}_\eta)\right) - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} && (80)\\
&= \boldsymbol{z}_1^T\left(\boldsymbol{\nu}_r + \boldsymbol{R}(\psi)^T(\Phi_1\hat{\boldsymbol{\vartheta}} - \dot{r}_\eta)\right) + \boldsymbol{z}_1^T\boldsymbol{R}(\psi)^T\Phi_1\tilde{\boldsymbol{\vartheta}} - \tilde{\boldsymbol{\vartheta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}} && (81)\\
&= \boldsymbol{z}_1^T\left(\boldsymbol{\nu}_r + \boldsymbol{R}(\psi)^T(\Phi_1\hat{\boldsymbol{\vartheta}} - \dot{r}_\eta)\right) + \tilde{\boldsymbol{\vartheta}}^T\left(\Phi_1^T\boldsymbol{R}(\psi)\boldsymbol{z}_1 - \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) && (82)
\end{aligned}$$

It is seen that choosing the control law for the virtual input $\boldsymbol{\alpha}$

$$\boldsymbol{\alpha} = -\boldsymbol{K}_1\boldsymbol{z}_1 - \boldsymbol{R}(\psi)^T\left(\Phi_1\hat{\boldsymbol{\vartheta}} - \dot{r}_\eta\right) \tag{83}$$

will cancel all known terms and

Using 73, $\boldsymbol{\nu}_r$ is substituted by $\boldsymbol{z}_2 + \boldsymbol{\alpha}$, this changes $\dot{\boldsymbol{z}}_1$ to

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{S}(r)^T \boldsymbol{z}_1 + \boldsymbol{z}_2 + \boldsymbol{\alpha} + \boldsymbol{R}(\psi)^T (\Phi_1 \boldsymbol{\vartheta} - \dot{r}_\eta) \tag{84}$$

$$= \boldsymbol{S}(r)^T \boldsymbol{z}_1 - \boldsymbol{K}_1 \boldsymbol{z}_1 + \boldsymbol{z}_2 + \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} \tag{85}$$

and $\dot{V}_1$ to

$$\dot{V}_1 = \boldsymbol{z}_1^T \left(\boldsymbol{z}_2 + \boldsymbol{\alpha} + \boldsymbol{R}(\psi)^T (\Phi_1 \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta) \right) + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{R}(\psi) \boldsymbol{z}_1 - \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \right) \tag{86}$$

$$= \boldsymbol{z}_1^T \boldsymbol{z}_2 - \boldsymbol{z}_1^T \boldsymbol{K}_1 \boldsymbol{z}_1 + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{R}(\psi) \boldsymbol{z}_1 - \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \right) \tag{87}$$

$$\tag{88}$$

4.4 Step 2

$$V_2 \triangleq V_1 + \frac{1}{2} \boldsymbol{z}_2^T \boldsymbol{z}_2 \tag{89}$$

Which has the derivative

$$\dot{V}_2 = \dot{V}_1 + \boldsymbol{z}_2^T \dot{\boldsymbol{z}}_2 \tag{90}$$

In order to evaluate this. We first have to find the derivative of $\boldsymbol{z}_2$

$$\dot{\boldsymbol{z}}_2 = \dot{\boldsymbol{\nu}}_r - \dot{\boldsymbol{\alpha}} \tag{91}$$

$$\dot{\boldsymbol{\alpha}} = -\boldsymbol{K}_1 \dot{\boldsymbol{z}}_1 - \boldsymbol{R}(\psi)^T \boldsymbol{S}(r)^T (\Phi \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta) - \boldsymbol{R}(\psi)^T (\Phi \dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta) \tag{92}$$

$$= -\boldsymbol{K}_1 \left(\boldsymbol{S}(r)^T \boldsymbol{z}_1 - \boldsymbol{K}_1 \boldsymbol{z}_1 + \boldsymbol{z}_2 + \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} \right) - \boldsymbol{R}(\psi)^T \left(\boldsymbol{S}(r)^T (\Phi \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta) + \Phi \dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta \right) \tag{93}$$

$$= \boldsymbol{f}_\alpha - \boldsymbol{K}_1 \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} \tag{94}$$

With

$$\boldsymbol{f}_\alpha = -\boldsymbol{K}_1 \left(\boldsymbol{S}(r)^T \boldsymbol{z}_1 - \boldsymbol{K}_1 \boldsymbol{z}_1 + \boldsymbol{z}_2 \right) - \boldsymbol{R}(\psi)^T \left(\boldsymbol{S}(r)^T (\Phi \hat{\boldsymbol{\vartheta}} - \dot{r}_\eta) + \Phi \dot{\hat{\boldsymbol{\vartheta}}} - \ddot{r}_\eta \right) \tag{95}$$

Then we have

$$\dot{\boldsymbol{z}}_2 = \boldsymbol{f}(\boldsymbol{\nu}_r) + \boldsymbol{M}^{-1} \left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r) \boldsymbol{\vartheta} \right) - \boldsymbol{f}_\alpha + \boldsymbol{K}_1 \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} \tag{96}$$

Again we try to isolate the terms that depend on $\tilde{\boldsymbol{\vartheta}}$

$$\dot{\boldsymbol{z}}_2 = \boldsymbol{f}(\boldsymbol{\nu}_r) - \boldsymbol{f}_\alpha + \boldsymbol{M}^{-1} \left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r) \hat{\boldsymbol{\vartheta}} \right) + \boldsymbol{K}_1 \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} + \boldsymbol{M}^{-1} \Phi_2(\boldsymbol{\nu}_r) \tilde{\boldsymbol{\vartheta}} \tag{97}$$

We are now ready to find $\dot{V}_2$

$$\dot{V}_2 = \dot{V}_1 + \boldsymbol{z}_2^T \dot{\boldsymbol{z}}_2 \tag{98}$$

$$= \boldsymbol{z}_1^T \boldsymbol{z}_2 - \boldsymbol{z}_1^T \boldsymbol{K}_1 \boldsymbol{z}_1 + \tilde{\boldsymbol{\vartheta}}^T \left(\Phi_1^T \boldsymbol{R}(\psi) \boldsymbol{z}_1 - \boldsymbol{\Gamma}^{-1} \dot{\hat{\boldsymbol{\vartheta}}} \right) \tag{99}$$

$$+ \boldsymbol{z}_2^T \left(\boldsymbol{f}(\boldsymbol{\nu}_r) - \boldsymbol{f}_\alpha + \boldsymbol{M}^{-1} \left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r) \hat{\boldsymbol{\vartheta}} \right) + \boldsymbol{K}_1 \boldsymbol{R}(\psi)^T \Phi_1 \tilde{\boldsymbol{\vartheta}} + \boldsymbol{M}^{-1} \Phi_2(\boldsymbol{\nu}_r) \tilde{\boldsymbol{\vartheta}} \right) \tag{100}$$

Factoring out $\tilde{\boldsymbol{\vartheta}}^T$ from all terms depending on the error, we get

$$\dot{V}_2 = \boldsymbol{z}_1^T\boldsymbol{z}_2 - \boldsymbol{z}_1^T\boldsymbol{K}_1\boldsymbol{z}_1 + \boldsymbol{z}_2^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) - \boldsymbol{f}_\alpha + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right)\right) \tag{101}$$

$$+ \tilde{\boldsymbol{\vartheta}}^T\left(\Phi_2(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-1}\boldsymbol{z}_2 + \Phi_1^T\boldsymbol{R}(\psi)\boldsymbol{K}_1^T\boldsymbol{z}_2 + \Phi_1^T\boldsymbol{R}(\psi)\boldsymbol{z}_1 - \boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\vartheta}}}\right) \tag{102}$$

Choosing the adaptation law to be

$$\dot{\hat{\boldsymbol{\vartheta}}} = \boldsymbol{\Gamma}\left(\Phi_2(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-1}\boldsymbol{z}_2 + \Phi_1^T\boldsymbol{R}(\psi)\boldsymbol{K}_1^T\boldsymbol{z}_2 + \Phi_1^T\boldsymbol{R}(\psi)\boldsymbol{z}_1\right) \tag{103}$$

$$= \boldsymbol{\Gamma}\left(\Phi_1^T\boldsymbol{R}(\psi)(\boldsymbol{z}_1 + \boldsymbol{K}_1^T\boldsymbol{z}_2) + \Phi_2(\boldsymbol{\nu}_r)^T\boldsymbol{M}^{-1}\boldsymbol{z}_2\right) \tag{104}$$

The Lyapunov function derivative becomes

$$\dot{V}_2 = \boldsymbol{z}_1^T\boldsymbol{z}_2 - \boldsymbol{z}_1^T\boldsymbol{K}_1\boldsymbol{z}_1 + \boldsymbol{z}_2^T\left(\boldsymbol{f}(\boldsymbol{\nu}_r) - \boldsymbol{f}_\alpha + \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}}\right)\right) \tag{105}$$

And finally, choosing the control law as

$$\boldsymbol{\tau} = \boldsymbol{M}\left(-\boldsymbol{z}_1 - \boldsymbol{K}_2\boldsymbol{z}_2 + \boldsymbol{f}_\alpha - \boldsymbol{f}(\boldsymbol{\nu}_r)\right) - \Phi_2(\boldsymbol{\nu}_r)\hat{\boldsymbol{\vartheta}} \tag{106}$$

Changing the Lyapunov function derivative to

$$\dot{V}_2 = -\boldsymbol{z}_1^T\boldsymbol{K}_1\boldsymbol{z}_1 - \boldsymbol{z}_2^T\boldsymbol{K}_2\boldsymbol{z}_2 \tag{107}$$