1 The system

The system is describe by

$$
\begin{aligned}
\dot{\phi} &= r \\
\dot{r} &= (r_{1x} + r_{2x})\sin(\xi)(T_{nn}n^2 + T_{nv}V_A n) \\
\dot{\xi} &= \tau_\xi
\end{aligned}
$$

Assuming constant propeller speed

$$
\begin{aligned}
\dot{\phi} &= r \\
\dot{r} &= k_r \sin(\xi)(k_{nn} + k_{nv}V_A) \\
\dot{\xi} &= \tau_\xi
\end{aligned}
$$

Rewriting this

$$
\begin{aligned}
\dot{x_1} &= x_2 + \boldsymbol{\phi}_1^T \vartheta \\
\dot{x_2} &= k\sin(x_3) + \boldsymbol{\phi}_2^T \vartheta \\
\dot{x_3} &= u
\end{aligned}
$$

where $k = k_r + k_{nn}$. The control objective can then by formulated as a reference tracking of x_1.

$$
\begin{aligned}
z_1 &\triangleq x_1 - r \\
z_2 &\triangleq x_2 - \alpha_1(x_1, \hat{\vartheta}) \\
z_3 &\triangleq x_3 - \alpha_2(x_1, x_2, \hat{\vartheta})
\end{aligned}
$$

1.1 Step 1

The dynamics of z_1 is

$$
\begin{aligned}
z_1 &\triangleq x_1 - r && (1) \\
\dot{z_1} &= \dot{x_1} - \dot{r} && (2) \\
&= x_2 + \boldsymbol{\phi}_1^T \vartheta - \dot{r} && (3) \\
&= z_2 + \alpha_1(x_1, \hat{\vartheta}) + \boldsymbol{\phi}_1^T \vartheta - \dot{r} && (4)
\end{aligned}
$$

Considering x_2 as a virtual input we can find a stabilizing function for z_1

$$\alpha_1 = -k_1 z_1 - \boldsymbol{\phi}_1^T \hat{\vartheta} + \dot{r} \tag{5}$$

If x_2 takes the value of α_1 we have

$$\dot{z_1} = x_2 + \boldsymbol{\phi}_1^T \vartheta - \dot{r} = -k_1 z_1 - \boldsymbol{\phi}_1^T \hat{\vartheta} + \dot{r} + \boldsymbol{\phi}_1^T \vartheta - \dot{r} = -k_1 z_1 - \boldsymbol{\phi}_1^T \tilde{\vartheta} \tag{6}$$

As $\hat{\vartheta} = \vartheta + \tilde{\vartheta}$. We see that if the estimate $\hat{\vartheta}$ is perfect ($\tilde{\vartheta} = 0$), then z_1 will be GES for $k_1 > 0$. In practice we will of course never have a perfect estimate.

Choosing the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{\vartheta}^2 \tag{7}$$

$$\dot{V}_1 = z_1\dot{z}_1 + \frac{1}{\gamma}\tilde{\vartheta}\dot{\tilde{\vartheta}} \tag{8}$$

$$= z_1(z_2 + \alpha_1(x_1, \hat{\vartheta}) + \boldsymbol{\phi}_1^T\vartheta - \dot{r}) + \frac{1}{\gamma}\tilde{\vartheta}\dot{\hat{\vartheta}} \tag{9}$$

$$= z_1(z_2 - k_1z_1 - \boldsymbol{\phi}_1^T\hat{\vartheta} + \dot{r} + \boldsymbol{\phi}_1^T\vartheta - \dot{r}) + \frac{1}{\gamma}\tilde{\vartheta}\dot{\hat{\vartheta}} \tag{10}$$

$$= z_1z_2 - k_1z_1^2 - z_1\boldsymbol{\phi}_1^T\tilde{\vartheta} + \frac{1}{\gamma}\tilde{\vartheta}\dot{\hat{\vartheta}} \tag{11}$$

$$= z_1z_2 - k_1z_1^2 + \tilde{\vartheta}\left(-z_1\boldsymbol{\phi}_1^T + \frac{1}{\gamma}\dot{\hat{\vartheta}}\right) \tag{12}$$

Where the property $\dot{\tilde{\vartheta}} = \dot{\hat{\vartheta}}$ was used. This is due to ϑ being constant or slowly varying. We see that chosing the adaptation law

$$\dot{\hat{\vartheta}} = \gamma z_1\boldsymbol{\phi}_1^T \tag{13}$$

$$= \gamma(x_1 - r)\boldsymbol{\phi}_1^T \tag{14}$$

$$= \gamma\tau_1(x_1) \tag{15}$$

Will make the last terms vanish.

1.2 Step 2

The dynamics of z_2 is

$$z_2 \triangleq x_2 - \alpha_1(x_1, \hat{\vartheta}) \tag{16}$$

$$\dot{z}_2 = \dot{x}_2 - \frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial\alpha_1}{\partial\hat{\vartheta}}\dot{\hat{\vartheta}} \tag{17}$$

$$= kv + \boldsymbol{\phi}_2^T\vartheta + k_1\dot{x}_1 + \phi_1^T\dot{\hat{\vartheta}} \tag{18}$$

$$v = \frac{-\boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x}_1 - \phi_1^T\dot{\hat{\vartheta}} - k_2z_2}{k} \tag{19}$$

$$\dot{z}_2 = -k_2z_2 - \boldsymbol{\phi}_2^T\tilde{\vartheta} \tag{20}$$

$$\dot{z}_2 = kv + \boldsymbol{\phi}_2^T\vartheta + k_1\dot{x}_1 + \phi_1^T\dot{\hat{\vartheta}} \tag{21}$$

$$= -\boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x}_1 - \phi_1^T\dot{\hat{\vartheta}} - k_2z_2 + \boldsymbol{\phi}_2^T\vartheta + k_1\dot{x}_1 + \phi_1^T\dot{\hat{\vartheta}} \tag{22}$$

$$= -\boldsymbol{\phi}_2^T\hat{\vartheta} - \phi_1^T\dot{\hat{\vartheta}} - k_2z_2 + \boldsymbol{\phi}_2^T\vartheta + \phi_1^T\dot{\hat{\vartheta}} \tag{23}$$

$$\tag{24}$$

Choosing the Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2 \tag{25}$$

$$\dot{V_2} = z_1 z_2 - k_1 z_1^2 + \tilde{\vartheta}\left(-\tau_1(x_1) + \frac{1}{\gamma}\dot{\hat{\vartheta}}\right) + z_2\left(kv + \boldsymbol{\phi}_2^T\vartheta + k_1\dot{x_1} + \phi_1^T\dot{\hat{\vartheta}}\right) \tag{26}$$

$$v = \frac{-k_2 z_2 - z_1 - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}}}{k} \tag{27}$$

$$\dot{V_2} = -k_1 z_1^2 - k_2 z_2^2 + \tilde{\vartheta}\left(-\tau_1(x_1) + \frac{1}{\gamma}\dot{\hat{\vartheta}} - z_2\boldsymbol{\phi}_2^T\right) \tag{28}$$

We see that chosing the adaptation law

$$\dot{\hat{\vartheta}} = \gamma(z_1\boldsymbol{\phi}_1^T + z_2\boldsymbol{\phi}_2^T) \tag{29}$$

$$\tag{30}$$

Will make the last terms vanish.

$$v = \frac{-k_2(x_2 - (-k_1 z_1 - \boldsymbol{\phi}_1^T\hat{\vartheta} + \dot{r})) - (x_1 - r) - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}}}{k} \tag{31}$$

$$v = \frac{-k_2 x_2 - k_2 k_1 z_1 - k_2\boldsymbol{\phi}_1^T\hat{\vartheta} + k_2\dot{r} - x_1 + r - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}}}{k} \tag{32}$$

$$v = \frac{-k_2 x_2 - k_2 k_1 x_1 + k_2 k_1 r - k_2\boldsymbol{\phi}_1^T\hat{\vartheta} + k_2\dot{r} - x_1 + r - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}}}{k} \tag{33}$$

$$v = \frac{-k_2 x_2 + (r - x_1)(k_2 k_1 + 1) + k_2\dot{r} - k_1\dot{x_1} - \phi_1^T(k_2\hat{\vartheta} + \dot{\hat{\vartheta}}) - \boldsymbol{\phi}_2^T\hat{\vartheta}}{k} \tag{34}$$

$$-1 \leq v \leq 1 \tag{35}$$

$$-1 \leq \frac{-k_2 z_2 - z_1 - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}}}{k} \leq 1 \tag{36}$$

$$-k \leq -k_2 z_2 - z_1 - \boldsymbol{\phi}_2^T\hat{\vartheta} - k_1\dot{x_1} - \phi_1^T\dot{\hat{\vartheta}} \leq k \tag{37}$$