1 Linear Control of the Otter

This document describe the steps taken in order to go from a nonlinear 6 DOF model of the Otter to a linear 3 DOF model. The 3 DOF model is then used to control the Otter.

2 The Otter

The Otter is a small unmanned surface vehicle (USV) made by Maritime Robotics. It is a twin-hull vessel measuring 200cm x 108cm x 81.5cm and weighing 55 kg. The Otter is used as a the subject of control in this project. The main reason for using the otter is that differential equations of the vessel could be obtained from the MSS toolbox, a toolbox written by Thor I. Fossen. These equations describe the movement of the vessel in 6 degrees of freedom (6 DOF). In control they are more commonly know as the system equations, and obtaining these is fundamental to control. The system equations of a vessel like the otter can be very cumbersome to obtain as one has to account for the hydrodynamics that influence the movement of the vessel.

Figure 1: The Otter ASV

2.1 The system equations

The dynamic of a vessel can be described as follows, using a 6 DOF model.

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \tag{1}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{2}$$

where ν is the velocities and η is the positions. This description includes a velocity and a position in each of the 6 modes. The modes are translation in x, y and z and rotation around x, y and z. ν_r is the relative speed of the vessel to the water.

In the model from the MSS toolbox the system equations are given as

$$\dot{\nu} = \frac{\tau + \tau_{damp} + \tau_{crossflow} - C\nu_r - G\eta - g_0}{M} \tag{3}$$

$$\dot{\eta} = J(\eta)\nu \tag{4}$$

Expanding (3) we get a set of 6 coupled differential equations. The equations for surge speed u, sway speed v and yaw rate r is shown here.

$$\begin{aligned}\dot{u} = 0.0054\,\tau_q - 1.3169\,q + 0.0175\,\tau_u + 0.00053523\,\tau_w - 16.0798\,\theta \\ - 1.3574\,u - 0.2925\,w - 12.1092\,z + 0.3371\,p\,r - 0.0147\,p\,v - 0.0265\,q\,u - 1.8664\,q\,w \\ + 2.3477\,r\,v + 0.2649\,u\,w + 0.0177\,p^2 + 0.1866\,q^2 + 0.1690\,r^2\end{aligned} \quad (5)$$

$$\begin{aligned}\dot{v} = 0.2504\,p + 4.9601\,\phi + 0.0667\,r - 0.0046\,\tau_p - 0.0014\,\tau_r + 0.0078\,\tau_v \\ - 0.00059306\,v - 0.0364\,p\,q + 0.1167\,q\,r + 0.7867\,p\,w - 0.4028\,r\,u - 0.1266\,v\,w + 0.3298\,r\,|r| \\ + 0.1143\,r\,|v| - 0.9463\,v\,|v|\end{aligned} \quad (6)$$

$$\begin{aligned}\dot{r} = 0.2696\,p + 5.3406\,\phi - 1.0413\,r - 0.0050\,\tau_p + 0.0229\,\tau_r - 0.0014\,\tau_v \\ + 0.0013\,v - 0.4188\,p\,q + 0.1257\,q\,r + 0.0395\,p\,w - 0.1107\,r\,u - 0.1363\,v\,w - 10.3735\,r\,|r| \\ - 2.0239\,r\,|v| + 0.1699\,v\,|v|\end{aligned} \quad (7)$$

3 Modelling

3.1 3 DOF

Making a 3 DOF model using only surge, sway and yaw, is simply a matter of discarding the 3 differential equations that describe the behaviour of the other modes. In the remaining 3 differential equations heave, roll and pitch is assumed to be zero. That is $w = 0$, $p = 0$ and $q = 0$. Doing this yields

$$\dot{u} = 0.1690\,r^2 + 2.3477\,v\,r + 0.0175\,\tau_u - 1.3574\,u$$

$$\dot{v} = 0.0736\,r - 0.0014\,\tau_r + 0.0078\,\tau_v - 0.0949\,v - 0.4028\,r\,u + 0.6506\,r\,|r|$$

$$\dot{r} = 0.0229\,\tau_r - 1.1746\,r - 0.0014\,\tau_v + 0.0176\,v - 0.1107\,r\,u - 10.3762\,r\,|r|$$

The same is done for $\dot{\eta}$, resulting in

$$\dot{x} = u\,\cos(\psi) - v\,\sin(\psi)$$

$$\dot{y} = v\,\cos(\psi) + u\,\sin(\psi)$$

$$\dot{\psi} = r$$

This clearly describe a simple transformation of the velocities. The transformation is a rotation around the z-axis by ψ. The system equations of the 3 DOF model is then define as

$$\boldsymbol{f}_\nu = \dot{\boldsymbol{\nu}}(\boldsymbol{x}) = \begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{r}\end{bmatrix} \qquad \boldsymbol{f}_\eta = \dot{\boldsymbol{\eta}}(\boldsymbol{x}) = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\phi}\end{bmatrix} \quad (8)$$

3.2 State space representation

Having the system equations, the state space representation of the velocities ν

$$\dot{\boldsymbol{\nu}} = \boldsymbol{A}\boldsymbol{\nu} + \boldsymbol{B}\boldsymbol{\tau} \tag{9}$$

can be made by calculating the following Jacobian matrices

$$\boldsymbol{A} = \frac{d\boldsymbol{f}_\nu}{d\boldsymbol{\nu}} \qquad \boldsymbol{B} = \frac{d\boldsymbol{f}_\nu}{d\boldsymbol{\tau}} \tag{10}$$

And evaluating them at a certain linearization point. The Jacobian matrices are

$$\boldsymbol{A} = \begin{bmatrix} -1.3574 & 2.3477\,r & 0.3379\,r + 2.3477\,v \\ -0.4028\,r & -0.0949 & 1.3012\,|r| - 0.4028\,u + 0.0736 \\ -0.1107\,r & 0.0176 & -0.1107\,u - 20.7524\,|r| - 1.1746 \end{bmatrix} \tag{11}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.0175 & 0 & 0 \\ 0 & 0.0078 & -0.0014 \\ 0 & -0.0014 & 0.0229 \end{bmatrix} \tag{12}$$

If we choose the linearization point $u = 3\,\text{kn}$, $v = 0$ and $r = 0$, which is a simple forward movement. The state-space representation becomes

$$\boldsymbol{A} = \begin{bmatrix} -1.3574 & 0 & 0 \\ 0 & -0.0949 & -0.5480 \\ 0 & 0.0176 & -1.3454 \end{bmatrix} \tag{13}$$

The matrix $\boldsymbol{B}$ does of course not change, as it is not depending on the state.

4 Control

4.1 Control Allocation

Before designing a controller we need to address a problem with the linear state space model. This model takes a vector of control forces τ as input. Making a controller for this system will result in a controller that provides τ as the input to the Otter. However the Otter will need to know to how position its propellers and fast they should spin, not what the resulting force should be.

4.1.1 Dynamics of the propellers

First we can split τ into linear force and torque.

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_{linear} \\ \boldsymbol{\tau}_{torque} \end{bmatrix} \tag{14}$$

If we define the force T provided by a propeller as a vector in the vessels coordinate frame (surge, sway, heave).

$$\boldsymbol{T} = \begin{bmatrix} F_u \\ F_v \\ F_w \end{bmatrix} \tag{15}$$

The linear control force is found as the sum of forces from each propeller p.

$$\boldsymbol{\tau}_{linear} = \sum^{P} \boldsymbol{T}_p \tag{16}$$

While the torque of each propeller can be found as the cross product between the force $\boldsymbol{T}_p$ and the position at which it acts $\boldsymbol{r}_p$. The total torque is then the sum of torques from the propellers.

$$\boldsymbol{\tau}_{torque} = \sum^{P} \boldsymbol{r}_p \times \boldsymbol{T}_p = \sum^{P} S(\boldsymbol{r}_p)\boldsymbol{T}_p \tag{17}$$

$\boldsymbol{r}_p$ is simply the position vector of thruster p and $S(\boldsymbol{r}_p)$ is the skew-symetric matrix of vector $\boldsymbol{r}_p$ which simplify the cross product operation. In the case of two propellers we have

$$\boldsymbol{\tau}_{linear} = \boldsymbol{T}_1 + \boldsymbol{T}_2 \tag{18}$$

$$\boldsymbol{\tau}_{torque} = S(\boldsymbol{r}_1)\boldsymbol{T}_1 + S(\boldsymbol{r}_2)\boldsymbol{T}_2 \tag{19}$$

Which can be written as

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_{linear} \\ \boldsymbol{\tau}_{torque} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ S(\boldsymbol{r}_1) & S(\boldsymbol{r}_2) \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} \tag{20}$$

Each propeller can rotate with an angle ξ_p. The thrust of each propeller can be describe as a vector by splitting it into components that align with the vessels coordinate frame

$$\boldsymbol{T}_p = \begin{bmatrix} \cos(\xi_p) \\ \sin(\xi_p) \\ 0 \end{bmatrix} t_p \tag{21}$$

where t_p is the magnitude of the thrust from each propeller and $\boldsymbol{T}_p$ is the resulting force vector. This is done under the assumption that the propellers are only able to move in the horizontal plane, hence $F_w = 0$. If roll and/or pitch is included in the model, this should be adjusted accordingly. The last thing that is needed is the relationship between the thrust and the revolutions of the propeller. From [Mogens] this relationship is

$$t_p = T_{nn} n_p^2 + T_{nv} V_A n_p \quad , \quad T_{nn} > 0 > T_{nv} \tag{22}$$

Where T_{nn} and T_{nv} are scalar constants, n_p is the revolutions of the propellers and V_A is the apparent water velocity at the location of the propeller. T_{nn} and T_{nv} can be estimated from the psysical dimensions of the propeller.

4.1.2 Control allocation

The problem at hand is to find $\boldsymbol{\xi}$ and $\boldsymbol{n}$ given $\boldsymbol{\tau}$. First we can find the individual force vectors of each propeller from (20)

$$\begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ S(\boldsymbol{r}_1) & S(\boldsymbol{r}_2) \end{bmatrix}^{+} \boldsymbol{\tau} \tag{23}$$

Where $+$ denotes the pseudo inverse. Then, for each of the propellers, the angle and the magnitude of the thrust is found as

$$\xi_p = \text{atan2}(F_v, F_u) \tag{24}$$

$$t_p = \|\boldsymbol{T}_p\| \tag{25}$$

This will produce $\xi_p \in [\pi; -\pi)$. However the propeller is limited to $\xi_p \in [\frac{\pi}{2}; -\frac{\pi}{2})$, In the case where ξ_p is outside this interval we can simply turn it by π and flip the sign on t_p. This will happen when a force in the aft direction is required e.g. when the vessel is moving forwards and propellers need to stop it. Turning the propeller by π and flipping the sign on t_p is analogous to putting the propellers in reverse. Finally the required revolutions can by calculated as

$$n_p = \text{sgn}(t_p)\frac{\sqrt{Tnv^2 V_A^2 + 4T_{nn}|t_p|} - T_{nv}V_A}{2T_{nn}} \tag{26}$$

This expression is found by solving (22) for n_p. Notice that the term under the square root will always be positive as $T_{nn} > 0$. Finally we can stack the result in vectors.

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \boldsymbol{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \tag{27}$$

This

4.2 Kinematics

The dynamic of a vessel can be described as follows, using a 6 DOF model.

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_A(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \tag{28}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{29}$$

where ν is the velocities and η is the positions. This description includes a velocity and a position in each of the 6 modes. The modes are translation in x, y and z and rotation around x, y and z. ν_r is the relative speed of the vessel to the water

4.3 State space

Assuming $\nu = \nu_r$ (no movement of the water relative to the seabed), we can define

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A \tag{30}$$

$$\boldsymbol{C}(\boldsymbol{\nu}) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}) + \boldsymbol{C}_A(\boldsymbol{\nu}) \tag{31}$$

And write the system as

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \tag{32}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{33}$$

Solving for $\dot{\boldsymbol{\nu}}$ we get

$$\dot{\boldsymbol{\nu}} = \boldsymbol{A}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{G}\boldsymbol{\eta} + \boldsymbol{B}\boldsymbol{\tau} \tag{34}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \tag{35}$$

Where

$$\boldsymbol{A}(\boldsymbol{\nu}) = -\boldsymbol{M}^{-1}(\boldsymbol{C}(\boldsymbol{\nu}) + \boldsymbol{D}(\boldsymbol{\nu})) \tag{36}$$

$$\boldsymbol{B} = \boldsymbol{M}^{-1} \tag{37}$$

Writing (34) and (35) in matrix form gives us a state space description of the whole system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{A}(\boldsymbol{\nu}) & \boldsymbol{G} \\ \boldsymbol{J}(\eta) & \boldsymbol{0} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{\tau} \tag{38}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\nu} \\ \boldsymbol{\eta} \end{bmatrix} \tag{39}$$

$$\boldsymbol{\nu} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \qquad \boldsymbol{\eta} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix} \tag{40}$$

5 Measurements

The vessel is fitted with a satellite compass and GPS like the one seen in figure 2. This device can provide the following data

SoG	Speed over Ground also know as track
CoG	Course over Ground
RoT	Rate of Turn (in yaw)
Pos	Position of vessel in x and y
HDT	Heading of vessel

Figure 2: Furuno SC70. Differential GPS (DGPS) and compass

Assuming normal noise distribution the measurement equations can be written as

$$
\begin{aligned}
HDT &= \psi + w && (41)\\
Pos &= \begin{bmatrix} x \\ y \end{bmatrix} + w && (42)\\
SoG &= \left| \begin{bmatrix} u \\ v \end{bmatrix} \right| + w && (43)\\
CoG &= atan2(v, u) + w && (44)\\
RoT &= r + w && (45)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{C}_m \boldsymbol{x} && (46)\\
\boldsymbol{C}_m &= \boldsymbol{I} && (47)
\end{aligned}
$$

6 Control

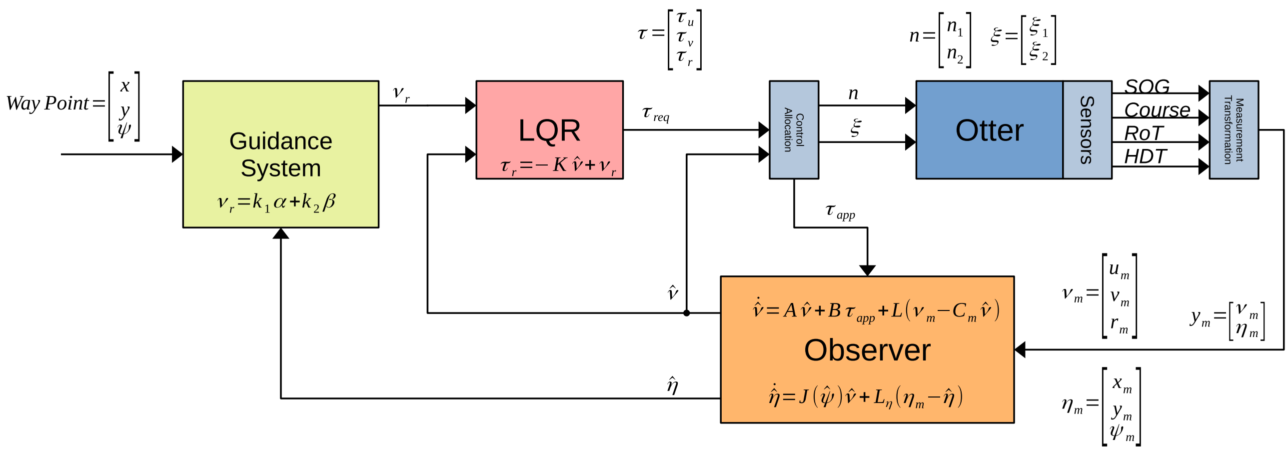

Figure 3:

7 Observer

7.1 Ordinary Kalman filter

7.2 Adaptive Kalman filter

Given the input u and the measurement y_m Data update

$$
\begin{aligned}
S_n &= C_m P C_m^T + R_{vv}\\
\kappa &= P C_m^T S_n^{-1}\\
\hat{y} &= C_m \hat{x}\\
\epsilon &= y_m - \hat{y}\\
\hat{x} &= \hat{x} + \kappa\epsilon\\
P &= P - \kappa S_n \kappa^T
\end{aligned}
$$

Time update

$$x = Ax + Bu$$
$$P = APA' + R_{ww}$$

8 Water current

$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$

$$\begin{aligned}\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}} - \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_c + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu} - \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_c \\ + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu} - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_c + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t)\end{aligned} \quad (48)$$

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \quad (49)$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \quad (50)$$

Using Property 8.1 of Fossen

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r \quad (51)$$

We can rewrite the system using only the relative velocity

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \quad (52)$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu} \quad (53)$$

$$\boldsymbol{J}(\eta)\boldsymbol{\nu}_c^l = \boldsymbol{\nu}_c^g \quad (54)$$

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{M}_A\dot{\boldsymbol{\nu}}_{\boldsymbol{r}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau} + \boldsymbol{w}(t) \quad (55)$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\eta)\boldsymbol{\nu}_r + \boldsymbol{\nu}_c^g \quad (56)$$