Solutions to Exercises in Steve Awodey's Book

Andreas Lynge

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Contents

Exercise 4.5.1 Regarding a group G as a category with one object and every arrow an isomorphism:

(a) Show that a categorical congruence $\sim$ on G is the same thing as (the equivalence relation on G determined by) a normal subgroup $N \subseteq G$, that is, show that the two kinds of things are in isomorphic correspondence.

(b) Show further that the quotient category $G/\sim$ and the factor group G/N coincide.

(c) Conclude that the homomorphism theorem for groups is a special case of the one for categories.

Solution. To solve this exercise I will make use of page 81 [1], where Awodey defines an equivalence relation $\sim_N$, determined by a normal subgroup $N \subseteq G$, such that

$$g \sim_N h \qquad \text{iff} \qquad g \cdot h^{-1} \in N \qquad\qquad \text{for } g, h \in G \tag{1}$$

I will begin with part (a). First, let $N \subseteq G$ be a normal subgroup in order to show that $\sim_N$ is a categorical congruence. If $r, s \in G$ and $r \sim_N s$ then $r \cdot s^{-1} \in N$, and since N is normal we have

$$(g \cdot r \cdot h) \cdot (h^{-1} \cdot s^{-1} \cdot g^{-1}) = g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N \qquad \text{for } g, h \in G$$

The above equation shows that

$$g \cdot r \cdot h \sim (h^{-1} \cdot s^{-1} \cdot g^{-1})^{-1} = g \cdot s \cdot h$$

and $\sim_N$ is therefore a categorical congruence. On the other hand, let $\sim$ be a categorical congruence on G regarded as a category. The goal is now to define a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying

$$g \sim_N h \qquad \text{iff} \qquad g \sim h \qquad\qquad \text{for } g, h \in G \tag{2}$$

Given definition (1), $N := \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$ seems to be a natural candidate. If $g \sim h$ then $g \cdot h^{-1} \in N$ and thus $g \sim_N h$ by definition (1). If $g \sim_N h$ then $g \cdot h^{-1} \in N$ so there exist $r, s \in G$ where $g \cdot h^{-1} = r \cdot s^{-1}$ and $r \sim s$. Since $\sim$ is a categorical congruence we get $g \cdot h^{-1} = r \cdot s^{-1} \sim s \cdot s^{-1} = u$, implying $g \sim h$. This shows that equation (2) is satisfied. It remains to show that $N \subseteq G$ is a normal subgroup. Let $g \cdot h^{-1} \in N$ and $r \cdot s^{-1} \in N$ such that $g \sim h$ and $r \sim s$, and we get

$$g \cdot h^{-1} \sim h \cdot h^{-1} = u = r \cdot r^{-1} \sim s \cdot r^{-1}$$

Transitivity implies that $(g \cdot h^{-1}) \cdot (s \cdot r^{-1})^{-1} = (g \cdot h^{-1}) \cdot (r \cdot s^{-1}) \in N$ and since $u \in N$ by reflexivity and N is closed under inverse by symmetry, N is a subgroup. Now, let $r \cdot s^{-1} \in N$ and $g \in G$. It follows that $g \cdot r \sim (s^{-1} g^{-1})^{-1} = g \cdot s$ because $r \sim s$ and $\sim$ is a congruence, hence $g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N$ and N is normal.

When I looked at the solution on page page 288 I discovered that Awodey defines the normal subgroup as $\overline{N} := \{g \in G \mid g \sim u\}$. Luckily, it turns out that $\overline{N} = N$, as I will show. If $g \in \overline{N}$ then $g \sim u$, so $g = g \cdot u^{-1} \in N$. If $r \cdot s^{-1} \in N$ then $r \sim s$, so $r \cdot s^{-1} \sim u$ and $r \cdot s^{-1} \in \overline{N}$ as required.

I will continue with part (b). Let $N \subseteq G$ be a normal subgroup determining equivalence relation $\sim_N$. By part (a), $g \sim h := g \sim_N h$ is a categorical congruence, hence $G/N = G/\sim$ by definition on page 81 and 84 [1] respectively. In the other direction.

Let $\sim$ be a categorical congruence, by part (a) there exists a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying equation (2). It follows that $G/N = G/\sim$ by definition.

Part (c). The homomorphism theorem for groups states that:
If $h : G \to H$ is a group homomorphism and $N \subseteq G$ a normal subgroup, then $N \subseteq \ker(h)$ iff there is a unique homomorphism $\overline{h} : G/N \to H$ with $\overline{h} \circ \pi = h$, where $\pi : G \twoheadrightarrow G/N$ is the quotient.
The homomorphism theorem for categories states that:
Every functor $F : \mathbf{C} \to \mathbf{D}$ has a kernel category $\ker(F)$, determined by a congruence $\sim_F$ on $\mathbf{C}$ such that given any congruence $\sim$ on $\mathbf{C}$ one has $f \sim g \Rightarrow f \sim_F g$ iff there is a functor $\overline{F} : \mathbf{C}/\sim \to \mathbf{D}$ with $\overline{F} \circ \pi = F$, where $\pi : \mathbf{C} \twoheadrightarrow \mathbf{C}/\sim$ is the quotient.
Assume that $h : G \to H$ is a homomorphism between groups, which is the same thing as a functor $F := h$ if $\mathbf{C} := G$ and $\mathbf{D} := H$ are regarded as categories. Using the correspondence from part (a), we have from part (b) that G/N coincide with $G/\sim$, hence it suffices to show:

(i) If $N \subseteq G$ is a normal subgroup then $N \subseteq \ker(h)$ implies $f \sim_N g \Rightarrow f \sim_h g$, where $\sim_N$ is given by definition (1).

(ii) If $\sim$ is a categorical congruence on G then $f \sim g \Rightarrow f \sim_h g$ implies $N \subseteq \ker(h)$, where $N = \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$.

In order to prove (i), assume $N \subseteq \ker(h)$ is a normal subgroup of G. If $f \sim_N g$ then $f \cdot g^{-1} \in N \subseteq \ker(h)$, hence $h(f) = h(g)$ and by definition on page 84 [1] we have $f \sim_h g$. To show (ii), assume $\sim$ is categorical congruence on G and $f \sim g \Rightarrow f \sim_h g$. Let $r \cdot s^{-1} \in N$ such that $r \sim s$ and by assumption $r \sim_h s$, by definition of $\sim_h$ we have $h(r) = h(s)$. This shows that $r \cdot s^{-1} \in \ker(h)$ and therefore $N \subseteq \ker(h)$. $\square$

Exercise 5.7.9 Suppose the category $\mathbf{C}$ has limits of type $\mathbf{J}$, for some index category $\mathbf{J}$. For diagrams F and G of type $\mathbf{J}$ in $\mathbf{C}$, a morphism of diagrams $\theta : F \to G$ consists of arrows $\theta_i : F_i \to G_i$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$, one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square).

(a) This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

(b) Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$$

(c) Infer that for any set I, there is a product functor,

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

Solution. I begin with (a). If $\theta : F \to G$ and $\phi : G \to H$ are morphisms of diagrams of type $\mathbf{J}$, then define the composite $\phi \circ \theta$ to consist of arrows $\phi_i \theta_i : F_i \to H_i$ for each $i \in \mathbf{J}$. Define the identity morphism 1_F of F to consist of arrows $1_{F_i} : F_i \to F_i$ for each $i \in \mathbf{J}$. The associativity and unit laws are inherited from those of $\mathbf{C}$. For (b), let

$$\varprojlim_{\mathbf{J}}(F) = \varprojlim_{i} F_i \qquad \text{where } F : \mathbf{J} \to \mathbf{C} \text{ is a diagram}$$

If there is a morphism $\theta : F \to G$ of diagrams of type $\mathbf{J}$, then $\varprojlim_i F_i$ is a cone to G, hence there exists a unique arrow $\bar{\theta} : \varprojlim_i F_i \to \varprojlim_i G_i$ from $\varprojlim_{\mathbf{J}}(F)$ to $\varprojlim_{\mathbf{J}}(G)$ since $\varprojlim_{\mathbf{J}}(G) = \varprojlim_i G_i$ is terminal object in the category of cones to G. Define the operation of $\varprojlim_{\mathbf{J}}$ on an arrow to be

$$\varprojlim_{\mathbf{J}}(\theta : F \to G) = \bar{\theta} : \varprojlim_{\mathbf{J}}(F) \to \varprojlim_{\mathbf{J}}(G)$$

By uniqueness of $\bar{\theta}$, $\varprojlim_{\mathbf{J}}$ is forced to preserve identity and composition. The product functor from (c) can be defined by

$$\prod_{i \in I}(X) = \prod_{i \in I} X_i \qquad \text{where } X : I \to \mathbf{Sets} \text{ is a diagram}$$

This is a special case of the functor $\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$ with $\mathbf{C} = \mathbf{Sets}$ and $\mathbf{J} = I$, where I is regarded as a discrete category. □

Exercise 5.7.11 Let $R \subseteq X \times X$ be an equivalence relation on a set X, with quotient $q : X \twoheadrightarrow Q$. Show that the following is an equaliser:

$$\begin{tikzcd} \mathcal{P}Q \arrow[r, "\mathcal{P}q"] & \mathcal{P}X \arrow[r, shift left, "\mathcal{P}r_1"] \arrow[r, shift right, "\mathcal{P}r_2"'] & \mathcal{P}R \end{tikzcd}$$

where $r_1, r_2 R \rightrightarrows X$ are the two projections of R, and $\mathcal{P}$ is the (contravariant) powerset functor. (Hint: $\mathcal{P}X \cong 2^X$).

Solution. Example 5.12 [1] shows that the contravariant powerset functor is representable by giving a natural isomorphism $\mathcal{P}(X) \cong 2^X$, and since contravariant representable functors map colimits to limits it suffices to show that the following diagram is a coequaliser

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

But this has already been shown in example 3.20 [1]. □

References

[1] Steve Awodey (2010), *Category Theory, Second Edition*