# Solutions to Exercises in Steve Awodey's Book

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**Contents**

**Exercise 4.5.1** Regarding a group $G$ as a category with one object and every arrow an isomorphism:

(a) Show that a categorical congruence $\sim$ on $G$ is the same thing as (the equivalence relation on $G$ determined by) a normal subgroup $N \subseteq G$, that is, show that the two kinds of things are in isomorphic correspondence.

(b) Show further that the quotient category $G/{\sim}$ and the factor group $G/N$ coincide.

(c) Conclude that the homomorphism theorem for groups is a special case of the one for categories.

*Solution.* To solve this exercise I will make use of page 81 [1], where Awodey defines an equivalence relation $\sim_N$, determined by a normal subgroup $N \subseteq G$, such that

$$g \sim_N h \qquad \text{iff} \qquad g \cdot h^{-1} \in N \qquad\qquad \text{for } g, h \in G \tag{1}$$

I will begin with part (a). First, let $N \subseteq G$ be a normal subgroup in order to show that $\sim_N$ is a categorical congruence. If $r, s \in G$ and $r \sim_N s$ then $r \cdot s^{-1} \in N$, and since $N$ is normal we have

$$(g \cdot r \cdot h) \cdot (h^{-1} \cdot s^{-1} \cdot g^{-1}) = g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N \qquad \text{for } g, h \in G$$

The above equation shows that

$$g \cdot r \cdot h \sim (h^{-1} \cdot s^{-1} \cdot g^{-1})^{-1} = g \cdot s \cdot h$$

and $\sim_N$ is therefore a categorical congruence. On the other hand, let $\sim$ be a categorical congruence on $G$ regarded as a category. The goal is now to define a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying

$$g \sim_N h \qquad \text{iff} \qquad g \sim h \qquad\qquad \text{for } g, h \in G \tag{2}$$

Given definition (1), $N := \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$ seems to be a natural candidate. If $g \sim h$ then $g \cdot h^{-1} \in N$ and thus $g \sim_N h$ by definition (1). If $g \sim_N h$ then $g \cdot h^{-1} \in N$ so there exist $r, s \in G$ where $g \cdot h^{-1} = r \cdot s^{-1}$ and $r \sim s$. Since $\sim$ is a categorical congruence we get $g \cdot h^{-1} = r \cdot s^{-1} \sim s \cdot s^{-1} = u$, implying $g \sim h$. This shows that equation (2) is satisfied. It remains to show that $N \subseteq G$ is a normal subgroup. Let $g \cdot h^{-1} \in N$ and $r \cdot s^{-1} \in N$ such that $g \sim h$ and $r \sim s$, and we get

$$g \cdot h^{-1} \sim h \cdot h^{-1} = u = r \cdot r^{-1} \sim s \cdot r^{-1}$$

Transitivity implies that $(g \cdot h^{-1}) \cdot (s \cdot r^{-1})^{-1} = (g \cdot h^{-1}) \cdot (r \cdot s^{-1}) \in N$ and since $u \in N$ by reflexivity and $N$ is closed under inverse by symmetry, $N$ is a subgroup. Now, let $r \cdot s^{-1} \in N$ and $g \in G$. It follows that $g \cdot r \sim g \cdot s = (s^{-1}g^{-1})^{-1}$ because $r \sim s$ and $\sim$ is a congruence, hence $g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N$ and $N$ is normal.

When I looked at the solution on page 288 I discovered that Awodey defines the normal subgroup as $\overline{N} := \{g \in G \mid g \sim u\}$. Luckily, it turns out that $\overline{N} = N$, as I will show. If $g \in \overline{N}$ then $g \sim u$, so $g = g \cdot u^{-1} \in N$. If $r \cdot s^{-1} \in N$ where $r \sim s$, then $r \cdot s^{-1} \sim u$ and $r \cdot s^{-1} \in \overline{N}$ as required.

I will continue with part (b). Let $N \subseteq G$ be a normal subgroup determining equivalence relation $\sim_N$. By part (a), $g \sim h := g \sim_N h$ is a categorical congruence, hence $G/N = G/{\sim}$ by definition on page 81 and 84 [1] respectively. In the other direction.

Let $\sim$ be a categorical congruence, by part (a) there exists a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying equation (2). It follows that $G/N = G/\sim$ by definition.

Part (c). The homomorphism theorem for groups states that:
*If $h : G \to H$ is a group homomorphism and $N \subseteq G$ a normal subgroup, then $N \subseteq \ker(h)$ iff there is a unique homomorphism $\overline{h} : G/N \to H$ with $\overline{h} \circ \pi = h$, where $\pi : G \twoheadrightarrow G/N$ is the quotient.*
The homomorphism theorem for categories states that:
*Every functor $F : \mathbf{C} \to \mathbf{D}$ has a kernel category $\ker(F)$, determined by a congruence $\sim_F$ on $\mathbf{C}$ such that given any congruence $\sim$ on $\mathbf{C}$ one has $f \sim g \Rightarrow f \sim_F g$ iff there is a functor $\overline{F} : \mathbf{C}/\sim \to \mathbf{D}$ with $\overline{F} \circ \pi = F$, where $\pi : \mathbf{C} \twoheadrightarrow \mathbf{C}/\sim$ is the quotient.*

Assume that $h : G \to H$ is a homomorphism between groups, which is the same thing as a functor $F := h$ if $\mathbf{C} := G$ and $\mathbf{D} := H$ are regarded as categories. Using the correspondence from part (a), we have from part (b) that $G/N$ coincide with $G/\sim$, hence it suffices to show:

(i) If $N \subseteq G$ is a normal subgroup then $N \subseteq \ker(h)$ implies $f \sim_N g \Rightarrow f \sim_h g$, where $\sim_N$ is given by definition (1).

(ii) If $\sim$ is a categorical congruence on $G$ then $f \sim g \Rightarrow f \sim_h g$ implies $N \subseteq \ker(h)$, where $N = \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$.

In order to prove (i), assume $N \subseteq \ker(h)$ is a normal subgroup of $G$. If $f \sim_N g$ then $f \cdot g^{-1} \in N \subseteq \ker(h)$, hence $h(f) = h(g)$ and by definition on page 84 [1] we have $f \sim_h g$. To show (ii), assume $\sim$ is categorical congruence on $G$ and $f \sim g \Rightarrow f \sim_h g$. Let $r \cdot s^{-1} \in N$ such that $r \sim s$ and by assumption $r \sim_h s$, by definition of $\sim_h$ we have $h(r) = h(s)$. This shows that $r \cdot s^{-1} \in \ker(h)$ and therefore $N \subseteq \ker(h)$. □

**Exercise 5.7.9** Suppose the category $\mathbf{C}$ has limits of type $\mathbf{J}$, for some index category $\mathbf{J}$. For diagrams $F$ and $G$ of type $\mathbf{J}$ in $\mathbf{C}$, a morphism of diagrams $\theta : F \to G$ consists of arrows $\theta_i : F_i \to G_i$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$, one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square).

(a) This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

(b) Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$$

(c) Infer that for any set $I$, there is a product functor,

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

*Solution.* I begin with (a). If $\theta : F \to G$ and $\phi : G \to H$ are morphisms of diagrams of type $\mathbf{J}$, then define the composite $\phi \circ \theta$ to consist of arrows $\phi_i \theta_i : F_i \to H_i$ for each $i \in \mathbf{J}$. Define the identity morphism $1_F$ of $F$ to consist of arrows $1_{F_i} : F_i \to F_i$ for each $i \in \mathbf{J}$. The associativity and unit laws are inherited from those of $\mathbf{C}$. For (b), let

$$\varprojlim_{\mathbf{J}}(F) = \varprojlim_{i} F_i \qquad\qquad \text{where } F : \mathbf{J} \to \mathbf{C} \text{ is a diagram}$$

If there is a morphism $\theta : F \to G$ of diagrams of type $\mathbf{J}$, then $\varprojlim_i F_i$ is a cone to $G$, hence there exists a unique arrow $\bar{\theta} : \varprojlim_i F_i \to \varprojlim_i G_i$ from $\varprojlim_{\mathbf{J}}(F)$ to $\varprojlim_{\mathbf{J}}(G)$ since $\varprojlim_{\mathbf{J}}(G) = \varprojlim_i G_i$ is terminal object in the category of cones to $G$. Define the operation of $\varprojlim_{\mathbf{J}}$ on an arrow to be

$$\varprojlim_{\mathbf{J}}(\theta : F \to G) = \bar{\theta} : \varprojlim_{\mathbf{J}}(F) \to \varprojlim_{\mathbf{J}}(G)$$

By uniqueness of $\bar{\theta}$, $\varprojlim_{\mathbf{J}}$ is forced to preserve identity and composition. The product functor from (c) can be defined by

$$\prod_{i \in I}(X) = \prod_{i \in I} X_i \qquad\qquad \text{where } X : I \to \mathbf{Sets} \text{ is a diagram}$$

This is a special case of the functor $\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$ with $\mathbf{C} = \mathbf{Sets}$ and $\mathbf{J} = I$, where $I$ is regarded as a discrete category. □

**Exercise 5.7.11** Let $R \subseteq X \times X$ be an equivalence relation on a set $X$, with quotient $q : X \twoheadrightarrow Q$. Show that the following is an equaliser:

$$\mathcal{P}Q \xrightarrow{\mathcal{P}q} \mathcal{P}X \underset{\mathcal{P}r_2}{\overset{\mathcal{P}r_1}{\rightrightarrows}} \mathcal{P}R$$

where $r_1, r_2 R \rightrightarrows X$ are the two projections of $R$, and $\mathcal{P}$ is the (contravariant) powerset functor. (Hint: $\mathcal{P}X \cong 2^X$).

*Solution.* Example 5.12 [1] shows that the contravariant powerset functor is representable by giving a natural isomorphism $\mathcal{P}(X) \cong 2^X$, and since contravariant representable functors map colimits to limits it suffices to show that the following diagram is a coequaliser

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{q} Q$$

But this has already been shown in example 3.20 [1]. □

**Exercise 6.8.4** Is the category of monoids cartesian closed?

*Solution.* I will show that category monoids is not cartesian closed. To obtain a contradiction, suppose that monoids is CCC. Let $G$ be the cyclic group $\mathbb{Z}/2\mathbb{Z}$, which is a monoid. There exist homomorphisms $\widetilde{\pi}_1, \widetilde{\pi}_2 : G \rightrightarrows G^G$ making the following diagrams commute in the category of monoids:

$$\begin{tikzcd}
G^G \times G \arrow[r, "\varepsilon"] & G \\
G \times G \arrow[u, "\widetilde{\pi}_1 \times 1_G"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G^G \times G \arrow[r, "\varepsilon"] & G \\
G \times G \arrow[u, "\widetilde{\pi}_2 \times 1_G"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

The left diagram gives

$$0 = \pi_1(0,1) = \varepsilon((\widetilde{\pi}_1 \times 1_G)(0,1)) = \varepsilon(\widetilde{\pi}_1(0),1) = \varepsilon(u,1)$$

where $u \in G^G$ is the unit. The right diagram gives

$$1 = \pi_2(0,1) = \varepsilon((\widetilde{\pi}_2 \times 1_G)(0,1)) = \varepsilon(\widetilde{\pi}_2(0),1) = \varepsilon(u,1)$$

which is a contradiction since $0 \neq 1$. $\square$

**Exercise 6.8.16** Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of $I$-indexed sets ($I$ a poset) can be computed "pointwise". Show, moreover, that the same is true for all limits and colimits.

*Solution.* I will show that limits and colimits can be computed pointwise, it follows that products can be so as well. Let $A, B : I \rightrightarrows \mathbf{Sets}$ be functors and $I$ a poset, then:

$\alpha : A \to B$ is a natural transformation in $\mathbf{Sets}^I$ iff,

for all $I_1 \ni p : i \to j$ (i.e. $i \leq j$), the diagram
$$\begin{tikzcd}
A(i) \arrow[r, "\alpha_i"] \arrow[d, "A(p)"'] & B(i) \arrow[d, "B(p)"] \\
A(j) \arrow[r, "\alpha_j"] & B(j)
\end{tikzcd}$$
commutes in $\mathbf{Sets}$ iff,

for all $I_1^{\mathrm{op}} \ni p^{\mathrm{op}} : j \to i$, the diagram
$$\begin{tikzcd}
A^{\mathrm{op}}(i) & B^{\mathrm{op}}(i) \arrow[l, "\alpha_i^{\mathrm{op}}"'] \\
A^{\mathrm{op}}(j) \arrow[u, "A^{\mathrm{op}}(p^{\mathrm{op}})"] & B^{\mathrm{op}}(j) \arrow[l, "\alpha_j^{\mathrm{op}}"'] \arrow[u, "B^{\mathrm{op}}(p^{\mathrm{op}})"']
\end{tikzcd}$$
commutes in $\mathbf{Sets}^{\mathrm{op}}$ iff,

$\alpha^{\mathrm{op}} : B^{\mathrm{op}} \to A^{\mathrm{op}}$ is a natural transformation in $(\mathbf{Sets}^{\mathrm{op}})^{I^{op}}$.

Natural transformations $\alpha : A \to B$ in $\mathbf{Sets}^I$ are in a bijective correspondence to natural transformations $\alpha^{\mathrm{op}} : B^{\mathrm{op}} \to A^{\mathrm{op}}$ in $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$. The bijection respects identity $1_A^{\mathrm{op}} = 1_{A^{\mathrm{op}}}$ and if $\alpha, \beta \in \mathbf{Sets}^I$ are natural transformations then $(\beta \circ \alpha)^{\mathrm{op}} = \alpha^{\mathrm{op}} \circ \beta^{\mathrm{op}}$, implying that $(\mathbf{Sets}^I)^{\mathrm{op}}$ is equivalent to $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$. Then, by duality, a limit in $\mathbf{Sets}^I$ corresponds to a colimit in $(\mathbf{Sets}^{\text{op}})^{I^{\text{op}}}$. Suppose $(a_x)_{x\in\mathbf{J}_0}$ is a pointwise computed limit of a diagram $F : \mathbf{J} \to \mathbf{Sets}^I$ with limit object $L$, i.e.

$$L(i) = \varprojlim_{x\in\mathbf{J}} F_x(i) \text{ with limit cone } (a_{x,i})_{x\in\mathbf{J}_0} \text{ for all } i \in I_0 \ (i \text{ is object in } I).$$

Let $L^{\text{op}}$ be the corresponding colimit object in $(\mathbf{Sets}^{\text{op}})^{I^{\text{op}}}$, then $L^{\text{op}}(i)$ is the colimit object in $\mathbf{Sets}^{\text{op}}$ with cocone $(a^{\text{op}}_{x,i})_{x\in\mathbf{J}_0}$ and corresponding limit $L(i)$ in $\mathbf{Sets}$, so the colimit is computed pointwise as well. This means that it suffices to prove: If $S$ is a category with all limits and $I$ is a poset, then $S^I$ has all limits and they can be computed pointwise.

To prove this, let $F : \mathbf{J} \to S^I$ be a diagram and define $L : I_0 \to S$ by

$$L(i) := \varprojlim_{x\in\mathbf{J}} F_x(i)$$

Let $i \in I_0$, by definition of $L(i)$ there exists a limit cone $(a_{x,i})_{x\in\mathbf{J}_0}$ of arrows $a_{x,i} : L(i) \to F_x(i)$ where $x \in \mathbf{J}_0$, such that for all $\mathbf{J}_1 \ni g : x \to y$,

$$F_{g,i} \circ a_{x,i} = a_{y,i} \tag{3}$$

Let $x \in \mathbf{J}_0$, it is required that $L$ is a functor, thus an object in $S^I$, and that $a_x$ is a natural transformation in $S^I$ from $L$ to $F_x$. Let $I_1 \ni p : i \to j$, i.e. $i \leq j$, it must be shown that there exists an arrow $L(p) : L(i) \to L(j)$ making the following diagram commute:

$$\begin{tikzcd}
L(i) \arrow[r, "a_{x,i}"] \arrow[d, dashed, "L(p)"'] & F_x(i) \arrow[d, "F_x(p)"] \\
L(j) \arrow[r, "a_{x,j}"] & F_x(j)
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
& L(i) \arrow[dl, "a_{x,i}"'] \arrow[dr, "a_{y,i}"] & \\
F_x(i) \arrow[rr, "F_{g,i}"] \arrow[d, "F_x(p)"'] & & F_y(i) \arrow[d, "F_y(p)"] \\
F_x(j) \arrow[rr, "F_{g,j}"] & & F_y(j)
\end{tikzcd}$$

where $\mathbf{J}_1 \ni g : x \to y$ and $I_1 \ni p : i \to j$ are arbitrary. The upper triangle commutes by equation (3) and the bottom square commutes because $F_g$ is a natural transformation, hence $L(i)$ is cone to $x \mapsto F_x(j)$. Since $L(j)$ is limit there exists a unique arrow $L(p) : L(i) \to L(j)$ where

$$F_x(p) \circ a_{x,i} = a_{x,j} \circ L(p) \qquad \text{for all } x \in \mathbf{J}_0 \tag{4}$$

Furthermore, by uniqueness it follows that $L(q \circ p) = L(q) \circ L(p)$ and $L(1_i) = 1_{L(i)}$, where $p, q \in I_1$ and $i \in I_0$. This shows that $L$ is a functor so $L \in S^I$. Equation (4) shows that $a_x$ is a natural transformation and hence an arrow in $S^I$. Equation (3) shows that the family $(a_x)_{x \in \mathbf{J}_0}$ of arrows in $S^I$ is a cone to $F$. The only thing missing is that it is a limit cone, so let $(b_x)_{x \in \mathbf{J}_0}$, $b_x : K \to F_x$, be an arbitrary cone to $F$. If $\mathbf{J}_1 \ni g : x \to y$ and $i \in I_0$ then $F_{g,i} \circ b_{x,i} = b_{y,i}$ and thus the family $(b_{x,i})_{x \in \mathbf{J}_0}$ is a cone to $x \mapsto F_x(i)$, and there exists a unique arrow $\beta_i : K(i) \to L(i)$ satisfying that $b_{x,i} = a_{x,i} \circ \beta_i$ for any $x \in \mathbf{J}_0$. I will show that $\beta$ is a natural transformation by showing that the following diagram commutes:

$$\begin{tikzcd}
K(i) \arrow[r, "\beta_i"] \arrow[d, "K(p)"'] & L(i) \arrow[d, "L(p)"] \\
K(j) \arrow[r, "\beta_j"] & L(j)
\end{tikzcd}$$

where $I_1 \ni p : i \to j$. As $b_x$ is a natural transformation, for all $x \in \mathbf{J}_0$,

$$a_{x,j} \circ \beta_j \circ K(p) = b_{x,j} \circ K(p) = F_x(p) \circ b_{x,i} = F_x(p) \circ a_{x,i} \circ \beta_i$$

Then, by equation (4),

$$a_{x,j} \circ \beta_j \circ K(p) = a_{x,j} \circ L(p) \circ \beta_i$$

According to the universal property of the limit $L(j)$, then $\beta_j \circ K(p) = L(p) \circ \beta_i$. $\square$

**Exercise 7.11.17**

(a) Complete the proof that, for any set $I$, the category of $I$-indexed families of sets, regarded as the functor category $\mathbf{Sets}^I$, is equivalent to the slice category $\mathbf{Sets}/I$ of sets over $I$,

$$\mathbf{Sets}^I \simeq \mathbf{Sets}/I$$

(b) Show that reindexing of families along a function $f : J \to I$, given by precomposition,

$$\mathbf{Sets}^f((A_i)_{i \in I}) = (A_{f(j)})_{j \in J}$$

is represented by pullback, in the sense that the following diagram of categories and functors commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathbf{Sets}^I \arrow[r, "\simeq"] \arrow[d, "\mathbf{Sets}^f"'] & \mathbf{Sets}/I \arrow[d, "f^*"] \\
\mathbf{Sets}^J \arrow[r, "\simeq"] & \mathbf{Sets}/J
\end{tikzcd}$$

Here $f^* : \mathbf{Sets}^I \to \mathbf{Sets}^J$ is the pullback functor along $f : J \to I$.

(c) Finally infer that $\mathbf{Sets}/2 \simeq \mathbf{Sets} \times \mathbf{Sets}$, and similarly for any $n$ other than 2.

*Solution.* (a) Define $F : \mathbf{Sets}^I \to \mathbf{Sets}/I$ and $G : \mathbf{Sets}/I \to \mathbf{Sets}^I$ by

$$\begin{aligned}
F_0((A_i)_{i\in I}) &:= (x,k) \mapsto k && \text{for } (x,k) \in \textstyle\coprod_{i\in I} A_i,\ x \in A_k \\
F_1((g_i : A_i \to B_i)_{i\in I}) &:= (x,k) \mapsto (g_k(x),k) && \text{for } (x,k) \in \textstyle\coprod_{i\in I} A_i,\ x \in A_k \\
G_0(a : A \to I) &:= (a^{-1}(i))_{i\in I} \\
G_1(g : A \to B) &:= (g|_{a^{-1}(i)})_{i\in I} && \text{where } g|_{a^{-1}(i)} \text{ is restriction}
\end{aligned}$$

It follows that $F, G$ are functors and they satisfy

$$\begin{aligned}
(G_0 \circ F_0)((A_i)_{i\in I}) &= (A_i \times \{i\})_{i\in I} \\
(G_1 \circ F_1)((g_i)_{i\in I}) &= ((x,k) \mapsto (g_i(x),i))_{i\in I} \\
(F_0 \circ G_0)(a) &= (x,k) \mapsto k && \text{where } (x,k) \in \textstyle\coprod_{i\in I} a^{-1}(i) \\
(F_1 \circ G_1)(g) &= (x,k) \mapsto (g(x),k) && \text{where } (x,k) \in \textstyle\coprod_{i\in I} a^{-1}(i) \\
&&& \text{and } g \in \mathbf{Sets}/I \text{ arrow with domain } a
\end{aligned}$$

For $a : A \to I$ object in $\mathbf{Sets}/I$, the functions $\beta_a : A \rightleftarrows \coprod_{i\in I} a^{-1}(i) : \widetilde{\beta}_a$ where

$$\beta_a(x) := (x, a(x)) \qquad \text{and} \qquad \widetilde{\beta}_a(x,i) := x$$

show that $(F_0 \circ G_0)(a) \cong a$, since

$$\begin{aligned}
\widetilde{\beta}_a(\beta_a(x)) &= \widetilde{\beta}_a(x, a(x)) = x \\
\beta_a(\widetilde{\beta}_a(x,i)) &= \beta_a(x) = (x, a(x)) = (x,i) && \text{where } x \in a^{-1}(i)
\end{aligned}$$

For $g \in \mathbf{Sets}/I$ arrow from $a : A \to I$ to $b : B \to I$, i.e. $a = b \circ g$, the square in $\mathbf{Sets}/I$:

$$\begin{tikzcd}
a \arrow[r, "\beta_a"] \arrow[d, "g"'] & (F_0 \circ G_0)(a) \arrow[d, "(F_1 \circ G_1)(g)"] \\
b \arrow[r, "\beta_b"] & (F_0 \circ G_0)(b)
\end{tikzcd}$$

commutes because

$$\beta_b(g(x)) = (g(x), b(g(x))) = (g(x), a(x)) = (F_1 \circ G_1)(x, a(x)) = (F_1 \circ G_1)(\widetilde{\beta}_a(x))$$

This shows that $1_{\mathbf{Sets}/I}$ is naturally isomorphic to $F \circ G$ since $\beta_a$ and $\beta_b$ are isomorphisms. For each object $(A_i)_{i\in I} \in \mathbf{Sets}^I$ there is a family of functions $(\alpha_k : (A_i)_{i\in I} \to (A_i \times \{i\})_{i\in I})_{k\in I}$ defined by

$$\alpha_k(x) := (x,k)$$

In $\mathbf{Sets}^I$, the square

$$\begin{tikzcd}
(A_i)_{i\in I} \arrow[r, "(\alpha_i)_{i\in I}"] \arrow[d, "(g_i)_{i\in I}"'] & (G\circ F)((A_i)_{i\in I}) \arrow[d, "(G\circ F)((g_i)_{i\in I})"] \\
(B_i)_{i\in I} \arrow[r, "(\alpha_i)_{i\in I}"] & (G\circ F)((B_i)_{i\in I})
\end{tikzcd}$$

is comutative since for all $i \in I$ and $x \in A_i$,

$$(\gamma_i \circ \alpha_i)(x) = ((y,k) \mapsto (g_k(y),k))(\alpha_i(x)) = (g_i(x), i) = (\alpha_i \circ g_i)(x)$$

where $\gamma_i(x,k) = (g_i(x), i)$ for $(x,k) \in A_i \times \{i\}$, so that $(G\circ F)((g_i)_{i\in I}) = (\gamma_i)_{i\in I}$. Hence $1_{\mathbf{Sets}^I}$ is naturally isomorphic to $G\circ F$ which concludes that $\mathbf{Sets}^I \simeq \mathbf{Sets}/I$. To show that the square in (b) commutes up to natural isomorphism it suffices to find a natural isomorphism $\alpha$ making the following square commute in $\mathbf{Sets}/J$:

$$\begin{tikzcd}
f^*(a) \arrow[r, "\alpha_a"] \arrow[d, "f^*(g)"'] & \widetilde{a} \arrow[d, "\widetilde{g}"] \\
f^*(b) \arrow[r, "\alpha_b"] & \widetilde{b}
\end{tikzcd} \tag{5}$$

where, using generalised $F$ from part (a),

$$\begin{aligned}
a &= F((A_i)_{i\in I}) = \pi_2 : \textstyle\coprod_{i\in I} A_i \to I \\
b &= F((B_i)_{i\in I}) = \pi_2 : \textstyle\coprod_{i\in I} B_i \to I \\
g(x,k) &= F((g_i)_{i\in I})(x,k) = (g_k(x), k) && \text{for } g_k : A_k \to B_k \\
\widetilde{a} &= F((A_{f(j)})_{j\in J}) = \pi_2 : \textstyle\coprod_{j\in J} A_{f(j)} \to J \\
\widetilde{b} &= F((B_{f(j)})_{j\in J}) = \pi_2 : \textstyle\coprod_{j\in J} B_{f(j)} \to J \\
\widetilde{g}(x,k) &= F((g_{f(j)})_{j\in J})(x,k) = (g_{f(k)}(x), k) && \text{for } g_{f(k)} : A_{f(k)} \to B_{f(k)}
\end{aligned}$$

with $f^* : \mathbf{Sets}^I \to \mathbf{Sets}^J$ the pullback functor. Let $c := \pi_2 : \coprod_{k\in K} C_k \to I$ and $\widetilde{c} := \pi_2 : \coprod_{j\in J} C_{f(j)} \to J$, for arbitrary $(C_k)_{k\in I} \in \mathbf{Sets}^I$ object. To see that $\widetilde{c}$ is a pullback of $c$ along $f$, consider the commutative square:

$$\begin{tikzcd}
\coprod_{j\in J} C_{f(j)} \arrow[r, "h_c"] \arrow[d, "\widetilde{c}"'] & \coprod_{i\in I} C_i \arrow[d, "c"] \\
J \arrow[r, "f"] & I
\end{tikzcd}$$

where $h_c(x,j) = (x, f(j))$. If there is an object $Z$ with arrows $z_1, z_2$ satisfying $c \circ z_1 = f \circ z_2$, then

$$\begin{gathered}
e : Z \to \textstyle\coprod_{j\in J} C_{f(j)} \\
e(x,j) := (\pi_1(z_1(x,j)), z_2(x,j))
\end{gathered}$$

is an arrow where $\widetilde{c} \circ e = z_2$ and $h_c \circ e = z_1$, since

$$\begin{aligned}h_c(e(x,j)) &= h_c(\pi_1(z_1(x,j)), z_2(x,j)) = (\pi_1(z_1(x,j)), f(z_2(x,j)))\\ &= (\pi_1(z_1(x,j)), c(z_1(x,j))) = z_1(x,j)\end{aligned}$$

Any arrow $d : Z \to \coprod_{j \in J} C_{f(j)}$ with $\widetilde{c} \circ d = z_2$ and $h_c \circ d = z_1$ satisfies

$$\begin{aligned}&\pi_1(e(x,j)) = \pi_1(z_1(x,j)) = \pi_1(h_c(d(x,j))) = \pi_1(d(x,j))\\ &\pi_2(e(x,j)) = z_2(x,j) = \widetilde{c}(d(x,j)) = \pi_2(d(x,j))\end{aligned}$$

which implies that $d = e$. This shows that $\widetilde{c}$ is pullback of $c$ along $f$. Using that $f^*(c)$ is also a pullback of $c$ along $f$, pick $\alpha_c : f^*(c) \to \widetilde{c}$ to be the unique isomorphism where

$$\widetilde{c} \circ \alpha_c = f^*(c) \qquad \text{and} \qquad h_c \circ \alpha_c = h_c^* \tag{6}$$

writing $h_c^*$ for the other half of the pullback $f^*(c)$. To show that the diagram (5) commutes, Corollary 5.9 [1] tells that $h_b \circ \widetilde{g} = g \circ h_a$ and $h_b^* \circ f^*(g) = g \circ h_a^*$, so by using the second equation in (6),

$$h_b \circ \alpha_b \circ f^*(g) = g \circ h_a \circ \alpha_a = h_b \circ \widetilde{g} \circ \alpha_a$$

By virtue of being arrows from $f^*(a)$ to $\widetilde{b}$ in $\mathbf{Sets}/I$ it follows that

$$\widetilde{b} \circ \alpha_b \circ f^*(g) = f^*(a) = \widetilde{b} \circ \widetilde{g} \circ \alpha_a$$

Using the universal mapping property of the pullback $(\widetilde{b}, h_b)$, there is a unique arrow $e$ which satisfies both

$$\begin{aligned}h_b \circ e = h_b \circ \alpha_b \circ f^*(g) = h_b \circ \widetilde{g} \circ \alpha_a\\ \widetilde{b} \circ e = \widetilde{b} \circ \alpha_b \circ f^*(g) = \widetilde{b} \circ \widetilde{g} \circ \alpha_a\end{aligned}$$

Hence $\alpha_b \circ f^*(g) = \widetilde{g} \circ \alpha_a$ as required. (c) holds because equivalence of categories is an equivalence relation and

$$\mathbf{Sets}/I \simeq \mathbf{Sets}^I \simeq \prod_{i \in I} \mathbf{Sets}$$

□

## References


[1] Steve Awodey (2010), *Category Theory, Second Edition*