Solutions to Exercises in Steve Awodey's Book

Andreas Lynge

August 12, 2018

Contents

Exercise 4.5.1 Regarding a group G as a category with one object and every arrow an isomorphism:

(a) Show that a categorical congruence $\sim$ on G is the same thing as (the equivalence relation on G determined by) a normal subgroup $N \subseteq G$, that is, show that the two kinds of things are in isomorphic correspondence.

(b) Show further that the quotient category $G/\sim$ and the factor group G/N coincide.

(c) Conclude that the homomorphism theorem for groups is a special case of the one for categories.

Solution. To solve this exercise I will make use of page 81 [1], where Awodey defines an equivalence relation $\sim_N$, determined by a normal subgroup $N \subseteq G$, such that

$$g \sim_N h \qquad \text{iff} \qquad g \cdot h^{-1} \in N \qquad\qquad \text{for } g, h \in G \tag{1}$$

I will begin with part (a). First, let $N \subseteq G$ be a normal subgroup in order to show that $\sim_N$ is a categorical congruence. If $r, s \in G$ and $r \sim_N s$ then $r \cdot s^{-1} \in N$, and since N is normal we have

$$(g \cdot r \cdot h) \cdot (h^{-1} \cdot s^{-1} \cdot g^{-1}) = g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N \qquad \text{for } g, h \in G$$

The above equation shows that

$$g \cdot r \cdot h \sim (h^{-1} \cdot s^{-1} \cdot g^{-1})^{-1} = g \cdot s \cdot h$$

and $\sim_N$ is therefore a categorical congruence. On the other hand, let $\sim$ be a categorical congruence on G regarded as a category. The goal is now to define a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying

$$g \sim_N h \qquad \text{iff} \qquad g \sim h \qquad\qquad \text{for } g, h \in G \tag{2}$$

Given definition (1), $N := \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$ seems to be a natural candidate. If $g \sim h$ then $g \cdot h^{-1} \in N$ and thus $g \sim_N h$ by definition (1). If $g \sim_N h$ then $g \cdot h^{-1} \in N$ so there exist $r, s \in G$ where $g \cdot h^{-1} = r \cdot s^{-1}$ and $r \sim s$. Since $\sim$ is a categorical congruence we get $g \cdot h^{-1} = r \cdot s^{-1} \sim s \cdot s^{-1} = u$, implying $g \sim h$. This shows that equation (2) is satisfied. It remains to show that $N \subseteq G$ is a normal subgroup. Let $g \cdot h^{-1} \in N$ and $r \cdot s^{-1} \in N$ such that $g \sim h$ and $r \sim s$, and we get

$$g \cdot h^{-1} \sim h \cdot h^{-1} = u = r \cdot r^{-1} \sim s \cdot r^{-1}$$

Transitivity implies that $(g \cdot h^{-1}) \cdot (s \cdot r^{-1})^{-1} = (g \cdot h^{-1}) \cdot (r \cdot s^{-1}) \in N$ and since $u \in N$ by reflexivity and N is closed under inverse by symmetry, N is a subgroup. Now, let $r \cdot s^{-1} \in N$ and $g \in G$. It follows that $g \cdot r \sim g \cdot s = (s^{-1} g^{-1})^{-1}$ because $r \sim s$ and $\sim$ is a congruence, hence $g \cdot (r \cdot s^{-1}) \cdot g^{-1} \in N$ and N is normal.

When I looked at the solution on page 288 I discovered that Awodey defines the normal subgroup as $\overline{N} := \{g \in G \mid g \sim u\}$. Luckily, it turns out that $\overline{N} = N$, as I will show. If $g \in \overline{N}$ then $g \sim u$, so $g = g \cdot u^{-1} \in N$. If $r \cdot s^{-1} \in N$ where $r \sim s$, then $r \cdot s^{-1} \sim u$ and $r \cdot s^{-1} \in \overline{N}$ as required.

I will continue with part (b). Let $N \subseteq G$ be a normal subgroup determining equivalence relation $\sim_N$. By part (a), $g \sim h := g \sim_N h$ is a categorical congruence, hence $G/N = G/\sim$ by definition on page 81 and 84 [1] respectively. In the other direction.

Let $\sim$ be a categorical congruence, by part (a) there exists a normal subgroup $N \subseteq G$ with corresponding equivalence relation $\sim_N$ satisfying equation (2). It follows that $G/N = G/\!\sim$ by definition.

Part (c). The homomorphism theorem for groups states that:
If $h : G \to H$ is a group homomorphism and $N \subseteq G$ a normal subgroup, then $N \subseteq \ker(h)$ iff there is a unique homomorphism $\bar{h} : G/N \to H$ with $\bar{h} \circ \pi = h$, where $\pi : G \twoheadrightarrow G/N$ is the quotient.
The homomorphism theorem for categories states that:
Every functor $F : \mathbf{C} \to \mathbf{D}$ has a kernel category $\ker(F)$, determined by a congruence $\sim_F$ on $\mathbf{C}$ such that given any congruence $\sim$ on $\mathbf{C}$ one has $f \sim g \Rightarrow f \sim_F g$ iff there is a functor $\overline{F} : \mathbf{C}/\!\sim \to \mathbf{D}$ with $\overline{F} \circ \pi = F$, where $\pi : \mathbf{C} \twoheadrightarrow \mathbf{C}/\!\sim$ is the quotient.

Assume that $h : G \to H$ is a homomorphism between groups, which is the same thing as a functor $F := h$ if $\mathbf{C} := G$ and $\mathbf{D} := H$ are regarded as categories. Using the correspondence from part (a), we have from part (b) that G/N coincide with $G/\!\sim$, hence it suffices to show:

(i) If $N \subseteq G$ is a normal subgroup then $N \subseteq \ker(h)$ implies $f \sim_N g \Rightarrow f \sim_h g$, where $\sim_N$ is given by definition (1).

(ii) If $\sim$ is a categorical congruence on G then $f \sim g \Rightarrow f \sim_h g$ implies $N \subseteq \ker(h)$, where $N = \{g \cdot h^{-1} \mid g, h \in G \text{ and } g \sim h\}$.

In order to prove (i), assume $N \subseteq \ker(h)$ is a normal subgroup of G. If $f \sim_N g$ then $f \cdot g^{-1} \in N \subseteq \ker(h)$, hence $h(f) = h(g)$ and by definition on page 84 [1] we have $f \sim_h g$. To show (ii), assume $\sim$ is categorical congruence on G and $f \sim g \Rightarrow f \sim_h g$. Let $r \cdot s^{-1} \in N$ such that $r \sim s$ and by assumption $r \sim_h s$, by definition of $\sim_h$ we have $h(r) = h(s)$. This shows that $r \cdot s^{-1} \in \ker(h)$ and therefore $N \subseteq \ker(h)$. □

Exercise 5.7.9 Suppose the category $\mathbf{C}$ has limits of type $\mathbf{J}$, for some index category $\mathbf{J}$. For diagrams F and G of type $\mathbf{J}$ in $\mathbf{C}$, a morphism of diagrams $\theta : F \to G$ consists of arrows $\theta_i : F_i \to G_i$ for each $i \in \mathbf{J}$ such that for each $\alpha : i \to j$ in $\mathbf{J}$, one has $\theta_j F(\alpha) = G(\alpha)\theta_i$ (a commutative square).

(a) This makes $\mathbf{Diagrams}(\mathbf{J}, \mathbf{C})$ into a category (check this).

(b) Show that taking the vertex-objects of limiting cones determines a functor:

$$\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$$

(c) Infer that for any set I, there is a product functor,

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

Solution. I begin with (a). If $\theta : F \to G$ and $\phi : G \to H$ are morphisms of diagrams of type $\mathbf{J}$, then define the composite $\phi \circ \theta$ to consist of arrows $\phi_i \theta_i : F_i \to H_i$ for each $i \in \mathbf{J}$. Define the identity morphism 1_F of F to consist of arrows $1_{F_i} : F_i \to F_i$ for each $i \in \mathbf{J}$. The associativity and unit laws are inherited from those of $\mathbf{C}$. For (b), let

$$\varprojlim_{\mathbf{J}}(F) = \varprojlim_{i} F_i \qquad\qquad \text{where } F : \mathbf{J} \to \mathbf{C} \text{ is a diagram}$$

If there is a morphism $\theta : F \to G$ of diagrams of type $\mathbf{J}$, then $\varprojlim_i F_i$ is a cone to G, hence there exists a unique arrow $\bar{\theta} : \varprojlim_i F_i \to \varprojlim_i G_i$ from $\varprojlim_{\mathbf{J}}(F)$ to $\varprojlim_{\mathbf{J}}(G)$ since $\varprojlim_{\mathbf{J}}(G) = \varprojlim_i G_i$ is terminal object in the category of cones to G. Define the operation of $\varprojlim_{\mathbf{J}}$ on an arrow to be

$$\varprojlim_{\mathbf{J}}(\theta : F \to G) = \bar{\theta} : \varprojlim_{\mathbf{J}}(F) \to \varprojlim_{\mathbf{J}}(G)$$

By uniqueness of $\bar{\theta}$, $\varprojlim_{\mathbf{J}}$ is forced to preserve identity and composition. The product functor from (c) can be defined by

$$\prod_{i \in I}(X) = \prod_{i \in I} X_i \qquad\qquad \text{where } X : I \to \mathbf{Sets} \text{ is a diagram}$$

This is a special case of the functor $\varprojlim_{\mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \to \mathbf{C}$ with $\mathbf{C} = \mathbf{Sets}$ and $\mathbf{J} = I$, where I is regarded as a discrete category. □

Exercise 5.7.11 Let $R \subseteq X \times X$ be an equivalence relation on a set X, with quotient $q : X \twoheadrightarrow Q$. Show that the following is an equaliser:

$$\begin{tikzcd} \mathcal{P}Q \arrow[r, "\mathcal{P}q"] & \mathcal{P}X \arrow[r, shift left, "\mathcal{P}r_1"] \arrow[r, shift right, "\mathcal{P}r_2"'] & \mathcal{P}R \end{tikzcd}$$

where $r_1, r_2 R \rightrightarrows X$ are the two projections of R, and $\mathcal{P}$ is the (contravariant) powerset functor. (Hint: $\mathcal{P}X \cong 2^X$).

Solution. Example 5.12 [1] shows that the contravariant powerset functor is representable by giving a natural isomorphism $\mathcal{P}(X) \cong 2^X$, and since contravariant representable functors map colimits to limits it suffices to show that the following diagram is a coequaliser

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

But this has already been shown in example 3.20 [1]. □

Exercise 6.8.4 Is the category of monoids cartesian closed?

Solution. I will show that category monoids is not cartesian closed. To obtain a contradiction, suppose that monoids is CCC. Let G be the cyclic group $\mathbb{Z}/2\mathbb{Z}$, which is a monoid. There exist homomorphisms $\widetilde{\pi}_1, \widetilde{\pi}_2 : G \rightrightarrows G^G$ making the following diagrams commute in the category of monoids:

$$\begin{tikzcd} G^G \times G \arrow[r, "\varepsilon"] & G \\ G \times G \arrow[u, "\widetilde{\pi}_1 \times 1_G"] \arrow[ur, "\pi_1"'] & \end{tikzcd} \qquad \begin{tikzcd} G^G \times G \arrow[r, "\varepsilon"] & G \\ G \times G \arrow[u, "\widetilde{\pi}_2 \times 1_G"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

The left diagram gives

$$0 = \pi_1(0,1) = \varepsilon((\widetilde{\pi}_1 \times 1_G)(0,1)) = \varepsilon(\widetilde{\pi}_1(0),1) = \varepsilon(u,1)$$

where $u \in G^G$ is the unit. The right diagram gives

$$1 = \pi_2(0,1) = \varepsilon((\widetilde{\pi}_2 \times 1_G)(0,1)) = \varepsilon(\widetilde{\pi}_2(0),1) = \varepsilon(u,1)$$

which is a contradiction since $0 \neq 1$. $\square$

Exercise 6.8.16 Verify the claim in the text that the products $A \times B$ in categories $\mathbf{Sets}^I$ of I-indexed sets (I a poset) can be computed "pointwise". Show, moreover, that the same is true for all limits and colimits.

Solution. I will show that limits and colimits can be computed pointwise, it follows that products can be so as well. Let $A, B : I \rightrightarrows \mathbf{Sets}$ be functors and I a poset, then:

$\alpha : A \to B$ is a natural transformation in $\mathbf{Sets}^I$ iff,

for all $I_1 \ni p : i \to j$ (i.e. $i \leq j$), the diagram

$$\begin{tikzcd} A(i) \arrow[r, "\alpha_i"] \arrow[d, "A(p)"'] & B(i) \arrow[d, "B(p)"] \\ A(j) \arrow[r, "\alpha_j"] & B(j) \end{tikzcd}$$

commutes in $\mathbf{Sets}$ iff,

for all $I_1^{\mathrm{op}} \ni p^{\mathrm{op}} : j \to i$, the diagram

$$\begin{tikzcd} A^{\mathrm{op}}(i) & B^{\mathrm{op}}(i) \arrow[l, "\alpha_i^{\mathrm{op}}"'] \\ A^{\mathrm{op}}(j) \arrow[u, "A^{\mathrm{op}}(p^{\mathrm{op}})"] & B^{\mathrm{op}}(j) \arrow[l, "\alpha_j^{\mathrm{op}}"'] \arrow[u, "B^{\mathrm{op}}(p^{\mathrm{op}})"'] \end{tikzcd}$$

commutes in $\mathbf{Sets}^{\mathrm{op}}$ iff,

$\alpha^{\mathrm{op}} : B^{\mathrm{op}} \to A^{\mathrm{op}}$ is a natural transformation in $(\mathbf{Sets}^{\mathrm{op}})^{I^{op}}$.

Natural transformations $\alpha : A \to B$ in $\mathbf{Sets}^I$ are in a bijective correspondence to natural transformations $\alpha^{\mathrm{op}} : B^{\mathrm{op}} \to A^{\mathrm{op}}$ in $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$. The bijection respects identity $1_A^{\mathrm{op}} = 1_{A^{\mathrm{op}}}$ and if $\alpha, \beta \in \mathbf{Sets}^I$ are natural transformations then $(\beta \circ \alpha)^{\mathrm{op}} = \alpha^{\mathrm{op}} \circ \beta^{\mathrm{op}}$, implying that $(\mathbf{Sets}^I)^{\mathrm{op}}$ is equivalent to $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$. Then, by duality, a limit in $\mathbf{Sets}^I$ corresponds to a colimit in $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$. Suppose $(a_x)_{x\in\mathbf{J}_0}$ is a pointwise computed limit of a diagram $F : \mathbf{J} \to \mathbf{Sets}^I$ with limit object L, i.e.

$$L(i) = \underset{\substack{\longleftarrow \\ x\in\mathbf{J}}}{\lim} F_x(i) \text{ with limit cone } (a_{x,i})_{x\in\mathbf{J}_0} \text{ for all } i \in I_0 \text{ (i is object in I).}$$

Let L^{op} be the corresponding colimit object in $(\mathbf{Sets}^{\mathrm{op}})^{I^{\mathrm{op}}}$, then $L^{\mathrm{op}}(i)$ is the colimit object in $\mathbf{Sets}^{\mathrm{op}}$ with cocone $(a^{\mathrm{op}}_{x,i})_{x\in\mathbf{J}_0}$ and corresponding limit $L(i)$ in $\mathbf{Sets}$, so the colimit is computed pointwise as well. This means that it suffices to prove: If S is a category with all limits and I is a poset, then S^I has all limits and they can be computed pointwise.

To prove this, let $F : \mathbf{J} \to S^I$ be a diagram and define $L : I_0 \to S$ by

$$L(i) := \underset{\substack{\longleftarrow \\ x\in\mathbf{J}}}{\lim} F_x(i)$$

Let $i \in I_0$, by definition of $L(i)$ there exists a limit cone $(a_{x,i})_{x\in\mathbf{J}_0}$ of arrows $a_{x,i} : L(i) \to F_x(i)$ where $x \in \mathbf{J}_0$, such that for all $\mathbf{J}_1 \ni g : x \to y$,

$$F_{g,i} \circ a_{x,i} = a_{y,i} \tag{3}$$

Let $x \in \mathbf{J}_0$, it is required that L is a functor, thus an object in S^I, and that a_x is a natural transformation in S^I from L to F_x. Let $I_1 \ni p : i \to j$, i.e. $i \le j$, it must be shown that there exists an arrow $L(p) : L(i) \to L(j)$ making the following diagram commute:

$$\begin{tikzcd}
L(i) \arrow[r, "a_{x,i}"] \arrow[d, dashed, "L(p)"'] & F_x(i) \arrow[d, "F_x(p)"] \\
L(j) \arrow[r, "a_{x,j}"] & F_x(j)
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
& L(i) \arrow[dl, "a_{x,i}"'] \arrow[dr, "a_{y,i}"] & \\
F_x(i) \arrow[rr, "F_{g,i}"] \arrow[d, "F_x(p)"'] & & F_y(i) \arrow[d, "F_y(p)"] \\
F_x(j) \arrow[rr, "F_{g,j}"] & & F_y(j)
\end{tikzcd}$$

where $\mathbf{J}_1 \ni g : x \to y$ and $I_1 \ni p : i \to j$ are arbitrary. The upper triangle commutes by equation (3) and the bottom square commutes because F_g is a natural transformation, hence $L(i)$ is cone to $x \mapsto F_x(j)$. Since $L(j)$ is limit cone there exists a unique arrow $L(p) : L(i) \to L(j)$ where

$$F_x(p) \circ a_{x,i} = a_{x,j} \circ L(p) \qquad \text{for all } x \in \mathbf{J}_0 \tag{4}$$

Furthermore, by uniqueness it follows that $L(q \circ p) = L(q) \circ L(p)$ and $L(1_i) = 1_{L(i)}$, where $p, q \in I_1$ and $i \in I_0$. This shows that L is a functor so $L \in S^I$. Equation (4) shows that a_x is a natural transformation and hence an arrow in S^I. Equation (3) shows that the family $(a_x)_{x \in \mathbf{J}_0}$ of arrows in S^I is a cone to F. The only thing missing is that it is a limit cone, so let $(b_x)_{x \in \mathbf{J}_0}$, $b_x : K \to F_x$, be an arbitrary cone to F. If $\mathbf{J}_1 \ni g : x \to y$ and $i \in I_0$ then $F_g(i) \circ b_{x,i} = b_{y,i}$ and thus the family $(b_{x,i})_{x \in \mathbf{J}_0}$ is a cone to $x \mapsto F_x(i)$, and there exists a unique arrow $\beta_i : K(i) \to L(i)$. I will show that β is a natural transformation by showing that the following diagram commutes:

$$\begin{tikzcd}
K(i) \arrow[r, "\beta_i"] \arrow[d, "K(p)"'] & L(i) \arrow[d, "L(p)"] \\
K(j) \arrow[r, "\beta_j"] & L(j)
\end{tikzcd}$$

where $I_1 \ni p : i \to j$. Since $L(p) \circ \beta_i$ is an arrow from $K(i)$ to $L(j)$, $K(i)$ is cone to the diagram $x \mapsto F_x(j)$ having limit cone $L(j)$, so there exists just one arrow $K(i) \to L(j)$, hence $L(p) \circ \beta_i = \beta_j \circ K(p)$. □

References

[1] Steve Awodey (2010), *Category Theory, Second Edition*