Expert Systems with Applications 41 (2014) 7023–7031

Contents lists available at ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

An application of capital allocation principles to operational risk and the cost of fraud

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ARTICLE INFO

Article history:
Available online 28 May 2014

Keywords:
Risk management
Quantile
Value at risk
Unexpected losses

ABSTRACT

The costs of operational risk refer to the capital needed to cover the losses generated by a firm's ordinary activities. In this paper several capital allocation principles are examined to demonstrate how such principles can be used to distribute aggregated capital across the various constituents that generate operational risk. Proportional allocation, for example, allows a cost per unit to be calculated. An application to fraud risk in the banking sector is presented and correlation scenarios between business lines are compared.

1. Introduction and motivation

Risk management in business concerns itself with anticipating the potential losses a firm might suffer and with designing methods that can either mitigate such losses or compensate for them. It is a field of intense research given that security and protection are essential elements of quality control.

In ordinary business operations, risks of malfunction or operational risks - including, software failures, electricity cuts, human errors, internal and external fraud, etc. – are almost inevitable and as such are a constant burden on expected profits. Expected operational losses can be integrated as a fixed cost component of production, while it is necessary to reserve a capital sum to offset any unexpected operational losses and, thus, respond to exceptional operational risk events.

Here, we address the costs of operational risk and calculate the proportion that each unit of production should contribute to the total capital held to cover this risk. A *constant* allocation involves dividing the total capital by the number of production units regardless of the contribution each unit makes to aggregate operational risk; a *proportional* allocation increases the contribution of those units that represent a greater risk; and, an *optimal* allocation considers the contribution of each unit in relation to the dependence between risks within a firm.

We examine an example concerning the risk of fraud in banking. Our simplified scenario centers on a bank with just two lines of business: credit cards and savings accounts. Losses attributable to fraud occur in these two services and, as such, represent a key area of research. Managers can predict the annual average loss due to fraud in these two services independently and, thus, include the expected loss as part of the general management costs of credit cards and savings accounts, respectively. However, additional capital needs to be held to offset exceptional exposure to risk from fraud in either of the two lines and, here, there are various methods for determining how much capital should be provided by the credit card business and how much by the savings account business. Yet, assuming independence between lines of business is unrealistic. It has been widely documented that the propensity for fraud fluctuates with exogenous factors that create spurious correlations between business units (Viaene, Ayuso, Guillén, Van Gheel, & Dedene, 2007). Factors such as economic recession; social networking, where people share information about the *modus operandi* of successful fraudulent activities; and periods of the year when consumers are more prone to defraud affect all business lines at the same time (see, for instance, Caudill, Ayuso, & Guillén (2005)). We address how to deal with this dependence between fraud risks in the two-dimensional setting of credit card and savings account fraud (see Fig. 1). It is worth noting that similar applications have been examined in the context of the automobile insurance (Ai, Brockett, Golden, & Guillén, 2013; Artis, Ayuso, & Guillén, 1999, 2002; Viaene, Van Gheel, Ayuso, & Guillén, 2004).

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[1] This work was developed partly at the Technical University of Catalonia (UPC) and Universitat Rovira Virgili.

[2] ICREA Academia and the Ministry of Economy and Competitiveness/FEDER Grant ECO2010-21787-C01-03 are acknowledged. The authors wish to express their gratitude to the editor and to two anonymous reviewers who provided many helpful suggestions on an earlier version of this paper.

http://dx.doi.org/10.1016/j.eswa.2014.05.017
0957-4174/

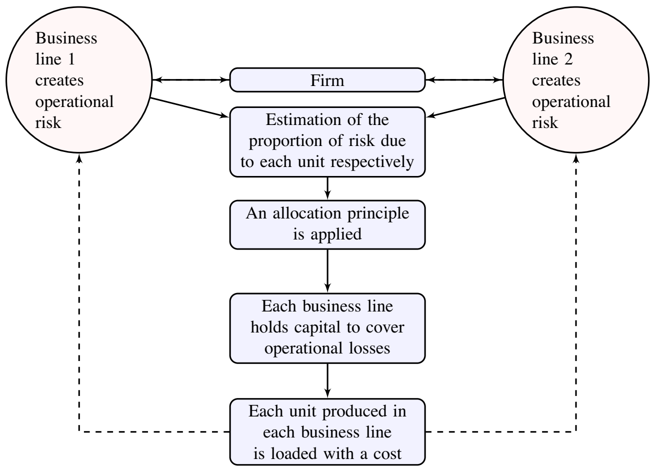

Fig. 1. Operational risk assessment system in a firm with two business lines.

In general, companies seek to allocate capital to their business units for reasons of solvency. Moreover, banks and insurance companies are legally required to set aside an amount of capital so as to guarantee their solvency and so they seek to associate this capital, and hence the loss of returns, to each single unit as a price loading or *risk premium*. The mere existence of operational risk means firms are advised to retain some capital, unless they prefer to purchase an insurance policy to cover operations failures. In this latter case, rather than capital, firms are required to take out an insurance policy, which in the terms described here can be considered an equivalent problem (see, Guillén, Gustafsson, & Nielsen, 2008).

The use of economic capital and its decomposition into a sum of single contributions of sub-businesses has become a standard approach in many banks (see, Rosen & Saunders, 2010) and insurance companies. Myers and Read (2001) and Das and Kratz (2012) propose alarm systems that signals possible ruin based on pattern of premium collection and demands for claim settlement.

The problem of operational risk arising from several risk sources is increasingly present in many areas. Buch, Dorfleitner, and Wimmer (2011) develop a procedure concerning capital allocation that is designed to maximize the Return On Risk-Adjusted Capital (RORAC) of a company. They consider conditions that are required for capital allocation to be a useful tool for obtaining the optimal value of a return function of a decentralized financial firm. They regard the maximization problem as a managerial control problem and embed it into a general systems framework. Zaks and Tsanakas (2014) extend the optimal capital allocation framework of Dhaene, Tsanakas, Valdez, and Vanduffel (2012) and they achieve a compromise between conflicting views of risk within the organization. They allow potentially diverging risk preferences in a hierarchical structure. where stakeholders at two organizational levels (e.g., board members vs line managers) may have conflicting objectives, preferences, and beliefs about risk.

Additionally, capital allocation for operational risk can be a useful tool or indicator for measuring performance and serve as the basis for management incentive schemes (Bolancé, Ayuso, & Guillén, 2012; Bolance, Guillén, Gustafsson, & Nielsen, 2013). Indeed, managerial performance can be assessed by the amount of capital allocated to a firm's respective business units.

Note that while capital allocation is the focus of this study, we do not seek to establish how a firm should determine the capital sum to be allocated, since this is dependent on other characteristics, such as risk aversion and regulatory rules. Thus, we assume the capital to be held for operational losses as given. The main problem we concern ourselves with is the so-called *allocation problem*. Based on the general framework proposed by Dhaene et al. (2012), we provide explicit formulations for different sources of risk of the proportion of capital a manager should allocate.

Our specific contribution is to provide an exact functional form of each allocation principle. In addition, we study the role of correlation. In this context, the correlation effect refers to changes in allocated capital resulting from the correlation between the losses arising in different sources of risk. We show that, in practice, these correlations influence capital allocation (Boucher & Guillén, 2011; Buch-Kromann, Guillén, Linton, & Nielsen, 2011; Englund, Guillén, Gustafsson, Nielsen, & Nielsen, 2008; Sarabia & Guillén, 2008).

Chavez-Demoulin, Embrechts, and Nešlehová (2006) assume that there is some dependence between segments of a firm. Cossette, Mailhot, and Marceau (2012) focus on the computation of the tail value at risk (TVaR) and the TVaR-based allocation for multivariate compound distributions, so they also consider dependence. They provide general formulas for the cumulative distribution function of total cost and the contribution to each risk and they only obtain closed-form expressions for these quantities for multivariate compound distributions in some particular cases, for instance, with gamma and mixed Erlang claim amounts.

In all those approaches the existence of dependence between segments is admitted but its consequences remain rather unexplored.

Many recent contribution propose fraud detection systems in a variety of situations. Jha, Guillén, and Westland (2012) explore fraud in credit cards. Ngai, Hu, Wong, Chen, and Sun (2011) present a comprehensive academic literature review of the data mining techniques that have been applied to financial fraud detection. However, the cost aggregate cost of fraud is usually neglected and how should customer cover this potential loss has not been addressed literature because it is generally assumed that the cost if part of the general budget of a company.

Our research is about the cost of operational risk and how this capital is assigned to each business segment, so we shown an implementation of capital allocation in firms where dependence between segments exists. In our contribution we provide explicit expressions to compute the allocation for some simple cases of risk measure and allocation criterion. We also show how to solve a problem where there are two sources of fraud that produce losses to a bank. We design a system to calculate how much proportion of the capital to cover operational risk arising from fraud should be reserved in each segment and we show that if dependence is taken into account then allocation is not strictly proportional to volume.

The remainder of this article is organized as follows. Section 2 discusses formally what the allocation problem is. The allocation principles are presented in Section 3 while the general framework for capital allocation, based on Dhaene et al. (2012), is also discussed there. An application to fraud is reported in Section 4 and some concluding remarks are provided in Section 5.

2. The general capital allocation problem in risk management

Capital allocation refers to the subdivision of aggregate capital held by a firm across its various constituents, which might be business lines, types of exposure, territories, or even individual products in a portfolio of insurance policies.

Consider a portfolio of n individual random losses $X_1, X_2, \ldots, X_n$ arising from different business lines, materializing at a fixed future date T. Assume that random vector $(X_1, X_2, \ldots, X_n)$ is defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that any loss X_i has a finite mean. The distribution function $\mathbb{P}(X_i \leqslant x)$ of X_i can be denoted by $F_{X_i}(x)$. The aggregate loss is formally defined as:

$$S = \sum_{i=1}^{n} X_i \quad i = 1, \ldots, n. \tag{1}$$

Following Dhaene et al. (2012) we define S as a corporation's total loss. For example, S may be the operational loss of a bank due to fraudulent practices in its credit card and savings account services, respectively. Similar interpretations can be found in McNeil, Frey, and Embrechts (2005).

Given that S is a random variable, we assume that the total capital K set aside to cover S is exogenously determined. The company seeks to allocate K across its various business lines, that is, to determine non-negative real numbers $K_1, \ldots, K_n$ satisfying the full allocation requirement:

$$\sum_{i=1}^{n} K_i = K. \tag{2}$$

This allocation is in some sense a notional exercise, since it does not mean that capital is physically shifted across a firm's various units, since its assets and liabilities continue to be pooled. The allocation exercise, however, could be conducted in order to rank the business units according to lines of profitability. This task might be performed, for example, by determining the returns on the capital allocated to the respective business lines. The units produced in each business line can then be charged with the operational risk costs for that respective line. Fig. 1 shows a typical work flow for capital allocation in operational risk.

The general approach to capital allocation raises the question as to what the appropriate risk capital for an individual investment opportunity might be. Thus the question of performance of the investment is intimately linked to the risk measurement chosen. In practice, a two-step procedure is used (McNeil et al., 2005).

1. Compute the overall risk capital K equal to $\rho(S)$, where S is defined in (1) and ρ is a particular risk measure, such as value at risk (VaR), expected shortfall (ES), or an economic capital (EC (p)) (see Dhaene et al. (2006), Guillén, Maria Sarabia, & Prieto (2013) and Abbasi & Guillén (2013) for detailed explanations and applications and (Alemany, Bolance, & Guillén, 2013) for estimation methods). Coherent measures are more appropriate than non-coherent measures as they guarantee subadditivity. A number of new measures have been proposed in this area and could serve to generalize the interpretation (Belles-Sampera, Merigo, Guillén, & Santolino, 2013).
2. Compute K as $\rho(S)$ and allocate capital K to the individual lines according to some mathematical *capital allocation principle* such that, if K_i denotes the capital allocated to i with potential loss X_i, the sum of K_i fulfills the requirement in (2).

We are interested in the second step of the above procedure.

Given that the correspondence with K_i can be computed in a countless number of ways, additional criteria must be established. A reasonable starting point is to require the allocated capital amounts K_i to be close to the risk created by loss X_i in some appropriately defined sense.

Allocation principles are methods designed to solve the allocation problem, i.e., finding K_i shown in (2) under different risk measures and assumptions as to how risk should be shared within a firm.

The key to obtaining the proportional allocation principles depends on which risk measure ρ is chosen to attribute capital K_i to loss X_i. This idea is formalized as:

$$K_i = \omega\rho(X_i), \quad i = 1, \ldots, n, \tag{3}$$

where K_i is the capital to be allocated to each business unit i, $\rho(\cdot)$ is a risk measure and factor ω is chosen such that the full allocation requirement in (2) is satisfied, this factor takes the following form:

Table 1
Proportional allocation principles.

Name	$\rho(X_i)$	K_i
Haircut	$F_{X_i}^{-1}(p)$	$\frac{K}{\sum_{i=1}^{n} F_{X_i}^{-1}(p)} F_{X_i}^{-1}(p), \quad i = 1, \ldots, n$
Covariance	$Cov(X_i, S)$	$\frac{K}{Var[S]} Cov(X_i, S), \quad i = 1, \ldots, n.$
Conditional Tail Expectation (CTE)	$E\left[X_i \mid S > F_S^{-1}(p)\right]$	$\frac{K}{CTE_p[S]} E\left[X_i \mid S > F_S^{-1}(p)\right], \quad i = 1, \ldots, n.$

p is a fixed probability level, $CTE_p[S] = E\left[S \mid S > F_S^{-1}(p)\right]$. E denotes mathematical expectation. *Var* and *Cov* stand for variance and covariance, respectively. X_i and S are random variables defined in (1). This table summarizes some proportional allocations presented in Dhaene et al. (2012), see also references there in.

$$\omega = \frac{K}{\sum_{i=1}^{n} \rho(X_i)}, \quad i = 1, \ldots, n. \tag{4}$$

Factor (4) can be seen as a weighting scheme for capital allocation, combining (4) and (3) we have the following explicit and general formulation:

$$K_i = \frac{K}{\sum_{i=1}^{n} \rho(X_i)} \rho(X_i), \quad i = 1, \ldots, n, \tag{5}$$

which encompasses the most common proportional allocation principles: Haircut, Covariance and Conditional Tail Expectation (CTE) allocation principle. Table 1 provides a summary for these allocation techniques.

Note that the Haircut allocation principle is based on VaR, which is also denoted as the inverse of the distribution function of $X_i, F_{X_i}^{-1}(p)$for a given fixed probability level p. The capital allocated by this principle does not rely on the structure dependence of the losses X_i of the different business units.

3. Optimal capital allocation

Proportional allocation is not the only criterion available for distributing K. Dhaene et al. (2012) note that "there seems to be a lack of a clear motivation for preferring [...] one method over another, although it appears obvious that different capital allocations must in some sense correspond to different questions that can be asked within the context of risk management". As such, capital allocation methods can be viewed as solutions to specific decision problems. If capital is allocated in such a way that, for each business line, the capital allocated and the loss are sufficiently close to each other, then it is convenient to start from,

$$S - K = \sum_{i=1}^{n} (X_i - K_i), \tag{6}$$

where the quantity $(X_i - K_i)$ expresses the loss minus the capital allocated for business line i. It should be borne in mind that in this setting, business lines cross-subsidize each other, in the sense that "$X_k > K_k$" for one particular line and k does not necessarily lead to "ruin". This unfavorable performance of k may be compensated for by a favorable outcome for one or more values $(X_l - K_l)$ in the other units.

3.1. The optimization problem

Definition 1 (*Optimal Capital Allocation Problem*). Given the aggregate capital $K > 0$, determine the allocated capitals K_i, $i = 1, \ldots, n$, from the following optimization problem:

$$\min_{K_1,\ldots,Kn} \sum_{i=1}^{n} \upsilon_i E\left[\zeta_i D\left(\frac{X_i - K_i}{\upsilon_i}\right)\right], such that, \sum_{i=1}^{n} K_i = K, \tag{7}$$

where the υ_i are non-negative real numbers such that $\sum_{i=1}^{n}\upsilon_i = 1$, the ζ_i are non-negative random variables such that $E(\zeta_i) = 1$, and D is a non-negative function.

Each of the components in the general optimal capital allocation problem in (7) are defined as follows:[3]

- υ_i: The non-negative real number υ_i is a measure of exposure or business volume of the ith unit. These scalar quantities are chosen so that they sum to 1.

  $D\left(\frac{X_i-K_i}{\upsilon_i}\right)$:

  The terms $D(X_i - K_i)$quantify the deviations of the outcomes of the losses X_i from their allocated capital K_i and υ_i normalizes the deviations $D(X_i - K_i)$.

- ζ_i: Non-negative random variables ζ_i with $E(\zeta_i) = 1$ that are used as weight factors for the different possible outcomes of $D(X_i - K_i)$. Different forms of ζ_i are selected and shown in subsequent sections.

The **Quadratic Optimization Criterion** is proposed by Dhaene et al. (2012) as the *General Solution of the Quadratic Allocation Problem* by letting

$$D(x) = x^2, \tag{8}$$

which reduces to

$$\min_{K_1,\ldots,K_n}\sum_{i=1}^{n}E\left[\zeta_i\frac{(X_i-K_i)^2}{\upsilon_i}\right], \quad \text{such that,} \quad \sum_{i=1}^{n}K_i = K. \tag{9}$$

The solution to this minimization problem is given in the following theorem.

Theorem 1. *The optimal allocation problem in (9) has the following unique solution:*

$$K_i = E(\zeta_i X_i) + \upsilon_i\left(K - \sum_{i=1}^{n}E(\zeta_i X_j)\right), \quad i = 1,\ldots,n. \tag{10}$$

A detailed proof of the solution of this minimization problem can be found in Dhaene et al. (2012).

By choosing different forms for ζ_i and using (10), two broad groups of capital allocations can be obtained: business unit driven allocations and aggregate portfolio driven allocations emerge for $\zeta_i = h_i(X_i)$and $\zeta_i = h(S)$, respectively.

Throughout Sections 3.2 and 3.3 we base our procedure on Dhaene et al. (2012), nevertheless our goal is aimed at providing explicit formulation for several allocation principles which are stated in Dhaene et al. (2012). These authors propose some functional forms for $\zeta_i$in order to obtain $E(X_i\zeta_i)$but they do not derive the exact form of $K_i$given an analytical expression for ζ_i. This constitute the goal to be accomplished in Section 3.2 and 3.3.

3.2. Business unit driven allocations

Following Dhaene et al. (2012), in this subsection, we consider the case where the weighting random variables ζ_i in the quadratic allocation problem in (9) are given by

$$\zeta_i = h_i(X_i), \tag{11}$$

with h_i being a non-negative and non-decreasing function such that $E[h_i(X_i)] = 1$, for $i = 1,\ldots,n$. Hence, for each business unit i, the states of the world to which we want to assign the heaviest weights are those under which the business unit performs the worst. Allocations based on (11) are called business unit driven allocations. In this case, the allocation rule in (10) can be rewritten as

$$K_i = E[X_i h_i(X_i)] + \upsilon_i\left(K - \sum_{i=1}^{n}E[X_i h_i(X_i)]\right), \quad i = 1,\ldots,n. \tag{12}$$

For an exogenously given value of K, the allocations K_i are not influenced by the mutual dependence structure between the losses X_i of the different business units. In this sense, we can say that the allocation principle (12) is independent of the portfolio context within which the X_is are embedded and, hence, is indeed business unit driven. Such allocations might be a useful instrument for determining the performance bonuses to be granted to the business unit managers, should it be assumed that each manager deserves to be rewarded for the performance of his own business unit but not additionally rewarded (or penalized) for the interrelationship that exists between the performance of his business unit and that of the other units of the company.

The law invariant risk measure $E[X_i h_i(X_i)]$ assigns to any loss X_i the expected value of the weighted outcomes of this loss, where higher weights correspond to larger outcomes of the loss, that is, to more adverse scenarios. Risk measures and premium principles of this general type are proposed and investigated in Heilmann (1989); Tsanakas (2007); and Furman and Zitikis (2008).

Defining the volumes $\upsilon_i = \frac{E[X_i h_i(X_i)]}{\sum_{i=1}^{n}E[X_i h_i(X_i)]}$, the allocation principle based on (12) can be written as:

$$K_i = \frac{K}{\sum_{i=1}^{n}E[X_i h_i(X_i)]}E[X_i h_i(X_i)]. \tag{13}$$

Different choices of $h_i(X_i)$ give rise to different business driven allocations.

3.2.1. (Pure) Conditional Tail Expectation principle

Having identified the principle for allocating K_i (that is by using the business unit driven principle), we can establish specific forms for $h_i(X_i)$. We can obtain several explicitly functional forms for K_i, for instance, by choosing $h_i(X_i) = \frac{\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right)}{1-F_{X_i}\left(F_{X_i}^{-1}(p)\right)}$, where K_i provides us with the *(Pure) Conditional Tail Expectation* principle.

This principle receives the name the *(Pure) Conditional Tail Expectation* because both the aggregate loss and the losses of each individual business unit are expected values conditional on the top $(1-p)$ losses. Since $CTE(\cdot)$ is applied to S and X_i then we call it the *(Pure) Conditional Tail Expectation* so as to distinguish it from the *Conditional Tail Expectation* principle based on Overbeck (2000), known as the *Overbeck type II allocation principle* which is a special case of the *Aggregate Portfolio Driven Allocation*, see Section 3.3.

By choosing $h_i(X_i) = \frac{\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right)}{1-F_{X_i}\left(F_{X_i}^{-1}(p)\right)}$ multiplying by X_i by considering the expectations, we have:

$$E[X_i h_i(X_i)] = E\left[X_i\frac{\mathbb{I}\left(X_i > F_{X_i}^{-1}(p)\right)}{1-p}\right] = \frac{1}{1-p}E\left[X_i | X_i > F_{X_i}^{-1}(p)\right],$$

where this expression reduces to the *Conditional Tail Expectation*, with $E[X_i h_i(X_i)] = CTE_p[X_i]$, therefore $K_i$is as follows:

$$K_i = \frac{K}{CTE_p(S)}CTE_p(X_i), \tag{14}$$

where $CTE_p(S) = \sum_{i=1}^{n}CTE_p(X_i) = CTE_p\left(\sum_{i=1}^{n}X_i\right)$ which follows from the additivity property of CTE.

[3] See Dhaene et al. (2012) for further details.

3.2.2. Standard deviation principle

The *standard deviation principle* (Bühlmann, 1970) is easily obtained by choosing $h_i(X_i) = 1 + a\frac{X_i - E(X_i)}{\sigma_{X_i}}$, $a \geqslant 0$. Then replacing it in $E[X_i h_i(X_i)]$ and plugging it into (13), we obtain the so-called *standard deviation principle*.

In order to obtain an expression for K_i based upon the *standard deviation principle* we proceed as follows:

$$E[X_i h_i(X_i)] = E\left[X_i + a\frac{X_i^2 - X_i E(X_i)}{\sigma_{X_i}}\right] = E(X_i) + a\sigma_{X_i}.$$

Thus $\sum_{i=1}^n E[X_i h_i(X_i)] = \sum_{i=1}^n E(X_i) + a\sum_{i=1}^n \sigma_{X_i}$ and if and only if $Cov(X_i, X_j) = 0\ \forall i \neq j$, $\sum_{i=1}^n E[X_i h_i(X_i)]$ reduces to $E(S) + a\sigma_S$. Consequently the form taken by K_i based upon the *standard deviation principle* is:

$$K_i = \frac{K}{E(S) + a\sigma_S}\left(E(X_i) + a\sigma_{X_i}\right). \tag{15}$$

3.2.3. Esscher principle

If we let $h_i(X_i)$ be $\frac{e^{aX_i}}{E[e^{aX_i}]}$ with $a > 0$ then K is allocated accordingly by the *Esscher Principle* (Gerber, 1981), as we show below:

$$E[X_i h_i(X_i)] = E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right].$$

Thus, the optimal K_i is similar to (16):

$$K_i = \frac{K}{\sum_{i=1}^n E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]. \tag{16}$$

Table 2 summarizes the business unit driven allocations when different choices of $h_i(X_i)$are made.

3.3. Aggregate portfolio driven allocations

In contrast with the *business unit driven allocation rule*, as in Dhaene et al. (2012) here we consider the case where

$$\zeta_i = h(S), \quad i = 1, \ldots, n, \tag{17}$$

with h being a non-negative and non-decreasing function such that $E[h(S)] = 1$. In this case, the states of the world to which we assign the heaviest weights are those under which the aggregate portfolio performs worst. The allocation rule (10) can now be rewritten as:

$$K_i = E[X_i h(S)] + \upsilon_i(K - E[Sh(S)]), \quad i = 1, \ldots, n. \tag{18}$$

Hence, the capital K_i allocated to unit $i$is determined using a weighted expectation of the loss X_i, with greater weights attached to states of the world that involve a large aggregate loss S. Notice that the allocation principle (18) can be reformulated as[4]

$$K_i = E(X_i) + Cov[X_i, h(S)] + \upsilon_i(K - E[Sh(S)]), \quad i = 1, \ldots, n. \tag{19}$$

This means that the capital allocated to the ith business unit is given by the sum of the expected loss $E[X_i]$, a loading that depends on the covariance between the individual and aggregate losses X_i and $h(S)$, plus a term proportional to the volume of the business unit. A strong positive correlation between X_i and $h(S)$, which reflects that X_i could be a substantial driver of the aggregate loss S, produces a higher allocated capital K_i.

Using aggregate portfolio driven allocations might be appropriate when we want to investigate each individual portfolio's contribution to the aggregate loss of the entire company.

[4] This follows from the fact that $Cov(X_i, h(S)) = E(X_i h(S)) - E(X_i)E(h(S))$solving for $E(X_i h(S))$ we end up with $E(X_i) + Cov[X_i, h(S)]$since $E(h(S)) = 1$.

Table 2
Business unit driven capital allocation.

Name	$h_i(X_i)$	K_i
(Pure) Conditional Tail Expectation (Overbeck, 2000)	$\frac{\mathbb{I}(X_i > F_{X_i}^{-1}(p))}{1 - F_{X_i}(F_{X_i}^{-1}(p))}$ $\quad p \in (0,1)$	$\frac{K}{CTE_p(S)} CTE_p(X_i)$
Standard deviation principle (Bühlmann, 1970)[a]	$1 + a\frac{X_i - E(X_i)}{\sigma_{X_i}}$, $\quad a \geqslant 0$	$\frac{K}{E(S) + a\sigma_S}\left(E(X_i) + a\sigma_{X_i}\right)$
Esscher principle (Gerber, 1981)	$\frac{e^{aX_i}}{E(e^{aX_i})}$, $\quad a > 0$	$K_i = \frac{K}{\sum_{i=1}^n E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]} E\left[\frac{X_i e^{aX_i}}{E[e^{aX_i}]}\right]$

[a] This is true iff $Cov(X_i, X_j) = 0\ \forall i \neq j$. See standard deviation principle.

Following Dhaene et al. (2012), if we define the exposures as $\upsilon_i = E[X_i h(S)]/E[Sh(S)]$, $i = 1, \ldots, n$,the proportional allocation rule is obtained as:

$$K_i = \frac{K}{E[Sh(S)]} E[X_i h(S)]. \tag{20}$$

Using the proportional allocation principle shown in (20) by choosing a structure for $h(S)$, the researcher/practitioner can adopt various ways to allocate K. For instance, let us consider a particular choice for $h(S)$ as being $h(S) = S - E(S)$. This yields the covariance allocation principle displayed in Table 1.

3.3.1. Covariance allocation principle

Here, we derive the *covariance allocation principle* from the general setting presented in the previous section. Let $h(S)$ be the deviation of S from its mean so that $h(S) = S - E(S)$, then $E[X_i h(S)] = Cov(X_i, S)$ and $E[Sh(S)] = Var(S)$. Hence, in order to obtain an expression for allocating capital K among the various business units (X_i with $i = 1, \ldots, n$), we have:

$$K_i = \frac{K}{Var[S]} Cov(X_i, S), \quad i = 1, \ldots, n. \tag{21}$$

This is exactly the same expression as that shown in (Table 1), which indicates that the *covariance principle* is a special case of the *aggregate portfolio driven allocation* when choosing $h(S) = S - E(S)$.

3.3.2. Overbeck allocation principles

The *Overbeck type I allocation principle* refers to the principle obtained by setting $h(S) = 1 + a\frac{S - E(S)}{\sigma_S}$, $a \geqslant 0$, while the *Overbeck type II allocation principle* refers to that obtained when using $h(S) = \frac{1}{1-p}\mathbb{I}\left(S > F_S^{-1}(p)\right)$, with $p \in (0,1)$.

As in the previous Sections, we now proceed to identify an explicit expression for K_i by setting $h(S) = 1 + a\frac{S - E(S)}{\sigma_S}$, $a \geqslant 0$. This implies $E[X_i h(S)] = E(X_i) + \frac{a}{\sigma_S} Cov(X_i, S)$ and $E[Sh(S)] = E(S) + a\sigma_S$. As such, the *Overbeck type I allocation principle* has the following form:

$$K_i = \frac{K}{E(S) + a\sigma_S}\left[E(X_i) + \frac{a}{\sigma_S} Cov(X_i, S)\right]. \tag{22}$$

On the other hand, the *Overbeck type II allocation principle* is determined by letting $h(S)$ be $\frac{1}{1-p}\mathbb{I}\left(S > F_S^{-1}(p)\right)$ with $p \in (0,1)$, where $F_S^{-1}(p)$represents the quantile function of S, for $E[X_i h(S)]$ and $E[Sh(S)]$, so that we have:

$$E[X_i h(S)] = \frac{1}{1-p} E\left[X_i \mathbb{I}\left(S > F_S^{-1}(p)\right)\right]$$

$$E[Sh(S)] = \frac{1}{1-p} E\left[S \mathbb{I}\left(S > F_S^{-1}(p)\right)\right] = CTE_p(S).$$

Therefore, K_i can be written as:

$$K_i = \frac{K}{CTE_p(S)} E\left[X_i | \mathbb{I}\left(S > F_S^{-1}(p)\right)\right]. \tag{23}$$

Note this principle is exactly the same as that presented above in Table 1.

A particularly interesting relationship between the *Overbeck type I allocation principle* examined here in (3.3.2) and the *standard deviation allocation principle*, (22) and (15), respectively, is given by:

$$K_i = \frac{K}{E(S) + a\phi}\left(E(X_i) + \frac{a}{\sigma}\gamma\right). \tag{24}$$

The *Overbeck type I* principle is retrieved by (24) when choosing $\phi = \sigma_S^2$ and $\gamma = Cov(X_i, S)$, whereas the *standard deviation principle* is recovered when setting $\phi = \sigma_S$ and $\gamma = Cov(X_i, X_i) = Var(X_i) = \sigma_{X_i}^2$.

3.3.3. Wang allocation principle

Let us consider $h(S) = \frac{e^{aS}}{E[e^{aS}]}$ with $a > 0$, we have $h(S) = \frac{e^{aS}}{E[e^{aS}]}$ and the expression for $E[X_i h(S)]$ is $\frac{E(X_i e^{aS})}{E(e^{aS})}$, then $E[Sh(S)] = \frac{E(Se^{aS})}{E(e^{aS})}$. Therefore, the allocation of the exogenously given aggregate capital K to n parts $K_1, \ldots, K_n$ corresponding to the different business units can be carried out using:

$$K_i = \frac{K}{E(Se^{aS})} E(X_i e^{aS}). \tag{25}$$

3.3.4. Tsanakas allocation principle

If we let $\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})}$ be $h(S)$ with $a > 0$, then this leads us to the Tsanakas (2009) principle.

Expressions for constructing K_i are as follows:

$$E[X_i h(S)] = E\left[X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right],$$

$$E[Sh(S)] = E\left[S \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right],$$

where the K_i to be allocated takes the following form:

$$K_i = \frac{K}{E\left[S\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]} E\left[X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]. \tag{26}$$

Letting Ψ be $\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma$, then K_i could be rewritten as:

$$K_i = \frac{K}{E(S\Psi)} E(X_i \Psi). \tag{27}$$

Table 3 summarizes the *aggregate portfolio driven allocations* by providing expressions for K_i.

4. An application to fraud in the banking sector

We present a practical example of capital allocation and discuss alternative approaches to dealing with operational risk. To do so, we draw on two sources: *Public data risk No. 1* and *Public data risk No. 2* from (Bolancé, Guillén, Gustafsson, & Nielsen, 2012). These sources comprise 1000 and 400 observed operational loss amounts, respectively, for two business lines over one year. In our application we assume that these data have been collected from a bank, which has to cover (1) the risk of losses due to fraudulent operational mistakes in savings accounts and (2) fraudulent transactions with credit cards.

We quantify the capital requirements using risk measures for these two types of operational loss. We assume that the total capital to be held K is calculated as the empirical VaR at 99% of the aggregate loss $S = X_1 + X_2$, where X_1 is the sum of losses due to fraud in savings accounts and X_2 is the sum of losses attributable to fraudulent transactions with credit cards. We compare the proportional allocation principle with the aggregate portfolio driven allocation principles: Haircut, Covariance and Overbeck type II and show that dependence cannot be ignored in risk management (Guillén, Prieto, & Maria Sarabia, 2011, 2012).

Some descriptive insights are provided in Table 4, where perhaps the most notable feature is the difference in the number of observations in each vector of losses: *Public data risk No. 1* has 1000 observations, whereas *Public data risk No. 2* has 400 observations. As such, the data are characterized by a strong right asymmetry. This behavior is typical in loss data analyses and has been mentioned by various authors (Bolancé, Guillén, & Nielsen, 2010; Buch-Larsen, Nielsen, Guillén, & Bolance, 2005).

The numerical exercises presented below comprise two cases: in the first, the dependence structure between the two lines of business is eliminated in the simulation procedure and, in the second, a strong correlation structure between losses is artificially created. The aim of these two scenarios is to verify the performance of the allocation principles when two extreme situations occur.

4.1. Case I: absence of dependence structure

In this subsection, we assess the performance of allocation principles when losses exhibit a low degree of linear dependence, indicating that the correlation coefficient between the yearly losses is close enough to zero.

Let $(x_{1,1}, \ldots, x_{1,1000})$ and $(x_{2,1}, \ldots, x_{2,400})$ be vectors consisting of 1000 and 400 observations on individual losses, moreover *Risk*

Table 4
Descriptive statistics for numerical example data.

	Public data risk No. 1	*Public data risk No. 2*
Number of Observations	1000	400
Minimum	0.00	0.00
Maximum	5122.14	1027.53
Mean	42.06	20.89
Median	3.47	4.29
Sum	42,059.41	8357.32
Standard deviation	291.96	95.91
Skewness	13.61	9.10
Kurtosis	210.87	89.20

Table 3
Aggregate portfolio driven capital allocation.

Name	$h(S)$	K_i
Covariance principle	$S - E(S)$	$\frac{K}{Var[S]} Cov(X_i, S)$
Overbeck Type I (Overbeck, 2000)	$1 + a\frac{S-E(S)}{\sigma_S}, \quad a \geqslant 0$	$\frac{K}{E(S)+a\sigma_S}\left[E(X_i) + \frac{a}{\sigma_S} Cov(X_i, S)\right]$
Overbeck Type II (Overbeck, 2000)	$\frac{1}{1-p}\mathbb{I}(S > F_S^{-1}(p)), \quad p \in (0,1)$	$\frac{K}{CTE_p(S)} E[X_i \vert S > F_S^{-1}(p)]$
Wang (Wang, 2007)	$\frac{e^{aS}}{E(e^{aS})}, \quad a > 0$	$\frac{K}{E(Se^{aS})} E(X_i e^{aS})$
Tsanakas (Tsanakas, 2009)	$\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma, \quad a > 0$	$\frac{K}{E\left[S\int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]} E\left[X_i \int_0^1 \frac{e^{\gamma aS}}{E(e^{\gamma aS})} d\gamma\right]$

No. 1 is now denoted by $X_1 = \sum_{j=1}^{1000} x_{1,j})$ and *Risk No. 2* is denoted by $X_2 = \sum_{j=1}^{400} x_{2,j})$. We estimate risk using the following Monte Carlo simulation method:

1. We draw 1000 random observations from vector *Public data risk No. 1* with replacement, we sum them and obtain a simulated value of X_1.
2. We draw 400 random observations from vector *Public data risk No. 2* with replacement, we sum them and obtain a simulated value of X_2.
3. We repeat steps (1) and (2) 10,000 times to obtain vectors of length equal to 10,000 for risk No. 1 and No. 2.

Note that we hold the number of losses constant throughout this process. The number of losses could fluctuate, so we could randomly choose the number of losses in each iteration. However, we introduce this simplification to avoid extra variability in this example.

An estimate of the 99% quantile of the sum of X_1 and X_2 can be obtained from the simulation above. In our example, it equals $75,573.96$. We assume that this capital, which is much greater than the sum of all the losses experienced over a year[5], should be high enough to cover all operational loss expenses in the next year. The allocation problem determines how much of this risk should be charged to savings accounts and how much should be charged to credit cards. We do not study whether or not each savings account or each credit card, should be charged the same. We only wish to determine how these two lines of business should share the risk.

Table 5 shows the capital allocated to the lines based on different capital allocation principles. These results show the amount of capital to be set aside for each source of risk. Here, proportional allocation is calculated simply on the proportion of aggregated losses observed in each risk line with respect to total losses. Note that the Haircut allocation principle results are very similar to those of proportional allocation as the correlation between losses is almost null. Our simulation assumed independence and therefore, the correlation between X_1 and X_2 is ≈ 0.00014. The covariance allocation principle assigns more capital than the other principles to the saving accounts line of business, i.e. to Risk No. 1.

In this part of the exercise, therefore, we can conclude that the covariance allocation principle performs worse than the other two principles. In the next section we introduce a strong dependence structure in order to assess the performance of the allocations that account for a correlation between the losses.

4.2. Case II: strong dependence structure

This section can be seen as the counterpoint to the previous one since we examine the case at the other extreme, that is, one involving a strong dependence framework.

In order to create two vectors of strongly correlated replications of X_1 and X_2 we base the sampling scheme on the Monte Carlo simulation of quantile-based extractions, based in turn on the same sequence of probability levels in steps (1) and (2) below. The steps are as follows:

1. We draw $M = 10,000$ values from a $U(0,1)$ uniform distribution, which we call p_k, where m refers to the replicate number $m = 1, \ldots, M$.
2. We generate two vectors of $10,000$ simulated values of risk No.1 and risk No. 2. We use the marginal empirical distributions of X_1 and X_2 obtained in the Monte Carlo simulation described in Case I. So, using an estimate of the empirical distribution of X_1 and X_2, we obtain $F_{X_1}^{-1}(p_k) + F_{X_2}^{-1}(p_k)$ and $S_m = F_X^{-1}(p_k) + F_Y^{-1}(p_k), m = 1, \ldots, M$.

[5] Note that the sum of losses in Table 4 equals $50,416.73 = 42,059.41 + 8,357.32$

Table 5
Case I. Capital allocation based on different principles under the independence hypothesis. Proportions are shown in parentheses.

	Risk No. 1	Risk No. 2
Proportional	63,058.91 (83.44%)	12,515.05 (16.56%)
Haircut	62,953.00 (83.30%)	12,620.96 (16.70%)
Covariance	72,497.53 (95.93%)	3076.43 (4.07%)
Overbeak II	66,070.96 (87.43%)	9503.00 (12.57%)

Total capital is 75,573.96 (100.00%).

Table 6
Case II. Capital allocation based on different principles assuming strong correlation between lines of business.

	Risk No. 1	Risk No. 2
Proportional	524,607.8 (83.44%)	104,116.8 (16.56%)
Haircut	412,897.2 (65.67%)	215,827.3 (34.33%)
Covariance	464,021.7 (73.80%)	164,702.9 (26.20%)
Overbeck II	414,842.6 (65.98%)	213,882.0 (34.02%)

Total capital is 628,724.6 (100.00%).

The correlation coefficient between X_1 and X_2 is now equal to 0.8875.

Adhering to the same idea as in the previous section, we consider the total capital to be allocated as the empirical VaR at 99% which is now $628,724.60$.

Table 6 presents the total capital and the amounts to be allocated to each business unit in the scenario in a situation of strong dependence. Note that the first business line is again the riskiest one and so more capital is allocated to it. Moreover, as the linear dependence between these two lines of business increases, all the allocation principles give similar outcomes w, with the exception of the proportional allocation.

Although the haircut principle is based on the idea of measuring stand-alone losses using a normal VaR, it performs well enough even if the correlation is high. The covariance allocation and the Overbeck type II allocation principles are based on coherent risk measures but, in our example, the two principles produce similar results.

In practice, a series of yearly data is necessary to assess the existence of a significant correlation between lines of business. Extreme scenarios, such as the one shown here, can help managers to make decisions about allocating risk costs under various scenarios.

5. Conclusions and future research

In this study we have presented a capital allocation problem in line with Dhaene et al. (2012) and we have provided explicit formulations for K_i when using various specifications for business unit driven principles and for aggregate portfolio driven allocations, which constitutes our main contribution.

Our contribution is to add the issue of fraudulent practices that arise from several sources to the existing body of knowledge concerning systems available for analyzing and detecting fraud (for example, Sahin, Bulkan, & Duman (2013), Bae & Lee (2012) and Duman & Ozcelik (2011)). We show that recent developments concerning capital allocation Xu and Mao (2013) can be usefully implemented in fraud management systems. In addition, we have given the explicit expressions for some basic capital allocation principles.

The numerical exercise conducted here shows that the configuration of capital allocations depends on the degree of linear dependence. The Haircut allocation principle, even though it is based on a non-coherent risk measure, performs well and is less influenced by the correlation effect. Haircut allocations are very similar to those suggested by the Overbeck type II principle when the correlation is high, which further confirms the good performance of the Haircut allocation principle.

We conclude that the failure to account for correlation may lead to risk management practices that treat the units contributing to risk unfairly. Our example, based on data from the banking sector, shows that operational risk evaluation and the allocation of costs due to events of this nature depend significantly on the choice of a decision principle.

We do not address conflicting risk preferences within a firm, so a limitation of our study is that we do not decide which allocation principle rule and which risk measure should be undertaken. We also do not decide how to determine how much dependency is used in a capital allocation method to quantify the contribution of each risk. However, we argue that the Haircut principle is more robust than the other are therefore we recommend this principle.

Future studies should help managers decide which allocation principle and risk measure is more appropriate to their business. We also think that combining allocation rules that set up a capital to be held by each business unit with fraud propensity scores, where each individual unit has a propensity to defraud, would lead to a full allocation of operational costs that would be personalized.

In practice many banks would consider all sources of internal fraud together. That means that there would be homogeneity and a flat rate would be paid by all customers. Our method shows that segments can be charged for operational risk according to a risk evaluation and a risk criterion that would take into consideration the dependence between lines of business. Therefore, in the future, sources of fraud should contribute to capital held for operational risk according to a capital allocation criterion. However, managers should decide which principle and which risk measures is considered.

For future research directions we point at the need to look at potential fraudulent operations in such a way that not all transactions contribute equally to operational risk. Therefore, we argue that credit card or savings accounts holders should pay for operational risk capital according to an allocation criterion. As a result, customers who have more propensity to defraud should contribute more than those with less propensity and they should also bear the costs of capital needed to cover operational risk due to fraudulent transactions accordingly.

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