ATS 421/521

Climate Modeling

Spring 2013

Lecture 7

- Numerics

April 24

Reading

- For Friday: Manabe and Strickler (1964)

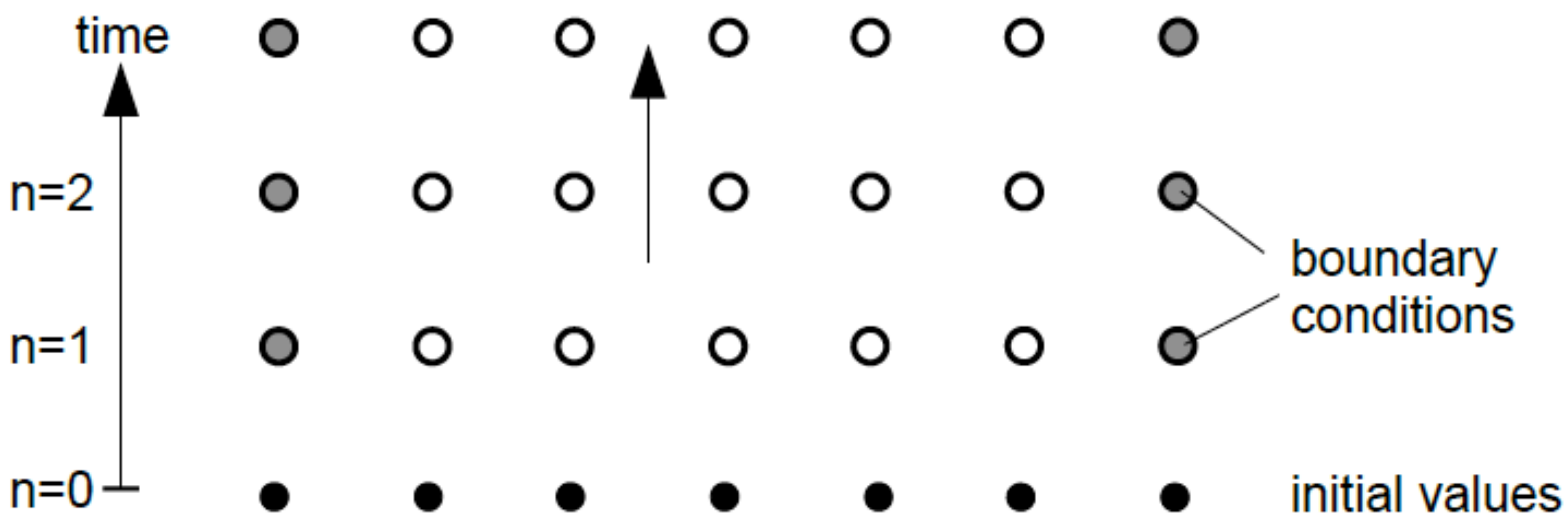
Numerics

Script chapter 2.6

Important criteria for numerical schemes:

1) Convergence for $\Delta x, \Delta t \to 0$
2) Stability
3) Accuracy
4) Conservation
5) Behavior of Amplitudes and Phases
6) Positive definite
7) No (or Small) Numerical Artifacts

Boundary Conditions



Two types of boundary conditions:

- Dirichlet: specify values
- Neuman: specify normal gradients

Of which type are our 1D EBM boundary conditions?

Develop *T* in Taylor series around *t*:

t may be replaced by any spatial dimension (e.g. x,y,z)

$$T(t+\Delta t)=T(t)+\frac{dT}{dt}|_t\Delta t+\frac{1}{2!}\frac{d^2T}{dt^2}|_t(\Delta t)^2+\ldots \qquad (2.23)$$

neglecting these terms gives the "Centered Differences" scheme

more accurate than Euler Forward since errors scale with

$$(\Delta t)^2$$

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$$\longrightarrow \boxed{\frac{dT}{dt}\Big|_t = \frac{T(t+\Delta t) - T(t-\Delta t)}{2 \cdot \Delta t}} \underbrace{- \frac{1}{3!}\frac{d^3T}{dt^3}\Big|_t (\Delta t)^2 - \ldots}_{\text{correction of order } (\Delta t)^2}$$

neglecting these terms gives the "Centered Differences" scheme

more accurate than Euler Forward since errors scale with $(\Delta t)^2$

Consider centered differences: $\frac{\partial C}{\partial x} \simeq \frac{C_{m+1} - C_{m-1}}{2\Delta x}$

with $C = \hat{C}\cos(kx)$ represented numerically as $C_m = \hat{C}\cos(kx)$

and $C_{m+1} = \hat{C}\cos(k(x+\Delta x))$

The exact solution is $\frac{\partial C}{\partial x} = -\hat{C}\, k \sin(kx)$

using $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

we get for the numerical solution $\frac{C_{m+1} - C_{m-1}}{2\Delta x} = -\frac{\hat{C}}{\Delta x}\sin(kx)\sin(k\Delta x) \underset{\Delta x \to 0}{\rightarrow} \frac{\partial C}{\partial x}$

Wave number

$k = 2\pi/(n\Delta x)\,;\, n = 2, 3, \ldots$

$$\frac{C_{m+1} - C_{m-1}}{2\Delta x} / \frac{\partial C}{\partial x} = \frac{\sin(k\Delta x)}{k\Delta x}$$

n	$\sin(k\Delta x)/(k\Delta x)$
3	0.41
4	0.64
6	0.82
8	0.9

For fixed Δx only waves with large n (large wavelengths) are well represented. Waves shorter than 8Δx have errors > 10 %.

Consider **advection equation**:

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}(uC) \tag{2.26}$$

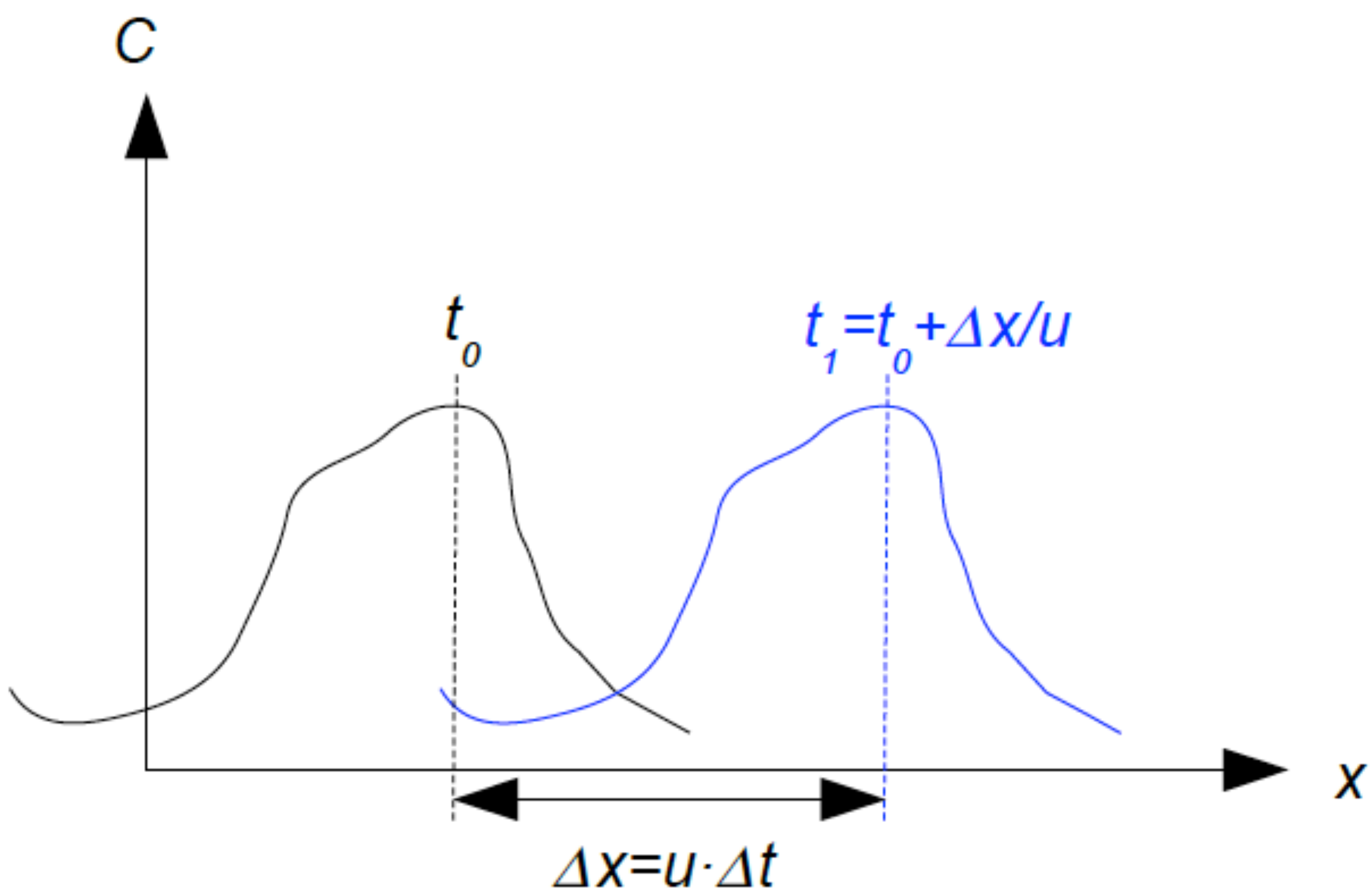
fluid velocity   fluid property (e.g. temperature)

Assume u = const.:

$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} \tag{2.27}$$

Arbitrary function f is solution if

$$C(x,t) = f(x-ut) \tag{2.28}$$



Von Neuman Stability Analysis

Assume wave function at time $t=0$:

$$C(x,0)=Ae^{ikx}=A(\cos(kx)+i\sin(kx)) \quad \text{, with } i^2=-1 \qquad (2.29)$$

At time t the solution is a plane wave: $C(x,t)=Ae^{ik(x-ut)}$

wave number	wavelength	angular frequency	period	frequency
$k=\frac{2\pi}{\lambda}$	$\lambda=\frac{2\pi}{k}=\frac{u}{\nu}$	$\omega=\frac{2\pi}{T}$	$T=\frac{2\pi}{\omega}=\frac{1}{\nu}$	$\nu=\frac{1}{T}=\frac{u}{\lambda}$

Now solve eq. (2.27) numerically by discretizing time and space:

$$t=n\Delta t \qquad n=0,1,2,...$$

$$x=m\Delta x \qquad m=0,1,2,...$$

$$C(x,t)=C(m\cdot\Delta x, n\cdot\Delta t)=C_{m,n}=\xi^n e^{ikm\Delta x} \quad , \qquad (2.31)$$

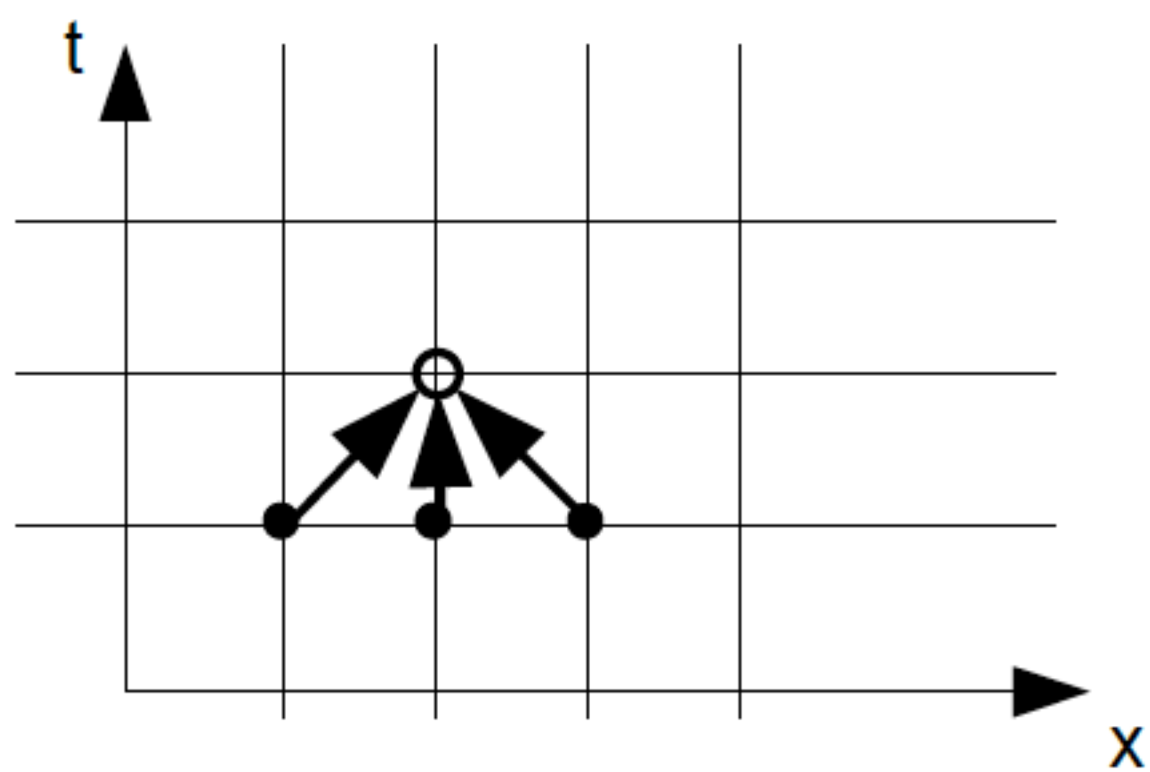
with the amplification factor $\xi(k)$. Each time step the solution is multiplied by ξ . Thus, if $\xi>1$ the solution will diverge (blow up) and if it is $\xi<1$ it will be damped.

Now let's examine the FTCS (forward in time centered in space) scheme:

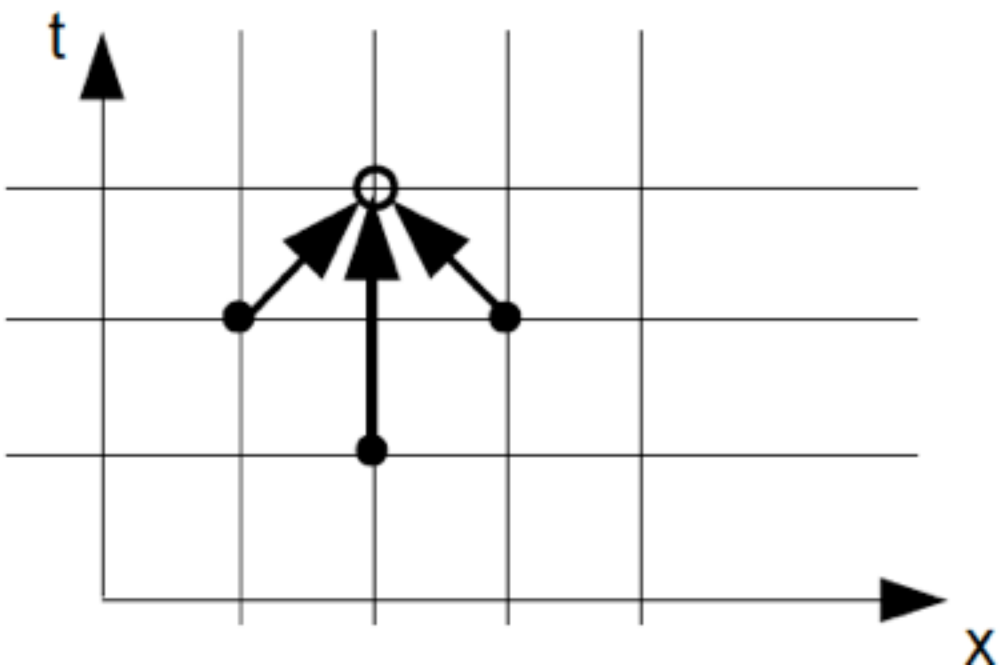
$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=-u\frac{C_{m+1,n}-C_{m-1,n}}{2\cdot\Delta x}$$

$$\rightarrow \xi=1-i\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$$

Thus $|\xi|>1$ for all *k*. The FTCS scheme is unconditionally unstable and therefore useless.



Now let's use centered differences (2.25) for eq. (2.26)



$$\frac{C_{m,n+1}-C_{m,n-1}}{2\cdot\Delta t}=-u\frac{C_{m+1,n}-C_{m-1,n}}{2\cdot\Delta x} \tag{2.32}$$

$$C_{m,n+1}=C_{m,n-1}-\frac{u\cdot\Delta t}{\Delta x}\left(C_{m+1,n}-C_{m-1,n}\right) \tag{2.33}$$

This is the CTCS (centered in time, centered in space), or "leap-frog" scheme. The first time step has to be taken by a Euler scheme and two time steps in the past need to be stored in memory.

Insert the analytical solution eq. (2.31) in (2.33):

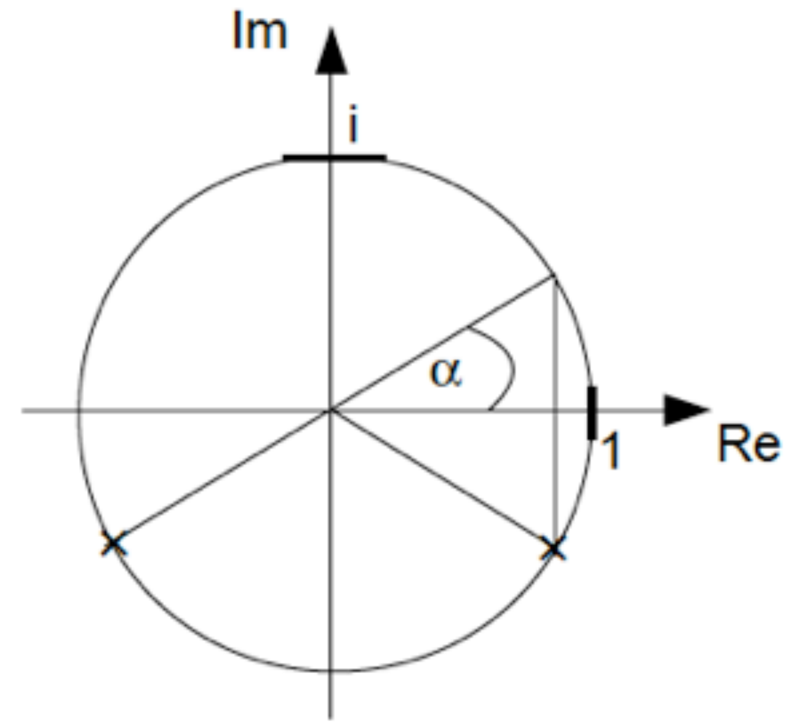
$$\xi=\xi^{-1}-\frac{u\Delta t}{\Delta x}\left(e^{ik\Delta x}-e^{-ik\Delta x}\right)$$

$$\Leftrightarrow \xi^2=1-2i\sigma\xi \tag{2.34}$$

with $\sigma=\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$. The solution of this quadratic equation is

$$\xi=-i\sigma\pm\sqrt{1-\sigma^2} \tag{2.35}$$

$$\xi = -i\,\sigma \pm \sqrt{1-\sigma^2} \qquad (2.35)$$



We distinguish two cases:

<u>Instable case</u> $|\sigma| > 1$:

$\xi = -i(\sigma \pm S)$, with $S = \sqrt{\sigma^2 - 1} > 0$.

If $\sigma > 1$ => $\sigma + S > 1$ => $|\xi^n| \to \infty$.

If $\sigma < -1$ => $\sigma - S < -1$ => $|\xi^n| \to \infty$.

<u>Stable case</u> $|\sigma| \leqslant 1$:

We can express sigma as a sine function $\sigma = \sin(\alpha)$ and using the trigonometric relation $\sin^2(\alpha) + \cos^2(\alpha) = 1$ we see that the solution of $\xi = -i\sin(\alpha) \pm \cos(\alpha)$ has an absolute value of one, it lies on the unit circle in the complex plane

$\xi = \left\{ \begin{matrix} e^{-i\alpha} \\ e^{i(\alpha+\pi)} \end{matrix} \right\}$, with

$$\boxed{C_{m.n} = \xi^n e^{ikm\Delta x} \quad , \qquad (2.31)}$$

Now insert this in eq. (2.31) we get

$$C_{m,n} = (Me^{-i\alpha n} + Ee^{i(\alpha+\pi)n})\, e^{ikm\Delta x} \qquad (2.36)$$

and

$$C_{m,0} = (M+E)\, e^{ikm\Delta x} \quad , \qquad (2.37)$$

$$C_{m,n}=(Me^{-i\alpha n}+Ee^{i(\alpha+\pi)n})e^{ikm\Delta x} \tag{2.36}$$

and

$$C_{m,0}=(M+E)e^{ikm\Delta x} \quad , \tag{2.37}$$

thus with (2.29) $A=M+E$ or

$$C_{m,n}=\underbrace{(A-E)e^{ik(m\Delta x-\frac{\alpha n}{k})}}_{P}+\underbrace{(-1)^n E\, e^{ik(m\Delta x+\frac{\alpha n}{k})}}_{N} \quad , \tag{2.38}$$

with a physical mode P, and a numerical mode N, which changes sign each time step. Now we only have to determine E. For the first time step we have

$$C_{m,1}=C_{m,0}-\frac{u\Delta t}{2\Delta x}(C_{m+1,0}-C_{m-1,0}) \tag{2.39}$$

with (2.37) we get

$$C_{m,1}=A(1-isin(\alpha))e^{ikm\Delta x}=(A-E)e^{ikm\Delta x-i\alpha}-Ee^{ikm\Delta x+i\alpha}$$

Solve for E and enter into eq. (2.38) yields

$$C_{m,n}=A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x-\frac{\alpha n}{k})}+(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x+\frac{\alpha n}{k})} \quad . \tag{2.40}$$

It can be shown that (2.40) converges to (2.30) provided $\Delta x\to 0$ it follows that $\sigma\to uk\Delta t$ and for $\Delta t\to 0$ it follows that $\sigma\ll 1$ and hence $\sigma=\sin(\alpha)\simeq\alpha$ and (2.40) converges to

$$C_{m,n}\to \underbrace{A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(x-ut)}}_{P}+\underbrace{(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(x+ut)}}_{N}\to A\,e^{k(x-ut)} \quad .$$

$$C_{m,n} \to \underbrace{A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(x-ut)}}_{P} + \underbrace{(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(x+ut)}}_{N} \to A e^{k(x-ut)} \quad .$$

Thus, the leapfrog scheme is stable (provided $|\sigma| \leqslant 1$) and it converges against the true solution. However, for finite time steps and finite grid spacing a numerical solution N appears, which is unphysical. The physical solution P describes a plane wave traveling towards the right, whereas N changes sign every time step and travels towards the left.

The condition for stability $|\sigma| = |(u\Delta t/\Delta x)\sin(k\Delta x)| \leqslant 1$ must hold for all wavelength, thus it follows that $|(u\Delta t)/(\Delta x)| \leqslant 1$, which can be regarded as a condition for the maximum time step

$$\boxed{\Delta t \leqslant \frac{\Delta x}{|u|} \quad .} \tag{2.41}$$

CFL criterion

(Courant-Friedrichs-Lewy, 1928)

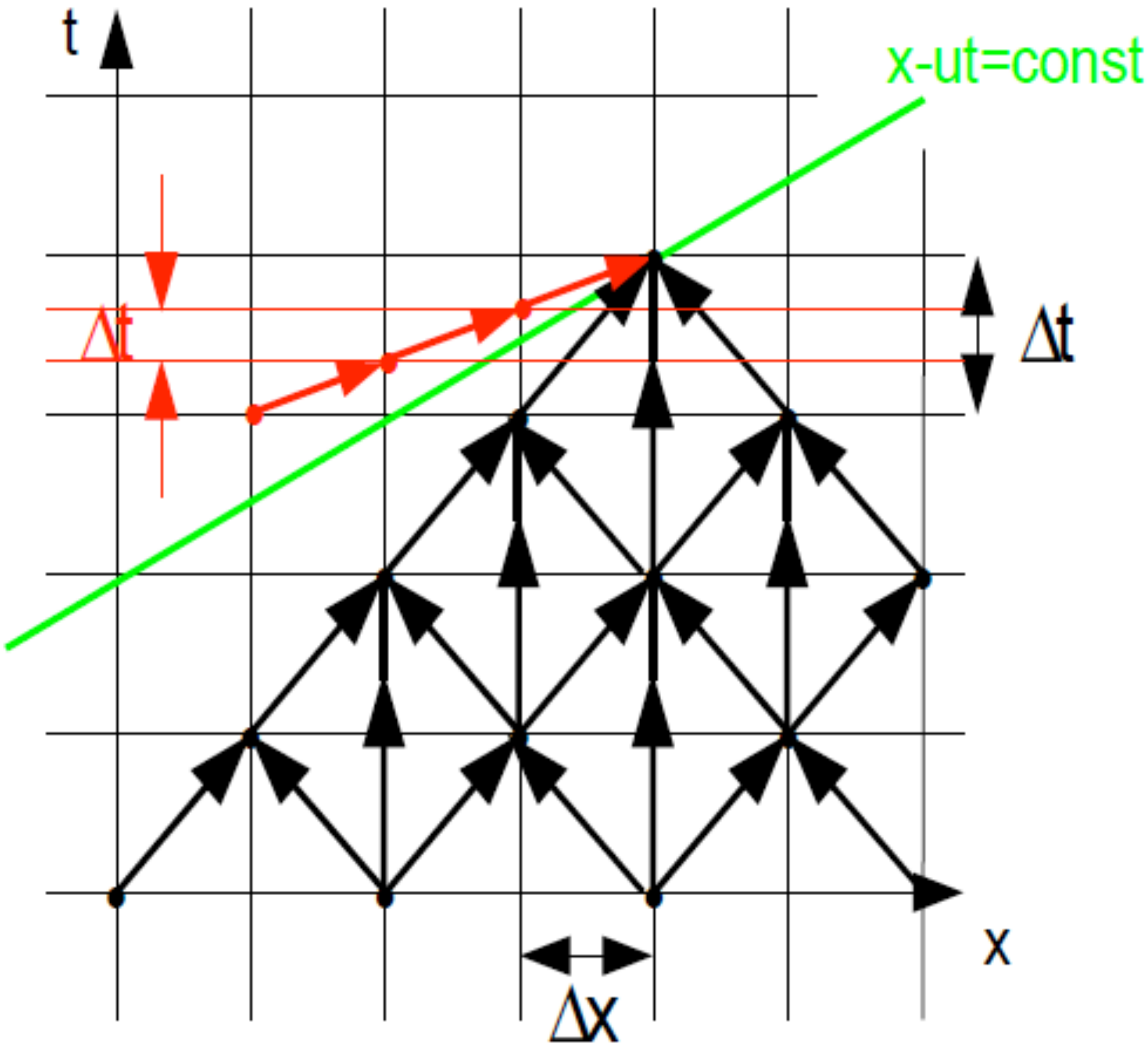
The CFL criterion limits the maximum possible time step.

For Δx = 300 km

ocean: max(u) = 1 m/s => $\Delta t < 3$ days

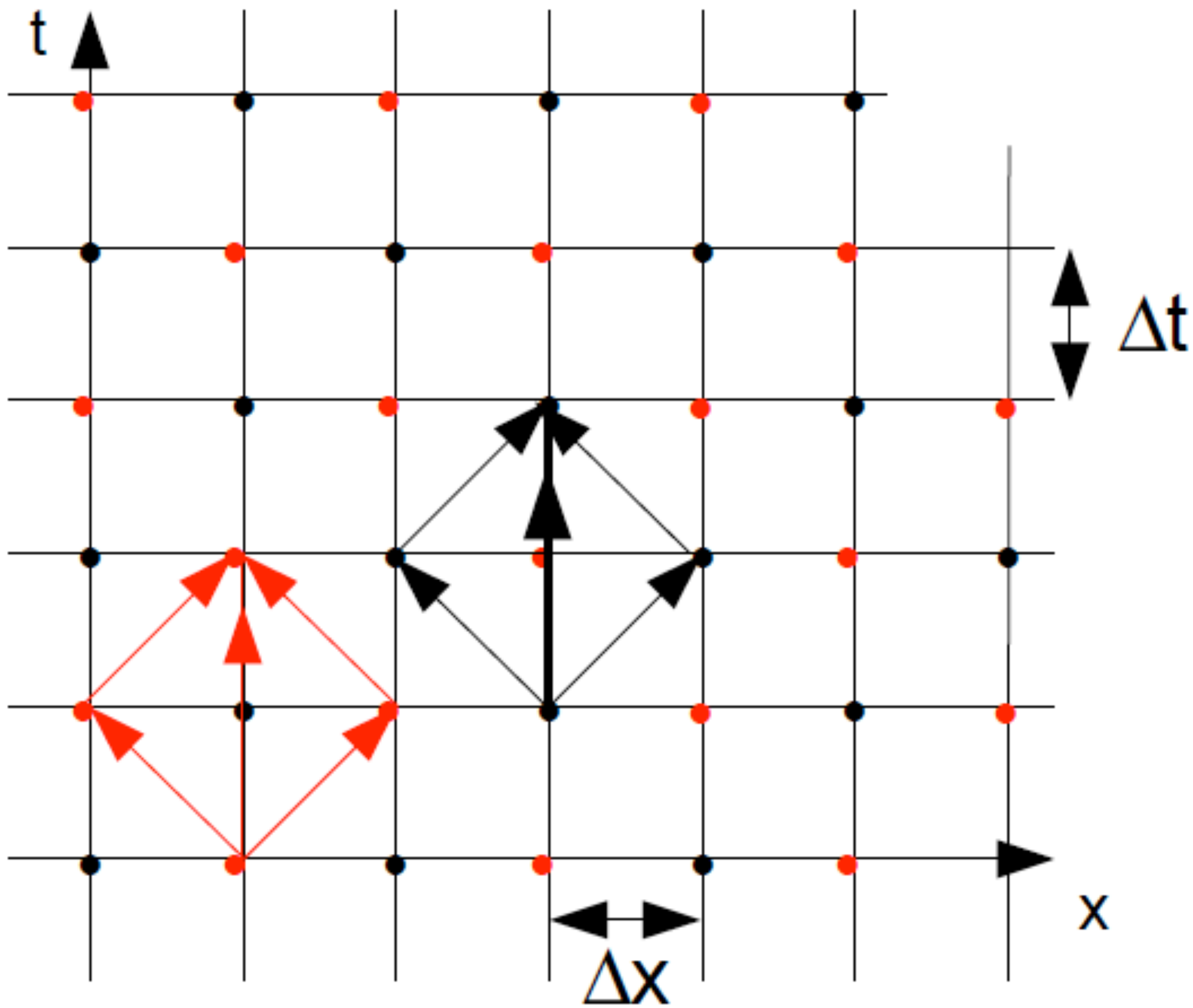
atmosphere: max(u) = 80 m/s => $\Delta t < 1$ hour

CFL criterion



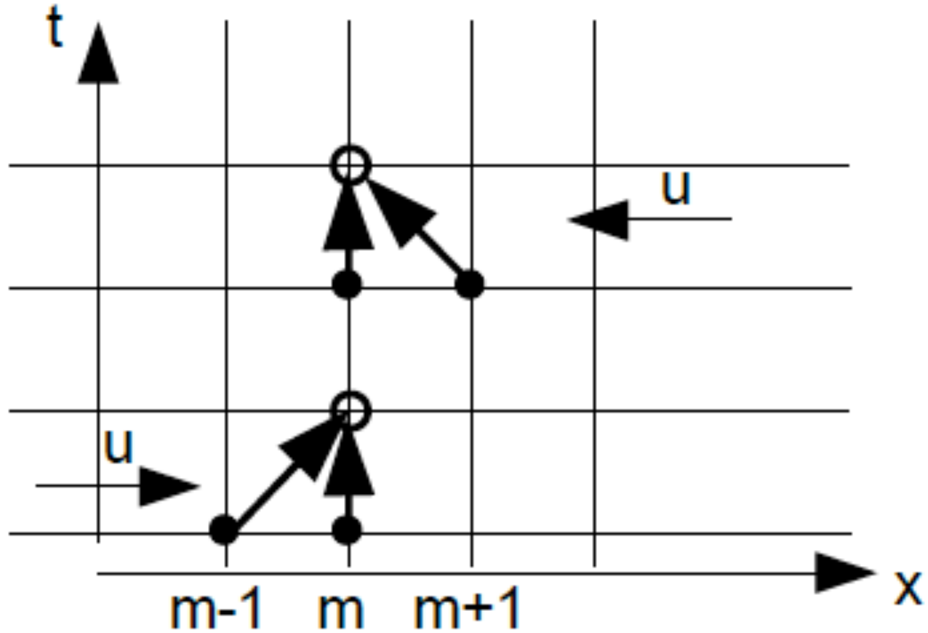
Signal propagates faster than the cone of influence for large time step Δt.
Signal propagates slower than the cone of influence for small time step Δt.

Numerical Mode (artifact)



Decoupling of red and black grid points.
Can be removed by using an Euler (FTCS) time step.

The Upwind Scheme



$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=-u\left\{\begin{matrix}\dfrac{C_{m,n}-C_{m-1,n}}{\Delta x}, u>0\\ \dfrac{C_{m+1,n}-C_{m,n}}{\Delta x}, u\leqslant 0\end{matrix}\right\}$$

$$\xi=1-\left|\frac{u\Delta t}{\Delta x}\right|(1-\cos(k\Delta x))-i\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$$

$$|\xi^2|=1-2\left|\frac{u\Delta t}{\Delta x}\right|(1-\left|\frac{u\Delta t}{\Delta x}\right|)(1-\cos(k\Delta x))$$

Again CFL criterion for stability.

Advantage: Positive definite

Disadvantage: only first order accurate (numerical diffusion)

Other Schemes

- Prather: higher order terms are calculated and stored (positive definite, very accurate, no numerical diffusion but requires more memory and computations)
- FCT (Flux corrected transport)

Consider **diffusion equation**: $$\frac{\partial C}{\partial t} = K \frac{\partial^2 C}{\partial x^2}$$

FTCS: $$\frac{C_{m,n+1} - C_{m,n}}{\Delta t} = K \frac{C_{m+1,n} - 2C_{m,n} + C_{m-1,n}}{\Delta x^2} , \qquad (2.42)$$

$$C_{m,n+1} = C_{m,n} + \frac{K \Delta t}{\Delta x^2} (C_{m+1,n} - 2C_{m,n} + C_{m-1,n})$$

$$\xi = 1 - \frac{4 K \Delta t}{(\Delta x)^2} \sin^2 (\frac{k \Delta x}{2})$$

$$\xi^2 = 1 - 2 \frac{4 K \Delta t}{(\Delta x)^2} \sin^2 (\frac{k \Delta x}{2}) + \left(\frac{4 K \Delta t}{(\Delta x)^2} \right)^2 \sin^2 (\frac{k \Delta x}{2})$$

$$|\xi| \leqslant 1 \longrightarrow \Delta t \leqslant \frac{(\Delta x)^2}{2 K}$$

Analogous to CFL criterion.

FTCS **stable** for diffusion equation.

FTCS: $$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n}-2C_{m,n}+C_{m-1,n}}{\Delta x^2}, \qquad (2.42)$$

Replace n with n+1:

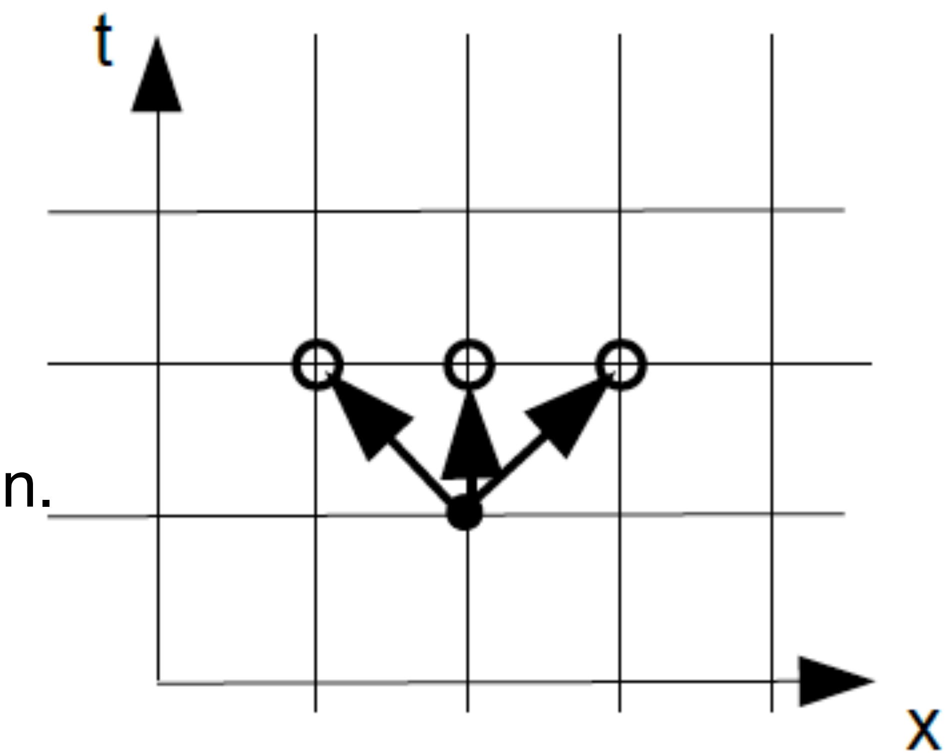
$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n+1}-2C_{m,n+1}+C_{m-1,n+1}}{\Delta x^2}$$

fully implicit (or backward in time) scheme

Can be solved by solving set of linear equations:

$$-\alpha C_{m-1,n+1}+(1+2\alpha)C_{m,n+1}-\alpha C_{m+1,n+1}=C_{m,n}$$

with $\alpha=K\Delta t/(\Delta x)^2$



Tridiagonal system can be solved by matrix inversion.
Unconditionally stable for any Δt !
Only first order accurate: numerical diffusion (not a big problem here since we're solving a diffusion equation, but for advection equation it is an issue).