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Multiple solutions in energy balance climate models

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ABSTRACT

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A review of multiple solutions in energy balance climate models is presented. The paper begins with the zero dimensional models with ice cap albedo feedback and shows that three solutions exist by elementary algebra. A proof of the slope stability theorem is presented for this case. Next, the one dimensional models are covered with special attention to the multiple branch structure. The appropriate form of the slope stability theorem is recounted for this case. Finally, an account of some recent studies of multiple solutions in seasonal climate models is presented. These studies indicate that some rather exotic behavior occurs when land masses are near the poles. In particular, cases with snow-free versus snow covered areas over the summer can occur as different solution branches for the same external conditions.

Introduction

The modern study of energy balance models (EBMs) was initiated by the remarkable papers by Budyko (1968, 1969) and Sellers 1969). These authors formulated one layer thermodynamic models of the earth's zonally averaged, mean annual surface temperature field as a balance between the net energy incoming to a strip around the earth and that outgoing. Each had a term for solar radiation absorbed, terrestrial radiation emitted and a simplified rule for divergence of heat for the strip. Budyko's model was somewhat simpler and lent itself to analytical solution. Seller's model incorporated slightly more physical formulations of the individual terms, utilizing for example a nonlinear formula for the outgoing radiation and a separation of latent and sensible heat transport. Both models included one extremely important nonlinear feedback mechanism: ice-albedo feedback. As a consequence of the latter both models produced solutions which were sensitive to solar constant changes and both even exhibited climatic catastrophes. Most dramatic was the plunge to an ice-covered planet when the solar constant is lowered by only a few percent.

These pioneering studies have opened the way to many further inquiries as to the meaning and relevance of the early results. First of all Schneider and Gal-Chen (1973) examined the stability of the present climate solution and the deep freeze solution numerically: both were stable. After that many other papers followed. An early review of the so-called hierarchy of climate models was written by Schneider and Dickinson (1976) and a later one which included many results of analytical studies was written by North et al. (1981). In this paper I summarize the results of the many studies that have concentrated on multiple solutions and stability properties of those solutions. While much remains to be done in this area, there is emerging a fairly clear picture of these phenomena and their origins. I will occasionally indulge in speculating on what might lie ahead.

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Zero dimensional models

The study of EBMs begins with the simplest of all models, the energy balance of the whole planet. We formulate here that model in order to illustrate what is to come. We take a very simple Budyko (1968, 1969) form for the outgoing terrestrial radiation, I:

$$I = A + BT_0 \tag{1}$$

where T_0 is the planetary average surface temperature (°C), A and B are empirical constants taken, for example, from satellite data (Short et al., 1984); for our purposes they can be taken to be 200.0 W m^{-2} and 2.10 W m^{-2} °C^{-1}, respectively. On a mean annual basis if the planet is in equilibrium, the outgoing radiation must be balanced by the absorbed solar radiation, which is given by $Qa_p(T_0)$, where Q is the solar constant divided by four (ratio of the area of a sphere to the earth's disk silhouette); Q can be taken to be 340 W m^{-2}. The planetary average of the coalbedo, $a_p(T_0)$ is allowed to be a function of the planetary average surface temperature, since the reflectivity might, for example, depend upon the amount of snow on the surface. The balance can be expressed in equation form:

$$A + BT_0 = Qa_p(T_0) \tag{2}$$

The left hand side can be plotted on the same graph as the right hand side so that roots can be found to see how such an equilibrium climate looks. This is shown in Fig. 1. Actually, a better way of viewing the relation is to rearrange eq.

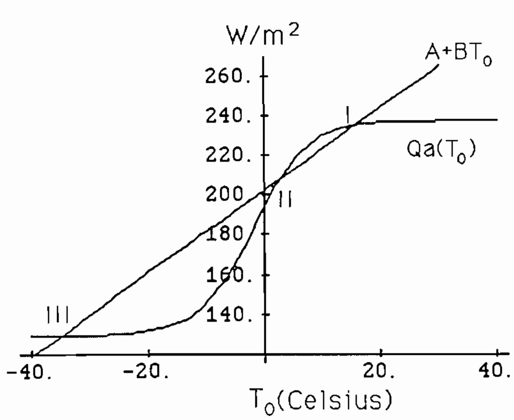


Fig. 1. Graphs of the left and right hand sides of Eq. 1 versus the global average temperature T_0. The intersections of the curves correspond to equilibrium climates.

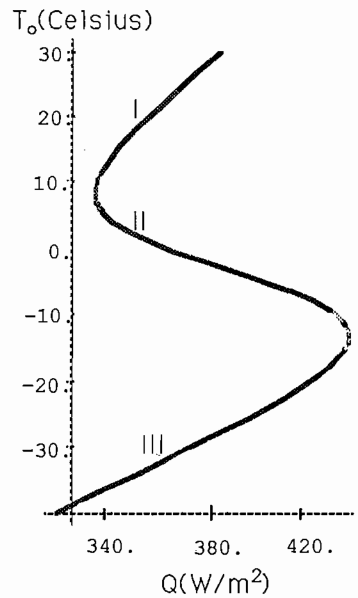


Fig. 2. Plot of solar constant (divided by 4) versus the equilibrium global temperature that is associated with it.

(2) to find what value of Q works for a given T_0:

$$Q = (A + BT_0)/a_p(T_0) \tag{3}$$

Figure 2 shows the result of evaluating Eq. 3; this kind of graph is called the *operating curve*. Note that three distinct solutions exist for a small range of solar constants, denoted *I*, *II*, and *III*. The one designated *I* is the closest to the present climate, while *III* is an ice-covered earth (so-called deep freeze) solution. Curiously, there is an intermediate solution designated *II*. In the early days it was thought that this solution might be related to the ice ages. Now we know otherwise. It is very interesting that if the earth resides on solution *I* and the solar constant is lowered by only a few percent the system finds only one solution available and that is the ice-covered planet, *III*.

Now consider the stability of the solutions. To do this we imagine that the system is in equilibrium at a value T_e corresponding to the value Q. Now allow the system to be perturbed away from equilibrium. Its new value at time t after the perturbation is $T(t) = T_e + \delta T(t)$. A new term must be added to account for the imbalance of the left and right sides of Eq. 2, it is the rate of storage $C\,\mathrm{d}T/\mathrm{d}t$, where C is the heat capacity/area of the globe.

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = -A - BT + Qa_{\mathrm{p}}(T) \tag{4}$$

To the first order in $\delta T(t)$ we can write

$$C\frac{\mathrm{d}}{\mathrm{d}t}\delta T(t) = \left[-B + Qa_{\mathrm{p}}'(T_{\mathrm{e}})\right]\,\delta T(t) \tag{5}$$

An alternative expression to that in the brackets can be found by differentiating the equilibrium condition (2) with respect to Q (considering Q now to be a function of T_{e} and vice-versa):

$$-a_{\mathrm{p}}\frac{\mathrm{d}Q}{\mathrm{d}T_{\mathrm{e}}} = \left[-B + Qa_{\mathrm{p}}'(T_{\mathrm{e}})\right] \tag{6}$$

which allows us to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta T = -\lambda\,\delta T \tag{7}$$

where

$$\lambda = \frac{a_{\mathrm{p}}(T_{\mathrm{e}})}{C}\frac{\mathrm{d}Q}{\mathrm{d}T_{\mathrm{e}}} \tag{8}$$

The sign of λ is determined solely by the sign of the slope of the operating curve $\mathrm{d}Q/\mathrm{d}T_{\mathrm{e}}$. We can tell immediately which solutions on an operating curve will be stable and which will not merely by checking the sign of the slope. For example, in Fig. 2 we see that solutions *I* and *III* are stable while *II* is not. This result is known as the slope-stability theorem (Cahalan and North, 1979). The simple proof given above first appeared in a letter to me from Robert Dickinson (ca. 1978).

In some cases such as in the zero and one dimensional models one can carry the stability analysis even further by constructing the potential function for the problem (North et al., 1979). That is, we can find a function $F(T)$ such that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{1}{C}\frac{\mathrm{d}F}{\mathrm{d}T} \tag{9}$$

where for the zero dimensional model

$$F(T) = AT + \tfrac{1}{2}T^2 - Q\int_0^T a_{\mathrm{p}}(T')\,\mathrm{d}T' \tag{10}$$

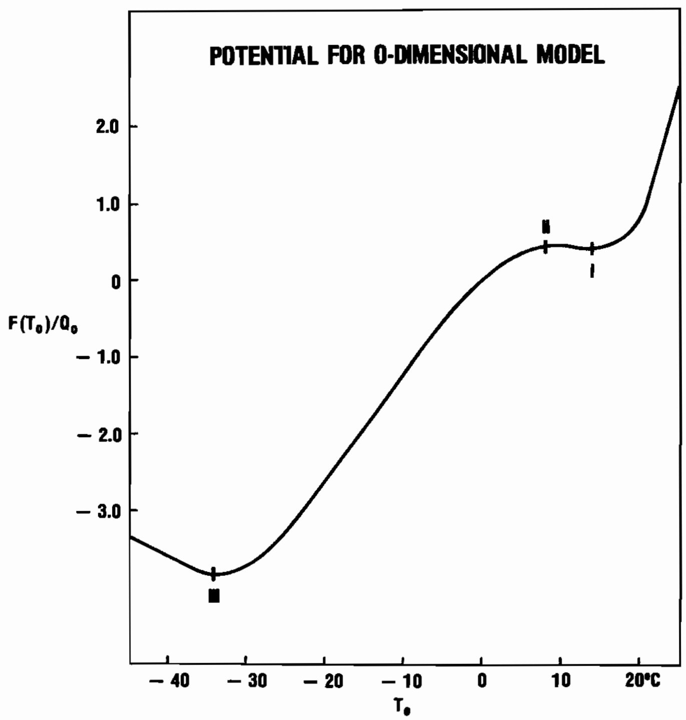


Fig. 3. Plot of the potential function for a zero dimensional climate model, Eq. 10.

The potential function is sketched in Fig. 3. The useful property of the potential function is that it is always decreasing as the climate relaxes after a perturbation. This is easily demonstrated (using the chain rule and (9)):

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t} = -C\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right)^2 \tag{11}$$

The significance of the last step is that one has a qualitative analysis of the stability problem even for finite amplitude perturbations. One learns how large the perturbation needs to be to push the system over the hill in Fig. 3 out of the shallow minimum representing the present climate and into the deep minimum corresponding to the deep freeze.

One dimensional models

Now we suppose that the surface temperature field is to be resolved in latitude but we retain the mean annual averaging. In making this step we would like to introduce as few new parameters as possible. We can proceed by using the same outgoing radiation formula as before, but we must now be a little more careful with the albedo, introducing some latitude dependence to account for the increased albedo with solar zenith angle. Taking $x = \sin(\textit{latitude})$, we use for the coalbedo over ice free areas, $a(x) = a_0 + a_2 P_2(x)$, where $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second Legendre polynomial and $a_0 = 0.70$ and $a_2 = -0.08$, which come from fairly old satellite data (North, 1975). Smaller values of $a(x)$ are taken over ice covered areas, indicating the increased reflectivity of ice. Recent studies have used a fraction, say one half of the ice free value (Mengel et al., 1988), $a_i(x) = a(x)/2$. The function $a(x)$ is taken to have a discontinuity at the ice edge, x_s; hence, we rename the coalbedo $a(x, x_s)$. One has to question the sharpness of the discontinuity in the real world where, after all, what we mean is that zonal and annual averages are to be taken. Much is to be learned by retaining the discontinuity and its significance will be brought out several times in what follows.

Each zonal strip of surface is to be in equilibrium regarding incoming and outgoing heat flux. We have learned how to include the radiative components of this budget and must now turn to the net heat flux carried into a strip by horizontal motions of the geophysical fluid system. We will introduce a diffusive approximation for the average horizontal flow of heat. If we group the terms of the energy budget for an infinitesimal strip $\mathrm{d}x$ at latitude x and divide through by four times its area $4\pi R^2\,\mathrm{d}x$ (R is earth radius), we obtain the energy balance equation in spherical coordinates

$$-\frac{\mathrm{d}}{\mathrm{d}x} D_0 (1 - x^2) \frac{\mathrm{d}}{\mathrm{d}x} T(x) + A + BT(x) = QS(x) a(x, x_s) \tag{12}$$

where we have introduced a constant diffusion coefficient D_0 and the ice line x_s is to be determined by the condition

$$T(x_s) = T_s = -10\,^\circ\mathrm{C} \tag{13}$$

A final condition that must be met is that the solution must be constrained by boundary conditions at the limits of the domain of x. If the globe is taken to be north–south symmetric this can be expressed as the condition that no net heat shall be conducted into the pole or across the equator:

$$\sqrt{1 - x^2}\,\frac{\mathrm{d}T}{\mathrm{d}x} = 0, \quad x = 0, 1 \tag{14}$$

Even though the equation is nonlinear because of the ice-albedo feedback, it can still be solved analytically (Held and Suarez, 1974; North, 1975a, b). The trick is to decompose the temperature field into Legendre Polynomials

$$T(x) = \sum_{n\ \mathrm{even}} T_n P_n(x) \tag{15}$$

This latter form insures that the boundary conditions will be satisfied for any truncation level of the series. Inserting the series into the energy balance equation, projecting out the Fourier-Legendre amplitudes and recomposing the series, we obtain

$$T(x) = Q \sum_{n\ \mathrm{even}} H_n(x_s) P_n(x)/L_n - A/B \tag{16}$$

$$L_n = n(n + 1) D_0 + B \tag{17}$$

where

$$H_n(x_s) = (2n+1)\int_0^1 S(x)a(x, x_s)P_n(x)\,dx \tag{18}$$

The ice-line condition (12) can now be satisfied by evaluating Eq. 15 at $x = x_s$ and forcing (12). This gives us a relation between x_s and Q; in fact, Q can be solved for in a manner analogous to Eq. 3. When this is done, we have an expression for the equilibrium value of the solar constant for which the ice line is at the given position x_s (for more detail see North et al., 1981). The result can be plotted as the operating curve for a one dimensional mean annual model and is shown in Fig. 4. This remarkable operating curve shows that there are *five* solutions for the present value of the solar constant. There are solutions for ice-free, ice-covered and three intermediate icy solutions.

As in the case of the zero dimensional models, there is a slope stability theorem for this class. Early special cases were discussed by many authors including Held and Suarez (1974), Ghil (1976), Drazin and Griffel (1976) and North (1975a, b), but the most complete discussion comes from Cahalan and North (1979). The theorem may be stated in terms of the operating curve just discussed. *Negative sloped branches are unstable while flat and positively sloped branches are stable.* The proof starts as before by assuming a small perturbation $\delta T(x, t)$ can be written in the form

$$\delta T(x, t) = \delta T(x)\, e^{-\lambda t} \tag{19}$$

After inserting this form in the time dependent equation (as in Eq. 4 dT/dt is added to the energy balance equation) and neglecting quadratic and higher terms in δT, one reaches (after much manipulation) an identity for the stability parameter λ:

$$\frac{dQ}{dx_s} = \lambda \sum_{n\ \mathrm{even}} \frac{b_n}{L_n(L_n - \lambda)} \tag{20}$$

where dQ/dx_s is the local slope of the operating curve and the b_n are a sequence of positive numbers. Equation 20 is to be solved for the roots λ_J, $J = 0, 1, \ldots$ These roots are the eigenvalues of the stability problem. Both sides of Eq. 20 are graphed in Fig. 5. Note that if the slope dQ/dx_s is negative, there will be one negative root λ_0, indicating one unstable mode. If

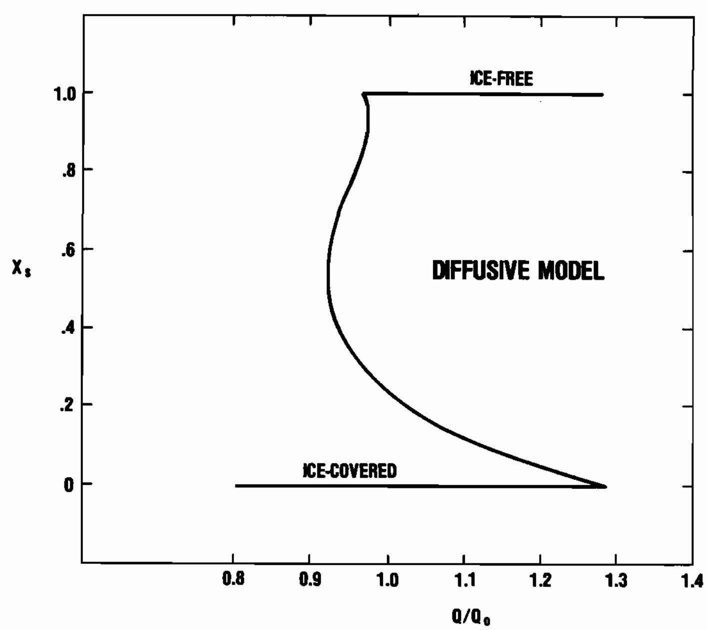


Fig. 4. Plot of the equilibrium value of ice line x_s for a given value of solar constant Q divided by its present value Q_0.

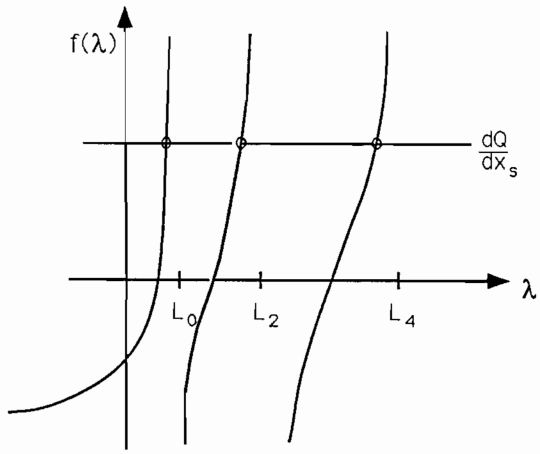


Fig. 5. Plot of the slope of the operating curve, dQ/dx_s, and the right hand side of Eq. 20 versus the stability parameter λ. Intersections of the curve correspond to the stability eigenvalues, positive roots meaning stable modes, negative meaning unstable. If the slope is negative, there will be one negative root.

the slope of the operating curve is positive, the roots will all be positive.

It is noteworthy that a potential function can also be derived for the one dimensional model paralleling the treatment for the one dimensional case (North et al., 1979). Unfortunately, the function is many dimensional. In principle, we would have an axis for each mode amplitude T_n as in the series (15). Such a figure is shown in North et al. (1979) for a two mode truncation. The resulting contour diagram is similar to that of Fig. 3 with a shallow depression for the present climate and a deep one for the aptly named deep freeze. A similar approach was discussed by Ghil (1976). If the time dependent form of the EBM is adopted and the solution is allowed to proceed toward equilibrium it could land on any of the stable solutions of the operating curve for a given value of the solar constant and depending on the initial conditions. Figure 7 shows the T_0, T_2 plane. Several solution trajectories are followed from arbitrarily chosen initial values of T_0 and T_2. Each trajectory proceeds until it finds its appropriate point in the diagram (the so-called attractor). The right-most attractor is the ice-free earth solution, the one in the center is the present climate solution and the one on the left is the ice-covered earth solution (these curves were kindly given to me by William T. Hyde).

Small ice caps

A very interesting feature of Fig. 4 needs further elaboration—the cusp at $x_s = 1$. An enlarged view of this portion of the operating curve is shown in Fig. 6 (taken originally from Drazin and Griffel (1977) and adapted by North (1984)). According to this operating curve, any ice cap smaller than one extending from the pole

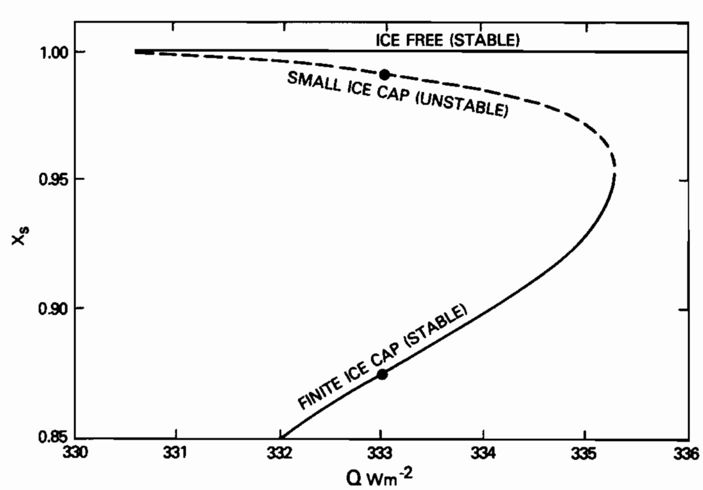


Fig. 6. An enlargement of the cusp region of the operating curve shown in Fig. 4 for a one dimensional climate model with ice-cap albedo feedback (originally from Drazin and Griffel, 1977; as adapted by North, 1984).

to about 70° (corresponding to $x_s \geq 0.95$) will be unstable. Most of the earlier studies of EBMs including my own dismissed the small ice cap instability (SICI) as being an unphysical artifact of the simple climate model parameterizations. Indeed, several studies showed that if the albedo at the ice edge is smoothed sufficiently, SICI goes away. This smoothing for the real earth is, of course, quite reasonable considering that we are averaging through the seasons and through the longitudes. The ice and snow edge being so irregular through these dimensions leads us to believe that such a peculiar feature should be dismissed. Before I explain why this attitude is premature, let us consider the meaning of SICI.

Lindzen and Farrell (1977) first pointed out that the simple one dimensional models have an inherent length scale $l \sim \sqrt{D/B}$ and that its size is about 20° on a great circle of the earth. It is reasonable to expect that features smaller than that would be unstable in a model solution. North (1984) elaborated further, noting that a physical interpretation of the length scale is possible: thermal anomaly at a point on the earth diffuses laterally by random walk a distance proportional to the square root of the time; if the time to be inserted in just taken to be one radiative relaxation time, then the dis-

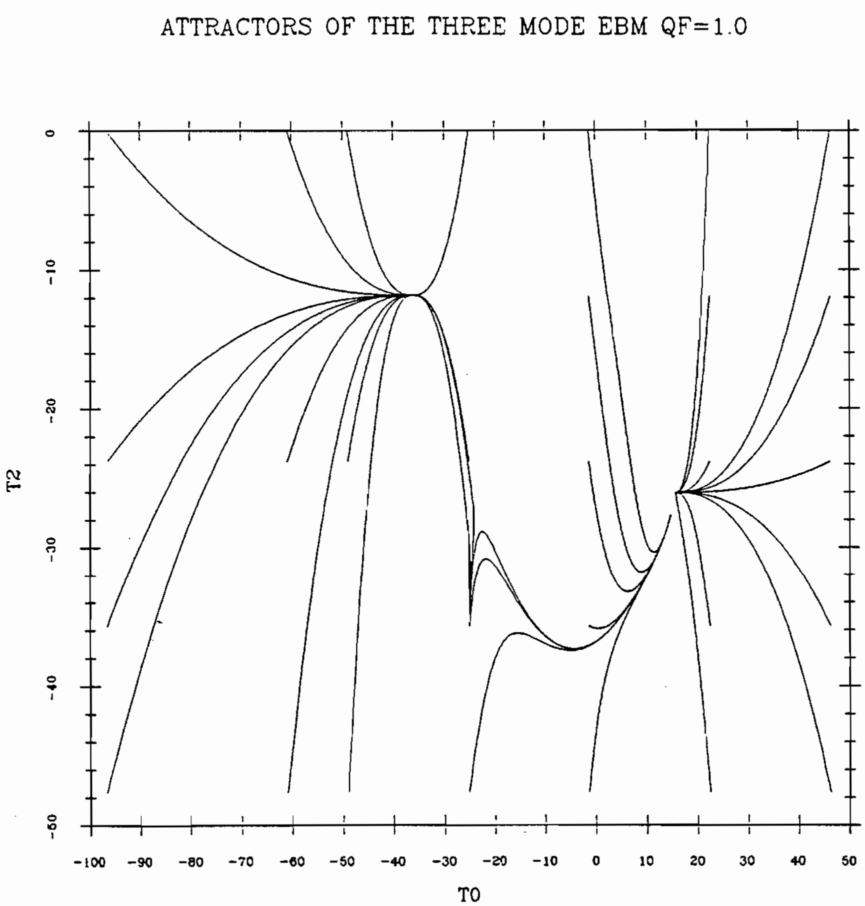


Fig. 7. Time dependent solution trajectories in the plane of the first two temperature mode amplitudes for a variety of initial conditions. The solar constant (divided by 4) is 330.0 W m^{-2} as in Fig. 6. There are three stable end points (attractors): the right most is the ice-free earth, the center one is the "present climate", and the left most is ice-covered earth (Courtesy of William T. Hyde, pers. comm.).

tance obtained is the length scale referred to. In fact, the influence function for a steady point source of heat is proportional to $K_0(r/l)$, the modified Bessel function, which for large values of its argument goes as $K_0(z) \sim (\pi/2z)^{1/2} \exp(-z)$. This last shows that a small feature has an appreciable influence out a distance of order l (keep in mind that the influence is also proportional to the strength of the heating or cooling source).

Based upon the above considerations it is clear that features far away ($\gg l$) will not have much influence on local conditions. For instance, in EBMs things happening locally in the Southern Hemisphere will have very little influence on the North polar ice cap. For that reason, it is not too surprising that one can have a small ice cap in one hemisphere and not in the other as found by Drazin and Griffel (1977) when they relaxed the boundary condition (14) to apply only at the poles (a net heat flux across the equator was allowed).

It is now fairly easy to see why there can be two stable solutions in the neighborhood of the cusp in Fig. 6. Imagine a solution with no ice cap (the linear problem when $x_s = 1$) in which the temperature at the pole is slightly above the critical point (this can be constructed by adjusting the solar constant downwards to the value $Q = 333.0$ W m^{-2} in Fig. 6). For that situation the temperature at the pole is $-9.0\,^\circ$C. This solution is steady and assuming no infinitesimal perturbations it will remain so. Now suppose a small ice cap is introduced at the pole (by hand). The addition of this white patch will cool the environment over a considerable region, possibly bringing the temperature below the critical $-10\,^\circ$C value and creating a real ice sheet. The analysis leads to two such values of ice sheet size (one unstable) which will be in equilibrium according to the energy balance equation (North, 1984). North also elaborated on why SICI went away under various conditions such as smoothing (e.g., most smoothing prescriptions would not allow for a very strong influence for a tiny test patch of ice).

Now what of the relevance of SICI considering its dependence on such features as the sharp albedo contrast between ice and ice-free areas? Actually, snow and ice boundaries are sharp as any view of the earth from space will show and this seems to be born out by recent albedo data taken from the Earth Radiation Budget Experiment (Ramanathan et al., 1989; and very recent data by G.L. Smith, pers. comm., 1990). The problem is not with the absence of discontinuity at the ice or snow margin but rather with our models which did not resolve the seasons or the longitudes. In other words, SICI may exist in the real world but it may not be a simple disk centered at the pole. Some preliminary findings will be shown in the next section where the seasonal cycle is resolved but a simple zonally symmetric geography is retained.

Multiple solutions in seasonal models

Seasonal climate models and their fit to the data for the present climate have been discussed by North and Coakley (1979), North et al. (1983) and Hyde et al. (1989). We confine ourselves here to the limited question of whether there are interesting features such as multiple solutions. While studies with nonlinear two dimensional seasonal models are underway at present, we shall limit this discussion to zonally symmetric land-sea distributions. The models are exactly the same as in (11) except for two modifications. A thermal storage rate term must be added, $C(x)\,\mathrm{d}T/\mathrm{d}t$, which takes into account the fact that the heat capacities over land and ocean differ by two orders of magnitude. The seasonal forcing has to be included in the solar distribution, $S(x, t)$. Finally, nonlinearity is introduced by allowing the snowline and its associated albedo discontinuity to be rigidly attached to the instantaneous seasonally varying $0\,^\circ$C isotherm (Mengel et al., 1988).

A diagram of the snowline cycle for a specific land-sea distribution (a land cap analogous to Antarctica) for two infinitesimally close values of the solar constant is shown in Fig. 8 (Mengel et al., 1988). One solution has snow-free summers over the land cap, while the other has the entire polar region snow free in summer (see also Suarez and Held, 1978; Watts and Hayder, 1984). Fig-

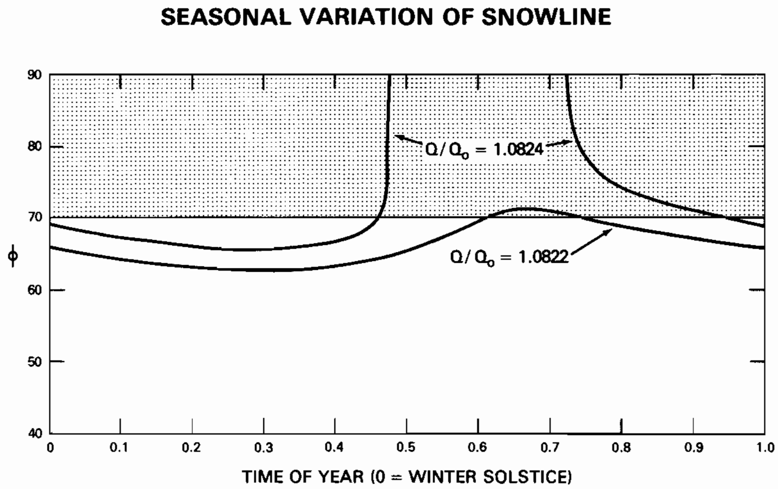


Fig. 8. Seasonal cycle of the snowline for a simple climate model with a land cap extending from the pole to 70° latitude (shown as the shaded area). The abscissa is time in units of years with $t = 0$ at the winter solstice. Note the two qualitatively different solutions for "infinitesimally" separated values of solar constant (Mengel et al., 1988).

ure 9 shows a kind of operating curve for this situation: the polar temperature one month after summer solstice versus the solar constant (Lin and North, 1989). The study of the stability of seasonal model solutions is much more complex than that of mean annual models because the

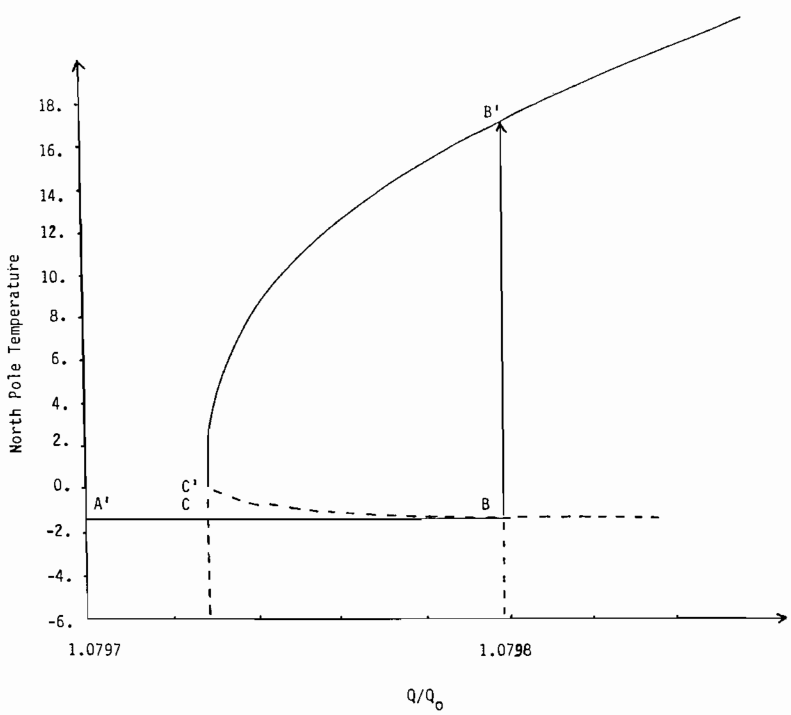
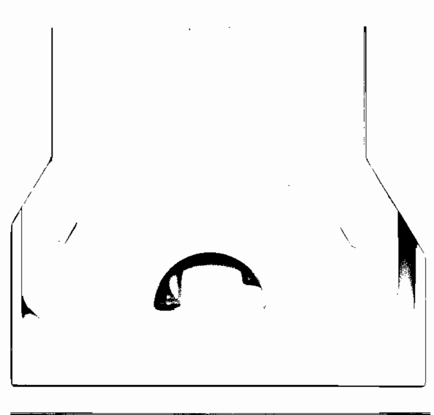


Fig. 9. Polar temperature one month after summer solstice for the geography shown in Fig. 8 versus solar constant (divided by four). The dashed branch is unstable (Lin and North, 1989).



basic state is periodic. One must use the Floquet theory of differential equations for this class of problems (Lin and North, 1989). In contrast to the approach taken by Mengel et al. (1988) who employed a time marching method to solve the seasonal EBM waiting for all transients to die out (this might take many decades), Lin and North (1989) used a Fourier series approach which solved for the Fourier coefficients directly at a considerably savings in computer time. After the Fourier coefficients for the steady state seasonal cycle were found, a system of linear disturbance equations could be studied for the stability eigenvalues. This approach leads to a very accurate determination of the bifurcation points. Lin and North (1989) investigated a number of other zonally symmetric land-sea configurations and found some rather exotic bifurcation phenomena. An example with a land band from 50° to 70° is illustrated in Fig. 10. Note that this case has two bifurcations as the solar constant is increased; the first leads to snow-free summers on the land band, finally the polar sea-ice abruptly disappears.

Concluding remarks

Energy balance climate models have a rich multiple solution structure. While they rather drastically simplify the physical processes operating in the real climate system, we gain considerable insight as to how idealized climate systems operate. In principle, this information could help us in assessing and understanding how our more complicated climate models work. For example, it requires deliberate planning to investigate some of the bifurcation phenomena discussed here in a general circulation model because of the long computer runs involved (near a bifurcation, most relaxation times increase dramatically, leading to much greater lengths of time series required for estimating means, etc., cf. Mengel et al., 1988). Yet if these phenomena do exist in the more complicated models it could have a profound impact on the way we view the problem of global change.

As examples of how these abrupt transitions associated with slow parameter changes past a critical point might have influenced paleocli-

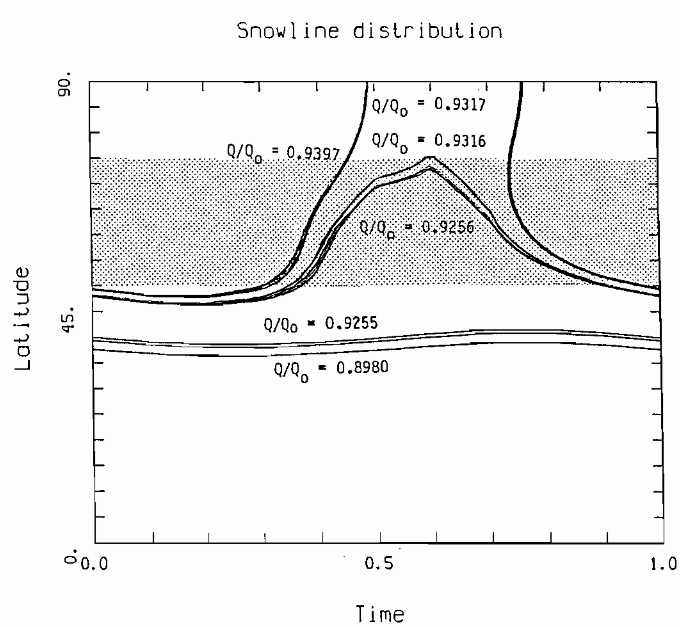


Fig. 10. Same as Fig. 8 except that the land-sea distribution is different (zonally symmetric land band shown shaded). The curves are labeled by their solar constants (normalized by the present value).

mate, we list a few conjectures that have recently been advanced. The initiation of ice sheets on Antarctica and Greenland are excellent candidates for sudden growth by SICI (North and Crowley, 1985; Crowley et al., 1986). Less certain are connections to the advance and growth of the Laurentide ice sheet. As a final example, Crowley and North (1988) have argued that abrupt climate changes that might be connected with the bifurcations discussed here could be responsible for some of the major extinctions that have occurred in earth history.

Acknowledgements

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