ATS 421/521

Climate Modeling

Spring 2015

Lecture 6

- 1D EBM
- Numerics I

April 20

Diffusive parameterization of meridional heat transport:

Lorenz, E. N. (1979) Forced and free variations of weather and climate, J. Atmos. Sci. 36, 1367-1376.

$$\vec{F}_m = -CK\vec{\nabla}T = -CK\frac{\partial T}{\partial y} \tag{2.18}$$

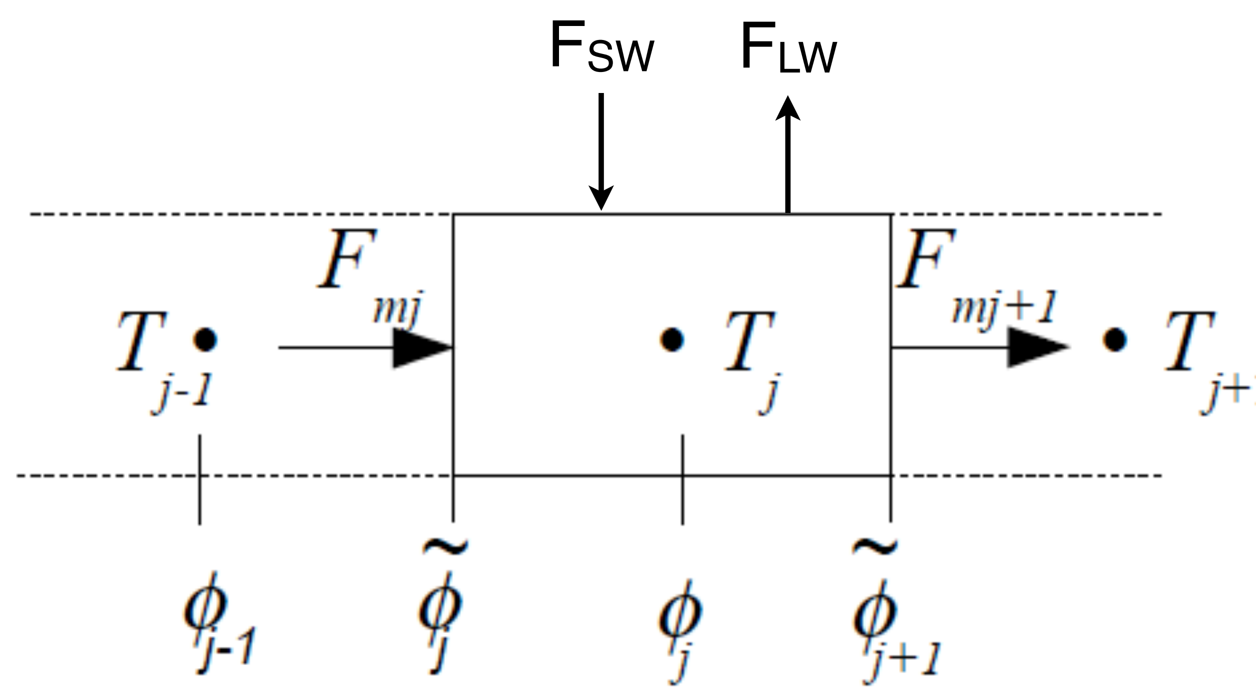
Heat Capacity

Diffusivity

Temperature Gradient

$$C\frac{\partial T}{\partial t} = -\vec{\nabla}\vec{F}_m + F_{SW} - F_{LW}$$

Meridional Heat Flux Convergence



in spherical coordinates

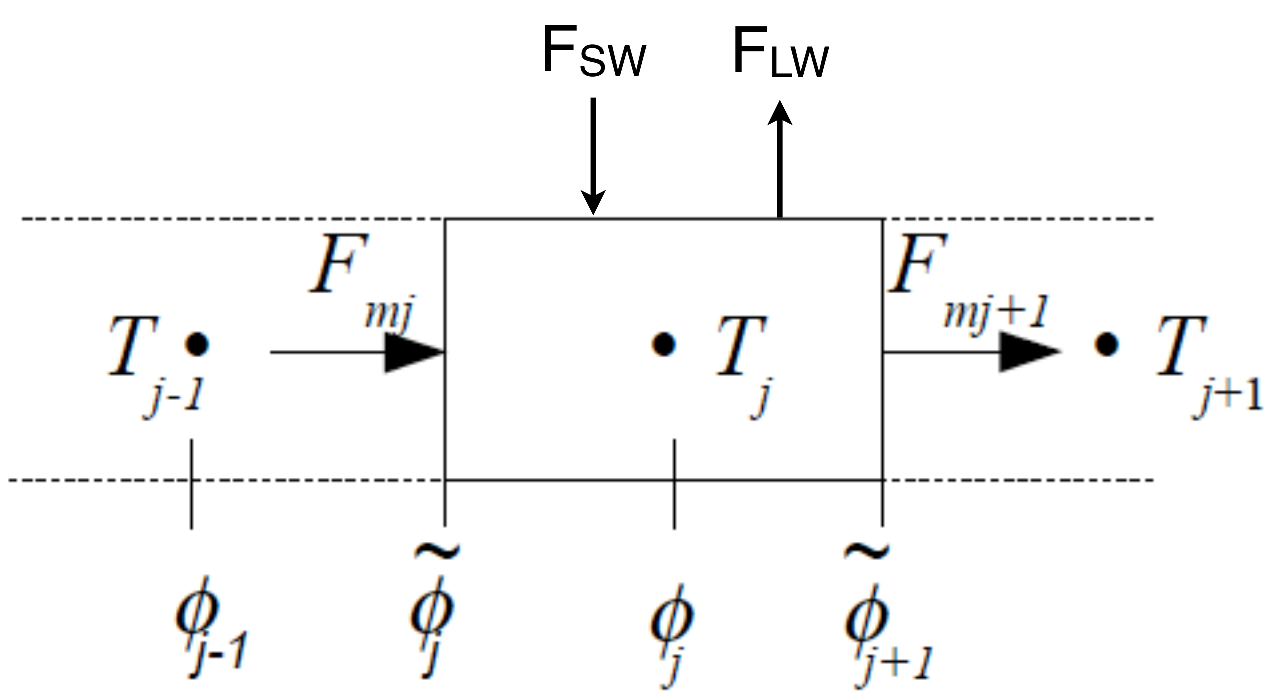
Meridional Heat Flux Divergence:

$$\vec{\nabla}\vec{F}_m = -\vec{\nabla}\left(CK\vec{\nabla}T\right) = \frac{-1}{R^2\cos\phi}\frac{\partial}{\partial\phi}\left(CK\cos\phi\frac{\partial T}{\partial\phi}\right) \qquad (2.20)$$

Discretized: latitude

$$-\vec{\nabla}\vec{F}_m = \frac{-1}{R\cos\phi}\frac{\Delta F_m}{\Delta\phi} = \frac{-1}{R\cos\phi}\frac{F_{mj+1}-F_{mj}}{\tilde{\phi}_{j+1}-\tilde{\phi}_j}$$

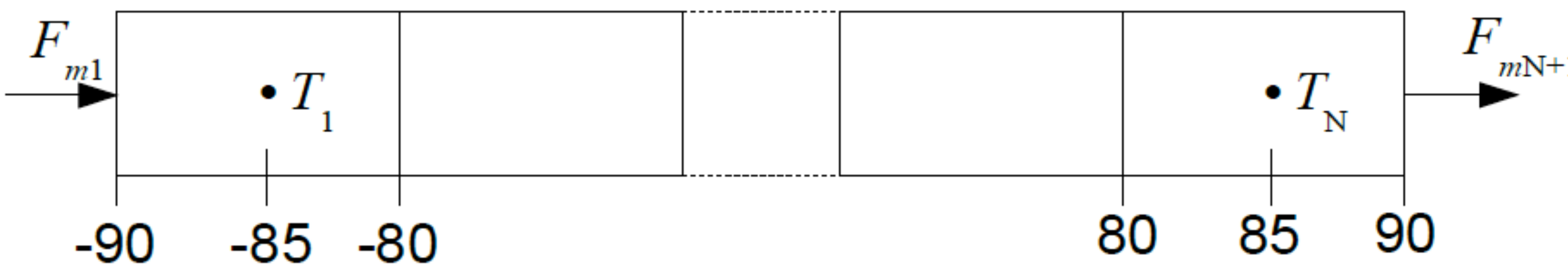
$$F_{mj} = -CK_j\frac{\cos\tilde{\phi}_j}{R}\frac{T_j-T_{j-1}}{\phi_j-\phi_{j-1}}$$



Set up 10º grid from pole to pole.

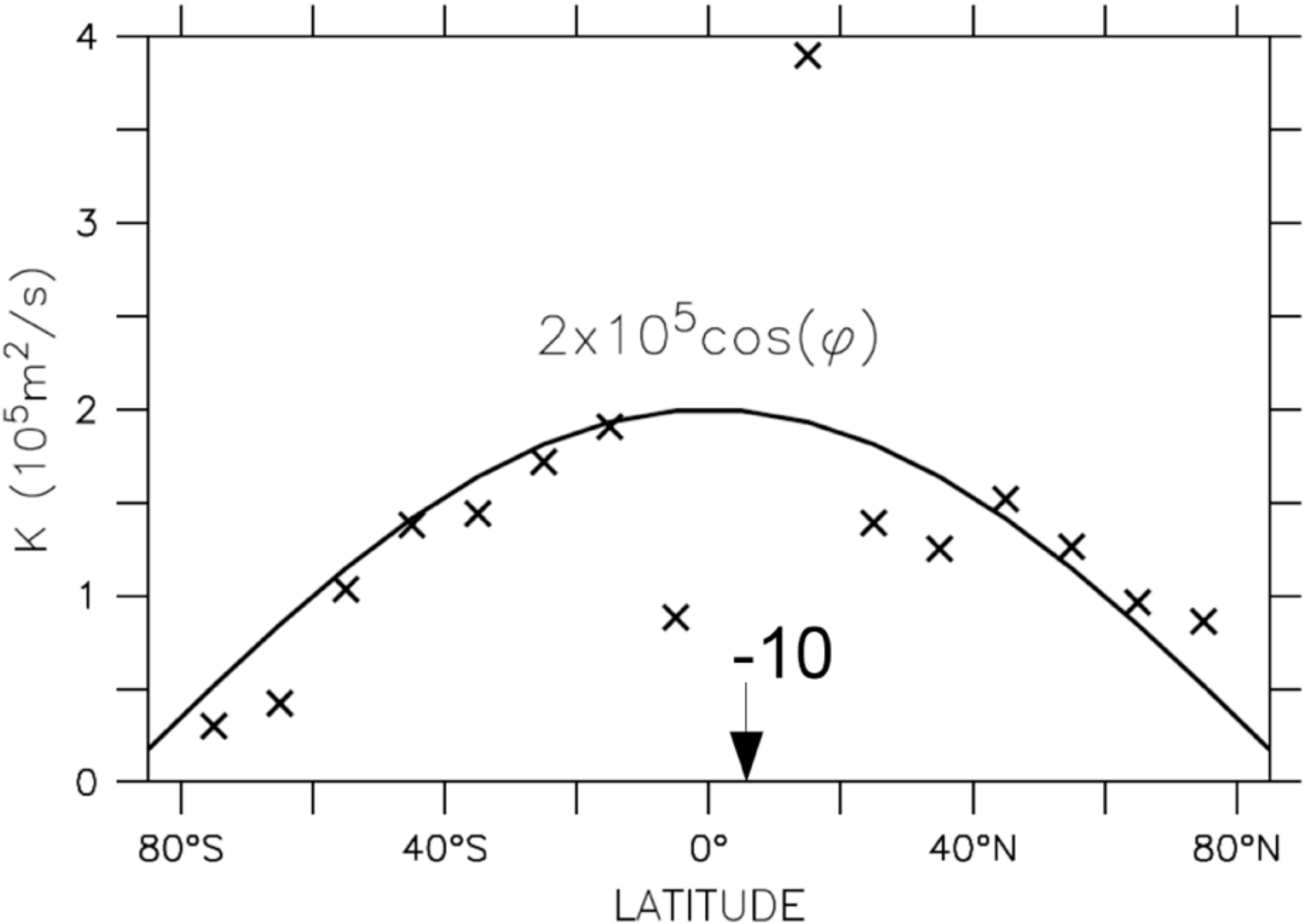
Boundary Conditions:

$$F_{m1} = F_{mN+1} = 0$$

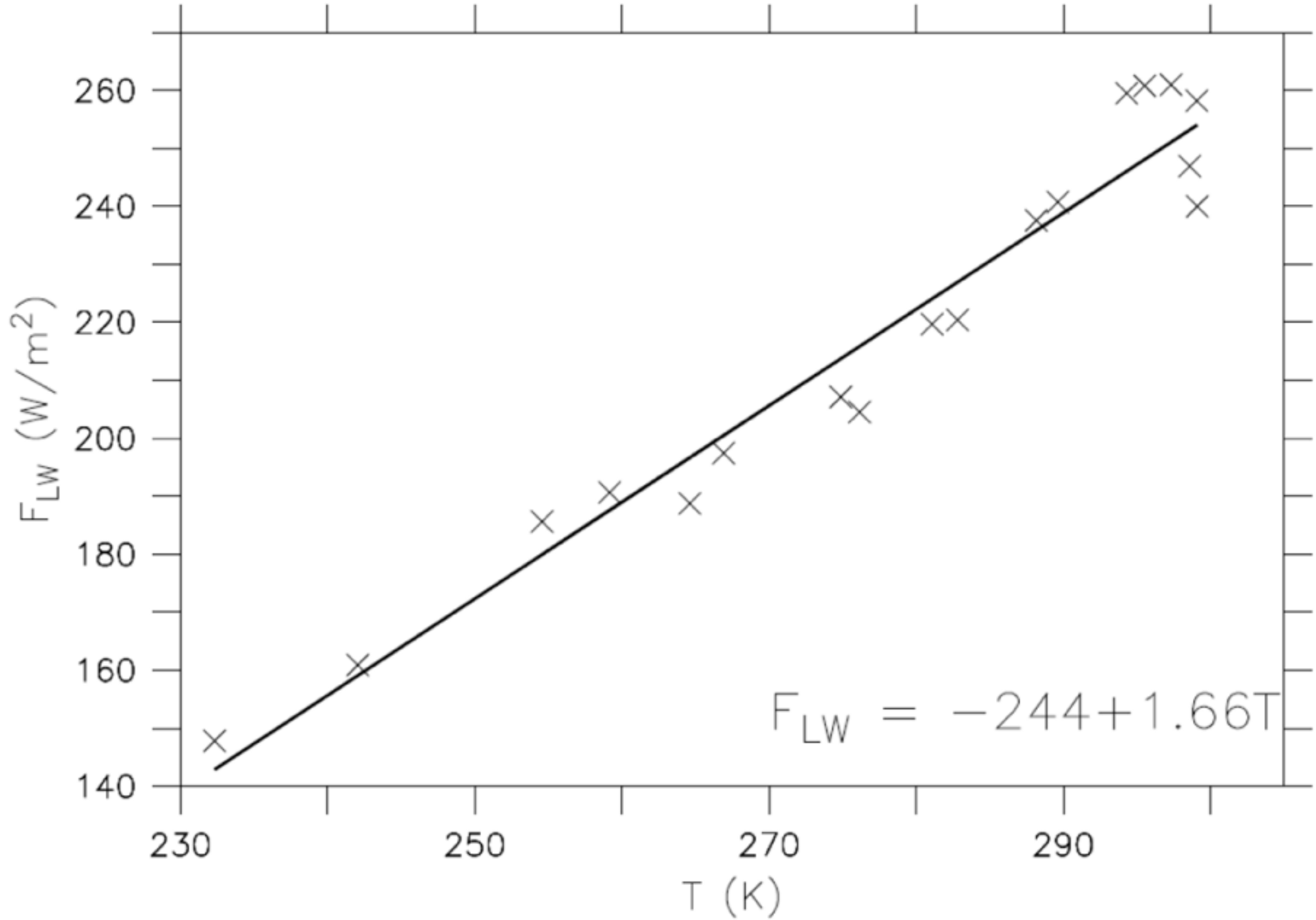


In FORTRAN use vectors:

```
parameter (jmax = 18) ! number of grid boxes
real temp(1:jmax), fm(1:jmax+1), phi(1:jmax), phim(1:jmax+1)
...
do i=1:imax ! time loop
   do j=1:jmax ! loop over latitudes
        ...
       temp(j) = temp(j) - divFm + FSW(j) - FLW(j)
```



4
3
2
1
0
K (10^5m^2/s)
2x10^5cos(φ)
-10
80°S
40°S
0°
40°N
80°N
LATITUDE



260
240
220
200
180
160
140
F_{LW} (W/m^2)
230
250
270
290
T (K)
$F_{LW} = -244+1.66T$

Update Albedo Parameters:

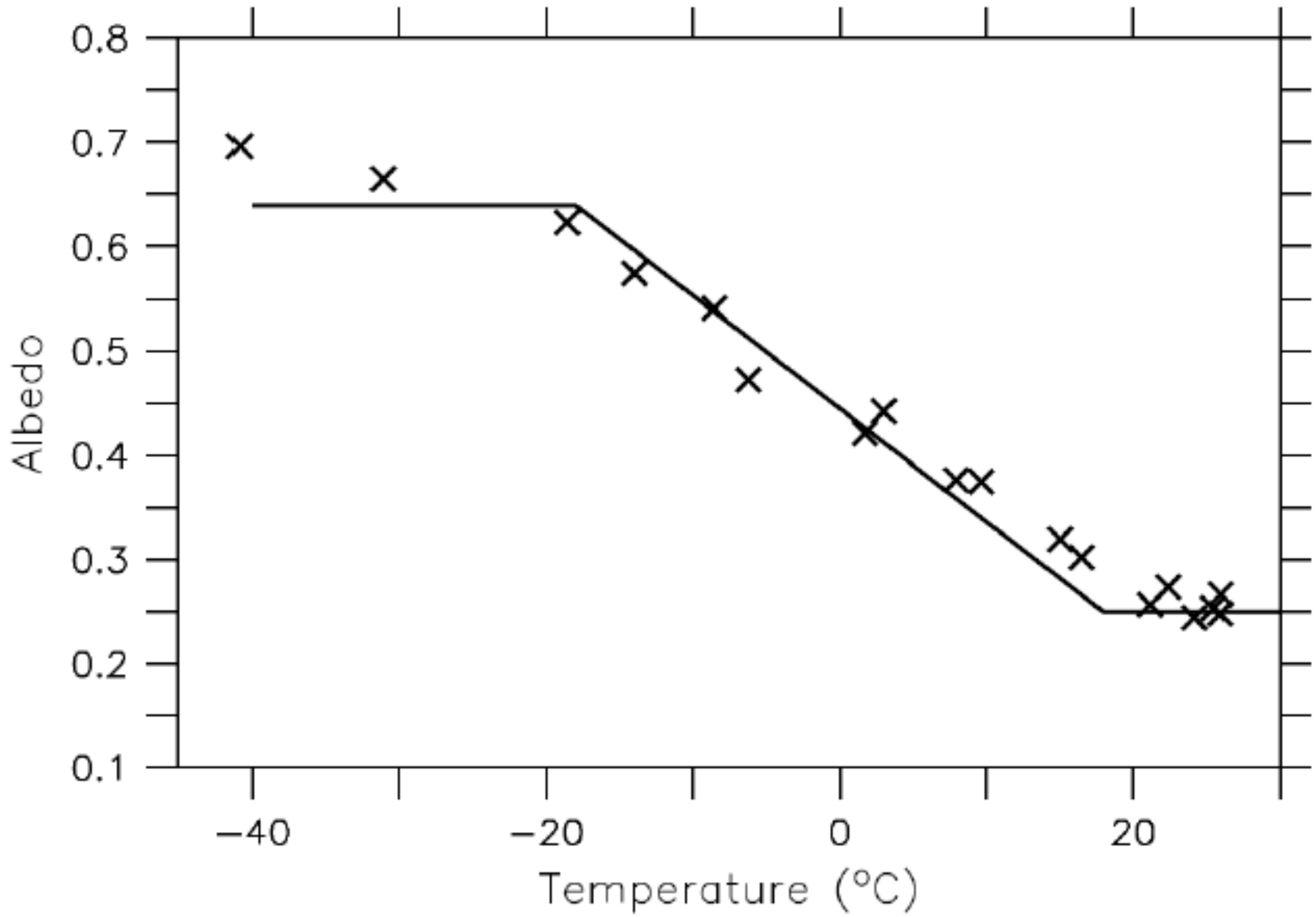


Figure 2.16: Albedo (from ERBE) as a function of surface air temperature (from NCEP) calculated from zonally averaged (on a 10° grid) data. The solid line shows a simple ramp function approximation (eq. 2.5) with T_L=-18°C, T_U =18°C, a_1=0.64 and a_2=0.25.

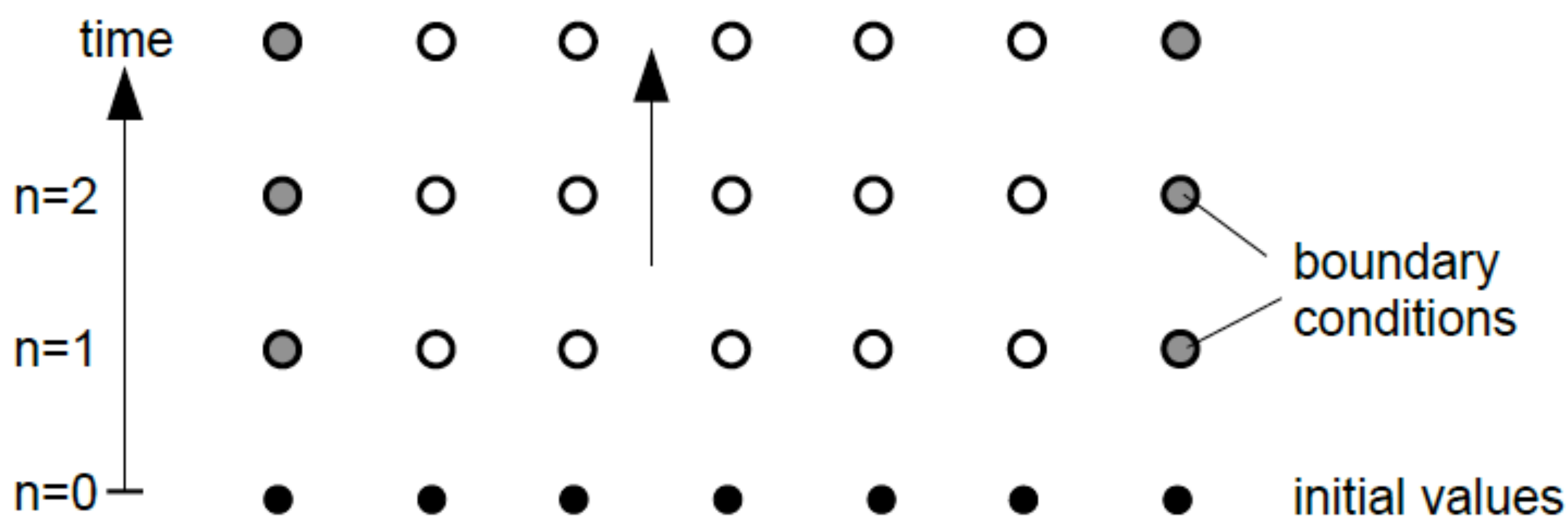
Numerics

Script chapter 2.6

Important criteria for numerical schemes:

1) Convergence for $\Delta x, \Delta t \rightarrow 0$
2) Stability
3) Accuracy
4) Conservation
5) Behavior of Amplitudes and Phases
6) Positive definite
7) No (or Small) Numerical Artifacts

Boundary Conditions



Two types of boundary conditions:

- Dirichlet: specify values
- Neuman: specify normal gradients

Of which type are our 1D EBM boundary conditions?

Develop T in Taylor series around t:

$$T(t+\Delta t)=T(t)+\frac{dT}{dt}\Big|_t \Delta t+\frac{1}{2!}\frac{d^2T}{dt^2}\Big|_t (\Delta t)^2+\ldots \qquad (2.23)$$

$$\longrightarrow \quad \boxed{\frac{dT}{dt}\Big|_t=\frac{T(t+\Delta t)-T(t)}{\Delta t}}\underbrace{-\frac{1}{2!}\frac{d^2T}{dt^2}\Big|_t \Delta t-\frac{1}{3!}\frac{d^3T}{dt^3}\Big|_t (\Delta t)^2-\ldots}_{correction\ of\ order\ \Delta t} \qquad (2.24)$$

neglecting these terms gives the "Euler forward" scheme

as $\Delta t \rightarrow 0$ the Euler scheme converges to the true solution

Now replace Δt with $-\Delta t$ in eq. (2.24) and add this new equation to (2.24)

$$\longrightarrow \quad \boxed{\frac{dT}{dt}\Big|_t=\frac{T(t+\Delta t)-T(t-\Delta t)}{2\cdot\Delta t}}\underbrace{-\frac{1}{3!}\frac{d^3T}{dt^3}\Big|_t (\Delta t)^2-\ldots}_{correction\ of\ order\ (\Delta t)^2}$$

neglecting these terms gives the "Centered Differences" scheme

more accurate than Euler Forward since errors scale with $(\Delta t)^2$

Consider centered differences: $$\frac{\partial C}{\partial x} \simeq \frac{C_{m+1} - C_{m-1}}{2\Delta x}$$

with $C = \hat{C}\cos(kx)$ represented numerically as $C_m = \hat{C}\cos(kx)$

and $C_{m+1} = \hat{C}\cos(k(x+\Delta x))$

The exact solution is $$\frac{\partial C}{\partial x} = -\hat{C}\, k \sin(kx)$$

using $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

we get for the numerical solution $$\frac{C_{m+1} - C_{m-1}}{2\Delta x} = -\frac{\hat{C}}{\Delta x}\sin(kx)\sin(k\Delta x) \underset{\Delta x \to 0}{\to} \frac{\partial C}{\partial x}$$

Wave number

$k = 2\pi/(n\Delta x);\, n = 2, 3, \ldots$

$$\frac{C_{m+1} - C_{m-1}}{2\Delta x} / \frac{\partial C}{\partial x} = \frac{\sin(k\Delta x)}{k\Delta x}$$

n	$\sin(k\Delta x)/(k\Delta x)$
3	0.41
4	0.64
6	0.82
8	0.9

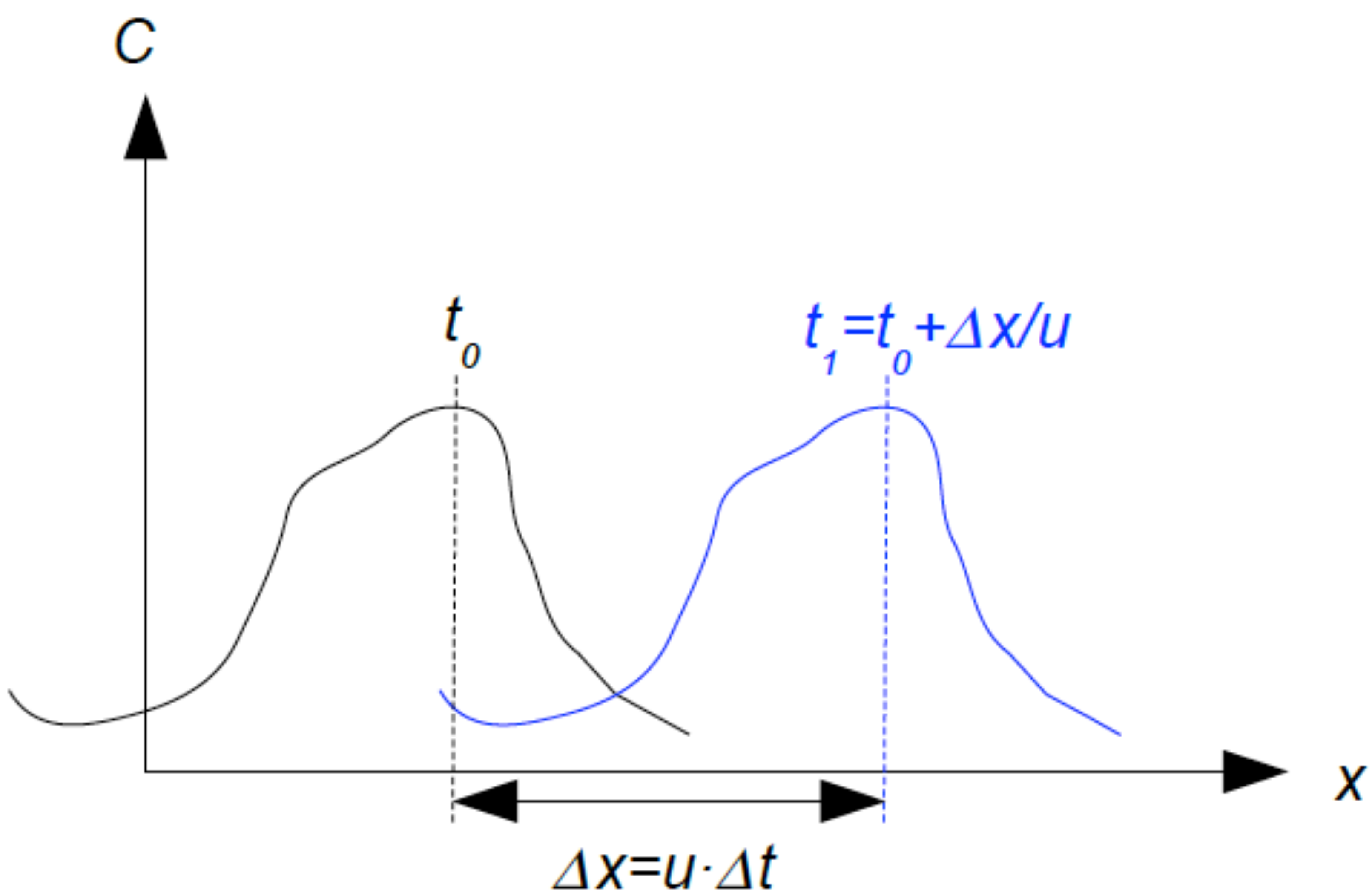
For fixed Δx only waves with large n (large wavelengths) are well represented. Waves shorter than 8Δx have errors > 10 %.

Consider **advection equation**:
$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}(uC) \tag{2.26}$$

fluid velocity　　fluid property (e.g. temperature)

Assume u = const.:
$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} \tag{2.27}$$

Arbitrary function f is solution if
$$C(x,t) = f(x-ut) \tag{2.28}$$



Von Neuman Stability Analysis

Assume wave function at time $t=0$:

$$C(x,0)=Ae^{ikx}=A(\cos(kx)+i\sin(kx)) \quad \text{, with } i^2=-1 \tag{2.29}$$

At time t the solution is a plane wave: $C(x,t)=Ae^{ik(x-ut)}$

wave number	wavelength	angular frequency	period	frequency
$k=\frac{2\pi}{\lambda}$	$\lambda=\frac{2\pi}{k}=\frac{u}{\nu}$	$\omega=\frac{2\pi}{T}$	$T=\frac{2\pi}{\omega}=\frac{1}{\nu}$	$\nu=\frac{1}{T}=\frac{u}{\lambda}$

Now solve eq. (2.27) numerically by discretizing time and space:

$$t=n\Delta t \qquad n=0,1,2,...$$

$$x=m\Delta x \qquad m=0,1,2,...$$

$$C(x,t)=C(m\cdot\Delta x, n\cdot\Delta t)=C_{m.n}=\xi^n e^{ikm\Delta x} \quad , \tag{2.31}$$

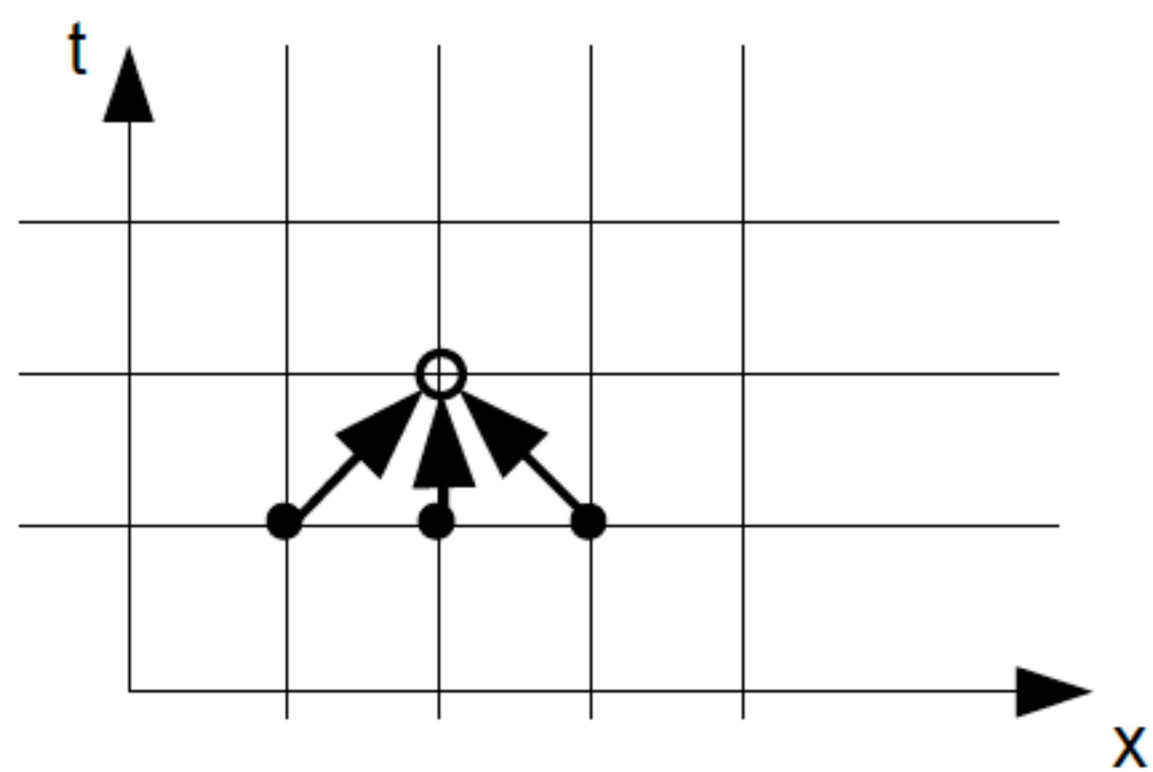
with the amplification factor $\xi(k)$. Each time step the solution is multiplied by ξ . Thus, if $\xi>1$ the solution will diverge (blow up) and if it is $\xi<1$ it will be damped.

Now let's examine the <u>FTCS (forward in time centered in space) scheme</u>:

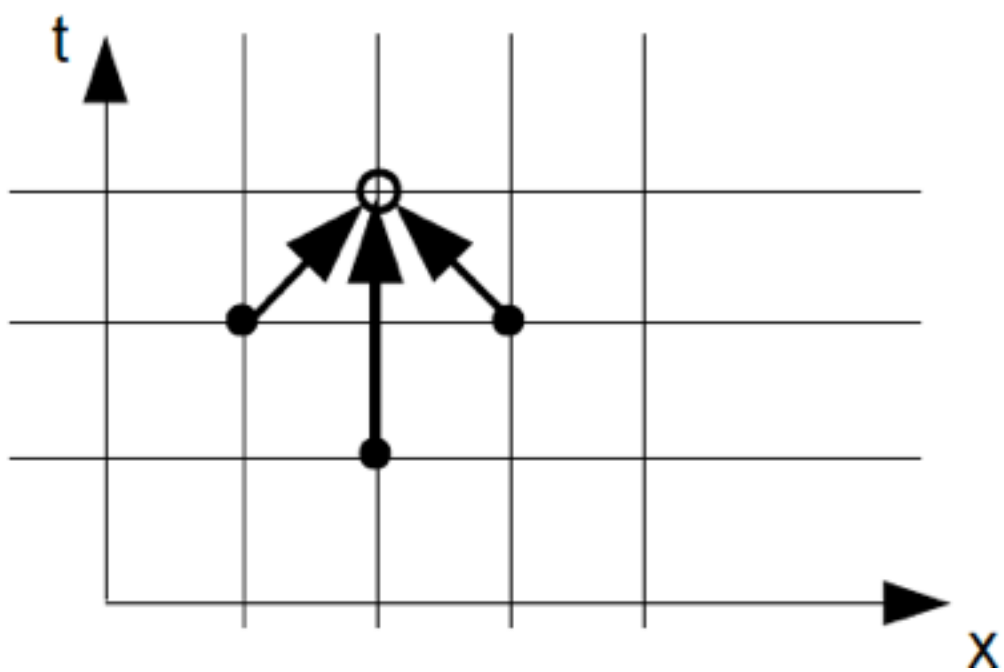
$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=-u\frac{C_{m+1,n}-C_{m-1,n}}{2\cdot\Delta x}$$

$$\rightarrow\xi=1-i\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$$

Thus $|\xi|>1$ for all *k*. The FTCS scheme is unconditionally unstable and therefore useless.



Now let's use centered differences (2.25) for eq. (2.26)



$$\frac{C_{m,n+1}-C_{m,n-1}}{2\cdot\Delta t}=-u\frac{C_{m+1,n}-C_{m-1,n}}{2\cdot\Delta x} \tag{2.32}$$

$$C_{m,n+1}=C_{m,n-1}-\frac{u\cdot\Delta t}{\Delta x}\left(C_{m+1,n}-C_{m-1,n}\right) \tag{2.33}$$

This is the <u>CTCS (centered in time, centered in space), or "leap-frog" scheme</u>. The first time step has to be taken by a Euler scheme and two time steps in the past need to be stored in memory.

Insert the analytical solution eq. (2.31) in (2.33):

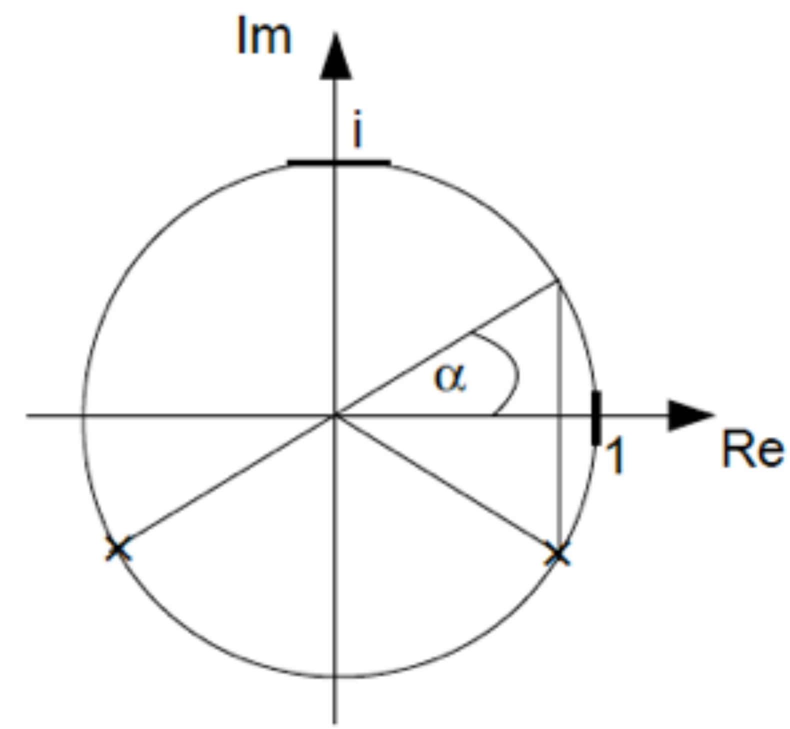
$$\xi=\xi^{-1}-\frac{u\Delta t}{\Delta x}\left(e^{ik\Delta x}-e^{-ik\Delta x}\right)$$

$$\Leftrightarrow\xi^2=1-2\mathrm{i}\,\sigma\,\xi \tag{2.34}$$

with $\sigma=\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$. The solution of this quadratic equation is

$$\xi=-i\,\sigma\pm\sqrt{1-\sigma^2} \tag{2.35}$$

$$\xi = -i\sigma \pm \sqrt{1-\sigma^2} \tag{2.35}$$



We distinguish two cases:

<u>Instable case</u> $|\sigma| > 1$:

$\xi = -i(\sigma \pm S)$, with $S = \sqrt{\sigma^2 - 1} > 0$.

If $\sigma > 1$ => $\sigma + S > 1$ => $|\xi^n| \to \infty$.

If $\sigma < -1$ => $\sigma - S < -1$ => $|\xi^n| \to \infty$.

<u>Stable case</u> $|\sigma| \leqslant 1$:

We can express sigma as a sine function $\sigma = \sin(\alpha)$ and using the trigonometric relation $\sin^2(\alpha) + \cos^2(\alpha) = 1$ we see that the solution of $\xi = -i\sin(\alpha) \pm \cos(\alpha)$ has an absolute value of one, it lies on the unit circle in the complex plane

$\xi = \left\{ \begin{array}{c} e^{-i\alpha} \\ e^{i(\alpha+\pi)} \end{array} \right\}$, with

$$\boxed{C_{m,n} = \xi^n e^{ikm\Delta x} \quad , \qquad (2.31)}$$

Now insert this in eq. (2.31) we get

$$C_{m,n} = \left(M e^{-i\alpha n} + E e^{i(\alpha+\pi)n}\right) e^{ikm\Delta x} \tag{2.36}$$

and

$$C_{m,0} = (M+E) e^{ikm\Delta x} \quad , \tag{2.37}$$

$$C_{m,n}=\left(Me^{-i\alpha n}+Ee^{i(\alpha+\pi)n}\right)e^{ikm\Delta x} \tag{2.36}$$

and

$$C_{m,0}=(M+E)e^{ikm\Delta x} \quad , \tag{2.37}$$

thus with (2.29) $A=M+E$ or

$$C_{m,n}=\underbrace{(A-E)e^{ik(m\Delta x-\frac{\alpha n}{k})}}_{P}+\underbrace{(-1)^n E\,e^{ik(m\Delta x+\frac{\alpha n}{k})}}_{N} \quad , \tag{2.38}$$

with a physical mode *P*, and a numerical mode *N*, which changes sign each time step. Now we only have to determine *E*. For the first time step we have

$$C_{m,1}=C_{m,0}-\frac{u\Delta t}{2\Delta x}\left(C_{m+1,0}-C_{m-1,0}\right) \tag{2.39}$$

with (2.37) we get

$$C_{m,1}=A(1-isin(\alpha))e^{ikm\Delta x}=(A-E)\,e^{ikm\Delta x-i\alpha}-Ee^{ikm\Delta x+i\alpha}$$

Solve for *E* and enter into eq. (2.38) yields

$$C_{m,n}=A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x-\frac{\alpha n}{k})}+(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x+\frac{\alpha n}{k})} \quad . \tag{2.40}$$

It can be shown that (2.40) converges to (2.30) provided $\Delta x\rightarrow 0$ it follows that $\sigma\rightarrow uk\Delta t$ and for $\Delta t\rightarrow 0$ it follows that $\sigma\ll 1$ and hence $\sigma=\sin(\alpha)\simeq\alpha$ and (2.40) converges to

$$C_{m,n}\rightarrow \underbrace{A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(x-ut)}}_{P}+\underbrace{(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(x+ut)}}_{N}\rightarrow A\,e^{k(x-ut)} \quad .$$

$$C_{m,n} \to \underbrace{A \frac{1+\cos(\alpha)}{2\cos(\alpha)} e^{ik(x-ut)}}_{P} + \underbrace{(-1)^n A \frac{1-\cos(\alpha)}{2\cos(\alpha)} e^{ik(x+ut)}}_{N} \to A e^{k(x-ut)} \quad .$$

Thus, the leapfrog scheme is stable (provided $|\sigma| \leqslant 1$) and it converges against the true solution. However, for finite time steps and finite grid spacing a numerical solution N appears, which is unphysical. The physical solution P describes a plane wave traveling towards the right, whereas N changes sign every time step and travels towards the left.

The condition for stability $|\sigma| = |(u \Delta t / \Delta x) \sin(k \Delta x)| \leqslant 1$ must hold for all wavelength, thus it follows that $|(u \Delta t)/(\Delta x)| \leqslant 1$, which can be regarded as a condition for the maximum time step

$$\Delta t \leqslant \frac{\Delta x}{|u|} \quad . \tag{2.41}$$

CFL criterion

(Courant-Friedrichs-Lewy, 1928)

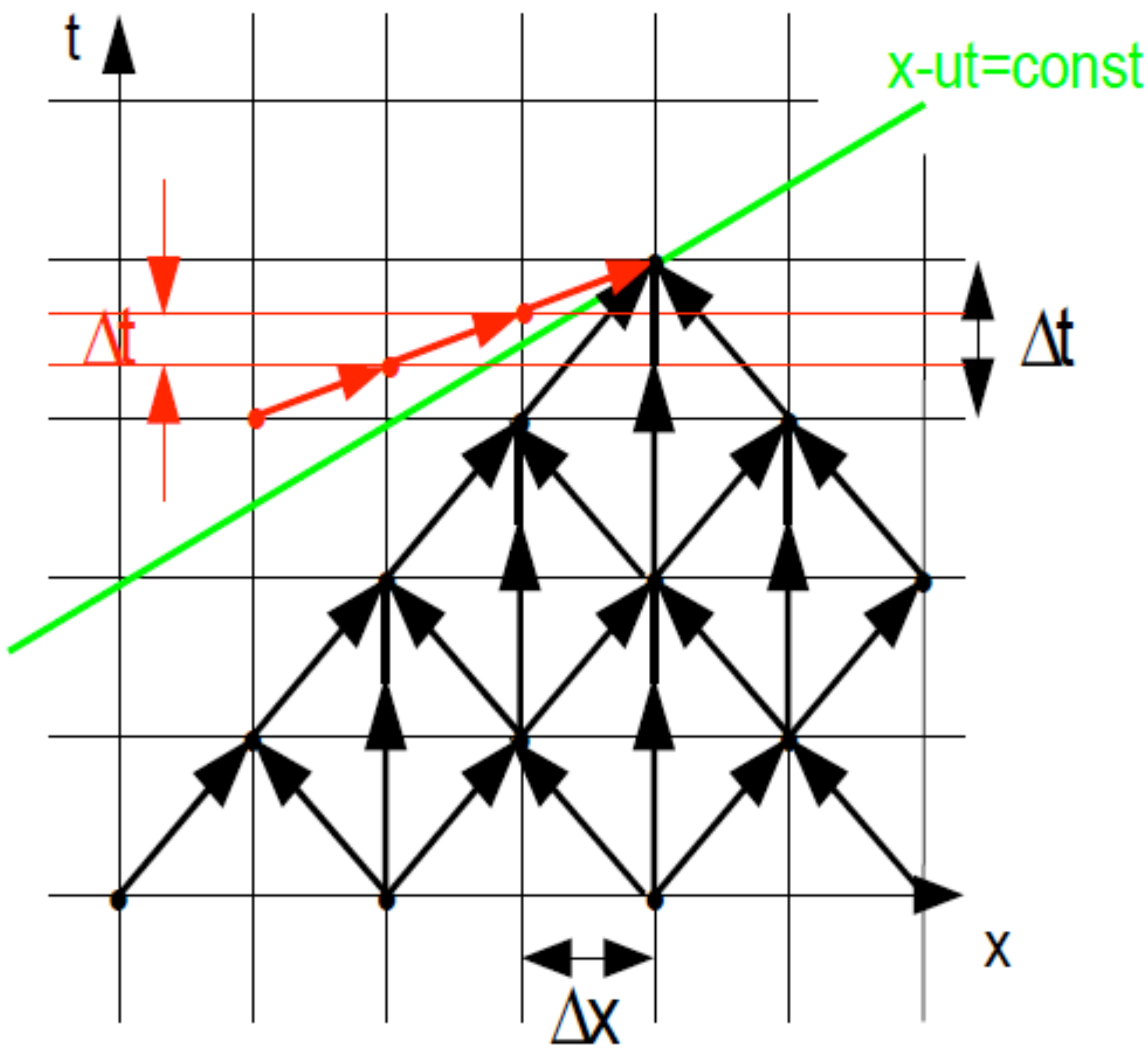
The CFL criterion limits the maximum possible time step.

For Δx = 300 km

ocean: max(u) = 1 m/s => $\Delta t < 3$ days

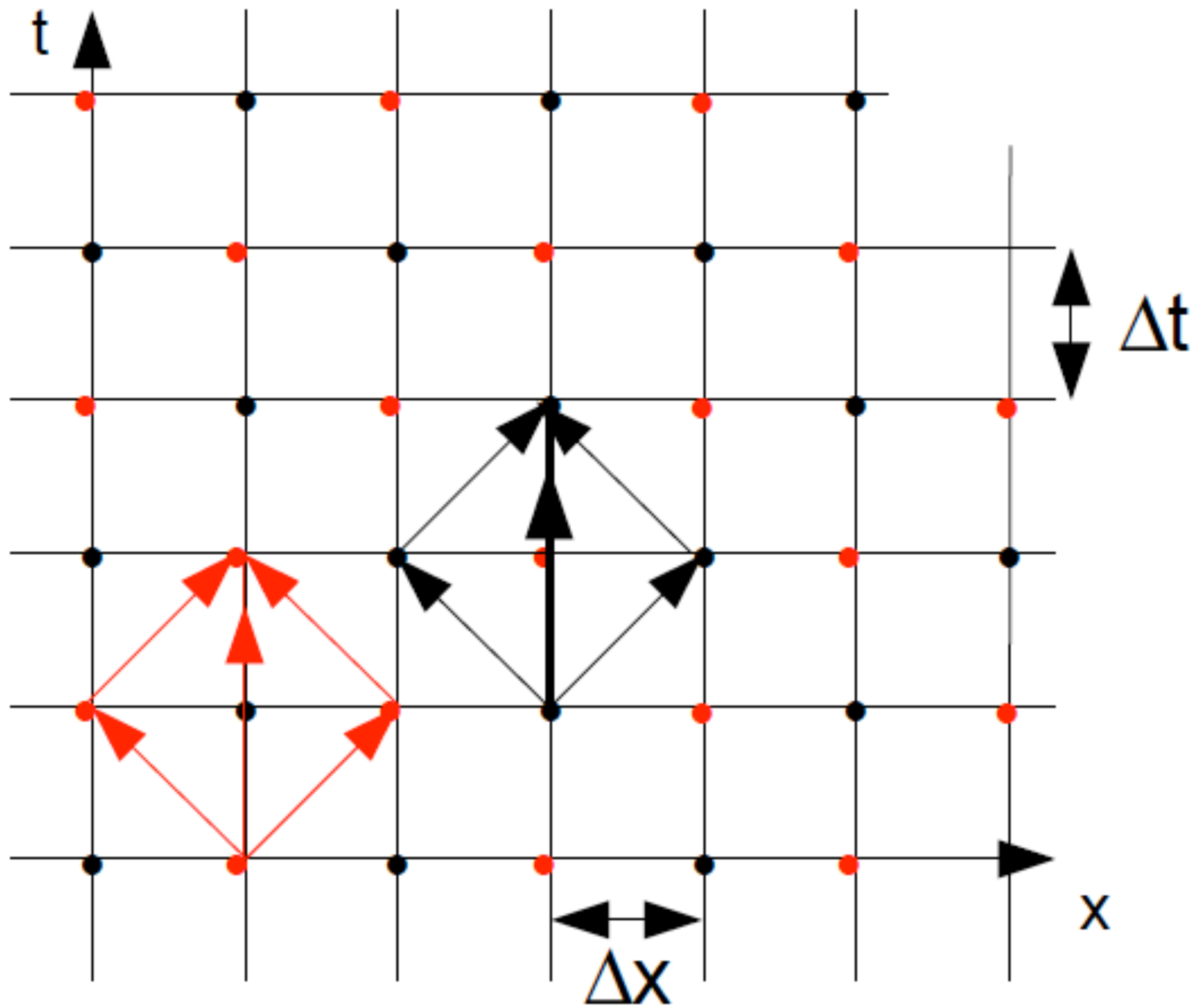
atmosphere: max(u) = 80 m/s => $\Delta t < 1$ hour

CFL criterion



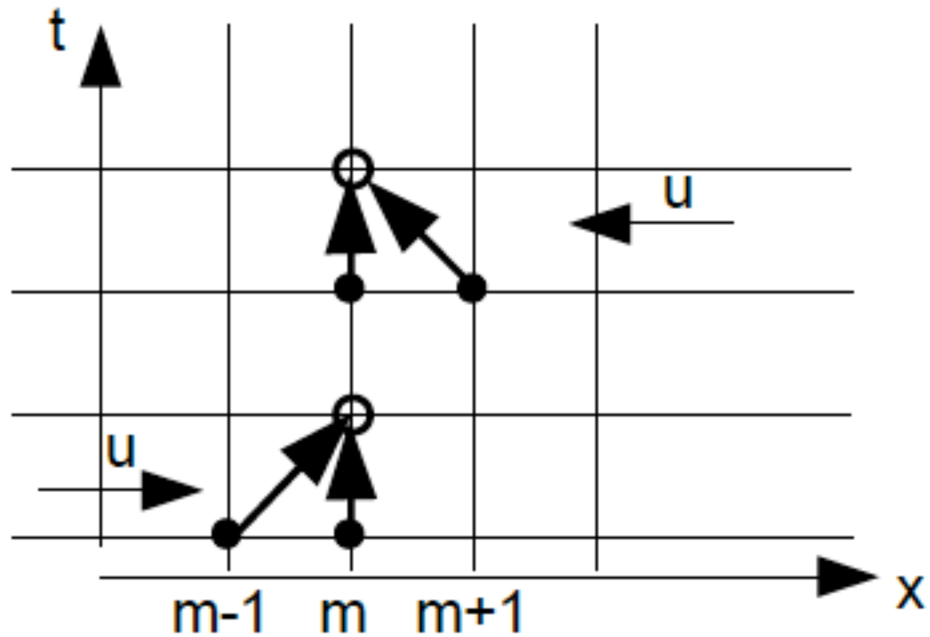
Signal propagates faster than the cone of influence for large time step **Δt**.
Signal propagates slower than the cone of influence for small time step **Δt**.

Numerical Mode (artifact)



Decoupling of red and black grid points.
Can be removed by using an Euler (FTCS) time step.

The Upwind Scheme



$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=-u\begin{cases}\dfrac{C_{m,n}-C_{m-1,n}}{\Delta x}, u>0\\[2ex]\dfrac{C_{m+1,n}-C_{m,n}}{\Delta x}, u\leqslant 0\end{cases}$$

$$\xi=1-\left|\frac{u\Delta t}{\Delta x}\right|(1-\cos(k\Delta x))-i\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$$

$$|\xi^2|=1-2\left|\frac{u\Delta t}{\Delta x}\right|\left(1-\left|\frac{u\Delta t}{\Delta x}\right|\right)(1-\cos(k\Delta x))$$

Again CFL criterion for stability.

Advantage: Positive definite

Disadvantage: only first order accurate (numerical diffusion)

Other Schemes

- Prather: higher order terms are calculated and stored (positive definite, very accurate, no numerical diffusion but requires more memory and computations)
- FCT (Flux corrected transport)

Consider **diffusion equation**: $$\frac{\partial C}{\partial t}=K\frac{\partial^2 C}{\partial x^2}$$

FTCS: $$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n}-2C_{m,n}+C_{m-1,n}}{\Delta x^2}\quad, \tag{2.42}$$

$$C_{m,n+1}=C_{m,n}+\frac{K\Delta t}{\Delta x^2}(C_{m+1,n}-2C_{m,n}+C_{m-1,n})$$

$$\xi=1-\frac{4K\Delta t}{(\Delta x)^2}\sin^2(\frac{k\Delta x}{2})$$

$$\xi^2=1-2\frac{4K\Delta t}{(\Delta x)^2}\sin^2(\frac{k\Delta x}{2})+\left(\frac{4K\Delta t}{(\Delta x)^2}\right)^2\sin^2(\frac{k\Delta x}{2})$$

$$|\xi|\leqslant 1 \longrightarrow \Delta t\leqslant\frac{(\Delta x)^2}{2K}$$

Analogous to CFL criterion.

FTCS **stable** for diffusion equation.

FTCS: $$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n}-2C_{m,n}+C_{m-1,n}}{\Delta x^2}\,, \qquad (2.42)$$

Replace n with n+1:

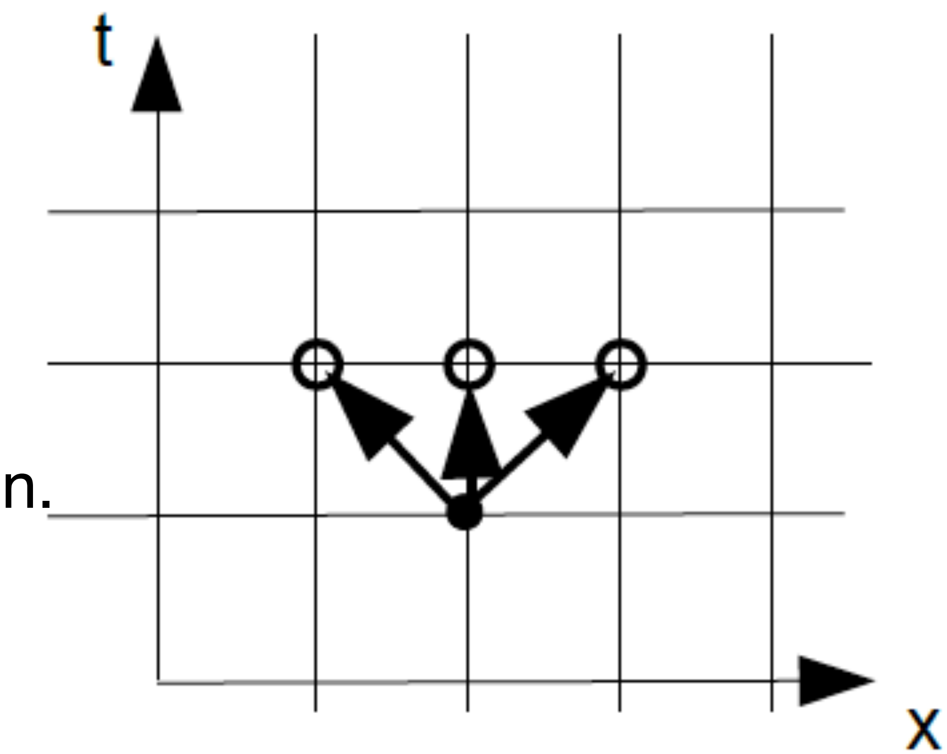
$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n+1}-2C_{m,n+1}+C_{m-1,n+1}}{\Delta x^2}$$

fully implicit (or backward in time) scheme

Can be solved by solving set of linear equations:

$$-\alpha C_{m-1,n+1}+(1+2\alpha)C_{m,n+1}-\alpha C_{m+1,n+1}=C_{m,n}$$

with $\alpha=K\Delta t/(\Delta x)^2$



Tridiagonal system can be solved by matrix inversion.
Unconditionally stable for any Δt !
Only first order accurate: numerical diffusion (not a big problem here since we’re solving a diffusion equation, but for advection equation it is an issue).