ATS 421/521

Climate Modeling

Spring 2013

Lecture 8

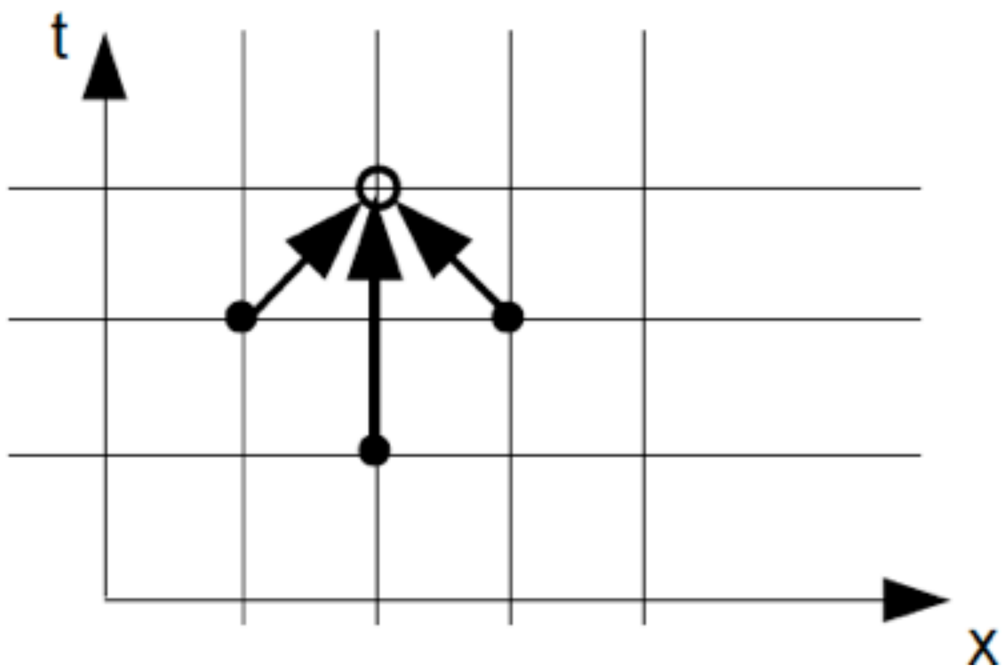
- Numerics II

April 26

Von Neuman Stability Analysis

$$C(x,t)=C(m\cdot\Delta x,n\cdot\Delta t)=C_{m.n}=\xi^n e^{ikm\Delta x} \quad , \qquad (2.31)$$

Now let's use centered differences (2.25) for eq. (2.26)



$$\frac{C_{m,n+1}-C_{m,n-1}}{2\cdot\Delta t}=-u\frac{C_{m+1,n}-C_{m-1,n}}{2\cdot\Delta x} \tag{2.32}$$

$$C_{m,n+1}=C_{m,n-1}-\frac{u\cdot\Delta t}{\Delta x}\left(C_{m+1,n}-C_{m-1,n}\right) \tag{2.33}$$

This is the <u>CTCS (centered in time, centered in space), or "leap-frog" scheme</u>. The first time step has to be taken by a Euler scheme and two time steps in the past need to be stored in memory.

Insert the analytical solution eq. (2.31) in (2.33):

$$\xi=\xi^{-1}-\frac{u\Delta t}{\Delta x}\left(e^{ik\Delta x}-e^{-ik\Delta x}\right)$$

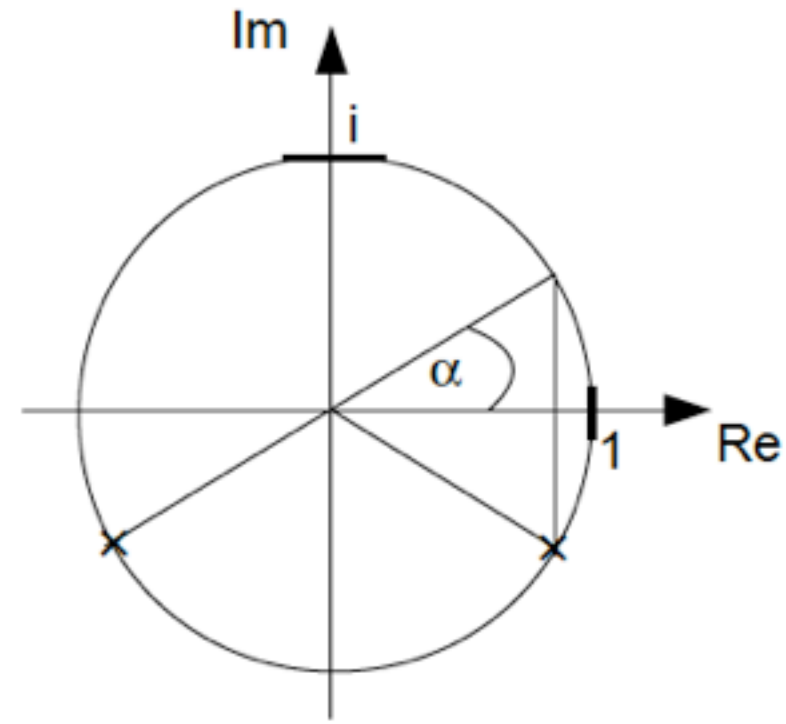
$$\Leftrightarrow\xi^2=1-2\mathrm{i}\,\sigma\,\xi \tag{2.34}$$

with $\sigma=\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$. The solution of this quadratic equation is

$$\xi=-i\,\sigma\pm\sqrt{1-\sigma^2} \tag{2.35}$$

$$\xi = -i\,\sigma \pm \sqrt{1-\sigma^2} \qquad (2.35)$$

We distinguish two cases:



<u>Instable case</u> $|\sigma|>1$:

$\xi = -i(\sigma \pm S)$, with $S=\sqrt{\sigma^2-1}>0$.

If $\sigma>1$ => $\sigma+S>1$ => $|\xi^n|\to\infty$.

If $\sigma<-1$ => $\sigma-S<-1$ => $|\xi^n|\to\infty$.

<u>Stable case</u> $|\sigma|\leqslant 1$:

We can express sigma as a sine function $\sigma=\sin(\alpha)$ and using the trigonometric relation $\sin^2(\alpha)+\cos^2(\alpha)=1$ we see that the solution of $\xi=-isin(\alpha)\pm\cos(\alpha)$ has an absolute value of one, it lies on the unit circle in the complex plane

$\xi=\begin{Bmatrix} e^{-i\alpha} \\ e^{i(\alpha+\pi)} \end{Bmatrix}$, with

$$\boxed{C_{m.n}=\xi^n e^{ikm\Delta x} \;, \qquad (2.31)}$$

Now insert this in eq. (2.31) we get

$$C_{m,n}=(Me^{-i\alpha n}+Ee^{i(\alpha+\pi)n})\,e^{ikm\Delta x} \qquad (2.36)$$

and

$$C_{m,0}=(M+E)\,e^{ikm\Delta x} \;, \qquad (2.37)$$

$$C_{m,n}=(Me^{-i\alpha n}+Ee^{i(\alpha+\pi)n})e^{ikm\Delta x} \tag{2.36}$$

and

$$C_{m,0}=(M+E)e^{ikm\Delta x} \quad , \tag{2.37}$$

thus with (2.29) $A=M+E$ or $C(x,0)=Ae^{ikx}=A(\cos(kx)+isin(kx))$

$$C_{m,n}=\underbrace{(A-E)e^{ik(m\Delta x-\frac{\alpha n}{k})}}_{P}+\underbrace{(-1)^n E e^{ik(m\Delta x+\frac{\alpha n}{k})}}_{N} \quad , \tag{2.38}$$

with a physical mode P, and a numerical mode N, which changes sign each time step. Now we only have to determine E. For the first time step we have

$$C_{m,1}=C_{m,0}-\frac{u\Delta t}{2\Delta x}(C_{m+1,0}-C_{m-1,0}) \tag{2.39}$$

with (2.37) we get

$$C_{m,1}=A(1-isin(\alpha))e^{ikm\Delta x}=(A-E)e^{ikm\Delta x-i\alpha}-Ee^{ikm\Delta x+i\alpha}$$

Solve for E and enter into eq. (2.38) yields

$$C_{m,n}=A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x-\frac{\alpha n}{k})}+(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(m\Delta x+\frac{\alpha n}{k})} \quad . \tag{2.40}$$

It can be shown that (2.40) converges to (2.30) provided $\Delta x\rightarrow 0$ it follows that $\sigma\rightarrow uk\Delta t$ and for $\Delta t\rightarrow 0$ it follows that $\sigma\ll 1$ and hence $\sigma=\sin(\alpha)\simeq\alpha$ and (2.40) converges to

$$C_{m,n}\rightarrow \underbrace{A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(x-ut)}}_{P}+\underbrace{(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(x+ut)}}_{N}\rightarrow A e^{k(x-ut)} \quad .$$

$$C_{m,n} \to \underbrace{A\frac{1+\cos(\alpha)}{2\cos(\alpha)}e^{ik(x-ut)}}_{P} + \underbrace{(-1)^n A\frac{1-\cos(\alpha)}{2\cos(\alpha)}e^{ik(x+ut)}}_{N} \to A\,e^{k(x-ut)} \quad .$$

Thus, the leapfrog scheme is stable (provided $|\sigma| \leqslant 1$) and it converges against the true solution. However, for finite time steps and finite grid spacing a numerical solution N appears, which is unphysical. The physical solution P describes a plane wave traveling towards the right, whereas N changes sign every time step and travels towards the left.

The condition for stability $|\sigma| = |(u\Delta t/\Delta x)\sin(k\Delta x)| \leqslant 1$ must hold for all wavelength, thus it follows that $|(u\Delta t)/(\Delta x)| \leqslant 1$, which can be regarded as a condition for the maximum time step

$$\boxed{\Delta t \leqslant \frac{\Delta x}{|u|} \quad .} \tag{2.41}$$

CFL criterion

(Courant-Friedrichs-Lewy, 1928)

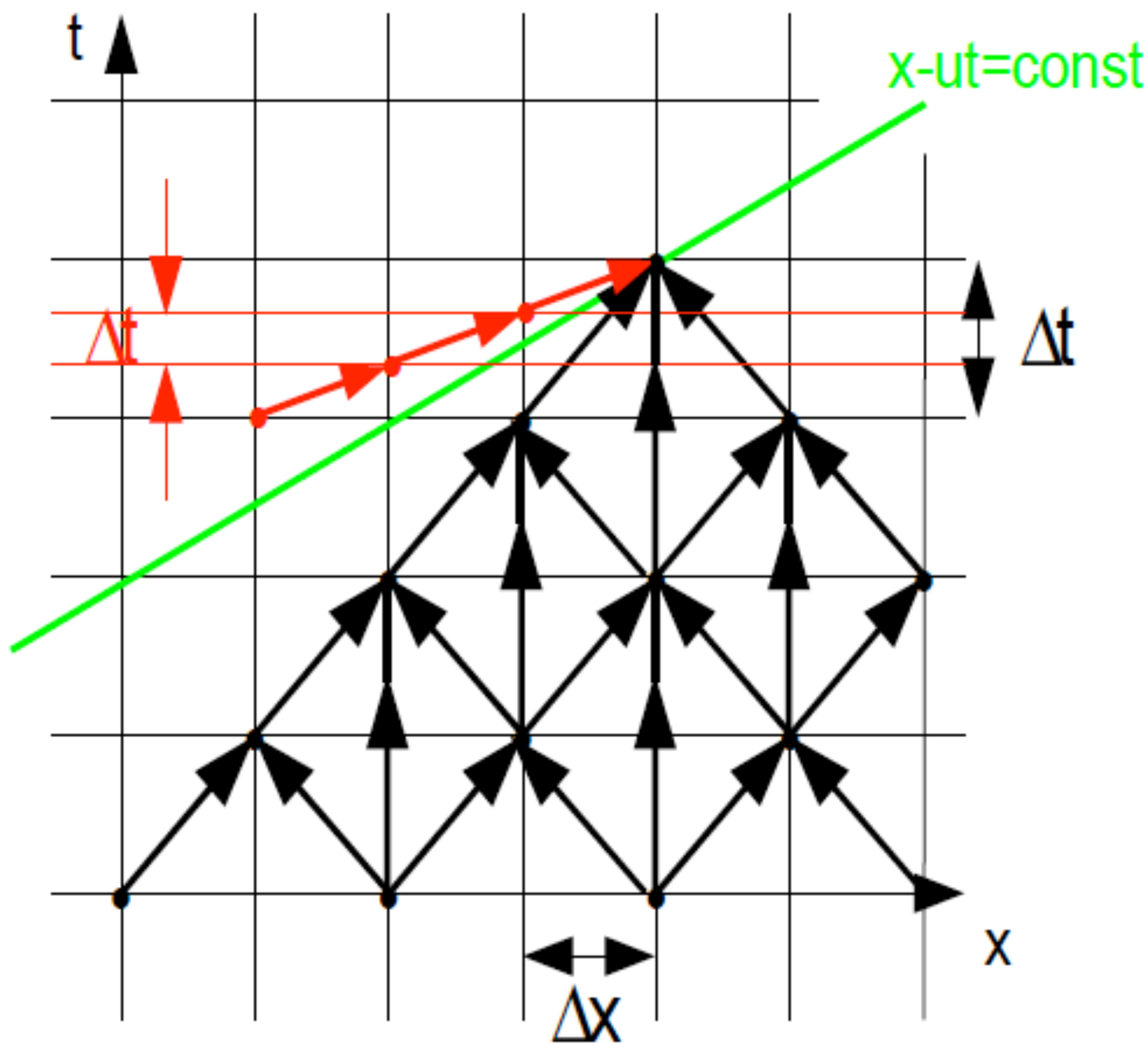
The CFL criterion limits the maximum possible time step.

For Δx = 300 km

ocean: max(u) = 1 m/s => $\Delta t < 3$ days

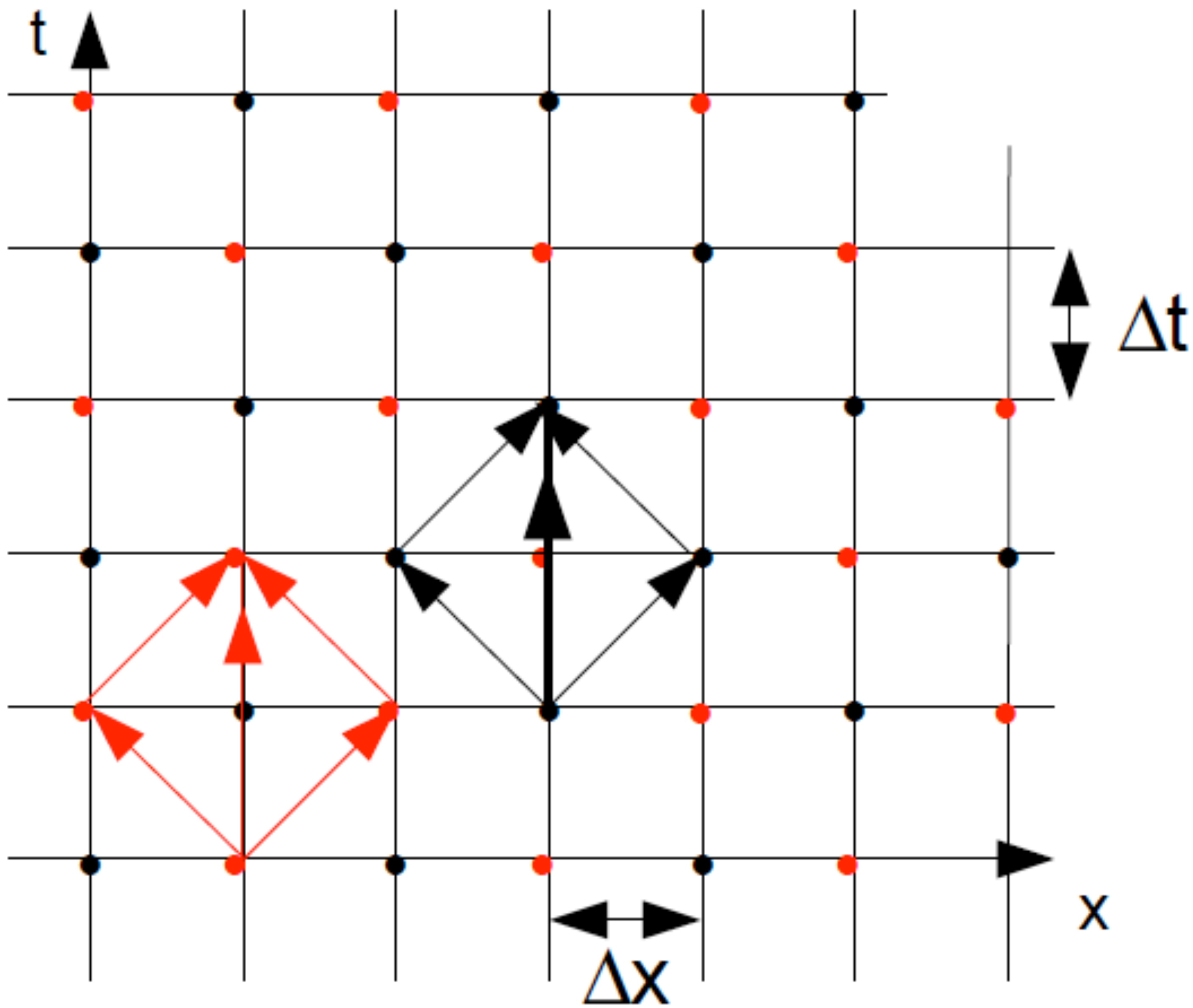
atmosphere: max(u) = 80 m/s => $\Delta t < 1$ hour

CFL criterion



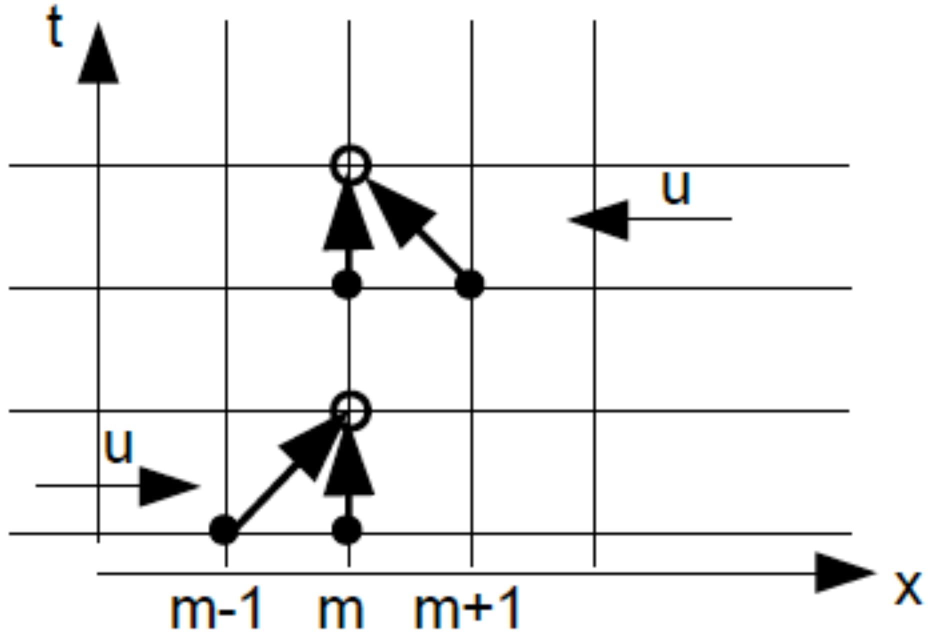
Signal propagates faster than the cone of influence for large time step $\mathbf{\Delta t}$.
Signal propagates slower than the cone of influence for small time step $\mathbf{\Delta t}$.

Numerical Mode (artifact)



Decoupling of red and black grid points.
Can be removed by using an Euler (FTCS) time step.

The Upwind Scheme



$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=-u\begin{cases}\dfrac{C_{m,n}-C_{m-1,n}}{\Delta x}, u>0\\ \dfrac{C_{m+1,n}-C_{m,n}}{\Delta x}, u\leqslant 0\end{cases}$$

$$\xi=1-\left|\frac{u\Delta t}{\Delta x}\right|(1-\cos(k\Delta x))-i\frac{u\Delta t}{\Delta x}\sin(k\Delta x)$$

$$|\xi^2|=1-2\left|\frac{u\Delta t}{\Delta x}\right|(1-\left|\frac{u\Delta t}{\Delta x}\right|)(1-\cos(k\Delta x))$$

Again CFL criterion for stability.

<u>Advantage:</u> Positive definite

<u>Disadvantage:</u> only first order accurate (numerical diffusion)

Other Schemes

- Prather: higher order terms are calculated and stored (positive definite, very accurate, no numerical diffusion but requires more memory and computations)
- FCT (Flux corrected transport)

Consider **diffusion equation**: $$\frac{\partial C}{\partial t} = K \frac{\partial^2 C}{\partial x^2}$$

FTCS: $$\frac{C_{m,n+1} - C_{m,n}}{\Delta t} = K \frac{C_{m+1,n} - 2C_{m,n} + C_{m-1,n}}{\Delta x^2} \quad , \qquad (2.42)$$

$$C_{m,n+1} = C_{m,n} + \frac{K \Delta t}{\Delta x^2} (C_{m+1,n} - 2C_{m,n} + C_{m-1,n})$$

$$\xi = 1 - \frac{4 K \Delta t}{(\Delta x)^2} \sin^2\left(\frac{k \Delta x}{2}\right)$$

$$\xi^2 = 1 - 2 \frac{4 K \Delta t}{(\Delta x)^2} \sin^2\left(\frac{k \Delta x}{2}\right) + \left(\frac{4 K \Delta t}{(\Delta x)^2}\right)^2 \sin^2\left(\frac{k \Delta x}{2}\right)$$

$$|\xi| \leqslant 1 \longrightarrow \Delta t \leqslant \frac{(\Delta x)^2}{2 K}$$

Analogous to CFL criterion.

FTCS **stable** for diffusion equation.

FTCS: $$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n}-2C_{m,n}+C_{m-1,n}}{\Delta x^2}, \tag{2.42}$$

Replace n with n+1:

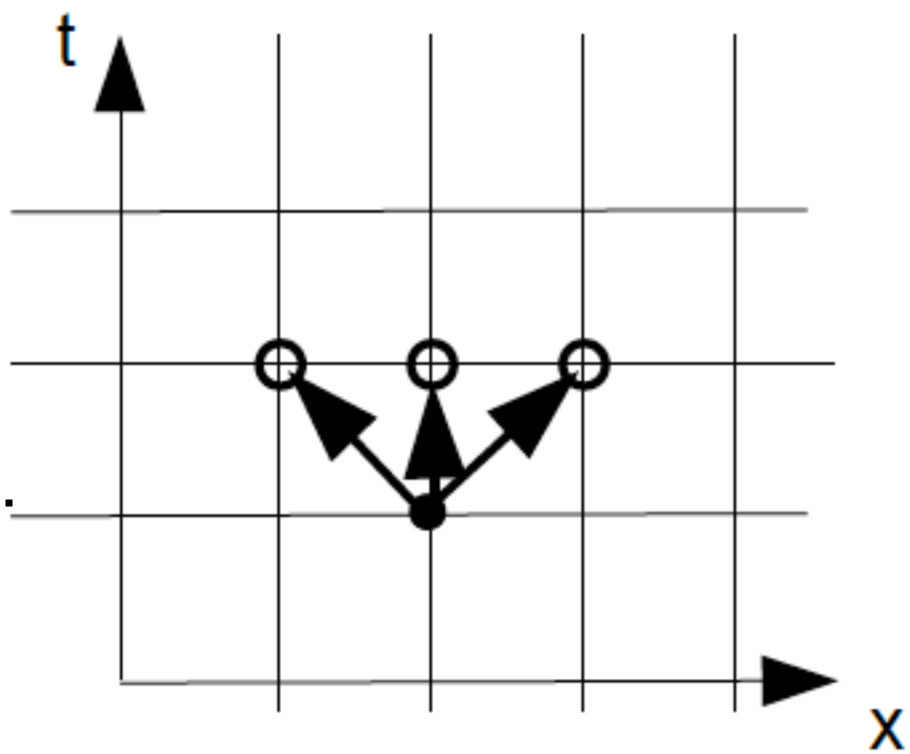
$$\frac{C_{m,n+1}-C_{m,n}}{\Delta t}=K\frac{C_{m+1,n+1}-2C_{m,n+1}+C_{m-1,n+1}}{\Delta x^2}$$

fully implicit (or backward in time) scheme

Can be solved by solving set of linear equations:

$$-\alpha C_{m-1,n+1}+(1+2\alpha)C_{m,n+1}-\alpha C_{m+1,n+1}=C_{m,n}$$

with $\alpha=K\Delta t/(\Delta x)^2$



Tridiagonal system can be solved by matrix inversion.
Unconditionally stable for any Δt !
Only first order accurate: numerical diffusion (not a big problem here since we're solving a diffusion equation, but for advection equation it is an issue).