CPSC 322: Introduction to Artificial Intelligence

CSPs: Arc Consistency and Domain Splitting

Textbook reference: [4.4, 4.5]

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Credit: These slides are adapted from the slides of the previous offerings of the course. Thanks to all instructors for creating and improving the teaching material and making it available!

Announcements

- Assignment 2 has been released and is due on **21 Oct 11:59pm**.
- Midterm **practice questions** are available on Piazza.
- Midterm time and location
  Time: Friday, Oct 25th, from 6pm to 7pm
  Location: Woodward 2
  (Instructional Resources Centre-IRC) (WOOD) - 2
- My office hours: Fridays from 11am to noon in ICCS 185

Lecture outline

- Recap CSPs (~5 mins) 👈🏻
- Solving CSPs
  - Arc consistency (~25 mins)
  - Domain splitting (~25 mins)
- Class activity (~15 mins)
- Summary and wrap up

CSP: Motivation

"…By applying constraint programming, Optimatch successfully provides prioritization lists and near-optimal assignments that take into account all resources and positions in the pool, as well as the complex constraints defining a good match…By applying the Optimatch assignment scheme, ***up to 15 percent more matches can be found*** *compared to a semi-automated first-come-first-serve scheme. In addition, Optimatch results show* ***an increase in the average quality of the matches****."*

https://www.research.ibm.com/haifa/dept/vst/csp_wf.shtml

Recap: CSPs definition

A **constraint satisfaction problem** (CSP) consists of

- a set of **variables** V
- a **domain** $dom(V_i)$ for each variable $V_i \in V$
- a set of **constraints** C

<u>Simple example:</u>

$V = \{V_1\}$

$dom(V_1) = \{2,4,8,16\}$

$C = \{c_1, c_2\}$

$c_1 : V_1 \neq 8$

$c_2 : V_1 > 2$

Recap: Possible worlds and models

A possible world of a CSP is an assignment of values to **all** of its variables.

A **model/solution** of a CSP is an assignment of values to all of its variables that **satisfies** all of its constraints.

Simple example:

$V = \{V_1\}$

$dom(V_1) = \{2,4,8,16\}$

$C = \{c_1, c_2\}$

$c_1 : V_1 \neq 8$

$c_2 : V_1 > 2$

Possible worlds and models

$\{V_1 = 2\}$

$\{V_1 = 4\}$ (a model)

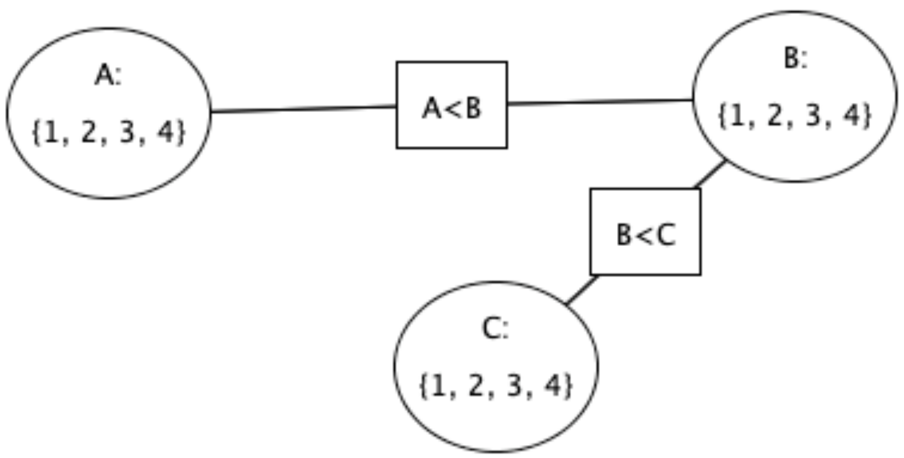
$\{V_1 = 8\}$

$\{V_1 = 16\}$ (a model)

Constraint networks

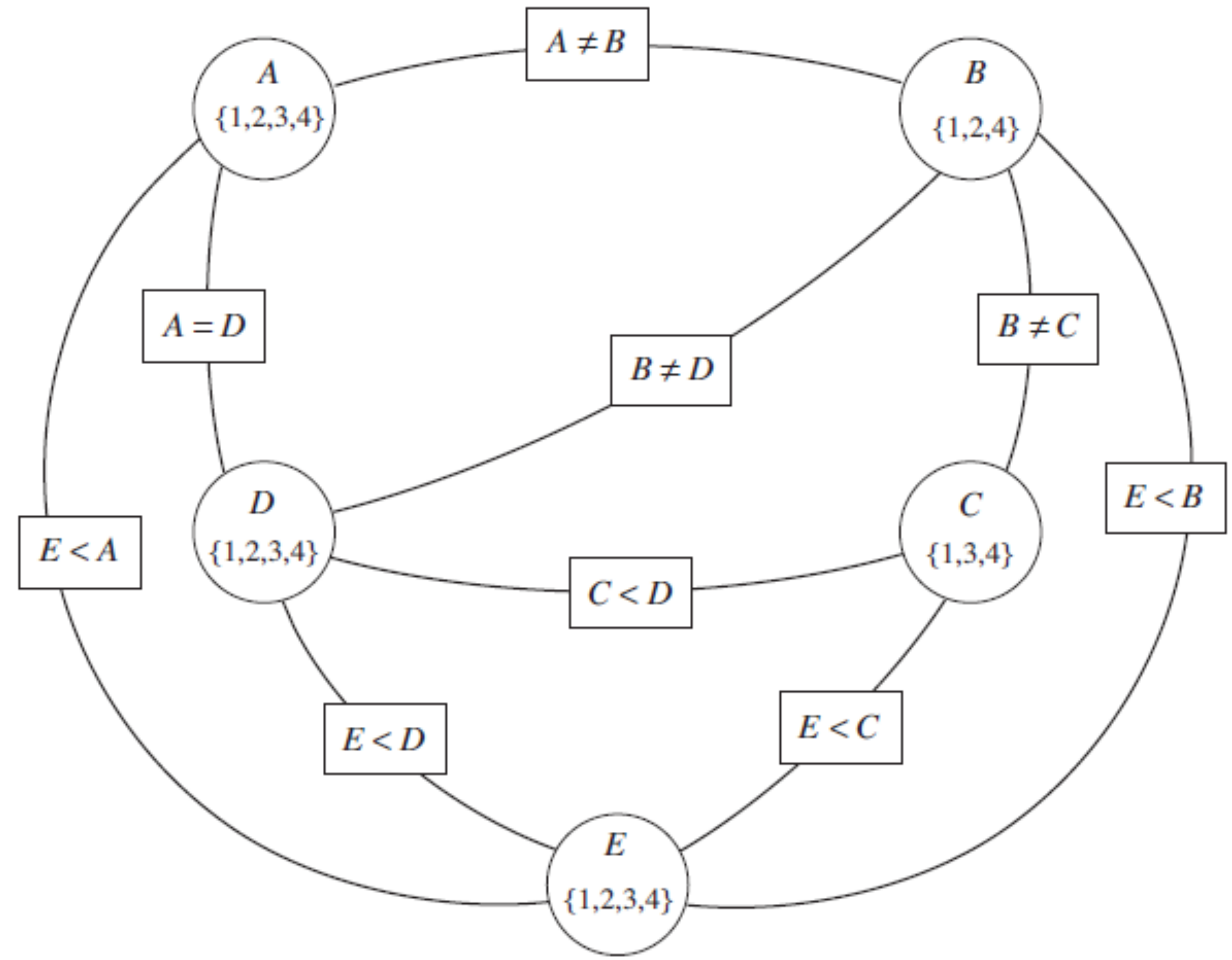
Definition: A **constraint network** is defined by a graph, with

- one node for every variable (drawn as circles or ovals)
- one node for every constraint (drawn as rectangles)
- undirected edges running between variable nodes and constraint nodes whenever a given variable is involved in a given constraint.



Recap: Constraint network

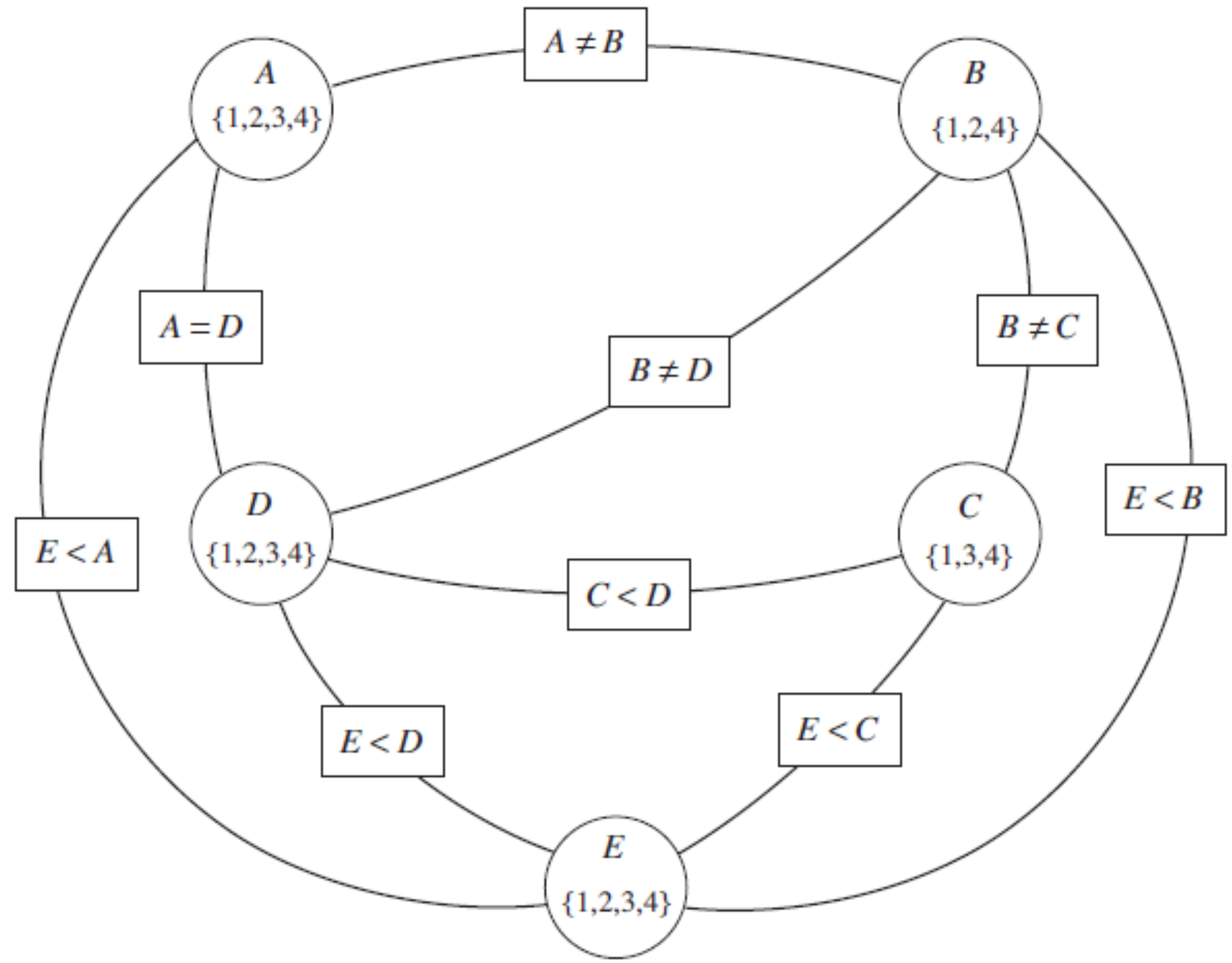
- How many variables?
- How many constraints?



Recap: Constraint network

- How many variables?
  5

- How many constraints?
  9



Solving CSPs

- Generate-and-Test (last lecture)
    - Time complexity $O(cd^k)$

$k \rightarrow$ **# variables**
$d \rightarrow$ **domain size**
$c \rightarrow$ **# constraints**

- Graph search (DFS with backtracking) (last lecture)
    - $O(d^k)$

Can we do better?

- **Today: Arc consistency with domain splitting**

Today: Learning outcomes

From this lecture, students are expected to be able to:

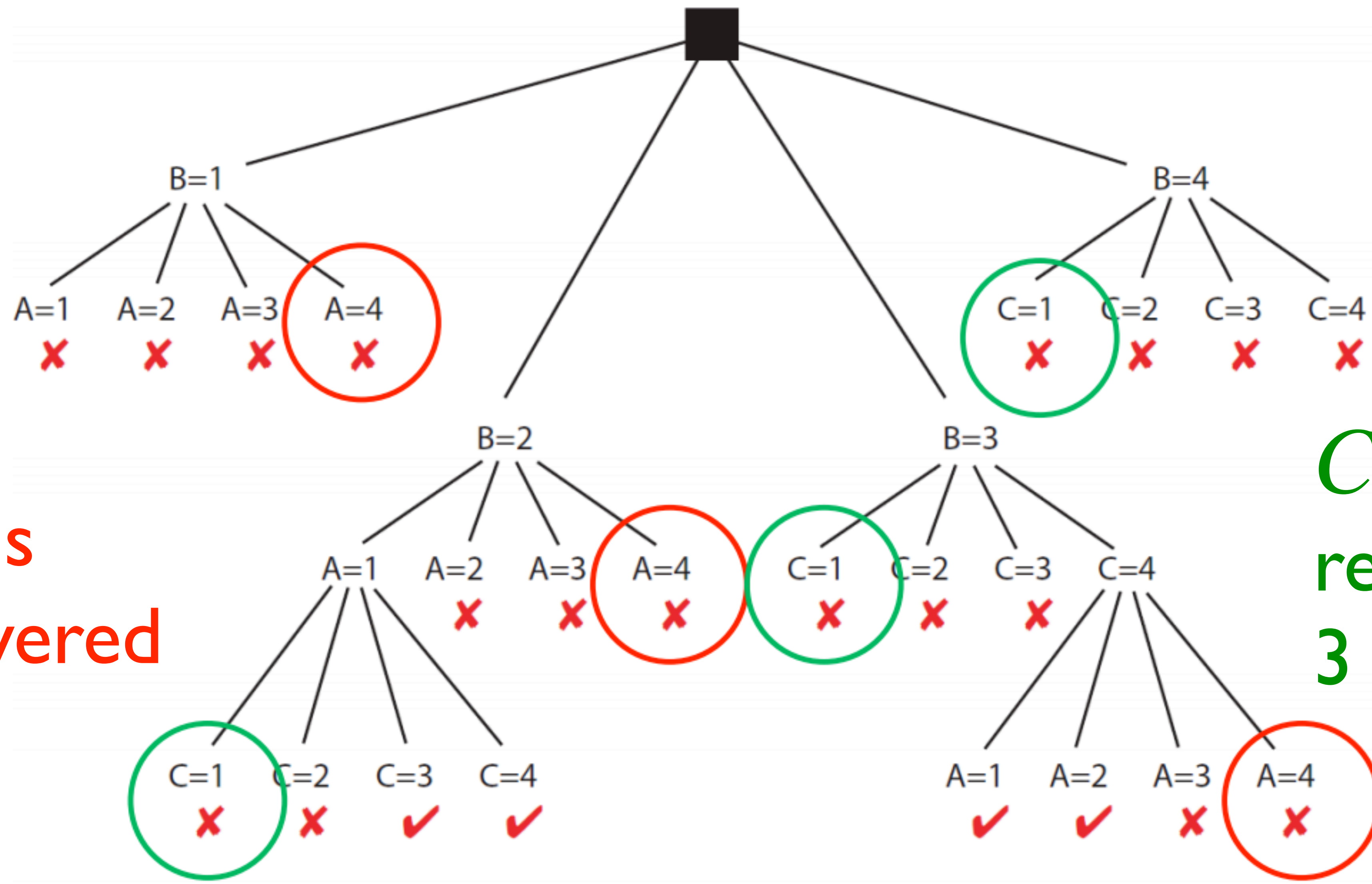
- Define/read/write/trace/debug the arc consistency algorithm. Compute its complexity and assess its possible outcomes.

- Define/read/write/trace/debug domain splitting and its integration with arc consistency.

Graph searching repeats work

<u>Toy problem</u>

Variables: A, B, C, Domains: {1,2,3,4}, Constraints: A<B, B<C



$A \neq 4$ is rediscovered 4 times

$C \neq 1$ is rediscovered 3 times

Can we do better than search?

Key idea: Prune the domains as much as possible before searching for a solution.

Pruning for unary constraints

Key idea: Prune the domains as much as possible before searching for a solution. For **unary constraints**, prune the values in domains that do not satisfy the constraints.

Definition: A variable is **domain consistent** if no value of its domain is ruled impossible by any unary constraint.

Example:
Suppose $dom(V_1) = \{1,2,3,4\}$ and a constraint is $V_1 \neq 4$, then V_1 is NOT domain consistent.

Arc consistency

Definition: An arc $< X, r(X, Y) >$ is **arc consistent** if for each value $x \in dom(X)$, there is some value $y \in dom(Y)$, such that $r(x, y)$ is satisfied.

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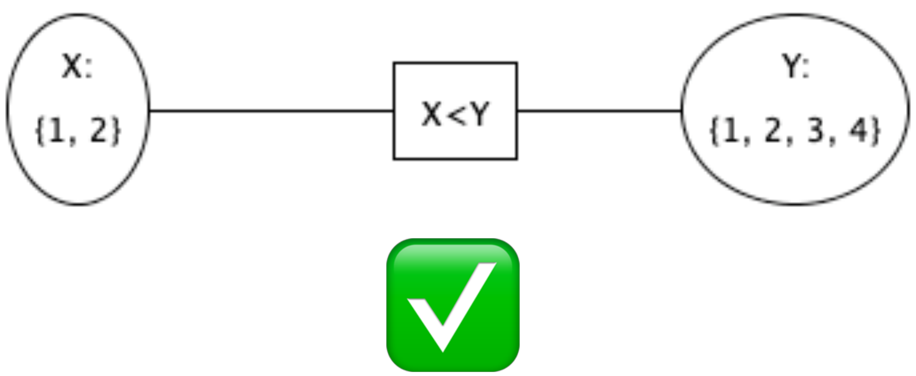

Arc $< X, r(X, Y) >$ is consistent.
$dom(X) = \{1,2\}, dom(Y) = \{1,2,3,4\}$
Arc consistent: Both $X = 1$ and $X = 2$ have OK values for Y.

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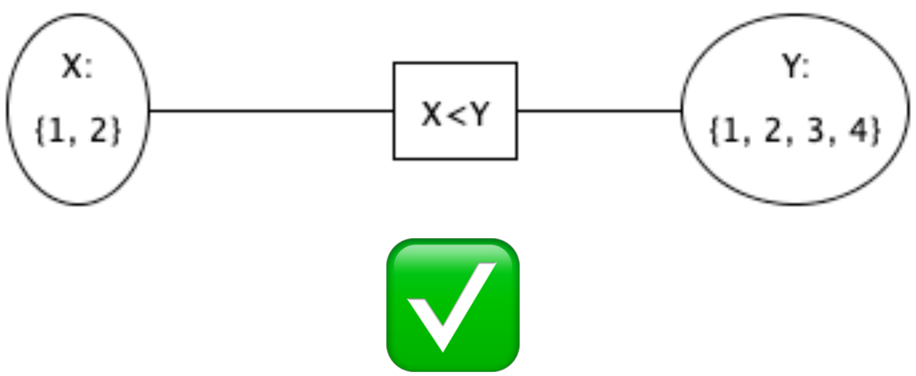

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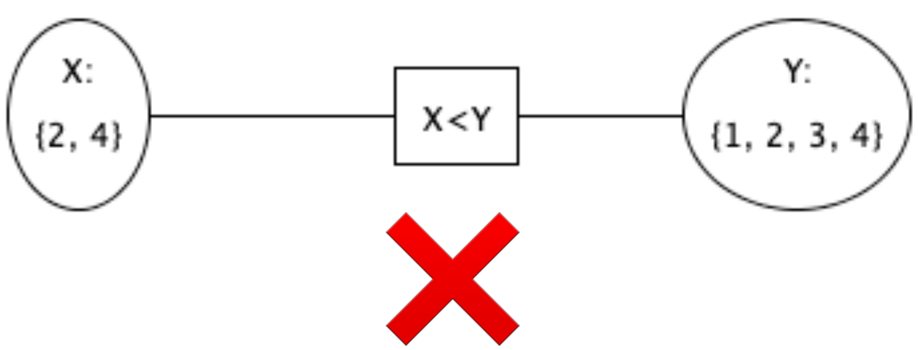
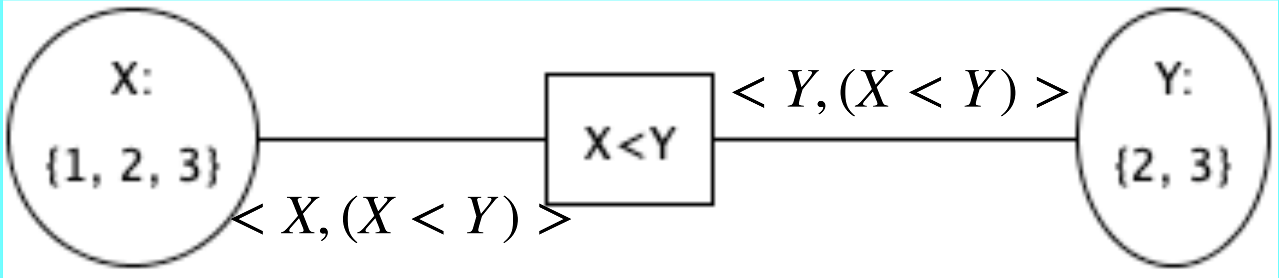


Arc $< X, r(X, Y) >$ is not consistent.
$dom(X) = \{2,4\}, dom(Y) = \{1,2,3,4\}$
Not arc consistent because no value in $dom(Y)$ that satisfies $X < Y$ if $X = 4$.

Arc consistency

Consider the two arcs $< X, (X < Y) >$ and $< Y, (X < Y) >$ in the constraint network below.



A. Both arcs are consistent

B. Left consistent and right inconsistent

C. Right consistent and left inconsistent

D. Both arcs are inconsistent

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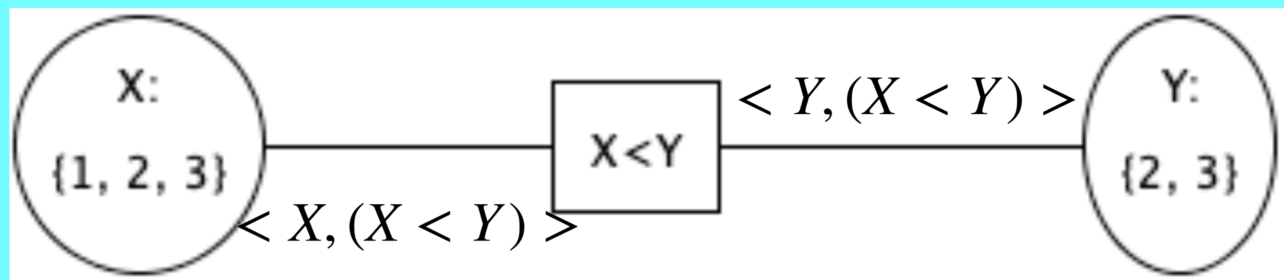


A. Both arcs are consistent

B. Left consistent and right inconsistent

C. Right consistent and left inconsistent ✅

D. Both arcs are inconsistent

Why arc consistency?

- Arc consistency is a desirable property because when an arc is non-consistent, the value that makes it inconsistent is never going to be part of the global solution.

- Waste of time to include that in backtracking.

Enforcing arc consistency

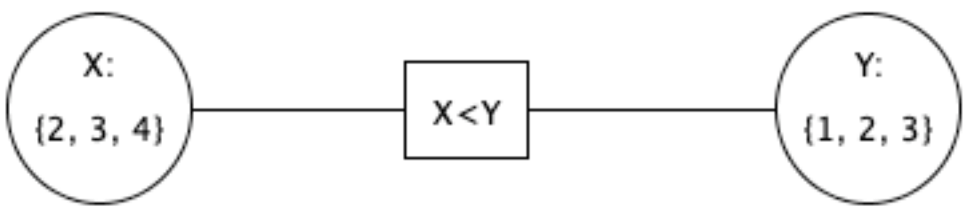
- If an $< X, r(X, Y) >$ is not arc consistent,
  - delete all values x in $dom(X)$ for which there is no corresponding value in $dom(Y)$ to make the arc $< X, r(X, Y) >$ consistent.
  - This removal can never rule out any models/solutions. Why?

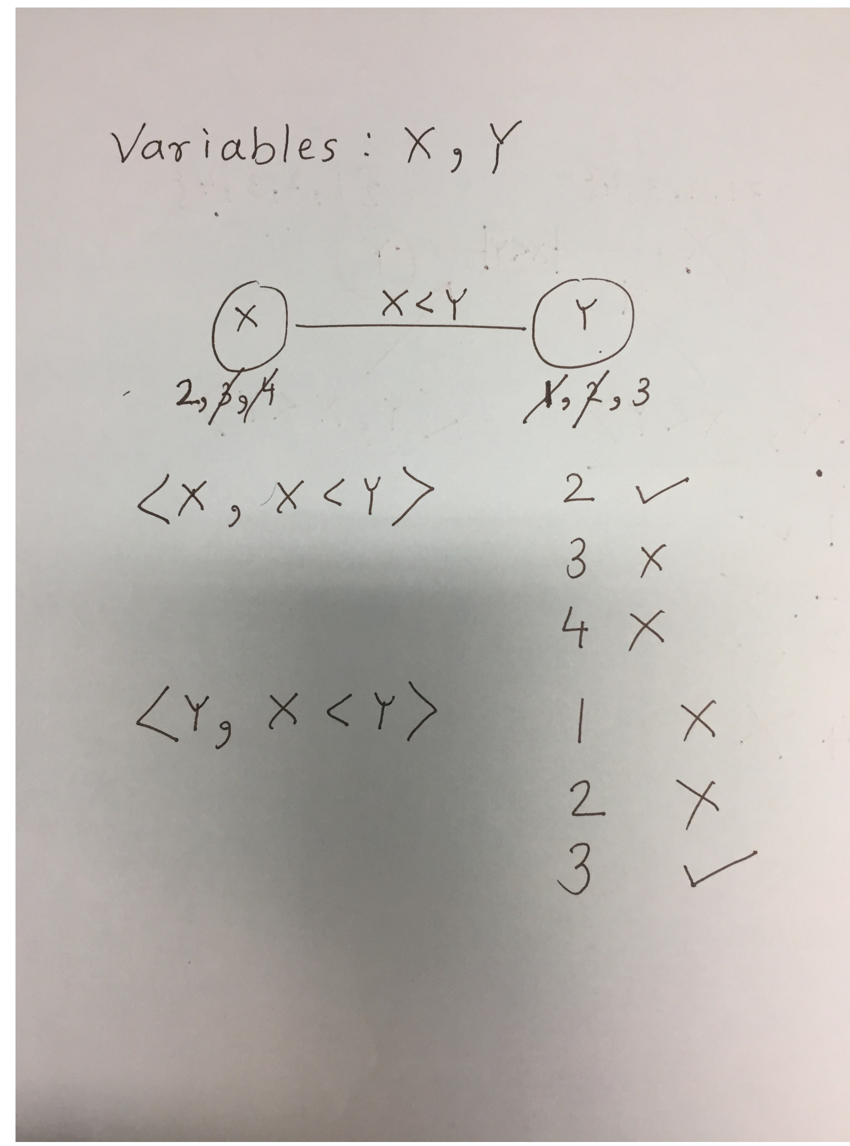
Example: enforcing arc consistency

<u>Toy problem</u>

Variables: X, Y; dom(X) = {2,3,4}; dom(Y) = {1,2,3};

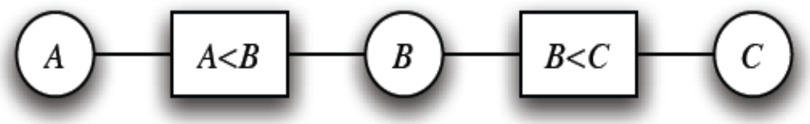
Constraints: X < Y

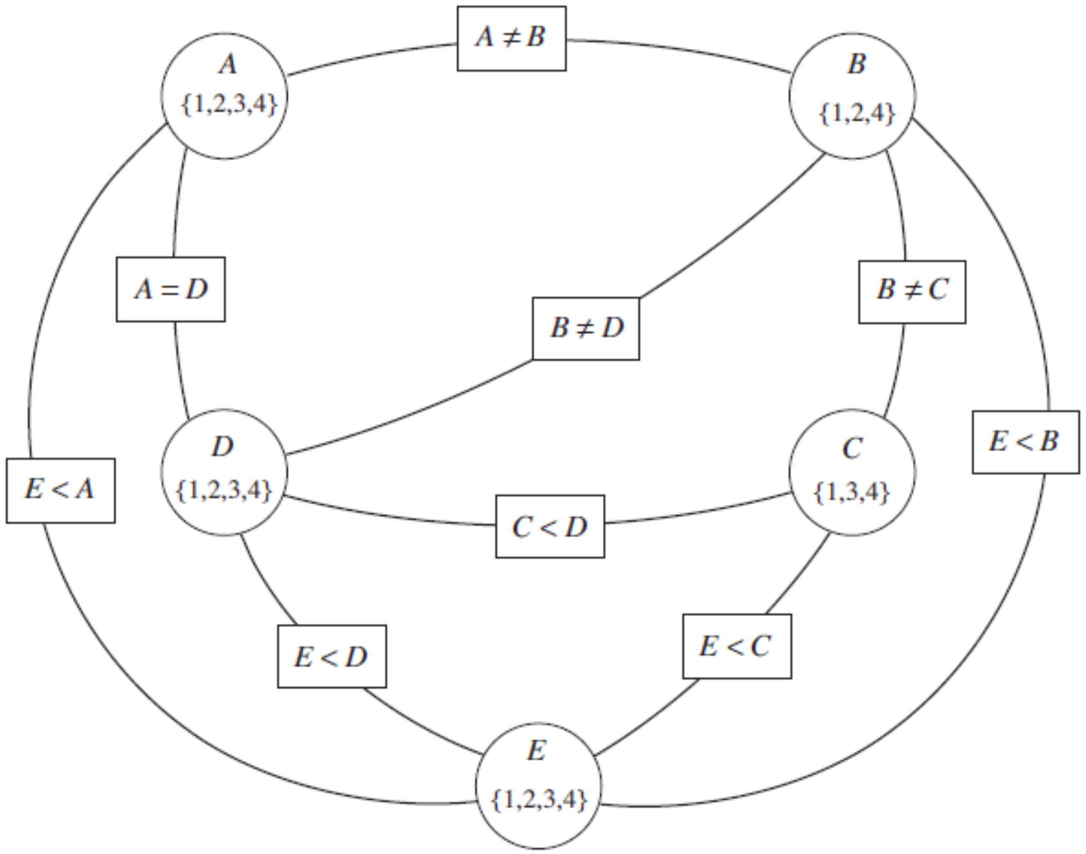




Enforcing arc consistency

How can we make a constraint network arc consistent?
Arc consistency algorithm





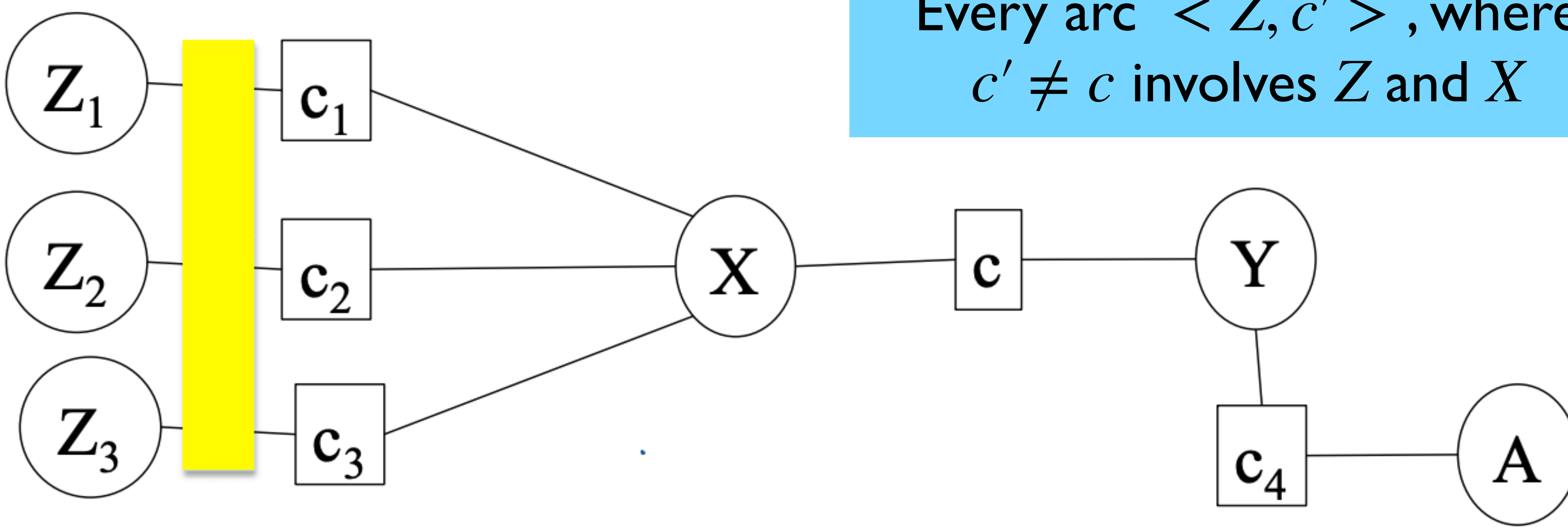
Arc consistency algorithm: high-level strategy

- Consider the arcs in turn, making each arc consistent.
- Reconsider arcs that could be made inconsistent again by this pruning of the domains
- Eventually reach a ‘fixed point’: all arcs consistent

Which arcs need to be considered?

When we reduce the domain of a variable X to make an arc $< X, c >$ arc consistent, which arcs do we need to consider?

Every arc $< Z, c' >$, where $c' \neq c$ involves Z and X



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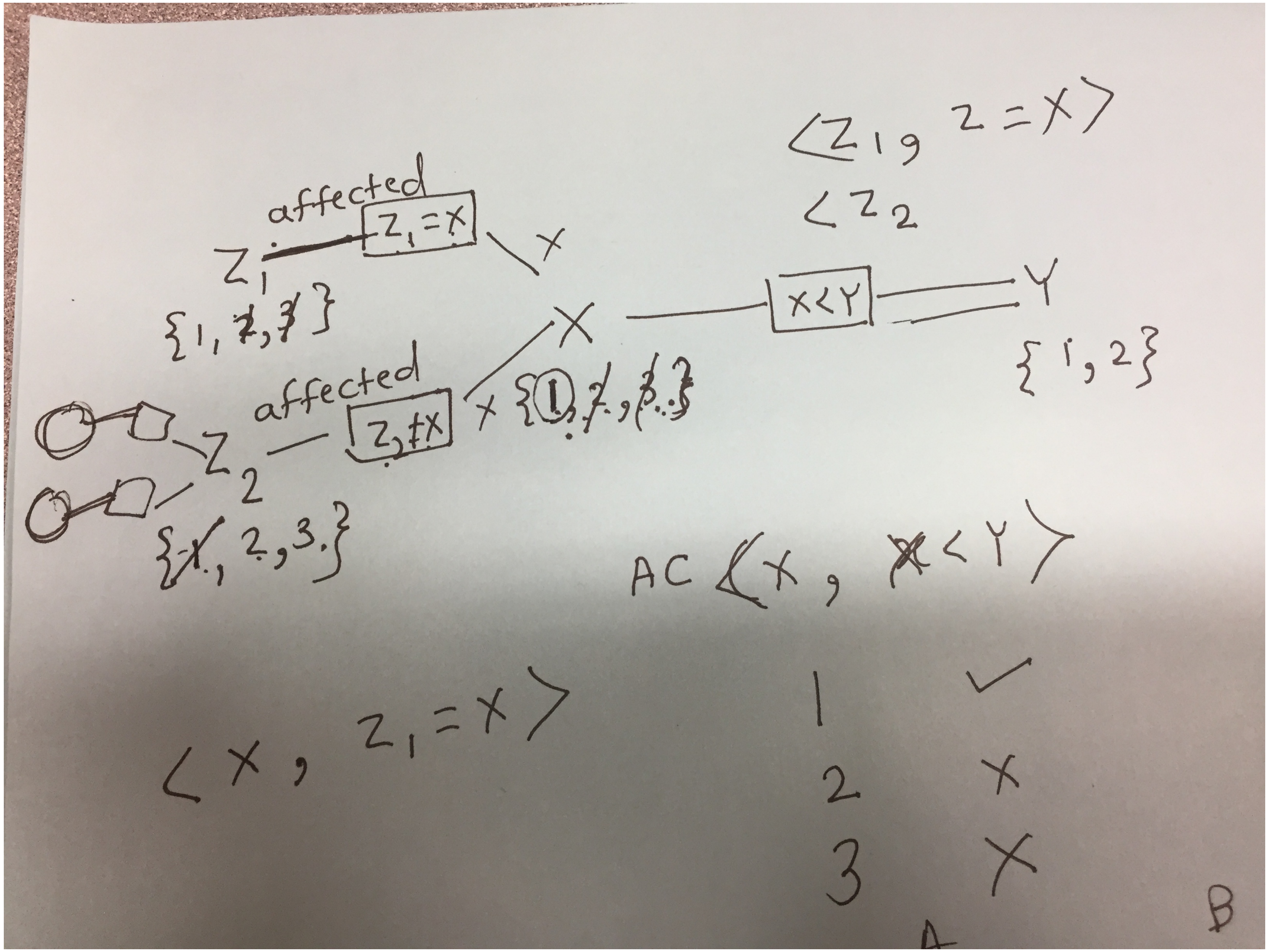

Arc consistency pseudo code

```
TDA → all arcs in constraint network

while (TDA is not empty):
    select arc a from TDA
    if a is not consistent then
        make a consistent
        add arcs to TDA that may now be
inconsistent
```

Arc consistency algorithm (for binary constrinats)

```
Procedure GAC(V,dom,C)
    Inputs
        V: a set of variables
        dom: a function such that dom(X) is the domain of variable X
        C: set of constraints to be satisfied
    Output
        arc-consistent domains for each variable
    Local
        DX is a set of values for each variable X
        TDA is a set of arcs

    for each variable X do
        DX ←dom(X)
    TDA ←{⟨X,c⟩| X ∈ V, c ∈ C  and X ∈ scope(c)}

    while (TDA  {})
        select ⟨X,c⟩ ∈TDA
        TDA ←TDA  \ {⟨X,c⟩}
        NDX ←{x| x ∈ DX and  y ∈ DY s.t. (x, y) satisfies c}
        if (NDX ≠  DX) then
            TDA ← TDA  ∪ { ⟨Z,c'⟩ | X ∈ scope(c'), c'  c, Z ∈ scope(c') \ {X} }
            DX ← NDX

     return {DX| X  is a variable}
```

TDA: ToDoArc, blue arcs in AIspace

Scope of constraint c is the set of variables involved in that constraint

X's domain changed $\Rightarrow$ arcs (Z, c') for variables Z sharing a constraint c' with X could become inconsistent .

ND_x: values x for X for which there a value y supporting x

Arc consistency algorithm: Interpreting outcomes

Three possible outcomes (when all arcs are arc consistent):

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Each domain has a single value.
We have a (unique) solution.
E.g., Variables = {A, B, C} with domains {1,2,3}; constraints: A < B, B <C

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At least one domain is empty.
No solution! All values are ruled out for this variable.
E.g., Variables = {A, B, C}; dom{A} = {1}, dom(B) = {1,2}, dom(C) = {1,2};
constraints: $A = B$, $B = C$, $A \neq C$

Arc consistency algorithm: Interpreting outcomes

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Some domains have more than one value
There may be a solution, multiple ones, or none. Need to solve this new CSP (usually simpler) problem: same constraints, domains have been reduced
E.g., built-in example "simple problem 2"

Arc consistency algorithm: Complexity

Let max size of a variable domain be d.
Let the number of variables be n.
Let the number of constraints be c
Let all constraints be **binary**.

How often will we prune the domain of variable V?

A. $O(n)$

B. $O(d)$

C. $O(n \times d)$

D. $O(c)$

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d possible values for each variable.

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Arc consistency algorithm: Complexity

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How many arcs will be put on the ToDoArc (TDA) list when pruning domain of variable V?
O(degree of variable V)
Across all variables, sum of degrees of all variables =
$2 \times$ number of constraints $= 2 \times c$
Together: we'll put $O(dc)$ arcs on the ToDoArc list

Arc consistency algorithm: Complexity

Let max size of a variable domain be d.
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Checking consistency for each arc.
Have to check constraints for each value in the head of the arc to the each value in the tail of the arc.

$O(d^2)$

Arc consistency algorithm: Complexity

Let max size of a variable domain be d.
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Overall complexity: arcs put on the TDA list × constraints consistency checking for each arc = $O(cd^3)$

Compare to $O(d^n)$ of DFS! Arc consistency is MUCH faster.

Arc consistency algorithm: Complexity

Let max size of a variable domain be d.
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Scales **linearly** with the number of constraints and **cubically** with the size of the domain

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Compare to $O(d^n)$ of DFS! Arc consistency is MUCH faster.

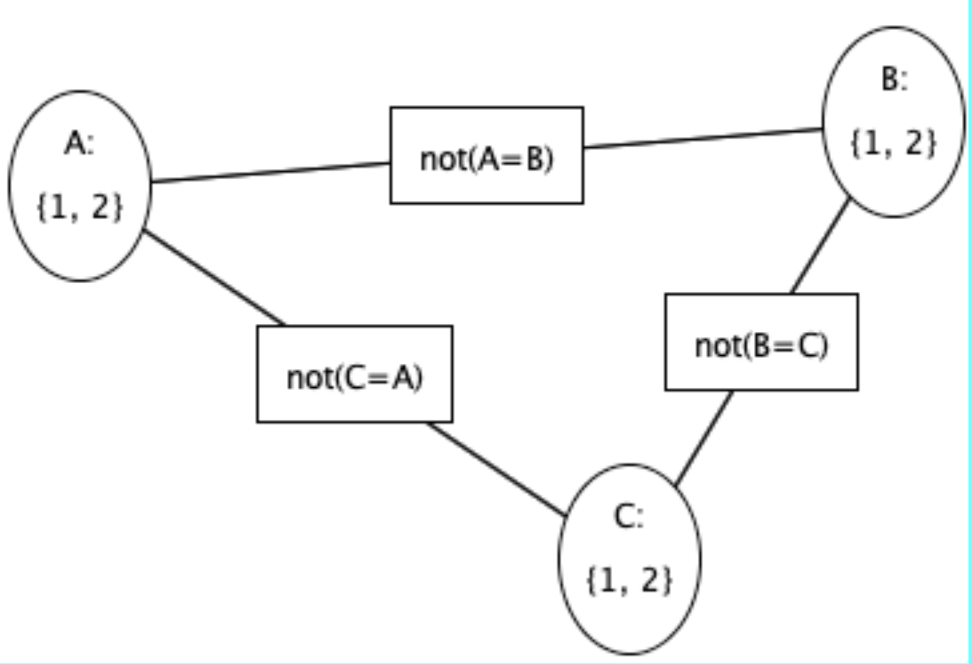
Scales **exponentially** with the number of variables

Domain splitting

Can we have an arc consistent network with non-empty domains that has no solution?

A. Yes

B. No



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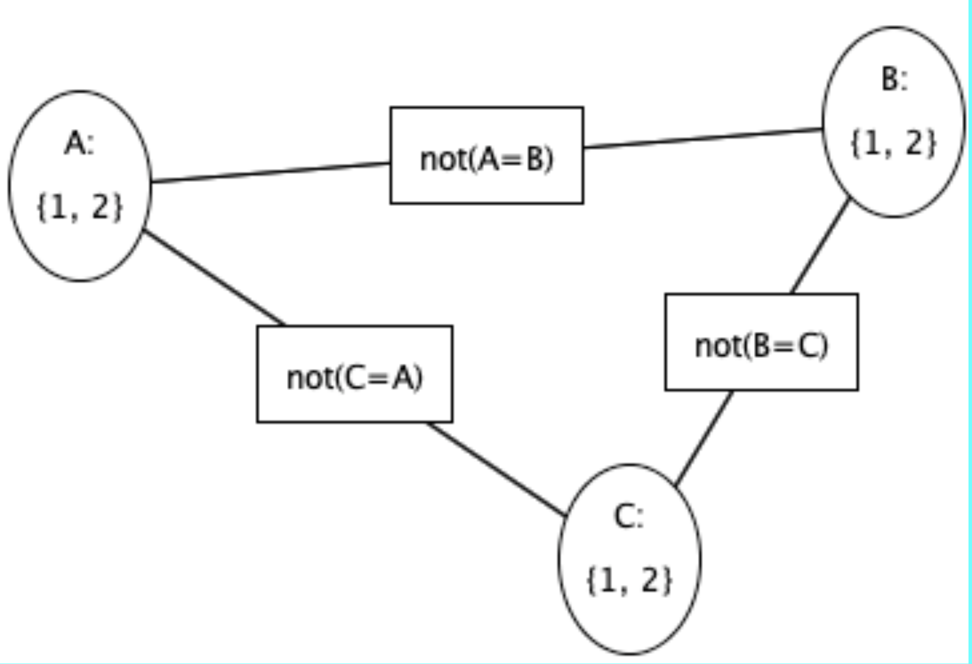

Domain splitting (case analysis)

- Arc consistency ends: Some domains have more than one value → may or may not have a solution
  - Apply Depth-First Search with Pruning or
  - Split the problem in a number of disjoint cases
  - Solution to CSP is the **union** of solutions to these disjoint cases

Example:
CSP with $dom(X) = \{x_1, x_2, x_3, x_4\}$ becomes
CSP_1 with $dom(X) = \{x_1, x_2\}$
CSP_2 with $dom(X) = \{x_3, x_4\}$
Solution is the union of solutions of CSP_1 and CSP_2

Domain splitting in action

Trace it on "simple problem 2"

Coming up

Readings for next class

- 4.5 Domain Splitting
- 4.7 Local Search