Bayesian nonparametric models for bipartite graphs

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Bipartite Networks

Definition

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- **Readers reading books**

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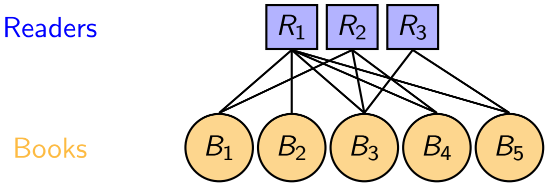
Definition

We say **degree of a vertex** the number of edges connected to that vertex.

Bipartite Networks

Bipartite Networks



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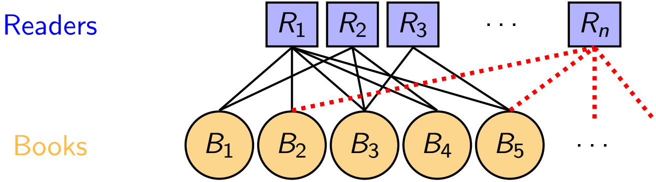

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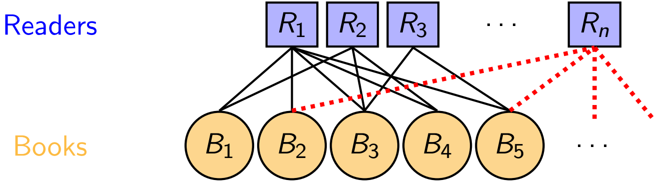
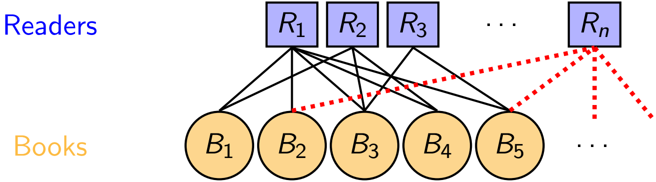


Bayesian nonparametric (BNP) models for network growth:

Bipartite Networks



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- Parameter of interest is **infinite-dimensional** (i.e. infinite number of books)

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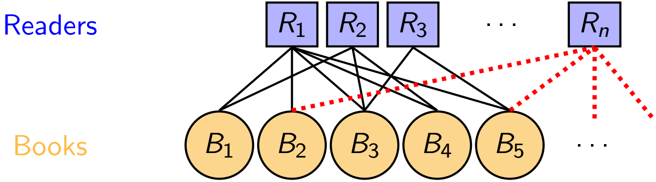

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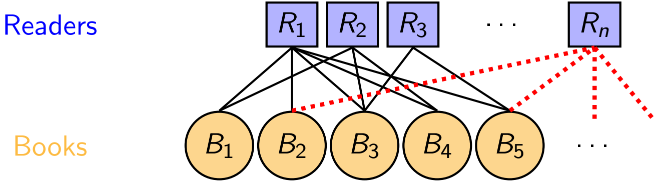

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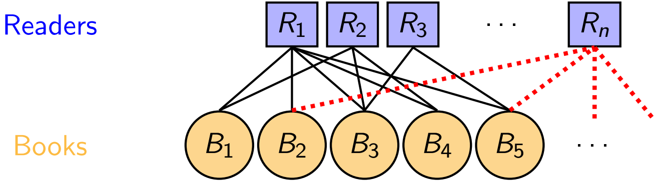

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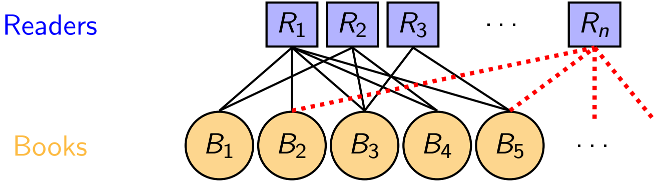

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- Bayesian nonparametric (BNP) models:
  - Indian Buffet Process (IBP), but does not induce power-law behaviour
  - Stable IBP, but induces Poissonian distribution for the degree of readers
- Flexible BNP model able to capture **power-law behaviour** for both books and readers, while retaining **computational tractability**

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Bayesian Model

Bipartite graph

Bayesian Model

Bipartite graph

- We represent a bipartite graph using a collection of atomic measure Z_i. For each reader $i = 1, \dots, n$ with books $j = 1, \dots, \infty$:

$$Z_i = \sum_{j=1}^{\infty} z_{ij} \delta_{\theta_j}$$

where $\{\theta\} \subset \Theta$ the set of books and z_{ij} equal 1 if reader i has read book j, 0 otherwise.

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  where $\{\theta\} \subset \Theta$ the set of books and z_{ij} equal 1 if reader i has read book j, 0 otherwise.
- For each reader we consider the latent process V_i:

$$V_i = \sum_{j=1}^{\infty} v_{ij} \delta_{\theta_j}$$

  where v_{ij} (inversely) controls the **probability of the existence of the edge** between reader i and book j.

Bayesian Model

Latent process

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- Assuming:

$$v_{ij} \mid w_j \sim Exp(w_j \gamma_i)$$

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  - A positive **popularity parameter** w_j assigned to each book
  - A positive **interest-in-reading parameter** γ_i assigned to each reader

- Then, the probability that reader i reads book j is:

$$p(z_{ij} = 1|w_j, \gamma_i) = 1 - exp(w_j\gamma_i)$$

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For tractability issues, we consider $\mathrm{u_{ij}} = \mathrm{min(v_{ij}, 1)}$ and the process $\mathrm{U_i}$. Z_i can be obtained deterministically from U_i.

Bayesian Model

Book popularity parameter

Bayesian Model

Book popularity parameter

Definition

Let Θ be a measurable space. A **completely random measure** (**CRM**) is a random measure G such that for any collection of disjoint measurable subsets $A_1, \ldots, A_n$ of Θ, the random masses of the subsets $G(A_1), \ldots, G(A_n)$ are independent.

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$\mathbf{G} \sim \mathbf{CRM}(\boldsymbol{\lambda}, \mathrm{h})$ with Levy measure:

$$\Lambda(\mathrm{d}w, \mathrm{d}\theta) = \lambda(w)h(\theta)\mathrm{d}w\mathrm{d}\theta$$

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G $\sim$ CRM($\boldsymbol{\lambda}$, h) with Levy measure:

$$\Lambda(\mathrm{d}w, \mathrm{d}\theta) = \lambda(w) h(\theta) \mathrm{d}w \mathrm{d}\theta$$

Realizations of G take the form of Poisson processes over $\{(w_j, \theta_j), j = 1, \ldots, \infty\} \subset \mathbb{R}_+ \times \Theta$:

$$G = \sum_{j=1}^{\infty} w_j \delta_{\theta_j}$$

Bayesian Model

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An example of CRM is the **generalized gamma process** (GGP), which includes the gamma process (GP), the inverse Gaussian process (IGP) and stable process as special cases:

$$\lambda(w;\, \alpha, \sigma, \tau) = \frac{\alpha}{\Gamma(1-\sigma)} w^{-\sigma-1} e^{-w\tau}$$

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$$\begin{cases} \int_0^\infty \lambda(w) dw = \infty \\ \int_0^\infty (1-e^{-w}) \lambda(w) dw < \infty \end{cases}$$

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We assume:

$$\begin{cases} \int_0^\infty \lambda(w) \mathrm{d}w = \infty \\ \int_0^\infty (1-e^{-w})\lambda(w)\mathrm{d}w < \infty \end{cases} \Rightarrow \mathbf{G}(\Theta) = \sum_{j=1}^\infty w_j \textbf{ finite and positive}$$

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Hierarchical model

Z_i **is a Poisson process**, obtained from transformations of Poisson processes.

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Proposition

Z_i is marginally characterized by a Poisson process. Furthermore, **the total mass** $Z_i(\Theta) = \sum_{j=1}^{\infty} z_{ij}$, which corresponds to the total number of books read by reader i, **is finite with probability one and admits a Poisson($\psi_\lambda(\gamma_i)$) distribution**, with:

$$\psi_\lambda(\gamma_i) = \int_0^\infty (1 - e^{-\gamma_i w})\lambda(w)\mathrm{d}w$$

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$$\psi_\lambda(\gamma_i) = \int_0^\infty (1 - e^{-\gamma_i w})\lambda(w)\mathrm{d}w$$

We can sum up the model in the following hierarchical form:

$$v_{ij} \mid G \sim \mathrm{Exp}(w_j \gamma_i)$$
$$G \sim \mathrm{CRM}(\lambda, h)$$

Bayesian Model
Posterior Characterization

We observe a set of edges $\{z_{ij}\}$ of a bipartite network $Z_1, \ldots, Z_n$ of n reader:

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- $m_j = \sum_{i=1}^{n} Z(\{\theta_j\}) = \sum_{i=1}^{n} z_{ij}$ **the degree of book** j

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Posterior distribution of the CRM given the latent process U coincides with the distribution of another CRM having a rescaled intensity and fixed observed points of discontinuity :

$$\mathbf{G} = \mathbf{G}^{*} + \sum_{j=1}^{K} w_j \delta_{\theta_j}$$

Bayesian Model

Posterior Characterization

- G^* and $\{w_j\}$ are mutually independent with:

$$G^* \sim \mathrm{CRM}(\lambda^*, h) \quad \text{and} \quad \lambda^*(w) = \lambda(w) \exp(-w \sum_{i=1}^{n} \gamma_i)$$

and the masses:

$$p(w_j | \, \text{rest}) \propto \lambda(w_j) w_j^{m_j} \exp(-w_j \sum_{i=1}^{n} \gamma_i U_{ij})$$

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- For the GGP, G^* is still a GGP with parameters $\alpha^* = \alpha$, $\sigma^* = \sigma$ and $\tau^* = \tau + \sum_i^n \gamma_i$ and:

$$w_j | \,\text{rest} \sim \mathrm{Gamma}(m_j - \sigma, \tau + \sum_{i=1}^{n} \gamma_i u_{ij})$$

Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $\mathrm{x_{ij}} = -\log(u_{ij})$ positive latent score

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Books

K_1

$\boxed{R_1}$

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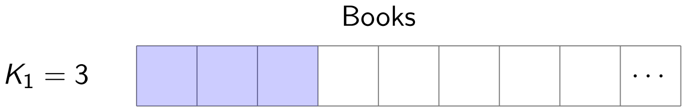

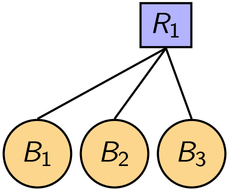

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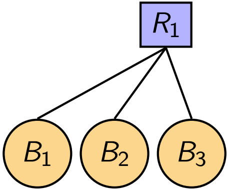
Books

$K_1 = 3$	18	4	14						$\cdots$



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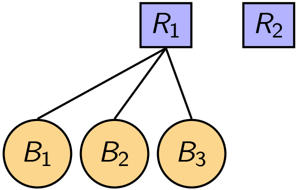

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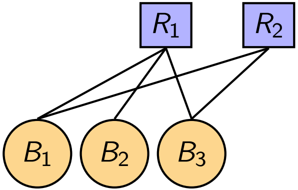

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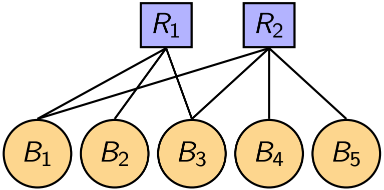
	Books								
$K_1 = 3$	18	4	14						$\cdots$
$K_2^+ = 2$									$\cdots$



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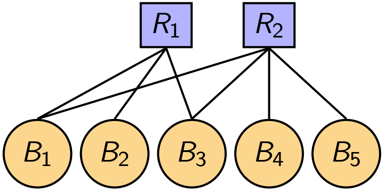
Books

$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$



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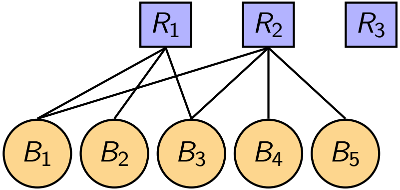

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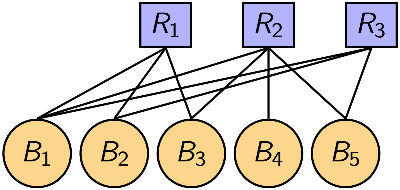

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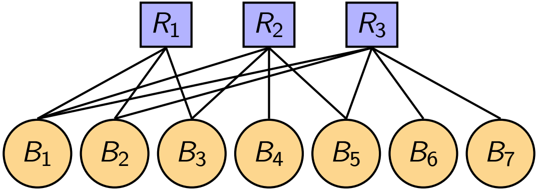

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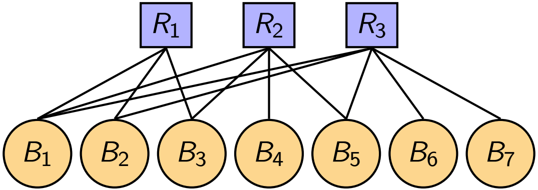
$K_2 = 4$	12	0	8	13	4				$\cdots$
$K_3 = 5$	16	10	0	0	14	9	6		$\cdots$



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For the GGP:

1. For $i = 1, \ldots, n$ and $j = 1, \ldots, K$ set $u_{ij} = 1$ if $z_{ij} = 0$, otherwise:

$$u_{ij} | z_{ij}, w_j, \gamma_i \sim \mathrm{rExp}(\gamma_i w_j, 1)$$

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2. For $j = 1, \ldots, K$:

$$w_j | U, \gamma_i \sim \mathrm{Gamma}(m_j - \sigma, \tau + \sum_i^n \gamma_i u_{ij})$$

and

$$G^*(\Theta) \sim \text{Exponentially tilted stable}^1$$

[1] For general cases $G^*(\Theta)$ follows $g^*(w) \propto g(w) \exp^{-w \sum_i^n \gamma_i}$ with $g(w)$ the distribution of $G(\Theta)$

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Update of γ_i

1. **Parametric**: γ_i to be unknown and estimate them from the graph by assigning a prior $\gamma_i \sim \text{Gamma}(a_\gamma, b_\gamma)$ and update:

$$\gamma_i |\, G, U \sim \text{Gamma}\left(a_\gamma + \sum_j^K z_{ij},\, b_\gamma + \sum_j^K w_j u_{ij} + G^*(\Theta)\right)$$

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   But $Z_i(\Theta)$ still have a (but more flexible) Poisson distribution!

2. **Nonparametric**: Let $\Gamma \sim \textbf{CRM}(\boldsymbol{\lambda}_\gamma, \mathrm{h}_\gamma)$ and a measurable space of readers $\tilde{\Theta}$, which we can represent in the form $\Gamma = \sum_{i=1}^\infty \gamma_i \delta_{\theta_j}$. Conditionally on $(U, w, G^*(\Theta))$, we update:

$$\Gamma = \Gamma^* + \sum_{i=1}^n \gamma_i \delta_{\tilde{\theta}_i}$$

Update of γ_i

1. **Parametric**: γ_i to be unknown and estimate them from the graph by assigning a prior $\gamma_i \sim \text{Gamma}(a_\gamma, b_\gamma)$ and update:

$$\gamma_i | G, U \sim \text{Gamma}\left(a_\gamma + \sum_j^K z_{ij},\ b_\gamma + \sum_j^K w_j u_{ij} + G^*(\Theta)\right)$$

   But $Z_i(\Theta)$ still have a (but more flexible) Poisson distribution!

2. **Nonparametric**: Let $\Gamma \sim \textbf{CRM}(\lambda_\gamma, \mathrm{h}_\gamma)$ and a measurable space of readers $\tilde{\Theta}$, which we can represent in the form $\Gamma = \sum_{i=1}^{\infty} \gamma_i \delta_{\theta_j}$. Conditionally on $(U, w, G^*(\Theta))$, we update:

$$\Gamma = \Gamma^* + \sum_{i=1}^{n} \gamma_i \delta_{\tilde{\theta}_i}$$

   We have more of flexibility in the modelling of the distribution of the degree of readers(**power-law behavior**)!

Posterior characterization for GGP for w_i and γ_i

Let G and Γ GGP distributed with parameters (α, σ, τ) and $(\alpha_\gamma, \sigma_\gamma, \tau_\gamma)$:

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$$\Gamma^* \sim \mathrm{CRM}(\lambda_\gamma^*, h_\gamma)$$

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$$\gamma_i | \, U, G \sim \text{Gamma}\Big(K_i - \sigma_\gamma, \tau_\gamma + \sum_{j=1}^{K} w_j u_{ij} + G^*(\Theta)\Big)$$

- Book update: $G = G^* + \sum_{i=1}^{K} w_i \delta_{\theta_i}$ with:

$$G^* \sim \text{CRM}(\lambda^*, h)$$

$$w_i | \, U, \Gamma \sim \text{Gamma}\Big(m_j - \sigma, \tau + \sum_{i=1}^{n} \gamma_i u_{ij} + \Gamma^*(\tilde{\Theta})\Big)$$

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Similar results are achievable results are achievable also with an S-IBP for the degree distribution of books, but not for readers for which it will always be Poisson!

Real world example – Book-crossing community network

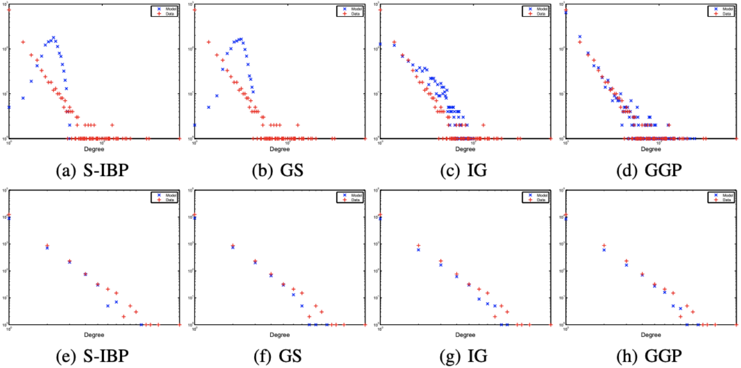

Figure 1: Degree distribution for readers (a-d) and books (e-h) with 4 models: a stable Indian Buffet Process (S-IBP); our model with $\gamma_i = \gamma$ and flat prior assigned (GS); our model with $\gamma_i \sim$ Gamma (a_γ, a_γ) and flat prior assigned to the parameters (IG); our model with GGP prior for γ_i (GGP). Data are presented in red and samples from the models in blue.

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