Bayesian nonparametric models for bipartite graphs

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September 5, 2022

Table of Contents

Bipartite Networks

Definition

A **bipartite graph** is a graph $g = (V, E)$, where vertices V are divided in two sets A and B and edges E can occur only between elements of two different sets.

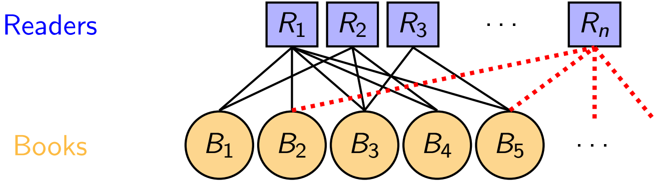
Real world examples:

- Scientists authoring papers
- Internet users posting messages on forums
- **Readers reading books**

Definition

We say **degree of a vertex** the number of edges connected to that vertex.

Bipartite Networks



Bayesian nonparametric (BNP) models for network growth:

- Parameter of interest is **infinite-dimensional** (i.e. infinite number of books)
- Bayesian nonparametric (BNP) models:
  - Indian Buffet Process (IBP), but does <u>not induce power-law behaviour</u>
  - Stable IBP, but <u>induces Poissonian distribution for the degree of readers</u>
- Flexible BNP model able to capture **power-law behaviour** for both books and readers, while retaining **computational tractability**

Table of Contents

Bayesian Model

Bipartite graph

- We represent a bipartite graph using a collection of atomic measure Z_i. For each reader $i = 1, \ldots, n$ with books $j = 1, \ldots, \infty$:

$$Z_i = \sum_{j=1}^{\infty} z_{ij} \delta_{\theta_j}$$

  where $\{\theta\} \subset \Theta$ the set of books and z_{ij} equal 1 if reader i has read book j, 0 otherwise.
- For each reader we consider the latent process V_i:

$$V_i = \sum_{j=1}^{\infty} v_{ij} \delta_{\theta_j}$$

  where v_{ij} (inversely) controls the **probability of the existence of the edge** between reader i and book j.

Bayesian Model

Latent process

- Assuming:

$$v_{ij} | w_j \sim Exp(w_j \gamma_i)$$

  - A positive **popularity parameter** w_j assigned to each book
  - A positive **interest-in-reading parameter** γ_i assigned to each reader

- Then, the probability that reader i reads book j is:

$$p(z_{ij} = 1 | w_j, \gamma_i) = 1 - exp(w_j \gamma_i)$$

For tractability issues, we consider $u_{ij} = \min(v_{ij}, 1)$ and the process U_i. Z_i can be obtained deterministically from U_i.

Bayesian Model

Book popularity parameter

Definition

Let Θ be a measurable space. A **completely random measure** (**CRM**) is a random measure G such that for any collection of disjoint measurable subsets $A_1, \ldots, A_n$ of Θ, the random masses of the subsets $G(A_1), \ldots, G(A_n)$ are independent.

G $\sim$ **CRM**$(\boldsymbol{\lambda}, \mathrm{h})$ with Levy measure:

$$\Lambda(\mathrm{d}w, \mathrm{d}\theta) = \lambda(w)h(\theta)\mathrm{d}w\mathrm{d}\theta$$

Realizations of G take the form of Poisson processes over $\{(w_j, \theta_j), j = 1, \ldots, \infty\} \subset \mathbb{R}_+ \times \Theta$:

$$G = \sum_{j=1}^{\infty} w_j \delta_{\theta_j}$$

Bayesian Model

Book popularity parameter

An example of CRM is the **generalized gamma process** (GGP), which includes the gamma process (GP), the inverse Gaussian process (IGP) and stable process as special cases:

$$\lambda(w; \alpha, \sigma, \tau) = \frac{\alpha}{\Gamma(1-\sigma)} w^{-\sigma-1} e^{-w\tau}$$

G is an **homogeneous CRM**:

- Atoms i.i.d from h (base density), independently from masses
- Masses distributed according to Poisson process over $\mathbb{R}^+$ with intensity λ (Levy intensity)

We assume:

$$\begin{cases} \int_0^\infty \lambda(w) dw = \infty \\ \int_0^\infty (1 - e^{-w}) \lambda(w) dw < \infty \end{cases} \Rightarrow \mathbf{G}(\Theta) = \sum_{j=1}^{\infty} w_j \textbf{ finite and positive}$$

Bayesian Model

Hierarchical model

Z_i **is a Poisson process**, obtained from transformations of Poisson processes.

Proposition

Z_i is marginally characterized by a Poisson process. Furthermore, **the total mass** $Z_i(\Theta) = \sum_{j=1}^{\infty} z_{ij}$, which corresponds to the total number of books read by reader i, **is finite with probability one and admits a Poisson($\psi_\lambda(\gamma_i)$) distribution**, with:

$$\psi_\lambda(\gamma_i) = \int_0^\infty (1 - e^{-\gamma_i w})\lambda(w)\mathrm{d}w$$

We can sum up the model in the following hierarchical form:

$$\begin{aligned} v_{ij} | \, G &\sim \mathrm{Exp}(w_j \gamma_i) \\ G &\sim \mathrm{CRM}(\lambda, h) \end{aligned}$$

We observe a set of edges $\{z_{ij}\}$ of a bipartite network $Z_1, \dots, Z_n$ of n reader:

- K books $\{\theta_1, \dots, \theta_K\}$
- $\mathrm{K_i} = Z_i(\Theta) = \sum_{j=1}^{\infty} z_{ij}$ **the degree of reader** i
- $\mathrm{m_j} = \sum_{i=1}^{n} Z(\{\theta_j\}) = \sum_{i=1}^{n} z_{ij}$ **the degree of book** j

Posterior distribution of the CRM given the latent process U coincides with the distribution of another CRM having a rescaled intensity and fixed observed points of discontinuity :

$$\mathbf{G} = \mathbf{G}^* + \sum_{j=1}^{K} w_j \delta_{\theta_j}$$

Bayesian Model

Posterior Characterization

- G* and $\{w_j\}$ are mutually independent with:

$$G^* \sim \text{CRM}(\lambda^*, h) \quad \text{and} \quad \lambda^*(w) = \lambda(w) \exp(-w \sum_{i=1}^{n} \gamma_i)$$

  and the masses:

$$p(w_j | \text{ rest}) \propto \lambda(w_j) w_j^{m_j} \exp(-w_j \sum_{i=1}^{n} \gamma_i U_{ij})$$

- For the GGP, G^* is still a GGP with parameters $\alpha^* = \alpha$, $\sigma^* = \sigma$ and $\tau^* = \tau + \sum_i^n \gamma_i$ and:

$$w_j | \text{ rest} \sim \text{Gamma}(m_j - \sigma, \tau + \sum_{i=1}^{n} \gamma_i u_{ij})$$

Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $\mathrm{x_{ij}} = -\log(u_{ij})$ positive latent score

Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

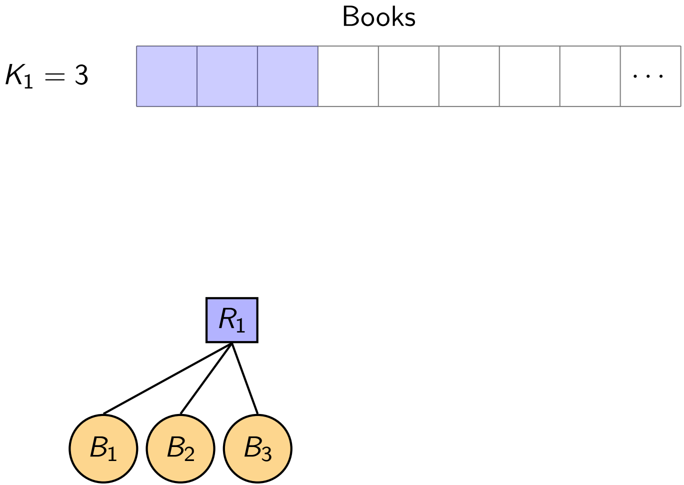
Books

K_1

$\boxed{R_1}$

Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

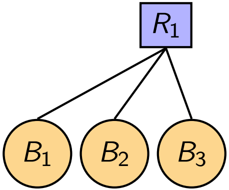


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

Books

$K_1 = 3$	18	4	14						$\cdots$

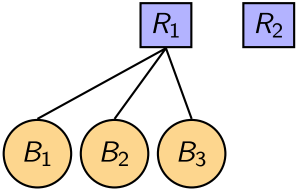


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $\mathrm{x_{ij}} = -\log(u_{ij})$ positive latent score

Books

$K_1 = 3$	18	4	14						$\cdots$
K_2									

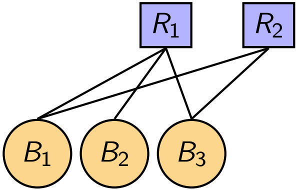


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

Books

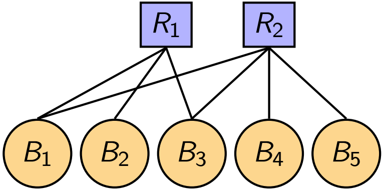
$K_1 = 3$	18	4	14						$\cdots$
K_2									



Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

	Books								
$K_1 = 3$	18	4	14						$\cdots$
$K_2^+ = 2$									$\cdots$

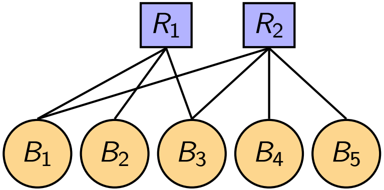


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $\mathrm{x_{ij}} = -\log(u_{ij})$ positive latent score

Books

$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$

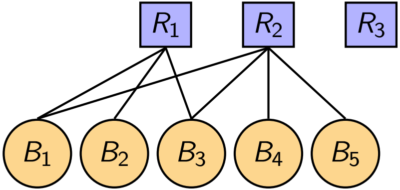


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

Books

$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$
K_3									

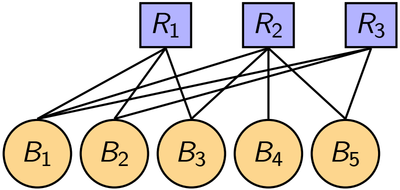


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

Books

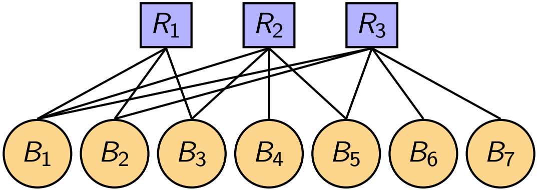
$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$
K_3									



Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

	Books								
$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$
$K_3^+ = 2$									$\cdots$

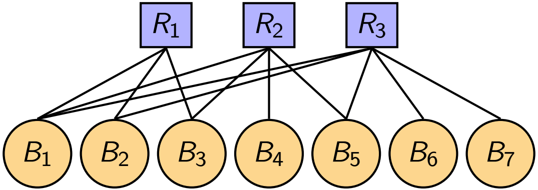


Generative Process

Distribution of $Z_n | U_1, \ldots, U_{n-1}$, with $x_{ij} = -\log(u_{ij})$ positive latent score

Books

$K_1 = 3$	18	4	14						$\cdots$
$K_2 = 4$	12	0	8	13	4				$\cdots$
$K_3 = 5$	16	10	0	0	14	9	6		$\cdots$



Gibbs sampling

We use Gibbs sampler to derive the posterior distribution of U, G| Z.

<u>For the GGP</u>:

1. For $i = 1, \dots, n$ and $j = 1, \dots, K$ set $u_{ij} = 1$ if $z_{ij} = 0$, otherwise:

$$u_{ij} | z_{ij}, w_j, \gamma_i \sim \mathrm{rExp}(\gamma_i w_j, 1)$$

2. For $j = 1, \dots, K$:

$$w_j | U, \gamma_i \sim \mathrm{Gamma}(m_j - \sigma, \tau + \sum_i^n \gamma_i u_{ij})$$

and

$$G^*(\Theta) \sim \text{Exponentially tilted stable}^1$$

[1] For general cases $G^*(\Theta)$ follows $g^*(w) \propto g(w) \exp^{-w \sum_i^n \gamma_i}$ with $g(w)$ the distribution of $G(\Theta)$

Table of Contents

Update of γ_i

1. **Parametric**: γ_i to be unknown and estimate them from the graph by assigning a prior $\gamma_i \sim \text{Gamma}(a_\gamma,\ b_\gamma)$ and update:

$$\gamma_i | \, G, U \sim \text{Gamma}\Big(a_\gamma + \sum_j^K z_{ij},\ b_\gamma + \sum_j^K w_j u_{ij} + G^*(\Theta)\Big)$$

   But $Z_i(\Theta)$ still have a (but more flexible) Poisson distribution!

2. **Nonparametric**: Let $\Gamma \sim \mathbf{CRM}(\lambda_\gamma, \mathrm{h}_\gamma)$ and a measurable space of readers $\tilde{\Theta}$, which we can represent in the form $\Gamma = \sum_{i=1}^{\infty} \gamma_i \delta_{\theta_j}$. Conditionally on $(U, w, G^*(\Theta))$, we update:

$$\Gamma = \Gamma^* + \sum_{i=1}^{n} \gamma_i \delta_{\tilde{\theta}_i}$$

   We have more of flexibility in the modelling of the distribution of the degree of readers (**power-law behavior**)!

Posterior characterization for GGP for w_i and γ_i

Let G and Γ GGP distributed with parameters (α, σ, τ) and $(\alpha_\gamma, \sigma_\gamma, \tau_\gamma)$:

- Reader update: $\Gamma = \Gamma^* + \sum_{i=1}^{n} \gamma_i \delta_{\tilde{\theta}_i}$ with:

$$\Gamma^* \sim \mathrm{CRM}(\lambda^*_\gamma, h_\gamma)$$

$$\gamma_i | \, U, G \sim \mathrm{Gamma}\Big(K_i - \sigma_\gamma, \tau_\gamma + \sum_{j=1}^{K} w_j u_{ij} + G^*(\Theta)\Big)$$

- Book update: $G = G^* + \sum_{i=1}^{K} w_i \delta_{\theta_i}$ with:

$$G^* \sim \mathrm{CRM}(\lambda^*, h)$$

$$w_i | \, U, \Gamma \sim \mathrm{Gamma}\Big(m_j - \sigma, \tau + \sum_{i=1}^{n} \gamma_i u_{ij} + \Gamma^*(\tilde{\Theta})\Big)$$

Table of Contents

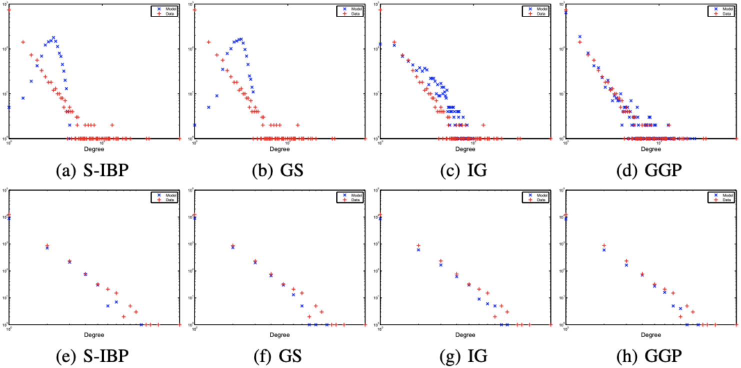
Power-law properties

For the GGP with $\sigma > 0$, we can achieve **power-law behavior of the network growth**:

- The total number of books read by n readers is $O(n^{\sigma})$
  $\rightarrow$ **Proof.** When $\gamma_i = \gamma$, the total number of books is Poisson($\psi_\lambda(n\gamma)$) distributed. Considering the GGP:
  $$\psi_\lambda(n\gamma) = \frac{\alpha}{\sigma}((n\gamma + \sigma)^{\sigma} - \tau^{\sigma})$$
  which for large n, is of order n^{σ}.

Similar results are achievable also with an S-IBP for the degree distribution of books, but not for readers for which it will always be Poisson!

Real world example – Book-crossing community network



(a) S-IBP (b) GS (c) IG (d) GGP

(e) S-IBP (f) GS (g) IG (h) GGP

Figure 1: Degree distribution for readers (a-d) and books (e-h) with 4 models: a stable Indian Buffet Process (S-IBP); our model with $\gamma_i = \gamma$ and flat prior assigned (GS); our model with $\gamma_i \sim$ Gamma (a_γ, a_γ) and flat prior assigned to the parameters (IG); our model with GGP prior for γ_i (GGP). Data are presented in red and samples from the models in blue.

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