Algorithms for Combinatorial Auction

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Problem Statement

Combinatorial Auction Problem (CAP), [NRTV07]

Let M, with $|M| = m$, a set of items to be sold to n bidders. Then we define

$$\mathrm{v_i} : S \subseteq M \rightarrow v_i(S) \in \mathbb{R}$$

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The objective of the problem is to find an *allocation* $S_1 \cdots S_n$ with $S_i \cap S_j = \varnothing$ for every $i \neq j$ that maximize the **common welfare**, namely:

$$\max_{S_1 \cdots S_n} \sum_i v_i(S_i)$$

Main problems for CAP

Examples of CAP problems:

- *Spectrum auctions*
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The main issues of the problem that we need to take care of are:

1. The optimization problem could be **computationally hard**.
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2. The input size is exponential, indeed each evaluation function v_i requires 2^m estimates to be well defined, it is a **combinatorial problem**.
3. How to **design an efficient auction**?
4. How to take into account the **strategic behaviour** of bidders?

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Single-Minded Case

Let us consider a **simplification of the CAP**, in particular we impose a restriction on the valuation functions:

Definition

We say a *valuation* v to be **single minded** if there exists a bundle of items S^* and a value $v^* \in \mathbb{R}$ s.t.:

$$v(S) = \begin{cases} v^* & \text{if } S \supseteq S^* \\ 0 & \text{otherwise} \end{cases}$$

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Single minded valuations are very simply represented and the algorithmic allocation problem is given by:

INPUT: (S_i^*, v_i^*) for each bidders $i = 1 \cdots n$

OUTPUT: A winner subset W $\subseteq \{1, \cdots, n\}$ such that $S_i \cap S_j = \varnothing$ for every $i \neq j$

Single-Minded Case – NP-hardness

Proposition

The allocation problem among single-minded bidders is **NP-Hard**

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Proof

Consider the **Independent Set Problem** (**ISP**), namely given a graph $\mathcal{G} = (\mathcal{E}, \mathcal{V})$ find the largest possible independent set. This problem is known to be NP-hard.

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Then consider the following graph $G = (E, V)$:

- The set of edges E to be the set of items.
- The set of vertexes V to be the bidders. For vertex $i \in V$, we will have the desired bundle of i to be the set of adjacent vertices, namely $S_i^* = \{e \in E : i \in e\}$ and the value will be $v_i^* = 1$.

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Now notice that a set W of winner in the CAP problem satisfies $S_j^* \cap S_i^* = \varnothing$ for every $i \neq j \in W$ if and only if the set of vertices is an independent set in G. The social welfare of W is exactly the size of the independent set in G. $\square$

Single-Minded Case – Example

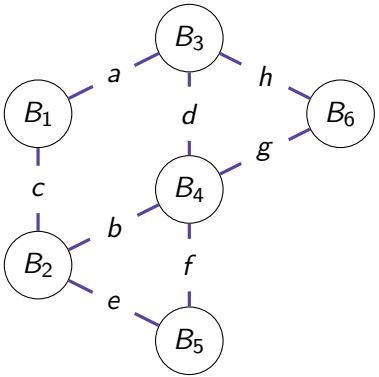

Figure: Example of CAP adapted to ISP

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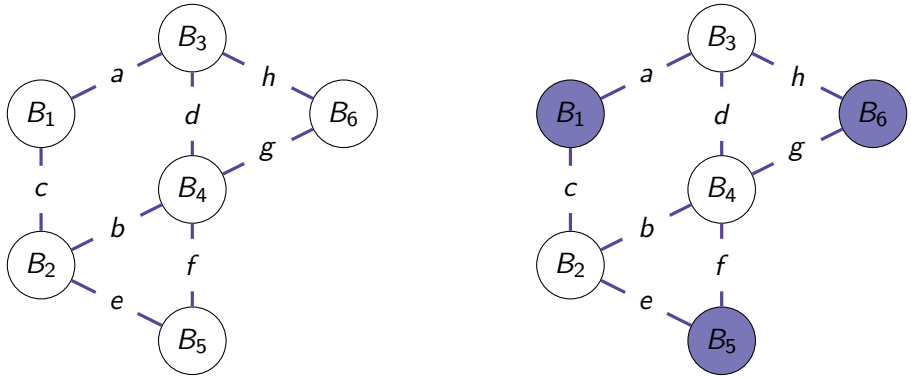

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Bidding Languages

With respect to the problem of representing bidders valuations, we have to consider the following problems:

- A naive approach asking for a valuation for each set would require a real value for each $2^m - 1$ **non-empty set** for each bidder, which could be computationally unmanageable even with few items.
- We are looking for **bidding languages** that allow bidders to encode **succinctly** and **effectively** their valuations and send them to the auctioneer.
- In designing bidding languages we face an **expressiveness–simplicity tradeoff**.

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- In designing bidding languages we face an **expressiveness–simplicity tradeoff**.

Commonly used bidding languages are:

1. **OR bid**
2. **XOR bid**
3. **OR/XOR bid**

Bidding Languages – OR bids

Common bidding languages are combinations of **atomic bids**. This simple evaluations are in the form (S, p), meaning an offer of p monetary units for any bundle T, with $T \supseteq S$.

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OR bids

OR language considers different bids as totally independent. Given an OR valuation for the $j-th$ bidder $v_j = (S_1, p_1)OR \ldots OR(S_k, p_k)$, the valuation for the bundle S is:

$$v(S) = \max_{W} \sum_{j \in W} p_i$$

where W is valid collection of pairs, meaning for all $i \neq j \in W, S_i \cap S_j = \varnothing$.

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OR can represent only **superadditive** valuations, namely:

$$v(S \cup T) \geq v(S) + v(T) \quad \forall S \cap T = \varnothing$$

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XOR bids

XOR language considers different bids as totally mutually exclusive. Given a XOR valuation for the $j-th$ bidder $v_j = (S_1, p_1)XOR \ldots XOR(S_k, p_k)$, the valuation for the bundle S is:

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XOR can directly represent **unit demand** valuations of this kind:

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It is possible to form general combinations of **OR/XOR**:

$$\begin{aligned} \text{e.g. } v(S) &= (u)OR(\{d\}, 5) \\ &= \big((\{a, b\}, 3)XOR(\{c\}, 2)\big)OR(\{d\}, 5) \end{aligned}$$

Bidding Languages – Example

e.g. $v(S) = (\{a, b\}, 3) OR (\{c, d\}, 5)$ $\quad\quad$ $u(S) = (\{a, b\}, 3) XOR (\{c, d\}, 5)$

Bidding Languages – Example

e.g. $v(S) = (\{a, b\}, 3) OR (\{c, d\}, 5)$
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We call this formulation OR*, defined on $M \cup D$, with D adequate set of dummy variables.

Bidding Languages – Closing words

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OR^* is a very appealing bidding language:

- OR^* bids look like a regular OR on a larger set of items.
- OR looks at an allocation algorithm just like a collection of **atomic bids from different players**. We can use the **same algorithms for single-minded bids** (it does not matter the number of bidders).

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Integer programming formulation of CAP

CAP ILP, [VV03]

Let N be the set of n bidders and M the set of m items. For every subset $S \subseteq M$ let $b_j(S)$ be the bid of the agent $j \in N$ for S. Let $b(S) = \max_{j \in N} b_j(S)$ and $x_S = 1$ when the set S is accepted, while $x_S = 0$ when the set is refused.

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Then the CAP problem can formulated as the following integer program:

$$\max \sum_{S \subset M} b(S) x_S$$

$$s.t. \quad \sum_{S \ni i} x_S \leq 1 \quad \forall i \in M$$

$$x_S \in \{0, 1\} \quad \forall S \subset M$$

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Consider this **example**:

$$x = \left[\begin{array}{ccccc} x_{\{a\}} & x_{\{b\}} & x_{\{c\}} & x_{\{a,b\}} & x_{\{a,b,c\}} \end{array} \right]^T$$
$$b = \left[\begin{array}{ccccc} \max_{j \in N} b_j\{a\} & \max_{j \in N} b_j\{b\} & \max_{j \in N} b_j\{c\} & \max_{j \in N} b_j\{a,b\} & \max_{j \in N} b_j\{a,b,c\} \end{array} \right]^T$$

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$$\mathrm{A} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccc} \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,b,c\} \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array} \right] \end{array}$$

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$$\begin{aligned} &\max \mathrm{b}^{\mathsf{T}}\mathrm{x} \\ &\text{s.t} \quad \mathrm{A}\mathrm{x} \leq 1 \\ &x_i \in \{0,1\} \ \forall i \end{aligned}$$

Integer programming

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The **combinatorial nature** of the problem, combined with the fact that general instances of the problem are **not polynomial** leads to a **very hard framework**(e.g. $2^{20} \approx 10^6$ columns).

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Approximation Single-Minded Bidders

Algorithm 1 Greedy Mechanism for Single-Minded Bidders

Require: Ordered set of single minded bids such that $\frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \cdots \geq \frac{v_n^*}{\sqrt{|S_n^*|}}$

Approximation Single-Minded Bidders

Algorithm 2 Greedy Mechanism for Single-Minded Bidders

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1: $W \leftarrow \varnothing$
2: **for** $i = 1$ to N **do**
3: **if** $S_i^* \cap \left(\bigcup_{j \in W} S_j^* \right) = \varnothing$ **then**
4: $W \leftarrow W \cup i$
5: **end if**
6: **end for**

Approximation Single-Minded Bidders

Algorithm 3 Greedy Mechanism for Single-Minded Bidders

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Ensure: The set of winners W.

Approximation Single-Minded Bidders

Algorithm 4 Greedy Mechanism for Single-Minded Bidders

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2: **for** $i = 1$ to N **do**
3: **if** $S_i^* \cap \left(\bigcup_{j \in W} S_j^* \right) = \varnothing$ **then**
4: $W \leftarrow W \cup i$
5: **end if**
6: **end for**

Ensure: The set of winners W.

- Using OR^* as bidding language, we can apply this algorithm to the CAP in ILP form and not only to Single-Minded Bidders ($i.e. |S_j^*| = \sum_{i \in m} aij$).

Approximation Single-Minded Bidders

Algorithm 5 Greedy Mechanism for Single-Minded Bidders

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- Using OR^* as bidding language, we can apply this algorithm to the CAP in ILP form and not only to Single-Minded Bidders ($i.e.\ |S_j^*| = \sum_{i \in m} aij$).
- It is **efficiently computable** in polynomial time.

Greedy Mechanism for Single-Minded Bidders

Proposition

The Greedy mechanism for Single-Minded Bidders **achieves a $\sqrt{m}$ approximation**.

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For each $i \in W$ let $OPT_i = \{j \in OPT, j \geq i | S_i^* \cap S_j^* \neq \varnothing\}$.

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Proof

For each $i \in W$ let $OPT_i = \{j \in OPT, j \geq i | S_i^* \cap S_j^* \neq \varnothing\}$. Then $OPT \subseteq \bigcup_{i \in W} OPT_i$ and thus it is enough to prove:

$$\forall i \in W, \sum_{j \in OPT_i} v_j^* \leq \sqrt{m} v_i^*$$

Greedy Mechanism for Single-Minded Bidders II

Note for every $j \in OPT_i$ $v_j^* \leq \frac{v_i^* \sqrt{|S_j^*|}}{\sqrt{|S_i^*|}}$. Then, we can sum over all $j \in OPT_i$:

$$\sum_{j \in OPT_i} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in OPT_i} \sqrt{|S_j^*|} \tag{1}$$

Greedy Mechanism for Single-Minded Bidders II

Note for every $j \in OPT_i$ $v_j^* \leq \frac{v_i^* \sqrt{|S_j^*|}}{\sqrt{|S_i^*|}}$. Then, we can sum over all $j \in OPT_i$:

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Using the Cauchy-Schwarz inequality we can bound the second member of the RHS of (1):

$$\begin{aligned} \sum_{j \in OPT_i} \sqrt{|S_j^*|} \cdot 1 &\leq \sqrt{\sum_{j \in OPT_i} 1} \sqrt{\sum_{j \in OPT_i} |S_j^*|} \\ &= \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |S_j^*|} \end{aligned}$$

Greedy Mechanism for Single-Minded Bidders III

From the definition of OPT_i it follows $|OPT_i| \leq |S_i^*|$ and since OPT is an allocation $\sqrt{\sum_{j \in OPT_i} |S_j^*|} \leq \sqrt{m}$.

Greedy Mechanism for Single-Minded Bidders III

From the definition of OPT_i it follows $|OPT_i| \leq |S_i^*|$ and since OPT is an allocation $\sqrt{\sum_{j \in OPT_i} |S_j^*|} \leq \sqrt{m}$. We can substitute these two results in the Cauchy-Schwarz inequality obtaining:

$$\sum_{j \in OPT_i} \sqrt{|S_j^*|} \leq \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |S_j^*|}$$
$$\leq \sqrt{|S_i^*|} \sqrt{m}$$

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$$\begin{aligned}\sum_{j \in OPT_i} \sqrt{|S_j^*|} &\leq \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |S_j^*|} \\ &\leq \sqrt{|S_i^*|} \sqrt{m}\end{aligned}$$

Plugging this result in (1) we obtain:

$$\begin{aligned}\sum_{j \in OPT_i} v_j^* &\leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in OPT_i} \sqrt{|S_j^*|} \\ &\leq \sqrt{m} v_i^* \quad \square\end{aligned}$$

Greedy Mechanism for Single-Minded Bidders

```
function greedy_solver(M::Vector{String}, S::Dict{Vector, Int64})
    # M is the ground set of items
    # S is a dictionary with some valuations
    l = sort(collect(keys(S)), by = x -> S[x]/sqrt(size(x)[1]), rev=true)
    W=[]
    z=0
    for set in l
        if intersect(M, set) == set
            append!(W, [set])
            setdiff!(M,set)
            z+=S[set]
        end
    end
    return W, z
end
```

Figure: Greedy algorithm in Julia

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   - Greedy Mechanism for Single-Minded Bidders
   - Solving Integer Programs in Julia

Exact Solutions for ILP

For obtaining exact solutions (or at least better approximations) we need more involved algorithms, mainly based on the simplex algorithm.

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- For solving LP it uses a parallelized version of the **revised dual simplex** algorithm.
- It uses branch and bound algorithms to solve ILP when there are few columns.
- Some applications can be found at `https://github.com/andreateruzzi/combinatorial_auction_ILP`.

Exact Solutions for ILP

```
function cap_solver(M::Vector{String}, S::Dict{Vector, Int64}, optmizer, display::Bool=false)

    # M is the ground set of items
    # S is a dictionary with some valuations

    #model construction
    l = collect(keys(S))
    model = Model(optmizer)
    if display==false
        set_silent(model)
    end
    @variable(model, x[l] >= 0, Bin)
    @objective(
        model,
        Max,
        sum(S[s] * x[s] for s in l),
    );

    for e in M
        V = [s for s in l if e in s]
        intake = @expression(
            model,
            sum(x[subset] for subset in V),
        )
        @constraint(model, intake <= 1)
    end

    #solving  model
    optimize!(model)
end
```

Figure: Integer program solver in Julia

Exact Solutions for ILP – Example

INPUT: $\mathrm{M} = \{a, b, c\}$

Exact Solutions for ILP – Example

INPUT: $M = \{a, b, c\} \rightarrow v_1 = (\{a\}, 3) OR (\{b\}, 3) OR (\{c\}, 3)$

$\rightarrow v_2 = (\{a, b\}, 5) OR (\{b\}, 4) OR (\{a, b, c\}, 6)$

$\rightarrow v_3 = (\{c\}, 5)$

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$$x = \begin{bmatrix} x_{\{a\}} & x_{\{b\}} & x_{\{c\}} & x_{\{a,b\}} & x_{\{a,b,c\}} \end{bmatrix}^T$$

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$$\max z = b^T x$$

$$\text{s.t} \quad Ax \leq 1$$

$$x_i \in \{0, 1\} \ \forall i$$

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$$\text{s.t} \quad \mathrm{A}\mathrm{x} \leq 1$$

$$x_i \in \{0, 1\} \ \forall i$$

OUTPUT: $\mathrm{x}^* = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$

$$z^* = 12$$

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