Estimation and Prediction for Stochastic Blockstructures

Krzysztof Nowicki and Tom A. B. Snijders, 2001

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Table of Contents

Introduction and Preview

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  - Inference on the probability law (Bayesian)

Table of Contents

Example

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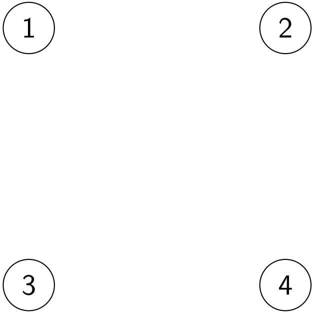
Set of vertices V



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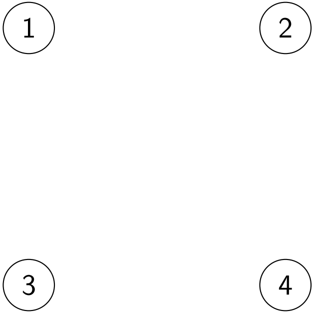
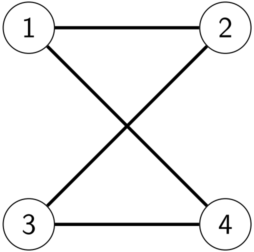


Graph G (e.g. assistance)



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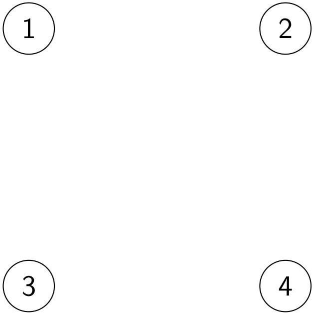

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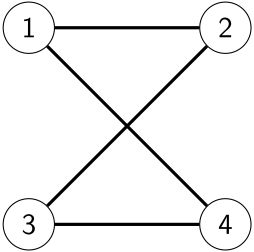
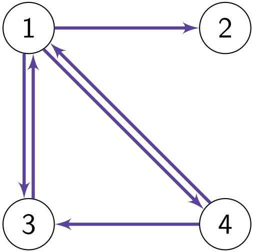


Digraph D (e.g. friendship)



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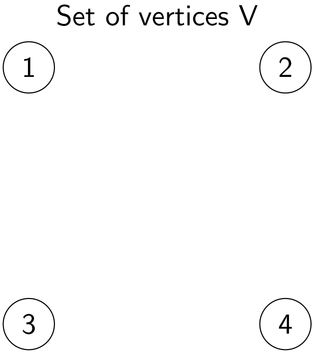
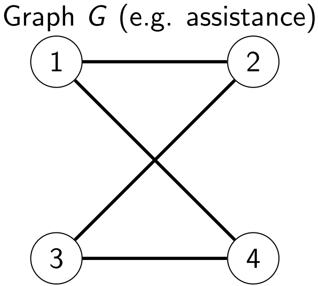

Set of vertices V
1
2
3
4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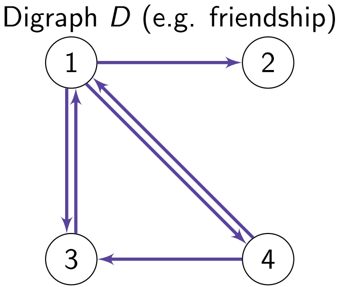
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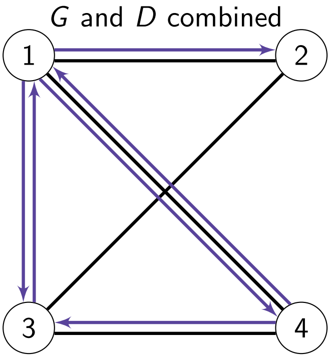
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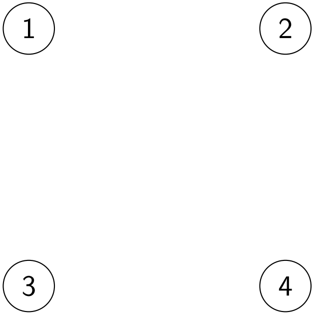

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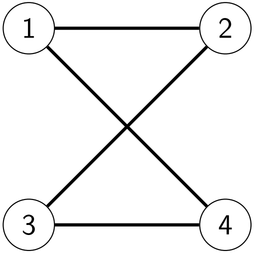

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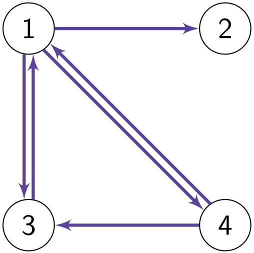

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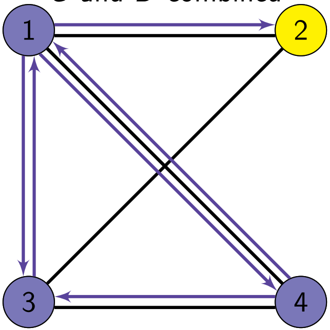

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- Given a set of n vertices labeled $1, \ldots, n$, we define the **set of pairs** for which the probability model applies:

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  The aims of our probability model is take into account the mutual dependence of relation from i to j and from j to i. We define the **set of dyadic relations**:

$$\boldsymbol{\alpha}^2 = \{\mathrm{a} = (a_t, a_v) | a_t, a_v \in \boldsymbol{\alpha}\}$$

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- We call $\mathcal{A} \subset \alpha^2$ the **alphabet** of pairwise relations. For every $(i, j) \in \mathcal{N}$ exists $\mathrm{a} = (a_t, a_v) \in \mathcal{A}$ describing the dyadic relation of between i and j.

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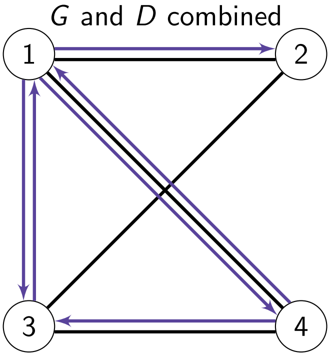
  - We define $\mathcal{A}' = \mathcal{A}_0 \cup \mathcal{A}_{10}$
  - We denote $r = |\mathcal{A}|$, $r_0 = |\mathcal{A}_0|$, $r_1 = \frac{1}{2}|\mathcal{A}_1| = |\mathcal{A}_{10}| = |\mathcal{A}_{11}|$

Example



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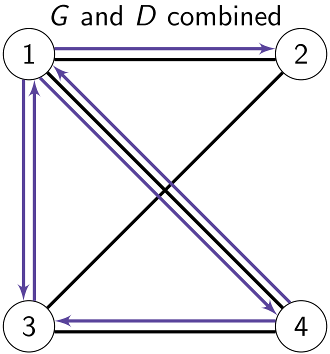

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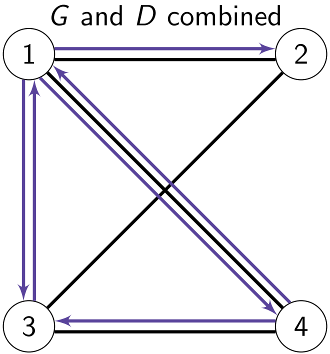

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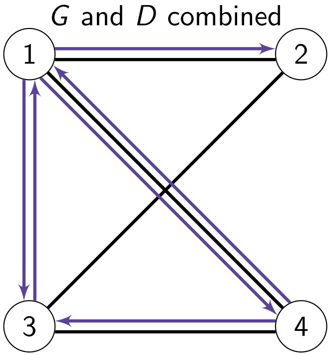

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$$r = |\mathcal{A}| = 8$$
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Relational Data: Colored Relational Structure

The second aspect of the structure is a **discrete vertex characteristic**. The set of vertices $\{1, \dots, n\}$ is partitioned into c colors $1, \dots, c$ and we denote the set of colors $\mathcal{C} = \{1, \dots, c\}$.

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Definition

It is called **colored relational structure** a set $\mathcal{N}$ of ordered pairs of vertices between which relations are given, with the vertices belonging to c categories stored in the vector $\mathbf{x} = (x_i)_{i=1}^n$ with $x_i \in \mathcal{C}$ and with a relational structure where the dyadic relations $\mathbf{y} = (y_{ij})_{(i,j)\in\mathcal{N}}$ take values in $\mathcal{A}$.
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In our particular framework, we will consider $\mathbf{Y} = \mathbf{y}$ observed, but $\mathbf{X}$ unknown (posteriori blockmodelling).

Table of Contents

Stochastic blockmodel: $P(X)$

The ultimate goal of our analysis is specify a probability model for colored relational structures. In particular we are interested in:

- The predictive density $P(\mathbf{X}|\mathbf{y})$
- Estimate the vector of parameters of the models.

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and the joint distribution of $\mathbf{X}$ to be:

$$P(X = x) = \theta_1^{m_1} \dots \theta_c^{m_c}$$
$$m_k = \sum_{i=1}^{n} I(x_i = k)$$

The vector $(m_k)_{k \in \mathcal{C}}$ is **sufficient statistic** for the probability law of X.

Stochastic blockmodel: $P(Y|x)$ I

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and the array $\boldsymbol{\eta}$ of **color-dependent dyad probabilities** satisfies:

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Stochastic blockmodel: $P(Y|x)$ I

Given the vector of colors $\mathbf{X} = \mathbf{x}$, the random vectors $\mathbf{Y}_{ij}$ for $(i,j) \in \mathcal{N}$ with $i < j$ are **independent** with probabilities:

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which implies a **redundancy in parameters** $\eta_a(k,h)$.

Stochastic blockmodel: $P(Y|x)$ II

A non-redundant parameterization is obtained as follows:

$$\begin{cases} \eta_{\mathrm{a}}(k,h) & \text{for a} \in \mathcal{A} \text{ and } k < h \\ \eta_{\mathrm{a}}(k,k) & \text{for a} \in \mathcal{A}' \text{ and } k \in \mathcal{C} \end{cases}$$

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The **conditional joint distribution** $P(\mathbf{y}|\mathbf{x})$ is given by:

$$P(y|\mathbf{x},\boldsymbol{\theta},\boldsymbol{\eta}) = \left(\prod_{\mathrm{a}\in\mathcal{A}} \prod_{1\leq k<h\leq c} (\eta_{\mathrm{a}}(k,h))^{e_{\mathrm{a}}(k,h)}\right) \times \left(\prod_{\mathrm{a}\in\mathcal{A}'} \prod_{k=1}^{c} (\eta_{\mathrm{a}}(k,k))^{e_{\mathrm{a}}(k,k)}\right)$$

where $e_{\mathrm{a}}(k,h)$ counts the number of relations of type a from vertices of color k to vertices of color h and is a **sufficient statistic**.

Stochastic blockmodel

The stochastic blockmodel is then given by the joint distribution of $(\mathbf{Y}, \mathbf{X})$:

$$
\begin{aligned}
P(\mathbf{y}, \mathbf{x} | \boldsymbol{\theta}, \boldsymbol{\eta}) &= P(\mathbf{x}) \times P(\mathbf{y} | \mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\eta}) \\
&= \theta_1^{m_1} \times \cdots \times \theta_c^{m_c} \\
&\quad \times \left(\prod_{\mathrm{a} \in \mathcal{A}} \prod_{1 \le k < h \le c} (\eta_{\mathrm{a}}(k, h)^{e_{\mathrm{a}}(k,h)}) \right) \\
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\end{aligned}
$$

Because our model consider a stochastic structure $(\mathbf{Y}, \mathbf{X})$, with $\mathbf{Y}$ observed and $\mathbf{X}$ latent variable, the probability of observing edge pattern $\mathbf{y}$ is:

$$
P(\mathbf{y} | \boldsymbol{\theta}, \boldsymbol{\eta}) = \sum_{\mathbf{x} \in \mathcal{C}^n} P(\mathbf{y}, \mathbf{x} | \boldsymbol{\theta}, \boldsymbol{\eta})
$$

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- We assume a prior density $f(\boldsymbol{\theta}, \boldsymbol{\eta})$ for the parameters $(\boldsymbol{\theta}, \boldsymbol{\eta})$, which we can update using $\mathbf{y}$ and **inference about the parameters**:

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- The **recovery of the block structure** is based on the posterior predictive distribution of of **X**:

$$P(x | y) = \int f(\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{x} | \mathbf{y}) d\boldsymbol{\theta} d\boldsymbol{\eta}$$

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$\Longrightarrow$ **Gibbs sampling** is used to obtain the conditional distribution $f(\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{x} | \mathbf{y})$, which we need to solve both problems.

Table of Contents

Identifiability

The parameters $(\boldsymbol{\theta}, \boldsymbol{\eta})$ in the joint distribution $f(\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{x}, \mathbf{y})$ **are not identifiable**. What matters is the partition defined by $\mathbf{x}$, not the colors labels $1, \ldots, c$.

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In particular, let $\mathcal{S}$ denote the group of permutation of $\{1, \ldots, c\}$ and define $h_s(\cdot)$ and $s(\cdot)$ such that:

$$P(\mathbf{Y} = \mathbf{y} | \, \boldsymbol{\theta}, \boldsymbol{\eta}) = P(\mathbf{Y} = \mathbf{y} | \, h_s(\boldsymbol{\theta}, \boldsymbol{\eta}))$$
$$s(\mathbf{x}) = (s(x_1), \ldots, s(x_n))$$

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for every permutation $s \in \mathcal{S}$. The transformation preserves the partition of the vertices and it implies:

$$P(\mathbf{x} | \, \mathbf{y}, \boldsymbol{\theta}, \boldsymbol{\eta}) = P(s(\mathbf{x}) | \, \mathbf{y}, h_s(\boldsymbol{\theta}, \boldsymbol{\eta}))$$

Invariance I

Because the model is invariant under the relabelling of the colors, **when the prior distribution is invariant, then the posterior is invariant**:

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independently from k. We can deal with the invariance problem in some ways:

1. Put **restrictions** on the parameters

$$e.g. \quad \theta_1 < \theta_2 \cdots < \theta_c$$

   However this approach is not always applicable and it may require some prior information (informative prior).

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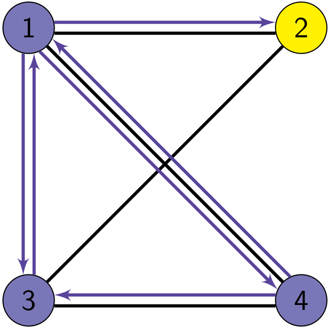
   We can apply the expectation to these functions and find the following matrix and the three-way array :

$$\left(P(X_i = X_j | \, \mathbf{y}) \right)_{1 \le i \ne j \le n}$$

$$\left(\mathbb{E}(\eta_a(X_i, X_j)|y \right)_{1 \le i \ne j \le n, \, a \in \mathcal{A}'}$$

Example

G and D combined



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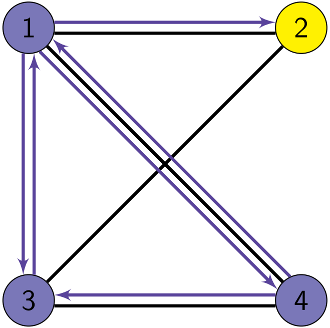

$$I\{X_i = X_j\}$$

X_i \ X_j	*1*	*2*	*3*	*4*
1	-	0	1	1
2	0	-	0	0
3	1	0	-	1
4	1	0	1	-

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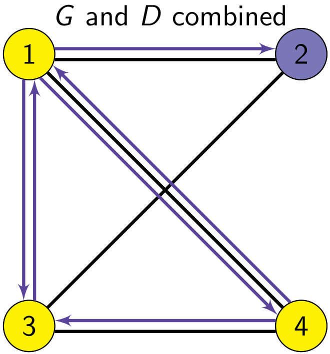

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Table of Contents

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We apply this scheme to $((\boldsymbol{\theta}, \boldsymbol{\eta}), X_1, X_2, \ldots, X_n)$.

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We apply this scheme to $((\boldsymbol{\theta}, \boldsymbol{\eta}), X_1, X_2, \ldots, X_n))$. At each iteration, given the current values $(\boldsymbol{\theta}^{(p)}, \boldsymbol{\eta}^{(p)}, \mathbf{X}^{(p)})$:

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We are left to specify the full conditional distribution for these two steps.

Step 1: Prior choice I

For step 1, we need to specify the **prior distribution** $f(\theta, \eta)$. If we do not have any particular prior information, it is also reasonable to assume **independence** between $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$.

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  - For $k < h$, $\boldsymbol{\eta}(k, h) = (\eta_{\mathrm{a}}(k, h))_{\mathrm{a} \in \mathcal{A}}$ is an unconstrained probability vector of dimension $r = |\mathcal{A}|$.

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transforming $\boldsymbol{\eta}(k,k)$ in a $(r_0 + r_1)$-dimensional (note $r_0 + r_1 < r$) vectors without redundant elements for every $k \in \mathcal{C}$.

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The posterior distribution of $(\boldsymbol{\theta}, \boldsymbol{\eta})$ given $(\mathbf{y}, \mathbf{x})$ is is largely tractable:

$$\begin{aligned} &(m_k + T_k)_{k \in \mathcal{C}} && \text{for } \boldsymbol{\theta} \\ &(e_{\mathsf{a}}(k,h) + E_{\mathsf{a}}(k,h))_{\mathsf{a} \in \mathcal{A}} && \text{for } \boldsymbol{\eta}(k,h),\ 1 \leq k < h \leq c \\ &(e_{\mathsf{a}}(k,k) + E_{\mathsf{a}}(k,k))_{\mathsf{a} \in \mathcal{A}'} && \text{for } \boldsymbol{\eta}^{(0)}(k,k),\ 1 \leq k \leq c \end{aligned}$$

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We can approximate $\boldsymbol{\eta}_{\mathrm{a}}(X_i, X_j)$ using the average of $\boldsymbol{\eta}_{\mathrm{a}}^{(p)}(X_i, X_j)$ drawn from the Gibbs sampler.

Step 2: $f(X_i|\boldsymbol{\theta}, \boldsymbol{\eta}, \mathbf{x}_{-i})$

From the probability law of the stochastic block-model $P(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta}, \boldsymbol{\eta})$ we can derive:

$$P(X_i = k|\, \mathbf{y}, \boldsymbol{\theta}, \boldsymbol{\eta}, \{X_j\}_{j\neq i}) = Q\theta_k \prod_{\mathrm{a}\in\mathcal{A}} \prod_{h=1}^{c} (\boldsymbol{\eta}_{\mathrm{a}}(k,h))^{d_{\mathrm{a}}(i,h)}$$

with $d_{\mathrm{a}}(i,k) = \sum_{j:(i,j)\in\mathcal{N}} I\{y_{ij} = \mathrm{a}\} I\{x_j = k\}$ counting number of dyadic relations of type a from vertex i to vertices of color k and with Q a constant not depending on k.

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The **relative frequency** of $X_i^{(p)} = X_j^{(p)}$ over the Gibbs sampler can be used to **approximate the matrix of posterior predictive probabilities** $(P(X_i = X_j|\,\mathbf{y}))_{1\leq i\neq j\leq n}$

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We can build some parameters for evaluating the adequacy of the obtained class structure:

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We can build some parameters for evaluating the adequacy of the obtained class structure:

- I_y is the information of the observed relations with $c \geq 2$:

$$I_y = -\frac{2}{|\mathcal{N}|} \sum_{(i,j)\in\mathcal{N}, i<j} log(\eta_{y_{ij}}(X_i, X_j))$$

This value is small when $\boldsymbol{\eta}$ and $\mathbf{X}$ determine to a large extent of the observed relations.

Adequacy of class structure

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  This value is small when $\boldsymbol{\eta}$ and $\mathbf{X}$ determine to a large extent of the observed relations.

- H_x measures the extent to which the distribution of $\mathbf{X}$ defines one clear-cut partition of vertices into classes.

$$H_x = \frac{4}{n(n-1)} \sum_{i,j=1}^{n} \pi_{ij}(1 - \pi_{ij})$$

  with $\pi_{ij} = P(X_i = X_j | \, \mathbf{y})$. If H_x is small it relatively clear if pair of vertices are in the same class or not.

Example – Adequacy parameters

c	I_y	H_x
2	.94	.24
3	.91	.21
4	.89	.26
5	.89	.27

Figure 1: Parameter for the class structure for Kapferer's dataset

Convergence and class assignment

To check to convergence of the Gibbs sampler:

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- Run multiple Gibbs sampler with different independent starting points and check that parameters H_x and I_y have similar values.
- For invariant prior, check that the the posterior predictive distribution of the sequence $\mathbf{X}^{(p)}$ is uniform.

Assign vertices to classes:

- For uniform prior, arbitrarily assign labels.
- For nonuniform prior, labels are identified by the prior.

Example – Estimated posterior probabilities I

i	Posterior probability $P(X_i = X_j \mid \mathbf{y})$																																					
1	—	9	9	9	9	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
2	9	—	9	9	9	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
3	9	9	—	9	9	6	9	9	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
5	9	9	9	—	9	6	9	9	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
7	9	9	9	9	—	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
9	6	6	6	6	6	—	6	6	6	5	5	7	6	4	5	5	4	4	4	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	0
12	9	9	9	9	9	6	—	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
13	9	9	9	9	9	6	9	—	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
14	9	9	9	9	9	6	9	9	—	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
16	8	8	9	9	8	5	8	9	8	—	9	5	9	9	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
19	9	9	9	9	9	5	9	9	9	9	—	5	9	9	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
21	6	6	6	6	6	7	6	6	6	5	5	—	6	4	7	6	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	0
24	9	9	9	9	9	6	9	9	9	9	9	6	—	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
25	8	8	8	8	8	4	8	8	8	9	9	4	8	—	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
30	8	8	8	8	8	5	8	8	8	7	7	7	8	7	—	8	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0
34	7	7	7	7	7	5	7	7	7	6	6	6	7	6	8	—	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
4	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
6	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
8	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
10	1	1	1	1	1	5	1	1	1	0	0	4	1	0	2	3	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
15	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
17	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
18	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
20	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
22	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	0
23	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	0
26	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	0
27	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	0
28	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	0
29	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	0
31	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	0
32	1	1	1	1	1	4	1	1	1	1	1	4	1	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	0
33	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	0
35	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	0
36	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	0
37	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	0
38	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	0
39	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	0
11	0	0	0	0	0	0	0	0	0	1	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—

Figure 2: Estimated posterior probabilities $P(X_i = X_j \mid \mathbf{y})$ for Kapferer's dataset

Example – Estimated posterior probabilities I

i	Posterior probability $P(X_i = X_j \mid \mathbf{y})$																																						
1	—	9	9	9	9	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
2	9	—	9	9	9	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
3	9	9	—	9	9	6	9	9	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
5	9	9	9	—	9	6	9	9	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
7	9	9	9	9	—	6	9	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
9	6	6	6	6	6	—	6	6	6	5	5	7	6	4	5	5	4	4	4	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	0
12	9	9	9	9	9	6	—	9	9	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
13	9	9	9	9	9	6	9	—	9	9	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
14	9	9	9	9	9	6	9	9	—	8	9	6	9	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
16	8	8	9	9	8	5	8	9	8	—	9	5	9	9	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
19	9	9	9	9	9	5	9	9	9	9	—	5	9	9	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
21	6	6	6	6	6	7	6	6	6	5	5	—	6	4	7	6	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	0
24	9	9	9	9	9	6	9	9	9	9	9	6	—	8	8	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
25	8	8	8	8	8	4	8	8	8	9	9	4	8	—	7	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
30	8	8	8	8	8	5	8	8	8	7	7	7	8	7	—	8	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0
34	7	7	7	7	7	5	7	7	7	6	6	6	7	6	8	—	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0
4	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
6	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
8	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
10	1	1	1	1	1	5	1	1	1	0	0	4	1	0	2	3	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
15	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
17	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
18	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
20	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	9	0
22	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	9	0
23	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	9	0
26	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	9	0
27	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	9	0
28	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	9	0
29	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	9	0
31	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	9	0
32	1	1	1	1	1	4	1	1	1	1	1	4	1	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	9	0
33	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	9	0
35	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	9	0
36	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	9	0
37	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	9	0
38	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	9	0
39	0	0	0	0	0	4	0	0	0	0	0	4	0	0	2	3	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	—	0
11	0	0	0	0	0	0	0	0	0	1	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—

Figure 2: Estimated posterior probabilities $P(X_i = X_j \mid \mathbf{y})$ for Kapferer's dataset

Example – Estimated posterior probabilities II

a	h	$k=1$	$k=2$	$k=3$
(0,0)	1	.27	.72	.14
	2	.72	.78	.31
	3	.14	.31	
(A,A)	1	.38	.19	.15
	2	.19	.15	.22
	3	.15	.22	
(F,F)	1	.03	.01	.07
	2	.01	.01	.04
	3	.07	.04	
(AF,AF)	1	.16	.03	.42
	2	.03	.05	.07
	3	.42	.07	
$(0,F)$	1	.01	.00	.05
	2	.03	.00	.27
	3	.04	.03	
(A,AF)	1	.07	.01	.08
	2	.02	.00	.04
	3	.05	.03	

Figure 3: Estimated posterior probabilities $\eta_a(X_i, X_j)$ for Kapferer's dataset

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