A solver for the Monodomain Heart Equations

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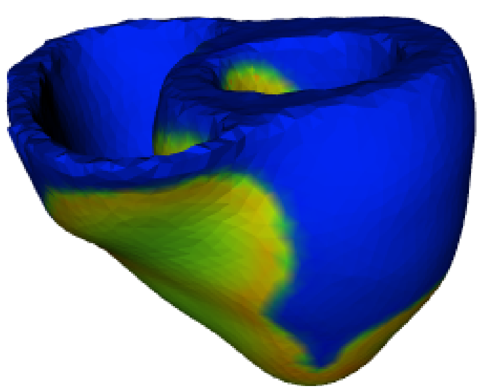

Preface

During the summer of 2013 i have written an object oriented code which solves the partial differential equation (PDE) terms of the monodomain heart equations. The code solves the PDE by using the FEniCS software[1], and recieves as input the solution of the coupled system of ODEs in the heart equation, typically as provided by the packages goss[2] and gotran[3]. The purpose of this report is to detail the work done, as well as to note some of the pitfalls encountered underway.

If you are reading this, you are most likely my employer, or someone looking to use the code developed, in order to not have to re-invent the wheel. This report is mostly aimed at the latter. We start with a brief review of the theory (a more full description is given in [5]), and then spend some time going into details on the produced code. There will also be notes explaining the other minor files found among the produced code. Beware! This code was worked on for a mere six weeks, and never polished properly. There might be parts of the code that is unclear, ineffective or even completely wrong.

If you wish to make changes to or improve the code in some way, feel free to contact me, in order to become a collaborator on the code. If you only wish to use the code, it can be found at the gitub repository [4].

Contents

1 Introduction and theory

This section will give a quick introduction to the problem we are looking to solve, and explain where the code produced will fit in a network of other packages.

1.1 Electrical activity in the heart

The heart beats as a result of electrical impulses produced by the heart cells themselves. This causes the intracellular electrical potential to increase, and this potential difference between the intracellular and the extracellular potential then spreads out like a diffusion process. It is useful to start off with some notation.

u_i denotes the intracellular heart potential
u_e denotes the extracellular heart potential
u_o denotes the potential outside the heart (sometimes u_T is also used)
$v = u_i - u_e$
H denotes the heart domain
∂H denotes the heart domain boundary (where the heart connects to the torso)
T denotes the torso domain (outside the heart)
∂T denotes the torso domain boundary (where the torso connects to the surroundings)

More will be introduced underway, but at the end these will be the most important ones. We will now go through a very brief derivation of the heart equations.

1.1.1 Warm-up: equations for the torso

We start with the spread of electrical potential in the torso. Assuming quasi-static conditions, we have from Maxwells equations that

$$\nabla \times E = 0, \tag{1}$$

which means that for some scalar potential u, we have

$$E = -\nabla u \tag{2}$$

The current is then given by the relation

$$J = ME \tag{3}$$

where M is the conductivity (this might be a tensor). This, of course, gives

$$J = -M\nabla u \tag{4}$$

If we assume that there is no sources or sinks for the potential inside the medium, and no build-up of charge, then for a small subvolume V with surface S, we have

$$\int_S n \cdot J \, dS = 0 \tag{5}$$

and so from the convergence theorem

$$-\int_V \nabla \cdot J \, dV = 0 \tag{6}$$

and since this is independent of the chosen domain, we have

$$\nabla \cdot J = 0, \tag{7}$$

and this means that

$$\nabla \cdot (M \nabla u) = 0 \tag{8}$$

What we have derived so far is just the standard heat equation. The torso should have the same potential on the boundary as the heart to ensure continuity of the potential, and if the torso is surrounded by air there should be no current flowing out of the body. The equations describing the torso is then

$$\nabla \cdot (M_T \nabla u) = 0 x \in T, \tag{9}$$

$$n \cdot M_T \nabla u_t = 0 x \in \partial T, \tag{10}$$

$$u_T = u_{\partial H} x \in \partial H. \tag{11}$$

1.1.2 Exitable tissue

The heart cells are called excitable, which means they can respond to an electrical stimulus. This ability enables an electric stimulation of one part of the heart to propagate through the muscle and activate the complete heart. This process happens through a depolarization: when the cells are at rest, there is a potential difference across the cell membrane. The stimulation causes the potential difference to go to zero or even further. This is a very fast process, and is followed by a slower repolarization that restores the difference. This is not handled by us at all, so I will skip the detailed on this, and just note that this creates a time dependent source term, and the equations describing this are

$$\nabla \cdot (M_i \nabla (u_e + v)) = \chi C_m \frac{\partial v}{\partial t} + \chi I_{ion} \tag{12}$$

$$\nabla \cdot (M_i \nabla v) + \nabla \cdot ((M_i + M_e) \nabla u) = 0 \tag{13}$$

where χ represents the area of cell surface per cell volume, and C_m is the capacitance of the cell membrane.

1.2 Bringing it all together

We have shown some of the relations that make out the heart equations. The complete system, in normalized units, is described by

$$\frac{\partial s}{\partial t} = f(s, v, t) \qquad x \in H \tag{14}$$

$$\nabla \cdot (M_i \nabla v) + \nabla \cdot (M_i \partial u_e) = \frac{\partial v}{\partial t} + I_{ion}(v, s) \qquad x \in H \tag{15}$$

$$\nabla \cdot (M_i \nabla v) + \nabla((M_i + M_e)\nabla u_e) = 0 \qquad x \in H \tag{16}$$

$$\nabla \cdot (M_o \nabla u_o) = 0 \qquad x \in T \tag{17}$$

$$u_e = u_o \qquad x \in \partial H \tag{18}$$

$$n \cdot (M_i \nabla v + (M_i + M_e)) = n \cdot (M_o \nabla u_o) \qquad x \in \partial H \tag{19}$$

$$n \cdot (M_i \partial v + M_i \nabla u_e) = 0 \qquad x \in \partial H \tag{20}$$

$$n \cdot M_o \nabla u_o = 0 \qquad x \in \partial T \tag{21}$$

This is called the **Bidomain model**. If we assume that the extra- and intracellular conductivity are linearly dependent, $M_e = \lambda M_i$, then the equations simplify greatly, and we are left with

$$\frac{\partial s}{\partial t} = f(s, v, t) \qquad x \in H \tag{22}$$

$$\frac{\lambda}{1+\lambda}\nabla(M_i \nabla v) = \frac{\partial v}{\partial t} + I_{ion}(v, s) \qquad x \in H \tag{23}$$

$$n \cdot (M_i \nabla v) = 0 \qquad x \in \partial H \tag{24}$$

This is called the **Monodomain model**, and this is what we will be looking to solve here. (22) and the source term in (23) is taken care of by the goss/gotran packages, and we will take care of the rest.

2 Computational Details

In this section we will go through some of more details on how to set up the problem in order to solve it.

2.1 Operator splitting

The meat of what we are looking to solve is the equation

$$\frac{\partial v}{\partial t} = \nabla \cdot (M_i \nabla) - I_{ion}(v, s) \tag{25}$$

References

[1] *FEniCS home page*. July 2013. URL: www.fenicsproject.org.

[2] *goss repository*. July 2013. URL: https://bitbucket.org/johanhake/goss.

[3] *gotran repository*. July 2013. URL: https://bitbucket.org/johanhake/gotran.

[4] *source code repository*. July 2013. URL: https://github.com/andreavs/simula_summer13.

[5] Joakim Sundnes et al. *Computing the Electrical Activity in the Heart*. Ed. by T. J. Barth et al. Springer, 2006. ISBN: 978-3-642-07005-1.