another (integrable) function q on D with $q(x) \geq p(x)$ such that we have a way to sample from the density

$$\hat{q}(x) \doteq \frac{q(x)}{\mathrm{I}[q]} = \frac{q(x)}{\int_D q(x) d\mu}$$

For simplicity, further assume that D has finite measure. That is, $\int_D d\mu < \infty$ where μ is the Lebesque measure. Randomly sample x'_n according to $\hat{q}$ and y_n according to the uniform distribution from 0 to 1. For every x'_n in our sample, we accept it as a realization of p if

$$0 < y_n < \frac{p(x'_n)}{q(x'_n)}$$

and reject it otherwise. The set of accepted points obtained from this process is a sample with density p according to the rejection method.

Let us show why the rejection method works. This proof is much more detailed than the one found in [Caf98]. Without loss of generality, we assume $q(x) > 0$ on D. Since $p(x) \leq q(x)$ for all x in the domain,

$$p(x) = \frac{p(x)}{q(x)}\hat{q}(x)\,\mathrm{I}[q] = \int_0^1 \chi\left(\left\{y \mid y < \frac{p(x)}{q(x)}\right\}\right) dy\, \hat{q}(x)\mathrm{I}[q]$$

where $\chi(A)$ is the characteristic function on a (measurable) set A given by

$$\chi(A)(y) = \begin{cases} 1, & y \in A \\ 0, & y \notin A \end{cases}$$

For simplicity of notation, let $A(x) = \left\{y \,|y < \frac{p(x)}{q(x)}\right\}$. This is a measurable set-valued function, since $\frac{p}{q}$ is measurable and thus $A(x) = \frac{p}{q}^{-1}(-\infty, x)$ is a measurable set. The previous equation becomes

$$p(x) = \int_0^1 \chi(A(x))\, dy\, \hat{q}(x)\mathrm{I}[q]$$

For f an integrable function on D,

$$\begin{aligned} \mathbb{E}(f(X)) &= \int_D f(x)p(x)dx \\ &= \int_D \int_0^1 f(x)\chi(A(x))(y)\hat{q}(x)\, dy\, \mathrm{I}[q]\, dx \\ &= \int_0^1 \int_D f(x)\chi(A(x))(y)\hat{q}(x)\, dx\, \mathrm{I}[q]\, dy \\ &= \int_0^1 \mathbb{E}_{\hat{q}}(f \cdot \chi(A(x)))\, dy\, \mathrm{I}[q] \end{aligned} \tag{4.14}$$