Dynamic Programming

“Those who do not remember the past are condemned to repeat it.”

Miloš Trujić

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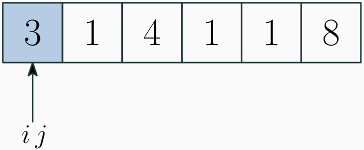

$$v_0 = 3 < k \longrightarrow \text{increase } j$$

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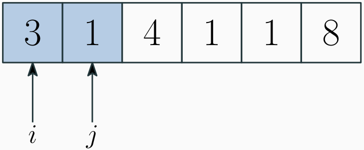

$$v_0 + v_1 = 4 < k \longrightarrow \text{increase } j$$

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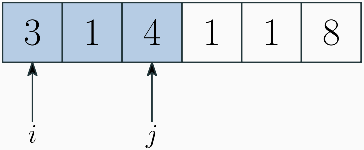

$v_0 + v_1 + v_2 = 8 > k \longrightarrow$ increase i

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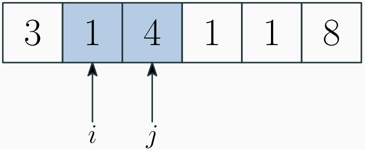

$$v_1 + v_2 = 5 < k \longrightarrow \text{increase } j$$

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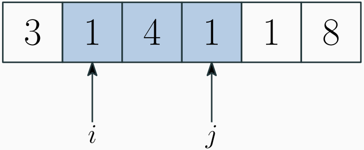

$$v_1 + v_2 + v_3 = 6 < k \longrightarrow \text{increase } j$$

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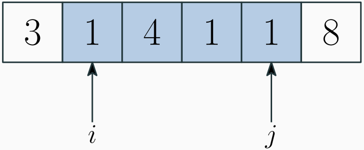

$$v_1 + v_2 + v_3 + v_4 = 7 = k \longrightarrow \textbf{YAY!}$$

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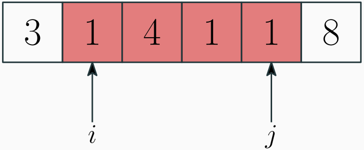

$v_1 + v_2 + v_3 + v_4 = 7 = k \longrightarrow$ **YAY!** The solution is $i = 1$ and $j = 4$

How to solve Deck of Cards in $O(n)$? **Sliding Window**!

```
int i = 0, j = 0;
int val = v[0];

while (j < n) {
    if (val == k) break;

    if (val < k) {
        j++;
        val += v[j];
    } else {
        val -= v[i];
        i++;
        if (i > j) {
            j = i;
            val = v[i];
        }
    }
}
```

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Exercise: Extend it to solve the 'real' problem.

Trick/technique (Sliding window)

Some problems in which you need to find some optimal interval can be solved in linear time using a similar sliding window approach.

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  - How to identify a DP problem?
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  - How to implement it?
- Today we start from scratch

Outline for today:

- Three examples (Fibonacci, Rod Cutting, LSI)
- Elements of Dynamic Programming on examples
- Common pitfalls
- Tips & Tricks

First Example: Fibonacci Numbers

Definition: $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

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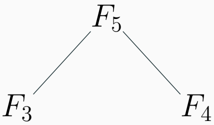

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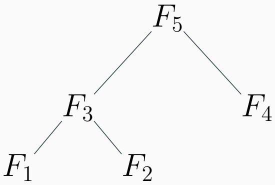

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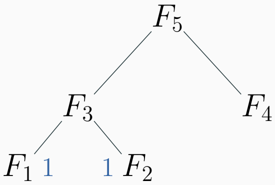

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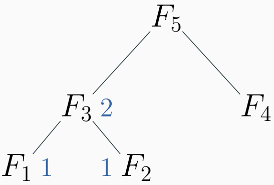

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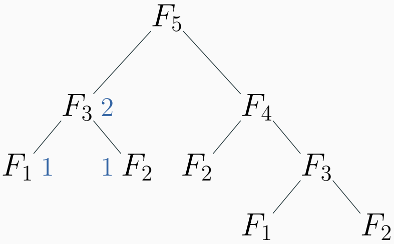

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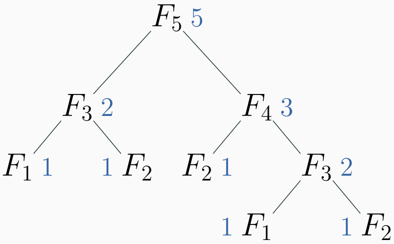

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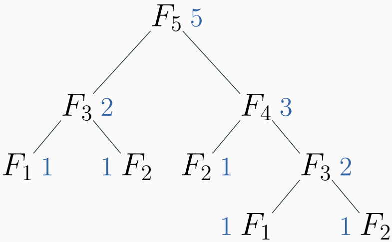

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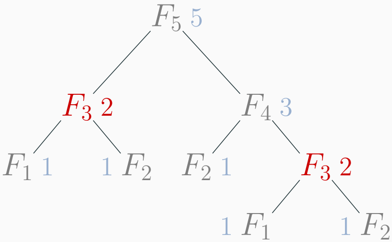

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Overlapping subproblems

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Memoization (or top-down DP) is **simple and powerful :)**

(not a typo, comes from *memo*)

Second Example: Rod Cutting

Input:

- a metal rod of length n
- values $p_1, \ldots, p_n$ s.t. p_i denotes the price for a rod of length i

Output: r_n, maximal possible revenue for a rod of length n (i.e. maximal sum of prices of pieces over all possible partitions)

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Example: $n = 4$

length i	1	2	3	4
price p_i	1	5	8	9

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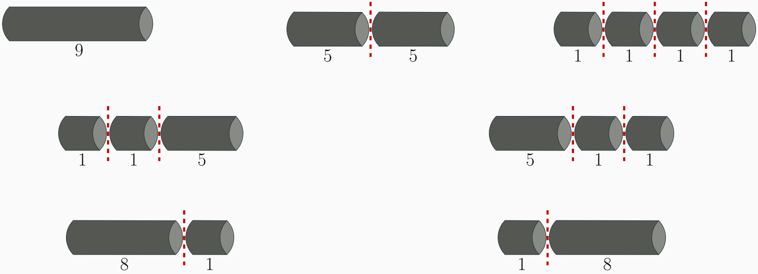

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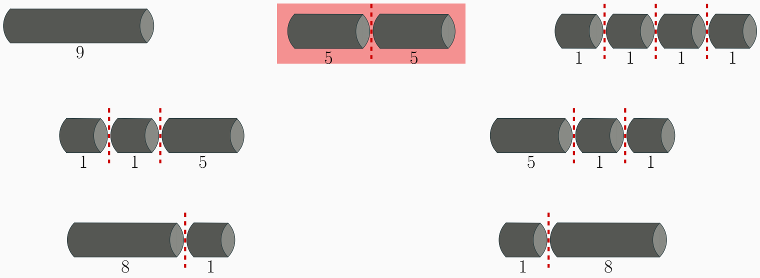
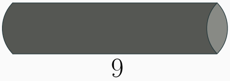


Recursive maximisation:

$$r_n = \max\{p_n$$



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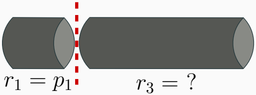
$$r_n = \max\{p_n, r_1 + r_{n-1}$$



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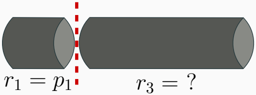

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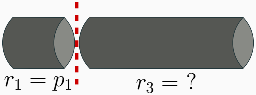
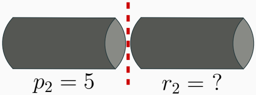


Reformulation: piece containing the left end + a partition of the rest

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$



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Time complexity: $\Theta(2^n)$

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Dynamic Programming

Dynamic Programming

- Top-Down (recursion + memo)
- Bottom-Up (fill up a table)

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

Top-Down DP

```
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
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Bottom-Up DP

```
vector<int> r(n + 1, -1);
r[0] = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i; j++) {
        r[i] = max(r[i], p[j] + r[i - j]; // IMPORTANT: Current subproblem
                                          // relies only on the solution of
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Reconstructing a Solution: What if we also want to know where to cut?

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No problem!

```
vector<int> cut(n + 1, 0); // cut[i] stores where to optimally cut a rod of
                           // lenght i
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) {
        return memo[n];
    }
    int res = -1;
    for (int i = 1; i <= n; i++) {
        if (p[i] + r(p, n - i) > res) {
            res = p[i] + r(p, n - i);
            cut[n] = i; // We should cut at position i
        }
    }
    memo[n] = res;
    return res;
}
```

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Deriving a recursive algorithm is the difficult part.

Usually problems are not given in a way that can straightforwardly be translated into a recursive definition. :(

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Example: Rod Cutting Problem

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r_n, maximal possible revenue for a rod of length n

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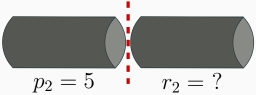

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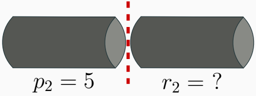

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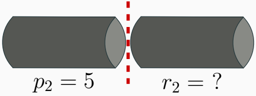

- Suppose the DP gods show you what is the ‘last’ choice to make.
- Look at which subproblems arise once making this choice.
- Show that the subproblems used in an optimal solution must themselves be optimal.

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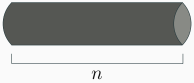

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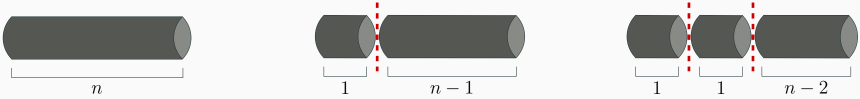

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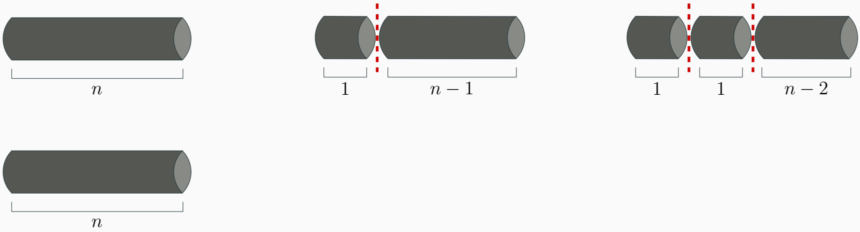

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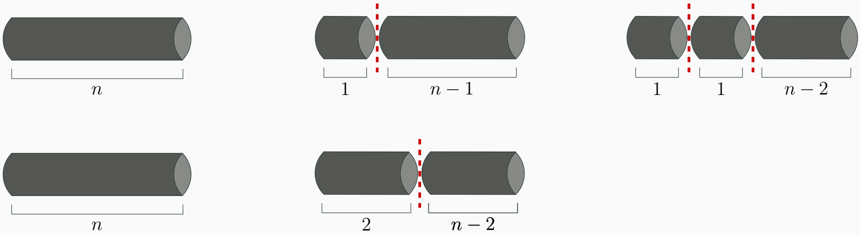

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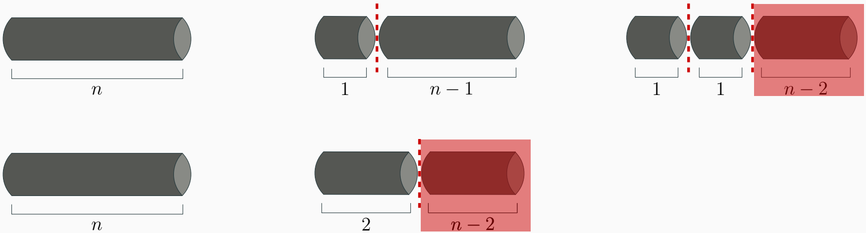

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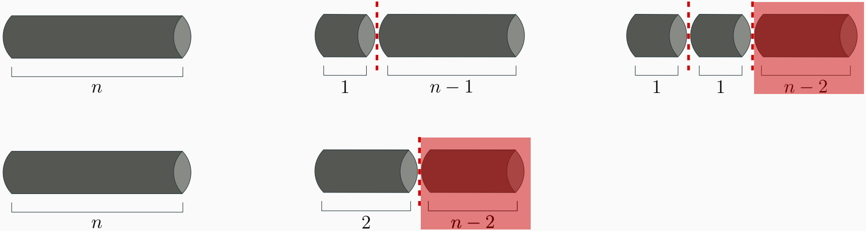

- Remember, do not recompute!
- *"Those who do not remember the past are condemned to repeat it."*

Common Pitfalls

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
vector<int> memo(n + 1, -1);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != -1) return memo[n];
    int res = -1;
    for (int i = 1; i <= n + 1; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
}
```

This results in a SEG FAULT/RUN ERROR, can you see why?

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This results in a SEG FAULT/RUN ERROR, can you see why?

Make sure that you stay 'within' the memo boundaries! Similarly, sometimes the memo table does not/cannot contain the base cases.

$$r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```cpp
vector<int> memo(n + 1, 0);

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo[n] != 0) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
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    }
    memo[n] = res;
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This results in a TIMELIMIT, can you see why?

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    }
    memo[n] = res;
    return res;
}
```

This results in a **TIMELIMIT**, can you see why?

Make sure that the **default** memo value is not a possible output!

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \qquad \text{with } r_0 = 0$$

```
map<int, int> memo;

int r(vector<int> &p, int n) {
    if (n == 0) return 0;
    if (memo.find(n) != memo.end()) return memo[n];
    int res = -1;
    for (int i = 1; i <= n; i++) {
        res = max(res, p[i] + r(p, n - i));
    }
    memo[n] = res;
    return res;
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This (possibly) results in a **TIMELIMIT**, can you see why?

`std::map` adds an $O(\log n)$ insert/find/access overhead.

Third Example: Longest Increasing Subsequence

Input: a sequence of n integers $a_1, \ldots, a_n$

Output: the length of a longest increasing subsequence (LIS)

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Example:

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Result: LIS $= 4$

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- Base cases: $f(1) = 1$
- $f(i) = \max_{j<i: a_j \leq a_i} (1 + f(j))$

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We had to reformulate the problem s.t. it admits a recursive formulation, this is **difficult**!

Time complexity: $O(n^2)$

Explanation: n function calls (with memo), i-th call takes $O(i)$ time.

(**Exercise:** Can you do it in $O(n \log n)$?)

Tips & Tricks

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- Time complexity sometimes not so obvious
- Overhead of function calls

- More effort to code
- Subproblems must be described by integers
- Always computes all subproblems
- Time complexity obvious
- Saves some constant factors

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How to determine the runtime?

Informally, a product of two factors: the overall number of subproblems and how many choices for each subproblem

Bottom-Up: Easy!

Top-Down: Sometimes harder to see immediately.

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Remember! `std::map` has a insert/find/access overhead of $O(\log n)$.

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- Add memo (usually does the trick) or construct a DP table
- Practice deriving recurrent relations on paper for standard DP problems (e.g. Knapsack, SubsetSum, Coin Change, LCS, Edit Distance, LIS, etc.)

That’s all for today!