Orthogonal Faces in the CHSH Scenario

Andrea Zingarofalo, Victor Barizien and Jean-Daniel Bancal

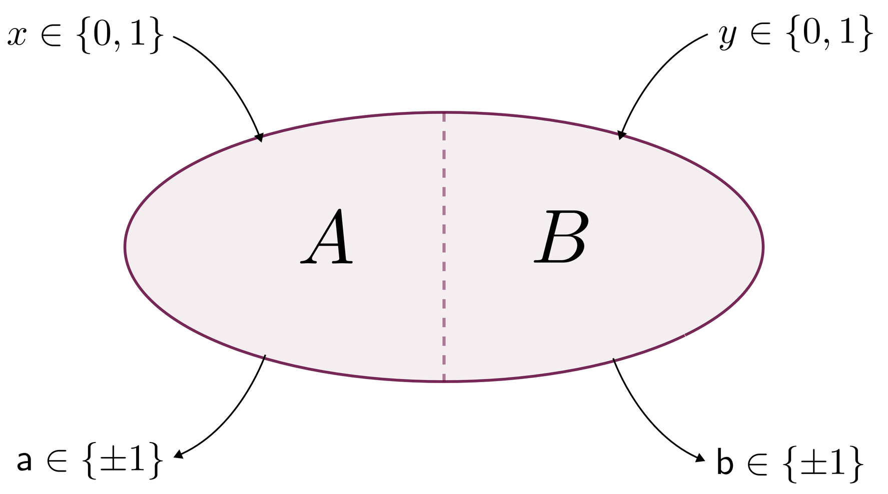
Université Paris-Saclay, CEA, CNRS, Institut de Physique Théorique

Goal

Investigate the geometry of the quantum correlation set from a **dual perspective**.

1. Background

In **Bell-type experiments** [1], joint probabilities $P(a, b|x, y)$ describe outcomes (a, b) given measurement settings (x, y).

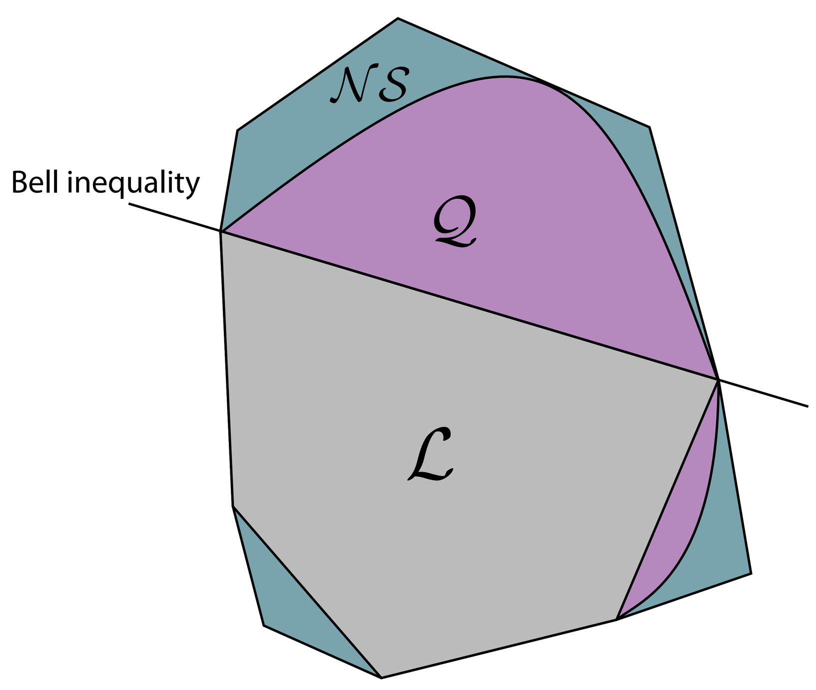


We focus on three fundamental sets of correlations:

- **No-signaling set $\mathcal{NS}$.** Contains all correlations that do not allow instantaneous communication.
- **Local set $\mathcal{L}$.** Comprises correlations explainable by LHVTs.
- **Quantum set $\mathcal{Q}$.** Consists of correlations achievable by QM. Formally,

$$P(a, b|x, y) = \langle\psi| \Pi^A_{a|x}\Pi^B_{b|y} |\psi\rangle ,$$

$\mathcal{Q}$ is a convex set, but not a polytope.



The following hierarchy holds:

$$\mathcal{L} \subsetneq \mathcal{Q} \subsetneq \mathcal{NS}.$$

By taking into account the no-signaling conditions, the 16 $P(a, b|x, y)$ can be parametrized as:

$$\boldsymbol{P} = \begin{array}{c|c|c} & \langle B_0\rangle & \langle B_1\rangle \\ \hline \langle A_0\rangle & \langle A_0B_0\rangle & \langle A_0B_1\rangle \\ \hline \langle A_1\rangle & \langle A_1B_0\rangle & \langle A_1B_1\rangle \end{array}$$

where

$$A_x := \Pi^A_{0|x} - \Pi^B_{1|x}$$
$$B_y := \Pi_{0|y} - \Pi^B_{1|y}$$

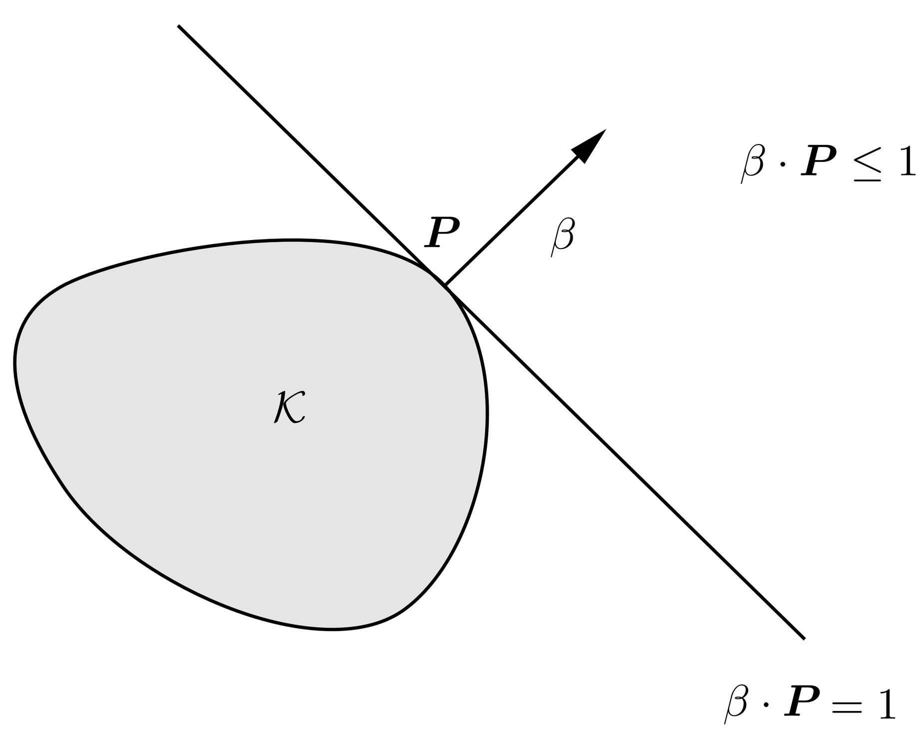
are unitary observables.

2. The dual approach

To study $\mathcal{Q}$, we leverage **convex duality**.

Def. Given a convex set $\mathcal{K} \subset \mathbb{R}^n$, its dual $\mathcal{K}^*$ is defined as:

$$\mathcal{K}^* := \{\boldsymbol{\beta} \in \mathbb{R}^n \colon \boldsymbol{\beta} \cdot \boldsymbol{P} \le 1, \forall \boldsymbol{P} \in \mathcal{K}\}.$$



$\mathcal{Q}^*$ coincides with the set of all Bell expressions β whose Tsirelson bound is ≤ 1:

$$\mathcal{Q}^* := \{\boldsymbol{\beta} \in \mathbb{R}^8 \colon \boldsymbol{\beta} \cdot \boldsymbol{P} \le 1, \forall \boldsymbol{P} \in \mathcal{Q}\}.$$

3. Orthogonal faces

Every extremal point in the CHSH scenario has a realization of the form [2]

$$|\phi_\theta\rangle := c_\theta |00\rangle + s_\theta |11\rangle ,$$
$$A_x := c_{a_x} Z_A + s_{a_x} X_A$$
$$B_y := c_{b_y} Z_B + s_{b_y} X_B.$$

In terms of correlators,

$$\boldsymbol{P}_{\theta,a_x,b_y} := \begin{array}{c|cc} 1 & c_{2\theta}c_{b_0} & c_{2\theta}c_{b_1} \\ \hline c_{2\theta}c_{a_0} & \multicolumn{2}{c}{\multirow{2}{*}{$c_{a_x}c_{b_y} + s_{2\theta}s_{a_x}s_{b_y}$}} \\ c_{2\theta}c_{a_1} & & \end{array},$$

with $\theta, a_x, b_y \in \mathbb{R}$. Moreover,

Th. A realization leads to a **nonlocal extremal point** iff it can be mapped to a realization on $|\phi_\theta\rangle$ with $\theta \in [0, \pi)$ and measurements satisfying $\forall(s, t) \in \{\pm 1\}^2$,

$$0 \le [\tilde{a}^s_0]_\pi \le b_0 \le [\tilde{a}^t_1]_\pi \le b_1 < \pi,$$

where

$$\tilde{a}^r_x := \operatorname{atan}\left(\tan\left(\frac{a_x}{2}\right)\tan(\theta)^r\right).$$

We consider

$$a_0 = \pi/4, \quad a_1 = -\pi/4, \quad b_0 = 0, \quad b_1 = \pi/2.$$

Fixed $\boldsymbol{P}_\theta \in \mathcal{Q}$, the set

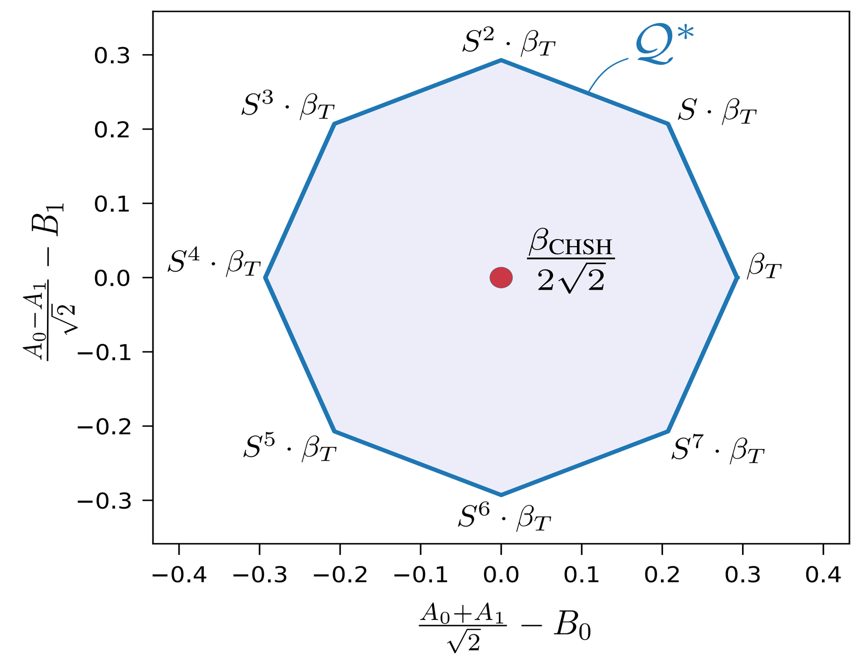
$$F^\perp_\theta := \{\boldsymbol{\beta} \in \mathcal{Q}^* \colon \boldsymbol{\beta} \cdot \boldsymbol{P}_\theta = 1\}$$

is a face of $\mathcal{Q}^*$, the **orthogonal face of $\boldsymbol{P}_\theta$**. $F^\perp_\theta$ provides a description of $\mathcal{Q}$ around the point $\boldsymbol{P}_\theta$ to first order.

Notice that we are looking for Bell expressions that are maximized by $\boldsymbol{P}_\theta$.

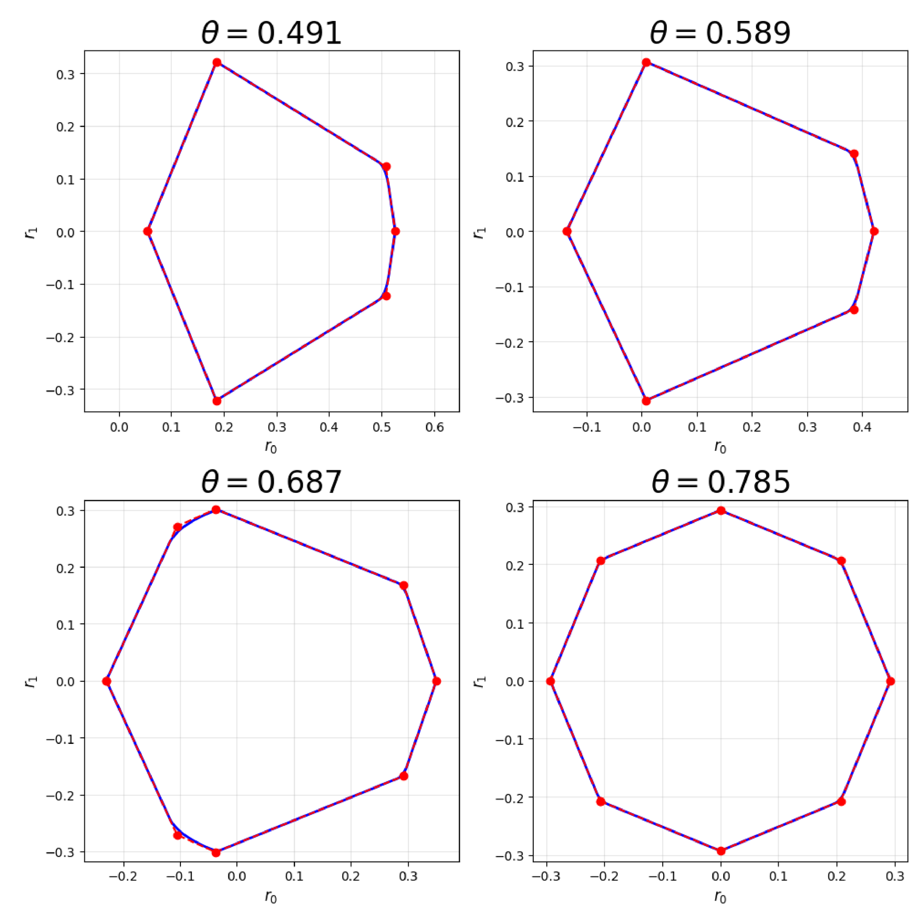
4. Results

For $\theta = \pi/4$, a face of $\mathcal{L}^*$ looks like [3]



In this plane, the boundaries of $\mathcal{L}^*$ coincide with the ones of $\mathcal{Q}^*$, (**NPA hierarchy** [4], **SOS decomposition** [5]).

The same phenomenology shows for $\theta \in [\pi/8, \pi/4]$:



Extremal Tsirelson inequalities for $|\phi_\theta\rangle$.

5. Conclusion

In this work, we studied $\mathcal{Q}$ from a dual perspective. We derived constructively all the Bell expressions that $\boldsymbol{P}_\theta$ maximizes, and characterized some orthogonal faces of $\mathcal{Q}^*$. This provides fresh insight on the geometry of $\mathcal{Q}$.

References

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[4] M. Navascués, S. Pironio, and A. Acín, PRL **98**, 010401 (2007).
[5] V. Barizien, P. Sekatski, and J.-D. Bancal, Quantum **8**, 1333 (2024).

Contact Information

- Web: https://quantum.paris
- Email: andrea.zingarofalo@ipht.fr

université PARIS-SACLAY cea cnrs IPhT SACLAY