Università del Salento

Dipartimento di Matematica e Fisica
"Ennio de Giorgi"

Corso di Laurea Triennale in Fisica

Tesi di Laurea Triennale

Study of one-dimensional quantum integrable systems through the Bethe Ansatz techniques

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Anno Accademico 2022/2023

FIS/02 - FISICA TEORICA, MODELLI E METODI MATEMATICI

My advice to young people entering science:
you should do it for the love of science...
You should enter because you are fascinated by it.
(J. Peebles)

Sommario

Questo lavoro di tesi introduce concetti e tecniche fondamentali legati allo studio di sistemi quantistici a molti corpi integrabili unidimensionali.

Dopo aver richiamato la nozione classica di integrabilità, affrontiamo il problema della definizione di sistema quantistico integrabile. Poi concentriamo la nostra attenzione alla speciale classe dei sistemi quantistici a molti corpi "soggetti a diffusione". Introduciamo la tecnica del Bethe Ansatz Asintotico e l'equazione di Yang-Baxter per sistemi che non presentano diffrazione. In seguito, consideriamo due particolari sistemi: il modello di Lieb e Liniger e la famiglia di modelli di Calogero-Moser-Sutherland, dimostrandone le proprietà di integrabilità.

Infine ne studiamo lo stato fondamentale, le eccitazioni e la termodinamica (a temperatura zero e finita) nel limite termodinamico, e accenniamo brevemente a esperimenti rilevanti nel campo degli atomi ultrafreddi.

Abstract

This dissertation introduces fundamental concepts and techniques related to the study of one-dimensional integrable quantum many-body systems.

We review classical integrability, and discuss the problem of a definition of a quantum integrable system. Then, we restrict ourselves to the special class of quantum many-body systems that support scattering. We introduce the Asymptotic Bethe Ansatz and the consistency conditions for systems that exhibit no diffraction. Afterward, we consider two quantum exactly solvable models in one dimension: the Lieb and Liniger model and the family of Calogero-Moser-Sutherland models, proving that they are nondiffractive.

We explore the ground states, excitations, and thermodynamics (at both zero and finite temperature) of these models in the thermodynamic limit, and briefly mention relevant experiments in the field of ultracold atoms.

Ringraziamenti

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List of Symbols

Symbol	Description
$A(\boldsymbol{x}_{\mathcal{Q}}, \boldsymbol{k}_{\mathcal{P}})$	Amplitude of the many-particle wavefunction
$\mathbb{C}, \mathbb{R}$	Sets of real and complex numbers
d	Density
E	Energy
e, e_0	Energy density
G_N	Group of permutations of N integers
$\mathcal{H}$	Classical Hamiltonian
H	Quantum Hamiltonian operator
$\mathscr{H}$	Hilbert space
I_j	Bethe quantum numbers
I_N	N-dimensional identity matrix
$\mathbb{I}$	Identity operator
(L, M)	Lax couple
ℓ	Length of the one-dimensional space
N	Number of particles
p	Pressure
$\mathcal{P}, \mathcal{Q}$	Permutations of integers
$\wp$	Weierstrass elliptic function
q	"Fermi" momentum
(q^n, p_n)	Position and momentum of the n-th particle
S, s	Entropy, entropy density
S_{nm}	S-matrix
$\mathbb{T}^N$	N-dimensional torus
v_s	Sound speed in the medium
(x_n, k_n)	Position and momentum operator of the n-th particle
Γ	Phase space
$\epsilon(k)$	Dressed energy per particle
$\theta(k - k')$	Two particles scattering phase shift
μ	Chemical potential
μ_B	Bohr magneton
$\rho(k)$	Number momentum density
$(\boldsymbol{\phi}, \boldsymbol{J})$	Action-angle variables
$\psi_n(x_n)$	Single particle wavefunction
$\Psi(\boldsymbol{x})$	Many particles wavefunction

Preface

Solvable models have a crucial role in the mathematical modeling of natural phenomena. They enable us to understand the fundamental nature of physical occurrences and guide us in finding suitable methods for dealing with more complex and realistic problems.

This dissertation aims to introduce important concepts and techniques related to the study of one-dimensional integrable quantum many-body systems. The history of exactly solvable models dates back to 1931, when H. Bethe [1] introduced a particular form of wavefunction to obtain the eigenspectrum of a one-dimensional Heisenberg spin chain. This novel technique remained essentially unrecognized until Lieb and Liniger's seminal work [2, 3]. The two scientists introduced and solved a many-body quantum system analytically, without any approximation, for both a finite system of bosons and the thermodynamic limit. This opened up a new research field, which has helped to understand the physics of strongly correlated quantum systems and the dynamics of dilute gases. Today Bethe's technique has applications in a wide variety of branches of physics, from superconductivity to string theory [4].

Furthermore, recent experimental realizations of these systems [5] have sparked intense efforts to study and reproduce new different one-dimensional systems, from both theoretical and experimental points of view. Over the last two decades, incredible strides in quantum engineering have enabled the creation of genuine quantum many-body systems using ultracold atoms. This breakthrough has resulted in the successful realization of some prototypical quantum integrable systems with exact solutions. By comparing these solutions to their experimental counterparts, we have an opportunity to test the sophisticated mathematical physics of Bethe ansatz integrable models.

Among quantum many-body systems, we focus on one-dimensional ones. A key difference between one-dimensional and multidimensional many-body systems is the way that they interact with each other. One-dimensional quantum systems can only interact with each other through pairwise interactions. Multidimensional quantum systems, on the other hand, can interact with each other through collective interactions. What is remarkable about such systems is that they are strongly

interacting. In one dimension, ordered phases exist at zero temperature: if we want to study phase transitions in this case, we need to look only at the ground state.

One-dimensional systems are essential to study because they are more likely to admit analytical solutions, making it easier to understand the dynamics of the system. It is currently unclear which qualitative features distinguish integrable from non-integrable quantum systems [6]: while this distinction is clear in classical physics, proposed answers for the quantum case remain unsatisfactory. Despite this, there exists a captivating subset of many-body quantum systems that is amenable to exact solutions [7]. This means that each eigenstate within these systems can be uniquely characterized using a set of quantum numbers. It is worth highlighting that the real advantage of these techniques goes beyond their efficacy, as they offer valuable insights into interpreting intricate systems.

The critical concept of integrability will be our starting point. After recalling the notion of classical integrability, we discuss the troublesome problem of defining what is a quantum integrable system. Then, we restrict ourselves to the special class of quantum many-body systems that support scattering.

In the second chapter, we introduce Bethe's method, giving a general description, and explaining why its solution implies the system's integrability. Bethe's method and its generalizations allow us to determine whether a system is integrable or not, as solvability by the ansatz is a clear signature of the system's integrability.

The essence of the Bethe Ansatz lies in the fact that any scattering event can be factorized into a sequence of two-body scatterings, and the order of these events does not affect the outcome. This pivotal concept constitutes the foundation of a fascinating mathematical physics construction. This property of integrability is unique to one-dimensional quantum systems.

We introduce the Asymptotic Bethe Ansatz and the consistency conditions for systems that exhibit no diffraction. Afterward, we consider two quantum exactly solvable models in one dimension: the one-dimensional Bose gas with delta function interaction (known as the Lieb and Liniger model) and the family of Calogero-Moser-Sutherland models, proving that they are nondiffractive. We also derive an essential result concerning the periodic boundary conditions of quantum integrable systems: the fundamental relation.

Then, equipped with this powerful tool, in the third chapter we determine the ground state of the Lieb-Liniger model and the Calogero-Moser system with inverse-squared potential. We proceed to study the general response of a system in the ground state to a perturbation and characterize the excitations. We investigate the thermodynamics of these models, both at zero and at finite temperature. We close by mentioning some experiments in the field of ultracold atoms.

The literature on the subject is vast. An excellent introduction to the subject is constituted by Sutherland's book [7], and Eckle [8] and Franchini's [9] monographs. Moreover, D.C. Mattis' [10] compilation of reprints on one-dimensional quantum and classical many-body problems serves as a valuable resource for general reference.

The main purpose of this work is to serve as an elementary introduction to this interesting topic.

Chapter 1

Quantum Integrability

This chapter begins by reviewing classical integrability. In this case, a classical integrable system is defined clearly, as the differential equations describing the system's time evolution can be explicitly integrated with the help of action-angle variables. We will also introduce various essential structures for integrable models that aid in exploring and solving the model. Subsequently, we will delve into the troublesome concept of quantum integrability and examine the primary challenges associated with its definition.

1.1 Classical integrability

A standard way to investigate the behaviour of a physical system is to construct a mathematical model of the system, solve it, use the outcomes to make predictions and compare theoretical results with real-world experiments. However, often differential equations provided by these models cannot be solved analytically and solutions can only be approximated numerically. Relatively few systems can be solved exactly with explicit analytic expressions. These systems are referred to as *integrable*.

The field of integrable systems is born together with Classical Mechanics [11], with a quest for exact solutions to Newton's equations of motion. Two centuries after Newton solved the Kepler problem, only a few other cases were found:

- geodesics on an ellipsoid (Jacobi, 1837);

- the harmonic oscillator on the sphere (Neumann, 1859);

- Lagrange, Euler and Kovalevskaya tops.

We need to clarify that integrability is not the same as *solvability*. There are integrable systems that cannot be expressed analytically, just as there are systems exactly solvable that are not integrable. Solvability ultimately depends on capacity and computational power. Integrability refers instead to the property of a system to exhibit regular behaviors (quasi-periodic) for every possible initial condition, as opposed to chaotic and irregular ones. For this reason, an inclusive definition of an integrable system is elusive, as pointed out in [12].

1.1.1   Hamiltonian systems

Let us recall some basic elements of Hamiltonian systems.[1] We start from the definition:

Definition 1.1. A dynamical system of $2N$, first order, ordinary differential equations

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}, \tag{1.1}$$

with $(\boldsymbol{q}, \boldsymbol{p}) = (q^1, \ldots, q^N, p_1, \ldots, p_N)$, is said an N-degree-of-freedom Hamiltonian system if $\mathcal{H}$, the Hamiltonian, is a smooth scalar function defined on the phase space, $\mathcal{H} \colon T^*\mathcal{M} \equiv \Gamma \longrightarrow \mathbb{R}$, where $\mathcal{M}$ is a N-dimensional smooth manifold (called configuration space) and $T^*\mathcal{M}$ its cotangent bundle [13].

In its natural form, for a system of N particles, one writes

$$\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} \sum_{n=1}^{N} p_n^2 + V(\boldsymbol{q}). \tag{1.2}$$

Here q^n is the position of the n-th particle and p_n is its canonical momentum $p_n = \dot{q}^n$. Masses are supposed to be unitary.

Hamilton's equations are equivalent to Newton's equation of motion.

Definition 1.2. Smooth real-valued functions on the phase space Γ are called classical observables. We denote the set of classical observables on the space Γ by $C^\infty(\Gamma)$.

Definition 1.3. We define the Poisson brackets for any couple of classical observables

$$\{\cdot, \cdot\} : C^\infty(\Gamma) \times C^\infty(\Gamma) \longrightarrow C^\infty(\Gamma)$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto \{\mathcal{F}, \mathcal{G}\} \equiv \sum_{n=1}^{N} \left(\frac{\partial \mathcal{F}}{\partial q^n} \frac{\partial \mathcal{G}}{\partial p_n} - \frac{\partial \mathcal{F}}{\partial p_n} \frac{\partial \mathcal{G}}{\partial q^n} \right). \tag{1.3}$$

Proposition 1.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} : C^\infty(\Gamma)$, k, $h \in \mathbb{R}$. The Poisson brackets obey the following properties:*

1. $\{\mathcal{A}, \mathcal{B}\} = -\{\mathcal{B}, \mathcal{A}\}$ *(skew-symmetry);*

2. $\{\mathcal{A}, k\mathcal{B} + h\mathcal{C}\} = k\{\mathcal{A}, \mathcal{B}\} + h\{\mathcal{A}, \mathcal{C}\}$ *(bilinearity);*

3. $\{\mathcal{A}, \mathcal{B}\mathcal{C}\} = \{\mathcal{A}, \mathcal{B}\}\mathcal{C} + \mathcal{B}\{\mathcal{A}, \mathcal{C}\}$ *(Leibniz rule);*

4. $\{\mathcal{A}, \{\mathcal{B}, \mathcal{A}\}\} + \{\mathcal{B}, \{\mathcal{C}, \mathcal{A}\}\} + \{\mathcal{C}, \{\mathcal{A}, \mathcal{B}\}\} = 0$ *(Jacobi identity).*

Coordinates $(\boldsymbol{q}, \boldsymbol{p})$ satisfy canonical relations:

$$\{q^n, q^m\} = \{p_n, p_m\} = 0, \qquad \{q^n, p_m\} = \delta_m^n. \tag{1.4}$$

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[1]For a complete discussion and insights, please refer to V. Arnol'd, *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics, Springer, 1997.

Remark 1.1. Hamilton's equations can be written as

$$\dot{\boldsymbol{q}} = \{\boldsymbol{q}, \mathcal{H}\}, \qquad \dot{\boldsymbol{p}} = \{\boldsymbol{p}, \mathcal{H}\}. \tag{1.5}$$

It yields the following

Proposition 1.2. *A function $\mathcal{F}$ on phase space, $\mathcal{F} \in C^\infty(\Gamma)$ is a first integral for the Hamiltonian system $\mathcal{H}$ if and only if*

$$\{\mathcal{F}, \mathcal{H}\} = 0. \tag{1.6}$$

Proof. Expanding the total derivative of $\mathcal{F}$ with respect to t, and using Hamilton's equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} &= \frac{\partial \mathcal{F}}{\partial t} + \sum_{n=1}^{N} \left(\frac{\partial \mathcal{F}}{\partial q^n} \frac{\partial q^n}{\partial t} - \frac{\partial \mathcal{F}}{\partial p_n} \frac{\partial p_n}{\partial t} \right) \\
&= \frac{\partial \mathcal{F}}{\partial t} + \sum_{n=1}^{N} \left(\frac{\partial \mathcal{F}}{\partial q^n} \frac{\partial \mathcal{H}}{\partial p_n} - \frac{\partial \mathcal{F}}{\partial p_n} \frac{\partial \mathcal{H}}{\partial q^n} \right) = \frac{\partial \mathcal{F}}{\partial t} + \{\mathcal{F}, \mathcal{H}\}.
\end{aligned}$$

Because $\mathcal{F}$ is defined on the phase space, $\partial \mathcal{F}/\partial t = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = 0 \iff \{\mathcal{F}, \mathcal{H}\} = 0.$$

$\square$

A natural question is which are the transformations that leave invariant in form the Hamilton equations. Such transformations are called *canonical*.

Definition 1.4. The coordinate transformation T

$$T : \begin{cases} \boldsymbol{q} &= \boldsymbol{q}(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}}) \\ \boldsymbol{p} &= \boldsymbol{p}(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}}) \end{cases} \tag{1.7}$$

is said to be canonical if, however, the Hamiltonian $\mathcal{H}: \Gamma \longrightarrow \mathbb{R}$ is taken, the Hamiltonian equations

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \tag{1.8}$$

are equivalent to the new equations

$$\dot{\widetilde{\boldsymbol{q}}} = \frac{\partial \widetilde{\mathcal{H}}}{\partial \widetilde{\boldsymbol{p}}}, \qquad \dot{\widetilde{\boldsymbol{p}}} = -\frac{\partial \widetilde{\mathcal{H}}}{\partial \widetilde{\boldsymbol{q}}}, \tag{1.9}$$

where $\widetilde{\mathcal{H}} = \mathcal{H} \circ T$, i.e. $\widetilde{\mathcal{H}}(\widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{p}}) = \mathcal{H}(T(\boldsymbol{q}, \boldsymbol{p}))$.

Remark 1.2. A canonical coordinate transformation T preserves the Poisson brackets. Given $\mathcal{F}, \mathcal{G} \in C^\infty(\Gamma)$, denoting by $\widetilde{\mathcal{F}} = \mathcal{F} \circ T$, $\widetilde{\mathcal{G}} = \mathcal{G} \circ T$ the transformed functions, it results

$$\{\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}\} = \{\mathcal{F}, \mathcal{G}\} \circ T. \tag{1.10}$$

1.1.2   Action-angle variables

Consider an autonomous Hamiltonian system with N degrees of freedom, of Hamiltonian $\mathcal{H}(\boldsymbol{q}, \boldsymbol{p})$. Now suppose there exists a canonical transformation

$$T\colon (\boldsymbol{q}, \boldsymbol{p}) \longrightarrow (\boldsymbol{\phi}, \boldsymbol{J})$$

such that $\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}) = \widetilde{\mathcal{H}}(\boldsymbol{J})$. With evidence, the integration of the equations of motion is completely trivial: in fact, posited $\omega^n(\boldsymbol{J}) \equiv \partial \widetilde{\mathcal{H}}(\boldsymbol{J})/\partial J_n$, $n = 1, \dots, N$, the equations of motion become

$$\dot{\boldsymbol{J}} = 0, \qquad \dot{\boldsymbol{\phi}} = \boldsymbol{\omega}(\boldsymbol{J}), \tag{1.11}$$

and thus the general integral is written

$$\boldsymbol{J}(t) = \boldsymbol{J}_0, \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 + \boldsymbol{\omega}(\boldsymbol{J}_0)t, \tag{1.12}$$

where $\boldsymbol{\phi}_0$, $\boldsymbol{J}_0$, denote the initial data. The motion in the new variables $(\boldsymbol{\phi}, \boldsymbol{J})$ then takes place on a torus $\mathbb{T}^N$ and is linear.

In the new variables the phase space appears naturally decomposed into the product $B \times \mathbb{T}^N$. For their geometric interpretation, $(\boldsymbol{\phi}, \boldsymbol{J})$ are called action-angle variables.

$$\mathbb{T}^N$$
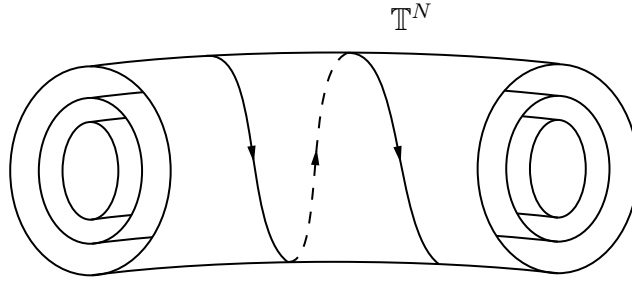
Figure 1.1: Liouville tori.

Let us consider the generating function $F = F(\boldsymbol{q}, \boldsymbol{\phi})$ of the canonical transformation into the action-angle variables. Its total differential reads

$$\mathrm{d}F(\boldsymbol{q}, \boldsymbol{\phi}) = \sum_{n=1}^{N} p_n \mathrm{d}q^n - \sum_{n=1}^{N} J_n \mathrm{d}\phi^n. \tag{1.13}$$

The motion is periodic in ϕ^m, hence the transformation $F(\boldsymbol{q}, \boldsymbol{\phi})$ must also be a periodic function of ϕ^m. We can decompose the N-dimensional torus $\mathbb{T}^N$ into a Cartesian product of one-dimensional tori (circles),

$$\mathbb{T}^N \equiv \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N,$$

where $\Gamma_n \cong \mathbb{S}^1 \ \forall n$.

Integrating the total differential of F along the m-th closed path Γ_m

$$0 = \oint_{\Gamma_m} \mathrm{d}F = \oint_{\Gamma_k} \sum_{n=1}^{N} p_n \mathrm{d}q^n - 2\pi J_m \tag{1.14}$$

and then

$$J_m = \frac{1}{2\pi} \oint_{\Gamma_m} \sum_{n=1}^{N} p_n \mathrm{d}q^n. \tag{1.15}$$

Remark 1.3. Using the Green's theorem, one writes

$$J_m = \frac{1}{2\pi} \oint_{\Gamma_m} \sum_{n=1}^{N} p_n \mathrm{d}q^n = \frac{1}{2\pi} \iint_{A_m} \sum_{n=1}^{N} \mathrm{d}p_n \mathrm{d}q^n. \tag{1.16}$$

Therefore, the action J_m is the area A_m enclosed by the phase space path Γ_m.

1.1.3 Complete integrability for classical systems

We can now define classical integrability:

Definition 1.5. The Hamiltonian $\mathcal{H} \colon \Gamma \longrightarrow \mathbb{R}$ is said to be integrable in the domain $D \subseteq \Gamma$ if there exists a canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) = T(\boldsymbol{\phi}, \boldsymbol{J})$, defined for $\boldsymbol{J} \in B \subseteq \mathbb{R}^N$ and $\boldsymbol{\phi} \in \mathbb{T}^N$, such that

1. $D = T(B \times \mathbb{T}^N)$,

2. the new Hamiltonian $\widetilde{\mathcal{H}} = \mathcal{H} \circ T$ is a function of actions only:

$$\widetilde{\mathcal{H}}(\boldsymbol{\phi}, \boldsymbol{J}) = \widetilde{\mathcal{H}}(\boldsymbol{J}), \qquad \widetilde{\mathcal{H}} \colon B \longrightarrow \mathbb{R}.$$

Example 1.1. The Hamiltonian for the classical one-dimensional harmonic oscillator with unitary mass and angular frequency ω reads

$$\mathcal{H}_{\mathrm{HO}} = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2, \tag{1.17}$$

with $\{q, p\} = 1$. Hamilton's equations are given by

$$\dot{q} = p, \qquad \dot{p} = -\omega^2 q. \tag{1.18}$$

We can consider a canonical transformation T in the new variables (ϕ, J)

$$T \colon \begin{cases} q = \frac{J}{\omega} \sin \phi \\ p = J \cos \phi. \end{cases} \tag{1.19}$$

Fixed the initial energy $\mathcal{H} = E$ of the system, Hamilton's equations become

$$J = \sqrt{2E}, \qquad \phi(t) = \phi_0 + \omega t \tag{1.20}$$

with ϕ_0 constant. Then (ϕ, J) are the action-angle variables for the harmonic oscillator, which is indeed integrable.

In the second half of the nineteenth century, J. Liouville [14] provided a general criterion characterizing the cases where the equations of motion are *solvable by quadratures*. This result was reported by V. I. Arnol'd [13] with a new proof based on symplectic geometry. This contribution was so important that now the theorem is known as the Liouville-Arnol'd theorem:

Theorem 1.1 (Liouville-Arnol'd). *If a Hamiltonian system with N degrees of freedom has N integrals of motion $\mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2, \ldots, \mathcal{H}_N$, ($\mathcal{H}_n \colon \Gamma \longrightarrow \mathbb{R} \;\forall n = 1, \ldots, N$)*

1. functionally independent and analytic in their variables,

2. in involution, i.e. $\{\mathcal{H}_n, \mathcal{H}_m\} = 0 \;\forall n \neq m$,

3. with the additional condition

$$\det \frac{\partial(\mathcal{H}_1, \ldots, \mathcal{H}_N)}{\partial(p_1, \ldots, p_N)} \neq 0,$$

then the system can be integrated by quadratures.

A traditional proof of the theorem [15] is presented in appendix A.

The characterization provided by the Liouville-Arnol'd theorem lets us state an equivalent definition of a classical integrable system: a classical Hamiltonian system is integrable according to Liouville if it possesses N commuting integrals of motion $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ functionally independent and analytical in their variables.

Remark 1.4. Summarizing, we distinguish three classes of evolution for a dynamical system:

- *Integrability.* A dynamical system is integrable if it has a complete set of independent first integrals. This means that the system can be "solved exactly", and its motion can be predicted for all time. Integrable systems are typically characterized by periodic or *quasi-periodic motion*, where the system oscillates between different states with rationally compatible or incompatible periods. As a consequence, classical integrable systems do not *thermalize*.

- *Superintegrability.* Some dynamical systems have more integrals than necessary for integrability. Such systems are called *superintegrable* [16]. In superintegrable systems, some of the periods of the quasi-periodic motion are rationally compatible, and the system's orbit partially closes. *Maximally superintegrable systems* have the maximum possible number of first integrals, and their orbits are completely closed. Some examples of superintegrable systems are the multidimensional harmonic oscillator with rationally compatible frequencies and Kepler's planetary motion.

- *Chaos.* Most dynamical systems with more than one degree of freedom are *chaotic* [17]. In chaotic systems, it is impossible to predict the system's motion for all time, even if we know the initial conditions perfectly. Chaotic systems are extremely sensitive to their initial conditions, and even a small change in the initial conditions can lead to a completely different long-term evolution.

1.1.4   Lax pairs, classical Yang-Baxter equation

A new concept that emerged from modern studies on integrable systems is the notion of *Lax pairs* [18], introduced by the Hungarian-born American mathematician P. Lax in 1968.

Definition 1.6. A classical Lax pair (L, M) consists of two square matrices, functions on the phase space of the system, such that the Hamiltonian evolution equations may be written as

$$\dot{L} = [L, M] \tag{1.21}$$

where $[L, M]$ denotes the commutator on the set of $N \times N$ square matrices over the complex field $\mathrm{Mat}_N(\mathbb{C})$.

Suppose that $\mathcal{H}$ is integrable, and that the integrals of motion $\mathcal{H}_1, \ldots, \mathcal{H}_N$ are known. Then

$$\Lambda = \mathrm{diag}\{\mathcal{H}_n : n = 1, \ldots, N\} \tag{1.22}$$

is the Lax representation of $\mathcal{H}$. Note that $\mathcal{H}_1, \ldots, \mathcal{H}_N$ are the eigenvalues of $\hat{L}$.

Because similarity transformations preserve eigenvalues, if

$$L = U \Lambda U^{-1} \tag{1.23}$$

where $U(t)$ is some invertible matrix, then the eigenvalues λ_n of L coincide with the ones of Λ: $\lambda_n = \mathcal{H}_n$ $(n = 1, ..., N)$. Consider now the eigenvalue equation

$$L u_n = \lambda_n u_n. \tag{1.24}$$

Taking the time derivative we get:

$$\dot{L} u_n + L \dot{u}_n = \dot{\lambda}_n u_n + \lambda_n \dot{u}_n = \lambda_n \dot{u}_n. \tag{1.25}$$

Supposing $\dot{u}_n = -M u_n$, where $M \in \mathrm{Mat}_N(\mathbb{C})$ we have

$$\dot{L} u_n - L M u_n = -\lambda_n M u_n = -M \lambda_n u_n = -M L u_n, \tag{1.26}$$

thus

$$\dot{L} u_n - L M u_n = -M L u_n \iff (\dot{L} - [L, M]) u_n = 0. \tag{1.27}$$

which implies $\dot{L} = [L, M]$.

Since the elements of L are functions on phase space, the classical Lax evolution equations must be equivalent to Hamilton's equations of motion for $\mathcal{H}$.

Remark 1.5. A Lax representation for an integrable system is not unique [11].

Theorem 1.2. *Given a classical Lax pair (L, M) for a dynamical system, then the functions*

$$\mathrm{Tr}(L^n), \qquad n = 1, 2, \ldots, \tag{1.28}$$

are all the integrals of motion of the dynamical system.

Proof. For $n = 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}(L) = \mathrm{Tr}(\dot{L}) = \mathrm{Tr}(LM - ML) = 0 \tag{1.29}$$

due to the cyclic property of the trace operator.

For the generic n

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}L^n &= \dot{L}L^{n-1} + L\dot{L}L^{n-2} + \cdots + L^{n-2}\dot{L}L + L^{n-1}\dot{L} = \\
&= LML^{n-1} - ML^k - LML^{n-1} + L^2ML^{n-2} + \dots \\
&\quad - L^{n-2}ML^2 + L^{n-1}ML - L^{n-1}ML + L^nM = \\
&= [L^n, M].
\end{aligned}
\tag{1.30}
$$

Then

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(L^n) = \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}L^n\right) = \mathrm{Tr}(L^nM - ML^n) = 0.
\tag{1.31}
$$

$\square$

Example 1.2. A Lax pair for the classical harmonic oscillator has the form:

$$
L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & \frac{\omega}{2} \\ -\frac{\omega}{2} & 0 \end{pmatrix}.
\tag{1.32}
$$

The characteristic equation for L

$$
(p - \lambda)(p + \lambda) + \omega^2 q^2 = 0
\tag{1.33}
$$

gives the eigenvalues

$$
\lambda_{1,2} = \pm\sqrt{p^2 + \omega^2 q^2},
\tag{1.34}
$$

that are integrals of motion of the system related to the energy.

We check immediately that the Lax evolution equation, $\dot{L} = [L, M]$, is equivalent to the equations of motion

$$
\dot{q} = p, \qquad \dot{p} = -\omega^2 p.
\tag{1.35}
$$

The integrals of motion are given by

$$
\mathrm{Tr}(L^1) = 0,
\tag{1.36}
$$
$$
\mathrm{Tr}(L^2) = 2p^2 + 2\omega^2 q^2 = 4\mathcal{H}_{\mathrm{HO}},
\tag{1.37}
$$
$$
\mathrm{Tr}(L^3) = 0,
\tag{1.38}
$$
$$
\mathrm{Tr}(L^4) = 2(p^2 + \omega^2 q^2)^2 = 8\mathcal{H}_{\mathrm{HO}}^2,
\tag{1.39}
$$

and so on. Here the first and the second ones are trivial and can be ignored. The only independent integral of motion is the Hamiltonian $\mathcal{H}_{\mathrm{HO}} = E$. The traces of the higher even powers of L are solely powers of the Hamiltonian which are not independent integrals of motion.

Now, let $\{e_{ij}\}$ a basis of $\mathrm{Mat}_N(\mathbb{C})$. Then every Lax matrix L can be written as

$$
L = \sum_{i,j} L_{ij}e_{ij},
\tag{1.40}
$$

where L_{ij} are classical observables.

We define

$$L_1 \equiv L \otimes I_N = \sum_{i,j} L_{ij}(e_{ij} \otimes I_N), \quad L_2 \equiv I_N \otimes L = \sum_{i,j} L_{ij}(I_N \otimes e_{ij}), \qquad (1.41)$$

with $A \otimes B$ the Kronecker product between the matrices A and B, I_N the N-dimensional identity matrix. We introduce the Poisson structure for the elements of the Lax matrix

$$\{L_1, L_2\} = \sum_{i,j,k,l} \{L_{ij}, L_{kl}\}(e_{ij} \otimes e_{kl}). \qquad (1.42)$$

Then the following holds [19]:

Theorem 1.3 (Babelon-Viallet). *The involution property of the eigenvalues of the Lax matrix $L \in \mathrm{Mat}_N(\mathbb{C})$ is equivalent to the existence of a function*

$$r\colon \Gamma \longrightarrow \mathrm{Mat}_N(\mathbb{C}) \otimes \mathrm{Mat}_N(\mathbb{C})$$

with elements defined on the phase space such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \qquad (1.43)$$

Here the indices 1 and 2 of r_{12} refers to the first and second matrix components of $\mathrm{Mat}_N(\mathbb{C}) \otimes \mathrm{Mat}_N(\mathbb{C})$. More specifically, as a matrix, $r_{12} \equiv (r_{ij,kl})$, where $i, j = 1, 2, ..., N$ refer to the first matrix space of the tensor product, and $k, l = 1, 2, ..., N$ refer to the second one.

Proof. By assumption, the eigenvalues of L commute. Let $U \in \mathrm{Mat}_N(\mathbb{C})$ an invertible matrix such that $\Lambda = U^{-1}LU$ is diagonal. We define with a compact notation

$$\Lambda_1 \equiv \Lambda \otimes I_N, \qquad \Lambda_2 \equiv I_N \otimes \Lambda, \qquad (1.44)$$

and

$$U_1 \equiv U \otimes I_N, \qquad U_2 \equiv I_N \otimes U. \qquad (1.45)$$

Let us compute the Poisson brackets between the operators L_1 and L_2. Expanding according to the Leibniz rule we get

$$\begin{aligned}
\{L_1, L_2\} &= \{U_1\Lambda_1 U_1^{-1}, U_2\Lambda_2 U_2^{-1}\} = \\
&= \{U_1, U_2\}\Lambda_1 U_1^{-1}\Lambda_2 U_2^{-1} + U_1\{\Lambda_1, U_2\}U_1^{-1}\Lambda_2 U_2^{-1} \\
&\quad - U_1\Lambda_1 U_1^{-1}\{U_1, U_2\}U_1^{-1}\Lambda_2 U_2^{-1} + U_2\{U_1, \Lambda_2\}\Lambda_1 U_1^{-1}U_2^{-1} \\
&\quad - U_1\Lambda_1 U_2 U_1^{-1}\{U_1, \Lambda_2\}U_1^{-1}U_2^{-1} - U_2\Lambda_2 U_2^{-1}\{U_1, U_2\}U_2^{-1}\Lambda_1 U_1^{-1} \\
&\quad - U_2\Lambda_2 U_2^{-1}U_1\{\Lambda_1, U_2\}U_2^{-1}U_1^{-1} + U_1\Lambda_1 U_1^{-1}U_2\Lambda_2 U_2^{-1}\{U_1, U_2\}U_1^{-1}U_2^{-1}.
\end{aligned} \qquad (1.46)$$

Now we introduce the quantities

$$k_{12} = \{U_1, U_2\}U_1^{-1}U_2^{-1}, \qquad q_{12} = U_2\{U_1, \Lambda_2\}U_1^{-1}U_2^{-1} \qquad (1.47)$$

and

$$k_{21} = \{U_2, U_1\} U_1^{-1} U_2^{-1}, \qquad q_{21} = U_1 \{U_2, \Lambda_1\} U_1^{-1} U_2^{-1}. \tag{1.48}$$

We can rewrite eq. (1.46) as

$$\begin{aligned} \{L_1, L_2\} = &k_{12} L_1 L_2 + L_1 L_2 k_{12} - L_1 k_{12} L_2 \\ &- L_2 k_{12} L_1 - q_{21} L_2 + q_{12} L_1 - L_1 q_{12} + L_2 q_{21}, \end{aligned} \tag{1.49}$$

and we observe that $k_{21} = -k_{12}$ by definition. Thus

$$\begin{aligned} \{L_1, L_2\} &= [k_{12} L_2 - L_2 k_{12}, L_1] + [q_{12}, L_1] - [q_{21}, L_2] \\ &= \frac{1}{2} [[k_{12}, L_2], L_1] - \frac{1}{2} [[k_{21}, L_1], L_2] + [q_{12}, L_1] - [q_{21}, L_2], \end{aligned} \tag{1.50}$$

and finally

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{1.51}$$

where

$$r_{12} = q_{12} + \frac{1}{2} [k_{12}, L_2]. \tag{1.52}$$

On the other side, assume that eq. (1.51) is satisfied. Thus, one can write

$$\{L_1^n, L_2^m\} = [a_{12}^{n,m}, L_1] + [b_{12}^{n,m}, L_2] \tag{1.53}$$

with

$$a_{12}^{n,m} = \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1} L_2^{m-q-1} r_{12} L_1^p L_2^q, \tag{1.54}$$

$$b_{12}^{n,m} = -\sum_{p=0}^{n-1} \sum_{q=0}^{m-1} L_1^{n-p-1} L_2^{m-q-1} r_{21} L_1^p L_2^q. \tag{1.55}$$

Lastly, using the fact that the trace of a commutator is zero, we verify that the quantities $\mathrm{Tr}(L^k)$ commute, for $k = 1, 2, \ldots$. $\qquad \square$

Now, applying the Jacobi identity, we get

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyclic permutations} = 0. \tag{1.56}$$

If r has constant elements, the condition becomes

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0, \tag{1.57}$$

which is called the *classical Yang-Baxter equation* (CYBE) in the case where r is antisymmetric ($r_{12} = -r_{21}$). The matrix r is called *dynamical r-matrix*. The resolution of the eq. (1.57) is of considerable interest as it makes it possible to characterize integrable Hamiltonian systems.

Example 1.3. For the one-dimensional harmonic oscillator discussed above, the Lax operator L in eq. (1.32), expressed in the action-angle variables (ϕ, J) is diagonalized by

$$U = U^{-1} = \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ \sin\phi/2 & -\cos\phi/2 \end{pmatrix}. \tag{1.58}$$

Now r_{12} is defined by construction in eq. (1.52). This time it holds that $\{U_1, U_2\} = 0$ and we have

$$r_{12} = \frac{\omega}{2J^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes L. \tag{1.59}$$

It can be shown that the matrix (1.59) satisfies eq. (1.51).

1.2   Quantum integrability

We now discuss the definition of quantum integrability.

1.2.1   Quantum many-body systems

Let us begin recalling some quantum mechanics basics. In a quantum context, a physical state is represented by a vector in a complex Hilbert space $\mathscr{H}$. For a system with N degrees of freedom, we can build a model in which state vectors are complex square-integrable functions $\psi(\boldsymbol{x}, t)$ on $\mathbb{R}^N$:

$$\psi \colon \mathbb{R}^N \longrightarrow \mathbb{C}, \quad \psi \in \mathscr{H} \equiv L^2(\mathbb{R}^N) \tag{1.60}$$

with the inner product between state vectors

$$\langle \cdot, \cdot \rangle \colon \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{C}$$

$$(\psi, \phi) \longmapsto \langle \psi, \phi \rangle = \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{x} \; \psi^\dagger(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t). \tag{1.61}$$

Every measurable physical quantity is described by a linear Hermitian operator $O \colon \mathscr{H} \longrightarrow \mathscr{H}$. Such an operator is called an observable. We denote the set of all observable operators of the Hilbert space $\mathscr{H}$ as $\mathscr{L}(\mathscr{H})$.

Definition 1.7. We define the commutators (Lie brackets) for any couple of operators A and B on the Hilbert space $\mathscr{H}$:

$$[\cdot, \cdot] \colon \mathscr{L}(\mathscr{H}) \times \mathscr{L}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H})$$

$$(A, B) \longmapsto [A, B] = AB - BA. \tag{1.62}$$

Proposition 1.3. *Let A, B, $C \colon \mathscr{H} \longrightarrow \mathscr{H}$, k, $h \in \mathbb{C}$. The commutators obey the following properties:*

1. *$[A, B] = -[B, A]$ (skew-symmetry);*

2. *$[A, kB + hC] = k[A, B] + h[A, C]$ (bilinearity);*

3. *$[A, BC] = [A, B]C + B[A, C]$ (Leibniz rule);*

4. *$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacobi identity).*

Commutators satisfy the same relations as the Poisson brackets

$$[x_n, x_m] = [p_n, p_m] = 0, \qquad [x_n, p_m] = i\hbar\delta_{nm}. \tag{1.63}$$

where $\hbar = h/2\pi$, h the Planck's constant.

The time evolution of the state vector ψ is governed by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x}, t) = H\psi(\boldsymbol{x}, t), \tag{1.64}$$

where H, the Hamiltonian operator, is the observable associated with the total energy of the system. For a particle of mass m and position operator $\boldsymbol{x}$, interacting with a potential $V(\boldsymbol{x})$, the quantum Hamiltonian operator H takes the form

$$H = -\frac{\hbar^2}{2m} \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + V(\boldsymbol{x}). \tag{1.65}$$

From now on, we will use natural units ($c = \hbar = 1$) and unitary masses.

1.2.2 The problem of quantum integrability

What is quantum integrability and why is it important? To answer this question, it's helpful to consider the classical case. As we know, in classical systems integrability has a significant impact on physical behavior.

However, we currently lack a proper understanding of quantum integrability, which means we may be missing out on some important properties and features. For instance, the absence of a quantum equivalent to the integrability property poses difficulties in making conclusive assertions regarding the equilibration and thermalization of many-body quantum systems [17, 20].

To state a meaningful definition of quantum integrability, we need to meet several requirements [6]:

- It should be unambiguous.

- It should divide all possible quantum models into distinct classes.

- Different classes of models should exhibit distinguishable physical behavior.

Additionally, we should aim for a natural connection to the classical limit.

It may seem natural to extend the classical integrability definition to quantum systems, simply by replacing Poisson brackets with commutators. However, it turns out the matter is not that simple. As written by L. Faddeev [21]:

> *The notion of complete integrability assumes a central place in classical mechanics. Almost all great names in mechanics are associated with the development of this notion and it is vividly discussed in textbooks. It is not so in quantum mechanics. . . quantization is rather an art than a science. Thus, the transfer of the features of classical mechanics into their quantum analog is not automatic. Moreover, these features could be modified in this passage.*

In the search for an extension of the concept of integrability to quantum systems, some critical problems immediately arise [22, 23]. For systems without a classical analog, for example, it is unclear how to assign the parameter N: we think for example to a spin chain of n spins, living in a state space

$$\mathscr{H} = \bigotimes_{j=1}^{n} \mathscr{H}_j, \tag{1.66}$$

where $\mathscr{H}_j$ are the state spaces of the single spins. A seemingly natural extension of the classical definition of integrability to quantum systems could be the following [6]:

Definition 1.8 (Liouville quantum integrability). A quantum system is integrable if it possesses a maximal set of independent N commuting quantum operators $\{O_n\}_{n=1,\dots,N}$, where $N = \dim \mathscr{H}$.

Such definition has a few loopholes: for example, it is not clear what is the meaning of the expression "independent operators". A theorem by von Neumann [24] states that for any number of commuting operators $\{O_n\}_{n=1,2,\dots,N}$, there exist one Hermitian operator O such that all others can be written as functions of O.

Let us consider two commuting Hermitian operators $A, B \in \mathscr{L}(\mathscr{H})$ with discrete, non-degenerate spectra. Given an orthonormal basis of common eigenvectors $\{\psi_n\}_{n=1,2,\dots,N}$ the corresponding spectral decomposition reads

$$A = \sum_{n=1}^{N} a_n P_n, \qquad B = \sum_{n=1}^{N} b_n P_n \tag{1.67}$$

where P_n are projection operators, $P_n \equiv \langle \psi_n, \cdot \rangle \psi_n$ and

$$A \psi_n = a_n \psi_n, \qquad B \psi_n = b_n \psi_n. \tag{1.68}$$

The projector operator P_n can be represented as

$$A = \sum_{n=1}^{N} a_n \prod_{m \neq n}^{N} \frac{B - b_m}{b_n - b_m} = \sum_{m=1}^{N} c_m B^{m-1}. \tag{1.69}$$

where the coefficients c_m are determined in terms of a_n and b_n. In this way, we have shown that any pair of commuting operators A and B are algebraically dependent. On the other hand, there are at most N linearly independent operators constituting a complete set of commuting observables.

Example 1.4. Let us consider the well-known hydrogen atom [25]. Then the Hamiltonian H, the total angular momentum $\boldsymbol{L}^2$ and the third component of the angular momentum L_z constitute a complete set of commuting observables. Measuring H, one finds that the eigenvalue E could be written as

$$E = \frac{E_0}{n^2}, \tag{1.70}$$

with E_0 a negative constant and n an integer. However, an energy measure does not determine the state uniquely, unless $n = 1$. To fully characterize the state of the (spinless) particle, we must perform a measure of $\boldsymbol{L}^2$ and L_z. Hence we need a total number of three measurements to completely determine the state of the system with three quantum numbers (n, l, m).

Now consider the operator

$$O = \frac{E_0}{H} + \frac{\sqrt{2}}{\hbar^2} \boldsymbol{L}^2 + \frac{\sqrt{3}}{\hbar} L_z. \tag{1.71}$$

A measure of the operator O in the state $|\psi\rangle$ gives

$$O |\psi\rangle = (n^2 + \sqrt{2} l + \sqrt{3} m) |\psi\rangle. \tag{1.72}$$

Hence, with a single operator, we can completely characterize the state of the system: only one measure is needed. In general, given a non-degenerate system, one needs only one operator to determine the state of the system.

Despite its problems, the former definition is the most popular in the literature (see for example [16]).

Chapter 2

The Bethe Ansatz

In Chapter 1 we discuss the problem of defining integrability in quantum mechanics. We learnt that the mere existence of conserved charges cannot be a sufficient criterion to classify quantum models into separate classes. Still today there is no consensus on the definition of integrability for quantum systems.

Now, we restrict to systems that support scattering. We introduce the Asymptotic Bethe Ansatz and the consistency condition for systems that exhibit no diffraction. We then present two peculiar nondiffractive models: the Bose gas with δ-potential, known as the Lieb-Liniger model, and the family of quantum Calogero-Moser-Sutherland models.

The chapter closes with the derivation of an essential result concerning periodic boundary conditions of quantum integrable systems: the fundamental relation.

2.1 The Asymptotic Bethe Ansatz

For convenience, let us restrict to systems that support scattering, in the sense that they will *fly apart into pieces unless constricted* [7]. Let us consider a one-dimensional system of N identical (spinless) particles interacting with a pair potential $v(x)$. The Hamiltonian operator reads

$$H = \frac{1}{2} \sum_{n=1}^{N} k_n^2 + \sum_{n,\, m=1,\, n<m}^{N} v(x_m - x_n). \tag{2.1}$$

where we suppose the potential symmetric,

$$v(x_m - x_n) = v(x_n - x_m) \tag{2.2}$$

with the additional requirements

$$\lim_{x \to \infty} v(|x|) = 0, \qquad \lim_{x \to 0} v(|x|) = \infty, \qquad v'(|x|) < 0. \tag{2.3}$$

Example 2.1. Consider two particles ($N = 2$). When the particles are far apart, the wavefunction is the product of two plane waves. Let k_1, k_2 be the initial asymptotic momenta of the particles, with $k_1 > k_2$. Then, since momentum K and energy E are local operators, their spectra are additive. Therefore the total momentum and energy of the system are given by the following expressions:

$$K = k_1 + k_2, \tag{2.4}$$

$$E = \epsilon(k_1) + \epsilon(k_2). \tag{2.5}$$

During the scattering process, total momentum and energy must be conserved. Let then k'_1, k'_2 be the final asymptotic momenta of the particles. Due to the conservation laws, the only possibility is

$$\begin{aligned} k'_1 &= k_2 \\ k'_2 &= k_1. \end{aligned} \tag{2.6}$$

Then the scattering amplitude coincides with a phase shift. The Hamiltonian operator for the two-body problem reads

$$H = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + v(x_1 - x_2). \tag{2.7}$$

We can consider relative coordinates in the reference frame of the center of mass

$$X = \frac{x_1 + x_2}{2}, \qquad x = x_1 - x_2 \tag{2.8}$$

and analogously for the momenta

$$K = k_1 + k_2, \qquad k = \frac{k_1 - k_2}{2}. \tag{2.9}$$

Rewriting the Hamiltonian we get:

$$H = -\frac{1}{2}\frac{1}{2m}\frac{\partial^2}{\partial X^2} - \frac{1}{2}\frac{1}{m/2}\frac{\partial^2}{\partial x^2} + v(x) = H_{\text{CM}}(X) + H_{\text{rel}}(x). \tag{2.10}$$

In the asymptotic regions, we can ignore the potential. Thus, the wavefunction is

$$\Psi(x_1, x_2) = e^{iKX}\left(e^{ikx} - e^{-i\theta(k_1 - k_2)}e^{-ikx}\right). \tag{2.11}$$

Substituting according to eq. (2.8) and eq. (2.9), we get

$$\Psi(x_1, x_2) = e^{i(k_1 + k_2)(x_1 - x_2)}\left(e^{i(k_1 - k_2)(x_1 - x_2)/2} - e^{-i\theta(k_1 - k_2)}e^{-i(k_1 - k_2)(x_1 - x_2)/2}\right). \tag{2.12}$$

Expanding, the asymptotic wavefunction for two-body scattering is then

$$\Psi(x_1, x_2) = e^{i(k_1 x_1 + k_2 x_2)} - e^{-i\theta(k_1 - k_2)}e^{i(k_2 x_1 + k_1 x_2)}. \tag{2.13}$$

The scattering amplitude corresponds to a phase shift $\theta(k_1 - k_2)$. We observe that because of the time-reversal symmetry of the system, when $t \mapsto -t$ and $k \mapsto -k$, $\Psi \mapsto \Psi^\dagger$. Then $\theta(k)$ has to be skew-symmetric:

$$\theta(-k) = -\theta(k). \tag{2.14}$$

Now, we introduce a tool that will be useful in the coming pages. Let us consider again the two-body scattering wavefunction $\Psi(x_1, x_2)$. When particles are far apart, single-particle wave functions are plane waves. Using relative coordinates

we can write two independent asymptotic solutions to the problem $\psi_1(x)$, $\psi_2(x)$, parametrized by the relative momentum k

$$\psi_1(x) = \begin{cases} e^{-ikx} + R(k)e^{ikx}, & x \to -\infty \\ T(k)e^{ikx}, & x \to +\infty \end{cases} \tag{2.15}$$

and

$$\psi_2(x) = \begin{cases} T(k)e^{ikx} & x \to -\infty \\ e^{ikx} + R(k)e^{-ikx} & x \to +\infty, \end{cases} \tag{2.16}$$

where $R(k)$, $T(k)$ are respectively the reflection and transmission coefficients, with

$$|R(k)|^2 + |T(k)|^2 = 1 \tag{2.17}$$

due to the conservation of probability. Returning to the original variables we get

$$\Psi_1(x_1, x_2) = \begin{cases} e^{i(k_1 x_1 + k_2 x_2)} + R(k)e^{i(k_2 x_1 + k_1 x_2)} & x_1 \ll x_2 \\ T(k)e^{i(k_1 x_1 + k_2 x_2)} & x_1 \gg x_2 \end{cases} \tag{2.18}$$

and

$$\Psi_2(x_1, x_2) = \begin{cases} T(k)e^{i(k_1 x_1 + k_2 x_2)} & x_1 \ll x_2 \\ e^{i(k_2 x_1 + k_1 x_2)} + R(k)e^{i(k_1 x_1 + k_2 x_2)} & x_1 \gg x_2. \end{cases} \tag{2.19}$$

The general asymptotic solution has to be a linear combination of Ψ_1, Ψ_2:

$$\Psi(x_1, x_2) = \begin{cases} c_1 e^{i(k_1 x_1 + k_2 x_2)} + (c_1 R(k) + c_2 T(k))e^{i(k_2 x_1 + k_1 x_2)}, & x_1 \ll x_2 \\ c_2 e^{i(k_2 x_1 + k_1 x_2)} + (c_2 R(k) + c_1 T(k))e^{i(k_1 x_1 + k_2 x_2)}, & x_1 \gg x_2 \end{cases} \tag{2.20}$$

where $c_1, c_2 \in \mathbb{C}$. It is convenient to rewrite the expression in terms of new parameters, A, B, C, D, called amplitudes:

$$\Psi(x_1, x_2) = \begin{cases} Ae^{i(k_1 x_1 + k_2 x_2)} + Be^{i(k_2 x_1 + k_1 x_2)}, & x_1 \ll x_2 \\ Ce^{i(k_2 x_1 + k_1 x_2)} + De^{i(k_1 x_1 + k_2 x_2)}, & x_1 \gg x_2 \end{cases} \tag{2.21}$$

Introducing the S-matrix, or scattering matrix S in the reflection-diagonal representation [19]

$$S(k) = \begin{pmatrix} R(k) & T(k) \\ T(k) & R(k) \end{pmatrix}, \tag{2.22}$$

the scattering process is described by the equation

$$\begin{pmatrix} B \\ D \end{pmatrix} = S(k) \begin{pmatrix} A \\ C \end{pmatrix}. \tag{2.23}$$

The S-matrix contains all the information concerning the process under analysis.

Example 2.2. Now, let us consider three particles ($N = 3$). Let k_1, k_2, k_3 be the initial asymptotic momenta of the particles, with $k_1 > k_2 > k_3$. The total momentum and energy of the system are given by

$$K = k_1 + k_2 + k_3 \tag{2.24}$$

$$E = \epsilon(k_1) + \epsilon(k_2) + \epsilon(k_3). \tag{2.25}$$

This time the number of degrees of freedom is increased, but not the number of conservation laws: we cannot say anything about the final asymptotic wavefunction but

$$\Psi(\boldsymbol{x}) = \sum_{\mathcal{P} \in G_3} A(\boldsymbol{k}_{\mathcal{P}}) \exp[i(k_{\mathcal{P}_1} x_1 + k_{\mathcal{P}_2} x_2 + k_{\mathcal{P}_3} x_3)] +$$
$$\iiint_\Omega \mathrm{d}k_1' \, \mathrm{d}k_2' \, \mathrm{d}k_3' \, S(\boldsymbol{k}') \exp[i(k_1' x_1 + k_2' x_2 + k_3' x_3)], \tag{2.26}$$

where the triple integral has to be performed over the region

$$\Omega \equiv \{\boldsymbol{k}' \equiv (k_1', k_2', k_3') \mid k_1' < k_2' < k_3', \ K, \ E \text{ fixed}\}. \tag{2.27}$$

and $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) \in G_3$ denote the index of the ordered particles in a given possible permutation of 1, 2, 3. In general, we denote by G_N the symmetric group of the permutations of N integers.

We can consider an experiment in which two particles collide first, then, after reaching the asymptotic region, they collide with the third one. In this case,

$$\frac{A(k_2, k_1, k_3)}{A(k_1, k_2, k_3)} = -\exp[-i\theta(k_1 - k_2)]. \tag{2.28}$$

The final result is independent of the order of the collisions. Let S_{12}, S_{23} the scattering operators of the first pair and the last pair of particles respectively, the following commutative diagram is recovered:

$$
\begin{array}{ccc}
A(k_1, k_2, k_3) \xrightarrow{S_{12}} A(k_2, k_1, k_3) \xrightarrow{S_{23}} A(k_2, k_3, k_1) \\
\Big\downarrow S_{23} \qquad\qquad\qquad\qquad\qquad \Big\downarrow S_{12} \\
A(k_1, k_3, k_2) \xrightarrow{S_{12}} A(k_3, k_1, k_2) \xrightarrow{S_{23}} A(k_3, k_2, k_1).
\end{array} \tag{2.29}
$$

The outgoing amplitude $A(k_3, k_2, k_1)$ is given by:

$$\frac{A(k_3, k_2, k_1)}{A(k_1, k_2, k_3)} = -\exp[-i(\theta_{12} + \theta_{23} + \theta_{13})] \tag{2.30}$$

where θ_{nm} represent the scattering phase for the interaction between the n-th and the m-th particle. The second term of the wavefunction in eq. (2.26) is due to diffraction.

In general, we are not able to treat the diffractive term in eq. (2.26) for a quantum many-body system. Furthermore, in the presence of diffractive scattering, we cannot expect a third independent conserved quantity. This leads to the conclusion that a system that presents a diffractive behaviour is a non-integrable quantum system. On the other hand, a nondiffractive system can be a good candidate for an integrable quantum system [6, 7]:

Definition 2.1 (Nondiffractive integrability). A quantum system is integrable if the scattering it supports is nondiffractive.

Remark 2.1. Given a nondiffractive quantum system composed of N identical particles, then the asymptotic wavefunction has the form

$$\Psi(\boldsymbol{x}) = \sum_{\mathcal{P} \in G_N} A(\boldsymbol{k}_{\mathcal{P}}) \exp\left[i \sum_{j=1}^{N} k_{\mathcal{P}_j} x_j\right] \tag{2.31}$$

where the $N!$ amplitudes $A(\boldsymbol{k}_{\mathcal{P}})$ are related by

$$\frac{A(\boldsymbol{k}_{\mathcal{P}'})}{A(\boldsymbol{k}_{\mathcal{P}})} = -\exp[-i\theta(k - k')]. \tag{2.32}$$

with $\mathcal{P}, \mathcal{P}' \in G_N$ permutations of incoming momenta between the particles, differing each other by one exchange in the sequence.

The hypothesis on the form of the wavefunction in eq. (2.31) is called the *Asymptotic Bethe Ansatz*.

In the next section we will show that indeed such nondiffractive systems exist.

2.2   Quantum Yang-Baxter equation

Recalling the commutative diagram (2.29), in the presence of non-diffractive scattering, it must hold that

$$\begin{aligned} A(k_3, k_2, k_1) &= S_{12}S_{23}S_{12}A(k_1, k_2, k_3) \\ &= S_{23}S_{12}S_{23}A(k_1, k_2, k_3). \end{aligned} \tag{2.33}$$

Since the amplitudes are arbitrary, then we have

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23} \tag{2.34}$$

which is known as the *consistency condition*, or *quantum Yang-Baxter equation* (QYBE), as independently discovered by C. N. Yang and R. J. Baxter [26, 27]. Eq. (2.34) is depicted in fig. 2.1.
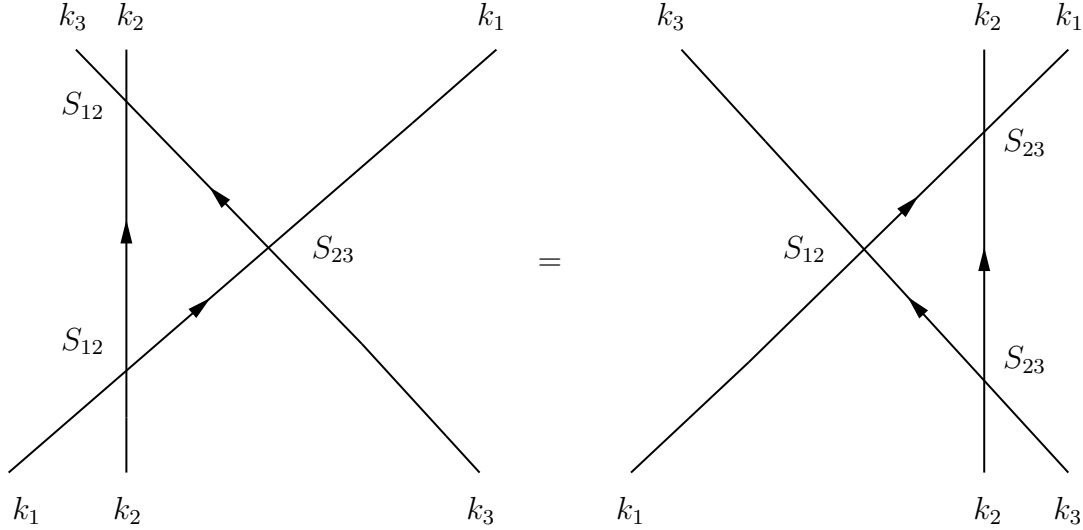
Figure 2.1: The Yang-Baxter equation for the three-body scattering problem.

The expression can be directly generalized to the case of N particles [7]. If $S_{j(j+1)}$ represents the scattering operator for a two-body scattering between the j-th particle and the $(j+1)$-th one, then

$$(S_{j(j+1)}S_{k(k+1)})^{n(j,k)} = I_N. \tag{2.35}$$

with

$$n(j,k) = \begin{cases} 1, & |j-k| = 0 \\ 3, & |j-k| = 1 \\ 2, & |j-k| > 1. \end{cases} \tag{2.36}$$

Remark 2.2. Eq. (2.36) is explained as follows:

1. The first condition is equivalent to the involution property of the scattering operator:

$$S^2_{j(j+1)}(k,k') = S_{j(j+1)}(k',k)S_{j(j+1)}(k,k') = I_N. \tag{2.37}$$

2. The second one reproduces eq. (2.34), providing us with the consistency conditions.

3. The third relation tells us that a two-body scattering of a disjoint pair of non-contiguous particles ($|j-k| > 1$) does commute.

In conclusion, let us briefly mention that nowadays various specifications of the general Bethe Ansatz technique are known

- The *Coordinate Bethe Ansatz*. The original "flavour", introduced by H. Bethe [1] to solve the Heisenberg model.

- The *Nested Bethe Ansatz*. An extension of the coordinate Bethe Ansatz for models with internal degrees of freedom, where scattering entails alterations to the internal states of scatters. It was first proposed by C. N. Yang and M. Gaudin.

- The *Asymptotic Bethe Ansatz.* The one we are interested in. It was introduced and extensively studied by B. Sutherland [7].

- The *Algebraic Bethe Ansatz.* It was realized later that the Bethe Ansatz could be rephrased such that it can be interpreted as the quantum equivalent of the *Classical Inverse Scattering Method* for integrable models. The Algebraic Bethe Ansatz, known as the *Quantum Inverse Scattering Method*, was mainly developed by Faddeev and the Leningrad School [28].

- The *Thermodynamic Bethe Ansatz.* Introduced by C. N. Yang and C. P. Yang [29], this method allows to study the thermodynamics of integrable quantum systems.

2.3   Nondiffractive systems

2.3.1   The Lieb-Liniger model

Let us consider a quantum gas composed of N particles satisfying the Bose-Einstein statistics, whose Hamiltonian is

$$H_{LL} = \frac{1}{2} \sum_{n=1}^{N} k_n^2 + c \sum_{n,\, m=1,\, n \neq m}^{N} \delta(x_n - x_m) \tag{2.38}$$

and $c > 0$. Such a system is known as *the Lieb-Liniger model* [2, 3]. Physically, this is a fairly realistic model for one-dimensional particles with short-range interactions, and it serves as a valuable analytical tool for interpreting experimental data. Its principal limit lies in the fact that the Bethe Ansatz solution requires translational invariance, which is compromised by the trapping utilized in current cold atoms experimental implementations. In fact, because of the external potential, actual systems are inhomogeneous, with particle densities that vary in space.

We state that

Proposition 2.1. *The Bose gas model in one dimension with a δ-function inter-particle potential interaction is nondiffractive.*

Proof. We will show that the Bethe wavefunction is a solution of the Schrödinger equation

$$H_{LL}\Psi(\boldsymbol{x}) = E\Psi(\boldsymbol{x}). \tag{2.39}$$

Let us consider the interaction Hamiltonian for the n-th and the $(n+1)$-th particle. The stationary wavefunction reads

$$\Psi(\boldsymbol{x}) = e^{i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})} \psi_n(x_n, x_{n+1}) e^{i(k_{n+2} x_{n+2} + \cdots + i k_N x_N)} \tag{2.40}$$

and ψ_n satisfies

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2}\right)\psi_n(x_n, x_{n+1}) + c\delta(x_{n+1} - x_n) = E_n \psi_n(x_n, x_{n+1}). \tag{2.41}$$

To simplify the problem, we introduce relative coordinates $y_n = x_{n+1} - x_n$. Then the time-independent Schrödinger equation becomes

$$-\frac{1}{2}\frac{\partial^2}{\partial y_n^2}\psi_n(y_n) + c\delta(y_n) = E_n\psi_n(y_n). \tag{2.42}$$

Integrating in the interval $[y_n - \epsilon, y_n + \epsilon]$, where $\epsilon > 0$ and using the properties of the δ function, we get the condition

$$\frac{\partial\psi_n}{\partial y_n}\bigg|_{\epsilon} - \frac{\partial\psi_n}{\partial y_n}\bigg|_{-\epsilon} = 2c\psi_n(y_n = 0). \tag{2.43}$$

We recall that for a system of bosons, the wavefunction must be symmetric. This implies that the partial derivative with respect to y_j has to be antisymmetric:

$$\frac{\partial\psi_n}{\partial y_n}\bigg|_{\epsilon} = -\frac{\partial\psi_n}{\partial y_n}\bigg|_{-\epsilon} \tag{2.44}$$

and

$$\frac{\partial\psi_n}{\partial y_n}\bigg|_{\epsilon\to 0^+} = c\psi_n(y_n = 0). \tag{2.45}$$

Returning back to the original coordinates, for $\epsilon \to 0^+$, we get the boundary conditions

$$\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_{n+1}}\right)\psi_n\bigg|_{x_{n+1}-x_n\to 0^+} = c\psi_n\bigg|_{x_{n+1}=x_n} \tag{2.46}$$

for $n = 1, 2, \ldots, N - 1$.

Suppose now that the wavefunction has the form of eq. (2.31). Then one could pair the terms relative to the n-th and the $(n + 1)$-th particle as follows

$$\begin{aligned}\Psi(\boldsymbol{x}) &= \cdots + A(\boldsymbol{k}_{\mathcal{P}})\exp[\cdots + ikx_j + ik'x_{j+1} + \ldots] \\ &\quad + A(\boldsymbol{k}_{\mathcal{P}'})\exp[\cdots + ik'x_j + ikx_{j+1} + \ldots] + \ldots,\end{aligned} \tag{2.47}$$

with $\mathcal{P}, \mathcal{P}' \in G_N$ the same permutations but $\mathcal{P}_n = \mathcal{P}'_{n+1}$ and $\mathcal{P}_{n+1} = \mathcal{P}'_n$. Applying the boundary conditions in eq. (2.46), we get

$$[i(k - k') - c]A(\boldsymbol{k}_{\mathcal{P}}) + [i(k' - k) - c]A(\boldsymbol{k}_{\mathcal{P}'}) = 0 \tag{2.48}$$

from which

$$\frac{A(\boldsymbol{k}_{\mathcal{P}'})}{A(\boldsymbol{k}_{\mathcal{P}})} = \frac{k - k' - ic}{k - k' + ic} \equiv -e^{-i\theta(k-k')} \tag{2.49}$$

where

$$\theta(k - k') = -2\arctan\left(\frac{k - k'}{c}\right) \tag{2.50}$$

and $-\pi < \theta(k - k') < \pi$.

We obtained the expression of the two-body scattering amplitude between the n-th and the $(n + 1)$-th particle. This implies that the system is nondiffractive. $\quad\square$

One can write the wavefunction $\Psi(\boldsymbol{x})$ as a sum over all the factorized permutations and amplitudes [8]:

$$\Psi(\boldsymbol{x}) \sim \sum_{\mathcal{P} \in G_N} \text{sgn}(\mathcal{P}) \prod_{n>m} (k_{\mathcal{P}_n} - k_{\mathcal{P}_m} - ic) \exp\left[i \sum_{n=1}^{N} k_{\mathcal{P}_n} x_n\right]. \qquad (2.51)$$

Remark 2.3. In the limit $c \longrightarrow +\infty$, we are dealing with impenetrable bosons or hard-core bosons (*Tonks-Girardeau regime* [30]). Their behaviour is similar to that of free fermions.
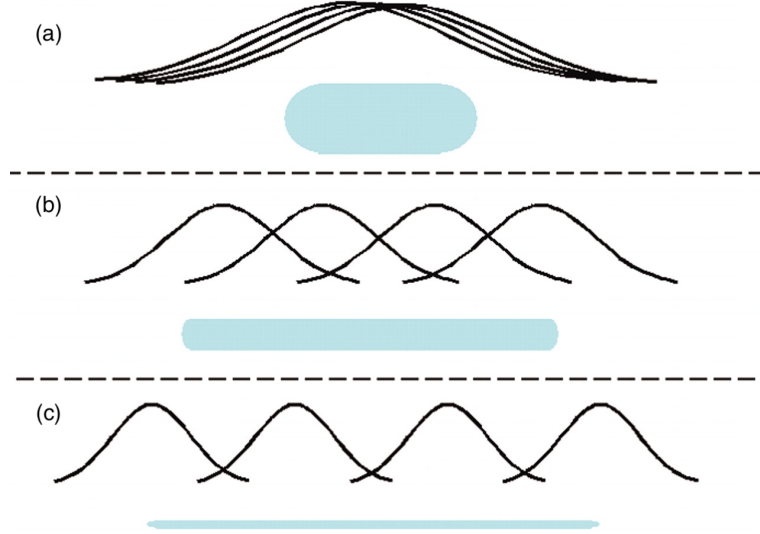

Figure 2.2: Pictorial representation of particle distributions in three c-regimes. When $c \ll 1$ (a), the gas behaves like a fluid. At $c \sim 1$ (b), particles start to separate. When $c \gg 1$, in the Tonks-Girardeau regime (c), bosons become spatially distinct like fermions. Reproduced from [31].

2.3.2 Calogero-Moser-Sutherland models

Let $(\boldsymbol{q}, \boldsymbol{p}) = (q^1, \ldots, q^N, p_1, \ldots, p_N)$ a set of canonical coordinates, with $\{q^n, p_m\} = \delta_m^n$. A dynamical system is said to be a *classical Calogero-Moser system* if it is characterized by the Hamiltonian

$$\mathcal{H}_{CMS} = \frac{1}{2} \sum_{n=1}^{N} p_j^2 + \sum_{n,\,m=1,\,n \neq m}^{N} v(q^n - q^m), \qquad (2.52)$$

with the interparticle potential $v(q)$ satisfying the differential equation

$$v''(q)v(q) - g_1[v'(q)]^2 + g_2[v(q)]^2 = 0, \qquad (2.53)$$

with g_1, g_2 constants. The general solution of eq. (2.53) is given by

$$v(q) = \wp(aq|\omega_1, \omega_2) + \text{const.} \qquad (2.54)$$

where a is an appropriate constant and $\wp(x|\omega_1, \omega_2)$ is the *Weierstrass elliptic $\wp$-function* with periods ω_1, ω_2 (see appendix B for a detailed discussion).

In complete analogy with the classical case, *quantum Calogero-Moser systems* are defined by the Hamiltonian operator

$$H_{CMS} = \frac{1}{2}\sum_{n=1}^{N} k_n^2 + \sum_{n,m=1,\; n\neq m}^{N} v(x_n - x_m), \tag{2.55}$$

with x_n, k_n respectively the position and momentum operators relative to the n-th particle, $[x_n, k_m] = i\delta_{nm}$.

F. Calogero was the first to introduce and solve in a quantum context the 1-dimensional Calogero-Moser system with arbitrary N [32]. The analogous model on the circle was treated by B. Sutherland in [33]. For this reason, such systems are also known as Calogero-Moser-Sutherland (CMS) models. C. Marchioro studied the classical analog on the line [34, 35]. In 1975 J. Moser proved that the classical system is integrable for arbitrary N [36].

To prove that these models are nondiffractive, we need to introduce first the *quantum Lax representations*. In analogy with the classical case, one can define a *quantum Lax pair* for a system with a given Hamiltonian operator H [19]:

Definition 2.2. A quantum Lax representation (L, M) for a system with a given Hamiltonian operator H consists of two square matrices, with operator-valued entries, such that

$$\dot{L} = i[H, L_{nm}] = i[L, M]_{nm} \quad \forall n, m \tag{2.56}$$

where $[H, L]$ denotes the commutator between the two operators, while on the LHS one has the commutator of two matrices.

Calogero's ansatz for Lax representation of Calogero-Moser system reads

$$\begin{cases} M_{nm} = \delta_{nm}\sum_{l\neq n}\gamma(x_n - x_l) + (1 - \delta_{nm})\beta(x_n - x_m). & (2.57a) \\ L_{nm} = \delta_{nm}k_n + i(1 - \delta_{nm})\alpha(x_n - x_m) & (2.57b) \end{cases}$$

where $\alpha(x), \beta(x), \gamma(x)$ are functions to be determined. The study of the functional form of the potential is deferred to the appendix B.

We claim that

Proposition 2.2. *The quantum Calogero-Moser systems are nondiffractive.*

Proof. Let η be an N-dimensional vector with $\eta_j = 1$ $\forall j$. Then

$$M\eta = \eta^\dagger M = 0. \tag{2.58}$$

We can construct constants of motion by

$$L_n = \eta^\dagger L^n \eta \tag{2.59}$$

In fact:

$$[H, L_n] = \eta^\dagger [H, L^n] \eta = \eta^\dagger \sum_{j=0}^{n-1} [L^j [H, L] L^{n-1-j}] \eta$$

$$= \eta^\dagger \sum_{j=0}^{n-1} [L^j (ML - LM) L^{n-1-j}] \eta = \eta^\dagger [ML^{n-1} - L^{n-1} M] \eta = 0. \tag{2.60}$$

Hence, using the Jacobi's relation

$$[H, [L_m, L_n]] + [L_n, [H, L_m]] + [L_m, [L_n, H]] = 0, \tag{2.61}$$

we see that

$$[H, [L_m, L_n]] = 0. \tag{2.62}$$

Thus, the commutator of $[L_n, L_m]$ is a constant of motion for the system. We then consider the Lax matrix

$$L_{nm} = \delta_{nm} k_n + i(1 - \delta_{nm}) \alpha(x_n - x_m). \tag{2.63}$$

Since the system supports scattering, $\alpha(x_n - x_m) \to 0$ as $t \to \pm\infty$. Hence asymptotically L depends on the asymptotic momenta k_n, and the constants of motion L_n are symmetric polynomials in k_n. Therefore,

$$[L_n, L_m] \xrightarrow{t \to \pm\infty} 0. \tag{2.64}$$

But we have proven that $[L_n, L_m]$ are constants of motion for the system: thus, they are identically zero $\forall t$. We then have proven that L_n, $n = 1, 2, \ldots, N$ are N commuting operators for the system under analysis.

Now, we will use the following lemma (whose proof is given in appendix C).

Lemma 2.1. *Fixing the constant of motion L_n for $n \leq N$ for a quantum many-body system composed by N particles of the Calogero-Moser type, the asymptotic momenta $\{k_j\}_{j=1}^N$ are conserved.*

If the asymptotic momenta are forced to be simply rearrangements of the initial ones, it means there's no diffraction. Hence quantum Calogero-Moser systems with inverse-square, trigonometric, and hyperbolic potentials are nondiffractive. $\square$

As in the case of the Lieb-Liniger model, we can determine the two-body scattering phase shift $\theta(k)$. We are especially interested in the case of the inverse-squared potential.

The Schrödinger equation in relative coordinates reads

$$-\frac{d^2}{dx^2} \psi(x) + \frac{\lambda(\lambda - 1)}{x^2} \psi(x) = \left(\frac{k}{2}\right)^2 \psi(x). \tag{2.65}$$

By substituting $z = kx/2$, $\psi(x) = f(z)\sqrt{x}$, we get the differential equation

$$f''(z) + \frac{f'(z)}{z} + f(z) - \frac{(\lambda - 1/2)^2}{z^2} f(z) = 0 \tag{2.66}$$

whose solution is provided by the spherical Bessel function $J_{\lambda - 1/2}(z)$:

$$\psi(x) = J_{\lambda-1/2}\left(\frac{kx}{2}\right)\sqrt{x}. \tag{2.67}$$

In the asymptotic limit, for $x \to +\infty$

$$\lim_{x\to\infty} \psi(x) = \frac{2}{\sqrt{\pi k}} \cos\left(\frac{kx - \pi\lambda}{2}\right) \equiv \frac{1}{\sqrt{\pi k}}\left(e^{i(kx-\pi\lambda/2)} + e^{-i(kx-\pi\lambda/2)}\right). \tag{2.68}$$

Rewriting in the form of eq. (2.13)

$$\psi(x) \sim e^{ikx}e^{-i\pi\lambda/2} + e^{-ikx}e^{i\pi\lambda/2} = e^{-i\pi\lambda/2}\left(e^{ikx} + e^{i\pi\lambda}e^{-ikx}\right)$$
$$\sim e^{ikx} - e^{i\pi(\lambda-1)}e^{-ikx}. \tag{2.69}$$

Hence the phase shift is a constant. Since we are searching for an odd function of k, we multiply the constant for the sign function and finally get

$$\theta(k) = \pi(\lambda - 1)\mathrm{sign}(k). \tag{2.70}$$

2.4   Periodic boundary conditions

We are interested in the study of the thermodynamic limit for systems under analysis, i. e. in the limit of a very large number of particles in an infinite volume, with finite density.

We can think at the thermodynamic limit as considering N particles in a cube of side ℓ or along a ring of circumference, as in fig. 2.3. We then need solutions with periodic boundary conditions.
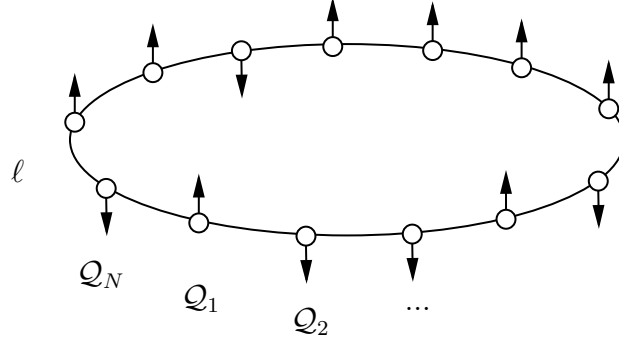

Figure 2.3: N particles with spin along a ring of circumference.

Let us consider N indistinguishable particles in one dimension, confined in a circumference of length ℓ. Particles are ordered, such that

$$x_1 < x_2 < \cdots < x_N. \tag{2.71}$$

Let $\mathcal{Q} = (\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_N)$ be a permutation of integers, by which we wish to indicate a possible ordering of particles (particle $\mathcal{Q}_1$ is at x_1, etc.). Suppose that the first particle is moved along the ring until it returns to the exact same position ($x_1' = x_1 + \ell$). The new configuration is described by $\mathcal{Q}' = (\mathcal{Q}_2, \mathcal{Q}_3, \ldots, \mathcal{Q}_N, \mathcal{Q}_1)$.

We recall the expansion of the asymptotic wavefunction for a system of N ordered identical particles, which scatter without diffraction, shown in eq. (2.31):

$$\Psi(\boldsymbol{x}_{\mathcal{Q}}) = \sum_{\mathcal{P} \in G_N} \operatorname{sgn}(\mathcal{P}) A(\boldsymbol{x}_{\mathcal{Q}} | \boldsymbol{k}_{\mathcal{P}}) \exp\left[\sum_{n=1}^{N} i k_{\mathcal{P}_n} x_m\right] \tag{2.72}$$

where $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N)$ and $\operatorname{sgn}(\mathcal{P})$ represents the signature of the permutation. Analogously, the final wavefunction reads

$$\Psi(\boldsymbol{x}_{\mathcal{Q}'}) = \sum_{\mathcal{P}' \in G_N} \operatorname{sgn}(\mathcal{P}') A(\boldsymbol{x}_{\mathcal{Q}} | \boldsymbol{k}_{\mathcal{P}'}) \exp\left[\sum_{n=1}^{N} i k_{\mathcal{P}'_n} x_n + i k_{\mathcal{P}'_N} L\right] \tag{2.73}$$

with $\mathcal{P}' = (\mathcal{P}_N, \mathcal{P}_1, \ldots, \mathcal{P}_{N-1})$. Since we have periodic boundary conditions, the wavefunction has to be the same:

$$\Psi(\boldsymbol{x}_{\mathcal{Q}}) = \Psi(\boldsymbol{x}_{\mathcal{Q}'}). \tag{2.74}$$

Hence, term by term,

$$A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}}) \exp\left[i\sum_{n=1}^{N} k_{\mathcal{P}_n} x_n\right] = A(\boldsymbol{x}_{\mathcal{Q}'}|\boldsymbol{k}_{\mathcal{P}'}) \exp\left[i\sum_{n=1}^{N} k_{\mathcal{P}'_n} x_n + i k_{\mathcal{P}'_N} L\right]. \qquad (2.75)$$

Since

$$\sum_{n=1}^{N} k_{\mathcal{P}_n} x_n = \sum_{n=1}^{N} k_{\mathcal{P}'_n} x_n, \qquad (2.76)$$

simplifying the appropriate terms we recover

$$A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}}) = A(\boldsymbol{x}_{\mathcal{Q}'}|\boldsymbol{k}_{\mathcal{P}'}) e^{i\ell k_{\mathcal{P}'_N}}. \qquad (2.77)$$

During the transition from $\mathcal{Q}$ to $\mathcal{Q}'$ the system has suffered $N-1$ permutations. Due to the statistics of the identical particles:

$$A(\boldsymbol{x}_{\mathcal{Q}'}|\boldsymbol{k}_{\mathcal{P}'}) = (\pm 1)^{N-1} A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}'}) \qquad (2.78)$$

where the plus or minus sign refers to bosons and fermions respectively. Substituting we get

$$A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}}) = (\pm 1)^{N-1} A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}'}) e^{i\ell k_{\mathcal{P}'_N}} \qquad (2.79)$$

Now, we have to consider the phase shifts. Recalling the result in eq. (2.32), for $N-1$ scatterings the ratio between $A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}'})$ and $A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}})$ is given by

$$\frac{A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}'})}{A(\boldsymbol{x}_{\mathcal{Q}}|\boldsymbol{k}_{\mathcal{P}})} = (\pm 1)^{N-1} \prod_{n=1}^{N-1}\left[-\exp\left[-i\theta(k_{\mathcal{P}'_N} - k_{\mathcal{P}'_n})\right]\right]. \qquad (2.80)$$

Thus, for $\mathcal{P}'_N = n$ we get

$$1 = e^{i k_j \ell} \prod_{m=1,\, m\neq n}^{N}\left[\mp \exp[-i\theta(k_n - k_m)]\right] \qquad (2.81)$$

where the negative sign of the exponential function relates to bosons and the positive one to fermions.

Taking the natural logarithm of both sides, we get for fermions

$$0 = i k_n \ell - 2i\pi h_n - i \sum_{m=1,\, m\neq n}^{N} \theta(k_n - k_m). \qquad (2.82)$$

where h_n are integer quantum numbers due to the periodicity of the complex exponential function.

For bosons

$$0 = i k_n \ell - 2i\pi h_n - i \sum_{m=1,\, m\neq n}^{N} \theta(k_n - k_m) + i(N-1)\pi \qquad (2.83)$$

from which, with a simple manipulation, we write

$$ik_n L = i \sum_{m=1,\, m \neq n}^{N} \theta(k_n - k_m) - 2i\pi\left(h_n - \frac{N-1}{2}\right) \tag{2.84}$$

which let us see that the quantity in brackets is an integer for an odd number of bosons, otherwise half-odd integer.

We can write a single equation for the two cases:

$$k_n \ell = 2\pi I_n + \sum_{m=1,\, m \neq n}^{N} \theta(k_n - k_m). \tag{2.85}$$

where I_n are integers or half-odd integers (for an even number of bosons). This is our fundamental relation: it consists of N coupled equations relating momenta[1] with the so-called Bethe quantum numbers I_n. Eqs. (2.85) are also called Bethe equations.

The phase shift $\theta(k)$ is zero for either free fermions or hard-core bosons ($c \longrightarrow +\infty$ in the Lieb-Liniger model, or $\lambda = 1$ for Calogero-Moser systems). In these limits, the fundamental equations reads

$$k_n = \frac{2\pi}{\ell} I_n \tag{2.86}$$

and the total momentum of a given state is

$$K = \sum_{n=1}^{N} k_n = \frac{2\pi}{\ell} \sum_{n=1}^{N} I_n. \tag{2.87}$$

---

[1]Sometimes they are called quasi-momenta, to stress the fact that they are not observables, in contrast to the physical momentum.

Chapter 3

The Ground State

In the previous chapter, we have seen that nondiffractive systems exist. Arguably, the interest in these models lies in the fact that, in principle, they can be solved exactly. In the following discussion we will show how. Equipped with the powerful tool of fundamental relation, we determine the ground state of the Lieb-Liniger model and the Calogero-Moser system with inverse-squared potential.

Then, we proceed to study the general response of a system in the ground state to a perturbation and characterize the excitations. We investigate the thermodynamics of these models, both at zero and at finite temperature. We close by mentioning some interesting experiments in the field of ultracold atoms.

The complete solution of a one-dimensional quantum-mechanical model requires several steps.

1. First of all, we relate the problem of determining the spectrum of the Hamiltonian to solving the Bethe equations: the original problem of exponential complexity is mapped to one of polynomial complexity.

2. We find the solution of the fundamental relation which corresponds to the ground state. To simplify the discussion, one considers the system in the thermodynamic limit, with $N, V \to \infty$ at constant density.

3. Then, we construct excitations near the ground state and compute the expression for their energy in the thermodynamic limit.

4. The last step is the study of the thermodynamics of the system at zero and finite temperature.

3.1   The ground state

In order to determine the ground state of a nondiffractive system, we use the fact that the quantum numbers of the ground state are the same both for the interacting system and the one in the free fermion or hard-core bosons limit [7].

At the free fermion point, the I_n are the smallest allowed by the Fermi-Dirac statistics:

$$I_n = -\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}-1, \frac{N-1}{2}. \tag{3.1}$$

In the thermodynamic limit, the single-particle momenta k_js are replaced by a continuous variable k and distributed symmetrically in the interval $(-q, q)$. We then introduce a momentum number density distribution $\rho(k)$ such that

$$\rho(k) \equiv \frac{1}{\ell} \frac{\mathrm{d}I(k)}{\mathrm{d}k}. \tag{3.2}$$

Thus, for $N, \ell \to \infty$, with $N/\ell \equiv d$ constant

$$\sum_k \to \ell \int_{-q}^{q} \mathrm{d}k\, \rho(k), \tag{3.3}$$

with the property $\rho(-k) = \rho(k)$. Similarly, the discrete index j in I_j is replaced by a continuous dependence $I = I(k)$.

Written with a single continuous variable, the fundamental equation reads

$$k\ell = 2\pi I(k) + \sum_k \theta(k - k'). \tag{3.4}$$

Taking the derivative with respect to k, by using eq. (3.3) and eq. (3.2) in the thermodynamic limit, we get

$$\frac{1}{2\pi} = \rho(k) + \frac{1}{2\pi} \int_{-q}^{q} \mathrm{d}k'\, \theta'(k - k')\rho(k'). \tag{3.5}$$

This is the fundamental equation of the ground state in the thermodynamic limit. In compact form

$$\frac{1}{2\pi} = \rho + \mathcal{K}\rho = (\mathbb{I} + \mathcal{K})\rho, \tag{3.6}$$

where

$$\mathcal{K}y(k) = \frac{1}{2\pi} \int_{-q}^{q} \mathrm{d}k'\, \theta'(k - k')y(k') \tag{3.7}$$

for a continuous function $y \colon [-q, q] \longrightarrow \mathbb{C}$.

Functions of k are considered as vectors. We denote the unit vector as $\eta(k) = 1$. The space is equipped with the commutative product

$$f \star g \equiv \int_{-q}^{q} \mathrm{d}k\, f(k)g(k). \tag{3.8}$$

We can compute other properties, for example, the particle number density d,

$$d \equiv \lim_{N, \ell \to \infty} \frac{N}{\ell} = \int_{-q}^{q} \mathrm{d}k\, \rho(k) \equiv \eta \star \rho, \tag{3.9}$$

and the ground state energy density e_0,

$$e_0 \equiv \lim_{N, \ell \to \infty} \frac{E_0}{\ell} = \frac{1}{2} \int_{-q}^{q} \mathrm{d}k\, k^2 \rho(k) \equiv \left(\frac{k^2}{2}\right) \star \rho, \tag{3.10}$$

while the momentum density is

$$k_0 \equiv \lim_{N, \ell \to \infty} \frac{K_0}{\ell} = \int_{-q}^{q} \mathrm{d}k \; k\rho(k) \equiv k \star \rho = 0. \tag{3.11}$$

K_0 and E_0 are respectively the total momentum and energy of the ground state.

3.1.1 Ground state of the Lieb-Liniger model

Recalling the two-body scattering phase shift (2.50) determined in the previous chapter for the Lieb-Liniger model

$$\theta(k) = -2\arctan\left(\frac{k}{c}\right), \tag{3.12}$$

the fundamental equation in the thermodynamic limit becomes

$$\rho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \int_{-q}^{q} \mathrm{d}k' \frac{\rho(k')}{c^2 + (k - k')^2}. \tag{3.13}$$

The latter is called the *Lieb-Liniger equation*, and is classified as an *inhomogeneous Fredholm equation of the second type*. It cannot be solved analytically [19]. We can solve it perturbatively around the Tonks-Girardeau limit $c \longrightarrow \infty$. Expanding the kernel $\theta'(k - k')$ into power series in inverse powers of c:

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=0}^{\infty} \int_{-q}^{q} \mathrm{d}k' \frac{(-1)^n (k - k')^{2n}}{c^{2n+1}} \rho(k'), \tag{3.14}$$

one looks for solutions in the form

$$\rho(k) = \sum_{n=0}^{\infty} \frac{\rho_n(k)}{c^n}, \qquad \rho_0(k) = \frac{1}{2\pi}. \tag{3.15}$$

We obtain the perturbative series [19]

$$\rho(k) = \frac{1}{2\pi} \left[1 + 2\left(\frac{q}{\pi c}\right) + 4\left(\frac{q}{\pi c}\right)^2 + \left(8 - \frac{2\pi^2}{3} - \frac{2k^2\pi^2}{q^2}\right)\left(\frac{q}{\pi c}\right)^3 + \ldots \right] \tag{3.16}$$

We can compute the energy density e_0, finding a dependence on the particle density

$$e_0 = \frac{\pi^2 d^3}{3} \left[1 - 4\frac{d}{c} + 12\left(\frac{d}{c}\right)^2 + \ldots \right]. \tag{3.17}$$

It can be shown that the series converges for $d/c < 1/2$.

3.1.2 Ground state of the inverse-squared potential

In the previous chapter we see that, for the inverse-squared potential, the two-body scattering phase shift has the form of eq. (2.70)

$$\theta(k) = 2\pi(\lambda - 1)\mathrm{sign}(k), \tag{3.18}$$

with derivative in the sense of distributions

$$\theta'(k - k') = 2\pi(\lambda - 1)\delta(k - k'). \tag{3.19}$$

Substituting in eq. (3.5) we get

$$1 = 2\pi\rho(k) + 2\pi(\lambda - 1) \int_{-q}^{q} dk' \, \delta(k - k')\rho(k') \tag{3.20}$$

which, due to the properties of the δ-function gives

$$\rho(k) = \frac{1}{2\pi\lambda}, \tag{3.21}$$

for $|k| < q$. Analogously, we can determine d and e_0 for the Calogero-Moser system under analysis:

$$d = \frac{q}{\pi\lambda} \tag{3.22}$$

and

$$e_0 = \frac{q^3}{6\pi\lambda} = \frac{\pi^2\lambda^2 d^3}{6}. \tag{3.23}$$

Except for the rational potential case, the fundamental equation for the momentum number density $\rho(k)$ of the ground state cannot be solved exactly for any other CMS model [7, 19].

Remark 3.1. In the limit for $q \longrightarrow +\infty$, the eq. (3.6) is translationally invariant, and the eigenvectors are plane waves. Then the spectrum of the eigenvalues of the K operator is given by the Fourier transform of the kernel

$$\tilde{\mathcal{K}}(s) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \theta'(k)e^{-iks}. \tag{3.24}$$

For the δ-function potential

$$\tilde{\mathcal{K}}(s) = -e^{-c|s|} \tag{3.25}$$

for $c > 0$. We note that $-1 \le \tilde{\mathcal{K}}(s) \le 0$. For the inverse-squared potential

$$\tilde{\mathcal{K}}(s) = \lambda - 1 > -1. \tag{3.26}$$

Therefore in cases of interest, $\tilde{\mathcal{K}}(s) \ge -1$, and the operator $(\mathbb{I} + \mathcal{K})$ has no zero eigenvalues. This means that there exists an operator $\mathcal{J}$, called resolvent, such that

$$(\mathbb{I} + \mathcal{J}) = (\mathbb{I} + \mathcal{K})^{-1}. \tag{3.27}$$

Consequently, we may solve the eq. (3.6) in terms of ρ:

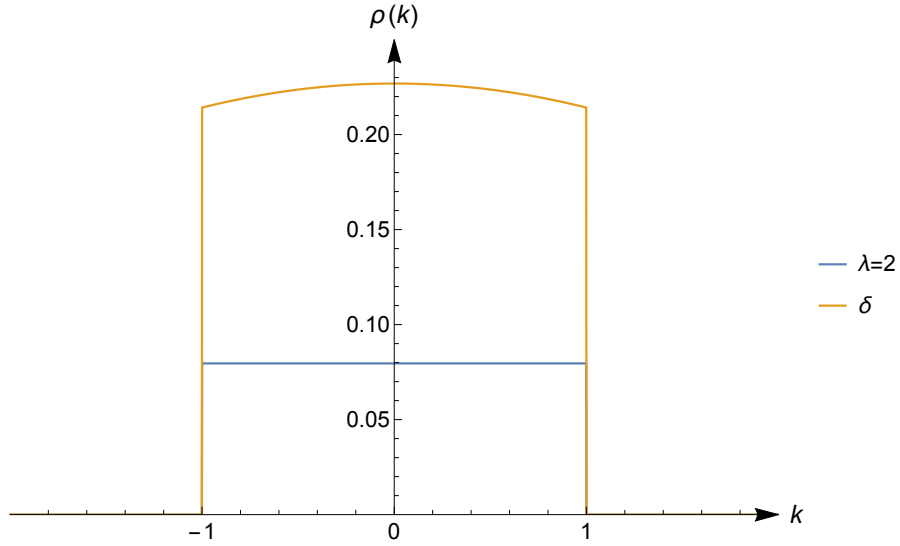$$\rho = \frac{1}{2\pi}(1 + \mathcal{J})\eta. \tag{3.28}$$

Figure 3.1: The density of quasimomenta in the ground state $\rho(k)$ for the Calogero-Moser system with inverse-squared potential with $\lambda = 2$, and the Lieb-Liniger model with $c = 2$. We set $q = 1$.
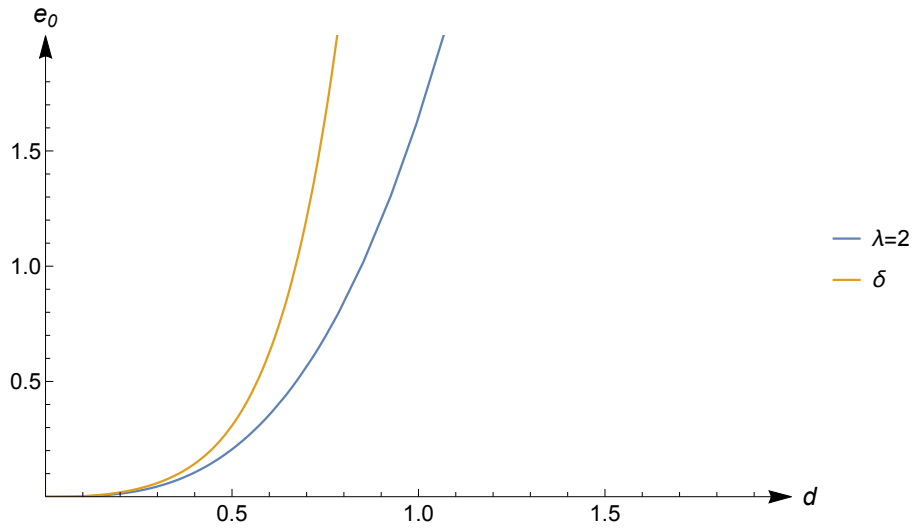

Figure 3.2: The energy density in the ground state as a function of density d, for the Calogero-Moser system with inverse-squared potential with $\lambda = 2$, and the Lieb-Liniger model with $c = 2$. We set $q = 1$.

3.2   Response of the ground state

Now we want to investigate the response of the general nondiffractive system to an external perturbation, at zero temperature. To do this, we will add a perturbation corresponding to a phase shift $\phi(k)$ to our fundamental equation for the ground state. Then, the momenta will shift from the ground state value by a small quantity $\Delta k = \Delta k(k)$. Eq. (3.5) becomes

$$\ell(k + \Delta k(k)) = 2\pi I(k) + \sum_k \theta(k + \Delta k - k' - \Delta k') + \phi(k + \Delta k). \tag{3.29}$$

Expanding in Δk,

$$\ell\Delta k = \sum_k \theta'(k - k')(\Delta k - \Delta k') + \phi(k) \tag{3.30}$$

or, in the thermodynamic limit

$$\ell\Delta k = \ell\Delta k \int_{-q}^{q} dk' \, \theta'(k - k')\rho(k') - \ell \int_{-q}^{q} dk'\theta'(k - k')\rho(k')\Delta k(k') + \phi(k). \tag{3.31}$$

Using eq. (3.5) we may rewrite the former as

$$2\pi\rho(k)\Delta k\ell + \ell \int_{-q}^{q} dk' \, \theta'(k - k')\rho(k')\Delta k(k') = \phi(k), \tag{3.32}$$

and defining

$$\gamma(k) \equiv \rho(k)\Delta k(k)\ell, \tag{3.33}$$

we have

$$(\mathbb{I} + \mathcal{K})\gamma = \frac{1}{2\pi}\phi, \tag{3.34}$$

whose formal solution reads

$$\gamma = \frac{1}{2\pi}(\mathbb{I} + \mathcal{J})\phi. \tag{3.35}$$

We can evaluate the variation in the total momentum

$$\Delta K = \sum_k \Delta k \to \ell \int_{-q}^{q} dk \, \rho(k)\Delta k(k) = \int_{-q}^{q} dk \, \gamma(k) \equiv \eta \star \gamma, \tag{3.36}$$

and then

$$\Delta K = \eta \star \frac{1}{2\pi}(\mathbb{I} + \mathcal{J})\phi = \phi \star \frac{1}{2\pi}(\mathbb{I} + \mathcal{J}) = \phi \star \rho. \tag{3.37}$$

From the total energy

$$E = \frac{1}{2}\sum_k (k + \Delta k)^2 \simeq \frac{1}{2}\sum_k k^2 + \sum_k k\Delta k, \tag{3.38}$$

we derive the variation from the ground state energy E_0, to the first order

$$\Delta E \simeq \sum_k k \Delta k \rightarrow \ell \int_{-q}^{q} \mathrm{d}k \, \rho(k) k \Delta k(k) = \int_{-q}^{q} \mathrm{d}k \, k \gamma(k)$$

$$= k \star \gamma = k \star \frac{1}{2\pi}(\mathbb{I} + \mathcal{J})\phi = \phi \star \frac{1}{2\pi}(\mathbb{I} + \mathcal{J})k = \frac{1}{2\pi}\phi \star \epsilon'$$

$$(3.39)$$

where $\epsilon'(k)$ is the solution of

$$(\mathbb{I} + \mathcal{K})\epsilon'(k) = k. \tag{3.40}$$

We observe that $\epsilon'(k)$ is an odd function of k. In fact

$$(\mathbb{I} + \mathcal{K})\epsilon'(-k) = -k = -(\mathbb{I} + \mathcal{K})\epsilon'(k) = (\mathbb{I} + \mathcal{K})[-\epsilon'(k)]. \tag{3.41}$$

We integrate both sides of the equation in the k variable

$$\int_{-q}^{k} \mathrm{d}k' \, \epsilon'(k') + \int_{-q}^{k} \mathrm{d}k' \, \mathcal{K}\epsilon'(k') = \int_{-q}^{k} \mathrm{d}k' \, k'. \tag{3.42}$$

Now, integrating by parts

$$\mathcal{K}\epsilon'(k) = \int_{-q}^{q} \mathrm{d}t \, \theta'(k - t) \, \epsilon'(t)$$

$$= \theta'(k - t)\epsilon(t)\Big|_{-q}^{q} - \int_{-q}^{q} \mathrm{d}t \, \theta''(k - t)\epsilon(t) \tag{3.43}$$

$$= -\int_{-q}^{q} \mathrm{d}t \, \theta''(k - t)\epsilon(t).$$

using $\epsilon(-q) = \epsilon(q) = 0$. Then

$$\int_{-q}^{k} \mathrm{d}y \, \mathcal{K}\epsilon'(y) = -\int_{-q}^{k} \mathrm{d}y \int_{-q}^{q} \mathrm{d}t \, \theta''(y - t)\epsilon(t) = -\int_{-q}^{q} \mathrm{d}t \, \epsilon(t) \int_{-q}^{k} \mathrm{d}y \, \theta''(y - t)$$

$$= -\int_{-q}^{q} \mathrm{d}t \, \epsilon(t) \int_{-q-t}^{k-t} \mathrm{d}\xi \, \theta''(\xi) = -\int_{-q}^{q} \mathrm{d}t \, \epsilon(t)\theta'(\xi)\Big|_{-q-t}^{k-t}$$

$$= -\int_{-q}^{q} \mathrm{d}t \, \epsilon(t)\theta'(k - t) + \int_{-q}^{q} \mathrm{d}t \, \epsilon(t)\theta'(-q - t)$$

$$= \mathcal{K}\epsilon + \int_{-q}^{q} \mathrm{d}t \, \epsilon(t)\theta'(-q - t).$$

Finally, we get

$$(\mathbb{I} + \mathcal{K})\epsilon = \frac{k^2}{2} - \bar{\mu}, \tag{3.44}$$

with the constant

$$\bar{\mu} \equiv \frac{q^2}{2} + \int_{-q}^{q} \mathrm{d}t \, \theta'(-q - t)\epsilon(t). \tag{3.45}$$

Solving eq. (3.44) formally

$$\epsilon(k) = (\mathbb{I} + \mathcal{J})\frac{k^2}{2} - (\mathbb{I} + \mathcal{J})\overline{\mu} = (\mathbb{I} + \mathcal{J})\frac{k^2}{2} - 2\pi\overline{\mu}\rho(k) \tag{3.46}$$

via eq. (3.28).

Using eq. (3.10) we determine the energy density of the ground state

$$e_0 = \left(\frac{k^2}{2}\right) \star \rho = \left(\frac{k^2}{2}\right) \star \frac{1}{2\pi}(1 + \mathcal{J})\eta = \eta \star \frac{1}{2\pi}(\mathbb{I} + \mathcal{J})\frac{k^2}{2} \tag{3.47}$$

and remembering eq. (3.9) we write

$$e_0 = \eta \star \frac{1}{2\pi}(\epsilon + 2\pi\overline{\mu}\rho) = \frac{1}{2\pi}\eta \star \epsilon + \overline{\mu}d. \tag{3.48}$$

From the fundamental equation (3.4), we see that by increasing the length ℓ of the system from $\ell \rightarrow \ell + \Delta\ell$, we build a perturbation $\phi(k) = -k\Delta\ell$:

$$k\ell = 2\pi I(k) + \sum_k \theta(k - k') - k\Delta\ell. \tag{3.49}$$

From eq. (3.39)

$$\begin{aligned}
\Delta E &= -\frac{1}{2\pi}\Delta\ell k \star \epsilon' \equiv -\frac{1}{2\pi}\Delta\ell \int_{-q}^{q} \mathrm{d}k \; k\epsilon'(k) = \\
&= k\epsilon(k)\Big|_{-q}^{q} + \frac{1}{2\pi}\Delta\ell \int_{-q}^{q} \mathrm{d}k \; \epsilon(k) = \frac{1}{2\pi}\Delta\ell\eta \star \epsilon,
\end{aligned} \tag{3.50}$$

since $\epsilon(q) = \epsilon(-q) = 0$.

Remark 3.2. In a system at zero temperature, with zero entropy, the first principle of thermodynamics reads

$$\Delta E = -p\Delta\ell + \mu\Delta N, \tag{3.51}$$

where p is the pressure and μ the chemical potential. Hence comparing eq. (3.50) with eq. (3.51), we identify the pressure

$$p \equiv \frac{1}{2\pi}\eta \star \epsilon. \tag{3.52}$$

Lastly, comparing the Gibbs-Duhem relation

$$E = -p\ell + \mu N \tag{3.53}$$

with eq. (3.48)

$$E = \frac{1}{2\pi}\eta \star \epsilon\ell + \overline{\mu}N, \tag{3.54}$$

we find that the constant $\overline{\mu}$ previously defined is the chemical potential of the system.

Remark 3.3. Summing up we have determined the response equations of the ground state:

$$\Delta N = 0, \tag{3.55}$$

$$\Delta K = \phi \star \rho, \tag{3.56}$$

$$\Delta E = \frac{1}{2\pi} \phi \star \epsilon'. \tag{3.57}$$

All the equations obtained are linear. This means that a superposition principle applies: the response to a linear combination of perturbations is the linear combination of the single responses.

3.3   Low-lying excitations

3.3.1   Particles

Now suppose we add to our system in the ground state an extra particle with momentum k_p, with $|k_p| > q$. The system's response is a phase shift

$$\phi(k) = \theta(k - k_p). \tag{3.58}$$

The perturbation result is characterized by $\Delta N = 1$ and

$$\Delta K = k_p + \phi \star \rho, \tag{3.59}$$

$$\Delta E = \frac{k_p^2}{2} + \frac{1}{2\pi}\phi \star \epsilon' = \frac{k_p^2}{2} - \frac{1}{2\pi}\epsilon \star \phi'. \tag{3.60}$$

We can rewrite the last as follows:

$$
\begin{aligned}
\phi \star \epsilon' &\equiv \int_{-q}^{q} dk\; \phi(k)\epsilon'(k) = \phi(k)\epsilon(k)\Big|_{-q}^{q} - \int_{-q}^{q} dk\; \phi'(k)\epsilon(k) \\
&= -\int_{-q}^{q} dk\; \phi'(k)\epsilon(k) \equiv -\epsilon \star \phi',
\end{aligned}
\tag{3.61}
$$

using the fact that $\epsilon(q) = \epsilon(-q) = 0$. For bosons, the system stays in the ground state with $N+1$ particles, and it must be $\Delta K = 0$ and $\Delta E = \mu$, from the definition of the chemical potential. For fermions, the situation is different, due to the Pauli exclusion principle.

3.3.2   Holes

By a similar argument, supposing we remove a particle with momentum k_h from the system, results in applying the perturbation

$$\phi(k) = -\theta(k - k_h). \tag{3.62}$$

Now $\Delta N = -1$ and

$$\Delta K = -k_h + \phi \star \rho, \tag{3.63}$$

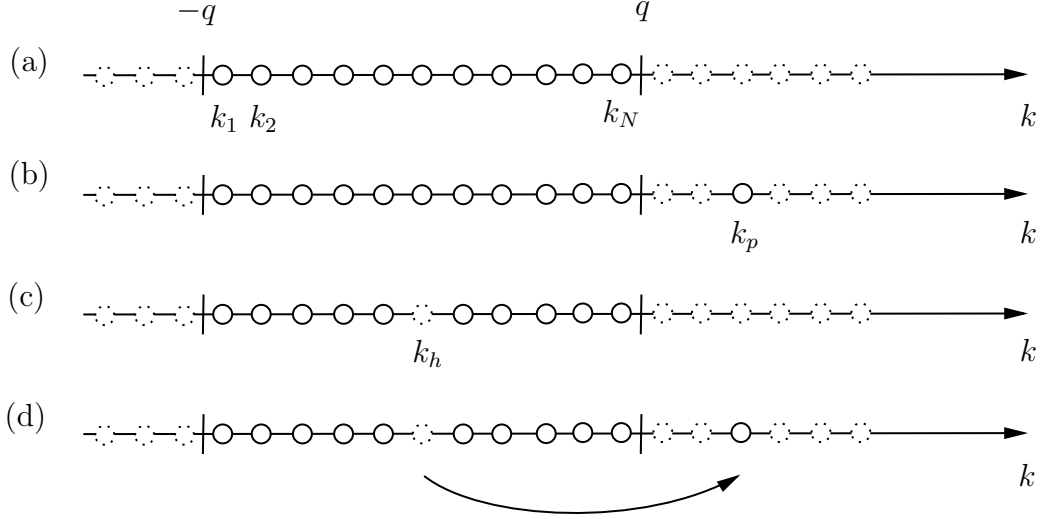$$\Delta E = -\frac{k_h^2}{2} - \frac{1}{2\pi}\epsilon \star \phi'. \tag{3.64}$$

Figure 3.3: Pictorial representation of the ground state and elementary excitations. (a) The ground state. The largest quasimomentum denotes the "Fermi points" $\pm q$. (b) A particle excitation k_p. (c) A hole excitation k_h. (d) A single particle–hole excitation.

3.3.3 Determining the response

We underline the fact that the derivative of the perturbation, ϕ' coincides with the kernel of the $\mathcal{K}$ operator. This greatly simplifies this expression. For particles, by eq. (3.62), we write the energy as

$$\begin{aligned}
\Delta E(k_p) &= \frac{k_p^2}{2} - \frac{1}{2\pi}\epsilon \star \phi' \equiv \frac{k_p^2}{2} - \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k' \; \epsilon(k')\phi'(k') \\
&= \frac{k_p^2}{2} - \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k' \; \epsilon(k')\theta'(k' - k_p) \equiv \frac{k_p^2}{2} - \mathcal{K}\epsilon.
\end{aligned} \tag{3.65}$$

Remembering eq. (3.44), $\mathcal{K}\epsilon = k^2/2 - \mu - \epsilon$, hence

$$\Delta E(k_p) = \mu + \epsilon(k_p). \tag{3.66}$$

We may also obtain a new expression for the momentum. From eq. (3.59) and eq. (3.58)

$$\begin{aligned}
\Delta K(k_p) &= k_p + \phi \star \rho = k_p + \int_{-q}^{q} \mathrm{d}k \; \phi(k)\rho(k) \\
&= k_p + \int_{-q}^{q} \mathrm{d}k \; \theta(k - k_p)\rho(k).
\end{aligned} \tag{3.67}$$

By eq. (3.6) it holds that

$$2\pi\rho = 1 - 2\pi\mathcal{K}\rho. \tag{3.68}$$

Integrating in the k variable

$$2\pi\int_0^k \mathrm{d}k' \; \rho(k') \equiv 2\pi f(k) = k - \int_{-q}^{q} \mathrm{d}k' \; \theta(k - k')\rho(k') \equiv k - \Theta\rho \tag{3.69}$$

with $f' = \rho$, $\Theta' = 2\pi\mathcal{K}$. Then, comparing the latter with eq. (3.67), we get

$$\Delta K(k_p) = k_p + \Theta\rho = 2\pi f(k_p), \tag{3.70}$$

using $-\theta(k - k') = \theta(k' - k)$. In the same way, one can find for holes

$$\Delta E(k_h) = -\mu - \epsilon(k_h) \tag{3.71}$$
$$\Delta K(k_h) = -2\pi f(k_h). \tag{3.72}$$

We can make use of the previous results to compute the group velocity $v(k)$ of the excitations (both particles and holes). Let ω be the angular wave number of the excitation, then using De Broglie's relations

$$v(k) \equiv \frac{d\omega}{dk} = \frac{d\hbar\omega}{d\hbar k} = \frac{\Delta E'(k)}{\Delta K'(k)} = \frac{\epsilon'(k)}{2\pi\rho(k)}, \tag{3.73}$$

with $|k| \geq q$ for particles, $|k| \leq q$ for holes.

As k approaches q, the excitations approach a common Fermi velocity v_f

$$v_f \equiv v(q) = \frac{\epsilon'(q)}{2\pi\rho(q)}. \tag{3.74}$$

This is also the speed of sound v_s in the medium [7].

Remark 3.4. We must pay attention to the fact that $\rho(k)$ and $\epsilon(k)$ are defined for $k \in [-q, q]$, but the particle momentum $k_p > q$. Therefore we define $\rho(k)$ and $\epsilon(k)$ by analytic continuation through the integral eq. (3.6) and eq. (3.44)

$$2\pi\rho(k) \equiv 1 - \int_{-q}^{q} dk'\, \theta'(k - k')\rho(k'), \quad |k| > q; \tag{3.75}$$

$$\epsilon(k) \equiv \frac{k^2}{2} - \mu - \frac{1}{2\pi}\int_{-q}^{q} dk'\, \theta'(k - k')\epsilon(k'), \quad |k| > q. \tag{3.76}$$

3.4 Zero-Temperature Thermodynamics

We are interested in determining the thermodynamic observables in terms of $\rho(q)$ and $\epsilon'(q)$ in the grand canonical ensemble description (i.e., fixing T and μ). We designate the pressure $p(T, \mu)$ as our thermodynamic potential. Then, at zero temperature with zero entropy, the only parameter left is the chemical potential μ. We are therefore interested in determining $\partial p/\partial \mu$ and $\partial^2 p/\partial \mu^2$.

From the Gibbs-Duhem relation

$$\ell \, \mathrm{d}p(T, \mu) = S \, \mathrm{d}T + N \, \mathrm{d}\mu \tag{3.77}$$

or

$$\mathrm{d}p(T, \mu) = s \, \mathrm{d}T + d \, \mathrm{d}\mu. \tag{3.78}$$

where $s \equiv S/\ell$ is the entropy density.

At zero temperature, with $s = 0$ we differentiate by μ, obtaining

$$\frac{\partial p}{\partial \mu} = d. \tag{3.79}$$

For the density

$$d = \eta \star \rho \equiv \int_{-q}^{q} \mathrm{d}k \, \rho(k) = k\rho(k) \Big|_{-q}^{q} - \int_{-q}^{q} \mathrm{d}k \, k\rho'(k) = 2q\rho(q) - k \star \rho', \tag{3.80}$$

using $\rho(k) = \rho(-k)$.

In order to express d in terms of $\rho(q)$ and $\epsilon'(q)$, we need an expression for $\rho'(q)$. We start from the fundamental relation for the ground state, eq. (3.6):

$$(\mathbb{I} + \mathcal{K})\rho = \frac{1}{2\pi}. \tag{3.81}$$

Taking the derivative with respect to k, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}(\mathbb{I} + \mathcal{K})\rho(k) &= \rho'(k) + \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}k}\int_{-q}^{q} \mathrm{d}k' \, \theta'(k - k')\rho(k') \\
&= \rho'(k) + \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k' \, \frac{\partial}{\partial k}\theta'(k - k')\rho(k').
\end{aligned} \tag{3.82}$$

Using $\partial\theta'(k - k')/\partial k = -\partial\theta'(k - k')/\partial k'$ and integrating by parts

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}(\mathbb{I} + \mathcal{K})\rho(k) &= \rho'(k) + \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k' \, \rho(k')\frac{\partial}{\partial k}\theta'(k - k') \\
&= \rho'(k) - \frac{1}{2\pi}\int_{-q}^{q} \mathrm{d}k' \, \rho(k')\frac{\partial}{\partial k'}\theta'(k - k') \\
&= -\frac{1}{2\pi}[\theta'(k - q) - \theta'(k + q)]\rho(q) + \rho'(k) + \mathcal{K}\rho' \\
&= \frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{2\pi} = 0.
\end{aligned} \tag{3.83}$$

Hence

$$(\mathbb{I} + \mathcal{K})\rho'(k) = \frac{1}{2\pi}[\theta'(k-q) - \theta'(k+q)]\rho(q). \tag{3.84}$$

We observe that

$$\mathcal{K}\delta(k) = \frac{1}{2\pi}\int_{-q}^{q} dk'\, \theta'(k-k')\delta(k') = \frac{1}{2\pi}\int_{q+k}^{q-k} dy\, \theta'(y)\delta(y-k) = \theta'(k). \tag{3.85}$$

Taking into account that

$$(\mathbb{I} + \mathcal{J})(\mathbb{I} + \mathcal{K}) = \mathbb{I}, \tag{3.86}$$

thus

$$(\mathbb{I} + \mathcal{J})\mathcal{K} = -\mathcal{J}, \tag{3.87}$$

or

$$(\mathbb{I} + \mathcal{J}) = -\mathcal{J}\mathcal{K}^{-1}. \tag{3.88}$$

We also define $j(k, q)$ as the kernel of the resolvent operator $\mathcal{J}$, namely

$$[\mathcal{J}y](k) = \int_{-q}^{q} dk'\, j(k, k')y(k'). \tag{3.89}$$

Now, we apply on both sides of eq. (3.84) the operator $(\mathbb{I} + \mathcal{J})$:

$$
\begin{aligned}
\rho'(k) &= (\mathbb{I} + \mathcal{J})\frac{1}{2\pi}[\theta'(k-q) - \theta'(k+q)]\rho(q) \\
&\overset{(3.88)}{=} -\mathcal{J}\mathcal{K}^{-1}\frac{1}{2\pi}[\theta'(k-q) - \theta'(k+q)]\rho(q) \\
&\overset{(3.85)}{=} -\mathcal{J}\mathcal{K}^{-1}\mathcal{K}[\delta(k-q) - \delta(k+q)]\rho(q) \\
&= -\mathcal{J}[\delta(k-q) - \delta(k+q)]\rho(q), \\
&\overset{(3.89)}{=} [j(k, -q) - j(k, q)]\rho(q).
\end{aligned}
\tag{3.90}
$$

Hence

$$\rho'(k) = [j(k, -q) - j(k, q)]\rho(q). \tag{3.91}$$

Therefore,

$$
\begin{aligned}
k \star \rho' &\equiv \int_{-q}^{q} dk'\, k'\rho'(k') \\
&\overset{(3.91)}{=} \left(\int_{-q}^{q} dk'\, k' j(k', -q) - \int_{-q}^{q} dk'\, k' j(k', q) \right)\rho(q) \\
&\overset{(3.89)}{=} ([\mathcal{J}k](q) - [\mathcal{J}k](-q))\rho(q).
\end{aligned}
\tag{3.92}
$$

Using eq. (3.40), we get

$$\mathcal{J}k = \mathcal{J}(\mathbb{I} + \mathcal{K})\epsilon' = -\mathcal{K}\epsilon' = \epsilon' - k, \tag{3.93}$$

and finally, combining eq. (3.80) with eq. (3.92) and eq. (3.93), we obtain

$$d = 2q\rho(q) + \rho(q)[\epsilon'(q) - q - [\epsilon'(-q) + q]] = 2\rho(q)\epsilon'(q), \tag{3.94}$$

where we have used the fact that $\epsilon'(q) = -\epsilon'(-q)$, as shown in eq. (3.41).

In the previous section, we have found that the speed of sound in the medium is given by

$$v_s = v_f \equiv \frac{\epsilon'(q)}{2\pi\rho(q)}. \tag{3.95}$$

However, it can be proved by standard thermodynamic arguments (see appendix C) that

$$\frac{1}{v_s^2} = \frac{1}{d}\frac{\partial^2 p}{\partial\mu^2}. \tag{3.96}$$

Then, combining eq. (3.94) with eq. (3.95) and eq. (3.96) we finally get

$$\frac{\partial^2 p}{\partial\mu^2} = \frac{8\pi^2\rho^3(q)}{\epsilon'(q)}. \tag{3.97}$$

Remark 3.5. We have determined

$$\frac{\partial p}{\partial\mu} = 2\rho(q)\epsilon'(q), \tag{3.98}$$

$$\frac{\partial^2 p}{\partial\mu^2} = \frac{8\pi^2\rho^3(q)}{\epsilon'(q)} \tag{3.99}$$

as functions of ρ, ϵ' only, as desired.

3.5   Finite-Temperature Thermodynamics

At $T > 0$, thermal energy will create a finite number of holes. As done previously, we could define an effective density of quantum numbers $f'(k)$ such that:

$$f(k) = \int_0^k \mathrm{d}k'\, f'(k') \equiv \frac{I(k)}{\ell}. \tag{3.100}$$

Similarly, we define momentum number densities $\rho(k)$ and $\rho_h(k)$ for particles and holes respectively, such that, for any interval $(k, k + \mathrm{d}k)$, $\ell\rho(k)\mathrm{d}k$ gives the number of particles and $\ell\rho_h(k)$ gives the number of holes. It must hold

$$\ell f'(k)\mathrm{d}k = \ell[\rho(k) + \rho_h(k)]\mathrm{d}k. \tag{3.101}$$

or

$$f'(k) = \rho(k) + \rho'(k). \tag{3.102}$$

The fundamental relation reads

$$2\pi\ell f(k) = \ell f(k) - \sum_{n=1}^N \theta(k - k_n), \tag{3.103}$$

with $\theta(0) = 0$. Taking the derivative with respect to k, in the thermodynamic limit one has

$$2\pi f' = 2\pi(\rho + \rho_h) = 1 - \int_{-\infty}^{\infty} \mathrm{d}k'\, \theta'(k - k')\rho(k') \equiv 1 - 2\pi\mathcal{K}\rho, \tag{3.104}$$

where the $\mathcal{K}$ operator has been extended to the whole real line.

At finite temperature, the entropy S is finite. We start calculating the degeneracy for free fermions

$$w(N, N_h, \ell) = \frac{(N + N_h)!}{N!\, N_h!} = \frac{[\ell f'(k)\mathrm{d}k]!}{[\ell\rho(k)\mathrm{d}k]!\, (\ell\rho_h(k)\mathrm{d}k]!}. \tag{3.105}$$

Using Stirling's approximation $N! = N \ln N - N + O(N)$, we compute the differential of the entropy

$$\mathrm{d}S \equiv \ln w(N, N_h, \ell) \approx \ell[f' \ln f' - \rho \ln \rho - \rho_h \ln \rho_h]\mathrm{d}k. \tag{3.106}$$

We compute the pressure p via the Euler relation $E = TS - p\ell + \mu N$:

$$\begin{aligned}
p(T, \mu) &= \frac{TS + \mu N - E}{\ell} = Ts + \mu d - e \\
&= \int_{-\infty}^{\infty} \mathrm{d}k \left[T(f' \ln f' - \rho \ln \rho - \rho_h \ln \rho_h) + \left(\mu - \frac{k^2}{2}\right)\rho \right],
\end{aligned} \tag{3.107}$$

where we used the fact that

$$d = \int_{-\infty}^{\infty} \mathrm{d}k\, \rho(k), \qquad e = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}k\, k^2 \rho(k). \tag{3.108}$$

The equilibrium distribution $\bar{\rho}$ is recovered by maximizing p with respect to ρ. We impose

$$\delta p(\rho) = 0, \tag{3.109}$$

obtaining

$$\int_{-\infty}^{\infty} dk \left[\delta\rho_h \ln \frac{f'}{\rho_h} + \delta\rho \left(\ln \frac{f'}{\rho} + \frac{\mu - k^2/2}{T} \right) \right] = 0. \tag{3.110}$$

From eq. (3.104), using the fact that $f' = \rho + \rho_h$, we get by differentiating

$$\delta\rho_h = -(\mathbb{I} + \mathcal{K})\delta\rho. \tag{3.111}$$

Now substituting eq. (3.111) into eq. (3.110) we find

$$T \ln \frac{\rho}{\rho_h} - T\mathcal{K} \ln \left(1 + \frac{\rho}{\rho_h} \right) + \mu - \frac{k^2}{2} = 0. \tag{3.112}$$

To simplify the former, we introduce the function

$$\epsilon(k) \equiv T \ln \frac{\rho_h(k)}{\rho(k)}. \tag{3.113}$$

Then eq. (3.112) reads

$$\epsilon - T\mathcal{K} \ln \left(1 + e^{-\epsilon/T} \right) + \mu - \frac{k^2}{2} = 0 \tag{3.114}$$

or, explicitly

$$\frac{k^2}{2} - \mu = \epsilon(k) - \frac{T}{2\pi} \int_{-\infty}^{\infty} dk' \, \theta'(k - k') \ln \left(1 + e^{-\epsilon(k')/T} \right) \tag{3.115}$$

Eq. (3.115) is called *the Yang-Yang equation* [29] and it constitutes the first step toward the so-called *Thermodynamic Bethe Ansatz* (TBA).

Remark 3.6. The interpretation of the function $\epsilon(k)$ is given adding $\rho(k)$ to both sides of the eq. (3.113)

$$e^{\frac{\epsilon(k)}{T}} \rho(k) + \rho(k) = \rho_h(k) + \rho(k) \tag{3.116}$$

from which we get

$$\frac{\rho(k)}{\rho(k) + \rho_h(k)} = \frac{1}{1 + e^{\frac{\epsilon(k)}{T}}}. \tag{3.117}$$

We recognize the RHS as the Fermi-Dirac distribution. The function $\epsilon(k)$ represents the dressed energy per particle excitation [9] and it is sometimes called *pseudo-energy*. In other words, $\epsilon(k)$ tells us how much energy we need to add to a system to increase its number of excitations by one. This is a useful quantity to know because it can help us understand how energy is distributed in a system and how it can be transferred between different levels of excitation.

We now compute the pressure using eq. (3.117):

$$
\begin{aligned}
p(T,\mu) &= \int_{-\infty}^{\infty} dk\, \rho(k)\left[\mu - \frac{k^2}{2} + T\left(\frac{f'}{\rho}\ln\frac{f'}{\rho} - \frac{\rho_h}{\rho}\ln\frac{\rho_h}{\rho}\right)\right] \\
&= \int_{-\infty}^{\infty} dk\, \rho(k)\left[\mu - \frac{k^2}{2} + T(1 + e^{\epsilon T})\ln\left(1 + e^{\epsilon/T}\right) - \epsilon e^{\epsilon/T}\right] \qquad (3.118) \\
&= \int_{-\infty}^{\infty} dk\, \rho(k)\left[\mu - \frac{k^2}{2} + \epsilon + T(1 + e^{\epsilon T})\ln\left(1 + e^{-\epsilon/T}\right)\right].
\end{aligned}
$$

We rewrite eq. (3.104) in the form

$$
(1 + e^{\epsilon/T})\rho = \frac{1}{2\pi} - \mathcal{K}\rho. \qquad (3.119)
$$

in order to simplify eq. (3.118):

$$
p(T,\mu) = \int_{-\infty}^{\infty} dk\, \rho(k)\left[\mu - \frac{k^2}{2} + \epsilon - \mathcal{K}\ln\left(1 + e^{-\epsilon T}\right)\right] + \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln\left(1 + e^{-\frac{\epsilon}{T}}\right). \qquad (3.120)
$$

Finally, using eq. (3.114), the first term is equal to zero, and we obtain a useful expression for the pressure

$$
p(T,\mu) = \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln\left(1 + e^{-\frac{\epsilon(k)}{T}}\right). \qquad (3.121)
$$

The other thermodynamic quantities can be derived from the pressure $p(T,\mu)$:

$$
\begin{aligned}
d &= \left(\frac{\partial p}{\partial \mu}\right)_T, & s &= \left(\frac{\partial p}{\partial T}\right)_\mu, \\
\kappa &= \left(\frac{\partial^2 p}{\partial \mu^2}\right)_T, & c_V &= \left(\frac{\partial^2 p}{\partial T^2}\right)_\mu,
\end{aligned} \qquad (3.122)
$$

where κ is the compressibility, and c_V the specific heat at constant volume.

Example 3.1. We apply the method to the rational CMS model, starting with the Yang-Yang equation. By appropriately substituting eq. (2.70) into eq. (3.114), one has

$$
\mu - \frac{k^2}{2} = T(\lambda - 1)\ln\left(1 + e^{-\frac{\epsilon(k)}{T}} - \epsilon(k)\right) \qquad (3.123)
$$

or

$$
e^{\mu/T - k^2/2T} = e^{-\epsilon/T}(1 + e^{-\epsilon/T})^{\lambda - 1} \qquad (3.124)
$$

which is transcendental.

We can determine the pressure at finite temperature

$$
p(T,\mu) = \frac{T}{2\pi}\int_{-\infty}^{\infty} dk\, \ln\left(1 + e^{-\epsilon(k)/T}\right) \qquad (3.125)
$$

Let the fugacity be $z \equiv e^{\mu/T}$. We define the following quantities [37]

$$\alpha \equiv e^{\frac{\mu}{T} - \frac{k^2}{2T}} = z e^{-\frac{k^2}{2T}}, \tag{3.126}$$

$$\zeta \equiv 1 + e^{-\frac{\epsilon(k)}{T}}, \tag{3.127}$$

$$w \equiv \ln \zeta(k, \mu). \tag{3.128}$$

In terms of the z, α, ζ, w parameters, eq. (3.124) becomes

$$\alpha = \zeta^{\lambda-1}(\zeta - 1) = 2 e^{w(\lambda - 1/2)} \sinh \frac{w}{2}. \tag{3.129}$$

We expand w in power series of α:

$$w \equiv \ln \zeta = \sum_{n=1}^{\infty} c_n \alpha^n. \tag{3.130}$$

Now, we rewrite eq. (3.125) in terms of the new parameters

$$\begin{aligned}
\frac{p}{T} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, w(k) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \sum_{n=1}^{\infty} c_n \alpha^n \equiv \frac{1}{2\pi} \sum_{n=1}^{\infty} c_n e^{\frac{n\mu}{T}} \int_{-\infty}^{\infty} dk \, e^{-\frac{n}{2T} k^2}.
\end{aligned} \tag{3.131}$$

Using

$$\int_{-\infty}^{\infty} dt \, e^{-at^2} = \sqrt{\frac{\pi}{a}}, \tag{3.132}$$

we get

$$\frac{p}{T^{3/2}} = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n}} e^{\frac{n\mu}{T}}. \tag{3.133}$$

We set $p_\lambda(z) \equiv p(T, \mu)/T^{3/2}$ (the temperature serves as a scaling factor) and the former becomes

$$p_\lambda(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} b_n z^n. \tag{3.134}$$

Comparing expansions in eq. (3.133) and eq. (3.134), it holds

$$b_n = \frac{c_n}{\sqrt{n}}. \tag{3.135}$$

Inverting eq. (3.129), we find [37]

$$c_n = \frac{(-1)^{n+1} \, \Gamma(\lambda n)}{\sqrt{n} \, n! \, \Gamma(n(\lambda - 1) + 1)}. \tag{3.136}$$

Finally, we recover the *virial expansion* for the pressure:

$$p_\lambda(z) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} b_n z^n = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \, \Gamma(\lambda n)}{\sqrt{n} \, n! \, \Gamma(n(\lambda - 1) + 1)} z^n, \tag{3.137}$$

with a radius of convergence R, given by the Cauchy-Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|b_n|} = \frac{\lambda^\lambda}{(\lambda-1)^{\lambda-1}}, \tag{3.138}$$

i.e.

$$R = \frac{(\lambda-1)^{\lambda-1}}{\lambda^\lambda}. \tag{3.139}$$

Once the pressure has been determined, one can compute the density as

$$d(T,\mu) = \frac{\partial p(T,\mu)}{\partial \mu} = \frac{1}{\sqrt{2\pi T}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}\, \Gamma(\lambda n)}{n!\, \Gamma(n(\lambda-1)+1)} e^{\frac{n\mu}{T}}, \tag{3.140}$$

with the same radius of convergence R.
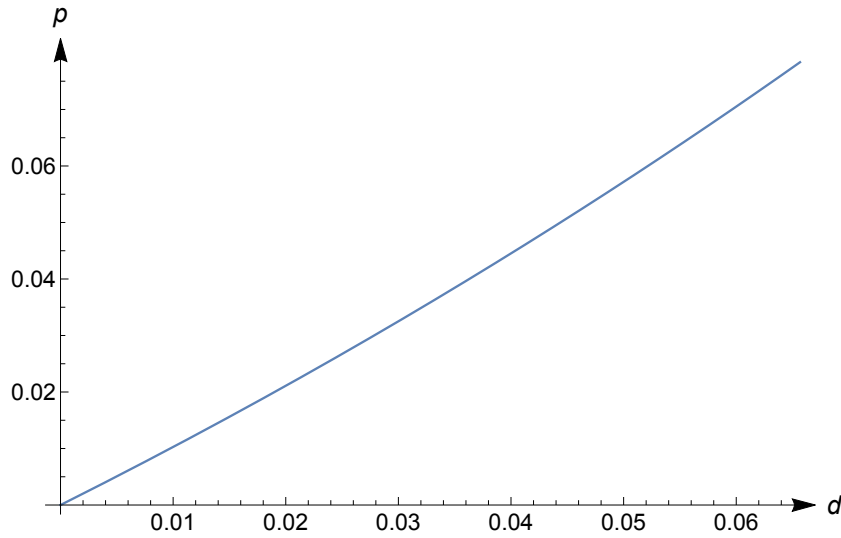
Figure 3.4: Plot of the pressure in terms of density for the inverse-squared potential (we set $T = 1$).

The plot was obtained with Wolfram Mathematica 13.0 using the following code:

```
We set  T = 1, z the fugacity .

In[1]:= p[z_, λ_] := ─────  Sum[ ──────────────────────  zⁿ, {n, 1, ∞}]
              1        (-1)ⁿ⁺¹ Gamma[λ n]
             √2 π      √n n! Gamma[n (λ - 1) + 1]

In[2]:= d[z_, λ_] := ─────  Sum[ ──────────────────────  zⁿ, {n, 1, ∞}]
              1       √n  (-1)ⁿ⁺¹ Gamma[λ n]
             √2 π        n! Gamma[n (λ - 1) + 1]

In[3]:= ListLinePlot[Table[{N[d[i, 2]], N[p[i, 2]]}, {i, 0, 0.24, 0.01}], Axes → True,
        PlotRange → All, AxesLabel → {Style["d", Italic, 18], Style["p", Italic, 18]},
        AxesStyle → Arrowheads[{0.0, 0.03}], ImageSize → Large]
```

One can indeed apply this machinery to the Lieb-Liniger model, as done in [38].

3.6 Experiments with ultracold atoms

The study of exactly solvable models has seen a renewed interest in recent decades due to the experimental realization of one-dimensional integrable models through *ultracold atoms* [39].

Over the last two decades, incredible strides in quantum engineering have enabled the creation of genuine quantum many-body systems using ultracold atoms. This breakthrough has resulted in the successful realization of some prototypical Yang-Baxter systems with exact solutions. By comparing these solutions to their experimental counterparts, we are presented with an opportunity to test the mathematical physics of Bethe ansatz integrable models.

(i)   (ii)
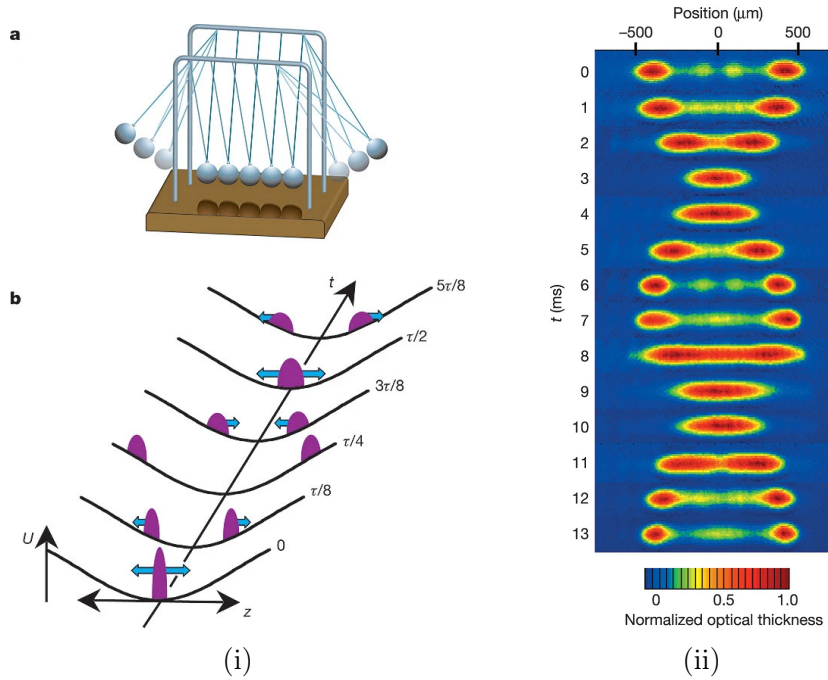
Figure 3.5: (i) A classical Newton's cradle (a) and a quantum one (b). The latter involves two parts of a cloud of quasi-1D trapped Bose gas in momentum space, which oscillate out-of-equilibrium for several dozens of milliseconds.
(ii) Momentum distributions of the quasi-1D trapped Bose gas read through absorption images during the first oscillation cycle. The gas was trapped and stimulated by the grating pulses, resulting in a superposition of $\pm \hbar k$ momentum. The separated clouds moved back and forth without any significant equilibration.
Reproduced from [40].

The use of ultracold atoms has provided a powerful tool for simulating quantum systems. This is because the parameters of ultracold atom systems, such as the interaction strength and dimension, can be finely tuned. This versatility has allowed researchers to simulate a wide range of quantum systems, including Bose-Einstein condensates and Tonks-Girardeau gases.

Thus far, the experimental outcomes have exhibited an incredible level of agreement with the predictions from the exactly solved models [5].

One of the most important contributions of ultracold atoms to the study of exactly solvable models is the experimental realization of a one-dimensional Bose

gas where the interaction strength is much greater than the kinetic energy (i.e., the Lieb-Liniger model in the limit of a Tonks-Girardeau gas).

The Lieb-Liniger Bose gas, which is one of the most studied exactly solvable models from an experimental viewpoint, has been used to simulate a variety of quantum phenomena, including ground state properties and Yang-Yang thermodynamics.

In a study involving ultracold bosons [40], researchers confined the particles to one dimension. Without great detail, the experiment involved creating a 2D array of 1D tubes with tight confinement in two directions and weak confinement in the third direction. Each tube contained between 40 to 250 atoms. After releasing the potential, the researchers measured the density profile, which corresponded to the momentum distribution $\rho_{\text{ex}}(k)$ after a time of flight. Over thousands of collisions between the clouds, they observed the separation of two clouds in the ensemble of quasi-1D trapped Bose gases of ^{87}Rb atoms. The measured distribution showed oscillations of two separated clouds with Newton's cradle dynamics in each measurement at different times.
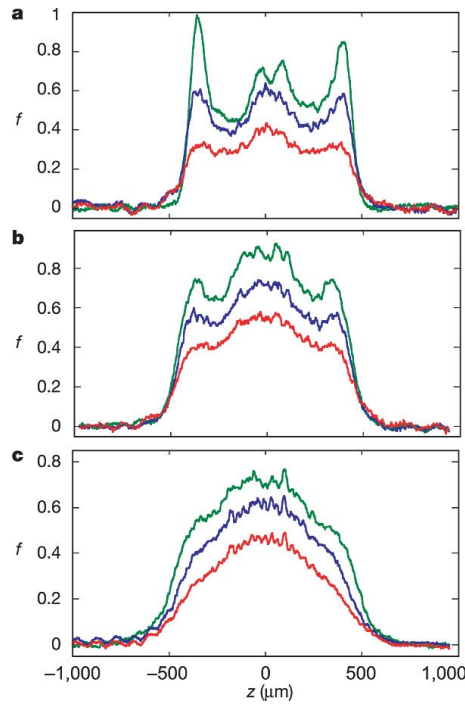
Figure 3.6: The momentum distribution, $f(k) \equiv \rho(k)$, for three values of c. The spatial position, z, is approximately proportional to the quasimomentum variable, k. (a), $c = 4$; (b), $c = 1$; and (c), $c = 0.62$. The green curve is the average of $f(k)$ from the first cycle for each set.
The lower curves in each set are the distributions $f(k)$ taken at single times. The changes in the distribution with time are attributable to known loss and heating. One can compare the experimental results with theoretical one (see fig. 3.1). Reproduced from [40].

The experiment has gained significant theoretical interest: in a research paper [41], the authors provided a numerical analysis of the time evolution of the trapped Lieb-Liniger gas to demonstrate the emergence of the quantum dynamics of Newton's cradle.

The above experiment was only the starting point of a rapidly developing strand. For an updated review, see [5].

On the other hand, there are no direct experiments with quantum Calogero-Moser-Sutherland models. However, there have been a number of experiments with systems that exhibit similar properties. For example, the *quantum Hall effect* can be understood as a realization of the Calogero-Sutherland model with a long-range interaction [42]. In the quantum Hall effect, electrons are confined to move in two dimensions in a strong magnetic field. This magnetic field causes the electrons to form quantized Landau levels, where the electrons are restricted to move in circles with a certain radius. As a result, the wave functions in the lowest Landau level can be expressed in one-dimensional form. This allows for a connection between the two-dimensional quantum Hall effect and the one-dimensional $1/x^2$ model.

In a 2014 paper [43], Y. Yu et al. proposed a way to realize the Calogero-Sutherland potential with cigar-shaped Bose-Einstein condensates in optical lattices. They showed that the magnetic dipole-dipole interaction between the atoms would give rise to an effective inverse-squared interaction between the *Wannier modes* of the BECs. This would make it possible to realize Calogero-Sutherland systems in a controlled way.

The paper examines a cold Bose alkali atom cloud with a small magnetic dipole moment ($\sim 1\mu_B$). In the presence of strong confinement in the z-direction, the cloud has a pancake shape trapped in the xy plane. By applying a periodic optical potential along the x-direction, the system is transformed into a lattice of cigar-shaped gases. If an external field is applied, all the dipoles will be polarized along the field direction, as in fig. 3.7i. One can show that the effective Hamiltonian [43]

$$H_{\text{eff}}^{(i)} = \sum_{n=1}^{N} \left(-\frac{\hbar^2}{2m_R}\frac{\mathrm{d}^2}{\mathrm{d}x_n^2} + \frac{m_R\omega_R^2}{2}x_n^2 \right) + \sum_{n<m}^{N} \frac{G}{(x_n - x_m)^2}, \qquad (3.141)$$

which coincides with the Hamiltonian operator of a quantum Calogero-Moser system with inverse-squared potential, with an additional external harmonic potential. m_R, ω_R, G are some constant parameters which depend on the setup.
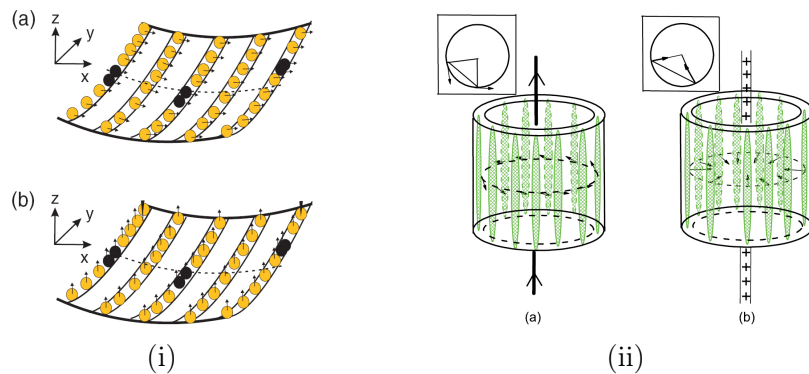

Figure 3.7: (i) The array of cigar-shaped cold atom clouds on an optical lattice with different polarizations of dipoles. The yellow spots are the BEC substrate and the black spots are the quasi-atom with the inverse-squared interaction.
(ii) Cigar-shaped BECs (green) confined to a cylinder. Magnetic (a) and electric (b) dipoles. Reproduced from [43].

The authors also consider the geometry in fig. 3.7ii. In this case, one may prove

that, up to a constant, the dipole-dipole interaction is a periodic inverse-squared interaction as in the Sutherland model ($\pm$ refers to fig. 3.7ii (a) and (b) respectively. L_x, λ are some constants):

$$V_{\text{eff}}(\boldsymbol{x}) \approx \pm \sum_{n \neq m} \frac{\pi^2 \lambda(\lambda - 1)}{L_x^2 \sin^2 \pi (x_n - x_m)/L_x}. \tag{3.142}$$

Such realization would be a major breakthrough in cold atom physics. It would allow physicists to study the properties of the Calogero-Sutherland model in a controlled experimental setting. This could lead to new insights into the physics of long-range interacting systems and quantum phase transitions.

We have thus shown how the topics introduced in this dissertation provide theoretical support for current physics experiments.

Conclusions

In this dissertation, we introduced fundamental concepts and techniques related to the study of one-dimensional integrable quantum many-body systems.

In the first chapter, we recalled the notion of Liouville integrability for classical systems and discussed the problem of a definition of quantum integrability. Although it may seem natural to extend the classical integrability definition to quantum systems, simply by replacing Poisson brackets with commutators, it turns out the matter is not that obvious. We reviewed the most popular definitions in the literature, stressing the problems of each proposal. We learnt that the mere existence of conserved charges cannot be a sufficient criterion to classify quantum models into separate classes. Still today there is no consensus on the definition of integrability for quantum systems.

We then restricted ourselves to the special class of quantum many-body systems that support scattering: Bethe's technique allows us to determine whether a system is integrable, as solvability by the ansatz is a clear signature of the system's integrability. We therefore posed the Bethe Ansatz as our new definition of quantum integrability: namely, a quantum system is integrable if the scattering it supports is nondiffractive. We also derived consistency conditions, or quantum Yang-Baxter equations, for systems that exhibit no diffraction. We then introduce the Lieb-Liniger model and the family of Calogero-Moser-Sutherland models as examples of quantum integrable systems, proving that they are nondiffractive and computing the two-body scattering phase shifts.

In the third chapter, we showed the power of Bethe's method: we were able to determine the ground state of the Lieb-Liniger model and the Calogero-Moser system with inverse-squared potential. Furthermore, we studied the general response of a system in the ground state to an external perturbation and characterized the elementary excitations. En passant, we also compute the velocity of sound for quantum systems under analysis. We determined a formula to compute the pressure for these models, both at zero and at finite temperature, and derived the Yang-Yang equation for the dressed energy per particle excitations. By these relations, we were able to completely characterize the thermodynamics of a quantum integrable system.

Finally, we briefly mentioned some interesting experiments in the field of ultracold atoms: over the last decades, significant progress in experimental control and manipulation capabilities has allowed researchers to create and study quantum systems that are difficult or impossible to study using other methods. Integrable Bethe ansatz quantum models have been successfully tested with ultracold atoms in several experiments. For example, researchers have used ultracold atoms to test the Lieb-Liniger model. These experiments have provided important insights into the behavior of these models and have helped to confirm the validity of the Bethe ansatz approach. On the other hand, while there have been experiments with sys-

tems similar to quantum Calogero-Moser-Sutherland models, no direct experiments have been conducted. However, some proposals exist to realize such systems.

Appendix A

The Liouville-Arnol'd theorem

In Chapter 1 we stated the Liouville-Arnol'd theorem on complete integrability:

Theorem 1.1 (Liouville-Arnol'd). *If a Hamiltonian system with N degrees of freedom has N integrals of motion $\mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2, \ldots, \mathcal{H}_N$, ($\mathcal{H}_n \colon \Gamma \longrightarrow \mathbb{R} \ \forall n = 1, \ldots, N$)*

1. *functionally independent and analytic in their variables,*

2. *in involution, i.e. $\{\mathcal{H}_n, \mathcal{H}_m\} = 0 \ \forall n \neq m$,*

3. *with the additional condition*

$$\det \frac{\partial(\mathcal{H}_1, \ldots, \mathcal{H}_N)}{\partial(p_1, \ldots, p_N)} \neq 0,$$

then the system can be integrated by quadratures.

We now present a standard proof, following [15]. As mentioned above, a different proof based on symplectic geometry was provided by Arnol'd, see [13].

A.1 Proof of the Liuoville-Arnol'd theorem

Proof. Suppose we have N integrals of motion $\mathcal{H}_1, \ldots, \mathcal{H}_N$ functionally independent and analytic in their variables.

The first requirement is equivalent to saying that at each point in the phase space the hypersurface defined by

$$\mathcal{H}_n(\boldsymbol{q}, \boldsymbol{p}) = J_n \tag{A.1}$$

cannot in any way overlap.

Now let us define *the level manifold $M_{\boldsymbol{J}}$* as the intersection of all level surfaces of the form (A.1):

$$M_{\boldsymbol{J}} \equiv \{(\boldsymbol{q}, \boldsymbol{p}) \mid \mathcal{H}_n(\boldsymbol{q}, \boldsymbol{p}) = J_m, \ m = 1, \ldots, N\} \subseteq \Gamma. \tag{A.2}$$

It must hold that $\dim M_{\boldsymbol{J}} = N$. Hence, one can uniquely identify a point on $M_{\boldsymbol{J}}$ by a N-dimensional vector of coordinates.

By hypothesis (3), we can invert the system of equations (A.1) and define the coordinates $\boldsymbol{\theta}$ on $M_{\boldsymbol{J}}$ as

$$\boldsymbol{p} = \boldsymbol{\theta}(\boldsymbol{q}, \boldsymbol{J}). \tag{A.3}$$

Now, on the space $(q_1, \ldots, q_N)$ it occurs that

$$\tilde{\mathcal{H}}_n(\boldsymbol{q}) \equiv \mathcal{H}_n(\boldsymbol{\theta}, \boldsymbol{J}) = J_n \qquad \forall n = 1, \ldots, N, \tag{A.4}$$

hence

$$\frac{\partial \tilde{\mathcal{H}}_n(\boldsymbol{q})}{\partial \boldsymbol{q}} \equiv 0 \qquad \forall n = 1, \ldots, N. \tag{A.5}$$

Explicitly, derivatives in eq. (A.5) take the form

$$\frac{\partial \mathcal{H}_n}{\partial q^m} + \sum_{j=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_j} \frac{\partial}{\partial q^m} [p_j - \theta^j(\boldsymbol{q}, \boldsymbol{J})]_{p_m = \phi^m}. \tag{A.6}$$

Taking into account that each of the $\mathcal{H}_n$ is an integral of motion and that, for each possible trajectory, the p_n can be expressed as certain analytic functions of the q_m, the former expression can be extended on the whole phase space Γ

$$\frac{\partial \mathcal{H}_n}{\partial q^m} + \sum_{j=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_j} \frac{\partial}{\partial q^m} [p_j - \theta^j(\boldsymbol{q}, \boldsymbol{J})]. \tag{A.7}$$

and analogously

$$\frac{\partial \mathcal{H}_n}{\partial p_m} + \sum_{j=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_j} \frac{\partial}{\partial p_m} [p_j - \theta^j(\boldsymbol{q}, \boldsymbol{J})]. \tag{A.8}$$

In this way, we can write

$$\begin{aligned}
\frac{\partial \mathcal{H}_n}{\partial q^m} \frac{\partial \mathcal{H}_j}{\partial p_m} &= \sum_{l,\,r=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_l} \frac{\partial \mathcal{H}_j}{\partial p_r} \frac{\partial}{\partial q^m} [p_l - \theta_l] \frac{\partial}{\partial p_m} [p_r - \theta^r] \\
&= \sum_{l,\,r=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_l} \frac{\partial \mathcal{H}_j}{\partial p_r} \frac{\partial}{\partial p_m} [p_l - \theta^l] \frac{\partial}{\partial q^m} [p_r - \theta_r].
\end{aligned} \tag{A.9}$$

Therefore, we can express Poisson brackets of integrals of motion in the form

$$\{\mathcal{H}_m, \mathcal{H}_j\} = \sum_{l,\,r=1}^{N} \frac{\partial \mathcal{H}_n}{\partial p_l} \frac{\partial \mathcal{H}_j}{\partial p_r} \{p_l - \theta^l, p_r - \theta^r\} \quad \forall m, j = 1, \ldots, N. \tag{A.10}$$

We consider the $N \times N$ matrix whose elements consist of all possible Poisson brackets between the N integrals of motion,

$$(\{\mathcal{H}_m, \mathcal{H}_j\}) = \left(\frac{\partial(\mathcal{H}_1, \ldots, \mathcal{H}_n)}{\partial(p_1, \ldots, p_N)}\right)^T (\{p_l - \theta^l(\boldsymbol{q}, \boldsymbol{J}), p_r - \theta^r(\boldsymbol{q}, \boldsymbol{J})\}) \left(\frac{\partial(\mathcal{H}_1, \ldots, \mathcal{H}_N)}{\partial(p_1, \ldots, p_N)}\right). \tag{A.11}$$

Taking into account hypothesis (3), the former yields

$$\{\mathcal{H}_m, \mathcal{H}_j\} = 0 \iff \{p_l - \theta^l(\boldsymbol{q}, \boldsymbol{J}), p_r - \theta^r(\boldsymbol{q}, \boldsymbol{J})\} = 0. \tag{A.12}$$

on the whole Γ.

Therefore, the two combined hypothesis of the theorem, it must hold

$$\{p_l - \theta^l(\boldsymbol{q}, \boldsymbol{J}), p_r - \theta^r(\boldsymbol{q}, \boldsymbol{J})\} = 0. \tag{A.13}$$

Computing explicitely the Poisson brackets, we get

$$\{p_l - \theta^l(\boldsymbol{q}, \boldsymbol{J}), p_r - \theta^r(\boldsymbol{q}, \boldsymbol{J})\} = \frac{\partial \theta^r}{\partial q^l} - \frac{\partial \theta^l}{\partial q^r} = 0. \tag{A.14}$$

The former condition is equivalent to the existence of a function $W(\boldsymbol{q}, \boldsymbol{J})$ such that

$$\boldsymbol{\theta} = \frac{\partial W(\boldsymbol{q}, \boldsymbol{J})}{\partial \boldsymbol{q}}. \tag{A.15}$$

Hence, by eq. (A.3), it has to be

$$\boldsymbol{p} = \frac{\partial W(\boldsymbol{q}, \boldsymbol{J})}{\partial \boldsymbol{q}}, \tag{A.16}$$

and thus

$$\det \frac{\partial^2 W}{\partial q^m \partial J_n} = \det \frac{\partial \theta^m}{\partial J_n} = \det \left(\frac{\partial(\mathcal{H}_1, \dots \mathcal{H}_N)}{\partial(p_1, \dots, p_N)} \right)^{-1} \neq 0 \tag{A.17}$$

by condition (3).

The matrix defines a canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) \longrightarrow (\boldsymbol{\phi}, \boldsymbol{J})$, with

$$\boldsymbol{\phi} = \frac{\partial W(\boldsymbol{q}, \boldsymbol{J})}{\partial \boldsymbol{J}}. \tag{A.18}$$

The new Hamilton's equations read

$$\dot{\boldsymbol{J}} = 0, \qquad \dot{\boldsymbol{\phi}} = \frac{\partial \tilde{\mathcal{H}}}{\partial \boldsymbol{J}}, \tag{A.19}$$

and the system can be integrated by quadratures:

$$\boldsymbol{J}(t) = \boldsymbol{J}_0, \qquad \boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 + \boldsymbol{\omega}(\boldsymbol{J}_0)t. \tag{A.20}$$

with $\boldsymbol{\phi}_0$, $\boldsymbol{J}_0$ constants, and $\boldsymbol{\omega}(\boldsymbol{J}_0) \equiv \partial\tilde{\mathcal{H}}/\partial\boldsymbol{J}$.

In this way, it was shown that the existence of N analytic integrals of motion, which satisfy the hypothesis of the theorem, is equivalent to the existence of action-angle variables and hence integrability. $\square$

Appendix B

Calogero's Potentials

In Chapter 2, we introduced classical and quantum Calogero-Moser systems, hinting that the peculiar functional forms of the potentials of these systems result from the choice of a convenient Lax representation. We now show how to determine such functional forms from Calogero's ansatz.

B.1 Lax representation of Calogero-Moser systems

Calogero's ansatz for Lax representation of Calogero-Moser systems reads

$$
\begin{cases}
M_{nm} = \delta_{nm} \sum_{l \neq n} \gamma(q^n - q^l) + (1 - \delta_{nm})\beta(q^n - q^m). & \text{(B.1a)} \\[2mm]
L_{nm} = \delta_{nm}p_n + i(1 - \delta_{nm})\alpha(q^n - q^m) & \text{(B.1b)}
\end{cases}
$$

where $\alpha(x), \beta(x), \gamma(x)$ are functions to be determined.

We insert this ansatz in the Lax evolution equation

$$
\dot{L} = [L, M]. \tag{B.2}
$$

The diagonal term is

$$
\dot{p}_n = \sum_{l \neq n} \Big[\alpha(q^n - q^l)\beta(q^l - q^n) - \alpha(q^l - q^n)\beta(q^n - q^l) \Big] \tag{B.3}
$$

while for $n \neq m$ one has

$$
(\dot{q}^n - \dot{q}^m)\alpha'(q^n - q^m) =
$$
$$
(p_n - p_m)\beta(q^n - q^m) + \alpha(q^n - q^m)\left[\sum_{l \neq m} \gamma(q^m - q^l) - \sum_{l \neq n} \gamma(q^n - q^l) \right]
$$
$$
+ \sum_{l \neq n,m} \Big[\alpha(q^n - q^l)\beta(q^l - q^m) - \alpha(q^l - q^m) - \alpha(q^l - q^m)\beta(q^n - q^l) \Big]. \tag{B.4}
$$

Now, from Hamilton's equations, $\dot{q}_n = \partial\mathcal{H}/\partial q^n \equiv p_n$. Then, eq. (B.4) holds if and only if

$$
\begin{cases}
\beta(q) = \alpha'(q), & \text{(B.5a)} \\
\gamma(q) = \gamma(-q), & \text{(B.5b)}
\end{cases}
$$

and

$$\alpha(q^n - q^m)[\gamma(q^m - q^l) + \gamma(q^n - q^l)]$$
$$= -\alpha(q^n - q^l)\beta(q^l - q^m) + \alpha(q^l - q^m)\beta(q^n - q^l). \tag{B.6}$$

Condition (B.5a) inserted into eq. (B.3) gives Newton's equation

$$\dot{p}_n = -\sum_{n \neq m} v'(q^n - q^m) \tag{B.7}$$

with $v(q) = v(-q) = \alpha(q)\alpha(-q)$. Setting $x = q^n - q^l$, $y = q^l - q^m$, applying condition (B.5b) to eq. (B.6) leads to the functional equation

$$\frac{\alpha(x)\alpha'(y) - \alpha(y)\alpha'(x)}{\alpha(x + y)} = \gamma(x) - \gamma(y). \tag{B.8}$$

We then make the critical assumption

$$\beta(x) = -\gamma(x) = \alpha'(x) \tag{B.9}$$

to finally get the functional equation

$$\frac{\alpha(x)\alpha'(y) - \alpha(y)\alpha'(x)}{\alpha(x + y)} = \alpha'(y) - \alpha'(x). \tag{B.10}$$

Remarkable solutions of the functional equation are

$$\alpha(x) = \frac{\lambda}{x}, \qquad v(x) = \frac{\lambda(\lambda - 1)}{x^2}, \tag{B.11}$$

that give rise to the rational Calogero-Moser system, and

$$\alpha(x) = \frac{\lambda}{\coth(x)}, \qquad v(x) = \frac{\lambda(\lambda - 1)}{\sinh^2(x)}, \tag{B.12}$$

which correspond to the hyperbolic CM systems.

B.2   General solution of the functional equation

Now we are interested in determining the general solution of the functional equation (B.8). Supposing $y = -x + \epsilon$ in eq. (B.8), we get

$$\alpha(x)\alpha'(-x + \epsilon) - \alpha(-x + \epsilon)\alpha'(x) = \alpha(\epsilon)[\gamma(x) - \gamma(-x + \epsilon)]. \tag{B.13}$$

For $\epsilon \to 0$, by a MacLaurin expansion, we recover

$$\alpha(x)\left[\alpha(-x) + \epsilon\alpha'(-x) + \frac{\epsilon^2}{2}\alpha''(x) + O(\epsilon^3)\right]$$
$$- \alpha'(x)\left[\alpha'(-x) + \epsilon\alpha''(-x) + \frac{\epsilon^2}{2}\alpha'''(-x) + O(\epsilon^3)\right] \tag{B.14}$$
$$= \alpha(\epsilon)\left[\epsilon\gamma'(x) - \frac{\epsilon^2}{2}\gamma''(x) + \frac{\epsilon^3}{6}\gamma'''(x) + O(\epsilon^4)\right],$$

having used $\gamma(x) = \gamma(-x)$. It must hold that

$$\alpha(\epsilon))\frac{c_{-1}}{\epsilon} + c_0 + c_1\epsilon + O(\epsilon^2), \tag{B.15}$$

with $c_{-1}, c_0 \in \mathbb{R}$, $c_{-1} \neq 0$. Applying eq. (B.15) to eq. (B.14) leads to

$$O(1): \quad c_{-1}\gamma'(x) = \alpha(x)\alpha'(x) - \alpha(-x)\alpha'(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\alpha(x)\alpha(-x), \tag{B.16a}$$

$$O(\epsilon): \quad c_0\gamma'(x) - \frac{c_{-1}}{2}\gamma''(x) = \alpha(x)\alpha''(-x) - \alpha'(-x)\alpha'(x), \tag{B.16b}$$

$$O(\epsilon^2): c_1\gamma'(x) - \frac{c_0}{2}\gamma''(x) + \frac{c_{-1}}{6}\gamma'''(x) = \alpha(x)\alpha'''(-x) - \alpha'(x)\alpha''(-x). \tag{B.16c}$$

Setting $c_0 = 0$, following Calogero [44], we get

$$c_{-1}\gamma'''(x) + 6c_1\gamma'(x) = 3[\alpha''(x)\alpha'(-x) - \alpha'''(x)\alpha(-x)] = 3\frac{\mathrm{d}}{\mathrm{d}x}\alpha''(x)\alpha(-x), \tag{B.17}$$

or

$$c_{-1}\gamma''(x) + 6c_1\gamma(x) + 3\alpha''(x)\alpha(-x) = 0. \tag{B.18}$$

Now $v(x) = -\alpha^2(x)$ and $v'(x) = -2\alpha'(x)\alpha(x)$, thus

$$-v''(x) = 2\alpha''(x)\alpha(x) + 2[\alpha'(x)]^2 \tag{B.19}$$

and

$$\alpha''(x)\alpha(-x) = \frac{1}{4}\left[2v''(x) - \frac{[v'(x)]^2}{v(x)}\right]. \tag{B.20}$$

We use the fact that $\gamma(x) = -v(x) + \text{const.}$, eq. (B.18)

$$-c_{-1}v''(x) - 6c_1v(x) + \frac{3}{4}\left[v''(x) - \frac{[v'(x)]^2}{v(x)}\right] = 0 \tag{B.21}$$

or, setting $c_{-1} = 1$ as done in [44],

$$2v''(x)v(x) - 3[v'(x)]^2 - 24c_1[v(x)]^2 = 0. \tag{B.22}$$

The ODE is satisfied by the potential [44]

$$v(x) = \wp\left(\sqrt{\frac{2c_1}{g}}x\,\middle|\,12g^2, 8g^3\right) + g, \tag{B.23}$$

where $g \in \mathbb{R}$, and $\wp(x|\omega_1, \omega_2)$ is the Weierstrass $\wp$-function with periods ω_1, ω_2, satisfying the differential equation [45]

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - \omega_1\wp(z) - \omega_2. \tag{B.24}$$

We also verify with the aid of Wolfram Mathematica 13.0 the solution (B.23):

Calogero's potentials

In[1]:= `Eq = 2 v''[x] ⋇ v[x] - 3 (v'[x])^2 - 24 c₁ v[x]^2`

Out[1]= $-24\, c_1\, v[x]^2 - 3\, v'[x]^2 + 2\, v[x]\, v''[x]$

In[2]:= `((Eq /. {v → Function[{x}, WeierstrassP[a x, {ω₁, ω₂}] + g]} // ExpandAll //`
`        Collect[#, WeierstrassP[a x, {ω₁, ω₂}]] &) /. {x → t / a}) / (-3 a^2) //`
`    Expand // Collect[#, WeierstrassP[t, {ω₁, ω₂}], FullSimplify] &`

Out[2]= $\dfrac{8\, g^2\, c_1}{a^2} + \dfrac{g\, \omega_1}{3} + \left(\dfrac{16\, g\, c_1}{a^2} + \dfrac{\omega_1}{3}\right)$ WeierstrassP[t, {ω₁, ω₂}] +

$\left(-4\, g + \dfrac{8\, c_1}{a^2}\right)$ WeierstrassP[t, {ω₁, ω₂}]² -

4 WeierstrassP[t, {ω₁, ω₂}]³ + WeierstrassPPrime[t, {ω₁, ω₂}]²

In[3]:= `sola = Solve[(8 c₁/a² - 4 g) == 0, a]⟦2⟧`

Out[3]= $\left\{a \to \dfrac{\sqrt{2}\ \sqrt{c_1}}{\sqrt{g}}\right\}$

In[4]:= `sol1 = Solve[((16 c₁ g/a² + ω₁/3) - ω₁ /. sola) == 0, ω₁]⟦1⟧`

Out[4]= $\left\{\omega_1 \to 12\, g^2\right\}$

In[5]:= `Solve[(1/3 g (24 c₁ g/a² + ω₁) - ω₂ /. sola /. sol1) == 0, ω₂]⟦1⟧`

Out[5]= $\left\{\omega_2 \to 8\, g^3\right\}$

The $\wp$-function reads [45]

$$\wp(z|\omega_1,\omega_2) \equiv \frac{1}{z^2} + \sum_{n,m\in\mathbb{Z}\backslash\{0\}} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right), \quad \text{(B.25)}$$

and one can verify that

$$\wp(x,\omega_1,\omega_2) = \begin{cases} 1/x^2 & \omega_1 = \infty,\ \omega_2 = \infty \\ a^2/\sin^2(ax) - a^2/3 & \omega_1 = \pi/2a,\ \omega_2 = i\infty \\ a^2/\sinh^2(ax) + a^2/3 & \omega_1 = \infty,\ \omega_2 = i\pi/2a. \end{cases} \quad \text{(B.26)}$$

Appendix C

Various results

C.1 Asymptotic momenta

In Chapter 2 we stated Lemma 2.1, concerning the conservation of the asymptotic momenta for Calogero-Moser particles. We now present the proof.

Lemma 2.1. *Fixing the constant of motion L_n for $n \leq N$ for a quantum many-body system composed by N particles of the Calogero-Moser type, the asymptotic momenta $\{k_j\}_{j=1}^N$ are conserved.*

Proof. We have shown that constant of motion L_n are symmetric polynomials

$$L_n = \eta^\dagger L^n \eta = \sum_{j,m=1}^N (L^n)_{jm} = \sum_{j=1}^N k_j^n + P_{n-1}(k_1, k_2, \ldots, k_N). \tag{C.1}$$

with P_{n-1} a symmetric polynomial in $k_1, \ldots, k_N$ of degree $n - 1$.

We now use the fact that every symmetric polynomial can be expressed by a combination of elementary symmetric polynomials of degree n

$$s_n(k_1, k_2, \ldots, k_N) \equiv \sum_{j=1}^N k_j^n \tag{C.2}$$

for $n = 1, 2, \ldots, N$. Hence, fixed the values of the integrals of motion

$$L_n(k_1, k_2, \ldots, k_N) = \lambda_n \in \mathbb{R} \tag{C.3}$$

and substituting, we write

$$\lambda_n = s_n + P_{n-1}(s_1, s_2, \ldots, s_{n-1}). \tag{C.4}$$

For $n = 1$

$$s_1 = \lambda_1 - P_0 \equiv \sigma_1, \tag{C.5}$$

and the value of the first symmetric polynomial is fixed. Then

$$s_2 = \lambda_2 - P_1(s_1) = \lambda_2 - P_1(\sigma_1) \equiv \sigma_2 \tag{C.6}$$

is determined as well. By induction, the N constants of motion will fix all the elementary symmetric polynomials $s_n(k_1, k_2, \ldots, k_n) \equiv \sigma_n \in \mathbb{R}$.

Let

$$s'_n(k_1, k_2, \ldots, k_{N-1}) \equiv \sum_{j=1}^{N-1} k_j^n \tag{C.7}$$

the reduced symmetric polynomial, for $n = 1, 2, \ldots, N$. Then, it must hold that

$$s'_n = \sigma_n - k_N^n \tag{C.8}$$

for $n = 1, 2, \ldots, N$. Let us focus on the reduced polynomial of degree N. We can always rewrite it in terms of other elementary reduced polynomials

$$s'_N = \sigma_N - k_N^N = s'_N(s'_1, \ldots, s'_{N-1}). \tag{C.9}$$

Substituting eq. (C.8) in eq. (C.9), we get

$$s'_N(\sigma_1 - k_N, \sigma_2 - k_N^2, \ldots, \sigma_{N-1} - k_N^{N-1}) = \sigma_N - k_N^N, \tag{C.10}$$

or

$$k_N^N + s'_N(\sigma_1 - k_N, \sigma_2 - k_N^2, \ldots, \sigma_{N-1} - k_N^{N-1}) - \sigma_N = 0. \tag{C.11}$$

Thus, k_N is a root of a polynomial of degree N with coefficients related to σ_n for $n = 1, \ldots, N-1$. Because of the fundamental theorem of algebra, there are at most N complex roots of eq. (C.11). If the N incoming momenta k_j^0, $j = 1, 2, \ldots, N$ are distinct, there are no other possibilities: k_N has to be one of the k_j^0s. $\qquad\square$

C.2   Sound speed in a medium

In Chapter 3 we used a particular expression regarding the speed of sound v_s in an ideal gas in terms of the chemical potential μ and the pressure $p = p(\boldsymbol{x}, t)$ of the system:

$$\frac{1}{v_s^2} = \frac{1}{d} \frac{\partial^2 p}{\partial \mu^2}. \tag{C.12}$$

Now we will derive it by standard thermodynamic arguments.

It can be shown [46] that an acoustic wave in an inviscid dry gas (we assume that gravitational effects are negligible) has the form

$$\nabla^2 p = \frac{1}{v_s^2} \frac{\partial^2 p}{\partial t^2} \tag{C.13}$$

and it is reasonable to suppose

$$v_s^2 \equiv \frac{\partial p}{\partial d}, \tag{C.14}$$

where $d = d(\boldsymbol{x}, t)$ the density and v_s the speed of sound in the considered system. Masses are supposed to be unitary.

From eq. (C.14) we can write

$$\frac{1}{v_s^2} = \frac{\partial d}{\partial p} = \frac{\partial d}{\partial \mu}\frac{\partial \mu}{\partial p}. \tag{C.15}$$

From the Maxwell relations,

$$\frac{\partial d}{\partial \mu} = \frac{\partial}{\partial \mu}\frac{\partial N}{\partial V}\bigg|_{T,\mu} = \frac{\partial}{\partial \mu}\frac{\partial p}{\partial \mu}\bigg|_{T,V} = \frac{\partial^2 p}{\partial \mu^2}. \tag{C.16}$$

By considering the Gibbs-Duhem equation

$$S\mathrm{d}T - V\mathrm{d}p + N\mathrm{d}\mu = 0, \tag{C.17}$$

we find that, at constant temperature

$$\frac{\partial \mu}{\partial p} = \frac{V}{N} \equiv \frac{1}{d}. \tag{C.18}$$

Then relation (C.12) is recovered from eqs. (C.15), (C.16) and (C.18).

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