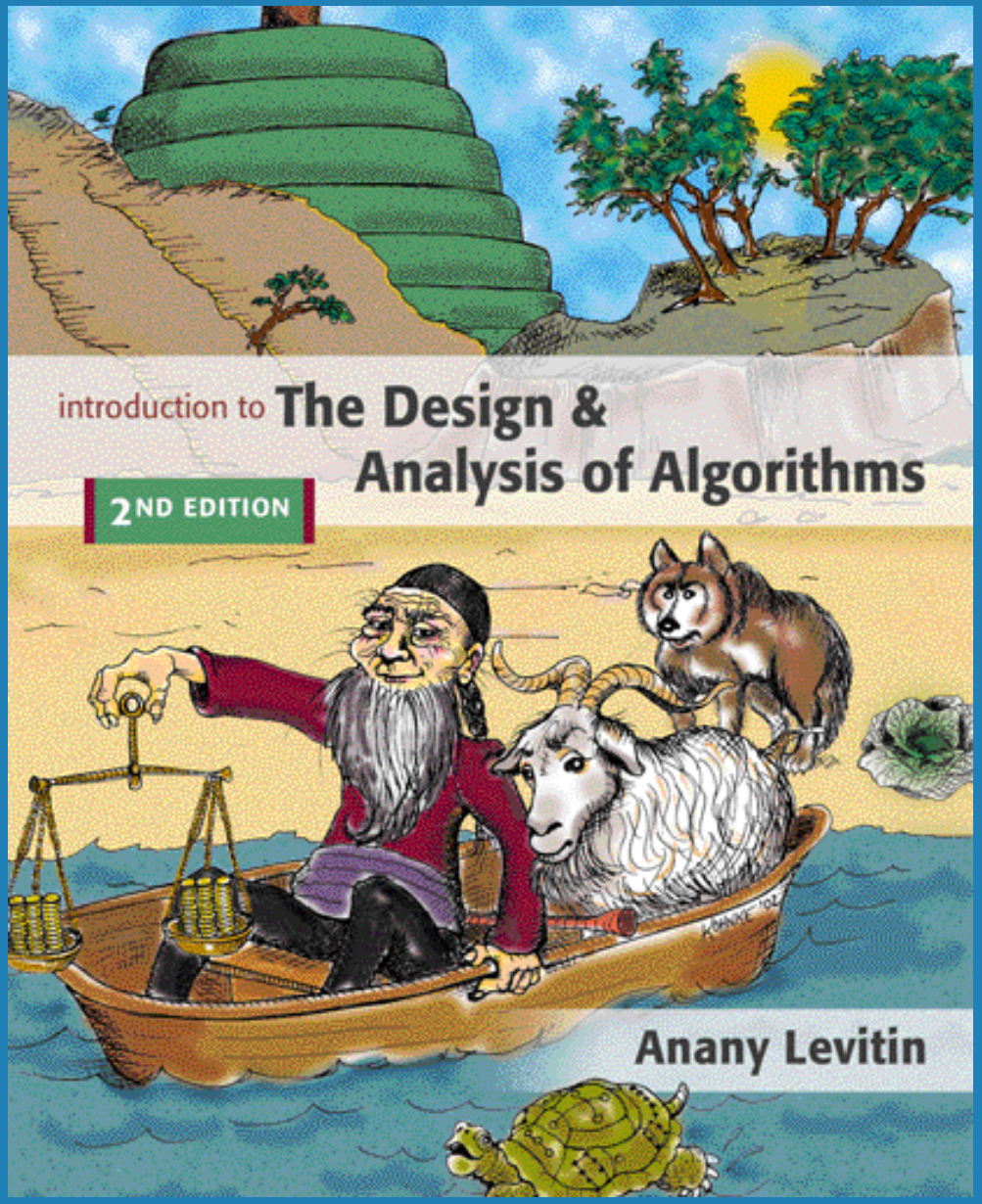
Chapter 1

Introduction

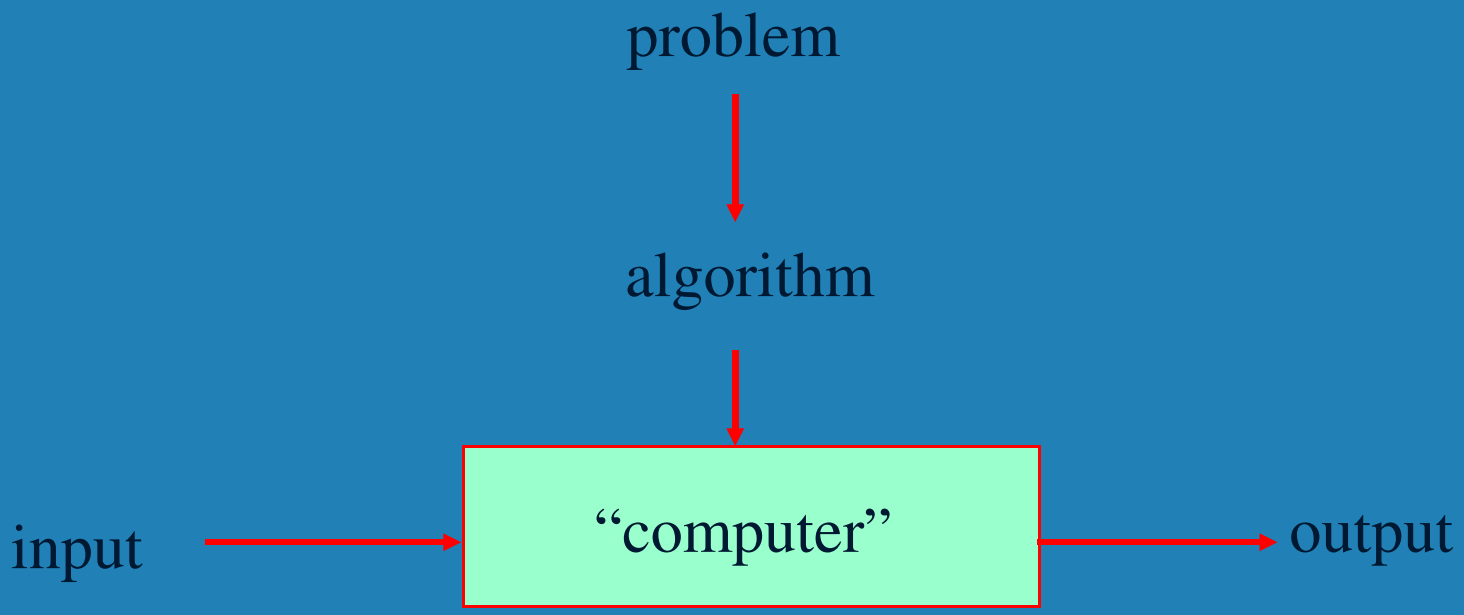


Why Study Algorithms?

- Al Khawarizmi
  - "A great Iranian mathematician, geographer and astronomer. He introduced the zero, negative numbers, algebra, and the decimal system to the West. He also invented mathematical programming using a set of instructions to perform complex calculations. The term algorithm is named after a variation of his name, Algorithmi. "
- What is it?
  - Briefly speaking, algorithms are procedure solution to problems.
  - Algorithms are not answers but rather precisely defined procedures for getting answers. (Example of sorting 3 numbers.)
- Cornerstone of computer science. Programs will not exist without algorithms.
- Algorithm design techniques, or problem-solving strategies, are useful in fields beyond computer science.

Algorithms

An algorithm is a sequence of unambiguous instructions for solving a computational problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.



Example of Computational Problem: Sorting

- **Statement of problem:**
  - ***Input:*** **A sequence of *n* numbers $\langle a_1, a_2, \ldots, a_n \rangle$**
  - ***Output:*** **A reordering of the input sequence $\langle a'_1, a'_2, \ldots, a'_n \rangle$ so that $a'_i \leq a'_j$ whenever $i < j$**
- **Instance: The sequence <5, 3, 2, 8, 3>**
- **Algorithms:**
  - **Selection sort**
  - **Insertion sort**
  - **Merge sort**
  - **(many others)**

Properties of Algorithms

- **What distinguish an algorithm from a recipe, process, method, technique, procedure, routine…?**
  - **Finiteness**

    terminates after a finite number of steps
  - **Definiteness**

    Each step must be rigorously and unambiguously specified.

    -e.g., "stir until lumpy"
  - **Input**

    Valid inputs must be clearly specified.
  - **Output**

    The data that result upon completion of the algorithm must be specified.
  - **Effectiveness**

    Steps must be sufficiently simple and basic.

    -e.g., check if 2 is the largest integer n for which there is a solution to the equation $x^n + y^n = z^n$ in positive integers x, y, and z

Examples

- Is the following a legitimate algorithm?

```
i ←1
While (i <= 10) do
    a ← i + 1
    Print the value of a
End of loop
Stop
```

Greatest Common Divisor of Two Integers

- gcd(m, n): the largest integer that divides both m and n
- First try -- Euclid's algorithm: gcd(m, n) = gcd(n, m mod n)
  - Step1: If n = 0, return the value of m as the answer and stop; otherwise, proceed to Step 2.
  - Step2: Divide m by n and assign the value of the remainder to r.
  - Step 3: Assign the value of n to m and the value of r to n. Go to Step 1.

gdc(60,24) = gdc(24,12) = gdc(12,0) = 12

Methods of Specifying an Algorithm

- **Natural language**
  - **Ambiguous**

    "Mike ate his sandwich on a bed of lettuce."
- **Pseudocode**
  - **A mixture of a natural language and programming language-like structures**
  - **Precise and succinct**
  - **Notation may**
    - **omit declarations of variables**
    - **use indentation to show the scope of such statements as for, if, and while.**
    - **use ← for assignment**

Pseudocode of Euclid's Algorithm

Algorithm *Euclid(m, n)*
//Computes gcd(m, n) by Euclid's algorithm
//Input: Two nonnegative, not-both-zero integers m and n
//Output: Greatest common divisor of m and n
while n ≠ 0 do
    r ← m mod n
    m ← n
    n ← r
return m

- **Questions:**
  - **Finiteness: how do we know that Euclid's algorithm actually comes to a stop?**
  - **Definiteness: nonambiguity**
  - **Effectiveness: effectively computable**

Second Try for gcd(m, n)

- Consecutive Integer Checking Algorithm
  - Step1: Assign the value of min{m, n} to t.
  - Step2: Divide m by t. If the remainder of this division is 0, go to Step3; otherwise, go to Step 4.
  - Step3: Divide n by t. If the remainder of this division is 0, return the value of t as the answer and stop; otherwise, proceed to Step4.
  - Step4: Decrease the value of t by 1. Go to Step2.
- Questions
  - Finiteness
  - Definiteness
  - Effectiveness
  - Which algorithm is faster, the Euclid's or this one?

Third try for gcd(m, n)

- **Middle-school procedure**
  - **Step1: Find the prime factors of m.**
  - **Step2: Find the prime factors of n.**
  - **Step3: Identify all the common factors in the two prime expansions found in Step1 and Step2. (If p is a common factor occurring Pm and Pn times in m and n, respectively, it should be repeated in min{Pm, Pn} times.)**
  - **Step4: Compute the product of all the common factors and return it as the gcd of the numbers given.**
- **gdc(6-,24) =**

Third try for gcd(m, n)

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- **60 = 2 . 2 . 3 . 5**
- **24 = 2 . 2 . 2. 3**
- **gdc(6-,24) = 2 . 2 . 3 = 12**

- **Question**
  - **Is this a legitimate algorithm?**
  - **Homework: algorithm for the Sieve of Erastosthenes**

What can we learn from the previous 3 examples?

- Each step of an algorithm must be unambiguous.
- The same algorithm can be represented in several different ways (different pseudocodes).
- There might exists more than one algorithm for a certain problem.
- Algorithms for the same problem can be based on very different ideas and can solve the problem with dramatically different speeds.

Fundamentals of Algorithmic Problem Solving

- Understanding the problem
  - Asking questions, do a few examples by hand, think about special cases, etc.
- Deciding on
  - Exact vs. approximate problem solving
  - Appropriate data structure
- Design an algorithm
- Proving correctness
- Analyzing an algorithm
  - Time efficiency : how fast the algorithm runs
  - Space efficiency: how much extra memory the algorithm needs.
  - Simplicity and generality
- Coding an algorithm

Algorithm design strategies

- Brute force
- Divide and conquer
- Decrease and conquer
- Transform and conquer
- Space and time tradeoffs
  - Sorting, Matching, Hashing, B-Trees
- Dynamic programming
- Greedy approach
- Iterative Improvement
- Limitations of Algorithm Power
  - Decision Trees, P/NP, Numerical Algorithms
- Coping with Limitations
  - Backtracking, Branch and bound, Approximation

Important Problem Types

- Sorting
- Searching
- String processing
- Graph problems

Sorting (I)

- **Rearrange the items of a given list in ascending order.**
  - **Input: A sequence of n numbers $<a_1, a_2, \ldots, a_n>$**
  - **Output: A reordering $<a'_1, a'_2, \ldots, a'_n>$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.**
- **Why sorting?**
  - **Help searching**
  - **Algorithms often use sorting as a key subroutine.**
- **Sorting key**
  - **A specially chosen piece of information used to guide sorting. I.e., sort student records by names.**

Sorting (II)

- Examples of sorting algorithms
  - Selection sort
  - Bubble sort
  - Insertion sort
  - Merge sort
  - Heap sort …
- Evaluate sorting algorithm complexity: the number of key comparisons.
- Two properties
  - Stability: A sorting algorithm is called stable if it preserves the relative order of any two equal elements in its input.
  - In place : A sorting algorithm is in place if it does not require extra memory, except, possibly for a few memory units.

Selection Sort

```
Algorithm SelectionSort(A[0..n-1])
//The algorithm sorts a given array by selection sort
//Input: An array A[0..n-1] of orderable elements
//Output: Array A[0..n-1] sorted in ascending order
for i ← 0 to n – 2 do
    min ← i
    for j ← i + 1 to n – 1 do
            if A[j] < A[min]
                    min ← j
    swap A[i] and A[min]
```

Searching

- Find a given value, called a search key, in a given set.
- Examples of searching algorithms
  - Sequential searching
  - Binary searching…

String Processing

- A string is a sequence of characters from an alphabet.
- Text strings: letters, numbers, and special characters.
- String matching: searching for a given word/pattern in a text.

Graph Problems

- **Informal definition**
  - **A graph is a collection of points called vertices, some of which are connected by line segments called edges.**
- **Modeling real-life problems**
  - **Modeling WWW**
  - **communication networks**
  - **Project scheduling …**
- **Examples of graph algorithms**
  - **Graph traversal algorithms**
  - **Shortest-path algorithms**
  - **Topological sorting**

Fundamental Data Structures

- Linear data structures
- Stacks, queues, and heaps
- Graphs
- Trees

Linear Data Structures

- **Arrays**
  - A sequence of n items of the same data type that are stored contiguously in computer memory and made accessible by specifying a value of the array's index.
- **Linked List**
  - A sequence of zero or more nodes each containing two kinds of information: some data and one or more links called pointers to other nodes of the linked list.
  - Singly linked list (next pointer)
  - Doubly linked list (next + previous pointers)

Arrays

fixed length (need preliminary reservation of memory)

contiguous memory locations

direct access

Insert/delete

Linked Lists

dynamic length

arbitrary memory locations

access by following links

Insert/delete

Stacks, Queues, and Heaps (1)

- **Stacks**
  - A stack of plates
    - insertion/deletion can be done only at the top.
    - LIFO
  - Two operations (push and pop)
- **Queues**
  - A queue of customers waiting for services
    - Insertion/enqueue from the rear and deletion/dequeue from the front.
    - FIFO
  - Two operations (enqueue and dequeue)

Stacks, Queues, and Heaps (2)

- Priority queues (implemented using heaps)
  - A data structure for maintaining a set of elements, each associated with a key/priority, with the following operations
    - Finding the element with the highest priority
    - Deleting the element with the highest priority
    - Inserting a new element
  - Scheduling jobs on a shared computer.

Graphs

- **Formal definition**
  - A graph $G = <V, E>$ is defined by a pair of two sets: a finite set V of items called vertices and a set E of vertex pairs called edges.
- **Undirected and directed graphs (digraph).**
- **What's the maximum number of edges in an undirected graph with |V| vertices?**
- **Complete, dense, and sparse graph**
  - A graph with every pair of its vertices connected by an edge is called complete. $K_{|V|}$

Graph Representation

- Adjacency matrix
  - n x n boolean matrix if |V| is n.
  - The element on the ith row and jth column is 1 if there's an edge from ith vertex to the jth vertex; otherwise 0.
  - The adjacency matrix of an undirected graph is symmetric.
- Adjacency linked lists
  - A collection of linked lists, one for each vertex, that contain all the vertices adjacent to the list's vertex.
- Which data structure would you use if the graph is a 100-node star shape?

Weighted Graphs

- **Weighted graphs**
  - **Graphs or digraphs with numbers assigned to the edges.**

Graph Properties -- Paths and Connectivity

- **Paths**
  - A path from vertex u to v of a graph G is defined as a sequence of adjacent (connected by an edge) vertices that starts with u and ends with v.
  - Simple paths: All edges of a path are distinct.
  - Path lengths: the number of edges, or the number of vertices – 1.
- **Connected graphs**
  - A graph is said to be connected if for every pair of its vertices u and v there is a path from u to v.
- **Connected component**
  - The maximum connected subgraph of a given graph.

Graph Properties -- Acyclicity

- Cycle
  - A simple path of a positive length that starts and ends a the same vertex.
- Acyclic graph
  - A graph without cycles
  - DAG (Directed Acyclic Graph)

Trees (I)

- **Trees**
  - A tree (or free tree) is a connected acyclic graph.
  - Forests: a graph that has no cycles but is not necessarily connected.
- **Properties of trees**
  - For every two vertices in a tree there always exists exactly one simple path from one of these vertices to the other. Why?
    - $|E| = |V| - 1$
    - Rooted trees: The above property makes it possible to select an arbitrary vertex in a free tree and consider it as the root of the so-called rooted tree.
    - Levels of rooted tree.

Trees (II)

- **ancestors**
  - For any vertex v in a tree T, all the vertices on the simple path from the root to that vertex are called ancestors.
- **descendants**
  - All the vertices for which a vertex v is an ancestor are said to be descendants of v.
- **parent, child and siblings**
  - If (u, v) is the last edge of the simple path from the root to vertex v (and $u \neq v$), u is said to be the parent of v and v is called a child of u.
  - Vertices that have the same parent are called siblings.
- **Leaves**
  - A vertex without children is called a leaf.
- **Subtree**
  - A vertex v with all its descendants is called the subtree of T rooted at v.

Trees (III)

- **Depth of a vertex**
  - **The length of the simple path from the root to the vertex.**
- **Height of a tree**
  - **The length of the longest simple path from the root to a leaf.**

Ordered Trees

- **Ordered trees**
  - An ordered tree is a rooted tree in which all the children of each vertex are ordered.
- **Binary trees**
  - A binary tree is an ordered tree in which every vertex has no more than two children and each children is designated s either a left child or a right child of its parent.
- **Binary search trees**
  - Each vertex is assigned a number.
  - A number assigned to each parental vertex is larger than all the numbers in its left subtree and smaller than all the numbers in its right subtree.
- $\lfloor \log_2 n \rfloor \leq h \leq n - 1$, where h is the height of a binary tree.