



Pós-Graduação em Ciência da Computação

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# **An Algebra of Temporal Faults**

**Ph.D. Thesis**



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A Ph.D. Thesis presented to the Center for Informatics of Federal University of Pernambuco in partial fulfillment of the requirements for the degree of Philosophy Doctor in Computer Science.

Federal University of Pernambuco

Center of Informatics

Graduate in Computer Science

Supervisor: Alexandre Cabral Mota

Co-supervisor: Alexander Romanovsky

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I dedicate this thesis to Juliana, Luciana (pipoquinha), and Bianca (snowflake).





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*“Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity.  
(Alan Turing)”*



# Resumo

A modelagem de falhas é essencial na antecipação de defeitos em sistemas críticos. Tradicionalmente, Árvores de Falhas Estáticas são empregadas para este fim, mas Árvores de Falhas Temporais e Dinâmicas têm ganhado evidência devido ao seu maior poder para modelar e detectar propagações complexas de falhas que levam a um defeito.

Em um trabalho anterior, mostramos uma estratégia baseada na álgebra de processos CSP e modelos Simulink para obter rastros (sequências) de falhas que levam a um defeito. A partir dos rastros de falhas nós descartamos a informação de ordenamento para obter expressões de estrutura para Árvores de Falhas Estáticas. Ao contrário de descartar tal informação de ordenamento, poderíamos usá-la para obter expressões de estrutura para Árvores de Falhas Temporais ou Dinâmicas.

No presente trabalho apresentamos: (i) uma álgebra temporal de falhas (com noção de propagação de falhas) para analisar defeitos em sistemas e provamos que ela é de fato uma álgebra Booleana, e (ii) uma lógica de ativação parametrizada para expressar comportamentos nominais e de falha, incluindo a modelagem de falhas a partir de uma álgebra e um conjunto de modos de operação. A álgebra permite herdar as propriedades de álgebras Booleanas, leis e técnicas de redução existentes, as quais são muito benéficas para a modelagem e análise de falhas. Com expressões na álgebra temporal de falhas nós permitimos a verificação de propriedades de segurança (*safety*) baseadas em Árvores de Falhas Estáticas, Temporais ou Dinâmicas. A lógica criada neste trabalho pode ser usada com outras álgebras além das apresentadas. Sendo usada em conjunto com a álgebra temporal de falhas, tem a intenção de ajudar os analistas a considerar todas as possíveis situações em expressões complexas com operadores relacionados ao ordenamento das falhas, evitando esquecer combinações de falhas sutis (porém relevantes). Além disso, nossa álgebra temporal de falhas trata operadores NOT, que têm sido deixados de fora em outros trabalhos. Nós ilustramos nosso trabalho com alguns estudos de caso simples, mas reais, fornecidos pelo nosso parceiro industrial, a EMBRAER.

Isabelle/HOL foi utilizado para a mecanização das provas dos teoremas da álgebra temporal de falhas.

**Palavras-chave:** Simulink, CSP, FDR, Fault Tree Analysis, Temporal Fault Trees, Dynamic Fault Trees, Isabelle/HOL, Pandora, Fault Injection



# Abstract

Fault modelling is essential to anticipate failures in critical systems. Traditionally, Static Fault Trees are employed to this end, but Temporal and Dynamic Fault Trees have gained evidence due to their enriched power to model and detect intricate propagation of faults that lead to a failure.

In a previous work, we showed a strategy based on the process algebra CSP and Simulink models to obtain fault traces that lead to a failure. From the fault traces we discarded the ordering information to obtain structure expressions for Static Fault Trees. Instead of discarding such an ordering information, it could be used to obtain structure expressions of Temporal or Dynamic Fault Trees.

In this work we present: (i) an algebra of temporal faults (with a notion of fault propagation) to analyse systems' failures, and prove that it is indeed a Boolean algebra, and (ii) a parametrized activation logic to express nominal and erroneous behaviours, including fault modelling, provided an algebra and a set of operational modes. The algebra allows us to inherit Boolean algebra's properties, laws and existing reduction techniques, which are very beneficial for fault modelling and analysis. With expressions in the algebra of temporal faults we allow the verification of safety properties based on Static, Temporal or Dynamic Fault Trees. The logic created in this work can be combined with other algebra beyond those shown here. Being used with the algebra of temporal faults it is intended to help analysts to consider all possible situations in complex expressions with order-related operators, avoiding missing subtle (but relevant) faults combinations. Furthermore, our algebra of temporal faults tackles NOT operators which have been left out in other works. We illustrate our work on simple but real case studies, some supplied by our industrial partner EMBRAER.

Isabelle/HOL was used to mechanize the theorems proofs of the algebra of temporal faults.

**Keywords:** Simulink, CSP, FDR, Fault Tree Analysis, Temporal Fault Trees, Dynamic Fault Trees, Isabelle/HOL, Pandora, Fault Injection





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# List of abbreviations and acronyms

AADL	Architecture Analysis and Design Language pp. 29, 113, 114
AL	Activation Logic pp. 15, 32–35, 93–99, 101, 104–106, 113, 114
AFP	archive of formal proofs p. 74
ATF	Algebra of Temporal Faults pp. 15, 25, 32–35, 43, 60, 71, 77–81, 84, 87–89, 92–94, 101–103, 107, 110, 111, 113, 114, 125, 140–142, 146–149, 151, 153, 155, 157, 159, 161, 163–167
BDD	Binary Decision Diagram pp. 15, 27, 30, 31, 43, 47, 53, 56–60, 62, 113
BN	Bayesian network p. 53
CML	COMPASS Modelling Language p. 40
CPN	coloured Petri-net p. 53
CSP	Communicating Sequential Processes p. 40
CSP <sub>M</sub>	Communicating Sequential Processes pp. 30, 33, 43, 68, 69, 71, 77, 91
CTMC	continuous-time Markov chain pp. 30, 53
DBN	dynamic bayesian network p. 31
DD	Dependence Diagram p. 41
DFT	Dynamic Fault Tree pp. 17, 28–32, 37, 41, 43–45, 48, 50–56, 59, 60, 71, 77, 84, 93, 113, 114
DNF	disjunctive normal form pp. 44, 50, 53, 58, 60, 84, 113
DRBD	Dynamic Reliability Block Diagram p. 41
DT	dependency tree pp. 30, 50, 51, 56, 58
DTMC	discrete-time Markov chain pp. 30, 41, 51, 56
EMP	electromagnetic pulse p. 41
FBA	Free Boolean Algebra pp. 27, 30, 31, 43, 55, 61, 74, 77, 78, 80, 81, 102, 113
FDR	Failures and Divergences Refinement pp. 30, 68, 69, 71
FMEA	Failure Modes and Effects Analysis pp. 31, 41
FSM	Finite State Machine p. 56
FT	fault tree pp. 15, 27–33, 35, 37, 38, 40, 43–46, 48, 50, 54, 55, 62–66, 71, 77, 87, 91
FTA	Fault Tree Analysis pp. 15, 27–35, 43–46, 64, 87
HCAS	cardiac assist system p. 54
HiP-HOPS	Hierarchically Performed Hazard Origin and Propagation Studies pp. 29–31, 38, 45, 71

HLPN	high-level Petri-net p. 56
HOL	higher-order logic p. 73
Isar	Intelligible semi-automated reasoning pp. 43, 73
ITL	Interval Temporal Logic p. 56
LTL	linear temporal logic p. 48
MCS	minimal cut set pp. 27, 44, 47, 50, 53
MCSeq	minimal cut sequence pp. 24, 29, 33, 49, 50, 53, 56, 58, 59, 87, 91
PN	Petri-net p. 39
ROBDD	Reduced Ordered Binary Decision Diagram pp. 56, 57
SBDD	Sequential Binary Decision Diagram pp. 31, 53, 56, 60
SFT	Static Fault Tree pp. 15, 28–30, 32, 37, 41, 43–48, 50, 53, 55, 56, 58, 59, 61, 62, 71, 77, 84, 93, 113
SoS	System of Systems pp. 32, 38
SWN	stochastic well-formed net p. 53
SysML	Systems Modelling Language pp. 32, 40
TFT	Temporal Fault Tree pp. 28–32, 37, 43–45, 48–50, 52, 53, 55, 56, 71, 77, 84, 93, 113, 114
TTT	Temporal Truth Table pp. 17, 30, 49, 50, 58
UML	Unified Modelling Language p. 40
Z	Z Notation pp. 56, 73
ZBDD	Zero-suppressed Binary Decision Diagram pp. 56, 59

# Fault tree gates

AND	$\wedge$ . Used in <a href="#">SFT</a> , <a href="#">TFT</a> , and <a href="#">DFT</a> . pp. <a href="#">27</a> , <a href="#">28</a> , <a href="#">43</a> , <a href="#">46</a> , <a href="#">48</a> , <a href="#">50</a> , <a href="#">53</a> , <a href="#">54</a> , <a href="#">60</a> , <a href="#">62</a> , <a href="#">64</a> , <a href="#">71</a> , <a href="#">72</a> , <a href="#">84–86</a> , <a href="#">89</a> , <a href="#">92</a>
CSp	cold spare. Used in <a href="#">DFT</a> . pp. <a href="#">29</a> , <a href="#">44</a> , <a href="#">52</a> , <a href="#">54</a> , <a href="#">56</a> , <a href="#">60</a>
FDEP	functional dependency. Used in <a href="#">DFT</a> . pp. <a href="#">29</a> , <a href="#">44</a> , <a href="#">51</a> , <a href="#">52</a> , <a href="#">54</a>
IBefore	inclusive-before. Used in structure expressions of <a href="#">DFT</a> . pp. <a href="#">53</a> , <a href="#">54</a> , <a href="#">60</a>
NIBefore	non-inclusive-before. Used in structure expressions of <a href="#">DFT</a> . pp. <a href="#">53</a> , <a href="#">54</a>
NOT	$\neg$ . Used in non-coherent trees. pp. <a href="#">29</a> , <a href="#">30</a> , <a href="#">32</a> , <a href="#">35</a> , <a href="#">43</a> , <a href="#">48</a> , <a href="#">62</a> , <a href="#">64</a> , <a href="#">66</a> , <a href="#">80</a> , <a href="#">84</a> , <a href="#">87</a> , <a href="#">88</a> , <a href="#">101</a> , <a href="#">113</a>
OR	$\vee$ . Used in <a href="#">SFT</a> , <a href="#">TFT</a> , and <a href="#">DFT</a> . pp. <a href="#">27</a> , <a href="#">28</a> , <a href="#">43</a> , <a href="#">46</a> , <a href="#">48</a> , <a href="#">50</a> , <a href="#">53</a> , <a href="#">54</a> , <a href="#">62</a> , <a href="#">71</a> , <a href="#">85</a> , <a href="#">86</a> , <a href="#">90</a>
PAND	priority-AND. Used in <a href="#">SFT</a> , <a href="#">TFT</a> , and <a href="#">DFT</a> . pp. <a href="#">28</a> , <a href="#">43</a> , <a href="#">44</a> , <a href="#">48–50</a> , <a href="#">52</a> , <a href="#">54</a> , <a href="#">56</a> , <a href="#">60</a> , <a href="#">88</a>
POR	priority-OR. Used in <a href="#">TFT</a> . pp. <a href="#">48</a> , <a href="#">50</a> , <a href="#">53</a>
SAND	simultaneous-AND. Used in <a href="#">TFT</a> . pp. <a href="#">48</a> , <a href="#">50</a> , <a href="#">52</a> , <a href="#">53</a>
SEQ	sequence enforcing. Used in <a href="#">DFT</a> . pp. <a href="#">29</a> , <a href="#">44</a> , <a href="#">52</a> , <a href="#">54</a>
SIMLT	simultaneous. Used in structure expressions of <a href="#">DFT</a> . pp. <a href="#">53</a> , <a href="#">54</a>
WSp	warm spare. Used in <a href="#">DFT</a> . pp. <a href="#">29</a> , <a href="#">60</a>
XBefore	exclusive-before. Proposed in this work. pp. <a href="#">25</a> , <a href="#">77–81</a> , <a href="#">83–87</a> , <a href="#">92</a> , <a href="#">102</a> , <a href="#">103</a> , <a href="#">113</a> , <a href="#">140</a> , <a href="#">141</a> , <a href="#">167</a>





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<sup>3</sup> This variant “M” is the machine-readable version of CSP.

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# 1 Introduction

The development process of critical control systems is based essentially on the rigorous execution of guides and regulations [4, 5, 6, 7]. Specialized agencies (like FAA, EASA and ANAC in the aviation field) use these guides and regulations to certify such systems.

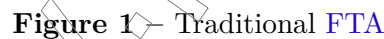
Safety is a system's attribute that plays a critical concern on critical systems and it is the responsibility of the safety assessment process. To employ such a process, dependable systems' taxonomy and safety assessment techniques must be well defined and understood. Clarification of concepts of dependable systems can be surprisingly difficult when systems are complex, because the determination of possible causes or consequences of failure can be a very subtle process [8].

ARP-4761 [7] defines several techniques to perform safety assessment. One of them is Fault Tree Analysis (FTA). It is a deductive method that uses trees to model faults and their dependencies and propagation. In such trees, the premises are the leaves (basic events) and the conclusions are the roots (top events). Intermediary events use gates to combine basic events and each kind of gate has its own combination semantics definition. Fault trees (FTs) that use only  $\vee$  (OR) and  $\wedge$  (AND) gates are called *coherent fault trees* [9, 10, 11, 12, 13]. They combine the events as *at least one shall occur* and *all shall occur*, respectively. To analyse FTs, their structures are abstracted as Boolean expressions called *structure expressions*. The analysis of coherent FTs uses a well-defined algorithm based on the Shannon's method to obtain minimal cut sets (MCSs) from the structure expressions, and a general formula to calculate the probability of top events. The MCSs are obtained by reducing structure expressions to a normal form, in which each term is a combination of variables (basic events) with conjunctive (AND) gates, and the terms are combined as disjunctive (OR) gates. These minimal terms are also called *prime implicants* or *minterms*. The Shannon's method originated a formalism to reduce structure expressions called Binary Decision Diagram (BDD) [14, 15]. Another approach to reduce structure expressions is to use a mathematical model—called Free Boolean Algebra (FBA) [16, pp. 256-266]—that uses sets of sets to represent Boolean expressions.

Although structure expressions are formulas with logical operators, they are formalisms to enable automatic FTA. As shown in [17], FTs are a much richer model enabling a visual indication of fault paths, and includes description of subsystems as intermediate events.

Figure 1 shows how FTA is traditionally performed. It starts with an architectural model, then faults are identified and modelled in an FT. System requirements are identified

are used, adding or modifying the original architecture to improve dependability. The analyses are executed until system requirements are met.



The work reported in [20] aims at performing the full implementation of the Fault Tree Handbook, adding temporal gates to its Pandora<sup>1</sup> methodology. It was this implementation that introduced the new concept of TFTs, cited previously. In such trees,

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<sup>1</sup> Pandora stands for: P-AND-ORA, which translates to Priority AND, Time.

events ordering is well-defined and an algebraic framework was proposed to reduce structure expressions to obtain **minimal cut sequences** (MCSeqs) and perform probabilistic analysis. Reducing expressions is also desirable to check for tautologies, for example.

DFTs introduce very different gates to capture dynamic configurations of systems: cold spare (CSp), functional dependency (FDEP), and sequence enforcing (SEQ). The semantics of the first is to add “backup” events, so the gate is active if the primary event and all spares are active. The second adds basic events’ dependency from a trigger event. The third forces the occurrence of events in a particular order. There is also a warm spare (WSp) gate that is slightly different from the CSp gate. They differ on the nature of sparing, whether fast (warm, always-on) or slow (cold, stand-by). The readiness of the backup system in a WSp gate is higher than in a CSp gate. The work reported in [22] shows an algebraic framework to compositionally reduce DFT gates to order-based gates and perform probabilistic analysis of structure expressions. Thus, despite some limitations for spare gates [23], the structure expressions used in TFTs and DFTs can be formulated in terms of a generic order-based operator.

The  $\neg$  (NOT) operator is absent in the algebras reported in [20, 21, 3, 24]. There is no consensus about the relevance of its use: (i) it can be misleading, generating non-coherent analysis [11], or (ii) it can be essential in practical use [9]. Our concern is that the decision of the relevance of its use should not be due to the choice of events-occurrence representation. The algebra created in this work defines the NOT operator and allows its use, as we show in Chapter 4.

Hierarchically Performed Hazard Origin and Propagation Studies<sup>2</sup> (HiP-HOPS) [25] is a set of methods and tools to analyse FTs. The semi-automatic generation of FTs has architectural models and failure expressions as inputs. The failure expressions are in fact structure expressions of components or subsystems. These expressions are annotated in components and subsystems and describe how they fail. The tool combines these expressions with regard to the architecture to generate FTs. The work reported in [19] shows a strategy to use the semi-automatic FT generation of HiP-HOPS with Pandora to generate structure expressions of TFTs.

Architecture Analysis and Design Language (AADL) [26] is a standard language to model (among other features) system structure and component interaction. AADL has several tools to perform different analyses to obtain SFT to perform FTA. But AADL’s assertion framework does not express order explicitly as needed for TFT and DFT analyses.

In previous work [27, 28], we proposed a systematic hardware-based faults identification strategy to obtain failure expressions as defined in HiP-HOPS for SFTs. We considered

<sup>2</sup> <<http://www.hip-hops.eu/>>

faults in components or subsystems, but if we obtain failure expressions of a whole system, they are in fact structure expressions of an FT. Our strategy throws away the ordering information of the fault event sequences to generate failure expressions for components or subsystems for SFTs. We focused on hardware faults because we assume that software does not fail as a function of time (wear, corrosion, etc). We inherited this view from our industrial partner, which assumes that functional behaviour is completely analysed by functional verification [29]. We followed industry common practices using Simulink diagrams [30] as a starting point. The work reported in [28] was based on Communicating Sequential Processes<sup>3</sup> (CSP<sub>M</sub>) to allow an automatic analysis using the model checker FDR. Thus, our strategy required the translation from Simulink to CSP<sub>M</sub> [31]. It then runs FDR to obtain several counter-examples (which are fault traces) ending in failures. For two case studies provided by our industrial partner we showed that our automatically created failure expressions match with the engineer's provided ones or are better (a weaker proposition).

## 1.1 Research questions

Both TFT and DFT lack a first-order logic mathematical model like the one defined for SFT. For SFTs, mathematical models to reduce structure expressions are either based on set inclusion, with FBA, or through tree search, with BDD. Both are efficient. One important concern on employing FTA is whether an FT indeed represents a system behaviour. The work reported in [32] exposes this concern for DFTs, and the HiP-HOPS framework—related to SFTs and TFTs—aims at getting this issue sorted out. Our contribution to this issue for SFT is shown in [28, 27].

The mathematical model for TFT has a discontinuity between two activation states: (i) non-occurrence, and (ii) occurrence some time later. Such a discontinuity has some drawbacks as, for example, the impossibility to use NOT gates, and handling the specific case of non-occurrence with zeros in Temporal Truth Tables (TTTs). The reduction of structure expressions in TFT is based on a combination of: (i) algebraic reduction—which can unfortunately result in an infinite application of rules—, (ii) modularisation of independent subtrees (subtrees not always are independent), and (iii) dependency tree (DT) [33]—which are limited to seven basic events, due to exponential growth.

Most mathematical models [34, 35, 36] for DFT are based on the formalisation of discrete-time Markov chain (DTMC) [37, 17] or continuous-time Markov chain (CTMC) [38, 39] because DFTs were initially conceived to be a visual representation of such models. As both DTMC and CTMC are state-based, they experience the state-space explosion problem. The works reported in [40, 41, 7] show techniques to overcome this problem.

<sup>3</sup> This variant “M” is the machine-readable version of CSP.



but the reduction can be infeasible because it depends on systems' models particularly, whether they are independent or not.

There are other approaches, however. For instance, a modified version of BDD to tackle events ordering, called Sequential Binary Decision Diagram (SBDD) [42, 43], to reduce structure expressions, and the work reported in [36], which proposes a conversion of DFT into dynamic bayesian network (DBN) [44] to perform probabilistic analysis.

The approach to tackle events ordering with SBDD [43] has two kinds of nodes: terminals and non-terminals (terminals are nodes with basic events, and non-terminals are nodes with two events and an operator). Although demonstrated in [45] that these unconventional nodes (non-terminals) generate correct and efficient Boolean analysis, the analysis is still dependent on the order-related operators because the relation of terminals and non-terminals is not established directly (non-terminals are seen as an independent node in [43]). For example, the occurrence of  $A \rightarrow B$  is related to the occurrence of  $A$  and  $B$ , but this relation is obtained in a further step, not in the SBDD.

The approach using the construction of DBNs [36] is automatic and handles time slices as  $t + \Delta t$ , which implies a notion of events ordering as well. As it is focused in probabilistic analysis, qualitative analysis is not directly supported.

The works reported in [3, 43] show that DFT's operators can be converted into order-related operators, simplifying DFT analysis. Although the mathematical model presented in [3] establishes a denotational semantics for order-related operators, it lacks a formal method for expression reduction based on such a model. It defines, instead, several algebraic laws to reduce expressions and an algorithm to minimize the structure function.

HiP-HOPS proposes a hierarchical approach to model systems and perform FTA (and Failure Modes and Effects Analysis (FMEA) [46]). Although there is a tool to model and analyse systems using HiP-HOPS, FTs construction is based on an algorithm.

Another concern, left untreated in the literature, is the undesirable possibility of non-determinism in structure expressions. For example, if a commission is observed when fault A is active and an omission is observed when faults A and B are active, then the system may behave non-deterministically with a commission or omission when both A and B are active (A and B implies A).

From the exposed in this section, our research questions are:

$RQ_1$ ) Is there a consistent mathematical model to analyse TFTs and DFTs that is set-based and similar to FBA?

$RQ_2$ ) What guarantees can we provide to detect non-determinism in erroneous behaviour?

Also, does such a model:



$RQ_3$ ) have the capacity of representing events ordering similar to TFT and DFT?

$RQ_4$ ) represent systems behaviour by construction?

$RQ_5$ ) allow both qualitative and quantitative analyses as supported by TFT and DFT?

## 1.2 Proposed solution

In this work we present an algebra, called Algebra of Temporal Faults (ATF), to express ordering of fault events (TFT and DFT), enabling analysis of acceptance criteria of FTs. The laws of ATF are given in a denotational semantics based on sets of lists of distinct elements. ATF aims at answering the research questions  $RQ_1$  and  $RQ_3$ . The analysis of acceptance criteria is a decision problem and we use first-order logic and Isabelle/HOL 2015<sup>4</sup> as verification tool. Indeed, ATF is part of a bigger strategy that relates fault injection on nominal models, fault modelling, FTA, and fault tolerance patterns.

System and fault modelling is an essential step towards safety analysis. Architectural modelling is the first step of the strategy and can be executed either in a graphical tool, or as requirements in natural language. For example, our work reported in [48, 49] uses fault modelling in Systems Modelling Language (SysML) [50] to verify fault tolerance of Systems of Systems (SoSs) [51].

Writing and analysing expressions with order-related operators is more complex than analysing expressions with Boolean operators only. We propose a logic, called Activation Logic (AL), which works together with an inner algebra to perform analysis of system structure and component interaction with a focus on fault modelling and fault propagation, tackling the complexity introduced by order-related operators. AL receives an algebra and the set of operational modes of a system as parameters. The choice of algebra defines which structure expressions can be obtained: if Boolean algebra is passed as a parameter, the AL can generate structure expressions with Boolean operators (SFT); if the ATF is passed as a parameter, the AL can generate structure expressions with order-related operators (TFT and DFT). The AL requires that the inner algebras provide a set of properties (tautology and contradiction) and semantic values. The use of the NOT is essential: besides its use in expressions, we use the complement to normalise the expressions to provide healthy expressions.

To obtain critical event expressions used in FTs and to denote faults propagation, the AL provides a predicates notation and verification of non-determinism. We show three different approaches to check the non-determinism and answer research question  $RQ_2$ :

<sup>4</sup> The 2002 tutorial is reported in [47], but there is a newer version published with the tool itself. The tool and the tutorial are available on their website at <http://isabelle.in.tum.de>.

(i) verify its existence, (ii) indicate which set of operational modes are active for a combination of faults, or (iii) **what is the** combination of faults ~~that~~ activates a set of operational modes.

In our proposed solution, depending on **how much clear are the** faults, the analyst may follow one of the paths: (i) model the system in Simulink to allow fault injection and discovery, or (ii) model faults using the Activation Logic. Both paths ends with structure expressions and the **FTA** is performed using **ATF**.

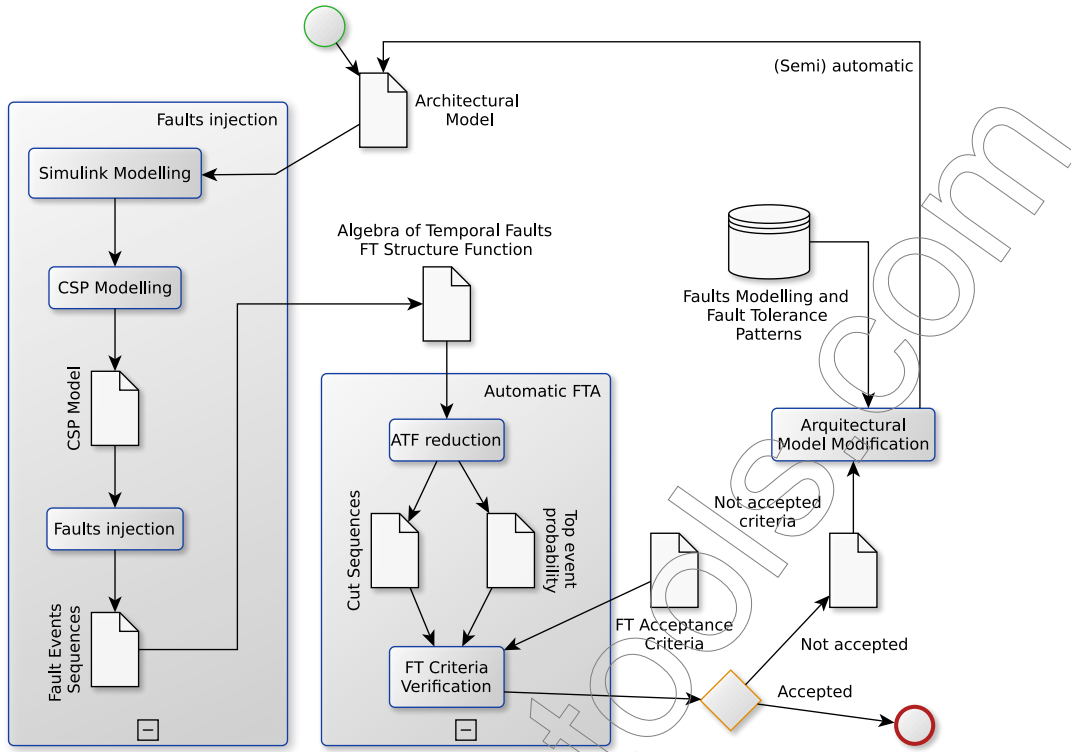
Figure 2 shows how to perform **FTA** using faults injection. The “Faults injection” block is obtained from part of our work reported in [28, 27]. It starts with Simulink modelling, converts the model to  $CSP_M$  and then obtains fault event sequences. The fault event sequences are then mapped to **ATF**, which **have** a denotational semantics based on sets of lists. This strategy aims at answering the research question **RQ<sub>4</sub>**.

Safety requirements are stated in terms of critical failures as, for example, “the probability of a complete failure of an airplane engine should be less than  $10^{-9}$ ” (quantitative), or “a complete failure of the propulsion system shouldn’t be caused by a single failure” (qualitative). In this work we call **acceptance criteria** the verification of a safety requirement on an **FT**, where **FT**’s top event is the undesired failure. Positive requirements as, for example, “the communications system should be operational 99.99% of the cruise phase” are treated as a complement (the complete failure should have a probability in less than 0.01% of the cruise phase). The acceptance criteria analysis aims at answering the **RQ<sub>5</sub>**.

From the model in **ATF** (Figure 2), the acceptance criteria are then verified. If the criteria are accepted, the process finishes. Otherwise, the system is modified, and the process continues, modifying system’s architecture, using fault tolerance patterns, improving system’s dependability.

Figure 3 shows a faults modelling strategy directly in the **AL**. The logic associates each operational mode with a faults expression. After modelling all faults, the top events are extracted in a predicates notation. For example, “**does** the behaviour of the system **is** the operational mode **X?**”, where **X** can omission, commission, etc. Given the flexibility of the **AL** notation **it is** used to reason about basic fault events and top-event failures, which are related to **RQ<sub>3</sub>**. Each predicate in **AL** generates an expression in **ATF**, which **are** reduced to obtain a normal form to obtain **MCSeqs** and to calculate top-events probability. With the system modelled in **AL**, the fault tolerance patterns can be applied directly on the model.

The complete proposed solution is summarized in Figure 4, joining the paths described in Figure 2 and Figure 3 (paths A and B, respectively).



**Figure 2** – Faults injection and ATF to perform FTA

### 1.3 Contributions

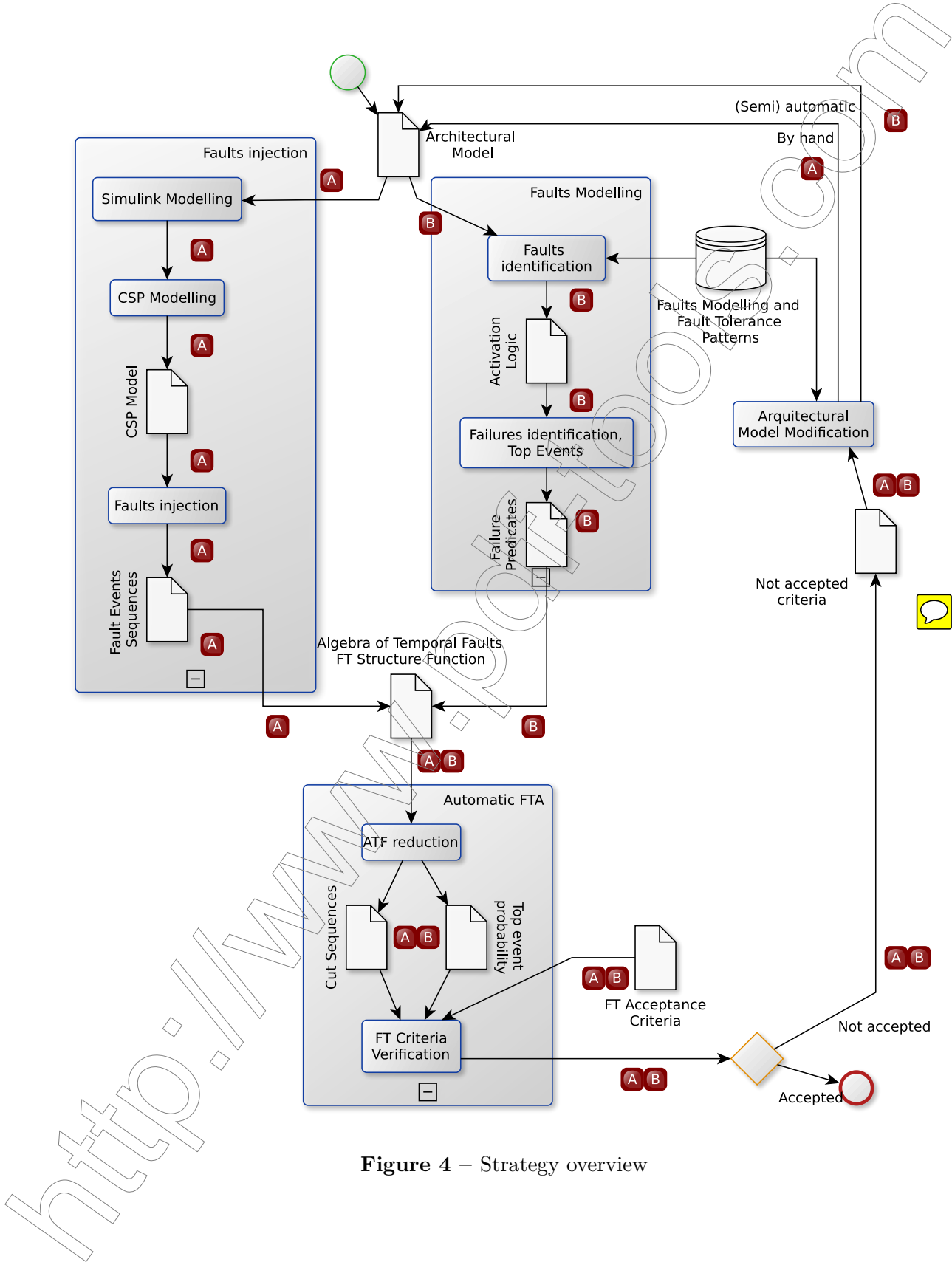
The main contributions of this work are:

- $C_1$ ) Define a denotational and algebraic model to express fault events order with ATF (Chapter 4);
- $C_2$ ) Define a new operator to express order explicitly and proving that the resulting algebra—(ATF) using this operator and Boolean operators—is a conservative extension of the Boolean algebra (also published in [52])—see Chapter 4;
- $C_3$ ) From Simulink models, obtain fault event sequences and mapping to ATF (Chapter 4);
- $C_4$ ) Reason about fault modelling in AL, to obtain formal expressions of critical failures (top-event failures, Chapter 5);
- $C_5$ ) Illustrate the application of the laws on a real case study, provided by our industrial partner (Chapter 6), and on a literature case study.

We use Isabelle/HOL, theories in Isabelle/HOL’s library, and a theory in the AFP library [53] to prove all theorems presented in this work.

The case studies cover the following scenarios, presented in Chapter 6:





**Figure 4** – Strategy overview

## 2 Basic concepts

Means to **dependability** are obtained by modelling and analysing a system. It is strongly related to fault modelling, which depends on the kinds of analyses we want to perform. **FTs** are present in several stages of systems' modelling. We introduce dependability and fault modelling in Section 2.1.

An **SFT** is a snapshot<sup>1</sup> of a faults topology of a system, subsystem or component. The time relation of fault events in **TFTs** and **DFTs** allows the analysis of different configurations (or snapshots) of a system, subsystem or component. We discuss these time relations in Section 2.2.

### 2.1 Systems, dependability, and fault modelling

Computing systems are characterized by **five properties**: functionality, performance, cost, dependability, security. The work reported in [54, p. 289–302] **explain** these properties—including dependability—with a focus on software. Hardware and software are connected, as software faults may cause a failure in a software-controlled hardware, and hardware faults may send incorrect data, causing a failure in the software.

The work reported in [8] summarizes all concepts of **and related to** dependability for computing systems that contain software and hardware. In the following, we show these concepts and highlight those used in this work.

#### 2.1.1 Systems

Before introducing systems' dependability, we first describe what a system is and its characteristics. A *system* is an entity that interacts with other systems (software and hardware as well), users (humans), and the physical world. These other entities are the *environment* of the given system, and its *boundary* is the frontier between the system and its environment.

The *function* of a system is what the system is intended to do, and its *behaviour* is what the system does to implement its function. The *total state* of a system are the means to implement its function and is defined as the set of the following states: computation, communication, stored information, interconnection, and physical condition. The *service* delivered by a system is its behaviour as it is perceived by its boundary. A system can both provide and consume services.

<sup>1</sup> Whether a top event indeed causes a catastrophic or major failure is out of the scope of this thesis; we consider that, if it is possible that such failure occur, then it will.



The *structure* of a system is how it is composed: a system is composed of components, and each component is another system, etc. This concept of hierarchical compositionality in systems is what originated the concept of SoS and is the object of analysis in HiP-HOPS. Such a recursion (of a system containing other systems) stops when a component—or a constituent system—is considered to be atomic. A system is the total state of its atomic components.

### 2.1.2 Dependability

The concepts that create the basis for dependability are: (i) threats to, (ii) attributes of, and (iii) means to attain.

*Threats to dependability* are the so-called *fault-error-failure* chain. A failure is a service deviation perceived on systems' boundary. An error is the part of the total state of a system that leads to subsequent service failure. Depending on how a system tolerate internal errors, many errors may not reach system's boundary. Finally, a fault is what causes an error. In this case, we say that the fault *occurred* (the fault is active). Otherwise, the fault is dormant, and has not occurred (yet). A *degraded* mode of a system is when there are active faults, so some functions of the system are inoperative, but the system still delivers its service.

There are two acceptable definitions of dependability reported in [8]. One is more general, difficult to measure: “the ability to deliver service that can justifiably be *trusted*”. A more precise definition that uses the definition of service failure is: “the ability to avoid service failures that are more frequent and more severe than is acceptable”. This definition has two implications about system's requirements: there should be defined how it can fail, and what are the acceptable severity and frequency of its failures.

The following systems' dependability attributes enlightens such requirements:

**Availability:** the readiness for correct service;

**Reliability:** continuity of correct service;

**Safety:** absence of catastrophic consequences on the environment (other systems, users, and the physical world). Safety can be verified using FTs, which is part of the objective of this work;

**Integrity:** absence of improper systems alterations;

**Maintainability:** ability to be modified and repaired.

A system description should mention all or most of these attributes, at least the first three of them.



The implementation of these attributes requires a deep analysis of system's models. The *means to attain dependability* are summarized as follows:

**Prevention** is about avoiding incorporating faults during development.

**Tolerance** deals with usage of mechanisms to still deliver a—possibly degraded—service even in the presence of faults.

**Removal** is about detecting and removing (or reducing severity of) failures from a system, both in the development and production stages.

**Forecasting** is about predicting likely faults so they can be removed, or tackling their effects.

The intersection of the current work with dependability is in fault removal during development and fault tolerance (analysis). Following the taxonomy presented in [8], there are some techniques for fault removal, summarized as follows:

a) Static verification:

– Structural model:

**Static analysis:** Range from inspection or walk-through, data flow analysis, complexity analysis, abstract interpretation, compiler checks, vulnerability search, etc.

**Theorem proving:** Check properties of infinite state models.

– Behaviour model:

**Model checking:** Usually the model is a finite state-transition model (Petri-nets (PNs), finite state automata). Model-checking verifies all possible states on a given system's model.

b) Dynamic verification:

– Symbolic inputs:

**Symbolic Execution:** It is the execution with respect to variables (symbols) as inputs.

– Actual inputs:

**Testing:** Selected input values are set on system's inputs and their outputs are compared to expected values. The verification outcomes are observed faults, in case of hardware testing or software mutation testing, and criteria-based, in case of software testing.

Verification methods are often used in combination. For example, symbolic execution may be used to obtain testing patterns, test inputs can be obtained by model-checking

as in [55], faults can be used as symbolic inputs, and system behaviour can be observed using model-checking as in [28, 27] (This technique is called fault injection; see also [56]).

The techniques to attain fault tolerance are summarized as follows:

**Error detection:** is used to identify the presence of an error. It can be a concurrent or a preemptive detection. Concurrent detection takes place during normal service, while preemptive detection takes place while normal service is suspended.

**Recovery:** transforms a system state that contains errors into a state without them. The behaviour of the system upon recovery is equivalent to the normal behaviour. Techniques range from rollback to a previously saved state without errors, error masking, isolation of faulty components, to reconfiguration using spare components.

In this work, we use a combination of: (i) fault-injection, (ii) theorem proving, and (iii) symbolic execution. We use these methods to obtain an erroneous behaviour of the system which is compared to the system dependability attributes (safety). We explain how these methods are combined in Chapter 4.

On the analyses of systems and its constituents, there is a distinction of operational modes and error events. Operational modes refer to the behaviour that is perceived on the boundaries of a system (*failure*). Error events, on the other hand, represent the behaviour detected in a constituent of a system. Such error events may relate to an operational mode, but not necessarily. Further in Chapter 4 we abstract these differences and leave the distinction as a parameter. We refer to such a set as *operational modes*.

### 2.1.3 Fault Modelling

Fault modelling plays an important role in reasoning about the fault-error-failure chain. They are the initial steps to perform the verification of a system, starting in the architectural model to reason about the critical failures, which are (in general) the top-events in FTs.

**SysML** is a profile for Unified Modelling Language (**UML**) that provides features to model structure and behaviour of systems. The works reported in [48, 49] define several structural and behavioural views in **SysML** to model the fault-error-failure chain and fault tolerance. Fault, error, failures, and fault propagation have structural views, which are related to behavioural views to describe fault activation and recovery. These works map **SysML** to two formal languages—COMPASS Modelling Language (**CML**) [57] and Communicating Sequential Processes (**CSP**) [58], respectively—to verify fault tolerance.

In [7] the safety assessment process for civil airborne systems and equipment describes development cycles and methods to “clearly identify each failure condition”. The

methods that involve failure identification are: (i) **SFT**, (ii) Dependence Diagram<sup>2</sup> (**DD**) [59, p. 198], (iii) **Markov chain**, and (iv) **FMEA**. The first three are top-down methods, that start with undesired failure conditions and moves to lower levels to obtain more detailed conditions that causes the top-level event. **DDs** are an alternative method of representing the data in **SFT**. **FMEA** is a bottom-up method that identifies failure modes of a component and determines the effects on the upper level. We detail **SFT** in Subsection 3.1.1.

**DFTs** are an extension of **SFTs** and models dynamic behaviour of system faults.



Similarly to the relation of **SFTs** and **DDs**, the work reported in [60] demonstrates the relation of **DFTs** to Dynamic Reliability Block Diagrams (**DRBDs**) [60]. As the models (**DFT** and **DRBD**) are equivalent, this work sticks to **DFT** due to the amount of work already published. We detail **DFTs** in Subsection 3.1.3.

## 2.2 Time relation of fault events

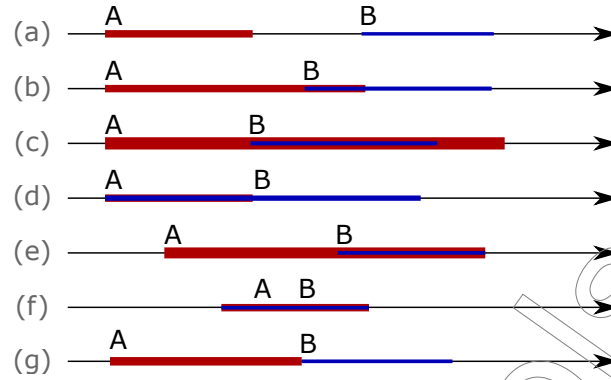
The most general case for time relations is to consider that each fault event has a continuous time duration. They are the basis on how fault events discretisation are defined. The point of view in this work is the analysis of the effects caused by a combination of faults in a snapshot of a system state. In Figure 5 we show all possibilities of events relations in a continuous time line from A to B (the converse relation is similar):

- a) A starts and ends before B starts;
- b) A starts before and ends after B has started, but before B has ended;
- c) A starts before B and ends after B has ended (A contains B);
- d) A and B start at the same time, but A ends before B;
- e) B starts after A, but they end at the same time;
- f) A and B start and end at the same time;
- g) A starts before B and ends when B starts.

Considering that fault occurrence corresponds to the start of a fault event and its duration, from Figure 5 we clearly identify which event comes first: A comes before B, except in the cases of items d) and f), where they start exactly at the same time. Even in the case of failures that have a common cause, there may be a slight fraction of time between failures. For example, an electromagnetic pulse (**EMP**) may cause a failure in all electronics, or a power shortage may cause a failure in all cooling systems in a power plant (see Fukushima accident [61]). There is a (temporal) causation relation of an **EMP** occurrence and the failures in all electronics, and also of a power shortage and the cooling systems's shutdown in Fukushima. On the other hand, there is no direct relation

<sup>2</sup> Also known as Reliability Block Diagram (**RBD**).

of the failure in each electronic, nor the failure in each cooling system. So, even if failure events have a common cause, and are not the same, they are *statistically independent*. The relations of items a) and g) shows the case that the system was repaired, thus A is not active when B starts.



**Figure 5** – Relation of two events with duration

In Chapter 4 we abstract the relation of events in continuous time as an *exclusive before* relation, based on fault *occurrence* (it is similar—at least implicitly—to what is reported in [20, 22]).

## 3 Analysis and tools

Structure expressions are used to analyse fault trees. In general, a structure expression comes from gates semantics and basic events. Basic events become variables and gates become operators (a gate may become one or more operators). In Section 3.1 we explain **SFTs**, **TFTs**, **DFTs**, and their respective structure expressions.

**FBA**s and **BDD**s are the basis to analyse structure expressions. Also, we were inspired by **FBA** concepts to create our Algebra of Temporal Faults (Chapter 4). We explain **BDD**s and derived techniques in Section 3.2, and **FBA**s in Section 3.3.

The use of the Boolean operator *NOT*: (i) can be misleading, generating non-coherent fault trees, or (ii) can be essential in practical use. We discuss such cases in Section 3.4.

In Section 3.5 we show the probability calculation of an **FT** with an explicit **NOT** operator.

To reuse a nominal model to analyse faults we need fault injection. In Section 3.6 we explain how we used Simulink and **CSP<sub>M</sub>** to inject faults and obtain failure expressions from a nominal model.

Finally, in Section 3.7 we present basic usage of Isabelle/HOL and Intelligible semi-automated reasoning (**Isar**), which were essential to carry out the proofs presented in this thesis.

### 3.1 Fault Tree Analysis and structure expressions

**FTA** was introduced in the Fault Tree Handbook [18] with Static Fault Trees. **FTA** is a deductive method that investigates what are the possible causes of an unwanted event. The method starts with the top-level event as the unwanted event and the combination of lower-level events that can cause it. Events are combined using gates, and each gate has a well defined semantics. It continues until basic (atomic) events are reached. An **SFT** represents, in a single view—very often considering faults outside of the boundaries of a system—different states in which a particular failure (top event) is active in a system. The most traditional gates are **AND** and **OR**, which are equivalent to Boolean operators. These gates are also called coherent gates because they construct coherent trees (see Section 3.4 about the use of **NOT** gates). The Fault Tree Handbook shows other gates as, for example, the **PAND** gate, but the **FTA** with these gates is not well defined. **SFT**'s gates and analysis are detailed in Subsection 3.1.1.

**TFTs** were created aiming at fully implementing the Fault Tree Handbook. The **PAND** gate was first defined for **SFTs**, but its analysis was left open in the handbook. The semantics (and analysis) of **TFTs** is defined in terms of a denotational semantics based on *sequence values* to express ordering of events, thus tackling **PAND**'s order. We explain **TFTs** and the sequence values in Subsection 3.1.2.

With component and system design evolution, **DFTs** were created to tackle dynamic behaviour: fault-tolerance-related components (**CSp**), functional dependency (**FDEP**), and analysis of particular order of occurrence of faults (**SEQ**). **SFT**'s gates are still present as **DFT**'s gates. We explain them and **DFT**'s analysis in Subsection 3.1.3.

The structure of an **FT** (or the structure of an **MCS**, explained further) is represented with a formula. The variables represent occurrences of basic events. Unary and binary relation symbols capture the semantics of gates. A formula with these characteristics is called *structure expression* or *structure function* (as the expression depends on the variables). The semantics of a structure expression is that the top-level event occurs if some combination of basic events occur.

The results obtained from the **FTAs** are shown in the Fault Tree Handbook. We summarize them as:

a) Qualitative

**MCSs:** **Smallest combinations** of component failures causing system failure. They are obtained from the reduction of structure expressions to a normal form. For example, in **SFTs**, structure expressions are reduced to disjunctive normal form (**DNF**). Each term in a reduced **DNF** is an **MCS**.

**Importance:** Qualitative rankings on contributions to system failure. A single fault causing a catastrophic failure is usually unacceptable. Ranking **MCSs** is the same as ordering them in ascending order of their size (smaller first).

b) Quantitative

**Numerical probabilities:** Probabilities of system and **MCS** failures. A system failure probability is obtained by assigning probabilities to basic events and then calculating it according to the gate semantics. **MCS** failure probability is the calculation of the probability of the occurrence of *all* basic events of a specific **MCS**.

**Importance:** Quantitative rankings on contributions to system failure. Ranking **MCSs** is the same as ordering them in descending order of some unreliability formula (higher first). These formulas used to **calculate importance** vary. The most common are: (i) system unavailability, and (ii) system failure occurrence rate.

**Sensitivity evaluation:** Modifying characteristics of components and evaluate their impact. For a particular event in a tree, a higher and a lower failure probability value are assigned. If the system unavailability is not changed, then such an event is not important—the system is not sensitive to such an event.

As stated in [62], there are other uses of FTA. One of great importance is using it to minimize and optimize resources, which has been object of study in HiP-HOPS [63]. Through importance measures, FTA not only identifies what is important but also what is unimportant. This removes components without impacting the overall failure probability, which is related to the quantitative importance and sensitivity evaluation.

In important stages of critical systems, FTA plays an essential role. At least three dependability means can be achieved using FTs:

**Removal.** An FTA calculates the probability of failure of a subsystem. If such a probability is higher than a certain maximum reference, such a subsystem should be removed or left to be incorporated in combination with a more reliable component.

**Tolerance.** An FTA indicates whether a single fault—or fewer combinations than expected—could lead to a catastrophic failure. In this case, a system should have replication, or stages of fault detection and error handling. Also, the probability of failure of the chosen fault tolerance method can be evaluated.

In Subsections 3.1.1 to 3.1.3 we briefly show FT symbology and means to analyse FTs. We detail its structure expression extraction because they are a common means to perform both qualitative and quantitative analysis.

### 3.1.1 Static Fault Trees

SFT gates and structure expressions were used as basis for other kinds of tree, as in TFTs and DFTs. We explain their symbology and semantics in this subsection.

The Fault Tree Handbook shows traditional symbols for gates and events. Basic events are usually drawn as rectangle (for the text) and a circle below it, as shown in Figure 6, or as a circle with the text of the basic event, as shown in Figure 7. Top-level and intermediary events are drawn as a rectangle (for the text) and a gate below it, as shown in Figures 6 and 7. When an FT becomes too large, transfer in and out symbols can be used. They are usually drawn as triangles with a letter or a number. Figure 7 depicts traditional gates as specified in the Fault Tree Handbook, and Figure 6 shows an FT using the Fault Tree Analyser<sup>1</sup>—a free commercial tool. In this work, to keep a visual identity with other FTs, and to avoid symbol confusion, we use gate symbols as shown in Figure 8.



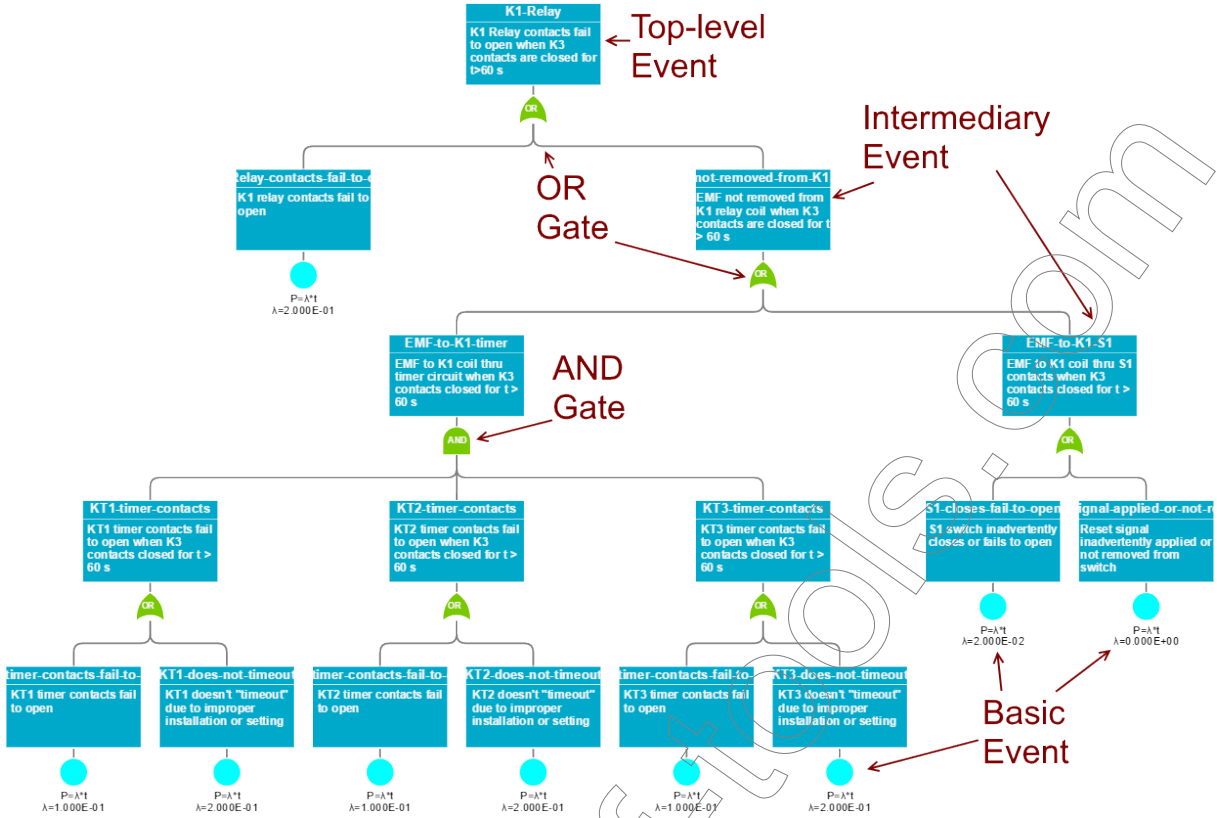


Figure 6 – SFT symbols using a free commercial tool

Structure expressions in FTA are defined in terms of set theory, using symbols for fault events occurrence. If a fault event symbol is in a set, then it means that this fault has occurred. A set is a combination of fault events that causes the occurrence of the top-level event of a tree. A structure expression of a tree is denoted by a set of sets of fault event combinations. The OR gate becomes the union operator between sets and the AND gate, the intersection. For example, if a system contains fault events  $a$ ,  $b$ , and  $c$ , fault trees for this system contain at most all these three events. The occurrence of the fault event  $a$  is denoted by a set of sets  $A$ , which contains the following sets:

- $\{a\}$ : only  $a$  occurs;
- $\{a, b\}$ :  $a$  and  $b$  occur in any order;
- $\{a, c\}$ :  $a$  and  $c$  occur in any order;
- $\{a, b, c\}$ : all three events occur in any order.

All sets of  $A$  contain the fault event  $a$ . Similarly, the sets of sets  $B$  that represents the occurrence of  $b$  contains all sets that contain the fault event  $b$  (it includes the set  $\{a, b, c\}$ , for example).

The fault tree in Figure 9 contains only two events and the resulting structure expression for this FT is the expression  $A \cap B (TOP)$ , where  $A$  and  $B$  are the sets of sets

<sup>1</sup> <<http://www.fault-tree-analysis-software.com>>, accessed 2/feb/2016



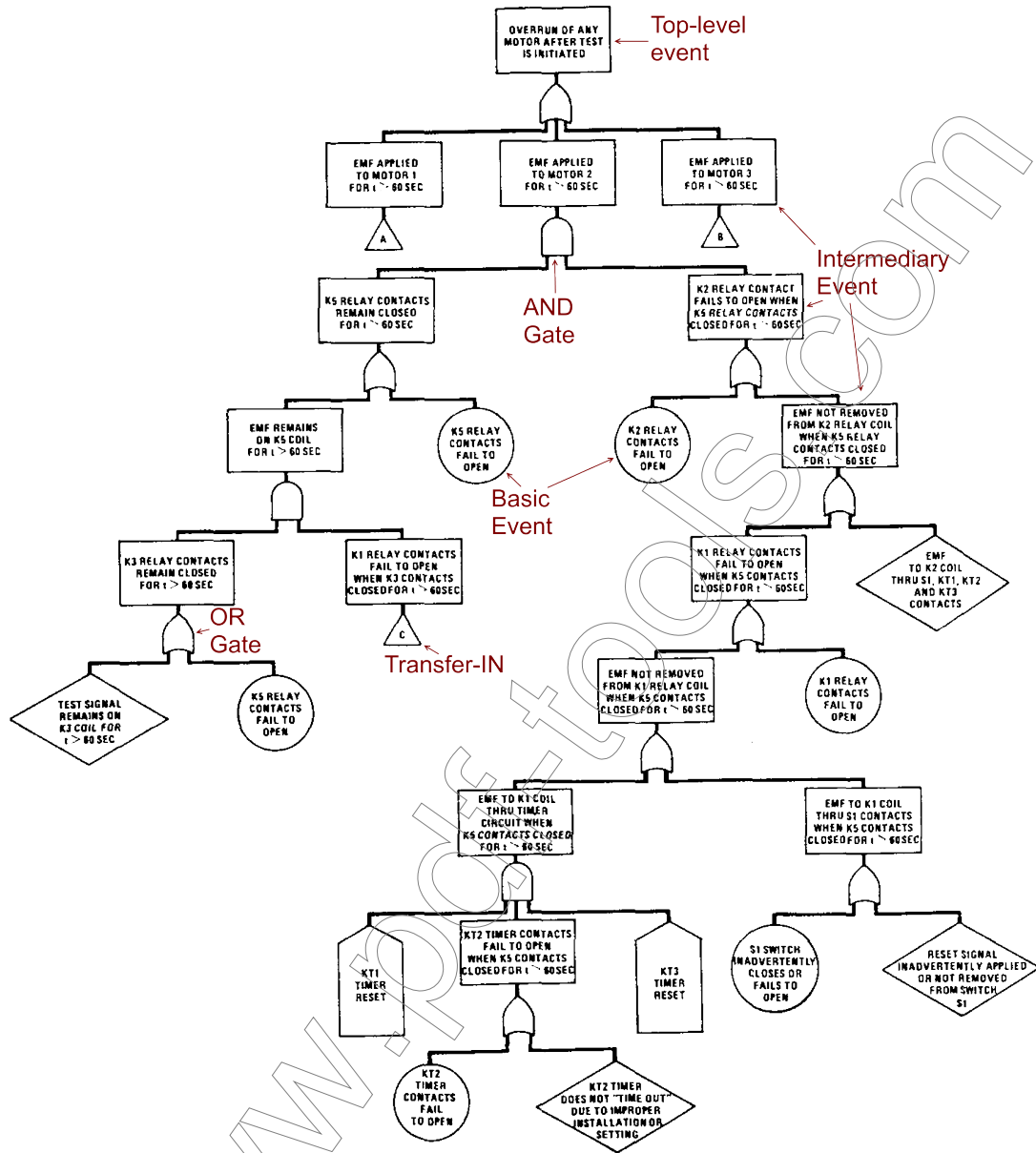


Figure 7 – SFT symbols as in the Fault Tree Handbook

that contain  $a$  and  $b$ , respectively. The resulting combinations for  $TOP$  are  $\{a, b\}$  and  $\{a, b, c\}$  (fault events  $a$  and  $b$  occur in all possibilities).

After obtaining structure expressions, the next step is to reduce the expressions to a normal form to obtain the  $MCS$ s, which are the sets that contain the minimum and sufficient events to activate the top-level failure. Probabilistic analysis is then performed on these events to obtain the overall probability of occurrence of the top-level event. The Fault Tree Handbook shows an algorithm based on Shannon's method to reduce structure expressions to obtain minimal cut sets. The Boolean expression of the tree shown in Figure 9 is  $TOP = A \wedge B$ . A technique called  $BDD$ —which derives from Shannon's method—is explained in Subsection 3.2.2.

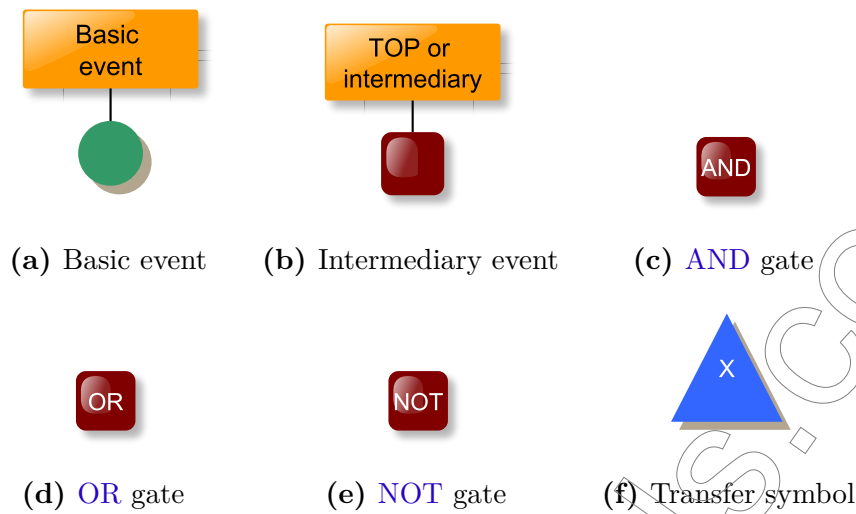


Figure 8 – SFT gates

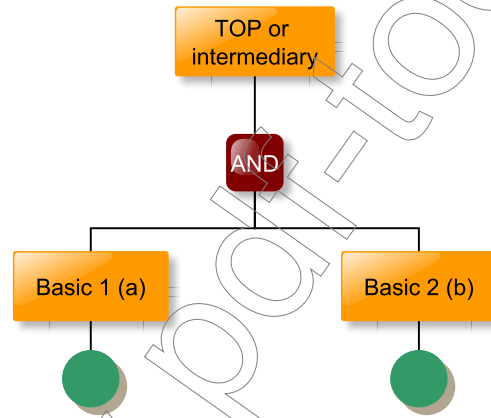


Figure 9 – Very simple example of a fault tree

### 3.1.2 Temporal Fault Trees

There are at least two versions of TFTs. One is described in [64] and uses a more traditional style of temporal logic (a variation of linear temporal logic (LTL)). The other version is called Pandora and is the one we refer to in the following.

TFTs express ordering of events by directly focusing on ordering relationships rather than different states of a system. Basically they extend SFT's PAND gates, allowing analysis of FT with such gates. It is simpler to express than DFT, but lacks the fault-tolerance-related gate of DFTs (which we show in Subsection 3.1.3).

Structure expressions are also present in TFTs [20, 21, 33], through the Pandora methodology. These expressions use the SFT operators OR and AND, and three new operators<sup>2</sup> related to events ordering: priority-AND (PAND), priority-OR (POR), and simultaneous-AND (SAND). The semantics of the PAND in TFTs is similar to the

<sup>2</sup> In formulas, the following symbols are used to represent the operators, respectively: “<”, “|”, and “&”

semantics of the **PAND** described in the Fault Tree Handbook. To avoid ambiguous expressions, the semantics in **TFTs** is stated in terms of natural numbers, using a *sequence value* function. For every possible combination of events ordering, it assigns a sequence value to each fault event. For example, if event A occurs before event B, then the sequence value of A is lower than the sequence value of B, and one formula to express this is  $A < B$ .

An invariant on sequence values is that there are no gaps for assigned values. For example, if faults A and B occur at the same time and there are only these two events, then they should both be assigned value 1. On the other hand, if A occurs before B, then the assigned values are 1 and 2, respectively. The possible values increases with the number of variables to express the cases that all events occur in different times, for example, A occurs before B, and B occurs before C. In this case, the assigned values are 1, 2, and 3, respectively. Value zero means that the event is not active on the combination. Similar to Boolean's truth tables, the Pandora methodology defines **TTTs**. They represent formula values for every combination of events. Table 1 shows the **TTT** of all **TFT** operators according to the semantics described in terms of a sequence value function  $S$  as follows:

$$S(A \wedge B) = \begin{cases} \max(S(A), S(B)) & \text{if } S(A) > 0 \wedge S(B) > 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.1a)$$

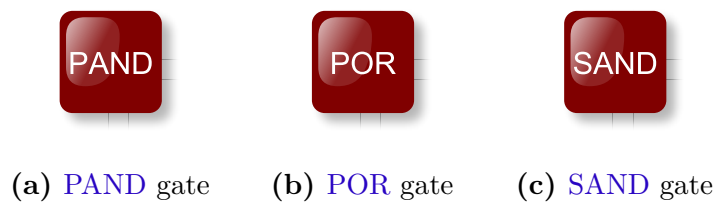
$$S(A \vee B) = \begin{cases} \min(S(A), S(B)) & \text{if } S(A) > 0 \wedge S(B) > 0 \\ \max(S(A), S(B)) & \text{otherwise} \end{cases} \quad (3.1b)$$

$$S(A < B) = \begin{cases} S(B) & \text{if } S(A) > 0 \wedge S(B) > 0 \wedge S(A) < S(B) \\ 0 & \text{otherwise} \end{cases} \quad (3.1c)$$

$$S(A | B) = \begin{cases} S(A) & \text{if } S(A) < S(B) \vee S(B) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.1d)$$

$$S(A \& B) = \begin{cases} S(A) & \text{if } S(A) > 0 \wedge S(B) > 0 \wedge S(A) = S(B) \\ 0 & \text{otherwise} \end{cases} \quad (3.1e)$$

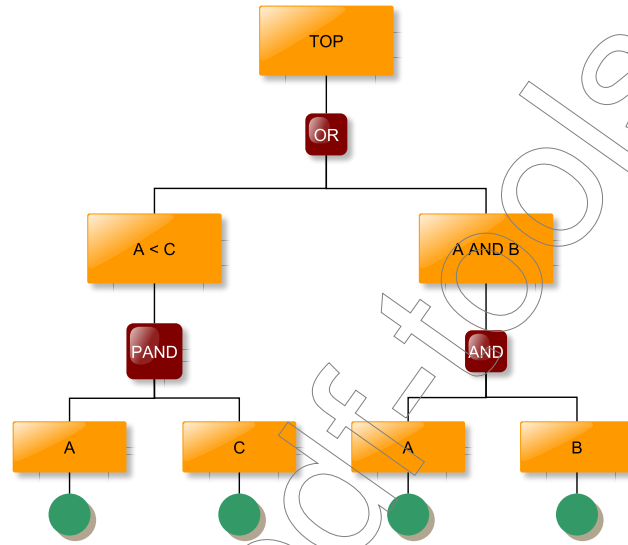
Figure 10 shows **TFT**-specific symbols used in this work. To illustrate **TFTs**, for the formula  $(A < C) \vee (A \wedge B)$ , we show: (i) the **TFT** in Figure 11, and (ii) its corresponding **TTT** in Table 2 (the column '#' indicates the **MCSeq** number).



**Figure 10** – **TFT**-specific gates

**Table 1** – TTT of TFT’s operators and sequence value numbers

A	B	AND	OR	PAND	POR	SAND
0	0	0	0	0	0	0
0	1	0	1	0	0	0
1	0	0	1	0	1	0
1	1	1	1	0	0	1
1	2	2	1	2	1	0
2	1	2	1	0	0	0

**Figure 11** – TFT small example

From structure expressions in order-sensitive FTs (TFT and DFT), MCSeqs are obtained. Several approaches represent MCSeq ordering differently. For the best of our knowledge they are introduced in the work [65] similarly to MCS, allowing set elements with arrows (“→”) to represent order.

For TFTs, in the work [21] MCSeqs are represented as a DNF using AND and the temporal operators (PAND, POR, and SAND) as doublets (a single temporal relation)—which are the minimal terms—or prime implicants—in the DNF. In a doublet, the expression is a product (AND) of temporal operators, and each temporal operator contains *exactly* two events. The conversion to doublets uses the temporal laws as shown in [21]. For example, the expression  $(X \& Y) | Z$  is a temporal relation (POR) of a temporal relation (SAND). To extract MCSeqs it needs to be converted to  $[X \& Y] \wedge [X | Z] \wedge [Y | Z]$  (the square brackets is the doublets notation and the conversion is the definition of the *Temporal Distributive Law* [21, p. 120]).

The normal form for TFT is similar to that for SFT: it is a DNF with temporal operators (PAND, POR, SAND) in the minimal terms. The reduction of TFT structure expressions is achieved using DT. In a DT, if all children of a tree node are true, then the

**Table 2** – TTT of a simple example

#	A	B	C	$A < C$	$A \wedge B$	$(A < C) \vee (A \wedge B)$
01	0	0	0	0	0	0
02	0	0	1	0	0	0
03	0	1	0	0	0	0
04	0	1	1	0	0	0
05	0	1	2	0	0	0
06	0	2	1	0	0	0
07	1	0	0	0	0	0
08	1	0	1	0	0	0
09	1	0	2	2	0	2
10	1	1	0	0	1	1
11	1	1	1	0	1	1
12	1	1	2	2	1	1
13	1	2	1	0	2	2
14	1	2	2	2	2	2
15	1	2	3	3	2	2
16	1	3	2	2	3	2
17	2	0	1	0	0	0
18	2	1	0	0	2	2
19	2	1	1	0	2	2
20	2	1	2	0	2	2
21	2	1	3	3	2	2
22	2	2	1	0	2	2
23	2	3	1	0	3	3
24	3	1	2	0	3	3
25	3	2	1	0	3	3

node is also true. Conversely, if a node is true, then all its children are also true. An issue with DTs is that they grow exponentially. According to the work reported in [33], it is already infeasible to deal with seven fault events in TFTs. Although there is a solution, it is based on a mixed application of DTs, modularisation of independent subtrees, and algebraic laws [20]. We show DTs in Subsection 3.2.3. Some of these algebraic laws are:

$$(X < Y) \vee (X \& Y) \vee (Y < X) = X \wedge Y \quad \text{Conjunctive Completion Law} \quad (3.2a)$$

$$(X | Y) \vee (X \& Y) \vee (Y | X) = X \vee Y \quad \text{Disjunctive Completion Law} \quad (3.2b)$$

$$(X | Y) \vee (X \& Y) \vee (Y < X) = X \quad \text{Reductive Completion Law 1st} \quad (3.2c)$$

$$(X \wedge Y) \vee (X | Y) = X \quad \text{Reductive Completion Law 2nd} \quad (3.2d)$$

### 3.1.3 Dynamic Fault Trees

Dynamic Fault Trees were designed with the goal of analysing complex systems with dynamic redundancy management and complex fault and recovery mechanisms [1]. The idea was to create easy-to-use and less error-prone modelling tools than using DTMCs—or simply Markov chains—directly. So, since the very beginning, DFTs were intended to be evaluated using Markov chains. Figure 12 depicts the original gate symbols as shown in [1, 2]. In this work, we use gate symbols as depicted in Figure 13. The informal semantics of them are:

**FDEP:** When the trigger event occurs, the dependent events are forced to occur. Timing in this gate between trigger event and dependent events occurrences can be instan-

taneous (like in **TFT**'s **SAND** gate), or a small amount of time, thus implying an order of occurrence, depending on the kind of dependency.

**CSp**: It is a specific gate to handle spare components. It is important to note that connected inputs are not components—they are fault events of connected components. If the  $i$ th input is already active (fault has occurred), then it is expected that the input  $(i + 1)$ th is not, following the specified order. The output becomes true after all connected inputs become true. A spare event can be connected to more than one **CSp** gate, representing the spare unit connection to one or more components.

**PAND**: The same as in **TFT**: when the connected input events occur in the specified order, it outputs true.

**SEQ**: The connected events *shall* occur in the specified order. It is different from the **PAND** gate, because the latter *detects* the specified order. The usage of this gate is usually associated with **FDEP**.

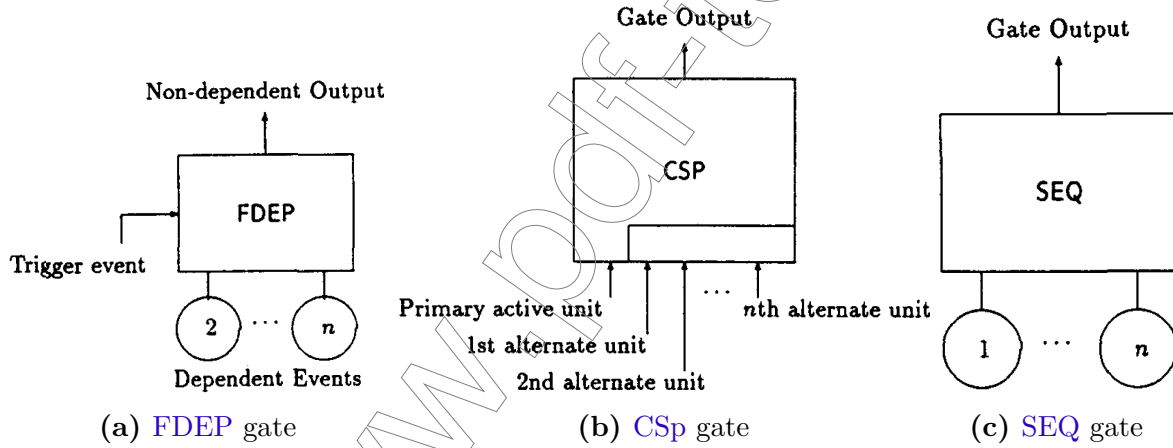


Figure 12 – DFT's original gates symbols

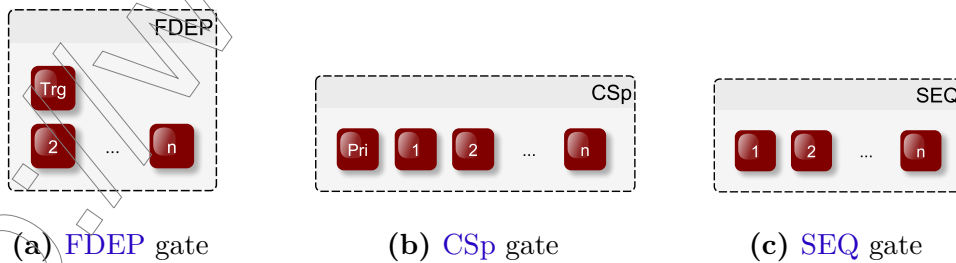


Figure 13 – DFT's gates symbols

There are several means to analyse DFT's qualitative and quantitatively. The works reported in [3, 66, 22, 23] use structure expressions to perform both qualitative and quantitative analysis, and the work reported in [23] summarizes other approaches. We increment their summary (Table 3) and categorize them as:

**Table 3** – DFT conversion to calculate probability of top-level event

Conversion	Calculation	Explained in
Automaton-like structure	CTMC	[35]
Bayesian network (BN) [67]	Inference algorithm (model-specific)	[36]
Stochastic well-formed net (SWN) [68] (a kind of coloured Petri-net (CPN) [69])	CTMC	[70]
SBDD (a modified version of BDD)	model-specific	[42, 43]

- a) Finding MCSeqs (qualitative analysis) is obtained by replacing DFT gates with SFT gates, using the text as its logical constraints. MCSs in the SFT are expanded using timing constraints from the texts into MCSeq. In this case, the behaviour of spare events cannot be correctly taken into account;
- b) Quantitative analysis consists in converting a DFT to a well-defined formalism to calculate the probability of its top-level event. Table 3 shows the conversion options, the calculation, and where the method is explained.

In [3, 66, 22] fault events occur in a specific time and are instantaneous (similar to detected faults), stated through a “date-of-occurrence” function. As the “date-of-occurrence” function is stated in continuous time, the probability of two events occurring at the same time is negligible. In fact, useful information is obtained from the possibilities of relation in time of the occurrence of the events. DFT gates’ algebraic model is summarized in Table 4. Structure expressions are written with an algebra that has operators OR and AND, and three new operators to express events ordering: (i) non-inclusive-before (NIBefore), (ii) simultaneous (SIMLT), and (iii) inclusive-before (IBefore). The NIBefore and the SIMLT operators are similar to TFT’s POR and SAND operators, respectively. The IBefore is a composition of NIBefore and SIMLT operators. Table 5 summarizes the date-of-occurrence function for all operators. An infinite value means the event never occurs.

MCSeqs are extracted from normal form of structure expressions written in a DNF. Minimal terms are products of variables and NIBefore operators (the other operators can be written as combinations of NIBefore). The reduction of DFT structure expressions uses algebraic laws as, for example:

$$(a \triangleleft b) \vee (a \triangle b) \vee (b \triangleleft a) = a \vee b \quad (3.3a)$$

$$(a \wedge (b \triangleleft a)) \vee (a \triangle b) \vee (b \wedge (a \triangleleft b)) = a \wedge b \quad (3.3b)$$

$$(a \triangleleft b) \wedge (b \triangleleft a) = a \triangle b \quad (3.3c)$$



**Table 4** – Algebraic model of DFT gates with inputs  $A$  and  $B$ 

Gate	Algebraic model of gate's output	Note
FDEP	$A_T = T \vee A$ and $B_T = T \vee B$	$A_T$ and $B_T$ replace $A$ and $B$ on the resulting expression
CSp	$(B_a \wedge (A \triangleleft B_a)) \vee (A \wedge (B_d \triangleleft A))$	$A$ is the active input, and $B$ is the spare. Subscripts $a$ and $d$ represent component's state— <i>active</i> and <i>dormant</i> , respectively, which are used on the failure distribution formulas
PAND	$B \wedge (A \trianglelefteq B)$	No distinction of active or dormant states.

**Table 5** – Date-of-occurrence function for operators defined in [3]

Operator	Expression	Expr. value if $d(a) < d(b)$	Expr. value if $d(a) = d(b)$	Expr. value if $d(a) > d(b)$
OR	$d(a \vee b)$	$d(a)$	$d(a)$	$d(b)$
AND	$d(a \wedge b)$	$d(b)$	$d(a)$	$d(a)$
NIBefore	$d(a \triangleleft b)$	$d(a)$	$+\infty$	$+\infty$
SIMLT	$d(a \triangle b)$	$+\infty$	$d(a)$	$+\infty$
IBefore	$d(a \trianglelefteq b)$	$d(a)$	$d(a)$	$+\infty$

Figure 14 shows an example of DFT extracted from [23]. It is a cardiac assist system (HCAS), which is divided in four modules: trigger, CPU unit, motor section, and pumps. The trigger is divided in two components, CS and SS. The failure of any CS or SS, triggers a CPU unit failure. The primary CPU (P) has a warm<sup>3</sup> spare (B). The motor module fails if both M and MC fail. In order for the pumps unit to fail, all three pumps need to fail, and the left-hand side spare gate needs to fail before (or at the same time as) the right-hand side spare gate (AND gate<sup>4</sup>). The top-level event structure expression is:

$$\begin{aligned}
 SYSTEM = & CS \vee SS \vee (M \wedge MC) \vee \\
 & (P \wedge (B_d \triangleleft P)) \vee (B_a \wedge (P \triangleleft B_a)) \vee \\
 & (BP_a \wedge (P2 \triangleleft P1) \wedge (P1 \triangleleft BP_a)) \vee (P2 \wedge (P1 \triangleleft BP_a) \wedge (BP_a \triangleleft P2))
 \end{aligned} \tag{3.4}$$

### 3.2 Structure expression analysis

In this section we explain the difference of stateful and stateless methods to analyse structure expressions and detail the stateless methods. Another common approach to analyse an FT is to perform structure expression analysis based on algebraic laws. Boolean

<sup>3</sup> Warm spare gates only differ from CSp on the activation time.

<sup>4</sup> Although the original example uses a PAND gate, accordingly to the informal description, a SEQ gate would fit better.



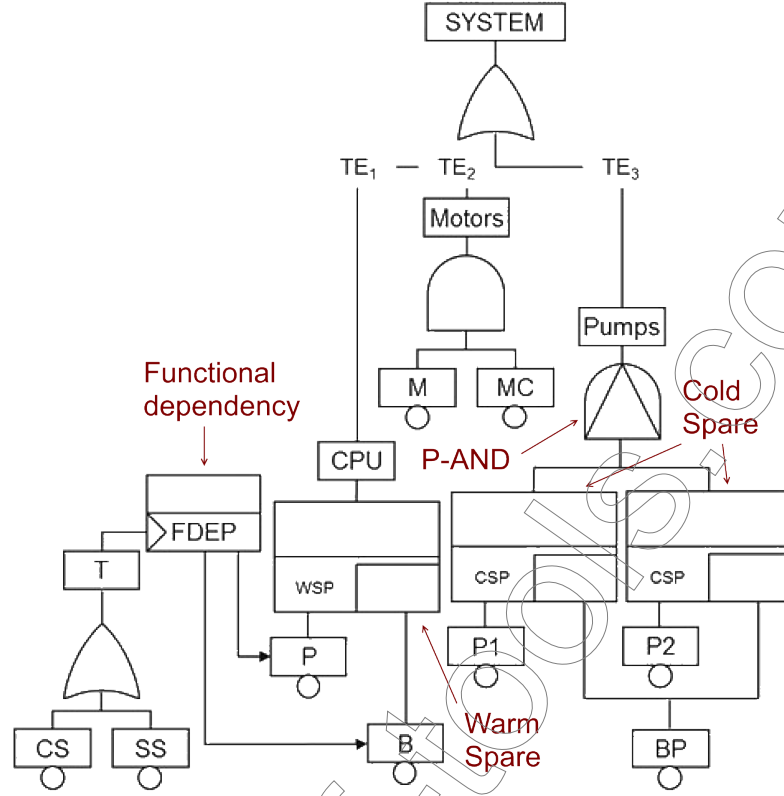


Figure 14 – DFT example

laws are well-known and are used for SFTs, temporal laws [21, 33] for TFTs, and the works reported in [3, 22] show laws for DFTs. An issue with algebraic laws is that, in some cases, the expression needs to be expanded before it gets reduced, so reduction automation is not trivial without a theorem prover. For example, the following TFT's structure expression needs to be expanded [33] before it gets reduced:

$$\begin{aligned}
 & (X \wedge Y) \vee ((X < Y) \wedge Z) \\
 &= (X < Y) \vee (X \& Y) \vee (Y < X) \vee ((X < Y) \wedge Z) && \text{by Eq. (3.2a)} \\
 &= (X < Y) \vee (X \& Y) \vee (Y < X) && \text{by Boolean absorption of } X < Y \\
 &= X \wedge Y && \text{by Eq. (3.2a)}
 \end{aligned}$$

A denotational semantics to Boolean expressions—and consequently to SFT—is given by FBAs (Section 3.3). Using denotational semantics in favour of axiomatic laws is preferred as the axioms must be very simple or must be generally accepted, because a subtlety in an axiom (an unfounded axiom) may invalidate a whole theory.

There are several works with stateful analysis methods for FTs (SFT, TFT, and DFT). We show some of them in Subsection 3.2.1.

### 3.2.1 Stateful methods and temporal logic analysis

The work reported in [71] shows a formal approach to analyse **SFT** using Interval Temporal Logic (**ITL**) [72]. Instead of tackling basic events ordering (as in **PAND**), it considers a causal relation over a gate, as for example, a relation of a basic event and a higher-level intermediary event.

For **TFTs**, the works reported in [73, 74] show an inverse solution. They map Finite State Machines (**FSMs**) to Pandora logic, then verify system properties. They show that such a mapping simplifies expression reduction, thus improving performance on the analysis.

Although there is formal modelling approaches to **DFTs**, they do not implement a direct modelling of the **DFT** itself. Instead, most of the works propose a formal modelling using a state-based approach. The work reported in [35] shows a formal model of **Markov chains** in the Z Notation (**Z**) [75] and each **DFT** element (basic events and gates). The analysis uses a quantifier on states of the resulting **Markov chain** automaton. The work reported in [76] shows a methodology to perform a modular analysis of **DFTs** based on **BDD** and **Markov chain**. As **DFT** extends **SFT**, it identifies subtrees that are purely **SFT** and uses **BDD**, otherwise. It performs **Markov chain** analysis. Still on the state-based approaches, the work reported in [77] maps **DFTs** to high-level Petri-net (**HLPN**) [78] to analyse false alarms.

In the following we show specific stateless methods that are designed to reduce structure expressions. In essence, the methods are very similar. Structure expressions for **SFTs** can be reduced using **BDDs** (Subsection 3.2.2), **TFTs** can be reduced using **DTs** (Subsection 3.2.3), **MCSeqs** of **DFTs** can be obtained using Zero-suppressed Binary Decision Diagram (**ZBDD**) [65] (Subsection 3.2.4), and the works reported in [42, 43] show the analysis of standby systems (**CSp** gates) using **SBDDs** (Subsection 3.2.5).

### 3.2.2 Binary Decision Diagrams

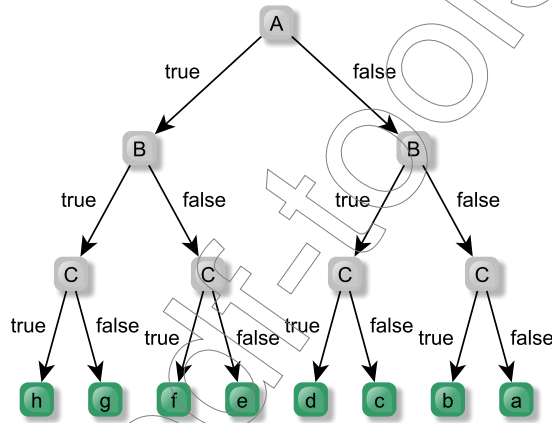
**BDDs** are directed acyclic graphs that represent a Boolean expression. They are still referred to as **BDD**, but the more **spread** version is the Reduced Ordered Binary Decision Diagram (**ROBDD**) [79], which is an optimisation. There are two ways to generate a **BDD** for an expression: (i) derive a diagram from the truth-table, or (ii) expand the paths based on Shannon's method (described in the Fault Tree Handbook).

To demonstrate the expressiveness of a **BDD**, Figure 15 shows a diagram for a truth table with three variables (Table 6). In a node, when a path is chosen, the variable of the node assumes the edge value. For example, the top-level node variable of Figure 15 is *A*. Following the right-hand side of the node, all leaf nodes correspond to the lines of the truth table that *A* has "0" values (the first four lines). The symbol nodes are replaced

**Table 6** – Truth table for a formula outputs with three variables (A, B, and C)

A	B	C	Formula
0	0	0	a
0	0	1	b
0	1	0	c
0	1	1	d
1	0	0	e
1	0	1	f
1	1	0	g
1	1	1	h

by the values assumed by a specific formula.

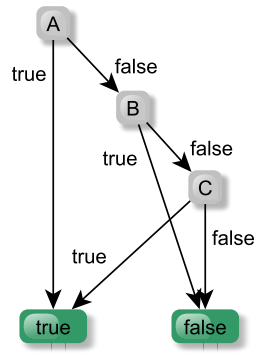
**Figure 15** – A diagram for a truth table

Following Shannon’s method, we choose a variable and build the lower level **BDD** assuming the edge value for the chosen variable. In the remainder of the path, the variable’s value is unchanged. For example, the expression  $A \vee (\neg B \wedge C)$  has value “0” in the lines *a* and *c*, and value “1” in the other lines. By choosing the variable *A* first, then *B* and *C*, the resulting **BDD** with the binary values nodes (called sink nodes “false” and “true”) for this formula is depicted in Figure 16. Starting from the top-level node *A*, the formula expressed by the **BDD** is true if *A* assumes value true. If *A* is false, and *B* is false, the expression is only true if *C* is true.

Figure 16 is an **ROBDD**. To be considered an **ROBDD**, the **BDD** must meet the following constraints [79]:

- the variables are assigned a constant ordering;
- every path to sink nodes visit the input variables in ascending order;
- each node represents a distinct logic function.

The size of an **ROBDD**, for a given expression, depends on the chosen variables ordering. The work reported in [80] shows initial findings on variable ordering, and the work reported



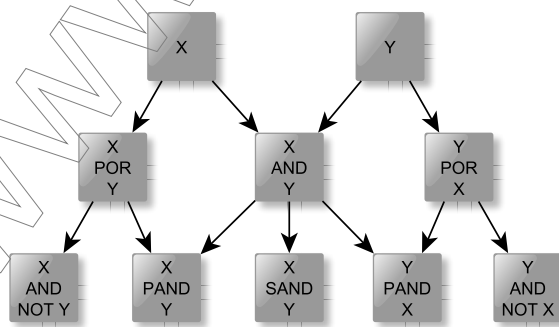
**Figure 16** – A BDD for the expression  $A \vee (\neg B \wedge C)$

in [81] shows heuristics to improve the performance for optimal order search.

For SFTs the evaluation of a BDD is the calculation of the probability of the paths ending in *true*. For example, the probability of the expression in Figure 16 is obtained from the formula:  $\Pr \{A \vee (\neg A \wedge \neg B \wedge C)\}$ . Note that the formula in the probability calculation is different from the formula that originated the diagram.

### 3.2.3 Dependency tree

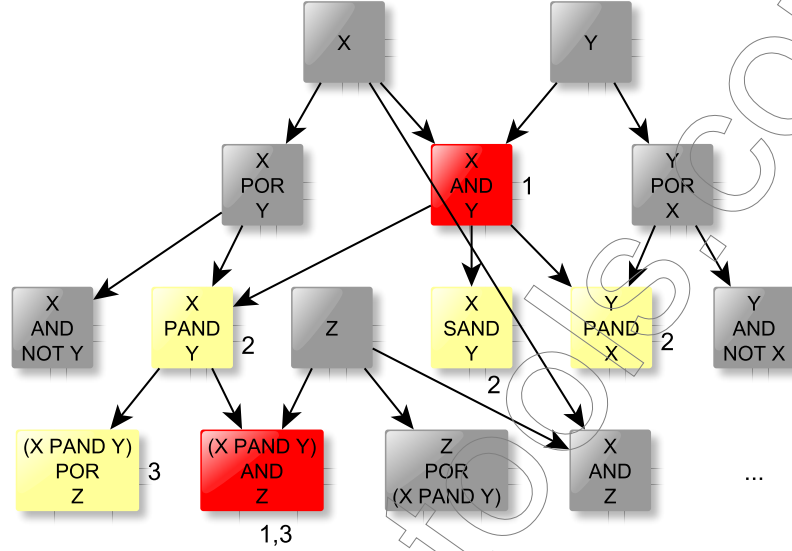
Dependency tree (DT) [33] is a hierarchical acyclic graph of expressions that shows all possible cut sequences for any given set of events. It is a graphical view of a TTT. At the top of a DT are the variables, that is, the single events that occur in an expression. On the lower levels are the increasingly complex expressions. Each node represents an MCSeq. Figure 17 shows a DT with all nodes for variables  $X$  and  $Y$ .



**Figure 17** – DT for variables  $X$  and  $Y$

The reduction of a structure expression is given by the activation (true values) of nodes. If a node is active (true), then all child nodes are also active; the converse is also true: if all node's children are active, then it is also active. The reduced expression is given by the DNF created with the expressions of higher active level nodes. To reduce the formula  $(X \wedge Y) \vee ((X < Y) \wedge Z)$ , given on the beginning of this subsection, we create the DT depicted in Figure 18. Nodes marked with “1” are those MCSeqs given directly by

the formula. Nodes marked with “2” are child nodes of the “1”’s nodes, and so forth. The node of the expression  $((X < Y) \wedge Z)$  is a grandchild of  $X \wedge Y$  and thus it is not necessary. The final expression is obtained by the active higher level node, which is  $X \wedge Y$ .



**Figure 18** – DT for the formula  $(X \wedge Y) \vee ((X < Y) \wedge Z)$

### 3.2.4 Zero-suppressed Binary Decision Diagrams

The work reported in [65] proposes Zero-suppressed Binary Decision Diagram, which is a variant of BDD, and uses set manipulations (union, intersection, difference, and product) to obtain MCSeqs of DFTs.

To reduce a BDD to a ZBDD, the nodes that have the “true” (‘1’) path pointing to the “false” (‘0’) sink node are removed, and the parent node is connected directly to the “false” subgraph of the removed node. Figure 19 shows an example of ZBDD for the combination set  $\{a, b\}$ , as shown in [65]. The idea of the reduction is to remove irrelevant variables and nodes. The irrelevant variables are set to “false”. The method obtains the MCSeqs by navigating the paths to sink node “true”.

Although the work reported in [65] shows ZBDD, the final solution does not use them directly. Instead, it defines a hierarchical manipulation of DFT to obtain the MCSeqs when traversing the a DFT. The order-related operators in a DFT are replaced by a new event, which takes ordering into account. The idea is to transform the DFT into an SFT, in a very similar way as the one shown in [42]. Finally, the MCSeqs are obtained using set manipulation with elements that are basic events alone or order-related operators. These order-related operators are event-to-event only, so they cannot be combined with other sets.

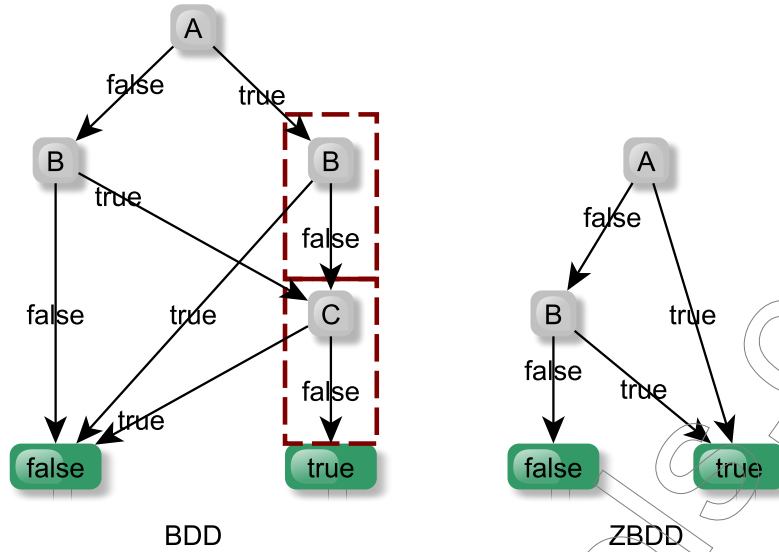


Figure 19 – ZBDD example of combination-set  $\{a, b\}$

The use of sets in [65] is very related to our ATF. We use sets of sequences to define the ATF, but keep the analysis with set operators. In ATF we do not create new events that represent an order-related operator. Our order-related operator has a set-based semantics that can be combined with other non-order-related (Boolean) operators.

### 3.2.5 Sequential Binary Decision Diagrams

SBDD is an extension of BDD to tackle ordering of events in DFTs for CS<sub>p</sub> and WSp gates. Ordering of events in CS<sub>p</sub> gates [43] is slightly different from WSp [42]. A backup system in CS<sub>p</sub> gets activated slower than in WSp, which implies that there are less failure possibilities in CS<sub>p</sub>, but its readiness is lower than in WSp. SBDD adds a new node kind that contains a binary operation of fault events, which allows to express the ordering of events. One kind of operation expresses the slowness of the relation of the fault events of CS<sub>p</sub>, and another one expresses the readiness of the WSp. The latter semantics is similar to the semantics of PAND and IBefore (combined with AND) gates.

SBDD creation has two steps: (i) CS<sub>p</sub> or WSp DFT conversion, and (ii) SBDD model generation. In (i), it is a DFT-to-DFT conversion. CS<sub>p</sub> and WSp gates are converted to a new, but equivalent DFT without CS<sub>p</sub> and WSp gates, where the operations appear as basic events and are combined using other gates. In (ii), the SBDD model is created. The model may contain nodes that are contradictory as, for example, nodes that assumes that an event  $A$  is false and a binary operation that contains  $A$  is true. This step ends when all contradictions are removed. The evaluation is similar to BDD's: each path ending in true is a minimal term in the DNF that may contain one of the binary operations and individual events.

### 3.3 Free Boolean Algebras

Another means to analyse SFTs is to use an FBA to perform set-theoretical operations (intersection, difference, etc.) to reduce expressions. In this section we briefly present the FBA theory and its elements.

Instead of using an axiomatic definition of a Boolean algebra directly, we follow its set-theoretical definition, as shown in [82, pp. 254–258] and [16, pp. 8–11]. This definition represents a Boolean algebra as an algebra of sets and does not rely on Boolean axioms (which can be misleading, case there is an unfounded axiom).

**Definition 3.1** (Boolean Algebra). *A Boolean algebra is defined as a triple  $\langle B, \cap, - \rangle$ , where  $B$  is a set with at least two elements,  $\cap$  is the intersection (also called meet or infimum) and  $-$  is the complement (also called negation).*

The other Boolean elements (union,  $\perp$ , and  $\top$ ) are derived from the previous two operators:

$\cup$  is the union (also called join or *supremum*):  $A \cup B = -(-A \cap -B)$

$\perp$  is the bottom (also called zero):  $\perp = A \cap \neg A$

$\top$  is the top (also called unit):  $\top = \neg \perp$

A Free Boolean Algebra is defined from a set  $E$  of generators. A generator can be represented as a proposition in statement calculus [82, p. 274]. For example, “valve A is stuck closed” and “motor M is malfunctioning” are valid statements. A Free Boolean Algebra is constructed from  $\mathbb{P}(E)$ , where  $\mathbb{P}$  is the power set. Note that if  $E$  has  $n$  symbols,  $\mathbb{P}(E)$  has  $2^n$  elements, called *atoms* of a finite Boolean algebra. For the two statements above, the atoms are:

- a) “Valve A is stuck closed” and “motor X is malfunctioning”
- b) “Valve A is stuck closed” and “motor X is *not* malfunctioning”
- c) “Valve A is *not* stuck closed” and “motor X is malfunctioning”
- d) “Valve A is *not* stuck closed” and “motor X is *not* malfunctioning”

Such a Boolean algebra has  $2^{2^n}$  formulas [16, p. 261]. For example, if  $E = \{a, b\}$ , then  $\mathbb{P}(E) = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$ , hence the Boolean algebra generated by  $E$  contains sixteen ( $2^{2^2}$ ) formulas:  $\{\{\}, \{\{\}\}, \{\{\}, \{a\}\}, \{\{\}, \{b\}\}, \dots, \{\{a\}, \{a, b\}\}, \dots, \{\{b\}, \{a, b\}\}, \dots, \{\{\}, \{a\}, \{b\}, \{a, b\}\}\}$ .

The Boolean algebra  $B$  can be inductively defined using some constructs.



**Definition 3.2** (Inductive Free Boolean Algebra). *Let  $s$  be a statement, then:*

$$\mathbf{var} s = \{X | s \in X\} \implies \mathbf{var} s \in B \quad (\text{variable}) \quad (3.5a)$$

$$X \in B \implies -X \in B \quad (\text{complement}) \quad (3.5b)$$

$$X \in B \wedge Y \in B \implies X \cap Y \in B \quad (\text{intersection}) \quad (3.5c)$$

The characterisation of a “free” Boolean algebra comes from that, for some valuation function  $a$ , **some** of the formulas evaluates to “1”. Given a function  $p : B \times \{0, 1\} \rightarrow B$ , such that:

$$p(i, j) = \begin{cases} i & j = 1 \\ -i & j = 0 \end{cases} \quad (3.6)$$

**Lemma 3.1** (Free generators (valuation)). *Let  $F$  be a finite set, such that  $F \subseteq E$ , and  $a : F \rightarrow \{0, 1\}$ , a necessary and sufficient condition for a set  $E$  of generators of a Boolean algebra  $B$  to be free is then:*

$$\bigwedge_{i \in F} p(i, a(i)) \neq 0 \quad (3.7)$$

Essentially, Lemma 3.1 states that there is no relation between generators, such as  $a = -b$ .

**Lemma 3.2** (Free generators (algebraic)). *Let  $i$  and  $j$  be statements, such that  $i, j \in E$ , hence from Definition 3.2 and Lemma 3.1 it is necessary and sufficient that:*

$$\mathbf{var} i = \mathbf{var} j \iff i = j \quad (3.8a)$$

$$\mathbf{var} i \neq -\mathbf{var} j \quad (3.8b)$$

$$-\mathbf{var} i \neq \mathbf{var} j \quad (3.8c)$$

### 3.4 Using the NOT operator in Static Fault Trees

Although the Fault Tree Handbook introduces several gates, the vast majority of SFT analyses would fit in FTs with only AND and OR gates (coherent FTs). Qualitative analysis requires the reduction of the structure expression of FTs and, when NOT gates are present (non-coherent FTs), such a reduction can cause the interpretation of failure expression to be misled [9, 11, 10, 12, 13]. The work reported in [11] shows three funny examples of this kind of problem, and the works reported in [9, 11, 12] show how to solve it using BDDs. In the following we show: (i) the second example presented in [11], which highlights the problem when using NOT gates (Subsection 3.4.1), and (ii) the second example presented in [9], which defends the usefulness of NOT gates in a multitasking system (Subsection 3.4.2).

Negated events in a non-coherent analysis are in fact the working state of a component. The failure probability contribution of a negated basic event is close to 1. The



problem with non-coherent FTs is that its analysis can cause impossible situations. The general formula to identify coherency is given in [9, 12] in terms of a structure function.

**Definition 3.3 (FT Coherency).** Let  $\Phi(x) : B^n \rightarrow B^1$  be a binary function of a vector of binary variables, such that  $x = [x_1, x_2, \dots, x_n]$ , representing the states of a system's components.

A binary structure function  $\Phi(x)$  is coherent if all the following hold:

- a)  $\Phi(x)$  is monotonic (non-decreasing) in each variable;
- b) Each  $x_i$  is relevant, which means that  $\Phi(x)[x_i/1] \neq \Phi(x)[x_i/0]$  for some vector  $x$ .

where  $B^1 = \{0, 1\}$ ,  $B^n = B^{n-1} \times B^1$ ,  $x_i = 1$  implies that component  $i$  failed, and  $\Phi(x) = 1$  implies the system failed. For  $y = [y_1, y_2, \dots, y_n]$ , monotonicity of  $\Phi$  means that for all  $i$ ,  $x_i \geq y_i$  ( $y_i = 1 \implies x_i = 1$ ), and for some  $i$ ,  $x_i > y_i$  ( $x_i = 1$  and  $y_i = 0$ ). Variable replacement ( $[a/b]$ ) is as usual:  $x[x_i/a] = [x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n]$

### 3.4.1 Non-coherent fault tree misleads

In this subsection we illustrate with the second example detailed in [11]—how non-coherent FT misleads.

A college student who wants to visit her mother in another city has two options: wake up early ( $x_3$ ) and take a ride with a friend ( $x_1$ ), or wake up late ( $\neg x_3$ ) and take the metro ( $x_2$ ). The top-event failure is “visit mother” with expression  $S = (x_1 \wedge x_3) \vee (x_2 \wedge \neg x_3)$ . Its fault tree is depicted in Figure 20. It is clear that the structure function is non-coherent in  $x_3$  accordingly to Definition 3.3:  $\Phi(1, 1, x_3)[x_3/1] = \Phi(1, 1, x_3)[x_3/0]$ .

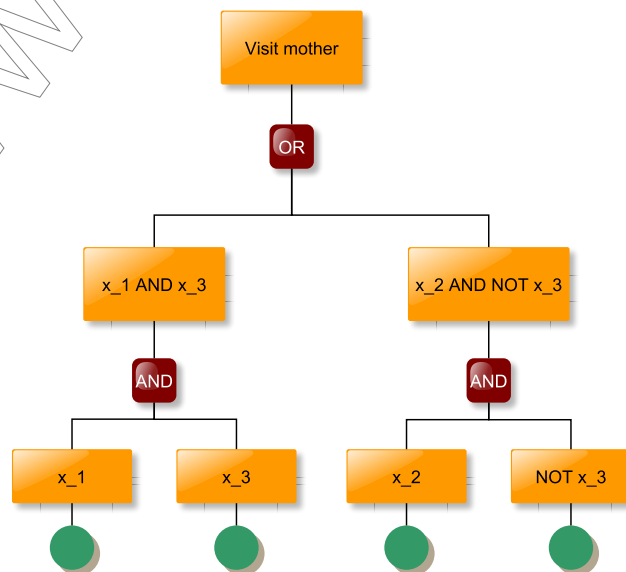


Figure 20 – Non-coherent FT college student's example

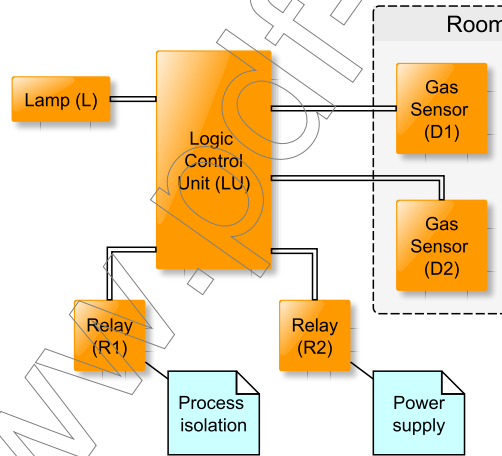
The problem with this tree is the interpretation of the qualitative results. One of the possibilities in this scenario is that the college student would take a ride **AND** take the metro ( $x_1 \wedge x_2$ ). Quantitatively, the analysis of the probabilities shows that this result is not negligible, but its interpretation is impossible.

### 3.4.2 Usefulness of **NOT** gates in **FTA**

In this subsection we show the second example detailed in [9].

The gas detection system depicted in Figure 21 has two sensors  $D_1$  and  $D_2$  which are used to detect a leakage in a confined space. When a leakage is detected, these sensors send a signal to the logic control unit  $LU$ , which performs three tasks:

- shuts-down the main system (process isolation) by de-energizing relay  $R_1$ ;
- informs the operator of the leakage by lamp and siren  $L$ ;
- deactivates all possible ignition sources, which is the interruption of power supply by de-energizing relay  $R_2$ .



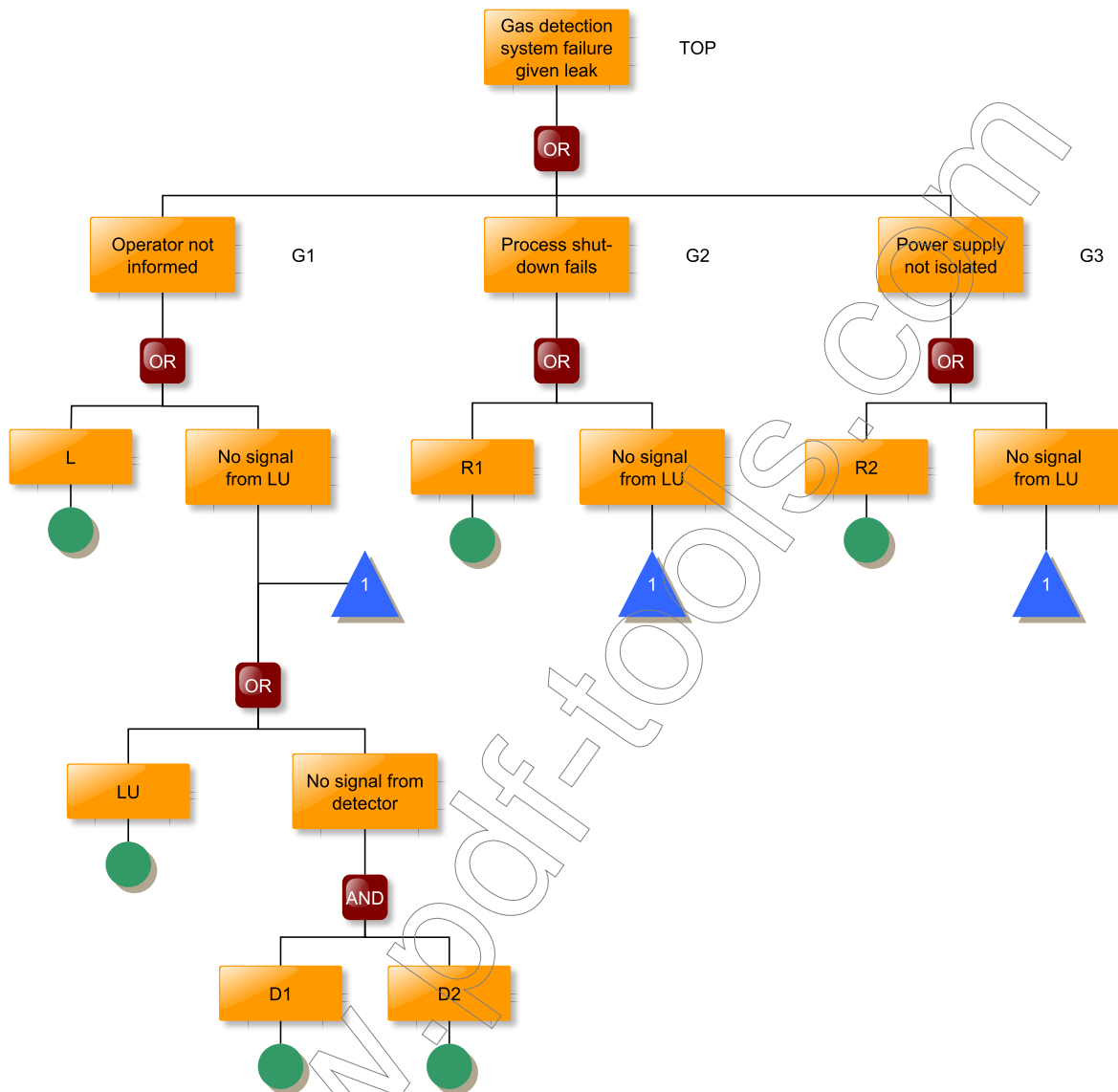
**Figure 21** – Gas detection system

The system is in a fail state if it does not perform one of these three tasks. The fault tree that represents this generic failure is depicted in Figure 22.  $G_1$ ,  $G_2$ , and  $G_3$  are subtrees that represents the three tasks “Operator not informed”, “Process shut-down fails”, and “Power supply not isolated”, respectively. All three tasks will fail if their respective main component fails ( $L$ ,  $R_1$ , and  $R_2$ ) or there is no signal from  $LU$  ( $LU$  fails or both  $D_1$  and  $D_2$  fail). The structure expressions for the subtrees are:

$$G_1 = L \vee LU \vee (D_1 \wedge D_2)$$

$$G_2 = R_1 \vee LU \vee (D_1 \wedge D_2)$$

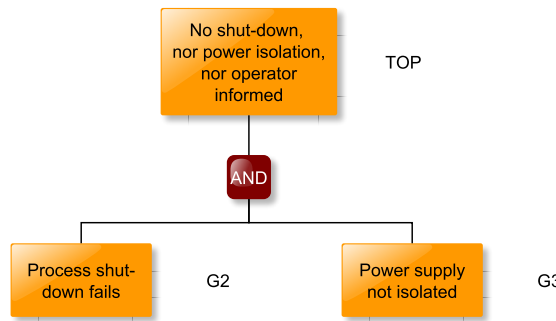
$$G_3 = R_2 \vee LU \vee (D_1 \wedge D_2)$$



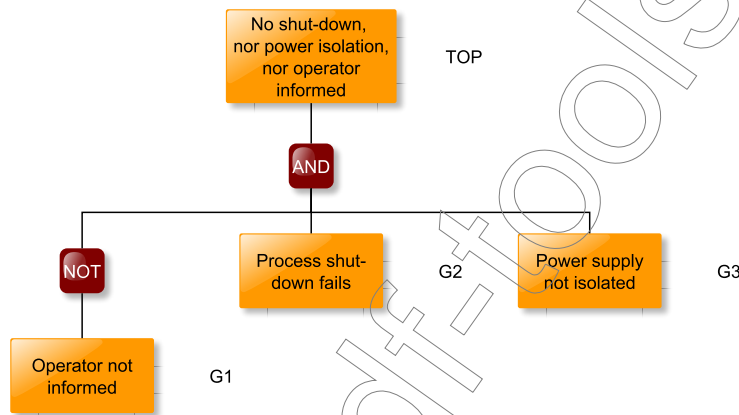
**Figure 22** – FT for a generic failure in the gas detection system

Analysing in more detail, there are different degrees of system failure. There are eight outcomes (given the three tasks) and the most critical one is when both process shut-down ( $G_2$ ) and power supply isolation ( $G_3$ ) fail keeping energized upon a leakage, and the operator is not informed ( $G_1$ ), but the operator information system is working (lamp and siren are off, but they are operational). The coherent FT of this outcome is depicted in Figure 23. The minimal cut sets obtained from this will be:  $\{R_1, R_2\}$ ,  $\{D_1, D_2\}$ , and  $\{LU\}$ .

Quantification of the coherent FT will overestimate the probability of the critical outcome unless the part of the system that is working (lamp and siren  $L$ ,  $LU$ , and sensors  $D_1$  and  $D_2$ ) is taken into account. The non-coherent FT with the working part is shown in Figure 24.



**Figure 23** – *Coherent FT* for the most critical outcome of the gas detection system



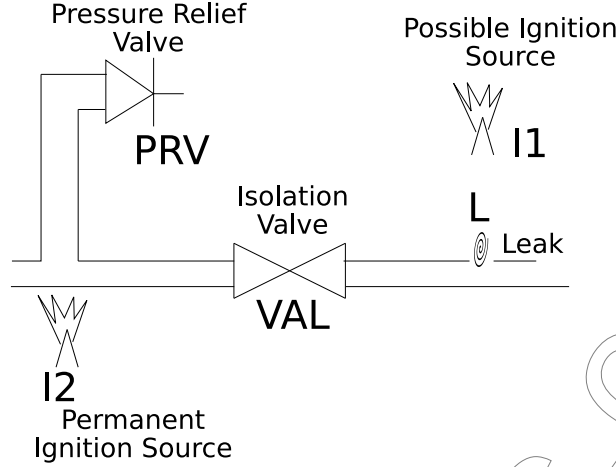
**Figure 24** – *Non-coherent FT* for the most critical outcome of the gas detection system

If the operator *can* be informed, then cut sets  $\{D_1, D_2\}$  and  $\{LU\}$  could not have occurred (see Figure 22), thus the correct qualitative analysis should consider only cut set  $\{R_1, R_2\}$ . Reducing the expressions of the non-coherent FT (Figure 24), we obtain the structure expression:  $\neg L \wedge \neg LU \wedge R_1 \wedge R_2 \wedge (\neg D_1 \vee \neg D_2)$ . The approximation for this expression, removing the negated events, gives the cut set  $\{R_1, R_2\}$ , which gives a correct quantitative analysis.

### 3.5 Probabilistic analysis of a non-coherent tree

The work reported in [9] shows an example of an FT with an explicit NOT operator, the importance of such an operator, and how to calculate the probability of a critical failure. The system is a leak protection system that has valves and sensors, and is depicted in Figure 25. Valve *VAL* closes a gas flow if a sensor detects a leak *L*. If the valve is closed for a certain amount of time, the pressure on the system may increase, and then, a relief valve *PRV* diverts the gas flow elsewhere.

The undesired top-event is an ignition in room 1 (*TOP*). As shown in [9], the



**Figure 25** – Leak Protection System architectural view

structure expression of  $TOP$  is:

$$TOP = L \wedge ((\neg VAL \wedge PRV) \vee (VAL \wedge I_1)) \quad (3.9)$$

For a coherent analysis, they use the consensus law to add a “missing” term. In this case, the missing term is  $L \wedge PRV \wedge I_1$ . This gives the final expression:

$$TOP = L \wedge ((\neg VAL \wedge PRV) \vee (VAL \wedge I_1) \vee (PRV \wedge I_1)) \quad (3.10)$$

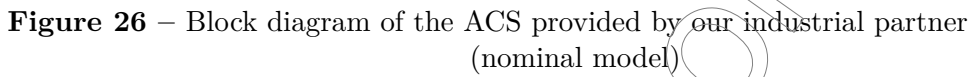
Finally, the probability for Eq. (3.9) is:

$$\Pr \{TOP\} = P_L \times (P_{PRV} + P_{VAL} \times P_{I_1} - P_{PRV} \times P_{VAL}) \quad (3.11)$$

where  $P_x$  is the failure probability of  $x$ ,  $x \in \{L, PRV, VAL, I_1\}$ .

### 3.6 Systems nominal model and fault injection

Control system modelling using Simulink block diagrams [83] is recommended in [30] and have been used by our industrial partner. It is a complementary tool of Matlab [84]. In fact, it works as a graphical interface to Matlab. A Simulink model has blocks and connections between these blocks, named signals. Each block has inputs and outputs and an internal behaviour expressed by its mathematical formula, which defines a function of the inputs for each output. There are many predefined blocks in the tool. It is also possible to create new blocks or use subsystems that encapsulate other blocks. A simulation adds extra parameters to a block diagram, like elapsed time and time between states. The elapsed time of a simulation is an abstraction for the quantity of possible simulation states and the time between states is related to the lowest common denominator of the sample time. Some components define different sample times, depending on their mode of operation. Usually, the value for this property is set to *auto*, allowing Simulink to choose a proper value automatically.



```

graph LR
    MonIn1((1)) --> HWB1[HW B1]
    MonIn2((2)) --> HWB2[HW B2]
    HWB1 --> RO[Relational Operator]
    HWB2 --> RO
    RO --> SW[Switch]
    SW --> MonOut((1))

```

Figure 27 shows the internal elements of the monitor component (Figure 26 (A)), which is used as a case study in Chapter 6 to illustrate our strategy. The outputs of the hardware elements are annotated with *HW*, which are the two power sources and an internal component of the monitor (switch command).

To perform a formal verification in a Simulink system model we use a model-checking tool, **FDR**. It is a refinement checker for formal models written in **CSP<sub>M</sub>**. To verify a **refinement**, it takes two specifications: (i) a specification with more abstract properties, and (ii) an implementation with more concrete properties. If a refinement does

not hold (the implementation fails to refine the specification), **FDR** shows counter-examples as traces of events. The **CSP<sub>M</sub>** language is suitable to model concurrent behaviour and is very expressive to model systems' states. The work reported in [31] translates a Simulink model to the **CSP<sub>M</sub>** language. The resulting **CSP<sub>M</sub>** code (implementation) is then used to check if it meets functional requirements also encoded in **CSP<sub>M</sub>** (specification).

In our previous work, reported in [28], we modified such a translation to perform fault injection using hardware annotations allowing a subsystem or part to “break” randomly. We designed a **CSP<sub>M</sub>** process to act as an observer (specification), watching outputs of the nominal version and comparing to the outputs of the “breakable” version (with injected faults—the implementation) of the system. When the **CSP<sub>M</sub>** process of the model and the observer are loaded into the **FDR** model-checker, counter-examples are generated for each output that differs from the nominal model, thus obtaining a *sequence* of injected fault combinations that leads to the unexpected output, which are indeed *fault traces*.

In what follows, injected faults and the top-level failure have generic names based on the names of the Simulink model blocks. It is out of the scope of [28] to define event names.

For the Simulink model shown in Figure 27, some representative fault traces are:

TRACE 1:

```
failure.Hardware.N04_RelationalOperator.1.EXP.B.true
failure.Hardware.N04_RelationalOperator.1.ACT.B.false
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION
```

TRACE 2:

```
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.true
failure.Hardware.N04_RelationalOperator.1.ACT.B.false
out.1.OMISSION
```

TRACE 3:

```
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION
```

TRACE 4:

```
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
out.1.OMISSION
```

TRACE 5:

```
failure.Hardware.N04_MonIn1.1.EXP.I.5
```

```
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
out.1.OMISSION
```

TRACE 6:

```
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION
```

TRACE 7:

```
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
```

TRACE 8:

```
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
```

TRACE 9:

```
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
```

TRACE 10:

```
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
```

TRACE 11:

```
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
```

where N04 is the subsystem name of the monitor in the Simulink diagram, MonIn1 (first input of the monitor), MonIn2 (second input of the monitor), and RelationalOperator



(switcher controller) are the names of the hardware components in the Simulink diagram.

We only show eleven counter-examples, but **FDR** generates a total of 64 counter-examples for this system. The other counter-examples are similar to the traces shown with different internal events.

To reuse **HiP-HOPS**, which is based on **SFTs**, we “remove” the ordering information of the traces to generate a failure expression. Each fault trace is abstracted as a conjunction (**AND** combination of the inner events, thus losing the ordering information), and the several conjunction-based fault events are combined using **ORs** (disjunctions). The result of the combination is a Boolean expression that represents the conditions that cause an undesirable output, the failure expression of the model. With the **ATF** proposed in this work we do not “remove” the ordering information, so we are able to use this information to generate or perform **DFT** and **TFT** analyses (**TFTs** have order-related operators, and it is shown in [3, 24, 22] that **DFTs** can be expressed by order-related operators).

If the failure expression is obtained for a whole system, it is indeed the structure expression of a fault tree for a general failure as the top-level event. Although it is possible to obtain the failure expression for a larger system, it may be impractical due to state-space explosion in **CSP<sub>M</sub>** model analysis. Thus it should be used for components and subsystems or small systems following **HiP-HOPS** compositional structure. Using failure expression as subsystem annotations in [25], it is possible to obtain structure expressions for a larger system. It is worth noting that the goal of the work reported in [28] was to connect with **HiP-HOPS**, which is based on static fault trees. But we already knew that we had a richer fault modelling information than that presented in [28] because we abstracted traces (which already capture fault events ordering) to create propositions (any fault events order combination).

To show how these traces become failure expression, let us abbreviate fault names as:

```
A = failure.Hardware.N04_MonIn1.1
B = failure.Hardware.N04_MonIn2.1
S = failure.Hardware.N04_RelationalOperator
```

**Table 7** – Annotations table of the ACS provided by our industrial partner

Component	Deviation	Port	Annotation
PowerSource	LowPower	Out1	PowerSourceFailure
Monitor	LowPower	Out1	(SwitchFailure AND (LowPower-In1 OR LowPower-In2)) OR (LowPower-In1 AND LowPower-In2)
Reference	OmissionSignal	Out1	ReferenceDeviceFailure OR LowPower-In1

So, for each trace, we obtain an expression:

$$\text{TRACE 1} = S \wedge B$$

$$\text{TRACE 2} = B \wedge S$$

$$\text{TRACE 3} = A \wedge B$$

$$\text{TRACE 4} = B \wedge A$$

$$\text{TRACE 5} = A \wedge S$$

$$\text{TRACE 6} = A \wedge S \wedge B$$

$$\text{TRACE 7} = A \wedge B \wedge S$$

$$\text{TRACE 8} = B \wedge A \wedge S$$

$$\text{TRACE 9} = S \wedge A \wedge B$$

$$\text{TRACE 10} = S \wedge B \wedge A$$

$$\text{TRACE 11} = B \wedge S \wedge A$$

And we combine them as a single Boolean expression:  $\text{TRACE 1} \vee \text{TRACE 2} \vee \text{TRACE 3} \vee \text{TRACE 4} \vee \text{TRACE 5} \vee \text{TRACE 6} \vee \text{TRACE 7} \vee \text{TRACE 8} \vee \text{TRACE 9} \vee \text{TRACE 10} \vee \text{TRACE 11}$ , which by a traditional Boolean reduction strategy results in:

$$(A \wedge B) \vee (S \wedge (A \vee B))$$

The above expression is exactly the same failure expression provided by our industrial partner if we use the following association (Table 7):

$$A = \text{LowPower-In1}$$

$$B = \text{LowPower-In2}$$

$$S = \text{SwitchFailure}$$

Note that when we combine each fault with **AND** gates, we lose the information about order<sup>5</sup>:  $S \wedge B$  and  $B \wedge S$  are equal, due to the commutative law of Boolean expressions.

<sup>5</sup> In our previous work we designed the observer to ignore order as well, by making similar traces—with

Our strategy finds fault combinations  $S$  and  $B$  (in the sense of  $S$  occurring before  $B$ ) as well as  $B$  and  $S$  (in the sense of  $B$  occurring before  $S$ ) but abstracts this ordering information obtaining  $B$  and  $S$ , which is equivalent to  $S$  and  $B$  in Boolean Algebra. If  $A$  fails before  $S$ , the system fails because it should switch to  $B$ , but the switcher is in a faulty state. On the other hand, if  $S$  fails before  $A$ , the switcher fails because it inadvertently switched to  $B$  when  $A$  was still operational. When  $A$  fails, nothing changes and the output of the system is obtained from  $B$ .

We also employed the strategy proposed in the work [28] in another case study and obtained a weaker failure expression (that is, our expression considers more cases). The failure expression provided by the engineers of our industrial partner was stronger because they considered that one component has a very low probability of failure and removed it from the failure analysis. Although acceptable, it may cause incorrect analysis. Our strategy avoids this kind of issue by being completely systematic.

### 3.7 Isabelle/HOL

We use the same words of the creators of this tool, retrieved from their website<sup>6</sup>:

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. The main application is the formalization of mathematical proofs and in particular formal verification, which includes proving the correctness of computer hardware or software and proving properties of computer languages and protocols.

Isabelle/HOL is the most widespread instance of Isabelle. HOL stands for higher-order logic. Isabelle/HOL provides a HOL proving environment ready to use, which includes: (co)datatypes, inductive definitions, recursive functions, locales, custom syntax definition, etc. Proofs can be written in both human<sup>7</sup> and machine-readable language based on *Isar*. The tool also includes the *sledgehammer*, a port to call external first-order provers to find proofs fully automatically. The user interface is based on jEdit<sup>8</sup>, which provides a text editor, syntax parser, shortcuts, etc. (see Figure 28).

Theories on Isabelle/HOL are based on a few axioms. Isabelle/HOL Library's theories—which comes with the installer—and user's theories are based on these axioms. This design decision avoids inconsistencies and paradoxes (similar as it is in Z).

---

different ordering—the same size. Here we modified the observer specification to make similar traces with different sizes.

<sup>6</sup> Accessed 27/jan/2016: <<https://isabelle.in.tum.de/overview.html>>

<sup>7</sup> By human we mean that anyone with mathematics and logic basic knowledge—it means that deep programming knowledge is not essential.

<sup>8</sup> Accessed 27/jan/2016: <<http://www.jedit.org/>>

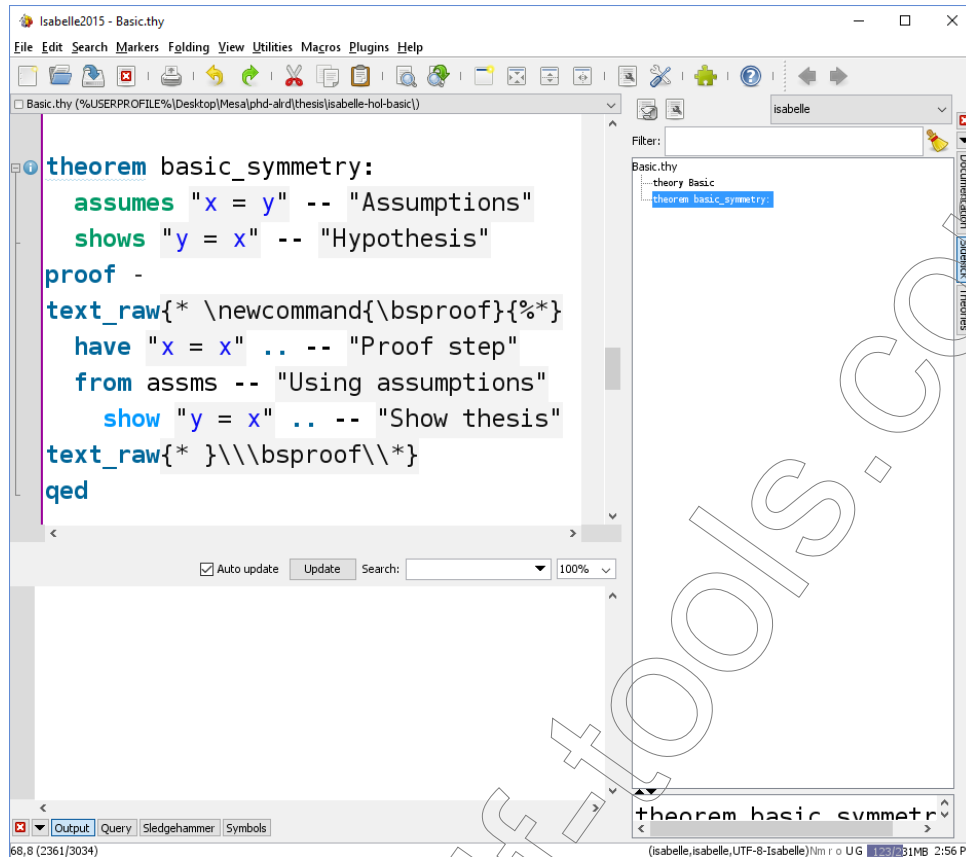


Figure 28 – Isabelle/HOL window, showing the basic symmetry theorem

Besides the provided theories, its active community provides a comprehensive archive of formal proofs<sup>9</sup> (AFP). Each entry in this archive can be cited and usually contains an *abstract*, a document, and a theory file. For example, a Free Boolean Algebra theory is available in [86]. To use it, it is enough to download and put on the same directory of your own theory files.

Bellow we show an example and explain the overall syntax of the human and machine-readable language.

```
theorem basic_symmetry:
  assumes "x = y" — Assumptions
  shows "y = x" — Hypothesis
proof -
  have "x = x" ∴ — Proof step
  from assms — Using assumptions
  show "y = x" .. — Show thesis
qed
```

Finally, Isabelle/HOL provides L<sup>A</sup>T<sub>E</sub>X syntax sugar and allow easy document preparation: this entire section was written in a theory file mixing Isabelle's and L<sup>A</sup>T<sub>E</sub>X's

<sup>9</sup> Accessed 27/jan/2016: <<http://afp.sourceforge.net/>>

syntax). The above theorem can be written using Isabelle's quotation and anti-quotations. For example, we can write it using usual L<sup>A</sup>T<sub>E</sub>X theorem environment:

**Theorem 3.1** (Basic symmetry). *Assuming  $x = y$ , thus:*

$$y = x$$

*Proof.*   **have** "x = x" .. — Proof step

**from** assms — Using assumptions

**show** "y = x" .. — Show thesis

□

Otherwise specified, in the next sections we will omit proofs because they are all verified using Isabelle/HOL. The complete listing is in [Appendix A](#).



## 4 A free algebra to express structure expressions of ordered events

Recall from Sections 2.2 and 3.1 that fault events are statistically independent of one another. The set-theoretical abstraction of structure expressions for SFTs [18, pp. VI-11] is very close to an FBA, where each generator in FBAs corresponds to a fault event symbol in fault trees. In FBAs, as generators are “free”, they are independent of one another and Boolean formulas are written as a set of sets of possibilities, which are similar to the structure expressions of SFTs.

We showed in Section 3.1 that there is a consistent presence of order-based operators to analyse TFTs and DFTs, and that each approach describes a new algebra based on different representations of events ordering with similar theorems to reduce expressions to a normal form.

From the need to tackle events ordering and from the ordering information we had from fault injection that we developed in [28], we defined a list-based algebra, called Algebra of Temporal Faults (ATF), to express and analyse systems considering events ordering. We also provide a mapping from fault traces [28] (from  $CSP_M$  models) to this algebra, shown in Section 4.4. The order-specific operations are expressed with a new operator ( $\rightarrow$ ) that we call exclusive-before (XBefore).

The set of sets for FBAs is our proposed denotational semantics for Boolean algebras. We use the concept of generators to propose the ATF with a denotational semantics of a set of lists without repetition (distinct lists<sup>1</sup>). The choice for lists is because this structure inherently associates a generator to an index, making implicit the representation of order. These lists are composed of non-repeated elements (distinct lists) because the events in fault trees are non-repairable. Thus, they do not occur more than once.

The elements of a list have an implicit order number, but such an order number is different from the Sequence Number function used in [20, 21]. Although different, the order number in lists is related to the concept that there should be no gaps of the indexes between consecutive events occurrence. The structure of the lists ensures this restriction. But it is different because order 0 (zero) in [20, 21] means non-occurrence. It may cause a discontinuity because 0 to 1 is different of 1 to 2. In FBAs the non-occurrence of an event is just the absence of the event. Thus we use the same representation of non-occurrence as absence of the event in ATF to avoid this discontinuity. For example, the following lists are all permutations of fault events  $a$  and  $b$  (the generators are  $a$  and  $b$ ):

<sup>1</sup> Although some may use the terminology “disjoint lists” to call the lists of non-repeated elements, we use the same terminology (distinct lists) of the theories built-in the Isabelle/HOL tool.

$[]$ : no fault occurs

$[a]$ : fault  $a$  occurs and  $b$  does not

$[b]$ : fault  $b$  occurs and  $a$  does not

$[a, b]$ : fault  $a$  occurs before  $b$ . Note that,  $a$  has index 0 and  $b$  has index 1.

$[b, a]$ : fault  $b$  occurs before  $a$ .

In the following we show the definitions and laws of our proposed ATF. To avoid repetition, let  $S$ ,  $T$  and  $U$  be sets of distinct lists. A list  $xs$  is distinct if it has no repeated element. So, if  $x$  is in  $xs$ , then it has a unique associated index  $i$  and we denote it as  $x = xs_i$ , where  $i \neq j$  and there is no  $xs_j$  such that  $x = xs_j$ . Furthermore, as we follow an FBA characterisation, we also need to show that the generators are independent.

The ATF form a free algebra similar to FBAs. *Infimum* and *Supremum* are denoted as set intersection ( $\cap$ ) and union ( $\cup$ ) respectively. The order within the algebra is denoted with set inclusion ( $\subseteq$ ).

To distinguish the permutations that are not defined in FBA, we need a new operator. We give the definition of XBefore ( $\rightarrow$ ) in terms of list concatenation, similar to the work reported in [87]:

$$\begin{aligned} \llbracket S \rightarrow T \rrbracket = \{ & zs \mid \exists xs, ys \bullet (\text{set } xs) \cap (\text{set } ys) = \{\} \wedge \\ & xs \in \llbracket S \rrbracket \wedge ys \in \llbracket T \rrbracket \wedge zs = xs @ ys \} \end{aligned} \quad (4.1)$$

where the **set** function returns the set of the elements of a list,  $@$  concatenates two lists, and  $\llbracket . \rrbracket$  obtains the denotational semantics of the formula.

In some cases it is more intuitive to use the XBefore definition in terms of lists slicing because it uses indexes explicitly. Lists slicing is the operation of taking or dropping elements, obtaining a sublist. In slicing, the starting index is inclusive, and the ending one is exclusive. Thus the first index is 0 and the last index is the list length. For example, the list  $xs_{[0..|xs|]}$  is equal to the  $xs$  list, where  $|xs|$  is the list length. We use the following notation for list slicing:

$$xs_{[i..j]} = \text{starts at } i \text{ and ends at } j - 1 \quad (4.2a)$$

$$xs_{[..j]} = xs_{[0..j]} \quad (4.2b)$$

$$xs_{[i..]} = xs_{[i..|xs|]} \quad (4.2c)$$

To simplify the use of list slicing, its definition includes the lower and upper bounds as 0 and its length, respectively:

$$xs_{[i..j]} = xs_{[L..U]} \quad (4.3)$$



where  $L = \max 0, i$  and  $U = \min j, |xs|$ .

List slicing and concatenation are complementary: concatenating two consecutive slices results in the original list:

$$\forall i \bullet xs_{[..i]} @ xs_{[i..]} = xs \quad (4.4)$$

There is an equivalent definition of **XBefore** with concatenation using lists slicing:

$$\llbracket S \rightarrow T \rrbracket = \left\{ zs \mid \exists i \bullet zs_{[..i]} \in \llbracket S \rrbracket \wedge zs_{[i..]} \in \llbracket T \rrbracket \right\} \quad (4.5)$$

A variable in **ATF** is defined by one generator, and denotes its occurrence:

$$\llbracket \mathbf{var} x \rrbracket = \{ zs \mid x \in zs \} \quad (4.6)$$

where  $set\ zs$  is the set of the elements of  $zs$ , and  $x \in zs$  is defined as  $x \in set\ zs$ .

For example, for generators  $a$  and  $b$ , we obtain the following denotational semantics:

$$\llbracket \mathbf{var} a \rrbracket = \{ [a], [a, b], [b, a] \} \quad (4.7a)$$

$$\llbracket \mathbf{var} b \rrbracket = \{ [b], [a, b], [b, a] \} \quad (4.7b)$$

Given this definition, we show a small example on how the **XBefore** operator works:

$$\text{For } zs = [], [] \notin \llbracket \mathbf{var} a \rrbracket \wedge [] \notin \llbracket \mathbf{var} b \rrbracket \implies$$

$$[] \notin \llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket$$

$$\text{For } zs = [a], [a] \in \llbracket \mathbf{var} a \rrbracket \wedge [a] \notin \llbracket \mathbf{var} b \rrbracket \wedge [] \notin \llbracket \mathbf{var} b \rrbracket \implies$$

$$[a] \notin \llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket$$

$$\text{For } zs = [b], [b] \in \llbracket \mathbf{var} b \rrbracket \wedge [b] \notin \llbracket \mathbf{var} a \rrbracket \wedge [] \notin \llbracket \mathbf{var} a \rrbracket \implies$$

$$[b] \notin \llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket$$

$$\text{For } zs = [a, b], [a] \in \llbracket \mathbf{var} a \rrbracket \wedge [b] \in \llbracket \mathbf{var} b \rrbracket \implies$$

$$[a, b] \in \llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket$$

$$\text{For } zs = [b, a], [b] \notin \llbracket \mathbf{var} a \rrbracket \wedge [] \notin \llbracket \mathbf{var} a \rrbracket \wedge [b, a] \in \llbracket \mathbf{var} a \rrbracket \wedge$$

$$[] \notin \llbracket \mathbf{var} b \rrbracket \wedge [a] \notin \llbracket \mathbf{var} b \rrbracket \wedge [b, a] \in \llbracket \mathbf{var} b \rrbracket \implies$$

$$[b, a] \notin \llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket$$

$$\llbracket \mathbf{var} a \rightarrow \mathbf{var} b \rrbracket = \{ [a, b] \} \quad (4.8)$$

Boolean operators are denoted as in **FBA**:

$$\llbracket S \wedge T \rrbracket = \llbracket S \rrbracket \cap \llbracket T \rrbracket \quad (4.9a)$$

$$\llbracket S \vee T \rrbracket = \llbracket S \rrbracket \cup \llbracket T \rrbracket \quad (4.9b)$$

$$\llbracket \neg S \rrbracket = \text{UNIV} - \llbracket S \rrbracket \quad (4.9c)$$

$$\llbracket \perp \rrbracket = \{ \} \quad (4.9d)$$

$$\llbracket \top \rrbracket = \text{UNIV} \quad (4.9e)$$

UNIV is the universal set. It contains all permutations of the generators of size 0 to the number of the generators. We denote the set of generators  $\text{Gen}(S)$  of a formula  $S$  as:

$$\text{Gen}(S) = \bigcup_{xs \in \llbracket S \rrbracket} \text{set } xs \quad (4.10)$$

The generators of **ATF** are  $\text{Gen}(\text{UNIV})$ , or simply  $\text{Gen}$ . For example, if the generators are  $a$  and  $b$ , then UNIV is:

$$\{ \llbracket \top \rrbracket, [a], [b], [a, b], [b, a] \}$$

The following expressions are sufficient to define the **ATF** in terms of an inductively defined set (**atf**):

$$\llbracket \text{var } x \rrbracket \in \text{atf} \quad \text{Variable} \quad (4.11a)$$

$$\llbracket S \rrbracket \in \text{atf} \implies \llbracket \neg S \rrbracket \in \text{atf} \quad \text{Complement, Negation} \quad (4.11b)$$

$$\llbracket S \rrbracket \in \text{atf} \wedge \llbracket T \rrbracket \in \text{atf} \implies \llbracket S \wedge T \rrbracket \in \text{atf} \quad \text{Intersection, Infimum} \quad (4.11c)$$

$$\llbracket S \rrbracket \in \text{atf} \wedge \llbracket T \rrbracket \in \text{atf} \implies \llbracket S \rightarrow T \rrbracket \in \text{atf} \quad \text{XBefore} \quad (4.11d)$$

Following these definitions, the expressions below are also valid for **atf**:

$$\text{UNIV} \in \text{atf} \quad \text{Universal set, True} \quad (4.11e)$$

$$\{ \} \in \text{atf} \quad \text{Empty set, False} \quad (4.11f)$$

$$\llbracket S \rrbracket \in \text{atf} \wedge \llbracket T \rrbracket \in \text{atf} \implies \llbracket S \vee T \rrbracket \in \text{atf} \quad \text{Union, Supremum} \quad (4.11g)$$

The **NOT** operator is given in terms of UNIV. For example, for generators  $a$  and  $b$ :

$$\begin{aligned} \llbracket \neg \text{var } a \rrbracket &= \text{UNIV} - \llbracket \text{var } a \rrbracket && \text{by Eq. (4.9c)} \\ &= \{ \llbracket \top \rrbracket, [a], [b], [a, b], [b, a] \} - \{ [a], [a, b], [b, a] \} && \text{by Eq. (4.7a)} \\ &= \{ \llbracket \top \rrbracket, [b] \} && (4.12) \end{aligned}$$

To avoid repetition, we abbreviate Eq. (4.9c) suppressing UNIV. For example:

$$\text{UNIV} - A \equiv -A$$

The following expressions are valid for generators  $a$  and  $b$  and are sufficient to show that the generators are independent:

$$\llbracket \mathbf{var} a \rrbracket = \llbracket \mathbf{var} b \rrbracket \iff a = b \quad (4.13a)$$

$$\llbracket \mathbf{var} a \rrbracket \not\subseteq -\llbracket \mathbf{var} b \rrbracket \quad (4.13b)$$

$$\llbracket \mathbf{var} a \rrbracket \neq -\llbracket \mathbf{var} b \rrbracket \quad (4.13c)$$

$$-\llbracket \mathbf{var} a \rrbracket \not\subseteq \llbracket \mathbf{var} b \rrbracket \quad (4.13d)$$

$$-\llbracket \mathbf{var} a \rrbracket \neq \llbracket \mathbf{var} b \rrbracket \quad (4.13e)$$

Expressions Eq. (4.11a) to Eq. (4.11g) and Eq. (4.13a) to Eq. (4.13e) implies that the ATF without the XBefore operator Eq. (4.1) forms a Boolean algebra based on sets of lists. And this is also equivalent to an FBA with the same generators.

In our previous work [87] we stated a relation of XBefore and *supremum*, provided the operands are variables Eq. (4.6). Now we generalise this relation in terms of abstract properties of the operands of the XBefore. We name these properties as *temporal properties*.

## 4.1 Temporal properties (*tempo*)

Temporal properties give a more abstract and less restrictive shape on the XBefore laws. They are abbreviations that some operators satisfy altogether or individually.

The first temporal property is about disjoint split. If the first part of a list is in a given set, then every remainder part is not. So, if a generator is in the beginning of a list, it must not be at the end (and vice-versa).

$$\mathbf{tempo}_1 S = \forall i, j, zs \bullet i \leq j \implies \neg (zs_{[..i]} \in \llbracket S \rrbracket \wedge zs_{[j..]} \in \llbracket S \rrbracket) \quad (4.14)$$

For example, let  $zs = [z_0, z_1, z_2, z_3, z_4]$ . If  $[z_0, z_1] \in \llbracket S \rrbracket$ , thus  $[z_3, z_4] \notin \llbracket S \rrbracket$ . Note that it is vague, but it is the first relation indication on the lists of  $S$ . In this property, as  $zs$  has no repeated elements, then there is no element that is in both sublists.

In what follows we show how variables satisfy **tempo** properties. To avoid repetition, the denotational semantics of  $\mathbf{var} z_1$  is considered for generators  $z_1, z_2, z_3$ :

$$\begin{aligned} \llbracket \mathbf{var} z_1 \rrbracket = \{ & [z_1], [z_1, z_2], [z_2, z_1], [z_1, z_3], [z_3, z_1], [z_1, z_2, z_3], [z_1, z_3, z_2], \\ & [z_2, z_1, z_3], [z_2, z_3, z_1], [z_3, z_1, z_2], [z_3, z_2, z_1] \} \end{aligned} \quad (4.15)$$

We demonstrate that the property **tempo**<sub>1</sub> is satisfied:

$$\begin{aligned}
[z_1] \in [S] &\implies [z_2] \notin [S] \wedge [z_3] \notin [S] \wedge [z_2, z_3] \notin [S] \wedge [z_3, z_2] \notin [S] \\
[z_1, z_2] \in [S] &\implies [z_3] \notin [S] \\
[z_2, z_1] \in [S] &\implies [z_3] \notin [S] \\
[z_1, z_3] \in [S] &\implies [z_2] \notin [S] \\
[z_3, z_1] \in [S] &\implies [z_2] \notin [S] \\
[z_1, z_2, z_3] \in [S] &\implies [] \notin [S] \\
&\dots
\end{aligned}$$

The second temporal property is about belonging to one sublist in the beginning or in the end:

$$\mathbf{tempo}_2 S = \forall i, zs \bullet zs \in [S] \iff zs_{[..i]} \in [S] \vee zs_{[i..]} \in [S] \quad (4.16)$$

For example, if a generator is in a list, then it must be at the beginning or at the end. If  $[z_0, z_1, z_2, z_3, z_4] \in [S]$ , thus either  $[z_0] \in [S]$ , or  $[z_1, z_2, z_3, z_4] \in [S]$ . If  $[z_0] \notin [S]$ , thus  $[z_1, z_2, z_3, z_4] \in [S]$ . Then, either  $[z_1] \in [S]$  or  $[z_2, z_3, z_4] \in [S]$ , and so forth. For variable **var**  $z_1$ :

$$\begin{aligned}
[z_1] \in [S] &\iff [z_1] \in [S] \vee [] \in [S] \\
[z_1, z_2] \in [S] &\iff [z_1] \in [S] \vee [z_2] \in [S] \\
[z_1, z_2, z_3] \in [S] &\iff ([z_1] \in [S] \vee [z_2, z_3] \in [S]) \wedge \\
&\quad ([z_1, z_2] \in [S] \vee [z_3] \in [S]) \\
[z_2, z_1, z_3] \in [S] &\iff ([z_2] \in [S] \vee [z_1, z_3] \in [S]) \wedge \\
&\quad ([z_2, z_1] \in [S] \vee [z_3] \in [S]) \\
&\dots
\end{aligned}$$

The third temporal property is about belonging to one sublist in the middle. If a generator belongs to a sublist between  $j$  and  $i$ , then it belongs to the sublist that starts at the first position and ends in the  $j^{th}$  and to the sublist that starts at  $i^{th}$  and ends at the last position (both sublists contain the sublist in the middle).

$$\begin{aligned}
\mathbf{tempo}_3 S = \forall i, j, zs \bullet j < i \implies \\
(zs_{[j..i]} \in [S] \iff zs_{[..j]} \in [S] \wedge zs_{[i..]} \in [S]) \quad (4.17)
\end{aligned}$$

For example, if  $[z_1, z_2, z_3] \in [S]$ , then both  $[z_0, z_1, z_2, z_3]$  and  $[z_1, z_2, z_3, z_4] \in [S]$ . For a

variable **var**  $z_1$ :

$$\begin{aligned}
[z_1] \in \llbracket S \rrbracket &\iff [z_1] \in \llbracket S \rrbracket \wedge [z_1] \in \llbracket S \rrbracket \wedge \\
&\quad [z_2, z_1] \in \llbracket S \rrbracket \wedge [z_1] \in \llbracket S \rrbracket \wedge \\
&\quad [z_3, z_1] \in \llbracket S \rrbracket \wedge [z_1] \in \llbracket S \rrbracket \wedge \\
&\quad [z_1] \in \llbracket S \rrbracket \wedge [z_1, z_2] \in \llbracket S \rrbracket \wedge \\
&\quad [z_1] \in \llbracket S \rrbracket \wedge [z_1, z_3] \in \llbracket S \rrbracket \wedge \\
&\quad [z_2, z_1] \in \llbracket S \rrbracket \wedge [z_1, z_3] \in \llbracket S \rrbracket \wedge \\
&\quad [z_3, z_1] \in \llbracket S \rrbracket \wedge [z_1, z_2] \in \llbracket S \rrbracket \\
[z_1, z_2] \in \llbracket S \rrbracket &\iff [z_3, z_1, z_2] \in \llbracket S \rrbracket \wedge [z_1, z_2] \in \llbracket S \rrbracket \\
&\quad [z_1, z_2] \in \llbracket S \rrbracket \wedge [z_1, z_2, z_3] \in \llbracket S \rrbracket \\
&\quad \dots
\end{aligned}$$

Finally, if a generator belongs to a list, then there is a sublist of size one that contains the generator.

$$\mathbf{tempo}_4 S = \forall zs \bullet zs \in \llbracket S \rrbracket \iff (\exists i \bullet zs_{[i..(i+1)]} \in \llbracket S \rrbracket) \quad (4.18)$$

If list  $zs = [z_0, z_1, z_2, z_3, z_4] \in \llbracket S \rrbracket$ , then one list  $[z_i] \in \llbracket S \rrbracket$ , where  $i \in \{0, \dots, 4\}$ . For a variable **var**  $z_1$ :

$$\begin{aligned}
[z_1] \in \llbracket S \rrbracket &\iff [z_1] \in \llbracket S \rrbracket \\
[z_1, z_2] \in \llbracket S \rrbracket &\iff [z_1] \in \llbracket S \rrbracket \vee [z_2] \in \llbracket S \rrbracket \\
[z_1, z_2, z_3] \in \llbracket S \rrbracket &\iff [z_1] \in \llbracket S \rrbracket \vee [z_2] \in \llbracket S \rrbracket \vee [z_3] \in \llbracket S \rrbracket \\
&\quad \dots
\end{aligned}$$

In general, for any generator  $z$ , the following is valid:

$$\mathbf{tempo}_1(\mathbf{var} z) \wedge \mathbf{tempo}_2(\mathbf{var} z) \wedge \mathbf{tempo}_3(\mathbf{var} z) \wedge \mathbf{tempo}_4(\mathbf{var} z) \quad (4.19)$$

We abbreviate Eq. (4.19) as:

$$\mathbf{tempo}_{1-4} S = \mathbf{tempo}_1 S \wedge \mathbf{tempo}_2 S \wedge \mathbf{tempo}_3 S \wedge \mathbf{tempo}_4 S \quad (4.20)$$

In our previous work [87] we used set difference to specify the **XBefore** operator. Provided  $\mathbf{tempo}_1 S$  and  $\mathbf{tempo}_1 T$ , **XBefore** in [87] is equivalent to Eq. (4.1):

$$\begin{aligned}
\llbracket S \rightarrow T \rrbracket &= \{zs \mid \exists xs, ys \bullet xs \in \llbracket S \rrbracket - \llbracket T \rrbracket \wedge ys \in \llbracket T \rrbracket - \llbracket S \rrbracket \wedge \\
&\quad \text{distinct } zs \wedge zs = xs @ ys\} \quad (4.21)
\end{aligned}$$

Other expressions also meet one or more temporal properties:

$$\mathbf{tempo}_1 S \wedge \mathbf{tempo}_1 T \implies \mathbf{tempo}_1 (S \wedge T) \quad (4.22a)$$

$$\mathbf{tempo}_3 S \wedge \mathbf{tempo}_3 T \implies \mathbf{tempo}_3 (S \wedge T) \quad (4.22b)$$

$$\mathbf{tempo}_2 S \wedge \mathbf{tempo}_2 T \implies \mathbf{tempo}_2 (S \vee T) \quad (4.22c)$$

$$\mathbf{tempo}_4 S \wedge \mathbf{tempo}_4 T \implies \mathbf{tempo}_4 (S \vee T) \quad (4.22d)$$

## 4.2 XBefore laws

We now show some laws to be used in the algebraic reduction of ATF formulas. The laws follow from the definition of XBefore, from events independence, and from the temporal properties.

We use a normal form similar to the DNF of Boolean algebra. In DNF each sub-expression is a minimal cut set for SFT. In our normal form, also called DNF, we allow ANDs, NOTs, and XBefores to be in the sub-expressions. Each sub-expression is a set of minimal cut sequences for TFT and DFT. The following formulas are in DNF:

$$(A \wedge \neg B) \vee ((A \rightarrow B) \wedge C)$$

$$A \vee B$$

$$A \rightarrow B$$

$$A \wedge B$$

$$A \rightarrow B \rightarrow C$$

The following formulas are *not* in DNF:

$$\neg (A \vee B)$$

$$A \wedge (B \vee C)$$

$$A \rightarrow (B \vee C)$$

$$A \rightarrow (B \wedge C)$$

But to transform the last two formulas into DNF, one can use Eqs. (4.27a) to (4.27d), for instance.

We define events independence ( $\triangleleft$ ) as the property that one operand does not imply the other. For example, we need to avoid that the operands of XBefore are  $\mathbf{var} a$  and  $\mathbf{var} a \vee \mathbf{var} b$  (it results in  $\{ \}$ , see Eq. (4.24g)).

$$S \triangleleft T = \forall i, zs \bullet \neg (zs_{[i..(i+1)]} \in \llbracket S \rrbracket \wedge zs_{[i..(i+1)]} \in \llbracket T \rrbracket) \quad (4.23)$$

For generators  $a$  and  $b$  and the denotational semantics of  $\mathbf{var} a$  and  $\mathbf{var} b$  (see Eqs. (4.7a) and (4.7b)), it is easy to check that  $\mathbf{var} a \triangleleft \mathbf{var} b$  is satisfied, and

$\mathbf{var} a \bowtie (\mathbf{var} a \vee \mathbf{var} b)$  is not. Essentially if the formulas contain independent events one must not imply on the other on the occurrence of individual events ( $zs_{[i..(i+1)]}$ ).

The absence of occurrences ( $\{\}$ , the empty set of **atf**) is a “0” for the *XBefore* operator. The negation of all events ( $\{\emptyset\}$ , a formula with the empty list) is a “1” for the *XBefore* operator.

$$\perp \rightarrow S = \perp \quad (4.24a)$$

$$S \rightarrow \perp = \perp \quad (4.24b)$$

$$\mathbf{tempo}_1 S \Rightarrow \mathbf{1} \rightarrow S = S \quad (4.24c)$$

$$\mathbf{tempo}_1 S \Rightarrow S \rightarrow \mathbf{1} = S \quad (4.24d)$$

$$(S \rightarrow T) \vee S = S \quad (4.24e)$$

$$(T \rightarrow S) \vee S = S \quad (4.24f)$$

$$\mathbf{tempo}_1 S \Rightarrow S \rightarrow S = \perp \quad (4.24g)$$

$$\begin{aligned} \mathbf{tempo}_1 S \wedge \mathbf{tempo}_1 T \wedge \mathbf{tempo}_1 U \Rightarrow \\ S \rightarrow (T \rightarrow U) = (S \rightarrow T) \rightarrow U \end{aligned} \quad (4.24h)$$

The *XBefore* is absorbed by one of the operands: if one of the operands may happen alone, thus the order with any other operand is irrelevant. However, an event cannot come before itself, thus *XBefore* is not idempotent. The *XBefore* is associative.

To allow formula reduction we need the relation of *XBefore* to the other Boolean operators. First we use the *XBefore* as operands of **OR** and **AND**.

$$\begin{aligned} \mathbf{tempo}_1 S \wedge \mathbf{tempo}_1 T \Rightarrow \\ (S \rightarrow T) \wedge (T \rightarrow S) = \perp \end{aligned} \quad (4.25a)$$

$$\begin{aligned} \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T \wedge S \bowtie T \Rightarrow \\ (S \rightarrow T) \vee (T \rightarrow S) = S \wedge T \end{aligned} \quad (4.25b)$$

As *XBefore* is not symmetric, the intersection of symmetrical *XBefores* is empty. The **OR** of symmetrical *XBefores* is a partition of the intersection of the operands. For example, given generators  $a$  and  $b$  ( $a \neq b$ ):

$$(\mathbf{var} a \rightarrow \mathbf{var} b) \vee (\mathbf{var} b \rightarrow \mathbf{var} a) = \mathbf{var} a \wedge \mathbf{var} b \quad (4.26)$$

because  $\mathbf{var} a$  and  $\mathbf{var} b$  satisfy all temporal properties (Eq. (4.19)) and are independent events.

In our previous work [87], we stated that  $S$  and  $T$  had to be variables, for example, of the form  $\mathbf{var} s$  and  $\mathbf{var} t$ . Now, each law requires that the operands satisfy some of the temporal properties, avoiding using variables explicitly.

Expressions with Boolean operators are used as operands of the **XBefore** in the following laws.

$$(S \vee T) \rightarrow U = (S \rightarrow U) \vee (T \rightarrow U) \quad (4.27a)$$

$$S \rightarrow (T \vee U) = (S \rightarrow T) \vee (S \rightarrow U) \quad (4.27b)$$

$$\begin{aligned} \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T \wedge S \diamond T &\Rightarrow \\ (S \wedge T) \rightarrow U &= (S \rightarrow T \rightarrow U) \vee \\ &\quad (T \rightarrow S \rightarrow U) \end{aligned} \quad (4.27c)$$

$$\begin{aligned} \mathbf{tempo}_{1-4} T \wedge \mathbf{tempo}_{1-4} U \wedge T \diamond U &\Rightarrow \\ S \rightarrow (T \wedge U) &= (S \rightarrow T \rightarrow U) \vee \\ &\quad (S \rightarrow U \rightarrow T) \end{aligned} \quad (4.27d)$$

$$\begin{aligned} \mathbf{tempo}_2 S &\Rightarrow S \wedge (T \rightarrow U) = ((S \wedge T) \rightarrow U) \vee \\ &\quad (T \rightarrow (S \wedge U)) \end{aligned} \quad (4.27e)$$

$$\begin{aligned} \mathbf{tempo}_1 T \wedge \mathbf{tempo}_3 T &\Rightarrow \\ S \rightarrow T \wedge T \rightarrow U &= (S \rightarrow T) \rightarrow U \end{aligned} \quad (4.27f)$$

$$\begin{aligned} S \diamond T \wedge \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T &\Rightarrow \\ S \wedge (S \rightarrow T) &= S \rightarrow T \end{aligned} \quad (4.27g)$$

$$\begin{aligned} S \diamond T \wedge \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T &\Rightarrow \\ S \wedge (T \rightarrow S) &= T \rightarrow S \end{aligned} \quad (4.27h)$$

**XBefore** is distributive over **OR**. On the other hand, the **AND** is not. The **AND** of an event with an **XBefore** states that such an event can occur in any order within the events in the **XBefore**. Lastly, the **XBefore** is transitive with preconditions over the intermediary variable.

The law name, unordered, of Eq. (4.27e) is clearer if we expand Eq. (4.27e) with Eq. (4.27c) and Eq. (4.27d):

$$\begin{aligned} \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T \wedge \\ \mathbf{tempo}_{1-4} U \wedge S \diamond T \wedge S \diamond U &\Rightarrow \\ S \wedge (T \rightarrow U) &= (S \rightarrow T \rightarrow U) \vee \\ &\quad (T \rightarrow S \rightarrow U) \vee \\ &\quad (T \rightarrow U \rightarrow S) \end{aligned} \quad \begin{array}{l} \text{expanded-unordered} \\ (4.28) \end{array}$$



In what follows we show properties of **XBefore** related to the **NOT** operators.

$$S \bowtie T \wedge \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T \implies \neg(S \rightarrow T) = (\neg S \vee \neg T) \vee (T \rightarrow S) \quad (4.29a)$$

$$\mathbf{tempo}_1 S \wedge \mathbf{tempo}_2 T \implies \neg S \rightarrow T = T \quad (4.29b)$$

$$\mathbf{tempo}_1 T \wedge \mathbf{tempo}_2 S \implies S \rightarrow \neg T = S \quad (4.29c)$$

$$S \bowtie T \wedge \mathbf{tempo}_{1-4} S \wedge \mathbf{tempo}_{1-4} T \implies (S \wedge \neg T) \vee (S \rightarrow T) = S \wedge \neg(T \rightarrow S) \quad (4.29d)$$

### 4.3 Qualitative and quantitative analyses

In Section 3.1 we showed the kind of results that are obtained in **FTA**. In this section we show how to formalize two kinds of properties of **FTs**: (i) **MCSeq**, the number of fault elements in the minimal sequences that cause a root failure, and (ii) the root probability, given the availability of basic fault occurrences probabilities. These properties are the most representative, but other criteria can be modelled similarly.

#### 4.3.1 Minimal cut sequence

Recall from the beginning of this chapter that the denotational semantics of **ATF** is a set of distinct lists. Thus each list has no repeated elements and represent a possible combination of faults that causes the root failure. **MCSeqs** are those distinct lists with the least length, defined in what follows.

**Definition 4.1** (Minimal cut sequences). *Let  $S$  be a formula in **ATF**. Its minimal cut sequences (**MCSeq** of  $S$ ) are:*

$$\mathbf{MCSeq}(S) = \{ xs \mid xs \in \llbracket S \rrbracket \bullet |xs| = \min_S \} \quad (4.30)$$

where

$$\min_S = \text{Min}(\{ |xs| \mid xs \in \llbracket S \rrbracket \})$$

and **Min** returns the least sequence length in the given set.

For example, the **MCSeqs** of  $\mathbf{var} a \rightarrow (\mathbf{var} b \vee \mathbf{var} c)$  (for generators  $a$ ,  $b$ , and  $c$ ) are:

$$\mathbf{MCSeq}(\mathbf{var} a \rightarrow (\mathbf{var} b \vee \mathbf{var} c)) = \{[a, b], [a, c]\} \quad (4.31)$$

Equation (4.31) states that it is sufficient that  $a$  occurs before  $b$ , or  $a$  occurs before  $c$ , to cause the top event. Other lists in the denotational semantics of the formula are  $[a, b, c]$  and  $[a, c, b]$ , but these are not minimal.

### 4.3.2 Root probability

The work reported in [3] shows how to calculate the probability of a **PAND** gate as:

$$\begin{aligned}
 P(t) &= \Pr \{T_1 \leq T_2\}^{<t} \\
 &= \int_0^t P'_2(t_2) \int_0^{t_2} P'_1(t_1) dt_1 dt_2 \\
 &= \int_0^t P'_2(t_2) P_1(t_2) dt_2
 \end{aligned} \tag{4.32}$$

where  $P_1$  and  $P_2$  are the probabilities of the occurrences of the first and the second faults (cumulative distribution function), respectively, and  $T_1$  and  $T_2$  are the times the first and the second faults occur. The general case (reported in [88]) of a **PAND** gate with  $n$  inputs is the probability of the sequence of faults  $f_i$ ,  $i \in \{1, \dots, n\}$ :

$$\begin{aligned}
 &\Pr \{[f_1, f_2, \dots, f_{n-1}, f_n]\} = \\
 &\int_0^t P'_n(t_n) \int_0^{t_n} P'_{n-1}(t_{n-1}) \dots \int_0^{t_3} P'_2(t_2) \int_0^{t_2} P'_1(t_1) dt_1 dt_2 \dots dt_{n-1} dt_n
 \end{aligned} \tag{4.33}$$

where  $P_i$  is the cumulative probability function of fault  $f_i$ .

For **ATF**, our proposal for probability calculation is defined in terms of the probability of **PAND** gates. The reason is that our semantics is defined in terms of a set of lists (or sequences) of fault occurrences. The main difference from the **PAND** calculation as reported in [88] is that repeated situations must be removed. For example, the probability of  $\llbracket \mathbf{var} f_1 \rightarrow \mathbf{var} f_2 \rrbracket$  contains the situations in which  $f_1$  occurs before  $f_2$  and  $f_3$  does not occur, or  $f_3$  occurs in some time:  $\Pr \{[f_1, f_2, f_3]\}$ ,  $\Pr \{[f_1, f_3, f_2]\}$ , and  $\Pr \{[f_3, f_1, f_2]\}$ .

Using Eqs. (4.10) and (4.33) we define the probability for a fault sequence as:

$$\Pr_{\text{FS}} \{xs\} = \Pr \{xs\} \times \prod_{f_j \in \text{Gen-set } xs} (1 - P_j(t)) \tag{4.34}$$

For example, for generators  $f_1$ ,  $f_2$ , and  $f_3$ ,  $\Pr_{\text{FS}} \{[f_1, f_2]\} = \Pr \{[f_1, f_2]\} \times (1 - P_3(t))$ , which is the probability of  $f_1$  occurring before  $f_2$  and  $f_3$  not occurring.

As the probability of each fault sequence is independent of each other, the probability of an **ATF** formula is the sum of the denotational probabilities of its constituent lists.

**Definition 4.2** (Probability of a formula in **ATF**). *Let  $S$  be a formula in **ATF**. Then the probability of  $S$ ,  $\text{FPr} \{S\}$ , is given by:*

$$\text{FPr} \{S\} = \sum_{xs \in \llbracket S \rrbracket} \Pr_{\text{FS}} \{xs\} \tag{4.35}$$

The interesting part behind the probabilistic calculus over the denotational semantics is that it is only about ordering of events. It means that even if a formula contains a

**NOT** operator, we still safely obtain a probability value, without worrying about complement probabilities as tackled in [9]. For example, the probability of  $\neg \mathbf{var} f_1$  (for generators  $f_1$  and  $f_2$ ) is:

$$\begin{aligned}
 \Pr \{ \neg \mathbf{var} f_1 \} &= \Pr_{\text{FS}} \{ [] \} + \Pr_{\text{FS}} \{ [f_2] \} && \text{by Eq. (4.35)} \\
 &= \Pr \{ [] \} \times (1 - P_1(t)) \times (1 - P_2(t)) \\
 &\quad + \Pr \{ [f_2] \} \times (1 - P_1(t)) && \text{by Eq. (4.34)} \\
 &= 1 \times (1 - P_1(t)) \times (1 - P_2(t)) \\
 &\quad + P_2(t) \times (1 - P_1(t)) && \text{by Eq. (4.33)} \\
 &= 1 - P_1(t) && (4.36)
 \end{aligned}$$

The empty list is **the special** and has value 1. It works as the universal complement probability of any other list. When the empty list appears in a denotational semantics it means that the top-event occurs if no fault occurs.

We use the traditional probability calculations [3] as reference to calculate the probabilities of formulas in **ATF**.



$$\text{FPr} \{ \mathbf{var} f_1 \wedge \mathbf{var} f_2 \} = P_1(t) \times P_2(t) \quad (4.37a)$$

$$\text{FPr} \{ \mathbf{var} f_1 \vee \mathbf{var} f_2 \} = P_1(t) + P_2(t) - P_1(t) \times P_2(t) \quad (4.37b)$$

For example, the formula  $(\mathbf{var} f_1 \rightarrow \mathbf{var} f_2) \vee (\mathbf{var} f_2 \rightarrow \mathbf{var} f_1)$ , considering only the two generators ( $f_1$  and  $f_2$ ), has denotational semantics  $\{[f_1, f_2], [f_2, f_1]\}$  and the probability of the formula is the probability of  $[f_1, f_2]$  or  $[f_2, f_1]$  (but not both):

$$\begin{aligned}
 &\text{FPr} \{ (\mathbf{var} f_1 \rightarrow \mathbf{var} f_2) \vee (\mathbf{var} f_2 \rightarrow \mathbf{var} f_1) \} \\
 &= \Pr_{\text{FS}} \{ [f_1, f_2] \} + \Pr_{\text{FS}} \{ [f_2, f_1] \} && \text{by Eq. (4.35)} \\
 &= \Pr \{ [f_1, f_2] \} + \Pr \{ [f_2, f_1] \} && \text{Eq. (4.34), both } \sum \text{ is 0} \\
 &= \int_0^t P_2'(x) P_1(x) dx + \int_0^t P_1'(x) P_2(x) dx && \text{by Eq. (4.32)} \\
 &= \int_0^t (P_2'(x) P_1(x) + P_1'(x) P_2(x)) dx && \text{by sum of } \int \\
 &\Rightarrow \int_0^t (P_1(x) P_2(x))' dx && \text{by inv. deriv. product} \\
 &= P_1(t) \times P_2(t) && (4.38)
 \end{aligned}$$

In Eq. (4.38) we demonstrated that the probability of a formula  $(\mathbf{var} f_1 \rightarrow \mathbf{var} f_2) \vee (\mathbf{var} f_2 \rightarrow \mathbf{var} f_1)$  is equal to the probability of the traditional **AND** probability  $(\mathbf{var} f_1 \wedge \mathbf{var} f_2)$ , shown in Eq. (4.37a). This is expected as these two formulas are equivalent, as shown in Eq. (4.25b).

Another example including complement is the formula  $\mathbf{var} f_1 \wedge \mathbf{var} f_2 \wedge \neg \mathbf{var} f_3$ :

$$\begin{aligned}
 S &= \mathbf{var} f_1 \wedge \mathbf{var} f_2 \wedge \neg \mathbf{var} f_3 && \text{by Eq. (4.35)} \\
 \Pr \{S\} &= \Pr_{\text{FS}} \{[f_1, f_2]\} + \Pr_{\text{FS}} \{[f_2, f_1]\} && \text{by Eq. (4.34)} \\
 &= \Pr \{[f_1, f_2]\} \times (1 - P_3(t)) + \Pr \{[f_2, f_1]\} \times (1 - P_3(t)) \\
 &= P_1(t) \times P_2(t) \times (1 - P_3(t)) && \text{by Eq. (4.38)} \quad (4.39)
 \end{aligned}$$

Using Eq. (4.35), and for generators  $f_1$  and  $f_2$ , we demonstrate the equivalence to Eq. (4.37b) (the probability of an OR operator), calculated using the denotational semantics:

$$\begin{aligned}
 S &= \mathbf{var} f_1 \vee \mathbf{var} f_2 \\
 \text{FPr} \{S\} &= \Pr_{\text{FS}} \{[f_1]\} + \Pr_{\text{FS}} \{[f_2]\} + \\
 &\quad \Pr_{\text{FS}} \{[f_1, f_2]\} + \Pr_{\text{FS}} \{[f_2, f_1]\} && \text{by Eq. (4.35)} \\
 &= \Pr \{[f_1]\} \times (1 - P_2(t)) + \Pr \{[f_2]\} \times (1 - P_1(t)) + \\
 &\quad P_1(t) \times P_2(t) && \text{by Eqs. (4.34) and (4.38)} \\
 &= P_1(t) \times (1 - P_2(t)) + P_2(t) \times (1 - P_1(t)) + \\
 &\quad P_1(t) \times P_2(t) \\
 &= P_1(t) + P_2(t) - P_1(t) \times P_2(t) \quad (4.40)
 \end{aligned}$$

We show an equivalence using the formula probability FPr for generators  $f_1$  and  $f_2$ :

$$\begin{aligned}
 \text{FPr} \{\mathbf{var} f_1 \wedge \mathbf{var} f_2\} &= \text{FPr} \{\mathbf{var} f_1\} \times \text{FPr} \{\mathbf{var} f_2\} \quad (4.41) \\
 \Pr_{\text{FS}} \{[f_1, f_2]\} + \Pr_{\text{FS}} \{[f_2, f_1]\} &= (\Pr_{\text{FS}} \{[f_1]\} + \Pr_{\text{FS}} \{f_1, f_2\} + \Pr_{\text{FS}} \{f_2, f_1\}) \times \\
 &\quad (\Pr_{\text{FS}} \{[f_2]\} + \Pr_{\text{FS}} \{f_1, f_2\} + \Pr_{\text{FS}} \{f_2, f_1\}) \\
 P_1(t) \times P_2(t) &= (P_1(t) \times (1 - P_2(t)) + P_1(t) \times P_2(t)) \times \\
 &\quad (P_2(t) \times (1 - P_1(t)) + P_1(t) \times P_2(t)) \\
 &= (P_1(t) - P_1(t) \times P_2(t) + P_1(t) \times P_2(t)) \times \\
 &\quad (P_2(t) - P_2(t) \times P_1(t) + P_1(t) \times P_2(t)) \\
 &= P_1(t) \times P_2(t)
 \end{aligned}$$

Finally, we propose that the formula probability calculation of a greater set of generators is the same of a smaller one. For example, for  $\mathbf{var} f_1 \wedge \mathbf{var} f_2$  and generators

$f_1$ ,  $f_2$ , and  $f_3$ :

$$\begin{aligned}
 \text{FPr} \{ \mathbf{var} f_1 \wedge \mathbf{var} f_2 \} &= \text{Pr}_{\text{FS}} \{ [f_1, f_2] \} + \text{Pr}_{\text{FS}} \{ [f_2, f_1] \} + \\
 &\quad \text{Pr}_{\text{FS}} \{ [f_1, f_2, f_3] \} + \dots + \text{Pr}_{\text{FS}} \{ [f_3, f_2, f_1] \} \\
 &= P_1(t) \times P_2(t) \times (1 - P_3(t)) + P_1(t) \times P_2(t) \times P_3(t) \\
 &= P_1(t) \times P_2(t) - P_1(t) \times P_2(t) \times P_3(t) + \\
 &\quad P_1(t) \times P_2(t) \times P_3(t) \\
 &= P_1(t) \times P_2(t)
 \end{aligned} \tag{4.42}$$

which is the same calculation as shown in Eq. (4.41).

### 4.3.3 Formal acceptance criteria

To enable the formal verification of structure expressions we use the concept of *acceptance criteria*. Safety requirements are written in terms of the properties of an FT, for example: (i) no single failure should cause a critical failure (the length the MCSeqs should be greater than 1), or (ii) the probability of all critical failure should be less than  $P_x$ . To check these requirements we translate these requirements into a value and verify in the theorem prover.

For the two analysis shown in this section, we define the two acceptance criteria:


$$|F|^{>n} = \text{Min}(\{ |xs| \mid xs \in \llbracket F \rrbracket \}) > n \quad \text{length of MCSeqs} \tag{4.43a}$$

$$\text{Pr} \{ F \}^{<P_x} = \text{FPr} \{ F \} < P_x \quad \text{root-event probability} \tag{4.43b}$$

Both equations have Boolean return value, which can be verified by a theorem prover.

Using the formal acceptance criteria, safety requirements as stated in the beginning of this section are:

$|F|^{>1}$ : the minimum length of the MCSeqs of  $F$  are higher than 1;

$\text{Pr} \{ F \}^{<10^{-9}}$ : the probability of the top nt of  $F$  shall be less than  $10^{-9}$ .

## 4.4 Mapping $CSP_M$ traces to ATF

In our previous work [28, 27] we reported a strategy to inject faults in a  $CSP_M$  model-generated from an architectural model of a system in Simulink. Using a model-checker we were able to obtain failure traces with the injected faults, which we produced failure expressions in Boolean algebra. Thus, the order-related information was lost, as there are no means to represent them in Boolean algebra. For example, a trace that represents the occurrence of  $f_1$  before  $f_2$  was written as the expression  $\mathbf{var} f_1 \wedge \mathbf{var} f_2$ , which is the same as  $\mathbf{var} f_2 \wedge \mathbf{var} f_1$ .

Using the same strategy, but now mapping the traces to **ATF** we are able to analyse order-related failures. In this section we show how to map the same failure traces obtained by the strategy reported in the work [28] to **ATF**.

The mapping function takes the set of traces as input and produces a structure expression in **ATF**. Each event in the traces becomes a variable, and each list becomes an **XBefore** with the conjunction (**ANDs**) of the variables of the remaining generators. The following recursive definitions describe the mapping function.

$$\langle [ ] \rangle_{XB} = \top \quad (4.44a)$$

$$\langle [ f ] \rangle_{XB} = \mathbf{var} f \quad (4.44b)$$

$$\langle [ f_1 ] @ tr \rangle_{XB} = \mathbf{var} f_1 \rightarrow \langle tr \rangle_{XB} \quad (4.44c)$$

$$\langle \{ tr_1, tr_2, \dots, tr_n \} \rangle_{XB} = \bigvee_{i \in \{1, \dots, n\}} \left( \langle tr_i \rangle_{XB} \wedge \neg \bigvee_{j \in \mathbf{Gen-set} tr_i} \mathbf{var} f_j \right) \quad (4.44d)$$

where @ concatenates two traces. The following examples show how it works for generators  $f_1, f_2, f_3$ :

$$\langle [ f_1 ] \rangle_{XB} = \mathbf{var} f_1 \wedge \neg (\mathbf{var} f_2 \vee \mathbf{var} f_3) \quad (4.45a)$$

$$\langle [ f_1, f_2 ] \rangle_{XB} = \mathbf{var} f_1 \rightarrow \mathbf{var} f_2 \wedge \neg \mathbf{var} f_3 \quad (4.45b)$$

$$\langle [ f_3, f_2, f_1 ] \rangle_{XB} = \mathbf{var} f_3 \rightarrow (\mathbf{var} f_2 \rightarrow \mathbf{var} f_1) \quad (4.45c)$$

$$\begin{aligned} \langle \{ [ f_1 ], [ f_1, f_2 ] \} \rangle_{XB} = & (\mathbf{var} f_1 \wedge \neg \mathbf{var} f_2 \wedge \neg \mathbf{var} f_3) \vee \\ & ((\mathbf{var} f_1 \rightarrow \mathbf{var} f_2) \wedge \neg \mathbf{var} f_3) \end{aligned} \quad (4.45d)$$

## 5 Reasoning about fault activation

The Activation Logic proposed in this work emerges from the need to analyse the behaviour of a system when a subset of the faults have been triggered in some order, and to provide completeness analysis of system behaviour. There are at least two strategies to use AL to obtain structure expressions of SFT, TFT, or DFT: (i) model systems directly in AL, and (ii) obtaining operational mode expressions extracted from failure traces, as shown in the work reported in [52]. In approaches as those reported in [20, 3], behavioural completeness is left for the analyst. Using tautology and the indication of undefined nominal values, we ensure that no situation is left forgotten.

The Activation Logic associates: (i) an operational mode, and (ii) the expression of fault events that *activates* the operational mode or error event. The expressions of fault events can be written in any algebra that provides tautology and contradiction properties. Boolean algebra and the ATF provide both. Thus, AL is parametrized by: (i) an algebra that provides at least tautology and contradiction, and (ii) operational modes. Figure 29 depicts an overview of AL.

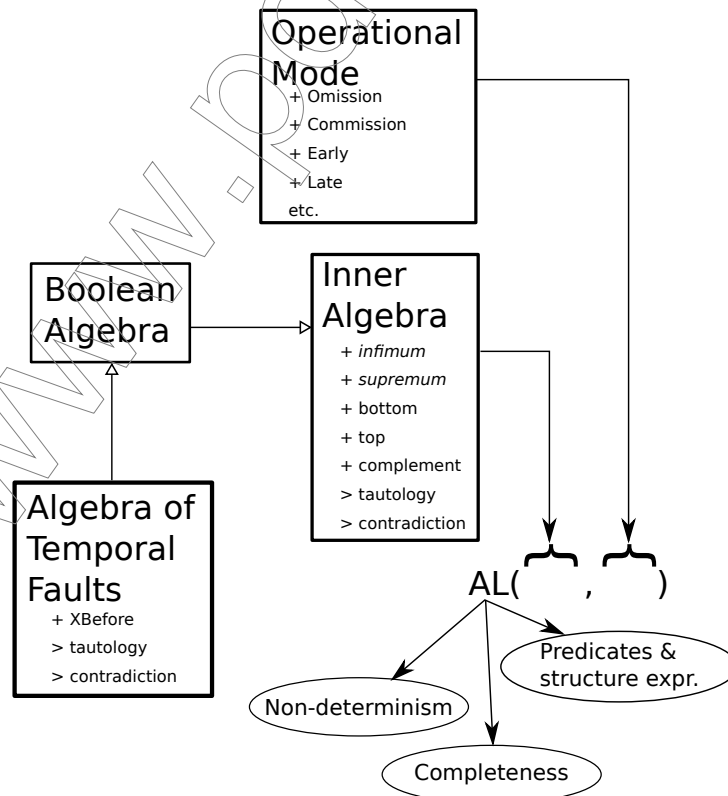


Figure 29 – AL overview

We summarise the properties of the AL as follows:

1. No expression predicate is a contradiction: there are no *false* predicates in activation expressions;
2. The predicates in the terms of an expression consider all possible situations: expression tautology;
3. There are no two terms with exactly the same operational mode: all expression terms are related to a unique operational mode.

These properties form the *healthiness conditions* [89] of an expression in the **AL**.

We show the general form of the **AL** to model faults in Section 5.1, the healthiness conditions to normalize expressions in Section 5.2, how to identify non-determinism in an expression in Section 5.3, and the predicates notation to analyse systems and model fault propagation in Section 5.4.

## 5.1 The Activation Logic Grammar

Each term in **AL** is a pair of a predicate and an operational mode. The predicate is written in either Boolean algebra, **ATF**, or any algebra that provides these properties: tautology and contradiction. We assume that the set of possible faults on a system is finite and that each variable declared in a predicate represents a fault event.

The operational mode has two generic values: (i) Nominal, and (ii) Failure. Nominal values either determine a value, or an undefined value (in this case, the constant value “*undefined*” is assumed). Failure values denote an erroneous behaviour, which can be a total failure (for example, signal omission) or a failure that causes degradation (for example, a signal below or above its nominal range). The choice of the operational modes depends on the system being analysed and its definition is generic and is left for the analyst. For the **AL**, it is sufficient to specify that it is an erroneous behaviour.

The grammar is parametrized by the syntax of an algebra (**Algebra**) and a set of operational modes (**OperModes**). The initial rules of the grammar are defined as follows:

```

AL(Algebra, OperModes)    = TERM(Algebra, OperModes)
                           | TERM(Algebra, OperModes)
                           | ' | ' AL(Algebra, OperModes)
TERM(Algebra, OperModes)  = '(' Algebra ',' OM(OperModes) ')'
OM(OperModes)             = 'Nominal' NominalValue
                           | 'Failure' OperModes
NominalValue              = 'undefined' | Number
Number                    = Integer | Bool | Decimal

```



The denotational semantics of the expressions in [AL](#) is a set of pairs. The predicate in each term of an expression depends on the semantics of the inner algebra. Thus the predicate evaluates to either *true* ( $\top$ ) or *false* ( $\perp$ ) depending on the valuation in the algebra. In what follows we show a sketch of the denotational semantics of [AL](#).

$$\begin{aligned} (P_1, O_1) &\mapsto \{(P_1, O_1)\} \\ (P_1, O_1) \mid (P_2, O_2) &\mapsto \{(P_1, O_1), (P_2, O_2)\} \\ \text{Nominal } 100 &\mapsto \text{Nominal } 100 \\ \text{Nominal undefined} &\mapsto \text{Nominal } \textit{undefined} \\ \text{Failure Omission} &\mapsto \text{Failure } \textit{Omission} \end{aligned}$$

In an expression, if the  $i^{\text{th}}$  predicate evaluates to *true* ( $\top$ ), we say that the  $i^{\text{th}}$  operational mode is *activated*. To simplify the presentation of the expressions and to ease the understanding, we use the denotational semantics in the remainder of this article (the right-hand side of the sketch above). Thus, instead of using  $\text{Exp} = (P_1, O_1) \mid (P_2, O_2)$  we will use  $\text{Exp} = \{(P_1, O_1), (P_2, O_2)\}$

In this section, to illustrate the properties and possible analyses, we use an example of a system with faults  $A$  and  $B$  and the following outputs:

$O_1$ : when  $A$  is active;

$O_2$ : when  $B$  is active;

$O_3$ : when  $A$  is active, but  $B$  is not;

$O_4$ : when  $A$  or  $B$  are active.

The expression for this example in [AL](#) is:

$$S = \{(A, O_1), (B, O_2), (A \wedge \neg B, O_3), (A \vee B, O_4)\} \quad (5.1)$$

We use Eq. (5.1) on the following sections in this chapter to illustrate [AL](#).

In this example we see that one of the healthiness conditions is not satisfied: when for instance,  $A$  and  $B$  are both inactive ( $\neg(A \wedge B)$ ), there is no explicit output defined. In Section 6.4 we show a more detailed case study to illustrate the reasoning about temporal faults. In the next section, we show how to normalise the expression, so that the three healthiness conditions are satisfied.

## 5.2 Healthiness Conditions

The healthiness conditions are fix points of a language. The property is defined as a function of an expression and returns another expression. For example, if a healthiness condition  $H$  is satisfied for an expression  $\text{Exp}$ , we have  $H(\text{Exp}) = \text{Exp}$ .

In what follows we show the three healthiness conditions for the *AL*. All definitions in this section requires an algebra that satisfies the following properties:

**contradiction:** the expression always evaluates to *false*;

**tautology:** the expression always evaluates to *true*.

### 5.2.1 $H_1$ : No predicate is a contradiction

This property is very simple and it is used to eliminate any term that has a predicate that always evaluates to false.

**Definition 5.1.** Let  $exp$  be an expression in the *AL*, then:



$$H_1(exp) = \{ (P, O) \mid (P, O) \in exp \bullet \neg \text{contradiction}(P) \} \quad (5.2)$$

where the operator  $\in$  indicates that a term is present in the expression.

Applying the first healthiness condition to our example results in:

$$H_1(S) = S$$

Thus, we conclude that  $S$  is  $H_1$ -healthy.

### 5.2.2 $H_2$ : All possibilities are covered

This property is used to make explicit that there are uncovered operational modes. In this case, there is a combination of variables in the algebra that was not declared in the expression. Very often the focus when modelling faults is on erroneous behaviour. So we assume that such an uncovered operational mode is nominal, but has an undefined value.

**Definition 5.2.** Let  $exp$  be an expression in the *AL*, and  $\tau$  is:

$$\tau = \neg \left( \bigvee_{(P,O) \in exp} P \right)$$

then:

$$H_2(exp) = \begin{cases} exp, & \text{if contradiction}(\tau) \\ exp \cup \{(\tau, \text{Nominal undefined})\}, & \text{otherwise} \end{cases} \quad (5.3)$$

This property checks erroneous behaviour completeness. If the expression is already complete, all possibilities are already covered, and the expression is healthy. Otherwise, a term containing the missing terms is introduced using the *Nominal undefined*.

Applying the second healthiness condition to our example results in the following expression after simplification:

$$H_2(S) = S \cup \{(\neg A \wedge \neg B, \text{Nominal } \textit{undefined})\}$$

Thus, we conclude that  $S$  is not  $H_2$ -healthy.

### 5.2.3 $H_3$ : There are no two terms with exactly the same operational mode

This property merges terms that contain the same operational mode. It avoids unnecessary formulas and may reduce the expression.

**Definition 5.3.** Let  $exp$  be an expression in the  $AL$ . Then:

$$\begin{aligned} H_3(exp) = \{ (P_1, O_1) \mid (P_1, O_1) \in exp \wedge \\ \forall (P_2, O_2) \in exp \bullet (P_1, O_1) = (P_2, O_2) \vee O_1 \neq O_2 \} \cup \\ \{ (P_1 \vee P_2, O_1) \mid (P_1, O_1), (P_2, O_2) \in exp \wedge O_1 = O_2 \} \end{aligned} \quad (5.4)$$

Applying  $H_3$  in the example in the beginning of the chapter, we conclude that  $S$  is  $H_3$ -healthy. On the other hand, if we consider an  $S'$  system being a copy of  $S$ , but making  $O_1 = O_2$ , then:

$$H_3(S') = \{(A \vee B, O_1), (A \wedge \neg B, O_3), (A \vee B, O_4)\}$$

Thus, we conclude that  $S'$  is not  $H_3$ -healthy.

### 5.2.4 Healthy expression

To obtain a healthy expression, we apply all three healthiness conditions. The order of application of each healthiness condition does not change the resulting expression. The healthiness function is written as composition of functions as follows:

$$H = H_1 \circ H_2 \circ H_3 \quad (5.5)$$

After applying the three healthiness conditions to  $S$ , the resulting expression is:

$$\begin{aligned} H(S) = \{ (A, O_1), (B, O_2), \\ (A \wedge \neg B, O_3), (A \vee B, O_4), \\ (\neg A \wedge \neg B, \text{Nominal } \textit{undefined}) \} \end{aligned}$$

The healthiness conditions are useful to faults modelling, aiding the faults analyst to check contradictions and completeness. Also, obtaining safe predicates is only possible in healthy expressions. In the next section, we show how to verify non-determinism in  $AL$  expressions.

### 5.3 Non-determinism

Non-determinism is usually an undesirable property. Thus, the analysis shall consider the activation of faults even if the fault might or not be active.

To identify non-determinism, we can check for the negation of a contradiction in a pair of predicates in the algebra.

**Definition 5.4** (Non-determinism). *Let  $exp$  be an expression in  $AL$ .*

$$\text{nondeterministic}(exp) = \exists (P_1, O_1), (P_2, O_2) \in exp \bullet \neg \text{contradiction}(P_1 \wedge P_2) \quad (5.6)$$

If there is at least one combination that evaluates  $P_1 \wedge P_2$  to true (it is not a contradiction), then  $exp$  is non-deterministic. Our example is clearly non-deterministic as at least  $A \wedge (A \vee B)$  is not a contradiction.

To analyse components and systems, and to model faults propagation, a predicates notation is shown in the next section. The predicates notation offers two additional ways to check non-determinism.

### 5.4 Predicates Notation

The Activation Logic needs a special notation to enable the analysis of: (i) a particular faults expression, or (ii) a propagation in components. Such a special notation extracts predicates in the algebra given an observable failure of the system (an undesired operational mode).

**Definition 5.5** (Predicate). *Let  $exp$  be an expression in  $AL$ , and  $O_x$  an operational mode. A predicate over  $exp$  that matches  $O_x$  is described as:*

$$\langle |out(exp) = O_x| \rangle \iff \exists (P, O) \in H(exp) \mid O = O_x \bullet P \quad (5.7)$$

The predicates notation function returns a predicate in the algebra. For the example in the beginning of this section, the predicate for  $O_2$  is obtained as follows:

$$\langle |out(S) = O_2| \rangle = B$$

To allow fault propagation of components we need another special notation. It expands the modes of an expression with a predicate in the inner algebra.

**Definition 5.6** (Modes). *Let  $exp$  be an expression in  $AL$ , and  $P$  a predicate in the inner algebra, then:*

$$\text{modes}(exp, P) = \{(P_i \wedge P, O_i) \mid (P_i, O_i) \in H(exp)\} \quad (5.8)$$

Finally, to check the possible outputs, we need a function to obtain a set of outputs given an expression.

**Definition 5.7** (Activation). *Let  $exp$  be an expression in  $AL$ , and  $P_x$  a predicate in the inner algebra, then:*

$$\text{activation}(exp, P_x) = \{O | (P, O) \in H(exp) \wedge \text{tautology}(P_x \Rightarrow P)\} \quad (5.9)$$

Non-determinism can also be checked using the predicates notation and the activation property:

$$\text{activation}(S, A \wedge \neg B) = \{O_1, O_3\} \quad (5.10a)$$

$$\langle |out(S) = O_1| \rangle \wedge \langle |out(S) = O_3| \rangle = A \wedge \neg B \quad (5.10b)$$

Equation (5.10a) shows that both  $O_1$  and  $O_3$  can be observed if  $A \wedge \neg B$  is *true*. Equation (5.10b) states that if the possible operational modes of healthy  $S$  are  $O_1$  and  $O_3$ , then the predicate is  $A \wedge \neg B$ . Non-determinism is the possibility of observing two different failures ( $O_1$  and  $O_3$ ) for the same failure expression ( $A \wedge \neg B$ ) in the algebra. In the next chapter, we show a practical case study using these properties and notations.



## 6 Case study



Our industrial partner provided us with the Simulink model of an Actuator Control System (depicted in Figure 26). The failure expression of this system (that is, for each of its constituent components) was also provided by our industrial partner (we show some of them in Table 7). In what follows we illustrate our strategy using the Monitor component.

A monitor component is a system commonly used for fault tolerance [90, 91]. Initially, the monitor connects the main input (power source on input port 1) with its output. It observes the value of this input port and compares it to a threshold. If the value is below the threshold, the monitor disconnects the output from the main input and connects to the secondary input. We present the Simulink model for this monitor in Figure 27.

Sensors and actuators are used to improve safety by taking measures to decrease potential failures, as the leak protection system shown in Section 3.5. A sensor is installed in a room that may have gas leakage. If the sensor detects a gas leak, then an actuator—a controlled valve—closes the gas flow. A second valve diverts the gas flow if a high pressure is detected due the first valve closing.

Now we show five contributions: (i) using ATF, but only with Boolean operators, thus ignoring ordering, we can obtain the same results obtained in [28], (ii) representing each of the fault traces reported in [28] as a term in our proposed ATF, using the mapping function shown in Section 4.4, (iii) modelling faults of the monitor using AL, using expressions in Boolean Algebra, (iv) modelling faults of the monitor with AL, but using ATF as the inner algebra, and (v) obtaining failure probability from a formula with explicit NOT operators without considering the consensus law nor the theory shown in [9]. Similarly to the association of fault events of Table 7 in Section 3.6, we associate the fault events as:

$$\begin{array}{ll}
 b_1 = \text{LowPower-In1} & B_1 = \mathbf{var} \, b_1 \\
 b_2 = \text{LowPower-In2} & B_2 = \mathbf{var} \, b_2 \\
 f = \text{SwitchFailure} & F = \mathbf{var} \, f
 \end{array}$$

and for the leak detection system, we associate fault events as:

$$\begin{array}{ll}
 prv = \text{the pressure relief valve fails} & PRV = \mathbf{var} \, prv \\
 i_1 = \text{there is an ignition source in room 1} & I_1 = \mathbf{var} \, i_1 \\
 l = \text{there is a gas leak in room 1} & L = \mathbf{var} \, l \\
 val = \text{the isolation valve fails} & VAL = \mathbf{var} \, val
 \end{array}$$

## 6.1 From traces to structure expressions with Boolean operators

In this section we show that the same result reported in [28] in terms of static failure expression (or Boolean propositions) can be obtained with our Boolean operator without using **XBefore**. Similarly to the mapping function shown in Section 4.4, we define a mapping function from traces to **ATF** with Boolean operators only as:

$$\langle [ ] \rangle_{bool} = \top \quad (6.1a)$$

$$\langle [ f ] \rangle_{bool} = \mathbf{var} f \quad (6.1b)$$

$$\langle [ f_1 ] @ tr \rangle_{bool} = \mathbf{var} f_1 \wedge \langle tr \rangle_{bool} \quad (6.1c)$$

$$\langle \{ tr_1, tr_2, \dots, tr_n \} \rangle_{bool} = \bigvee_{i \in \{1, \dots, n\}} \left( \langle tr_i \rangle_{bool} \wedge \neg \bigvee_{j \in \text{Gen}_{tr_i}} \mathbf{var} f_j \right) \quad (6.1d)$$

The only difference of the mapping function, when considering Boolean operators only, is Eq. (6.1c). Equations (6.1a), (6.1b) and (6.1d) are similar to Eqs. (4.44a), (4.44b) and (4.44d)

For each trace shown in Section 3.6, the mapping function generates the following expressions

$$\text{TRACE 1: } \langle [ f, b_2 ] \rangle_{bool} = F \wedge B_2$$

$$\text{TRACE 2: } \langle [ b_2, f ] \rangle_{bool} = B_2 \wedge F$$

$$\text{TRACE 3: } \langle [ b_1, b_2 ] \rangle_{bool} = B_1 \wedge B_2$$

$$\text{TRACE 4: } \langle [ b_2, b_1 ] \rangle_{bool} = B_2 \wedge B_1$$

$$\text{TRACE 5: } \langle [ b_1, f ] \rangle_{bool} = B_1 \wedge F$$

$$\text{TRACE 6: } \langle [ b_1, f, b_2 ] \rangle_{bool} = B_1 \wedge F \wedge B_2$$

$$\text{TRACE 7: } \langle [ b_1, b_2, f ] \rangle_{bool} = B_1 \wedge B_2 \wedge F$$

$$\text{TRACE 8: } \langle [ b_2, b_1, f ] \rangle_{bool} = B_2 \wedge B_1 \wedge F$$

$$\text{TRACE 9: } \langle [ f, b_1, b_2 ] \rangle_{bool} = F \wedge B_1 \wedge B_2$$

$$\text{TRACE 10: } \langle [ f, b_2, b_1 ] \rangle_{bool} = F \wedge B_2 \wedge B_1$$

$$\text{TRACE 11: } \langle [ b_2, f, b_1 ] \rangle_{bool} = B_2 \wedge F \wedge B_1$$

They represent the same faults shown in Section 3.6. Applying the mapping function, Eq. (6.1d), for the previously shown set of traces, we obtain the following expression in **ATF** (and in **FBA**):

$$M_{bool} = (B_1 \wedge B_2) \vee (F \wedge (B_1 \vee B_2)) \quad (6.2)$$

which is equivalent to our industrial partner's failure expression shown in Table 7. This shows that **ATF** can represent (static) failure expression as in our previous work [28].



## 6.2 From traces to structure expressions with XBefore

Now, by using ATF with the XBefore operator and the mapping function shown in Eqs. (4.44a) to (4.44d), we can capture each possible individual sequences as generated by the work [28]:

$$\begin{aligned}
\text{TRACE 1: } & \langle [f, b_2] \rangle_{XB} = (F \rightarrow B_2) \\
\text{TRACE 2: } & \langle [b_2, f] \rangle_{XB} = (B_2 \rightarrow F) \\
\text{TRACE 3: } & \langle [b_1, b_2] \rangle_{XB} = (B_1 \rightarrow B_2) \\
\text{TRACE 4: } & \langle [b_2, b_1] \rangle_{XB} = (B_2 \rightarrow B_1) \\
\text{TRACE 5: } & \langle [b_1, f] \rangle_{XB} = (B_1 \rightarrow F) \\
\text{TRACE 6: } & \langle [b_1, f, b_2] \rangle_{XB} = B_1 \rightarrow (F \rightarrow B_2) \\
\text{TRACE 7: } & \langle [b_1, b_2, f] \rangle_{XB} = B_1 \rightarrow (B_2 \rightarrow F) \\
\text{TRACE 8: } & \langle [b_2, b_1, f] \rangle_{XB} = B_2 \rightarrow (B_1 \rightarrow F) \\
\text{TRACE 9: } & \langle [f, b_1, b_2] \rangle_{XB} = F \rightarrow (B_1 \rightarrow B_2) \\
\text{TRACE 10: } & \langle [f, b_2, b_1] \rangle_{XB} = F \rightarrow (B_2 \rightarrow B_1) \\
\text{TRACE 11: } & \langle [b_2, f, b_1] \rangle_{XB} = B_2 \rightarrow (F \rightarrow B_1)
\end{aligned}$$

Using the mapping function, Eq. (4.44d), for the previously shown set of traces, we obtain:

$$\begin{aligned}
M_A &= (F \rightarrow B_2 \wedge \neg B_1) \vee (B_2 \rightarrow F \wedge \neg B_1) \vee (B_1 \rightarrow B_2 \wedge \neg F) \vee \\
&\quad (B_2 \rightarrow B_1 \wedge \neg F) \vee (B_1 \rightarrow F \wedge \neg B_2) \vee (B_1 \rightarrow (F \rightarrow B_2)) \vee \\
&\quad (B_1 \rightarrow (B_2 \rightarrow F)) \vee (B_2 \rightarrow (B_1 \rightarrow F)) \vee (F \rightarrow (B_1 \rightarrow B_2)) \vee \\
&\quad (F \rightarrow (B_2 \rightarrow B_1)) \vee (B_2 \rightarrow (F \rightarrow B_1)) \\
&= (F \wedge B_2 \wedge \neg B_1) \vee (B_1 \wedge B_2 \wedge \neg F) \vee (B_1 \rightarrow F \wedge \neg B_2) \vee \\
&\quad (B_1 \rightarrow (F \rightarrow B_2)) \vee (B_1 \rightarrow (B_2 \rightarrow F)) \vee (B_2 \rightarrow (B_1 \rightarrow F)) \vee \\
&\quad (F \rightarrow (B_1 \rightarrow B_2)) \vee (F \rightarrow (B_2 \rightarrow B_1)) \vee (B_2 \rightarrow (F \rightarrow B_1)) \quad \text{by Eq. (4.27c)} \\
&= (F \wedge B_2 \wedge \neg B_1) \vee (B_1 \wedge B_2 \wedge \neg F) \vee (B_1 \rightarrow F \wedge \neg B_2) \vee \\
&\quad (B_2 \wedge (B_1 \rightarrow F)) \vee (B_2 \wedge (F \rightarrow B_1)) \quad \text{by Eq. (4.28)} \\
&= (B_1 \wedge B_2) \vee (F \wedge B_2) \vee (B_1 \rightarrow F \wedge \neg B_2) \quad (6.3)
\end{aligned}$$

The semantics of the above expression is: (i) fault  $b_2$  (**var**  $b_2$ ) occurs and fault  $b_1$  (**var**  $b_1$ ) or fault  $f$  (**var**  $f$ ) occurs, or (ii) fault  $b_1$  occurs before fault  $f$  and fault  $b_2$  does not occur, which is more precise than the expression found without considering order of events.

Expanding Eq. (6.2), we have:

$$(B_1 \wedge B_2) \vee (F \wedge B_2) \vee (F \wedge B_1)$$

which differs from Eq. (6.3) only on terms:  $F \wedge B_1$  (of  $M_{bool}$ ) and  $F \rightarrow B_1 \wedge \neg B_2$  (of  $M_A$ ).

### 6.3 From AL to structure expressions with Boolean operators

The power source has only two possible operational modes: (i) the power source works as expected, providing a nominal value of 12V, and (ii) it has an internal failure  $B_i$ , and its operational mode is “low power”. In AL it is modelled as:

$$PowerSource_i = \{(B_i, LP), (\neg B_i, \text{Nominal } 12V)\} \quad (6.4)$$

where  $LP$  is the LowPower failure.  $PowerSource_i$  is healthy:

- H<sub>1</sub>-healthy: there is no contradiction in the expressions;
- H<sub>2</sub>-healthy: combining the expressions of the pairs in a disjunction, we obtain a tautology;
- H<sub>3</sub>-healthy: the operational modes of the pairs are distinct.

The monitor is a bit different because its behaviour depends not only on internal faults, but also on its inputs. We will now use the predicates notation defined in Section 5.4 to express fault propagation. As the monitor has two inputs and its behaviour is described in Figure 27, then it is a function of the expressions of both inputs:

$$\begin{aligned} Monitor_{bool}(in_1, in_2) = & \\ & \text{modes}(in_1, \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F) \cup \\ & \text{modes}(in_2, \neg \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F) \cup \\ & \text{modes}(in_2, \langle |out(in_1) = \text{Nominal } X| \rangle \wedge F) \cup \\ & \text{modes}(in_1, \neg \langle |out(in_1) = \text{Nominal } X| \rangle \wedge F) \end{aligned} \quad (6.5)$$

where  $X$  is an unbound variable and assumes any value. The expression states the following:

- The monitor output is the same as  $in_1$  if the output of  $in_1$  is nominal and there is no internal failure in the monitor:

$$\text{modes}(in_1, \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F)$$

- The monitor output is the same as  $in_2$  if the output of  $in_1$  is not nominal and there is no internal failure in the monitor:

$$\text{modes}(in_2, \neg \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F)$$

- The monitor output is the converse of the previous two conditions if the internal failure  $F$  is active:

$$\begin{aligned} & \text{modes}(in_2, \langle | \text{out}(in_1) = \text{Nominal } X | \rangle \wedge F) \cup \\ & \text{modes}(in_1, \neg \langle | \text{out}(in_1) = \text{Nominal } X | \rangle \wedge F) \end{aligned}$$

The operational modes (observed behaviour) of the monitor depend on: (i) its internal fault, and (ii) propagated errors from its inputs. Composing the monitor with the two power sources, we obtain the AL expression of a power supply subsystem  $System_{bool}$ :

$$\begin{aligned} System_{bool} &= \\ & \text{Monitor}_{bool}(\text{PowerSource}_1, \text{PowerSource}_2) \\ &= \text{modes}(in_1, \neg B_1 \wedge \neg F) \cup \text{modes}(in_2, \neg \neg B_1 \wedge \neg F) \cup \\ & \quad \text{modes}(in_2, \neg B_1 \wedge F) \cup \text{modes}(in_1, \neg \neg B_1 \wedge F) \quad \text{by Eq. (5.7)} \\ &= \text{modes}(in_1, \neg B_1 \wedge \neg F) \cup \text{modes}(in_2, B_1 \wedge \neg F) \cup \\ & \quad \text{modes}(in_2, \neg B_1 \wedge F) \cup \text{modes}(in_1, B_1 \wedge F) \quad \text{by simplification} \\ &= \{(P_i \wedge \neg B_1 \wedge \neg F, O_i) \mid (P_i, O_i) \in in_1\} \cup \\ & \quad \{(P_i \wedge B_1 \wedge \neg F, O_i) \mid (P_i, O_i) \in in_2\} \cup \\ & \quad \{(P_i \wedge \neg B_1 \wedge F, O_i) \mid (P_i, O_i) \in in_2\} \cup \\ & \quad \{(P_i \wedge B_1 \wedge F, O_i) \mid (P_i, O_i) \in in_1\} \quad \text{by Eq. (5.8)} \\ &= \{(B_1 \wedge \neg B_1 \wedge \neg F, LP), \\ & \quad (\neg B_1 \wedge \neg B_1 \wedge \neg F, \text{Nominal } 12V), \\ & \quad (B_2 \wedge B_1 \wedge \neg F, LP), \\ & \quad (\neg B_2 \wedge B_1 \wedge \neg F, \text{Nominal } 12V), \\ & \quad (B_2 \wedge \neg B_1 \wedge F, LP), \\ & \quad (\neg B_2 \wedge \neg B_1 \wedge F, \text{Nominal } 12V), \\ & \quad (B_1 \wedge B_1 \wedge F, LP), \\ & \quad (\neg B_1 \wedge B_1 \wedge F, \text{Nominal } 12V)\} \quad \text{replacing vars} \end{aligned}$$

Simplifying and applying  $H_1$ , we obtain:

$$\begin{aligned} H_1(System_{bool}) &= \\ & \{(\neg B_1 \wedge \neg F, \text{Nominal } 12V), (B_2 \wedge B_1 \wedge \neg F, LP), \\ & (\neg B_2 \wedge B_1 \wedge \neg F, \text{Nominal } 12V), (B_2 \wedge \neg B_1 \wedge F, LP), \\ & (\neg B_2 \wedge \neg B_1 \wedge F, \text{Nominal } 12V), (B_1 \wedge F, LP)\} \end{aligned}$$

Applying,  $H_3$ , we simplify to:

$$\begin{aligned}
 & H_3 \circ H_1 (System_{bool}) \\
 &= \left\{ \left( \begin{array}{l} (\neg B_1 \wedge \neg F) \vee \\ (B_1 \wedge \neg B_2 \wedge \neg F) \vee, \text{Nominal } 12V \\ (\neg B_1 \wedge \neg B_2 \wedge F) \end{array} \right), \right. \\
 &\quad \left. \left( \begin{array}{l} (B_1 \wedge B_2 \wedge \neg F) \vee \\ (\neg B_1 \wedge B_2 \wedge F) \vee, LP \\ (B_1 \wedge F) \end{array} \right) \right\} \\
 &= \{((\neg B_1 \wedge \neg B_2) \vee \neg F \wedge (\neg B_1 \vee \neg B_2), \text{Nominal } 12V), \\
 &\quad (F \wedge (B_1 \vee B_2) \vee (B_1 \wedge B_2), LP)\}
 \end{aligned}$$

The monitor expression is  $H_2$ -healthy (the predicates are complete), thus:

$$H_2 \circ H_3 \circ H_1 (System_{bool}) = H_3 \circ H_1 (System_{bool})$$

The resulting expression for the monitor after applying all healthiness conditions is:

$$\begin{aligned}
 H (System_{bool}) = \{ & ((\neg B_1 \wedge \neg B_2) \vee \neg F \wedge (\neg B_1 \vee \neg B_2), \text{Nominal } 12V), \\
 & (F \wedge (B_1 \vee B_2) \vee (B_1 \wedge B_2), LP) \}
 \end{aligned} \tag{6.6}$$

The operational modes of  $System_{bool}$  is either *Nominal 12V* or *LP* (low power).

Finally, we obtain the *low power* structure expression (see Table 7) using the predicates notation:

$$\langle |out (System_{bool}) = LP| \rangle \iff F \wedge (B_1 \vee B_2) \vee (B_1 \wedge B_2)$$

The monitor expression also indicates that if the switch is operational ( $\neg F$ ) and at least one PowerSource is operational ( $\neg B_1 \vee \neg B_2$ ), the monitor output is nominal. But if at least one PowerSource is faulty ( $B_1 \vee B_2$ ) and the monitor has an internal failure ( $F$ ) the system is not operational. These two sentences—written in **AL** using the predicates notation—are:

$$\begin{aligned}
 & \text{activation} (System_{bool}, \neg F \wedge (\neg B_1 \vee \neg B_2)) \\
 &= \{O \mid (P, O) \in H (System_{bool}) \wedge \\
 &\quad \text{tautology} (\neg F \wedge (\neg B_1 \vee \neg B_2) \implies P)\} \quad [\text{by Eq. (5.9)}] \\
 &= \{\text{Nominal } 12V\} \quad [\text{by simplification}] \tag{6.7a}
 \end{aligned}$$

$$\begin{aligned}
 & \text{activation} (System_{bool}, F \wedge (B_1 \vee B_2)) \\
 &= \{O \mid (P, O) \in H (System_{bool}) \wedge \\
 &\quad \text{tautology} (F \wedge (B_1 \vee B_2) \implies P)\} \quad [\text{by Eq. (5.9)}] \\
 &= \{LP\} \quad [\text{by simplification}] \tag{6.7b}
 \end{aligned}$$

## 6.4 From AL to structure expressions with XBefore

Now, let's consider the same system but with a subtle modification. As shown in [52], the order of the occurrence of faults may be relevant, and the qualitative and quantitative analyses results may be different than those results without considering the order of the occurrence of faults. Observing Figure 27, we see that if  $F$  activates before a failure in the first input of the monitor, then it would display a nominal behaviour. This is because the internal failure  $F$  anticipates switching to the second input. On the other hand, if the first input fails before  $F$ , then the monitor would switch to the second input, and then switch back due to the internal failure. We obtain the following expression for the monitor, now using the ATF:

$$\begin{aligned}
 \text{Monitor}_{XB}(in_1, in_2) = & \\
 & \text{modes}(in_1, \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F) \cup \\
 & \text{modes}(in_2, \neg \langle |out(in_1) = \text{Nominal } X| \rangle \wedge \neg F) \cup \\
 & \text{modes}(in_2, \langle |out(in_1) = \text{Nominal } X| \rangle \wedge F) \cup \\
 & \text{modes}(in_1, \neg \langle |out(in_1) = \text{Nominal } X| \rangle \rightarrow F) \cup \\
 & \text{modes}(in_2, F \rightarrow \neg \langle |out(in_1) = \text{Nominal } X| \rangle)
 \end{aligned} \tag{6.8}$$

where  $X$  is an unbound variable and assumes any value.

The difference to  $\text{System}_{\text{bool}}$  (Eq. (6.5)) is only the finer analysis of the cases of erroneous behaviours of the first input and an internal failure. Note that the finer analysis splits the predicate

$$\neg \langle |out(in_1) = \text{Nominal } 12V| \rangle \wedge F \quad (\text{activates } in_1)$$

into:

$$\neg \langle |out(in_1) = \text{Nominal } 12V| \rangle \rightarrow F \quad (\text{activates } in_1)$$

and

$$F \rightarrow \neg \langle |out(in_1) = \text{Nominal } 12V| \rangle \quad (\text{activates } in_2)$$

We can assure that such a split is complete because the predicates notation evaluates to  $B_1$ . As the operands satisfy all temporal properties (Eqs. (4.14) and (4.16) to (4.18)) and events independence (Eq. (4.23)), thus the law shown in Eq. (4.25b) is valid. For the first split item, the expected behaviour is the same as  $in_1$  because the system switches to  $in_2$ , but then an internal failure occurs, and it switches back to  $in_1$ . For the second split item, it switches to  $in_2$  due to an internal failure, then the first input fails, so the behaviour is similar to the nominal behaviour (see the second *modes* in Eq. (6.8)).

Following the similar expansions of Eq. (6.5), we obtain:

$$\begin{aligned}
 System_{XB} &= Monitor_{XB} (PowerSource_1, PowerSource_2) \\
 &= \{ (B_1 \wedge \neg B_1 \wedge \neg F, LP), \\
 &\quad (\neg B_1 \wedge \neg B_1 \wedge \neg F, \text{Nominal } 12V), \\
 &\quad (B_2 \wedge B_1 \wedge \neg F, LP), \\
 &\quad (\neg B_2 \wedge B_1 \wedge \neg F, \text{Nominal } 12V), \\
 &\quad (B_2 \wedge \neg B_1 \wedge F, LP), \\
 &\quad (\neg B_2 \wedge \neg B_1 \wedge F, \text{Nominal } 12V), \\
 &\quad (B_1 \wedge B_1 \rightarrow F, LP), \\
 &\quad (\neg B_1 \wedge B_1 \rightarrow F, \text{Nominal } 12V) \}, \\
 &\quad (B_2 \wedge F \rightarrow B_1, LP), \\
 &\quad (\neg B_2 \wedge F \rightarrow B_1, \text{Nominal } 12V) \}
 \end{aligned}$$

Simplifying and applying  $H_1$  to remove contradictions, we obtain:

$$\begin{aligned}
 H_1 (System_{XB}) &= \\
 &\{ (\neg B_1 \wedge \neg F, \text{Nominal } 12V), (B_2 \wedge B_1 \wedge \neg F, LP), \\
 &\quad (\neg B_2 \wedge B_1 \wedge \neg F, \text{Nominal } 12V), (B_2 \wedge \neg B_1 \wedge F, LP), \\
 &\quad (\neg B_2 \wedge \neg B_1 \wedge F, \text{Nominal } 12V), (B_1 \rightarrow F, LP), \\
 &\quad (B_2 \wedge F \rightarrow B_1, LP), (\neg B_2 \wedge F \rightarrow B_1, \text{Nominal } 12V) \}
 \end{aligned}$$

Applying  $H_3$  to remove redundant terms with identical operational modes and

using the rules shown in Section 4.2, we simplify to:

$$\begin{aligned}
& H_3 \circ H_1 (System_{XB}) \\
&= \left\{ \left( \begin{array}{l} (\neg B_1 \wedge \neg F) \vee \\ (B_1 \wedge \neg B_2 \wedge \neg F) \vee \\ (\neg B_1 \wedge \neg B_2 \wedge F) \vee \\ (\neg B_2 \wedge F \rightarrow B_1) \end{array} \right), \text{Nominal } 12V \right\}, \\
&\quad \left\{ \left( \begin{array}{l} (B_1 \wedge B_2 \wedge \neg F) \vee \\ (\neg B_1 \wedge B_2 \wedge F) \vee \\ (B_1 \rightarrow F) \vee \\ (B_2 \wedge F \rightarrow B_1) \end{array} \right), LP \right\} \\
&= \left\{ \left( \begin{array}{l} (\neg B_1 \wedge \neg F) \vee \\ (B_1 \wedge \neg B_2 \wedge \neg F) \vee \\ (\neg B_1 \wedge \neg B_2 \wedge F) \vee \\ (\neg B_2 \wedge F \rightarrow B_1) \end{array} \right), \text{Nominal } 12V \right\}, \\
&\quad \left\{ \left( \begin{array}{l} (B_1 \wedge B_2 \wedge \neg F) \vee \\ (\neg B_1 \wedge B_2 \wedge F) \vee \\ (B_2 \wedge B_1 \rightarrow F) \vee, LP \\ (\neg B_2 \wedge B_1 \rightarrow F) \vee \\ (B_2 \wedge F \rightarrow B_1) \end{array} \right) \right\} \\
&= \{ ((\neg B_1 \wedge \neg B_2) \vee \neg F \wedge (\neg B_1 \vee \neg B_2) \vee \\
&\quad \neg B_2 \wedge F \rightarrow B_1, \text{Nominal } 12V), \\
&\quad ((B_1 \wedge B_2) \vee (B_2 \wedge F) \vee (\neg B_2 \wedge B_1 \rightarrow F), LP) \}
\end{aligned}$$

The monitor expression is  $H_2$ -healthy. Simplifying Boolean operators as usual, the *XBefore* expression is:

$$(\neg B_2 \wedge F \rightarrow B_1) \vee (\neg B_2 \wedge B_1 \rightarrow F)$$

which simplifies to

$$\neg B_2 \wedge F \wedge B_1 \quad \text{by Eq. (4.25b)}$$

Thus:

$$H_2 \circ H_3 \circ H_1 (System_{XB}) = H_3 \circ H_1 (System_{XB})$$

The resulting expression for the monitor after applying all healthiness conditions

is:

$$\begin{aligned} H(System_{XB}) = \{ & ((\neg B_1 \wedge \neg B_2) \vee \neg F \wedge (\neg B_1 \vee \neg B_2) \vee \\ & \neg B_2 \wedge F \rightarrow B_1, \text{Nominal } 12V), \\ & ((B_1 \wedge B_2) \vee (B_2 \wedge F) \vee \\ & (\neg B_2 \wedge B_1 \rightarrow F), LP) \} \end{aligned} \quad (6.9)$$

Finally, we obtain the *low power* structure expression of the monitor using the predicates notation:

$$\langle |out(System_{XB}) = LP| \rangle \iff (B_1 \wedge B_2) \vee (B_2 \wedge F) \vee (\neg B_2 \wedge B_1 \rightarrow F)$$

Thus,  $System_{XB}$  fails with  $LP$  if:

- Both power sources fail;
- The monitor fails to detect the nominal state of the first power source and the second power source is in a failure state;
- The monitor fails to detect the failure state of the first power source (the monitor fails after the failure of the first power source).

Note that if the monitor fails before the failure of the first power source, it fails to detect the operational mode of the first power source and switches to the second power source, which is in a nominal state (see expression  $\neg B_2 \wedge F \rightarrow B_1$  in Eq. (6.9)).

## 6.5 Obtaining top-event probability with explicit NOT operators

In this section we show how to use ATF to obtain the same probability formula of Eq. (3.11).  $\diamond$

We use Eq. (4.41) to split the calculations of the top-event structure expression shown in Eq. (3.9):

$$\begin{aligned} \text{FPr} \{L \wedge ((\neg VAL \wedge PRV) \vee (VAL \wedge I_1))\} = \\ \text{FPr} \{L\} \times \text{FPr} \{(\neg VAL \wedge PRV) \vee (VAL \wedge I_1)\} \end{aligned} \quad (6.10)$$

Then, we obtain the formula probability of the top-event probability of the structure



expression shown in Eq. (3.9) in ATF:

$$\begin{aligned}
 \text{FPr}\{L\} &= \text{Pr}_{\text{FS}}\{[l]\} && \text{by Eq. (4.42)} \\
 &= P_l(t) && (6.11a) \\
 \text{FPr}\{(\neg VAL \wedge PRV) \vee \\
 (VAL \wedge I_1)\} &= \text{Pr}_{\text{FS}}\{[prv]\} + \text{Pr}_{\text{FS}}\{[i_1, prv]\} + \\
 &\quad \text{Pr}_{\text{FS}}\{[prv, i_1]\} + \text{Pr}_{\text{FS}}\{[val, i_1]\} + \\
 &\quad \text{Pr}_{\text{FS}}\{[i_1, val]\} + \\
 &\quad \text{Pr}_{\text{FS}}\{[val, i_1, prv]\} + \dots + \\
 &\quad \text{Pr}_{\text{FS}}\{[prv, i_1, val]\}
 \end{aligned}$$

Note that we use the expression without the consensus law, but the “missing” term  $PRV \wedge I_1$  appears in the denotational semantics used in the probability calculation.

$$\begin{aligned}
 \text{FPr}\{(\neg VAL \wedge PRV) \vee \\
 (VAL \wedge I_1)\} &= P_{prv}(t) \times (1 - P_{i_1}(t)) \times (1 - P_{val}(t)) \\
 &\quad P_{i_1}(t) \times P_{prv}(t) \times (1 - P_{val}(t)) \\
 &\quad P_{val}(t) \times P_{i_1}(t) \times (1 - P_{prv}(t)) \\
 &\quad P_{val}(t) \times P_{i_1}(t) \times P_{prv}(t) \\
 &= P_{prv}(t) + P_{val}(t) \times P_{i_1}(t) - P_{prv}(t) \times P_{val}(t) && (6.11b)
 \end{aligned}$$

From Eqs. (6.10), (6.11a) and (6.11b), we obtain:

$$\text{FPr}\{TOP\} = P_l(t) \times (P_{prv}(t) + P_{val}(t) \times P_{i_1}(t) - P_{prv}(t) \times P_{val}(t)) \quad (6.12)$$

which is equivalent to Eq. (3.11).



## 7 Conclusion

In this work we presented a foundational theory to support a more precise representation of fault events as compared to our previous strategy for injecting faults [28]. The failure logic is essential for system safety assessment because it is used as basic input for building fault trees [25, 31, 92]. Furthermore, we still connect the strategy presented in [93] with the works reported in [31] (functional analysis) and in [92, 25] (safety assessment) because our new algebra is at least a Boolean algebra.

We also proposed a parametrized logic, the **AL**, that enables the analysis of systems depending on the expressiveness of a given algebra and a given set of operational modes. If **ATF** is used as a parameter, then the order of occurrence of faults can be considered.



Other algebra, like ternary algebras [94] can be used, since they have tautology and contradiction properties. Although the **AL** is not as detailed as **AADL**, the predicates notation in conjunction with the **ATF** provides a richer assertion framework. Also, it is possible to verify non-determinism on the model, by: (i) verifying its existence with the nondeterministic function, (ii) providing an expression and obtaining the possible operational modes with the activation function, or (iii) using the predicates notation to obtain a predicate that enables two or more operational modes.



The work reported in [21, 20, 33] tackles simultaneity with “nearly simultaneous” events [95]. But we consider instantaneous events, like the work reported in [23], because we assume that simultaneity is probabilistically impossible.

The distinct lists representation in our algebra does not allow obtaining minimal cut sequences directly from the formula, similar to **FBA**s. The sets in an **FBA** formula are already the minimal cut sets. In our work, **ATF** allows us to find minimal cut sequences (with **XBefore**) from the formulas in **DNF** algebraically: each sub-expression is a minimal cut sequence.



Boolean formulas reduction can be achieved by: (i) application of Boolean laws, (ii) **BDD**s, or (iii) **FBA**s. We used Boolean and **XBefore** laws to reduce **ATF** formulas. The work reported in [42, 43] uses Sequential **BDD**s to reduce formulas with order-based operators. We plan to use similar concepts in a future work. A ternary tree with special nodes seems to be a solution, but we have not verified yet.

The works reported in [3, 66, 22, 21, 20] removed the **NOT** operator. Thus the algebras defined there (to analyse **TFT** and **DFT**) resembles a Boolean algebra, but are not complete. The **ATF** allows such trees to have **NOT** operators and the analysis could be performed similarly to **SFT**. Compared to **TFT**s, **ATF** does not allow simultaneous events. Compared to **DFT**s, **ATF** is equivalent to the algebra shown in the works reported in [24, 3],

although their algebra has an operator to represent simultaneity, because simultaneity is probabilistically impossible. The inclusion of an operator to represent simultaneity and the proofs of relation of **ATF** to the algebras of **TFT** and **DFT** are left as future work.

The **AADL** is extensible. The work reported in [96] shows an extension to perform dependability analysis through state machines and expressions on fault events and operational modes. Although such an extension captures system behaviour, operational mode activation conditions are expressed in state transitions in combination with an extension of Boolean expressions (not related to order). Our work relates operational modes and fault occurrences order explicitly.

As presented in [52], **TFTs** and **DFTs** structure expressions can be written as formulas in **ATF**. As the root events of **TFTs** and **DFTs** represent operational modes of a system, the **ATF** can be used to associate root events with operational modes, thus allowing the combination of two or more fault trees.

Although the properties of **AL** require that the inner algebra provides tautology and contradiction, and we used **ATF** in the case study, we did not show tautology and contradiction for **ATF**. Instead, we used a law to reduce the **ATF** expression to a Boolean expression. The methodology to check tautology and contradiction in **ATF** is related to expression reduction, which is a future work.

The original expression shown in the case study (Section 6.4) was already  $H_2$ -healthy. The second healthiness condition about completeness uses the concept of undefined value to make any expression  $H_2$ -healthy. Algebraically it is fine, but in practice, the property should be met initially, thus the initial expression is already  $H_2$ -healthy. This property should only be used as an alert to the analyst if it not met initially.

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## Appendix



# APPENDIX A – Formal proofs in Isabelle/HOL

In the following we list all theorems and proofs concerning the laws presented in Chapter 4. The complete set of verifiable theory files is available at <http://www.cin.ufpe.br/~alrd/phd/phd-alrd.zip> (password: 6Zvq\$5Vyj). We list only those files created in our work. Each theorem, proof or corollary is followed by its own proof.

The theory about lists of distinct elements (distinct lists) is available in [97] (we used the 2015 version that is available with Isabelle/HOL).

This Appendix is organized as follows: (i) Appendix A.1 presents the base lemmas and theorems for sliceable types; (ii) sublists (sliceable distinct lists) are shown in Appendix A.2; (iii) algebraic definitions and laws of the ATF are shown in Appendix A.3, and (iv) proofs using the denotational semantics of sets of distinct lists are shown in Appendix A.4.

## A.1 Sliceable

In this section we present a class to express sub-structures for a data type, and laws over such a class. For example, for lists, *sliceable* defines operators and theorems to obtain sublists.

```
class sliceable =
  fixes slice :: "'a ⇒ nat ⇒ nat ⇒ 'a" ("(3_†_.._)" [80,80,80] 80)
  fixes size :: "'a ⇒ nat" ("(1#_)" 65)
  fixes empty_inter :: "'a ⇒ 'a ⇒ bool"
  fixes disjoint :: "'a ⇒ bool"

  assumes slice_none: "x†0..(#x) = x"
  assumes empty_seq_inter [simp]:
    "disjoint x ⇒ c ≤ k ⇒ empty_inter (x†0..c) (x†k..(#x))"
  assumes size_slice: "size (x†i..j) = max 0 ((min j (size x))-i)"
  assumes slice_slice: "(x†i..j)†a..b = x†(i+a)..(min j (i+b))"
  assumes disjoint_slice_suc:
    "disjoint x ⇒ i ≠ j ⇒ i < (#x) ⇒ j < (#x) ⇒
     x†i..(Suc i) ≠ x†j..(Suc j)"
  assumes disjoint_slice[simp]: "disjoint x ⇒ disjoint (x†i..j)"
  assumes forall_slice_implies_eq: "(#x) = (#y) ∧ (∀ i j. (x†i..j) =
```

$$(y \dagger i..j) \longleftrightarrow (x = y)"$$

**notation** (latex output) *slice* ("(3\_ [..\_])" [80,80,80] 80)

Teste  $x_{[i..j]}$

**definition** *slice\_right* :: "'a::sliceable  $\Rightarrow$  nat  $\Rightarrow$  'a" ("(2\_†..\_)" [80,80] 80)  
**where** "slice\_right x i = x†0..i"

**notation** ("latex") *slice\_right* ("(2\_ [..\_])" [80,80] 80)

**definition** *slice\_left* :: "'a::sliceable  $\Rightarrow$  nat  $\Rightarrow$  'a" ("(2\_†..\_)" [80,80] 80)  
**where** "x†i.. = x†i..(# x)"

**notation** ("latex") *slice\_left* ("(2\_ [..\_])" [80,80] 80)

### A.1.1 Disjoint elements and sliceable

**lemma** (in sliceable) *slice\_right\_disjoint*[simp]: "disjoint xs  $\Rightarrow$   
 disjoint (slice\_right xs i)"  
**unfolding** *slice\_right\_def*  
**by** *simp*

The notation for  $x_{[..i]}$  is  $x_{[..i]}$

**lemma** (in sliceable) *slice\_left\_disjoint*[simp]: "disjoint xs  $\Rightarrow$   
 disjoint (xs†i..)"  
**unfolding** *slice\_left\_def*  
**by** *simp*

### A.1.2 n-th element in a sliceable

**abbreviation** *sliceable\_nth* :: "'a::sliceable  $\Rightarrow$  nat  $\Rightarrow$  'a"  
**where**  
 "sliceable\_nth 1 i  $\equiv$  1†i..(Suc i)"

### A.1.3 Theorems for sliceable

**theorem** (in sliceable) *empty\_seq\_inter\_eq* [simp]:  
 "disjoint x  $\Rightarrow$  empty\_inter (x†..i) (x†i..)"  
**by** (simp add: slice\_right\_def slice\_left\_def)



theorem (in sliceable) empty\_seq\_sliced\_inter [simp]:

"disjoint x  $\implies$  b  $\leq$  i  $\implies$  j  $\leq$  a  $\implies$  i  $\leq$  j  $\implies$  a  $\leq$  size x  $\implies$   
 empty\_inter (x**†**b..i) (x**†**j..a)"

proof-

let ?l = "x**†**b..a"

assume lt0: "i  $\leq$  j"

assume lt1: "j  $\leq$  a"

assume lt2: "b  $\leq$  i"

assume lt3: "a  $\leq$  size x"

assume lt4: "disjoint x"

have blta: "b  $\leq$  a" using lt0 lt1 lt2 by simp

have ilta: "i  $\leq$  a" using lt0 lt1 by simp

hence 2: "empty\_inter (?l**†**0..(i-b)) (?l**†**(j-b)..(#?l))"

using lt0 lt4 disjoint\_slice by simp

hence "empty\_inter ((x**†**b..a)**†**0..(i-b)) ((x**†**b..a)**†**(j-b)..(#?l))" by simp

hence 3: "empty\_inter (x**†**b..i) ((x**†**b..a)**†**(j-b)..(#(x**†**b..a)))" using ilta lt2

by (simp add: slice\_slice\_min\_absorb2)

hence 3: "empty\_inter (x**†**b..i) (x**†**j..a)"

using blta lt0 lt2 lt3

by (auto simp add: size\_slice\_slice\_slice\_min\_def)

thus ?thesis by simp

qed

theorem distinct\_slice\_lte\_inter\_empty[simp]:

"distinct l  $\implies$  i  $\leq$  j  $\implies$

set (take i (drop 0 l))

$\cap$  set (take (length l-i) (drop i l)) = {}"

by (simp add: set\_take\_disj\_set\_drop\_if\_distinct )

lemma (in sliceable) size\_slice\_right\_absorb: "(#(l**†**..i)) = min i (#l)"

by (simp add: slice\_right\_def sliceable\_class.size\_slice)

lemma (in sliceable) size\_slice\_left\_absorb: "(#(l**†**i..)) = (#l)-i"

by (simp add: slice\_left\_def sliceable\_class.size\_slice)

corollary (in sliceable) slice\_right\_slice\_left\_absorb: "(l**†**..i)**†**j.. = l**†**j..i"

unfolding slice\_left\_def slice\_right\_def

by (metis (mono\_tags, hide\_lams) add.left\_neutral add.right\_neutral max\_0L

min.left\_idem size\_slice\_right\_absorb slice\_right\_def

sliceable\_class.size\_slice sliceable\_class.slice\_none

sliceable\_class.slice\_slice)

corollary (in sliceable) slice\_right\_slice\_left\_absorb\_empty:

" $i \leq j \implies (\#((l \upharpoonright i) \upharpoonright j)) = 0$ "

by (simp add: size\_slice\_left\_absorb size\_slice\_right\_absorb)

corollary (in sliceable) slice\_left\_slice\_right\_absorb:

" $(l \upharpoonright i) \upharpoonright j = l \upharpoonright i..(i+j)$ "

unfolding slice\_left\_def slice\_right\_def

proof -

have " $(l \upharpoonright i..(\#l)) \upharpoonright 0..j = (l \upharpoonright 0..(\#l)) \upharpoonright i..(i+j)$ "

by (simp add: sliceable\_class.slice\_slice)

thus " $(l \upharpoonright i..(\#l)) \upharpoonright 0..j = l \upharpoonright i..(i+j)$ "

by (simp add: sliceable\_class.slice\_none)

qed

corollary (in sliceable) slice\_right\_slice\_right\_absorb:

" $(l \upharpoonright i) \upharpoonright j = (l \upharpoonright i..(\min i j))$ "

unfolding slice\_left\_def slice\_right\_def

by (simp add: sliceable\_class.slice\_slice)

corollary (in sliceable) slice\_left\_slice\_left\_absorb:

" $(l \upharpoonright i) \upharpoonright j.. = l \upharpoonright (i+j).. $"$$

unfolding slice\_left\_def slice\_right\_def

by (simp add: sliceable\_class.slice\_slice sliceable\_class.size\_slice min\_absorb1)

corollary (in sliceable) slice\_slice\_right\_absorb:

" $(l \upharpoonright i..j) \upharpoonright b = l \upharpoonright i..(\min j (i+b))$ "

unfolding slice\_left\_def slice\_right\_def

by (simp add: add.commute sliceable\_class.slice\_slice)

corollary (in sliceable) slice\_slice\_left\_absorb:

" $(l \upharpoonright i..j) \upharpoonright a.. = l \upharpoonright (i+a).. $"$$

unfolding slice\_left\_def slice\_right\_def

by (metis (mono\_tags, hide\_lams) add.assoc diff\_diff\_left max\_0L

slice\_left\_def slice\_left\_slice\_right\_absorb slice\_right\_def

slice\_slice\_right\_absorb sliceable\_class.size\_slice

sliceable\_class.slice\_none sliceable\_class.slice\_slice)

corollary (in sliceable) slice\_left\_slice\_absorb:

" $(l \upharpoonright i) \upharpoonright a..b = l \upharpoonright (i+a).. $"$$

unfolding slice\_left\_def slice\_right\_def

by (metis (mono\_tags, lifting) slice\_left\_slice\_right\_absorb slice\_right\_def

```

slice_right_slice_left_absorb slice_slice_left_absorb
sliceable_class.slice_none)

corollary (in sliceable) slice_right_slice_absorb:
  "(l†..j)†a..b = l†a..(min j b)"
unfolding slice_left_def slice_right_def
by (simp add: sliceable_class.slice_slice)

lemmas (in sliceable) slice_slice_simps =
  slice_left_slice_left_absorb slice_left_slice_right_absorb
  slice_right_slice_left_absorb slice_right_slice_right_absorb slice_slice
  slice_slice_right_absorb slice_slice_left_absorb slice_left_slice_absorb
  slice_right_slice_absorb

lemmas (in sliceable) size_slice_defs =
  size_slice size_slice_left_absorb size_slice_right_absorb

lemma (in sliceable) slice_f_min_neutral:
  "(P (l†i..(min f k)) ∧ f ≤ k) ↔ (P (l†i..f) ∧ f ≤ k)"
by linarith

lemma (in sliceable) slice_i_min_neutral:
  "(P (l†(min i k)..f) ∧ i ≤ k) ↔ (P (l†i..f) ∧ i ≤ k)"
by linarith

lemma (in sliceable) slice_i_min_neutral_lt:
  "(P (l†(min k i)..f) ∧ i < k) ↔ (P (l†i..f) ∧ i < k)"
by linarith

lemma (in sliceable) slice_foral_i_min_neutral:
  "(∀ i f . P (l†(min i k)..f) ∧ i ≤ k) ↔ (∀ i f . P (l†i..f) ∧ i ≤ k)"
using not_less by auto

lemma (in sliceable) slice_f_max_neutral:
  "(P (l†i..(max f k)) ∧ f ≥ k) ↔ (P (l†i..f) ∧ f ≥ k)"
by (metis max.orderE)

lemma (in sliceable) slice_i_max_neutral:
  "(P (l†(max i k)..f) ∧ i ≥ k) ↔ (P (l†i..f) ∧ i ≥ k)"
by (metis max.orderE)

```

```
lemma (in sliceable) empty_slice[simp]: "i ≤ j ⇒ (#(l†j..i)) = 0"
using local.size_slice by auto
```

```
corollary (in sliceable) forall_disjoint_slice_suc:
  "∀ i j . (disjoint x ∧ i ≠ j ∧ i < (#x) ∧ j < (#x)) →
    (x†i..(Suc i) ≠ x†j..(Suc j))"
by (simp add: local.disjoint_slice_suc)
```

```
lemma (in sliceable) empty_slice_none:
  "(#x) = 0 ⇒ (#(x†i..j)) = 0"
by (simp add: size_slice)
```

```
corollary (in sliceable) empty_slice_right_none:
  "(#x) = 0 ⇒ (#(x†..j)) = 0"
by (simp add: slice_right_def sliceable_class.empty_slice_none)
```

```
corollary (in sliceable) empty_slice_left_none:
  "(#x) = 0 ⇒ (#(x†i..)) = 0"
by (simp add: slice_left_def sliceable_class.empty_slice_none)
```

## A.2 Sliceable distinct lists

The following is the instantiation of the sliceable class for the dlist type.

```
instantiation dlist :: (type) sliceable
begin
```

**definition**

```
"l†i..f = Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l)))"
```

**definition**

```
"size l = length (list_of_dlist l)"
```

**definition**

```
"empty_inter l k =
  ((set (list_of_dlist l)) ∩ (set (list_of_dlist k)) = {})"
```

**definition**

```
"disjoint l = distinct (list_of_dlist l)"
```

```
lemma list_of_dlist_slice :
```

```

"list_of_dlist (l†i..f) = take (max 0 (f-i)) (drop i (list_of_dlist l))"
unfolding slice_dlist_def
by simp

```

```

lemma Dlist_slice_inverse :

```

```

  "list_of_dlist (Dlist (take (max 0 (c-i)) (drop i (list_of_dlist x))))
  = (take (max 0 (c-i)) (drop i (list_of_dlist x)))"

```

```

by simp

```

```

lemma Dlist_empty_seq_inter: "c ≤ k ⇒

```

```

  (
    set (take c (list_of_dlist x)) ∩
    set (drop k (list_of_dlist x))
  ) = {}"

```

```

by (simp add: set_take_disj_set_drop_if_distinct)

```

```

lemma Dlist_forall_slice_eq1:

```

```

  "(∀ i f. (Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l1)))) =
  Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l2)))) ⇒
  l1 = l2"

```

```

by (metis (mono_tags, hide_lams) Dlist_list_of_dlist

```

```

  Sliceable_dlist.list_of_dlist_slice drop_0 drop_take max_OL take_equalityI)

```

```

lemma Dlist_forall_slice_eq:

```

```

  "l1 = l2 ⇔
  (∀ i f. (Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l1)))) =
  Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l2))))"

```

```

using Dlist_forall_slice_eq1 by blast

```

```

lemma distinct_list_take_1_uniqueness:

```

```

  "distinct l ⇒ i ≠ j ⇒ i < length l ⇒ j < length l ⇒
  take 1 (drop i l) ≠ take 1 (drop j l)"

```

```

by (simp add: hd_drop_conv_nth nth_eq_iff_index_eq take_Suc)

```

```

lemmas list_of_dlist_simps = slice_left_def slice_right_def slice_dlist_def
size_dlist_def disjoint_dlist_def empty_inter_dlist_def Dlist_slice_inverse

```

```

instance proof

```

```

  fix l::"'a dlist"

```

```

  show "l†0..(#l) = l" by (simp add: dlist_eqI list_of_dlist_slice size_dlist_def)

```

```

  fix l::"'a dlist"

```

```

show "disjoint l" by (simp add: disjoint_dlist_def)
next
fix l::"'a dlist" and c::nat and k
assume "c ≤ k"
thus "empty_inter (l†0..c) (l†k..(#l))"
by (simp add: size_dlist_def empty_inter_dlist_def
    set_take_disj_set_drop_if_distinct list_of_dlist_slice )
next
fix l::"'a dlist" and i and j and a and b
show "size (l†i..j) = max 0 (min j (#l) - i)"
proof (cases "j ≤ #l")
  case True
  assume "j ≤ #l"
  thus ?thesis
    by (metis (no_types, hide_lams) list_of_dlist_simps(7) size_dlist_def
        drop_take length_drop length_take list_of_dlist_simps(3) max_0L
        min.commute)
  next
  case False
  assume "¬ (j ≤ #l)"
  hence "j > #l" by simp
  thus ?thesis
    by (metis (no_types, lifting) list_of_dlist_simps(3)
        list_of_dlist_simps(7) size_dlist_def length_drop length_take max_0L
        min.commute min_diff)
qed
next
fix l::"'a dlist" and i and j and a and b
show "(l†i..j)†a..b = l†(i + a)..(min j (i + b))"
proof -
  have f1: "∀n. max (0::nat) n = n"
    by (meson max_0L)
  hence "take b (take (max 0 (j - i)) (drop i (list_of_dlist l))) = drop i (take
    (i + b) (take j (list_of_dlist l)))"
    by (metis (no_types) diff_add_inverse drop_take)
  hence "take (max 0 (b - a)) (drop a (list_of_dlist (l†i..j))) = drop a (drop
    i (take (min (i + b) j) (list_of_dlist l)))"
    using f1 by (metis Sliceable_dlist.list_of_dlist_slice drop_take take_take)
  thus ?thesis
    using f1 by (metis (no_types) add.commute drop_drop drop_take list_of_dlist_simps(3)
        min.commute)
qed

```

```

next
fix l::"'a dlist" and i and j
assume "disjoint l" "i≠j" "i < (#l)" "j < (#l)"
hence "take 1 (drop i (list_of_dlist l)) ≠
      take 1 (drop j (list_of_dlist l))"
      using distinct_list_take_1_uniqueness size_dlist_def by auto
hence "take (Suc i - i) (drop i (list_of_dlist l)) ≠
      take (Suc j - j) (drop j (list_of_dlist l))"
      by simp
hence "take (max 0 (Suc i - i)) (drop i (list_of_dlist l)) ≠
      take (max 0 (Suc j - j)) (drop j (list_of_dlist l))"
      by simp
thus "l↑i..Suc i ≠ l↑j..Suc j"
by (metis list_of_dlist_slice)
next
fix l1::"'a dlist" and l2::"'a dlist"
show "(#l1) = (#l2) ∧ (∀ i j. l1↑i..j = l2↑i..j) ↔ (l1 = l2)"
      using Dlist_forall_slice_eq
      by (metis Sliceable_dlist.list_of_dlist_slice)
qed

end

```

### A.2.1 Properties of sliceable distinct lists

In the following we present lemmas, corollaries and theorems about sliceable distinct lists.

**abbreviation** `dlist_nth` :: "'a dlist ⇒ nat ⇒ 'a"

**where**

`"dlist_nth l i ≡ (list_of_dlist (sliceable_nth l i))!0"`

**theorem** `set_slice` :

```

"set (list_of_dlist l) =
  set (list_of_dlist (l↑..i)) ∪ set (list_of_dlist (l↑i..))"
unfolding slice_dlist_def slice_right_def slice_left_def size_dlist_def
apply (simp add: list_of_dlist_inject)
by (metis append_take_drop_id set_append)

```

**theorem** `take_slice_right`: `"take n (list_of_dlist l) = list_of_dlist (l↑..n)"`

**unfolding** `slice_right_def slice_dlist_def`

**by** `(metis Dlist_slice_inverse drop_0 max_0L minus_nat.diff_0)`

```

theorem slice_right_cons: "distinct (x # xs)  $\implies$ 
  (Dlist (x # xs)) $\dagger$ .. $\text{Suc } n$ ) = Dlist (x # (list_of_dlist ((Dlist xs) $\dagger$ .. $n$ )))"
unfolding slice_right_def slice_dlist_def
by (simp add: distinct_remdups_id)

theorem slice_append:
  " $\forall n. \text{Dlist } ((\text{list\_of\_dlist } (l\dagger..n)) @ (\text{list\_of\_dlist } (l\dagger n..))) = l$ "
unfolding size_dlist_def slice_left_def slice_right_def
by (simp add: list_of_dlist_inverse list_of_dlist_slice)

theorem slice_append_mid:
  " $\forall i \ s \ e. s \leq i \wedge i \leq e \longrightarrow$ 
   $((\text{list\_of\_dlist } (l\dagger s..i)) @ (\text{list\_of\_dlist } (l\dagger i..e))) =$ 
   $\text{list\_of\_dlist } (l\dagger s..e)$ "
unfolding size_dlist_def slice_left_def slice_right_def list_of_dlist_slice
by (smt Nat.diff_add_assoc2 drop_drop le_add_diff_inverse
  le_add_diff_inverse2 max_0L take_add)

theorem slice_append_3:
  " $\forall i \ j. i \leq j \longrightarrow$ 
   $((\text{list\_of\_dlist } (l\dagger..i)) @$ 
   $(\text{list\_of\_dlist } (l\dagger i..j)) @ (\text{list\_of\_dlist } (l\dagger j..))) = \text{list\_of\_dlist } l$ "
unfolding size_dlist_def slice_left_def slice_right_def list_of_dlist_slice
by (metis append_assoc append_take_drop_id drop_0 le_add_diff_inverse
  length_drop max.cobounded2 max_0L minus_nat.diff_0 take_add take_all)

theorem distinct_slice_lte_inter_empty [simp]:
  " $i \leq j \implies \text{set } (\text{list\_of\_dlist } (l\dagger..i)) \cap \text{set } (\text{list\_of\_dlist } (l\dagger j..)) = \{\}$ "
unfolding size_dlist_def slice_left_def slice_right_def
by (simp add: Dlist_empty_seq_inter list_of_dlist_slice)

corollary distinct_slice_inter_empty [simp]:
  " $\text{set } (\text{list\_of\_dlist } (l\dagger..i)) \cap \text{set } (\text{list\_of\_dlist } (l\dagger i..)) = \{\}$ "
by simp

corollary distinct_slice_lt_inter_empty [simp]:
  " $i < j \implies \text{set } (\text{list\_of\_dlist } (l\dagger..i)) \cap \text{set } (\text{list\_of\_dlist } (l\dagger j..)) = \{\}$ "
by simp

corollary distinct_slice_diff1:
  " $\text{set } (\text{list\_of\_dlist } (l\dagger..i)) - \text{set } (\text{list\_of\_dlist } (l\dagger i..)) =$ "

```



```

    set (list_of_dlist (l↑..i))"
by (simp add: Diff_triv)

```

corollary distinct\_slice\_diff2:

```

    "set (list_of_dlist (l↑i..)) - set (list_of_dlist (l↑..i)) =
    set (list_of_dlist (l↑i..))"
using distinct_slice_diff1 by fastforce

```

theorem distinct\_in\_set\_slice1\_not\_in\_slice2:

```

    "i ≤ j ⇒
    x ∈ set (list_of_dlist (l↑..i)) ∧ x ∈ set (list_of_dlist (l↑j..)) ⇒
    False"
using distinct_slice_lte_inter_empty by fastforce

```

corollary distinct\_in\_set\_slice1\_implies\_not\_in\_slice2:

```

    "i ≤ j ⇒ x ∈ set (list_of_dlist (l↑..i)) ⇒
    x ∈ set (list_of_dlist (l↑j..)) ⇒ False"
by (meson distinct_in_set_slice1_not_in_slice2)

```

```

lemma exists_sublist_or_not_sublist [simp]: "∃ i. l↑..i ∈ T ∨ l↑i.. ∉ T"
unfolding slice_right_def slice_left_def
by auto

```

lemma forall\_slice\_left\_implies\_exists [simp]:

```

    "∀ i . l↑i.. ∈ S ⇒ ∃ i . l↑(Suc i).. ∈ S"
unfolding slice_right_def slice_left_def
by (simp add: slice_dlist_def)

```

lemma forall\_slice\_right\_implies\_exists [simp]:

```

    "∀ i . l↑..i ∈ S ⇒ ∃ i . l↑..(i-1) ∈ S"
unfolding slice_right_def slice_left_def
by auto

```

lemma take\_Suc\_Cons\_hd\_tl: "length l > 0 ⇒

```

    take (Suc n) l = hd l # (take n (tl l))"

```

```

apply (induct l)

```

```

by auto

```

corollary take\_Suc\_Cons\_hd\_tl\_singleton:

"length l > 0  $\implies$  take (Suc 0) l = [hd l]"

apply (induct l)

by auto

lemma take\_drop\_suc: "i < length l  $\implies$  length l > 0  $\implies$

take (max 0 ((Suc i) - i)) (drop i l) = [!i]"

by (metis (no\_types, lifting) Suc\_diff\_Suc Suc\_eq\_plus1\_left add\_commute

append\_eq\_append\_conv cancel\_comm\_monoid\_add\_class.diff\_cancel

hd\_drop\_conv\_nth lessI max\_0L numeral\_1\_eq\_Suc\_0 numeral\_One take\_add

take\_hd\_drop)

lemma slice\_right\_take: "l!i.. = Dlist (take i (list\_of\_dlist l))"

unfolding slice\_right\_def slice\_dlist\_def

by auto

lemma slice\_left\_drop: "l!i.. = Dlist (drop i (list\_of\_dlist l))"

unfolding slice\_left\_def slice\_dlist\_def size\_dlist\_def

by auto

lemma take\_one\_singleton\_hd: "l  $\neq$  []  $\implies$  take (Suc 0) l = [hd l]"

apply (induct l, simp)

by auto

lemma take\_one\_singleton\_nth: "l  $\neq$  []  $\implies$  take (Suc 0) l = [!0]"

apply (induct l, simp)

by auto

lemma take\_one\_drop\_n\_append\_singleton\_nth:

"ys  $\neq$  []  $\implies$  take 1 (drop (length xs) (xs @ ys)) =

[(xs @ ys)!(length xs)]"

by (induct xs, auto simp add: take\_one\_singleton\_nth)

lemma append\_length\_nth\_hd: "ys  $\neq$  []  $\implies$  [(xs @ ys)!(length xs)] = [hd ys]"

by (induct ys, auto)

lemma take\_one\_drop\_n\_singleton\_nth: "l  $\neq$  []  $\implies$  n < length l  $\implies$

take 1 (drop n l) = [!n]"

proof-

assume 0: "l  $\neq$  []"

assume 1: "n < length l"

obtain xs where "xs = take n l" by simp

```

obtain ys where "ys = drop n l" by simp
have "take 1 (drop n l) = take 1 (drop (length xs) (xs @ ys))" using 0 1
  by (simp add: 'ys = drop n l')
also have "... = [(xs @ ys)!(length xs)]" using 0 1
  by (metis 'ys = drop n l' drop_eq_Nil not_le
    take_one_drop_n_append_singleton_nth)
also have "... = [l!(length xs)]"
  by (simp add: 'xs = take n l' 'ys = drop n l')
finally show ?thesis using 0 1
  by (simp add: hd_drop_conv_nth take_one_singleton_hd)
qed

```

```

lemma slice_singleton: "(list_of_dlist l) ≠ [] ⇒ i < (#l) ⇒
  list_of_dlist (l↑i..(Suc i)) = [(list_of_dlist l)!i]"
by (metis list_of_dlist_slice length_greater_0_conv size_dlist_def
  take_drop_suc)

```

```

lemma slice_right_zero_eq_empty: "list_of_dlist (l↑..0) = []"
by (simp add: slice_right_def slice_dlist_def)

```

```

lemma slice_left_size_eq_empty: "list_of_dlist (l↑(#l)..) = []"
by (simp add: slice_left_def slice_dlist_def)

```

```

lemma slice_right_singleton_eq_element: "list_of_dlist l ≠ [] ⇒
  list_of_dlist (l↑..1) = [(list_of_dlist l)!0]"
by (metis One_nat_def take_one_singleton_nth take_slice_right)

```

```

lemma slice_left_singleton_eq_element: "list_of_dlist l ≠ [] ⇒
  list_of_dlist (l↑((#l)-1)..) = [(list_of_dlist l)!((#l)-1)]"
by (metis (no_types, lifting) Cons_nth_drop_Suc list_of_dlist_slice
  Suc_diff_Suc Suc_leI diff_Suc_eq_diff_pred diff_less drop_0 drop_all
  drop_take length_greater_0_conv max_0L minus_nat.diff_0 size_dlist_def
  slice_left_def slice_none zero_less_one)

```

```

lemma dlist_empty_slice[simp]: "i ≤ j ⇒ (l↑j..i) = Dlist []"
by (simp add: slice_dlist_def)

```

```

lemma dlist_append_extreme_left:
  "i ≤ j ⇒ list_of_dlist (l↑..j) =
    (list_of_dlist (l↑..i)) @ (list_of_dlist (l↑i..j))"
by (metis list_of_dlist_slice le_add_diff_inverse max_0L take_add
  take_slice_right)

```

```

lemma dlist_append_extreme_right:
  "i ≤ j ⇒ list_of_dlist (l†i..) =
    (list_of_dlist (l†i..j)) @ (list_of_dlist (l†j..))"
unfolding list_of_dlist_slice slice_left_def slice_right_def
by (metis append_take_drop_id drop_drop le_add_diff_inverse2 length_drop
    max.cobounded2 max_0L size_dlist_def take_all)

lemma dlist_disjoint[simp]: "disjoint (l::'a dlist)"
by (simp add: disjoint_dlist_def)

lemma dlist_member_suc_nth1:
  "x ∈ set (list_of_dlist(l†i..(Suc i))) ⇒ x = (list_of_dlist l)!i"
proof-
  assume 0: "x ∈ set (list_of_dlist (l†i..(Suc i)))"
  obtain rl where 1: "rl = list_of_dlist l" by blast
  hence "x ∈ set (take (max 0 (Suc i - i)) (drop i rl))"
    using 0 by (metis list_of_dlist_slice )
  hence "x ∈ set (take 1 (drop i rl))" by simp
  hence "x = rl!i"
    by (metis drop_Nil drop_all empty_iff list.inject list.set(1)
        list.set_cases not_less take_Nil take_one_drop_n_singleton_nth)
  thus ?thesis using 1 by simp
qed

lemma dlist_member_suc_nth2:
  "i < (#l) ⇒ x = (list_of_dlist l)!i ⇒
    x ∈ set (list_of_dlist (l†i..(Suc i)))"
unfolding size_dlist_def slice_dlist_def
by (metis Dlist_slice_inverse drop_Nil drop_eq_Nil leD length_greater_0_conv
    list.set_intros(1) take_drop_suc)

lemma dlist_member_suc_nth: "i < (#l) ⇒
  (x = (list_of_dlist l)!i) ↔ (x ∈ set (list_of_dlist (l†i..(Suc i))))"
using dlist_member_suc_nth1 dlist_member_suc_nth2
by fastforce

corollary not_dlist_member_empty[simp]:
  "¬ Dlist.member (Dlist.empty) v"
  "¬ (Dlist.member (Dlist []) v)"
by (simp add: Dlist.member_def Dlist.empty_def List.member_def)+

```

```
lemma dlist_empty_slice_none: "(Dlist.empty↑i..j) = Dlist.empty"
by (simp add: Dlist.empty_def slice_dlist_def)
```

```
corollary dlist_empty_slice_right_none: "(Dlist.empty↑..j) = Dlist.empty"
by (simp add: dlist_empty_slice_none slice_right_def)
```

```
corollary dlist_empty_slice_left_none: "(Dlist.empty↑i..) = Dlist.empty"
by (simp add: dlist_empty_slice_none slice_left_def)
```

```
lemma dlist_member_slice_empty_none:
  "¬ (Dlist.member (Dlist.empty↑i..j) v)"
by (auto simp add: slice_dlist_def)
```

```
corollary dlist_member_slice_right_empty_none[simp]:
  "¬ (Dlist.member (Dlist.empty↑..j) v)"
by (simp add: slice_right_def dlist_empty_slice_none)
```

```
corollary dlist_member_slice_left_empty_none[simp]:
  "¬ (Dlist.member (Dlist.empty↑i..) v)"
by (simp add: slice_left_def dlist_empty_slice_none)
```

```
lemma dlist_member_slice_member_dlist:
  "∃ i j. Dlist.member (dl↑i..j) v ⇒ Dlist.member dl v"
unfolding Dlist.member_def List.member_def slice_dlist_def
using in_set_dropD in_set_taked by fastforce
```

```
corollary dlist_member_slice_right_member_dlist:
  "∃ j. Dlist.member (dl↑..j) v ⇒ Dlist.member dl v"
by (metis dlist_member_slice_member_dlist slice_right_def)
```

```
corollary dlist_member_slice_left_member_dlist:
  "∃ i. Dlist.member (dl↑i..) v ⇒ Dlist.member dl v"
by (metis dlist_member_slice_member_dlist slice_left_def)
```

```
lemma sliceable_nth_member1:
  "sliceable_nth dl i = Dlist [v] ⇒ Dlist.member dl v"
by (metis Dlist.member_def distinct_remdups_id distinct_singleton
  dlist_member_slice_member_dlist in_set_member list.set_intros(1) list_of_dlist_Dlist)
```

```
corollary sliceable_nth_member:
  "∃ i. sliceable_nth dl i = Dlist [v] ⇒ Dlist.member dl v"
by (auto simp add: sliceable_nth_member1)
```

```

lemma sliceable_nth_member_iff:
  "(\exists i. sliceable_nth dl i = Dlist [v]) \longleftrightarrow Dlist.member dl v"
apply (rule iffI, simp add: sliceable_nth_member)
by (metis Dlist.member_def empty_iff empty_set in_set_conv_nth in_set_member
  list_of_dlist_slice size_dlist_def slice_dlist_def slice_singleton)

```

### A.3 Algebra of Temporal Faults

In the following we present the algebraic laws for the [ATF](#).

#### A.3.1 Basic [ATF](#) operators and `tempo1`

```

class temporal_faults_algebra_basic = boolean_algebra +
  fixes neutral :: "'a"
  fixes xbefore :: "'a \Rightarrow 'a \Rightarrow 'a"
  fixes tempo1 :: "'a \Rightarrow bool"
  assumes xbefore_bot_1: "xbefore bot a = bot"
  assumes xbefore_bot_2: "xbefore a bot = bot"
  assumes xbefore_neutral_1: "tempo1 a \Longrightarrow xbefore neutral a = a"
  assumes xbefore_neutral_2: "tempo1 a \Longrightarrow xbefore a neutral = a"
  assumes xbefore_not_idempotent: "tempo1 a \Longrightarrow xbefore a a = bot"
  assumes inf_tempo1: "[tempo1 a; tempo1 b] \Longrightarrow tempo1 (inf a b)"
  assumes xbefore_not_sym:
    "[tempo1 a; tempo1 b] \Longrightarrow (xbefore a b) \leq -(xbefore b a)"

```

#### A.3.2 Definition of associativity of [XBefore](#)

```

class temporal_faults_algebra_assoc = temporal_faults_algebra_basic +
  assumes xbefore_assoc: "[tempo1 a; tempo1 b; tempo1 c] \Longrightarrow
    xbefore (xbefore a b) c = xbefore a (xbefore b c)"

```

#### A.3.3 Equivalences in the [ATF](#) and properties

```

class temporal_faults_algebra_equivs = temporal_faults_algebra_assoc +
  fixes independent_events :: "'a \Rightarrow 'a \Rightarrow bool"
  fixes tempo2 :: "'a \Rightarrow bool"
  fixes tempo3 :: "'a \Rightarrow bool"
  fixes tempo4 :: "'a \Rightarrow bool"

```

```

assumes xbefore_inf_equiv_bot:
  "[tempo1 a; tempo1 b]  $\implies$  inf (xbefore a b) (xbefore b a) = bot"
assumes xbefore_sup_equiv_inf:
  "independent_events a b  $\implies$  [tempo1 a; tempo1 b]  $\implies$ 
    [tempo2 a; tempo2 b]  $\implies$  [tempo3 a; tempo3 b]  $\implies$  [tempo4 a; tempo4 b]  $\implies$ 
    sup (xbefore a b) (xbefore b a) = inf a b"
assumes sup_tempo2: "[tempo2 a; tempo2 b]  $\implies$  tempo2 (sup a b)"
assumes inf_tempo3: "[tempo3 a; tempo3 b]  $\implies$  tempo3 (inf a b)"
assumes sup_tempo4: "[tempo4 a; tempo4 b]  $\implies$  tempo4 (sup a b)"

```

#### A.3.4 XBefore transitivity

```

class temporal_faults_algebra_trans = temporal_faults_algebra_equivs +
  assumes xbefore_trans:
    "[tempo1 a; tempo1 b; tempo1 c]  $\implies$  [tempo2 a; tempo2 b; tempo2 c]  $\implies$ 
      less_eq (inf (xbefore a b) (xbefore b c)) (xbefore a c)"
  assumes inf_xbefore_trans: "[tempo1 b; tempo3 b]  $\implies$ 
    inf (xbefore a b) (xbefore b c) = xbefore (xbefore a b) c"

```

#### A.3.5 Mixed operators in ATF

```

class temporal_faults_algebra_mixed_ops = temporal_faults_algebra_trans +
  assumes xbefore_sup_1:
    "xbefore (sup a b) c = sup (xbefore a c) (xbefore b c)"
  assumes xbefore_sup_2:
    "xbefore a (sup b c) = sup (xbefore a b) (xbefore a c)"
  assumes not_xbefore: "
    independent_events a b  $\implies$ 
    [tempo1 a; tempo1 b]  $\implies$ 
    [tempo2 a; tempo2 b]  $\implies$ 
    [tempo3 a; tempo3 b]  $\implies$ 
    [tempo4 a; tempo4 b]  $\implies$ 
    - (xbefore a b) = sup (sup (- a) (- b)) (xbefore b a)"
  assumes inf_xbefore_equiv_sups_xbefore: "tempo2 a  $\implies$ 
    inf a (xbefore b c) = sup (xbefore (inf a b) c) (xbefore b (inf a c))"
  assumes not_1_xbefore_equiv: "[tempo1 a; tempo2 b]  $\implies$  xbefore (- a) b = b"
  assumes not_2_xbefore_equiv: "[tempo1 b; tempo2 a]  $\implies$  xbefore a (- b) = a"

class temporal_faults_algebra = temporal_faults_algebra_mixed_ops

```

### A.3.6 Theorems in the context of ATF

The following theorems are valid for ATF. They are valid for any instantiation of the ATF class as, for example, for the sets of distinct lists type.

**context** *temporal\_faults\_algebra*  
**begin**

The following theorem proves Eq. (4.27c).

**theorem** *xbefore\_inf\_1*:

"independent\_events a b  $\Rightarrow$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \Rightarrow \llbracket$ tempo3 a; tempo3 b $\rrbracket \Rightarrow \llbracket$ tempo4 a; tempo4 b $\rrbracket \Rightarrow$   
 xbefore (inf a b) c =  
 sup (xbefore (xbefore a b) c) (xbefore (xbefore b a) c)"

**proof-**

assume "independent\_events a b" "tempo1 a" "tempo1 b"  
 "tempo2 a" "tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"  
 hence "xbefore (inf a b) c = xbefore (sup (xbefore a b) (xbefore b a)) c"  
 by (simp add: xbefore\_sup\_equiv\_inf)  
 thus ?thesis by (simp add: xbefore\_sup\_1)

**qed**

The following theorem proves Eq. (4.27d).

**theorem** *xbefore\_inf\_2*:

"independent\_events b c  $\Rightarrow$   $\llbracket$ tempo1 b; tempo1 c $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 b; tempo2 c $\rrbracket \Rightarrow \llbracket$ tempo3 b; tempo3 c $\rrbracket \Rightarrow \llbracket$ tempo4 b; tempo4 c $\rrbracket \Rightarrow$   
 xbefore a (inf b c) =  
 sup (xbefore a (xbefore b c)) (xbefore a (xbefore c b))"

**proof-**

assume "independent\_events b c" "tempo1 b" "tempo1 c" "tempo2 b" "tempo2 c"  
 "tempo3 b" "tempo3 c" "tempo4 b" "tempo4 c"  
 hence "xbefore a (inf b c) = xbefore a (sup (xbefore b c) (xbefore c b))"  
 by (simp add: xbefore\_sup\_equiv\_inf)  
 thus ?thesis by (simp add: xbefore\_sup\_2)

**qed**

The following lemma proves Eq. (4.24f).

**lemma** *xbefore\_sup\_absorb\_1b*:

"independent\_events a b  $\Rightarrow$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \Rightarrow \llbracket$ tempo3 a; tempo3 b $\rrbracket \Rightarrow \llbracket$ tempo4 a; tempo4 b $\rrbracket \Rightarrow$   
 sup (xbefore b a) a = a"

by (metis inf\_le1 order\_trans sup.absorb2 sup.cobounded2  
 xbefore\_sup\_equiv\_inf)



**lemma** *xbefore\_sup\_absorb\_2*:

"independent\_events a b  $\Rightarrow$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \Rightarrow \llbracket$ tempo3 a; tempo3 b $\rrbracket \Rightarrow \llbracket$ tempo4 a; tempo4 b $\rrbracket \Rightarrow$   
 sup a (xbefore a b) = a"

by (metis dual\_order.trans inf.cobounded1 sup.absorb1 sup.cobounded1  
 xbefore\_sup\_equiv\_inf)

The following corollary proves Eq. (4.24e).

**corollary** *xbefore\_sup\_absorb\_1*:

"independent\_events a b  $\Rightarrow$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \Rightarrow \llbracket$ tempo3 a; tempo3 b $\rrbracket \Rightarrow \llbracket$ tempo4 a; tempo4 b $\rrbracket \Rightarrow$   
 sup (xbefore a b) a = a"

**proof-**

assume 0: "independent\_events a b" "tempo1 a" "tempo1 b" "tempo2 a"

"tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"

hence "sup a (xbefore a b) = sup (xbefore a b) a"

by (simp add: sup commute)

thus ?thesis using 0 by (simp add: xbefore\_sup\_absorb\_2)

qed

**corollary** *xbefore\_sup\_absorb\_2b*:

"independent\_events a b  $\Rightarrow$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \Rightarrow$   
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \Rightarrow \llbracket$ tempo3 a; tempo3 b $\rrbracket \Rightarrow \llbracket$ tempo4 a; tempo4 b $\rrbracket \Rightarrow$   
 sup a (xbefore b a) = a"

**proof-**

assume 0: "independent\_events a b" "tempo1 a" "tempo1 b" "tempo2 a"

"tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"

hence "sup a (xbefore b a) = sup (xbefore b a) a"

by (simp add: sup commute)

thus ?thesis using 0 by (simp add: xbefore\_sup\_absorb\_1b)

qed

The following corollary proves Eq. (4.27g).

**theorem** *xbefore\_inf\_absorb\_1*: "independent\_events a b  $\Rightarrow$

$\llbracket$  tempo1 a; tempo1 b  $\rrbracket \Rightarrow$

$\llbracket$  tempo2 a; tempo2 b  $\rrbracket \Rightarrow$

$\llbracket$  tempo3 a; tempo3 b  $\rrbracket \Rightarrow$

$\llbracket$  tempo4 a; tempo4 b  $\rrbracket \Rightarrow$

inf a (xbefore a b) = xbefore a b"

by (simp add: local.inf\_absorb2 local.le\_iff\_sup xbefore\_sup\_absorb\_1)

The following corollary proves Eq. (4.27h).

```

theorem xbefore_inf_absorb_2: "independent_events a b  $\Rightarrow$ 
  [ tempo1 a; tempo1 b ]  $\Rightarrow$ 
  [ tempo2 a; tempo2 b ]  $\Rightarrow$ 
  [ tempo3 a; tempo3 b ]  $\Rightarrow$ 
  [ tempo4 a; tempo4 b ]  $\Rightarrow$ 
  inf a (xbefore b a) = xbefore b a"
by (simp add: local.inf.absorb2 local.sup.absorb_iff1 xbefore_sup_absorb_2b)

```

The following lemma proves Eq. (4.28).

```

lemma inf_xbefore_equiv_sup_xbefore_expanded:
  "independent_events a b  $\Rightarrow$  independent_events a c  $\Rightarrow$ 
  [ tempo1 a; tempo1 b; tempo1 c ]  $\Rightarrow$  [ tempo2 a; tempo2 b; tempo2 c ]  $\Rightarrow$ 
  [ tempo3 a; tempo3 b; tempo3 c ]  $\Rightarrow$  [ tempo4 a; tempo4 b; tempo4 c ]  $\Rightarrow$ 
  inf a (xbefore b c) =
  sup (sup (xbefore (xbefore a b) c)
    (xbefore (xbefore b a) c))
    (xbefore (xbefore b c) a)"

```

**proof-**

```

assume "independent_events a b" "independent_events a c"
  "tempo1 a" "tempo1 b" "tempo1 c"
  "tempo2 a" "tempo2 b" "tempo2 c"
  "tempo3 a" "tempo3 b" "tempo3 c"
  "tempo4 a" "tempo4 b" "tempo4 c"
hence "inf a (xbefore b c) =
  sup (xbefore (inf a b) c) (xbefore b (inf a c))"
  "xbefore (inf a b) c =
  sup (xbefore (xbefore a b) c) (xbefore (xbefore b a) c)"
  "xbefore b (inf a c) =
  sup (xbefore (xbefore b a) c) (xbefore (xbefore b c) a)"
by (auto simp add: inf_xbefore_equiv_sup_xbefore xbefore_inf_1
  xbefore_inf_2 xbefore_assoc)
thus ?thesis by (simp add: sup.assoc)

```

**qed**

**corollary** inf\_xbefore\_equiv\_sup\_xbefore\_1:

```

  "independent_events a b  $\Rightarrow$  independent_events a c  $\Rightarrow$ 
  [ tempo1 a; tempo1 b; tempo1 c ]  $\Rightarrow$  [ tempo2 a; tempo2 b; tempo2 c ]  $\Rightarrow$ 
  [ tempo3 a; tempo3 b; tempo3 c ]  $\Rightarrow$  [ tempo4 a; tempo4 b; tempo4 c ]  $\Rightarrow$ 
  inf a (xbefore b c) =
  sup (sup (xbefore (xbefore a b) c) (xbefore (xbefore b a) c))
    (xbefore (xbefore b c) a)"

```

by (simp add: inf\_xbefore\_equiv\_sup\_xbefore\_expanded)

The following lemma proves Eq. (4.29d).

lemma xbefore\_sup\_compl\_inf\_absorb1:

```

"independent_events a b  $\implies$   $\llbracket$ tempo1 a; tempo1 b $\rrbracket \implies$ 
 $\llbracket$ tempo2 a; tempo2 b $\rrbracket \implies \llbracket$ tempo3 a; tempo3 b $\rrbracket \implies \llbracket$ tempo4 a; tempo4 b $\rrbracket \implies$ 
sup (inf a (-b)) (xbefore a b) = inf a (- (xbefore b a))"
proof -
  assume a1: "independent_events a b"
  assume a2: "tempo1 a"
  assume a3: "tempo1 b"
  assume a4: "tempo2 a"
  assume a5: "tempo2 b"
  assume a6: "tempo3 a"
  assume a7: "tempo3 b"
  assume a8: "tempo4 a"
  assume a9: "tempo4 b"
  then have f10: "- xbefore a b = sup (sup (- a) (- b)) (xbefore b a)"
    using a8 a7 a6 a5 a4 a3 a2 a1 by (meson local.not_xbefore)
  have f11: " $\forall$  a aa ab. inf (a::'a) (sup aa ab) = sup (inf a aa) (inf a ab)"
    using local.distrib_imp2 local.sup_inf_distrib1 by force
  then have f12: "sup bot (xbefore b a) = inf b (sup (- b) (xbefore b a))"
    using a7 a3 by (metis local.inf_compl_bot local.inf_xbefore_trans local.xbefore_neutr)
  have f13: "inf (sup (- a) (- (- b))) (sup (- a) (- b)) = sup (- a) (inf b (- b))"
    using local.double_compl local.sup_inf_distrib1 by presburger
  have f14: "inf a (xbefore b a) = xbefore b a"
    using a9 a8 a7 a6 a5 a4 a3 a2 a1 by (meson xbefore_inf_absorb_2)
  have f15: "sup (sup (inf (sup (- a) (- b)) (- a)) (inf (xbefore b a) (- a))) bot
= sup (- a) (inf b (- b))"
    using f10 a9 a8 a7 a6 a5 a4 a3 a2 a1 by (metis (no_types) local.compl_sup local.inf_compl_inf
local.inf_sup_distrib2 xbefore_sup_absorb_1)
  have "inf a (- a) = sup (inf (xbefore b a) (- a)) bot"
    using f14 f11 by (metis (no_types) local.compl_inf local.inf_compl_bot)
  then have "sup (- sup (- a) (- (- b))) (- sup (- a) (- b)) = - inf (sup (sup (- a) (- b)) a) (- a)"
    using f15 f13 by (metis (full_types) local.compl_inf local.inf_sup_distrib2
local.sup_assoc)
  then have "sup (- sup (- a) (- (- b))) (- sup (- a) (- b)) = a"
    by (simp add: local.sup_assoc)
  then show ?thesis
    using f13 f12 f10 by (metis (no_types) local.compl_inf local.compl_sup local.double_compl_inf
local.inf_compl_bot local.sup_assoc local.sup_inf_distrib1)

```

qed

end

end

## A.4 Denotational semantics for ATF

In the following we present the denotation semantics for ATF in terms of sets of distinct lists.

### A.4.1 Formula: distinct lists

The definition of a formula in the ATF is a set of sets of distinct lists (dlist).

```
typedef 'a formula = "UNIV::'a dlist set set" by simp
```

#### A.4.1.1 Formula as Boolean algebra

In the following we instantiate the formula as a Boolean algebra and prove that Boolean operators are valid.

```
instantiation formula :: (type) boolean_algebra
begin
```

definition

```
"x  $\sqcap$  y = Abs_formula (Rep_formula x  $\cap$  Rep_formula y)"
```

definition

```
"x  $\sqcup$  y = Abs_formula (Rep_formula x  $\cup$  Rep_formula y)"
```

definition

```
" $\top$  = Abs_formula UNIV"
```

definition

```
" $\bot$  = Abs_formula {}"
```

definition

```
"x  $\leq$  y  $\longleftrightarrow$  Rep_formula x  $\subseteq$  Rep_formula y"
```

definition

" $x < y \iff \text{Rep\_formula } x \subset \text{Rep\_formula } y$ "

**definition**

" $\neg x = \text{Abs\_formula } (\neg (\text{Rep\_formula } x))$ "

**definition**

" $x - y = \text{Abs\_formula } (\text{Rep\_formula } x - \text{Rep\_formula } y)$ "

**lemma** Rep\_formula\_inf:

" $\text{Rep\_formula } (x \sqcap y) = \text{Rep\_formula } x \cap \text{Rep\_formula } y$ "

**unfolding** inf\_formula\_def

**by** (simp add: Abs\_formula\_inverse Rep\_formula)

**lemma** Rep\_formula\_sup:

" $\text{Rep\_formula } (x \sqcup y) = \text{Rep\_formula } x \cup \text{Rep\_formula } y$ "

**unfolding** sup\_formula\_def

**by** (simp add: Abs\_formula\_inverse Rep\_formula)

**lemma** Rep\_formula\_top[simp]: " $\text{Rep\_formula } \top = \text{UNIV}$ "

**unfolding** top\_formula\_def

**by** (simp add: Abs\_formula\_inverse)

**lemma** Rep\_formula\_bot[simp]: " $\text{Rep\_formula } \perp = \{\}$ "

**unfolding** bot\_formula\_def

**by** (simp add: Abs\_formula\_inverse)

**lemma** Rep\_formula\_compl: " $\text{Rep\_formula } (\neg x) = \neg \text{Rep\_formula } x$ "

**unfolding** uminus\_formula\_def

**by** (simp add: Abs\_formula\_inverse Rep\_formula)

**lemma** Rep\_formula\_diff:

" $\text{Rep\_formula } (x - y) = \text{Rep\_formula } x - \text{Rep\_formula } y$ "

**unfolding** minus\_formula\_def

**by** (simp add: Abs\_formula\_inverse Rep\_formula)

**lemmas** eq\_formula\_iff = Rep\_formula\_inject [symmetric]

**lemmas** Rep\_formula\_boolean\_algebra\_simps =

less\_eq\_formula\_def less\_formula\_def eq\_formula\_iff

Rep\_formula\_sup Rep\_formula\_inf Rep\_formula\_top Rep\_formula\_bot

Rep\_formula\_compl Rep\_formula\_diff

**instance proof**

**qed** (unfold Rep\_formula\_boolean\_algebra\_simps, auto)

The instantiation and this proof shows that **ATF** is a Boolean algebra as shown in Eqs. (4.11a) to (4.11g).

**end**

#### A.4.1.2 Tempo properties

In this section we define the tempo properties.

Tempo1: disjoint split

**definition** dlist\_tempo1 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo1 S  $\equiv \forall i j l. i \leq j \longrightarrow \neg ((S (l \upharpoonright .. i) \wedge S (l \upharpoonright j ..))"$

Tempo2: belonging iff

**definition** dlist\_tempo2 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo2 S  $\equiv \forall i l. S l \longleftrightarrow (S (l \upharpoonright .. i) \vee S (l \upharpoonright i ..))"$

**definition** dlist\_tempo3 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo3 S  $\equiv \forall i j l. j < i \longrightarrow (S (l \upharpoonright j .. i) \longleftrightarrow (S (l \upharpoonright .. i) \wedge S (l \upharpoonright j ..)))"$

**definition** dlist\_tempo4 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo4 S  $\equiv \forall l. S l \longleftrightarrow (\exists i. S (l \upharpoonright i .. (Suc i)))"$

**definition** dlist\_tempo5 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo5 S  $\equiv$   
 $\forall i j l. (i \neq j \wedge i < (\#l) \wedge j < (\#l)) \longrightarrow$   
 $\neg (S (l \upharpoonright i .. (Suc i)) \wedge S (l \upharpoonright j .. (Suc j)))"$

**definition** dlist\_tempo6 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"  
**where**  
 "dlist\_tempo6 S  $\equiv \forall l. (\forall i j. \neg S (l \upharpoonright i .. j)) \longleftrightarrow \neg S l"$

**definition** dlist\_tempo7 :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"

where

```
"dlist_tempo7 S  $\equiv \forall l. (\exists i j. i < j \wedge S (l \upharpoonright i..j)) \longleftrightarrow S l$ "
```

**definition** `dlist_tempo` :: "(`'a` `dlist`  $\Rightarrow$  `bool`)  $\Rightarrow$  `bool`"

where

```
"dlist_tempo S  $\equiv$  dlist_tempo1 S  $\wedge$  dlist_tempo2 S  $\wedge$ 
  dlist_tempo3 S  $\wedge$  dlist_tempo5 S  $\wedge$  dlist_tempo4 S  $\wedge$  dlist_tempo6 S  $\wedge$ 
  dlist_tempo7 S"
```

```
lemmas tempo_defs = dlist_tempo_def dlist_tempo1_def dlist_tempo2_def
  dlist_tempo3_def dlist_tempo5_def dlist_tempo4_def dlist_tempo6_def
  dlist_tempo7_def
```

**lemma** `dlist_tempo_1_no_gap`:

```
"dlist_tempo1 S  $\Rightarrow \forall i l. \neg ((S (l \upharpoonright ..i) \wedge S (l \upharpoonright i..))"$ 
```

**unfolding** `dlist_tempo1_def`

**by** `auto`

**corollary** `dlist_tempo_1_no_gap_append`:

```
"dlist_tempo1 S  $\Rightarrow$ 
   $\forall zs xs ys. \text{list\_of\_dlist } zs = \text{list\_of\_dlist } xs @ \text{list\_of\_dlist } ys \longrightarrow$ 
   $\neg ((S xs \wedge S ys))"$ 
```

**using** `dlist_tempo_1_no_gap`

**by** (`metis` `Dlist_list_of_dlist` `append_eq_conv_conj` `slice_left_drop` `take_slice_right`)

#### A.4.1.3 Tempo properties for list member

We use the naming convention of variable, but in fact, a variable is equivalent to a list membership: `var a = {xs | a  $\in$  list_of_dlist xs}`.

**lemma** `dlist_tempo1_member`: "`dlist_tempo1` ( $\lambda xs. \text{Dlist.member } xs \ a$ )"

**unfolding** `dlist_tempo1_def` `Dlist.member_def` `List.member_def`

**by** (`meson` `distinct_in_set_slice1_not_in_slice2`)

**lemma** `dlist_tempo2_member`: "`dlist_tempo2` ( $\lambda xs. \text{Dlist.member } xs \ a$ )"

**unfolding** `dlist_tempo2_def` `Dlist.member_def` `List.member_def`

**by** (`metis` (`no_types`, `lifting`) `Un_iff` `set_slice` )

**lemma** `dlist_tempo3_member`: "`dlist_tempo3` ( $\lambda xs. \text{Dlist.member } xs \ a$ )"

**unfolding** `dlist_tempo3_def` `Dlist.member_def` `List.member_def`

**by** (`metis` `DiffD2` `Un_iff` `distinct_slice_diff2` `dlist_append_extreme_left` `dlist_append_extreme_right` `less_imp_le_nat` `set_append`)

```

lemma dlist_tempo5_member: "dlist_tempo5 ( $\lambda$ xs. Dlist.member xs a)"
unfolding dlist_tempo5_def Dlist.member_def List.member_def
by (metis Dlist_list_of_dlist Suc_leI disjoint_dlist_def disjoint_slice_suc
    distinct_list_of_dlist dlist_empty_slice dlist_member_suc_nth1 empty_slice
    less_Suc_eq_0_disj not_less_eq slice_singleton)

lemma dlist_tempo4_member: "dlist_tempo4 ( $\lambda$ xs. Dlist.member xs a)"
unfolding dlist_tempo4_def Dlist.member_def List.member_def

by (metis dlist_member_suc_nth in_set_conv_nth in_set_dropD in_set_takeD
    list_of_dlist_Dlist set_remdups size_dlist_def slice_dlist_def)

lemma dlist_tempo6_member: "dlist_tempo6 ( $\lambda$ xs. Dlist.member xs a)"
unfolding dlist_tempo6_def Dlist.member_def List.member_def
by (metis append_Nil in_set_conv_decomp in_set_conv_nth in_set_dropD
    in_set_takeD length_pos_if_in_set list_of_dlist_slice take_drop_suc)

lemma dlist_tempo7_member: "dlist_tempo7 ( $\lambda$ xs. Dlist.member xs a)"
unfolding dlist_tempo7_def Dlist.member_def List.member_def
by (metis Un_iff dlist_append_extreme_left dlist_member_suc_nth2
    in_set_conv_nth lessI less_imp_le_nat set_append set_slice size_dlist_def)

theorem dlist_tempo_member: "dlist_tempo ( $\lambda$ xs. Dlist.member xs a)"
unfolding dlist_tempo_def
by (simp add: dlist_tempo1_member dlist_tempo2_member dlist_tempo3_member
    dlist_tempo5_member dlist_tempo4_member dlist_tempo6_member
    dlist_tempo7_member)

```

#### A.4.1.4 Tempo properties for other operators

```

lemma dlist_tempo1_inf: "[dlist_tempo1 a; dlist_tempo1 b]  $\Rightarrow$ 
    dlist_tempo1 ( $\lambda$ zs. a zs  $\wedge$  b zs)"
unfolding dlist_tempo1_def
by simp

lemma dlist_tempo3_inf: "[dlist_tempo3 a; dlist_tempo3 b]  $\Rightarrow$ 
    dlist_tempo3 ( $\lambda$ zs. a zs  $\wedge$  b zs)"
unfolding dlist_tempo3_def
by auto

lemma dlist_tempo2_sup: "[dlist_tempo2 a; dlist_tempo2 b]  $\Rightarrow$ 

```



```

dlist_tempo2 ( $\lambda zs. a\ zs \vee b\ zs$ )"
unfolding dlist_tempo2_def
by auto

```

```

lemma dlist_tempo4_sup: "[[dlist_tempo4 a; dlist_tempo4 b]]  $\Rightarrow$ 
  dlist_tempo4 ( $\lambda zs. a\ zs \vee b\ zs$ )"
unfolding dlist_tempo4_def
by blast

```

#### A.4.2 XBefore of distinct lists

```

definition dlist_xbefore :: "('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  ('a dlist  $\Rightarrow$  bool)  $\Rightarrow$ 
  'a dlist  $\Rightarrow$  bool"
where
  "dlist_xbefore a b xs  $\equiv \exists i. a\ (xs\!^{\dagger}\!..i) \wedge b\ (xs\!^{\dagger}\!i..)"$ 
```

##### A.4.2.1 XBefore and temporal properties

```

lemma dlist_tempo1_xbefore: "[[dlist_tempo1 a; dlist_tempo1 b]]  $\Rightarrow$ 
  dlist_tempo1 (dlist_xbefore a b)"
unfolding dlist_tempo1_def dlist_xbefore_def slice_slice_simps
by (smt le_add1 min.absorb2 min.cobounded1 slice_right_slice_left_absorb
  slice_right_slice_right_absorb)

```

##### A.4.2.2 XBefore and appending

```

lemma Rep_slice_append:
  "list_of_dlist zs = (list_of_dlist (zs $^{\dagger}$ ..i)) @ (list_of_dlist (zs $^{\dagger}$ i..))"
by (metis distinct_append distinct_list_of_dlist distinct_slice_inter_empty
  list_of_dlist_Dlist remdups_id_iff_distinct slice_append)

lemma dlist_xbefore_append:
  "dlist_xbefore a b zs  $\longleftrightarrow$ 
  ( $\exists xs\ ys. \text{set } (list\_of\_dlist\ xs) \cap \text{set } (list\_of\_dlist\ ys) = \{\}$ 
 $\wedge a\ xs \wedge b\ ys \wedge$ 
  list_of_dlist zs = ((list_of_dlist xs) @ (list_of_dlist ys)))"
unfolding dlist_xbefore_def
by (metis Rep_slice_append append_Nil2 append_eq_conv_conj
  distinct_slice_inter_empty dlist_xbefore_def drop_take max_0L
  size_dlist_def slice_append slice_dlist_def slice_left_def slice_right_def
  take_slice_right)

```

A.4.2.3 **XBefore**, bot, top and idempotency

```
lemma dlist_xbefore_bot_1: "dlist_xbefore ( $\lambda$ xs. False) b zs = False"
unfolding dlist_xbefore_def
by simp
```

```
corollary dlistset_xbefore_bot_1:
  "Collect (dlist_xbefore ( $\lambda$ xs. False) b) = {}"
by (simp add: dlist_xbefore_bot_1)
```

```
lemma dlist_xbefore_bot_2: "dlist_xbefore a ( $\lambda$ xs. False) zs = False"
unfolding dlist_xbefore_def
by simp
```

```
lemma dlistset_xbefore_bot_2:
  "Collect (dlist_xbefore a ( $\lambda$ xs. False)) = {}"
by (simp add: dlist_xbefore_bot_2)
```

```
lemma dlist_xbefore_idem:
  "dlist_tempo1 a  $\implies$  dlist_xbefore a a zs = False"
unfolding dlist_xbefore_def dlist_tempo1_def
by blast
```

```
lemma dlistset_xbefore_idem:
  "dlist_tempo1 a  $\implies$  Collect (dlist_xbefore a a) = {}"
by (simp add: dlist_xbefore_idem)
```

```
lemma dlist_xbefore_implies_idem:
  " $\forall$ xs. b xs  $\rightarrow$  a xs  $\implies$  dlist_tempo1 a  $\implies$  dlist_xbefore a b zs = False"
unfolding dlist_tempo1_def dlist_xbefore_def
by blast
```

A.4.2.4 **XBefore** neutral

```
lemma dlist_xbefore_neutral_1:
  "dlist_tempo1 a  $\implies$  dlist_xbefore ( $\lambda$ xs. xs = dlist_of_list []) a zs = a zs"
by (metis (full_types) Dlist_list_of_dlist Rep_slice_append append.simps(1)
  dlist_of_list dlist_xbefore_def take_0 take_slice_right)
```

```
corollary dlistset_xbefore_neutral_1:
  "dlist_tempo1 a  $\implies$  Collect (dlist_xbefore ( $\lambda$ xs. xs = Dlist []) a) = Collect a"
```

using dlist\_xbefore\_neutral\_1 by auto

lemma dlist\_xbefore\_neutral\_2:

"dlist\_tempo1 a  $\implies$  dlist\_xbefore a ( $\lambda xs. xs = \text{Dlist } []$ ) zs = a zs"  
by (smt Dlist\_list\_of\_dlist append\_Nil2 distinct\_append distinct\_list\_of\_dlist dlist\_of\_l

dlist\_xbefore\_append list\_of\_dlist\_empty)

corollary dlistset\_xbefore\_neutral\_2:

"dlist\_tempo1 a  $\implies$  Collect (dlist\_xbefore a ( $\lambda xs. xs = \text{Dlist } []$ )) = Collect a"  
using dlist\_xbefore\_neutral\_2 by auto

#### A.4.2.5 XBefore associativity

theorem dlist\_xbefore\_assoc1:

"dlist\_tempo1 S  $\implies$  dlist\_tempo1 T  $\implies$  dlist\_tempo1 U  $\implies$   
(dlist\_xbefore (dlist\_xbefore S T) U zs)  $\longleftrightarrow$   
(dlist\_xbefore S (dlist\_xbefore T U) zs)"

unfolding dlist\_xbefore\_def slice\_slice\_simps dlist\_tempo\_def

apply auto

apply (metis diff\_is\_0\_eq less\_imp\_le max\_0L min\_def not\_le  
ordered\_cancel\_comm\_monoid\_diff\_class.le\_iff\_add slice\_dlist\_def  
take\_eq\_Nil)

by (metis le\_add1 min.absorb2)

corollary dlist\_xbefore\_assoc:

"dlist\_tempo1 S  $\implies$  dlist\_tempo1 T  $\implies$  dlist\_tempo1 U  $\implies$   
(dlist\_xbefore (dlist\_xbefore S T) U) =  
(dlist\_xbefore S (dlist\_xbefore T U))"

using dlist\_xbefore\_assoc1 by blast

corollary dlistset\_xbefore\_assoc:

"dlist\_tempo1 S  $\implies$  dlist\_tempo1 T  $\implies$  dlist\_tempo1 U  $\implies$   
Collect (dlist\_xbefore (dlist\_xbefore S T) U) =  
Collect (dlist\_xbefore S (dlist\_xbefore T U))"

by (simp add: dlist\_xbefore\_assoc)

#### A.4.2.6 XBefore equivalences

lemma dlist\_tempo1\_le\_uniqueness:

"dlist\_tempo1 S  $\implies$  S (l†..i)  $\implies$  i  $\leq$  j  $\implies$   $\neg$  S (l†j..)" and  
"dlist\_tempo1 S  $\implies$  S (l†j..)  $\implies$  i  $\leq$  j  $\implies$   $\neg$  S (l†..i)"

unfolding dlist\_tempo1\_def

by auto

lemma dlist\_xbefore\_not\_sym:

"dlist\_tempo1 S  $\Rightarrow$  dlist\_tempo1 T  $\Rightarrow$  dlist\_xbefore S T xs  $\Rightarrow$   
dlist\_xbefore T S xs  $\Rightarrow$  False"

by (metis dlist\_xbefore\_def le\_cases dlist\_tempo1\_le\_uniqueness)

corollary dlist\_xbefore\_and:

"dlist\_tempo1 S  $\Rightarrow$  dlist\_tempo1 T  $\Rightarrow$   
((dlist\_xbefore S T zs)  $\wedge$  (dlist\_xbefore T S zs)) = False"

using dlist\_xbefore\_not\_sym by blast

corollary dlistset\_xbefore\_and:

"dlist\_tempo1 S  $\Rightarrow$  dlist\_tempo1 T  $\Rightarrow$   
(Collect (dlist\_xbefore S T))  $\cap$  (Collect (dlist\_xbefore T S)) = {}"

using dlist\_xbefore\_and

by auto

lemma dlist\_tempo2\_left\_absorb: "dlist\_tempo2 S  $\Rightarrow$  S (l†i..)  $\Rightarrow$  S l"

unfolding dlist\_tempo2\_def

by auto

lemma dlist\_tempo2\_right\_absorb: "dlist\_tempo2 S  $\Rightarrow$  S (l†..i)  $\Rightarrow$  S l"

unfolding dlist\_tempo2\_def

by auto

lemma dlist\_xbefore\_implies\_member1[simp]:

"dlist\_tempo2 S  $\Rightarrow$  dlist\_xbefore S T l  $\Rightarrow$  S l"

by (meson dlist\_xbefore\_def dlist\_tempo2\_right\_absorb)

lemma dlist\_xbefore\_implies\_member2[simp]:

"dlist\_tempo2 T  $\Rightarrow$  dlist\_xbefore S T l  $\Rightarrow$  T l"

by (meson dlist\_xbefore\_def dlist\_tempo2\_left\_absorb)

lemma dlist\_xbefore\_or1:

"dlist\_tempo2 S  $\Rightarrow$  dlist\_tempo2 T  $\Rightarrow$   
dlist\_xbefore S T l  $\vee$  dlist\_xbefore T S l  $\Rightarrow$  S l  $\wedge$  T l"

using dlist\_xbefore\_implies\_member1 dlist\_xbefore\_implies\_member2 by blast

definition dlist\_independent\_events ::

```
"('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  ('a dlist  $\Rightarrow$  bool)  $\Rightarrow$  bool"
```

```
where
```

```
"dlist_independent_events S T  $\equiv$ 
```

```
( $\forall i$  l.  $\neg$  (S (l†i..(Suc i))  $\wedge$  T (l†i..(Suc i))))"
```

```
lemma dlist_indepentent_events_member: "a  $\neq$  b  $\Rightarrow$ 
```

```
dlist_independent_events ( $\lambda$  dl. Dlist.member dl a) ( $\lambda$  dl. Dlist.member dl b)"
```

```
apply (simp add: dlist_independent_events_def Dlist.member_def List.member_def)
```

```
by (metis dlist_member_suc_nth1)
```

```
lemma dlist_and_split9:
```

```
"dlist_independent_events S T  $\Rightarrow$ 
```

```
dlist_tempo2 S  $\Rightarrow$  dlist_tempo2 T  $\Rightarrow$ 
```

```
dlist_tempo3 S  $\Rightarrow$  dlist_tempo3 T  $\Rightarrow$ 
```

```
dlist_tempo4 S  $\Rightarrow$  dlist_tempo4 T  $\Rightarrow$ 
```

```
S l  $\wedge$  T l  $\longleftrightarrow$  ( $\exists i$  j.  $i \leq j \wedge$   
((S (l†i..i)  $\wedge$  T (l†j..j))  $\vee$  (S (l†j..j)  $\wedge$  T (l†i..i))))"
```

```
unfolding dlist_independent_events_def
```

```
dlist_tempo2_def dlist_tempo3_def dlist_tempo4_def
```

```
by (metis le_refl not_less not_less_eq_eq)
```

```
lemma dlist_tempo_equiv_xor:
```

```
"dlist_tempo1 S  $\Rightarrow$  dlist_tempo2 S  $\Rightarrow$ 
```

```
 $\forall l$ . S l  $\longleftrightarrow$  ( $\forall i$ . (S (l†i..i)  $\wedge$   $\neg$  S (l†i..i))  $\vee$  ( $\neg$  S (l†i..i)  $\wedge$  S (l†i..i)))"
```

```
unfolding tempo_defs
```

```
by (meson order_refl)
```

```
corollary dlist_tempo_equiv_not_eq: "dlist_tempo1 S  $\Rightarrow$  dlist_tempo2 S  $\Rightarrow$ 
```

```
 $\forall l$ . S l  $\longleftrightarrow$  ( $\forall i$ . S (l†i..i)  $\neq$  S (l†i..i))"
```

```
using dlist_tempo_equiv_xor
```

```
by auto
```

```
lemma dlists_xbefore_or2:
```

```
"dlist_independent_events S T  $\Rightarrow$ 
```

```
dlist_tempo1 S  $\Rightarrow$  dlist_tempo1 T  $\Rightarrow$ 
```

```
dlist_tempo2 S  $\Rightarrow$  dlist_tempo2 T  $\Rightarrow$ 
```

```
dlist_tempo3 S  $\Rightarrow$  dlist_tempo3 T  $\Rightarrow$ 
```

```

dlist_tempo4 S  $\impl$  dlist_tempo4 T  $\impl$ 
S l  $\wedge$  T l  $\impl$  dlist_xbefore S T l  $\vee$  dlist_xbefore T S l"

```

```

unfolding dlist_xbefore_def dlist_tempo_def
by (metis dlist_and_split9 dlist_tempo_equiv_not_eq
    dlist_tempo1_le_uniqueness)

```

**theorem** dlist\_xbefore\_or\_one\_list:

```

"dlist_independent_events S T  $\impl$ 
dlist_tempo1 S  $\impl$  dlist_tempo1 T  $\impl$ 
dlist_tempo2 S  $\impl$  dlist_tempo2 T  $\impl$ 
dlist_tempo3 S  $\impl$  dlist_tempo3 T  $\impl$ 
dlist_tempo4 S  $\impl$  dlist_tempo4 T  $\impl$ 
dlist_xbefore S T l  $\vee$  dlist_xbefore T S l  $\longleftrightarrow$  S l  $\wedge$  T l"
using dlist_xbefore_or1 dlists_xbefore_or2 dlist_tempo_def
by blast

```

**corollary** dlist\_xbefore\_or:

```

"dlist_independent_events S T  $\impl$ 
dlist_tempo1 S  $\impl$  dlist_tempo1 T  $\impl$ 
dlist_tempo2 S  $\impl$  dlist_tempo2 T  $\impl$ 
dlist_tempo3 S  $\impl$  dlist_tempo3 T  $\impl$ 
dlist_tempo4 S  $\impl$  dlist_tempo4 T  $\impl$ 
( $\lambda$ zs. (dlist_xbefore S T zs)  $\vee$  (dlist_xbefore T S zs)) =
  ( $\lambda$ zs. S zs  $\wedge$  T zs)"
using dlist_xbefore_or_one_list
by blast

```

**corollary** dlistset\_xbefore\_or:

```

"dlist_independent_events S T  $\impl$ 
dlist_tempo1 S  $\impl$  dlist_tempo1 T  $\impl$ 
dlist_tempo2 S  $\impl$  dlist_tempo2 T  $\impl$ 
dlist_tempo3 S  $\impl$  dlist_tempo3 T  $\impl$ 
dlist_tempo4 S  $\impl$  dlist_tempo4 T  $\impl$ 
(Collect (dlist_xbefore S T))  $\cup$  (Collect (dlist_xbefore T S)) =
  Collect S  $\cap$  Collect T"
using dlist_xbefore_or
by (smt Collect_cong Collect_conj_eq Collect_disj_eq)

```

#### A.4.2.7 XBefore transitivity

**theorem** dlist\_xbefore\_trans: "

```

[[dlist_tempo1 a; dlist_tempo1 b; dlist_tempo1 c]]  $\impl$ 

```

```

[[dlist_tempo2 a; dlist_tempo2 b; dlist_tempo2 c]]  $\Rightarrow$ 
dlist_xbefore a b zs  $\wedge$  dlist_xbefore b c zs  $\Rightarrow$ 
dlist_xbefore a c zs"

```

```
using dlist_xbefore_not_sym
```

```
by (metis dlist_tempo2_def dlist_xbefore_def)
```

```
corollary dlistset_xbefore_trans: "
```

```
[[dlist_tempo1 a; dlist_tempo1 b; dlist_tempo1 c]]  $\Rightarrow$ 
```

```
[[dlist_tempo2 a; dlist_tempo2 b; dlist_tempo2 c]]  $\Rightarrow$ 
```

```
(Collect (dlist_xbefore a b)  $\cap$  Collect (dlist_xbefore b c))  $\subseteq$ 
Collect (dlist_xbefore a c)"
```

```
using dlist_xbefore_trans
```

```
by auto
```

#### A.4.2.8 Boolean operators mixed with XBefore

```
theorem mixed_dlist_xbefore_or1: "
```

```
dlist_xbefore ( $\lambda$ xs. a xs  $\vee$  b xs) c zs =
((dlist_xbefore a c zs)  $\vee$  (dlist_xbefore b c zs))"
```

```
unfolding dlist_xbefore_def by auto
```

```
corollary mixed_dlistset_xbefore_or1: "
```

```
Collect (dlist_xbefore ( $\lambda$ xs. a xs  $\vee$  b xs) c) =
Collect (dlist_xbefore a c)  $\cup$  Collect (dlist_xbefore b c)"
```

```
proof-
```

```
have "Collect ( $\lambda$ zs. (dlist_xbefore a c zs)  $\vee$  (dlist_xbefore b c zs)) =
(Collect (dlist_xbefore a c)  $\cup$  Collect (dlist_xbefore b c))"
```

```
by (simp add: Collect_disj_eq)
```

```
thus ?thesis using mixed_dlist_xbefore_or1 by blast
```

```
qed
```

```
theorem mixed_dlist_xbefore_or2: "
```

```
dlist_xbefore a ( $\lambda$ xs. b xs  $\vee$  c xs) zs =
((dlist_xbefore a b zs)  $\vee$  (dlist_xbefore a c zs))"
```

```
unfolding dlist_xbefore_def by auto
```

```
corollary mixed_dlistset_xbefore_or2: "
```

```
Collect (dlist_xbefore a ( $\lambda$ xs. b xs  $\vee$  c xs)) =
Collect (dlist_xbefore a b)  $\cup$  Collect (dlist_xbefore a c)"
```

```
proof-
```

```
have "Collect ( $\lambda$ zs. (dlist_xbefore a b zs)  $\vee$  (dlist_xbefore a c zs)) =
Collect (dlist_xbefore a b)  $\cup$  Collect (dlist_xbefore a c)"
```

```

    by (simp add: Collect_disj_eq)
  thus ?thesis using mixed_dlist_xbefore_or2 by blast
qed

```

lemma and\_dlist\_xbefore\_equiv\_or\_dlist\_xbefore:

```

"dlist_tempo2 a  $\implies$ 
(a zs  $\wedge$  dlist_xbefore b c zs)  $\longleftrightarrow$ 
(dlist_xbefore ( $\lambda$  xs. a xs  $\wedge$  b xs) c zs  $\vee$ 
dlist_xbefore b ( $\lambda$ xs. a xs  $\wedge$  c xs) zs)"

```

proof-

```

assume "dlist_tempo2 a"
hence 0: " $\forall i$  xs. a xs  $\longleftrightarrow$  (a (xs $\dagger$ ..i)  $\vee$  a (xs $\dagger$ i.. $\infty$ ))"
  using dlist_tempo2_def by auto
have "a zs  $\wedge$  dlist_xbefore b c zs  $\longleftrightarrow$ 
a zs  $\wedge$  ( $\exists i$ . b (zs $\dagger$ ..i)  $\wedge$  c (zs $\dagger$ i.. $\infty$ ))"
  by (auto simp add: dlist_xbefore_def)
thus ?thesis using 0 by (auto simp add: dlist_xbefore_def)
qed

```

corollary and\_dlistset\_xbefore\_equiv\_or\_dlistset\_xbefore:

```

"dlist_tempo2 a  $\implies$ 
((Collect a)  $\cap$  (Collect (dlist_xbefore b c))) =
(Collect (dlist_xbefore ( $\lambda$  xs. a xs  $\wedge$  b xs) c)  $\cup$ 
Collect (dlist_xbefore b ( $\lambda$ xs. a xs  $\wedge$  c xs)))"

```

```

by (smt Collect_cong Collect_conj_eq Collect_disj_eq dlist_tempo2_def
dlist_xbefore_def)

```

lemma dlist\_xbefore\_implies\_not\_sym\_dlist\_xbefore: "

```

[[dlist_tempo1 a; dlist_tempo1 b]]  $\implies$ 
dlist_xbefore a b zs  $\implies \neg$  dlist_xbefore b a zs"

```

unfolding dlist\_xbefore\_def dlist\_tempo1\_def

by (meson nat\_le\_linear)

corollary dlistset\_xbefore\_implies\_not\_sym\_dlistset\_xbefore:

```

"[[dlist_tempo1 a; dlist_tempo1 b]]  $\implies$ 
Collect (dlist_xbefore a b)  $\subseteq$  - Collect (dlist_xbefore b a)"

```

using dlist\_xbefore\_implies\_not\_sym\_dlist\_xbefore

by (metis (mono\_tags, lifting) CollectD ComplI subsetI)

theorem mixed\_not\_dlist\_xbefore: "dlist\_independent\_events a b  $\implies$

```

[[dlist_tempo1 a; dlist_tempo1 b]]  $\implies$ 

```

```

[[dlist_tempo2 a; dlist_tempo2 b]]  $\implies$ 

```



```

[[dlist_tempo3 a; dlist_tempo3 b]] ==>
[[dlist_tempo4 a; dlist_tempo4 b]] ==>
(¬ (dlist_xbefore a b zs)) =
((¬ a zs) ∨ (¬ b zs) ∨ (dlist_xbefore b a zs))"
using dlist_xbefore_implies_not_sym_dlist_xbefore dlist_xbefore_or_one_list
by blast

```

```

corollary mixed_not_dlistset_xbefore: "dlist_independent_events a b ==>
[[dlist_tempo1 a; dlist_tempo1 b]] ==>
[[dlist_tempo2 a; dlist_tempo2 b]] ==>
[[dlist_tempo3 a; dlist_tempo3 b]] ==>
[[dlist_tempo4 a; dlist_tempo4 b]] ==>
(¬ Collect (dlist_xbefore a b)) =
((¬ Collect a) ∪ (¬ Collect b) ∪ Collect (dlist_xbefore b a))"

```

proof-

```

assume 0: "dlist_independent_events a b" "dlist_tempo1 a" "dlist_tempo1 b"
"dlist_tempo2 a" "dlist_tempo2 b" "dlist_tempo3 a" "dlist_tempo3 b"
"dlist_tempo4 a" "dlist_tempo4 b"
have "((¬ Collect a) ∪ (¬ Collect b) ∪ Collect (dlist_xbefore b a)) =
((Collect (λzs. ¬ a zs ∨ ¬ b zs)) ∪ Collect (dlist_xbefore b a))"
by blast
also have "... = (Collect (λzs. ¬ a zs ∨ ¬ b zs ∨ dlist_xbefore b a zs))"
by blast
hence "Collect (λzs. (¬ a zs) ∨ (¬ b zs) ∨ (dlist_xbefore b a zs)) =
((¬ Collect a) ∪ (¬ Collect b) ∪ Collect (dlist_xbefore b a))"
"Collect (λzs. ¬ (dlist_xbefore a b zs)) =
¬ Collect (dlist_xbefore a b)"
by blast+

```

thus ?thesis using 0 mixed\_not\_dlist\_xbefore by blast

qed

theorem not\_1\_dlist\_xbefore:

```

"[[ dlist_tempo1 a; dlist_tempo2 b]] ==>
dlist_xbefore (λxs. ¬ a xs) b zs = b zs"
by (metis Dlist_list_of_dlist dlist_tempo1_no_gap dlist_xbefore_def dlist_xbefore_implies
drop_0 slice_left_drop slice_right_take take_0)

```

corollary not\_1\_dlistset\_xbefore:

```

"[[ dlist_tempo1 a; dlist_tempo2 b]] ==>
Collect (dlist_xbefore (λxs. ¬ a xs) b) = Collect b"
using not_1_dlist_xbefore by blast

```

**theorem** *not\_2\_dlist\_xbefore*:

" $\llbracket \text{dlist\_tempo1 } b; \text{dlist\_tempo2 } a \rrbracket \implies \text{dlist\_xbefore } a (\lambda xs. \neg b \text{ } xs) \text{ } zs = a \text{ } zs$ "

by (metis *Dlist.empty\_def* *append\_Nil2* *dlist\_tempo\_1\_no\_gap*

*dlist\_xbefore\_append* *dlist\_xbefore\_implies\_member1* *drop\_0* *inf commute*

*inf\_bot\_left* *list.set(1)* *list\_of\_dlist\_empty* *slice\_left\_drop*

*slice\_right\_take* *take\_0*)

**corollary** *not\_2\_dlistset\_xbefore*:

" $\llbracket \text{dlist\_tempo1 } b; \text{dlist\_tempo2 } a \rrbracket \implies$

*Collect* (*dlist\_xbefore* *a* ( $\lambda xs. \neg b \text{ } xs$ )) = *Collect* *a*"

using *not\_2\_dlist\_xbefore* by *blast*

**lemma** *empty\_dlist\_implies\_false[simp]*:

" $\llbracket \text{dlist\_tempo1 } a; \text{dlist\_tempo2 } a \rrbracket \implies a \text{ } (\text{Dlist } []) \implies \text{False}$ "

unfolding *dlist\_tempo1\_def* *dlist\_tempo2\_def* *dlist\_tempo3\_def* *dlist\_tempo4\_def*

*slice\_left\_def* *slice\_right\_def* *size\_dlist\_def* *slice\_dlist\_def*

by (metis *Dlist.empty\_def* *list.size(3)* *list\_of\_dlist\_empty* *nat\_le\_linear*)

**lemma** *dlist\_inf\_xbefore\_trans*:

" $\llbracket \text{dlist\_tempo1 } b; \text{dlist\_tempo3 } b \rrbracket \implies ((\text{dlist\_xbefore } a \text{ } b \text{ } zs) \wedge (\text{dlist\_xbefore } b \text{ } c \text{ } zs)) \longleftrightarrow$

$(\text{dlist\_xbefore } (\text{dlist\_xbefore } a \text{ } b) \text{ } c) \text{ } zs$ "

**proof-**

assume 0: "*dlist\_tempo1* *b*" "*dlist\_tempo3* *b*"

hence 1: " $\exists i. (\exists j. a \text{ } (zs \upharpoonright..i) \wedge b \text{ } (zs \upharpoonright i..)) \wedge b \text{ } (zs \upharpoonright..j) \wedge c \text{ } (zs \upharpoonright j..)) \longleftrightarrow$   
 $a \text{ } (zs \upharpoonright..i) \wedge b \text{ } (zs \upharpoonright i..j) \wedge c \text{ } (zs \upharpoonright j..))$ "

by (metis *slice\_left\_def* *slice\_right\_def*)

have 2: " $(\exists x \ y. a \text{ } (zs \upharpoonright..x) \wedge b \text{ } (zs \upharpoonright x..)) \wedge b \text{ } (zs \upharpoonright..y) \wedge c \text{ } (zs \upharpoonright y..)) \longleftrightarrow$   
 $(\exists x \ y. a \text{ } (zs \upharpoonright..x) \wedge b \text{ } (zs \upharpoonright x..y) \wedge c \text{ } (zs \upharpoonright y..))$ "

using 0

by (metis (no\_types, hide\_lams) *diff\_zero* *dlist\_empty\_slice* *dlist\_tempo1\_le\_uniqueness* *dlist\_tempo3\_def* *dlist\_tempo\_1\_no\_gap* *drop\_0* *list\_of\_dlist\_empty* *list\_of\_dlist\_simps(3)* *max\_0L* *not\_le* *slice\_left\_drop* *slice\_right\_def* *take\_0*)

have 3: " $((\exists i. a \text{ } (zs \upharpoonright..i) \wedge b \text{ } (zs \upharpoonright i..)) \wedge (\exists j. b \text{ } (zs \upharpoonright..j) \wedge c \text{ } (zs \upharpoonright j..))) \longleftrightarrow$

$(\exists i \ j. a \text{ } (zs \upharpoonright..i) \wedge b \text{ } (zs \upharpoonright i..)) \wedge b \text{ } (zs \upharpoonright..j) \wedge c \text{ } (zs \upharpoonright j..))$ "

$(\exists i. (\exists j. a \text{ } (zs \upharpoonright.. \min i \ j) \wedge b \text{ } (zs \upharpoonright j..i)) \wedge c \text{ } (zs \upharpoonright i..)) \longleftrightarrow$

$(\exists i \ j. a \text{ } (zs \upharpoonright.. \min i \ j) \wedge b \text{ } (zs \upharpoonright j..i) \wedge c \text{ } (zs \upharpoonright i..))$ "

by *auto*

have 4: " $(\exists x \ y. a \text{ } (zs \upharpoonright.. \min x \ y) \wedge b \text{ } (zs \upharpoonright x..y) \wedge c \text{ } (zs \upharpoonright y..)) \longleftrightarrow$

$(\exists x \ y. a \text{ } (zs \upharpoonright..x) \wedge b \text{ } (zs \upharpoonright x..y) \wedge c \text{ } (zs \upharpoonright y..))$ "

```

using 0
by (metis (no_types, lifting) Dlist.empty_def append_Nil2 dlist_empty_slice
dlist_tempo_1_no_gap_append list_of_dlist_empty min.cobounded1 min_def)
have "( $\exists i j. a (zs \uparrow..i) \wedge b (zs \uparrow i..j) \wedge c (zs \uparrow j..)$ )  $\longleftrightarrow$ 
( $\exists i j. a (zs \uparrow..min\ i\ j) \wedge b (zs \uparrow i..j) \wedge c (zs \uparrow j..)$ )"
using 4 by simp
hence "( $\exists i j. a (zs \uparrow..i) \wedge b (zs \uparrow i..)$ )  $\wedge b (zs \uparrow..j) \wedge c (zs \uparrow j..)$ )  $\longleftrightarrow$ 
( $\exists i j. a (zs \uparrow..min\ i\ j) \wedge b (zs \uparrow i..j) \wedge c (zs \uparrow j..)$ )"
using 0 2 by simp
hence "(( $\exists i. a (zs \uparrow..i) \wedge b (zs \uparrow i..)$ )  $\wedge (\exists j. b (zs \uparrow..j) \wedge c (zs \uparrow j..))$ )  $\longleftrightarrow$ 
( $\exists i j. a (zs \uparrow..min\ i\ j) \wedge b (zs \uparrow i..j) \wedge c (zs \uparrow j..)$ )"
using 0 3 by simp
hence "(( $\exists i. a (zs \uparrow..i) \wedge b (zs \uparrow i..)$ )  $\wedge (\exists j. b (zs \uparrow..j) \wedge c (zs \uparrow j..))$ )  $\longleftrightarrow$ 
( $\exists j. (\exists i. a (zs \uparrow..min\ i\ j) \wedge b (zs \uparrow i..j)) \wedge c (zs \uparrow j..)$ )"
using 3 by auto
hence "(dlist_xbefore a b zs  $\wedge$  dlist_xbefore b c zs)  $\longleftrightarrow$ 
( $\exists j. (\exists i. a (zs \uparrow..min\ i\ j) \wedge b (zs \uparrow i..j)) \wedge c (zs \uparrow j..)$ )"
using dlist_xbefore_def by auto
hence "(dlist_xbefore a b zs  $\wedge$  dlist_xbefore b c zs)  $\longleftrightarrow$ 
( $\exists j. (\exists i. a ((zs \uparrow..j) \uparrow..i) \wedge b ((zs \uparrow..j) \uparrow i..) \wedge c (zs \uparrow j..))$ )"
by (simp add: min.commute slice_right_slice_left_absorb slice_right_slice_right_absorb)
thus ?thesis unfolding dlist_xbefore_def by simp
qed

lemma dlistset_inf_xbefore_trans:
  "[[ dlist_tempo1 b; dlist_tempo3 b ]  $\Rightarrow$  (Collect (dlist_xbefore a b)  $\cap$  Collect
(dlist_xbefore b c)) =
  Collect (dlist_xbefore (dlist_xbefore a b) c)"
using dlist_inf_xbefore_trans
using Collect_cong Collect_conj_eq by blast

```

### A.4.3 Formulas as ATF

In the following we prove that a formula is a valid type instantiation for all ATF classes.

#### A.4.3.1 Basic properties of ATF

```

instantiation formula :: (type) temporal_faults_algebra_basic
begin

```

definition

```
"neutral = Abs_formula { dlist_of_list [] }"
```

definition

```
"xbefore a b = Abs_formula { zs .
  dlist_xbefore (λxs. xs ∈ Rep_formula a) (λys. ys ∈ Rep_formula b) zs }"
```

definition

```
"tempo1 a = dlist_tempo1 (λxs. xs ∈ Rep_formula a)"
```

lemma Rep\_formula\_xbefore\_to\_dlist\_xbefore:

```
"Rep_formula (xbefore a b) =
  Collect (dlist_xbefore (λx. x ∈ Rep_formula a) (λy. y ∈ Rep_formula b))"
```

```
unfolding dlist_xbefore_def xbefore_formula_def
by (simp add: Abs_formula_inverse)
```

```
lemma Rep_formula_xbefore_bot_1: "Rep_formula (xbefore bot a) =
  Rep_formula bot"
```

```
unfolding xbefore_formula_def
by (simp add: Abs_formula_inverse dlist_xbefore_bot_1)
```

```
lemma Rep_formula_xbefore_bot_2: "Rep_formula (xbefore a bot) =
  Rep_formula bot"
```

```
unfolding xbefore_formula_def
by (simp add: Abs_formula_inverse dlist_xbefore_bot_2)
```

```
lemma Rep_formula_xbefore_neutral_1: "tempo1 a ⇒ Rep_formula (xbefore neutral
a) = Rep_formula a"
```

```
unfolding xbefore_formula_def neutral_formula_def tempo1_formula_def
apply (simp add: Abs_formula_inverse)
using dlistset_xbefore_neutral_1 by auto
```

```
lemma Rep_formula_xbefore_neutral_2: "tempo1 a ⇒ Rep_formula (xbefore a neutral)
= Rep_formula a"
```

```
unfolding xbefore_formula_def neutral_formula_def tempo1_formula_def
apply (simp add: Abs_formula_inverse)
using dlistset_xbefore_neutral_2 by auto
```

lemma Rep\_formula\_xbefore\_not\_idempotent:

```
"tempo1 a ⇒ Rep_formula (xbefore a a) = Rep_formula bot"
```

```
unfolding xbefore_formula_def tempo1_formula_def
```

by (simp add: Abs\_formula\_inverse dlist\_xbefore\_idem)

lemma Rep\_formula\_xbefore\_not\_sym:

" $\llbracket \text{tempo1 } a; \text{ tempo1 } b \rrbracket \Rightarrow$   
 $\text{Rep\_formula } (\text{xbefore } a \ b) \subseteq \text{Rep\_formula } (\neg \text{xbefore } b \ a)$ "  
 unfolding xbefore\_formula\_def tempo1\_formula\_def uminus\_formula\_def  
 by (simp add: Abs\_formula\_inverse  
 dlistset\_xbefore\_implies\_not\_sym\_dlistset\_xbefore)

instance proof

fix a::"'a formula"  
 show "xbefore bot a = bot"  
 unfolding eq\_formula\_iff Rep\_formula\_xbefore\_bot\_1 by auto  
 next  
 fix a::"'a formula"  
 show "xbefore a bot = bot"  
 unfolding eq\_formula\_iff Rep\_formula\_xbefore\_bot\_2 by auto  
 next  
 fix a::"'a formula"  
 assume "tempo1 a"  
 thus "xbefore neutral a = a"  
 unfolding eq\_formula\_iff  
 using Rep\_formula\_xbefore\_neutral\_1 by auto  
 next  
 fix a::"'a formula"  
 assume "tempo1 a"  
 thus "xbefore a neutral = a"  
 unfolding eq\_formula\_iff  
 using Rep\_formula\_xbefore\_neutral\_2 by auto  
 next  
 fix a::"'a formula"  
 assume "tempo1 a"  
 thus "xbefore a a = bot"  
 unfolding eq\_formula\_iff  
 using Rep\_formula\_xbefore\_not\_idempotent by auto  
 next  
 fix a::"'a formula" and b::"'a formula"  
 assume "tempo1 a" "tempo1 b"  
 thus "xbefore a b  $\leq$  - xbefore b a"  
 unfolding eq\_formula\_iff less\_eq\_formula\_def  
 using Rep\_formula\_xbefore\_not\_sym by simp  
 fix a::"'a formula" and b::"'a formula"

```

assume "tempo1 a" "tempo1 b"
thus "tempo1 (inf a b)"
unfolding tempo1_formula_def
by (simp add: dlist_tempo1_inf Rep_formula_inf)
qed

```

The above proof shows basic laws about [ATF](#), as shown in Eqs. (4.22a), (4.24a) to (4.24d) and (4.24g).

end

#### A.4.3.2 Associativity of [ATF](#)

```

instantiation formula :: (type) temporal_faults_algebra_assoc
begin

```

```

instance proof

```

```

  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo1 a" "tempo1 b" "tempo1 c"
  thus "xbefore (xbefore a b) c = xbefore a (xbefore b c)"
  unfolding xbefore_formula_def tempo1_formula_def
  by (simp add: Abs_formula_inverse dlist_xbefore_assoc)
qed

```

The above proof shows associativity law about [ATF](#), as shown in Eq. (4.24h).

end

#### A.4.3.3 Equivalences in [ATF](#)

```

instantiation formula :: (type) temporal_faults_algebra_equivs
begin

```

```

definition

```

```

  "independent_events a b =
    dlist_independent_events
      (λxs. xs ∈ Rep_formula a) (λxs. xs ∈ Rep_formula b)"

```

```

definition

```

```

  "tempo2 a = dlist_tempo2 (λxs. xs ∈ Rep_formula a)"

```

```

definition

```

```

  "tempo3 a = dlist_tempo3 (λxs. xs ∈ Rep_formula a)"

```

```

definition

```

```
"tempo4 a = dlist_tempo4 ( $\lambda$ xs. xs  $\in$  Rep_formula a)"
```

instance proof

```
fix a::"'a formula" and b::"'a formula"
assume "tempo1 a" "tempo1 b"
thus "inf (xbefore a b) (xbefore b a) = bot"
unfolding xbefore_formula_def tempo1_formula_def bot_formula_def
  inf_formula_def
by (simp add: dlistset_xbefore_and Abs_formula_inverse)
next
fix a::"'a formula" and b::"'a formula"
assume "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a" "tempo2 b"
  "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
thus "sup (xbefore a b) (xbefore b a) = inf a b"
unfolding xbefore_formula_def tempo1_formula_def tempo2_formula_def
  tempo3_formula_def tempo4_formula_def independent_events_formula_def
  sup_formula_def inf_formula_def
by (simp add: dlistset_xbefore_or Abs_formula_inverse)
next
fix a::"'a formula" and b::"'a formula"
assume "tempo2 a" "tempo2 b"
thus "tempo2 (sup a b)"
unfolding tempo2_formula_def
by (simp add: dlist_tempo2_sup Rep_formula_sup)
next
fix a::"'a formula" and b::"'a formula"
assume "tempo3 a" "tempo3 b"
thus "tempo3 (inf a b)"
unfolding tempo3_formula_def
by (simp add: dlist_tempo3_inf Rep_formula_inf)
next
fix a::"'a formula" and b::"'a formula"
assume "tempo4 a" "tempo4 b"
thus "tempo4 (sup a b)"
unfolding tempo4_formula_def
by (simp add: dlist_tempo4_sup Rep_formula_sup)
```

qed

The above proof shows equivalences in ATF, as shown in Eqs. (4.22b) to (4.22d), (4.25a) and (4.25b).

end

## A.4.3.4 Transitivity in ATF

```

instantiation formula :: (type) temporal_faults_algebra_trans
begin
instance proof
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo1 a" "tempo1 b" "tempo1 c" "tempo2 a" "tempo2 b" "tempo2 c"
  thus "inf (xbefore a b) (xbefore b c) ≤ xbefore a c"
  unfolding tempo1_formula_def tempo2_formula_def xbefore_formula_def
    less_eq_formula_def inf_formula_def
  by (simp add: dlistset_xbefore_trans Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo1 b" "tempo3 b"
  thus "inf (xbefore a b) (xbefore b c) = xbefore (xbefore a b) c"
  unfolding xbefore_formula_def inf_formula_def tempo1_formula_def
    tempo3_formula_def
  by (simp add: Abs_formula_inverse dlistset_inf_xbefore_trans)
qed

```

The above proof shows transitivity in ATF, as shown in Eq. (4.27f).

end

## A.4.3.5 Mixed operators in ATF

```

instantiation formula :: (type) temporal_faults_algebra_mixed_ops
begin
instance proof
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  show "xbefore (sup a b) c = sup (xbefore a c) (xbefore b c)"
  unfolding xbefore_formula_def sup_formula_def
  by (simp add: mixed_dlistset_xbefore_or1 Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  show "xbefore a (sup b c) = sup (xbefore a b) (xbefore a c)"
  unfolding xbefore_formula_def sup_formula_def
  by (simp add: mixed_dlistset_xbefore_or2 Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a" "tempo2 b"
    "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
  thus "(- xbefore a b) = (sup (sup (- a) (- b)) (xbefore b a))"
  by (simp add: Abs_formula_inverse xbefore_formula_def uminus_formula_def)

```



```

sup_formula_def independent_events_formula_def tempo1_formula_def
tempo2_formula_def tempo3_formula_def tempo4_formula_def
mixed_not_dlistset_xbefore)
next
fix a::"'a formula" and b::"'a formula" and c::"'a formula"
assume "tempo2 a"
thus "inf a (xbefore b c) =
  sup (xbefore (inf a b) c) (xbefore b (inf a c))"
apply (auto simp add: xbefore_formula_def sup_formula_def inf_formula_def
tempo2_formula_def Abs_formula_inverse)
using and_dlistset_xbefore_equiv_or_dlistset_xbefore Abs_formula_inverse
by fastforce
next
fix a::"'a formula" and b::"'a formula"
assume "tempo1 a" "tempo2 b"
thus "xbefore (- a) b = b"
unfolding xbefore_formula_def tempo1_formula_def tempo2_formula_def
uminus_formula_def
by (auto simp add: Abs_formula_inverse not_1_dlistset_xbefore
Rep_formula_inverse)
next
fix a::"'a formula" and b::"'a formula"
assume "tempo1 b" "tempo2 a"
thus "xbefore a (- b) = a"
unfolding xbefore_formula_def tempo1_formula_def tempo2_formula_def
uminus_formula_def
by (auto simp add: Abs_formula_inverse not_2_dlistset_xbefore
Rep_formula_inverse)
qed

```

The above proof shows laws with mixed Boolean and **XBefore** operators, as shown in Eqs. (4.27a), (4.27b), (4.27e) and (4.29a) to (4.29c).

end

#### A.4.4 Equivalence of the new definition of **XBefore** with the old one

**definition** old\_dlist\_xbefore

**where**

```

"old_dlist_xbefore S T zs ≡
  (∃ xs ys. S xs ∧ ¬ T xs ∧ T ys ∧ ¬ S ys ∧
    set (list_of_dlist xs) ∩ set (list_of_dlist ys) = {} ∧
    list_of_dlist zs = (list_of_dlist xs) @ (list_of_dlist ys))"

```

```

theorem old_dlist_xbefore_equals_new_xbefore:
  "[[ dlist_tempo1 S; dlist_tempo1 T ]]  $\Rightarrow$ 
    dlist_xbefore S T zs = old_dlist_xbefore S T zs"
unfolding dlist_xbefore_append old_dlist_xbefore_def
using dlist_tempo_1_no_gap_append
by blast

```

#### A.4.5 Soundness and completeness on the mapping rules

```

theorem temporal_faults_algebra_mapping_soundness:
  " $\forall$  ( $f_1::$ 'a formula) ( $f_2::$ 'a formula).  $\exists$  S. ((Rep_formula  $f_1$  = S  $\wedge$  Rep_formula
 $f_2$  = S)  $\longleftrightarrow$   $f_1$  =  $f_2$ )"
by blast

```

```

theorem temporal_faults_algebra_mapping_completeness:
  " $\forall$  (S::'a dlist set).  $\exists$  f::'a formula. Rep_formula f = S"
using Abs_formula_inverse by auto

```