Pós-Graduação em Ciência da Computação

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An Algebra of Temporal Faults

Deveria usar template da qualificação.

Ph.D. Thesis

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I dedicate this thesis to Juliana, Luciana (pipoquinha), and Bianca (snowflake).

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If I were afraid of the path, I wouldn't have gotten here.

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"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity.

(Alan Turing)

Resumo

Não está claro aqui o que isso significa.

A modelagem de defeitos é essencial na antecipação de falhas em sistemas críticos. Tradicionalmente, Árvores de Defeitos Estáticas são empregadas para este fim, mas Árvores de Defeitos Temporais e Dinâmicas têm ganhado evidência devido ao seu maior poder para modelar e detectar propagações complexas de defeitos que levam a uma falha.

Em um trabalho anterior, mostramos uma estratégia baseada na álgebra de processos CSP e modelos Simulink para obter rastros (sequências) de defeitos que levam a uma falha. A partir dos rastros de defeitos nós descartamos a informação de ordenamento para obter expressões de estrutura para Ávores de Defeitos Estáticas. Ao contrário de descartar tal informação de ordenamento, poderíamos usá-la para obter expressões de estrutura para Árvores de Defeitos Temporais ou Dinâmicas. No presente trabalho, apresentamos uma álgebra temporal de defeitos (com noção de propagação de defeitos) para analisar falhas em sistemas e provamos que ela é de fato uma álgebra Booleana. Isso permite herdar as propriedades de álgebras Booleanas, leis e técnicas de redução existentes, as quais são muito benéficas para a modelagem e análise de defeitos. Com expressões na álgebra temporal de defeitos nós permitimos a verificação de propriedades de segurança (*safety*) baseadas em Árvores de Defeitos Estáticas, Temporais ou Dinâmicas. Nós ilustramos nosso trabalho com alguns estudos de caso simples, mas reais, fornecidos pelo nosso parceiro industrial, a EMBRAER.

Palavras-chave: Simulink, CSP, FDR, Fault Tree Analysis, Temporal Fault Trees, Dynamic Fault Trees, Pandora, Fault Injection

Abstract

Faults modelling is essential to anticipate failures in critical systems. Traditionally, Static Fault Trees are employed to this end, but Temporal and Dynamic Fault Trees have gained evidence due to their enriched power to model and detect intricate propagation of faults that lead to a failure.

In previous work, we showed a strategy based on the process algebra CSP and Simulink models to obtain fault traces that lead to a failure. From the fault traces we discarded the ordering information to obtain structure expressions for Static Fault Trees. Instead of discarding such an ordering information, it could be used to obtain structure expressions of Temporal or Dynamic Fault Trees. In this work we present an algebra of temporal faults (with a notion of fault propagation) to analyse systems' failures, and prove that it is indeed a Boolean algebra. This allows us to inherit Boolean algebra's properties, laws and existing reduction techniques, which are very beneficial for faults modelling and analysis. With expressions in the algebra of temporal faults we allow the verification of safety properties based on Static, Temporal or Dynamic Fault Trees. We illustrate our work on simple but real case studies, some supplied by our industrial partner EMBRAER.

Keywords: Simulink, CSP, FDR, Fault Tree Analysis, Temporal Fault Trees, Dynamic Fault Trees, Pandora, Fault Injection

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List of abbreviations and acronyms

ActA	Activation Algebra pp. 31, 33, 82–84, 91, 92
AFP	archive of formal proofs p. 72
ATF	Algebra of Temporal Faults pp. 24, 25, 31–33, 43, 59, 69, 77–80, 82–87, 91–93, 105, 118–120, 123–125, 127, 129, 131, 133, 135–141
BDD	Binary Decision Diagram pp. 15, 27, 29, 30, 43, 47, 53, 56–60, 62, 93
BN	Bayesian network p. 53
CML	COMPASS Modelling Language p. 40
CPN	coloured Petri-net p. 53
CSP	Communicating Sequential Processes p. 40
CSP_M	Communicating Sequential Processes pp. 29, 31, 43, 67, 68, 70, 77
CTMC	continuous-time Markov chain pp. 29, 30, 53
DBN	dynamic bayesian network p. 30
DD	Dependence Diagram pp. 40, 41
DFT	Dynamic Fault Tree pp. 17, 27–31, 37, 41, 43–45, 48, 49, 51–56, 59, 60, 69, 77, 80, 93
DNF	disjunctive normal form pp. 44, 50, 53, 58, 60, 80, 81, 91, 93
DRBD	Dynamic Reliability Block Diagram p. 41
DTMC	discrete-time Markov chain pp. 29, 30, 40, 51, 56, 93
FBA	Free Boolean Algebra pp. 27, 29, 30, 43, 55, 60, 61, 72, 77–79, 86, 93
FDR	Failures and Divergences Refinement pp. 29, 67–69
FMEA	Failure Modes and Effects Analysis pp. 30, 40
FSM	Finite State Machine p. 56
FT	fault tree pp. 15, 27–32, 37, 38, 40, 43–46, 48, 49, 54, 55, 62–66, 70, 77, 83, 91
FTA	Fault Tree Analysis pp. 27, 29–32, 43–46, 64
HCAS	cardiac assist system p. 53
HiP-HOPS	Hierarchically Performed Hazard Origin and Propagation Studies pp. 28–30, 38, 45, 69, 70
HLPN	high-level Petri-net p. 56
HOL	higher-order logic p. 72
Isar	Intelligible semi-automated reasoning pp. 43, 72

ITL	Interval Temporal Logic p. 55
LTL	linear temporal logic p. 48
MCS	minimal cut set pp. 27, 32, 44, 47, 50, 52
MCSeq	minimal cut sequence pp. 28, 32, 49, 50, 52, 53, 56, 58, 59
PN	Petri-net p. 39
ROBDD	Reduced Ordered Binary Decision Diagram pp. 56–58
SBDD	Sequential Binary Decision Diagram pp. 30, 53, 56, 60
SFT	Static Fault Tree pp. 15, 28, 29, 37, 40, 41, 43–48, 50, 52, 54–56, 58–60, 62, 69, 77, 80, 93
SoS	System of Systems pp. 31, 38
SWN	stochastic well-formed net p. 53
SysML	Systems Modelling Language pp. 31, 40
TDT	dependency tree pp. 29, 50, 51, 56, 58
TFT	Temporal Fault Tree pp. 27–31, 37, 43–45, 48–56, 69, 77, 80
TTT	Temporal Truth Table pp. 17, 29, 49, 58
UML	Unified Modelling Language p. 40
Z	Z Notation pp. 56, 72
ZBDD	Zero-suppresed Binary Decision Diagram pp. 56, 59

Fault tree gates

AND $\wedge$. Used in SFT, TFT, and DFT. pp. 27, 43, 46, 48–50, 53, 54, 60, 62, 63, 69, 71, 80, 86, 93

CSp cold spare. Used in DFT. pp. 28, 44, 52, 54, 56, 60

FDEP functional dependency. Used in DFT. pp. 28, 44, 51, 52, 54

IBefore inclusive-before. Used in structure expressions of DFT. pp. 53, 54, 60

NIBefore non-inclusive-before. Used in structure expressions of DFT. pp. 53, 54

NOT $\neg$. Used in non-coherent trees. pp. 28, 29, 43, 48, 62, 64, 80, 91, 93

OR $\vee$. Used in SFT, TFT, and DFT. pp. 27, 43, 46, 48, 49, 53, 54, 62, 69, 86, 87, 93

PAND priority-AND. Used in SFT, TFT, and DFT. pp. 27, 43, 44, 48–50, 52, 54, 55, 60

POR priority-OR. Used in TFT. pp. 48–50, 53

SAND simultaneous-AND. Used in TFT. pp. 48–51, 53

SEQ sequence enforcing. Used in DFT. pp. 28, 44, 52, 54

SIMLT simultaneous. Used in structure expressions of DFT. pp. 53, 54

WSp warm spare. Used in DFT. pp. 28, 60

XBefore exclusive-before. Proposed in this work. pp. 24, 77–82, 85–87, 91, 93, 119

Contents

1 Introduction

The development process of critical control systems is based essentially on the rigorous execution of guides and regulations [4, 5, 6, 7]. Specialized agencies (like FAA, EASA and ANAC in the aviation field) use these guides and regulations to certify such systems.

Safety is a system's attribute that plays a crucial concern on critical systems and it is the responsibility of the safety assessment process. To employ such a process, dependable systems taxonomy and safety assessment techniques must be well defined and understood. Clarification of concepts of dependable systems can be surprisingly difficult when systems are complex, because the determination of possible causes or consequences of failure can be a very subtle process [8].

ARP-4761 [7] defines several techniques to perform safety assessment. One of them is Fault Tree Analysis (FTA). It is a deductive method that uses trees to model faults and their dependencies and propagation. In such trees, the premises are the leaves (basic events) and the conclusions are the roots (top events). Intermediary events use gates to combine basic events and each kind of gate has its own combination semantics definition. Fault trees (FTs) that use only $\vee$ (OR) and $\wedge$ (AND) gates are called *coherent fault trees* [9, 10, 11, 12, 13]. They combine the events as *at least one shall occur* and *all shall occur*, respectively. To analyse FTs, their structures are abstracted as Boolean expressions called *structure expressions*. The analysis of coherent FTs uses a well-defined algorithm based on the Shannon's method to obtain minimal cut sets (MCSs) from the structure expressions, and a general formula to calculate the probability of top events. The MCSs are obtained by reducing structure expressions to a normal form, in which each term is a combination of variables (basic events) with AND gates, and the terms are combined as OR gates. These minimal terms are also called *prime implicants* or *minterms*. The Shannon's method originated a formalism to reduce structure expressions called Binary Decision Diagram (BDD) [14, 15]. Another approach to reduce structure expressions is to use a mathematical model—called Free Boolean Algebra (FBA) [16, pp. 256-266]—that uses sets of sets to represent Boolean expressions.

Besides the traditional OR and AND gates, the Fault Tree Handbook [17] defines other gates. For example the priority-AND (PAND) gate, which considers the order of occurrence of events. Although the Fault Tree Handbook defines these new gates, there is no algorithm to perform the analysis of trees that contain such new gates. This and the need of analysis of dynamic aspects of increasingly complex systems motivated the introduction of two new kinds of fault trees: Dynamic Fault Trees (DFTs) [1, 2] and Temporal Fault Trees

(TFTs) [18, 19, 20]. These variant trees can capture sequence dependencies of fault events in a system. The difference from TFT to DFT is that TFTs use temporal gates directly, while DFT does not—DFTs gates are an abstraction of temporal gates. To differentiate the fault trees as defined in the Fault Tree Handbook from the other two, we will call them Static Fault Trees (SFTs).

The work reported in [19] aims at performing the full implementation of the Fault Tree Handbook, adding temporal gates to its Pandora[1] methodology. It was this implementation that introduced the new concept of TFTs, cited previously. In such trees, events ordering is well-defined and an algebraic framework was proposed to reduce structure expressions to obtain minimal cut sequences (MCSeqs) and perform probabilistic analysis. Reducing expressions is also desirable to check for tautologies, for example.

DFTs introduce very different gates to capture dynamic configurations of systems: cold spare (CSp), functional dependency (FDEP), and sequence enforcing (SEQ). The semantics of the first is to add "backup" events, so the gate is active if the primary event and all spares are active. The second adds basic events dependency from a trigger event. The third forces the occurrence of events in a particular order. There is also a warm spare (WSp) gate that is slightly different from the CSp gate. They differ on the nature of sparing, whether fast (warm, always-on) or slow (cold, stand-by). The readiness of the backup system in a WSp gate is higher than in a CSp gate. The work reported in [21] shows an algebraic framework to compositionally reduce DFT gates to order-based gates and perform probabilistic analysis of structure expressions. Thus, despite some limitations for spare gates [22], the structure expressions used in TFTs and DFTs can be formulated in terms of a generic order-based operator.

The $\neg$ (NOT) operator is absent in the algebras reported in [19, 20, 3, 23]. There is no consensus about the relevance of its use: (i) it can be misleading, generating non-coherent analysis [11], or (ii) it can be essential in practical use [9]. Our concern is that the decision of the relevance of its use should not be due to the choice of events-occurrence representation. The algebra created in this work defines the NOT operator and allows its use, as we show in Chapter 4.

Hierarchically Performed Hazard Origin and Propagation Studies[2] (HiP-HOPS) [24] is a set of methods and tools to analyse FTs. The semi-automatic generation of FTs has architectural models and failure expressions as inputs. The failure expressions are in fact structure expressions of components or subsystems. These expressions are annotated in components and subsystems and describe how they fail. The tool combines these expressions with regard to the architecture to generate FTs. The work reported in [18] shows a strategy to use the semi-automatic FT generation of HiP-HOPS with Pandora to

[1] Pandora stands for: P-AND-ORA, which translates to Priority AND, Time.
[2] <http://www.hip-hops.eu/>

generate structure expressions of TFTs.

In previous work [25, 26], we proposed a systematic hardware-based faults identification strategy to obtain failure expressions as defined in HiP-HOPS for SFTs. We considered faults in components or subsystems, but if we obtain failure expressions of a whole system, they are in fact structure expressions of an FT. Our strategy throws away the ordering information of the fault events sequences to generate failure expressions for components or subsystems for SFTs. We focused on hardware faults because we assume that software does not fail as a function of time (wear, corrosion, etc). We inherited this view from our industrial partner (EMBRAER), which assumes that functional behaviour is completely analysed by functional verification [27]. We followed industry common practices using Simulink diagrams [28] as a starting point. The work [26] was based on Communicating Sequential Processes[3] (CSP_M) to allow an automatic analysis using the model checker FDR. Thus, our strategy required the translation from Simulink to CSP_M [29]. It then runs FDR to obtain several counter-examples (which are fault traces) ending in failures. For two case studies provided by our industrial partner, EMBRAER, we showed that our automatically created failure expressions match with the engineer's provided ones or are better (a weaker proposition).

1.1 Research questions

Both TFT and DFT lack a first-order logic mathematical model like the one defined for SFT. For SFTs, mathematical models to reduce structure expressions are either based on set inclusion, with FBA, or through tree search, with BDD. Both are efficient. One important concern on employing FTA is whether an FT indeed represents a system behaviour. The work reported in [30] exposes this concern for DFTs, and the HiP-HOPS framework—related to SFTs and TFTs—aims at getting this issued sorted. Our contribution to this issue for SFT is shown in [26, 25].

The mathematical model for TFT has a discontinuity between two activation states: (i) non-occurrence, and (ii) occurrence some time later. Such a discontinuity has some drawbacks, as for example, the impossibility to use NOT gates, and handling the specific case of non-occurrence with zeros in Temporal Truth Tables (TTTs). The reduction of structure expressions in TFT is based on a combination of: (i) algebraic reduction—which can unfortunately result in an infinite application of rules—, (ii) modularisation of independent subtrees (subtrees not always are independent), and (iii) dependency tree (TDT) [31]—which are limited to seven basic events, due to exponential growth.

Most mathematical models [32, 33, 34] for DFT are based on the formalisation of discrete-time Markov chain (DTMC) [35, 36] or continuous-time Markov chain (CTMC) [37,

[3] This variant "M" is the machine-readable version of CSP.

38] because DFTs were initially conceived to be a visual representation of such models. As both DTMC and CTMC are state-based, they experience the state-space explosion problem. The works reported in [39, 40, 7] show techniques to overcome this problem, but the reduction can be infeasible because it depends on systems' models, whether they are independent or not.

There are other approaches, however. For instance, a modified version of BDD to tackle events ordering, called Sequential Binary Decision Diagram (SBDD) [41, 42], to reduce structure expressions, and the work reported in [34], which proposes a conversion of DFT into dynamic bayesian network (DBN) [43] to perform probabilistic analysis.

The approach to tackle events ordering with SBDD [42] has two kinds of nodes: terminals and non-terminals (terminals are nodes with basic events, and non-terminals are nodes with two events and an operator). Although demonstrated in [44] that these unconventional nodes (non-terminals) generate correct and efficient Boolean analysis, the analysis is still dependent on the order-related operators because the relation of terminals and non-terminals is not established directly (non-terminals are seen as an independent node in [42]). For example, the occurrence of $A \rightarrow B$ is related to the occurrence of A and B, but this relation is obtained in a further step, not in the SBDD.

The approach using the construction of DBNs [34] is automatic and handles time slices as $t + \Delta t$, which implies a notion of events ordering as well. As it is focused in probabilistic analysis, qualitative analysis is not directly supported.

The works reported in [3, 42] show that DFT's operators can be converted into order-related operators, simplifying DFT analysis. Although the mathematical model presented in [3] establishes a denotational semantics for order-related operators, it lacks a formal method for expression reduction based on such a model. It defines, instead, several algebraic laws to reduce expressions and an algorithm to minimize the structure function.

HiP-HOPS proposes a hierarchical approach to model systems and perform FTA (and Failure Modes and Effects Analysis (FMEA) [45]). Although there is a tool to model and analyse systems using HiP-HOPS, FTs construction is based on an algorithm, without proofs for soundness nor completeness.

From the exposed in this section, our research question is:

RQ_1) Is there a mathematical model that is set-based and similar to FBA?

Also, does such a model:

RQ_2) have the capacity of representing events ordering similar to TFT and DFT?

RQ_3) represent systems behaviour by construction?

RQ_4) allow both qualitative and quantitative analyses as supported by TFT and DFT?

RQ_5) perform reduction of structure expressions to a normal form at least as efficient as current approaches?

In this version of the thesis we propose the theory that answers research questions RQ_1 to RQ_3. Research questions RQ_4 and RQ_5 are not answered in this version of the thesis.

1.2 Proposed solution

In this work we present an algebra, called Algebra of Temporal Faults (ATF), to analyse acceptance criteria of FTs with ordering of fault events (TFT and DFT). The laws of ATF are given in a denotational semantics based in sets of lists of distinct elements. ATF aims at answering the research questions RQ_1 and RQ_2. The analysis of acceptance criteria is a decision problem and we use first-order logic and Isabelle/HOL 2015[4] as verification tool. Indeed, ATF is part of a bigger strategy the relates fault injection on nominal models, fault modelling, FTA, and fault tolerance patterns. In Figure 1 the strategy starts in the top (green) node and ends in the bottom (red) node. Fault events are either extracted from a nominal model with injected faults (Figure 1, path A), or modelled using a proposed notation, called Activation Algebra (ActA) (Figure 1, path B). We depict traditional FTA in path C to show that we still need the acceptance criteria, which are the expected properties of system's FTs.

System and fault modelling is an essential step towards safety analysis. Architectural modelling is the first step of the strategy and can be executed either in a graphical tool, or as requirements in natural language. For example, our work reported in [47, 48] uses fault modelling in Systems Modelling Language (SysML) [49] to verify fault tolerance of Systems of Systems (SoSs) [50].

"Faults injection" block in Figure 1, path A, is obtained from part of our work reported in [26, 25]. It starts with Simulink modelling, converts the model to CSP_M and then obtains fault events sequences. The fault events sequences are then mapped to ATF, which have notational semantics based on sets of lists. As fault names are obtained directly from components and subsystems in a Simulink model, ActA (in the "Faults Modelling" group) allows them to be modified or complemented. ActA also allows reasoning about faults that are not modelled in Simulink as, for example, common cause or environmental faults. Path A aims at answering the research question RQ_3. Given the flexibility of the ActA notation, it can be used directly (path B) to model faults formally, reasoning about basic fault events and top-event failures, which are related to RQ_2. Each predicate in ActA generates an expression in ATF, which are reduced to obtain a canonical

[4] The 2002 tutorial is reported in [46], but there is a newer version published with the tool itself. The tool and the tutorial are available on their website at <http://isabelle.in.tum.de>.

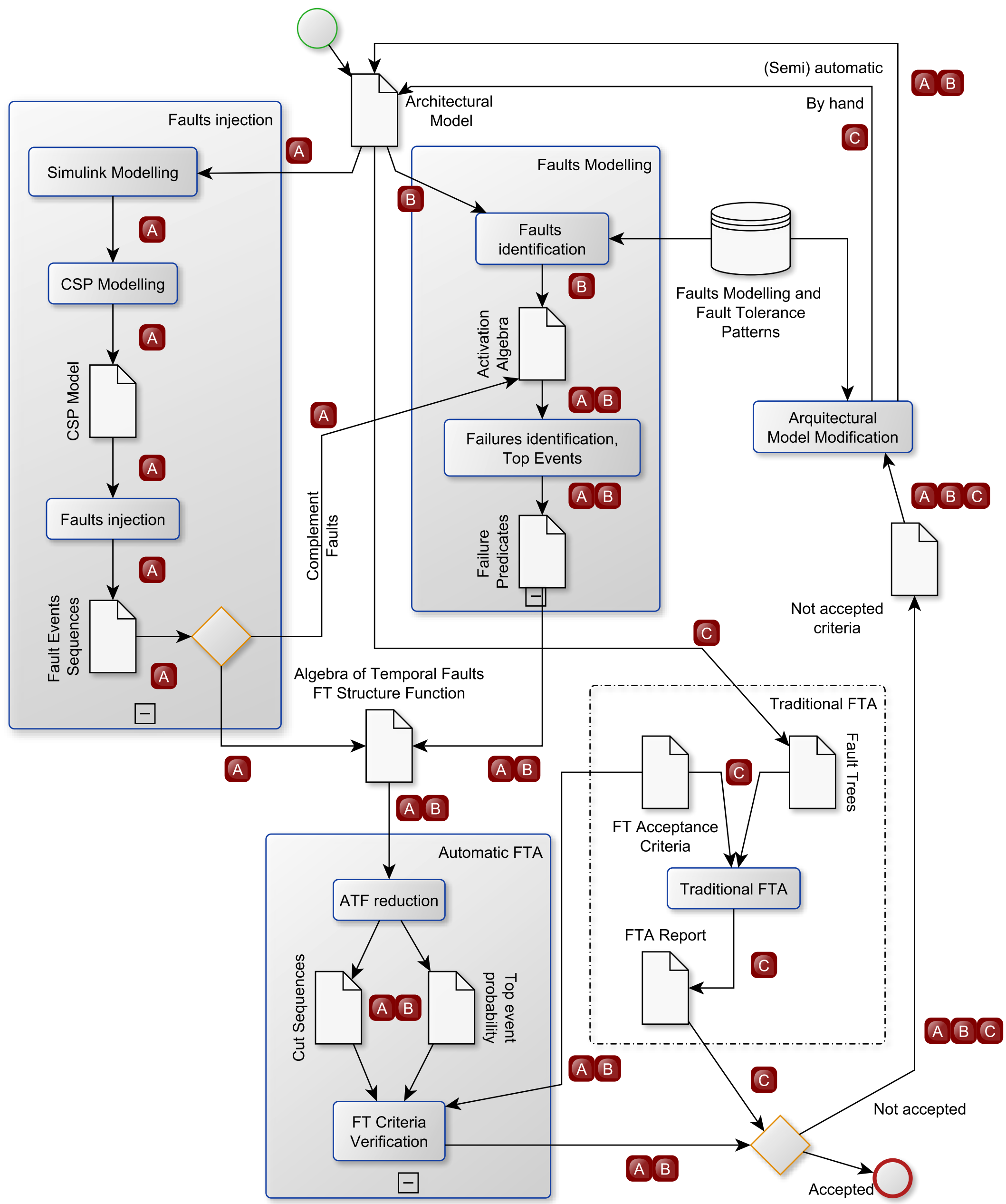


Figure 1 – Strategy overview

form to obtain MCSeqs and to calculate top-events probability.

FTA has associated non-functional system requirements which are in fact acceptance criteria for FTs. Once acceptance criteria are modelled as expressions in ATF, we formally check whether they are accepted by system models' expressions. The acceptance criteria can be either qualitative or quantitative. An example of qualitative acceptance criteria is: "an FT cannot have MCS with less than three basic events". A quantitative acceptance criteria example is: "the top-level event probability shall be less than 10^{-8}". The acceptance criteria analysis aims at answering the RQ_4.

1.3 Contributions

The main contributions of this work are:

C_1) Define a denotational and algebraic model to express fault events order with ATF—see Chapter 4;

C_2) Reuse Simulink models, obtaining fault event sequences and mapping to ATF—partially done, see Chapter 4;

C_3) Reason about faults modelling in ActA, to obtain formal expressions of critical failures (top-event failures)—see discussion in Subsection 4.3.2;

C_4) Illustrate the application of the laws on a real case study, provided by our industrial partner, EMBRAER—see Chapter 5.

C_5) Define a new operator to express order explicitly and proving that the resulting algebra—(ATF) using this operator and Boolean operators—is a conservative extension of the Boolean algebra (also published in [51])—see Chapter 4;

We use Isabelle/HOL, theories in Isabelle/HOL's library, and a theory in the AFP library [52] to prove all theorems presented in this work.

1.4 Thesis organization

This thesis is organized as follows: in Part I we show the concepts and tools used as basis for this work. Part II describes the results: Chapter 4 presents our strategy, Chapter 5 the case study and the application of the proposed strategy, and we present our conclusions and future work in Part III. The contributions presented in this work are summarized in terms of proved results. To facilitate the understanding of the presented strategy the effort to build laws and theirs (mechanized) proofs are shown in Appendix A.

Isabelle/HOL's theory files with all proofs are available at <http://www.cin.ufpe.br/~alrd/phd/phd-alrd.zip> (password: 6Zvq$5Vyj).

Part I

Background

2 Basic concepts

Means to dependability are obtained by modelling and analysing a system. It is strongly related to faults modelling, which depends on the kinds of analyses we want to perform. FTs are present in several stages of systems' modelling. We introduce dependability and faults modelling in Section 2.1.

An SFT is a snapshot[1] of a faults topology of a system, subsystem or component. The time relation of fault events in TFTs and DFTs allows the analysis of different configurations (or snapshots) of a system, subsystem or component. We discuss these time relations in Section 2.2.

2.1 Systems, dependability and fault modelling

Computing systems are characterized by five properties: functionality, performance, cost, dependability, security. The work reported in [53, p. 289–302] explain these properties—including dependability—with a focus in software. Hardware and software are connected, as software faults may cause a failure in a software-controlled hardware, and hardware faults may send incorrect data, causing a failure in the software.

The work reported in [8] summarizes all concepts of and related to dependability for computing systems that contains software and hardware. In the following, we show these concepts and highlight those used in this work.

2.1.1 Systems

Before introducing systems' dependability, we first describe what a system is and its characteristics. A *system* is an entity that interacts with other systems (software and hardware as well), users (humans), and the physical world. These other entities are the *environment* of the given system, and its *boundary* is the frontier between the system and its environment.

The *function* of a system is what the system is intended to do, and its *behaviour* is what the system does to implement its function. The *total state* of a system are the means to implement its function and is defined as the set of the following states: computation, communication, stored information, interconnection, and physical condition. The *service* delivered by a system is its behaviour as it is perceived by its boundary. A system can both provide and consume services.

[1] Whether a top event indeed causes a catastrophic or major failure is out of the scope of this thesis; we consider that, if it is possible that such failure occur, then it will.

The *structure* of a system is how it is composed: a system is composed of components, and each component is another system, etc. This concept of hierarchical compositionality in systems, is what originated the concept of SoS and is [illegible]ject of analysis in HiP-HOPS. Such a recursion (of a system containing other systems) stops when a component—or a constituent system—is considered to be atomic. A system is the total state of its atomic components.

2.1.2 Dependability

The concepts that creates the basis for dependability are: (i) threats to, (ii) attributes of, and (iii) means to attain.

Threats to dependability are the so-called *fault-error-failure* chain. A failure is a service deviation perceived on systems' boundary. An error is the part of the total state of a system that leads to subsequent service failure. Depending on how a system tolerate internal errors, many errors may not reach system's boundary. Finally, a fault is what causes an error. In this case, we say that the fault *occurred* (the fault is active). Otherwise, the fault is dormant, and has not occurred (yet). A *degraded* mode of a system is when there are active faults, so some functions of the system are inoperative, but the system still delivers its service.

There are two acceptable definitions of dependability reported in [8]. One is more general, difficult to measure: "the ability to deliver service that can justifiably be *trusted*". A more precise definition that uses the definition of service failure is: "the ability to avoid service failures that are more frequent and more severe than is acceptable". This definition has two implications about system's requirements: there should be [illegible]w it can fail, and what are the acceptable severity and frequency of its failures.

The following systems' dependability attributes enlightens such requirements:

Availability: the readiness for correct service;

Reliability: continuity of correct service;

Safety: absence of catastrophic consequences on the environment (other systems, users, and the physical world). Safety can be verified using FTs, which is part of the objective of this work;

Integrity: absence of improper systems alterations;

Maintainability: ability to be modified and repaired.

A system description should mention all or most of these attributes, at least the first three of them.

The implementation of these attributes requires a deep analysis of system's models. The *means to attain dependability* are summarized as follows:

Prevention is about avoiding incorporate faults during development.

Tolerance deals with usage of mechanisms to still deliver a—possibly degraded—service even in the presence of faults.

Removal is about detecting and removing (or reducing severity of) failures from a system, as in the development stage~~, as~~ in production stage.

Forecasting is about predicting likely faults so they can removed, or tackling their effects.

The intersection of the current work with dependability is in fault removal during development and fault tolerance (analysis). Following the taxonomy presented in [8], there are some techniques for fault removal, summarized as follows:

a) Static verification:

- Structural model:

  Static analysis: Range from inspection or walk-through, data flow analysis, complexity analysis, abstract interpretation, compiler checks, vulnerability search, etc.

  Theorem proving: Check properties of infinite state models.

- Behaviour model:

  Model checking: Usually the model is a finite state-transition model (Petri-nets (PNs), finite state automata). Model-checking verifies all possible states on a given system's model.

b) Dynamic verification:

- Symbolic inputs:

  Symbolic Execution: It is the execution with respect to variables (symbols) as inputs.

- Actual inputs:

  Testing: Selected input values are set on system's inputs and their outputs are compared to expected values. Outputs in this case are observed faults, in case of hardware testing or software's mutation testing, and criteria-based, in case of software testing.

Verification methods are often used in combination. For example, symbolic execution may be used to obtain testing patterns, test inputs can be obtained by model-checking as in [54], faults can be used as symbolic inputs, and system behaviour can be observed using model-checking as in [26, 25] (This technique is called fault injection; see also [55]).

The techniques to attain fault tolerance are summarized as follows:

Error detection: is used to identify the presence of an error. It can be a concurrent or a preemptive detection. Concurrent detection takes place during normal service, while preemptive detection takes place while normal service is suspended.

Recovery: transforms a system state that contains errors into a state without them. The behaviour of the system upon recovery is equivalent to the normal behaviour. Techniques range from rollback to a previously saved state without errors, error masking, isolation of faulty components, to reconfiguration using spare components.

In this work, we use a combination of: (i) fault-injection, (ii) theorem proving, and (iii) symbolic execution. We use these methods to obtain an erroneous behaviour of the system which is compared to system's dependability attributes (safety). We explain how these methods combine in Chapter 4.

2.1.3 Fault Modelling

Fault modelling plays an important role in reasoning about the fault-error-failure chain. They are the initial steps to perform the verification of a system, starting in the architectural model to reason about the critical failures, which are (in general) the top-events in FTs.

SysML is a profile for Unified Modelling Language (UML) that provides features to model structure and behaviour of systems. The works reported in [47, 48] define several structural and behavioural views in SysML to model the fault-error-failure chain and fault tolerance. Fault, error, failures, and fault propagation have structural views, which are related to behavioural views to describe fault activation and recovery. These works map SysML to two formal languages—COMPASS Modelling Language (CML) [56] and Communicating Sequential Processes (CSP) [57], respectively—to verify fault tolerance.

In [7] the safety assessment process for civil airborne systems and equipment describes development cycles and methods to "clearly identify each failure condition". The methods that involves failure identification are: (i) SFT, (ii) Dependence Diagram[2] (DD) [58, p. 198], (iii) Markov chain, and (iv) FMEA. The first three are top-down methods, that starts with undesired failure conditions and moves to lower levels to obtain more detailed conditions that causes the top-level event. DDs are an alternate method of representing the data in SFT. FMEA is a bottom-up method that identifies failure modes

[2] Also known as Reliability Block Diagram (RBD).

of a component and determines the effects on the next higher level. We detail SFT in Subsection 3.1.1.

DFTs are an extension of SFTs and models dynamic behaviour of system's faults. Similarly to the relation of SFTs and DDs, the work reported in [59] demonstrates the relation of DFTs to Dynamic Reliability Block Diagrams (DRBDs) [59]. As the models (DFT and DRBD) are equivalent, this work sticks to DFT due to the amount of work already published. We detail DFTs in Subsection 3.1.3.

2.2 Time relation of fault events

The most general case for time relations is to consider that each fault event has a continuous time duration. They are the basis on how fault events discretisation are defined. In Figure 2 we show all possibilities of events relations in a continuous time line (from A to B; the converse relation is similar):

a. A starts and ends before B starts;

b. A starts before and ends after B has started, but before B has ended;

c. A starts before B and ends after B has ended (A contains B);

d. A and B start at the same time, but A ends before B;

e. B starts after A, but they end at the same time;

f. A and B start and end at the same time;

g. A starts before B and ends when B starts.

Although the occurrence of fault events has at least seven possibilities, what really matters when analysing systems is when a fault is *detected*. Considering that fault detection corresponds to the start of a fault event, from Figure 2 we clearly identify which event comes first: A comes before than B, except in the cases (d) and (f), where they start exactly at the same time. If fault events are independent (they are not susceptible to have a common cause) then the probability of they are starting at the same time is very low. In Chapter 4 we abstract events relation in continuous time as an *exclusive before* relation, based on fault *detection* (it is similar—at least implicitly—to what is reported in [19, 21]).

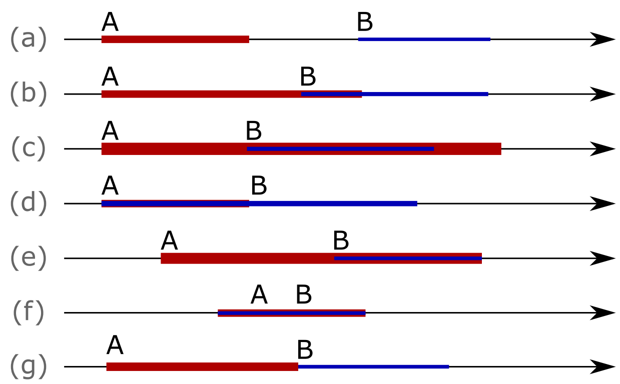


Figure 2 – Relation of two events with duration

3 Analysis and tools

Structure expressions are used to analyse fault trees. In general, a structure expression comes from gates semantics and basic events. Basic events become variables and gates become operators (a gate may become one or more operators). In Section 3.1 we explain SFTs, TFTs, DFTs, and their respective structure expressions.

FBAs and BDDs are the basis to analyse structure expressions. Also, we were inspired by FBA concepts to create our Algebra of Temporal Faults (Chapter 4). We explain BDDs and derived techniques in Section 3.2, and FBAs in Section 3.3.

The use of the Boolean operator *NOT*: (i) can be misleading, generating non–coherent fault trees, or (ii) can be essential in practical use. We discuss such cases in Section 3.4.

To reuse a nominal model to analyse faults we need fault injection. In Section 3.5 we explain how we used Simulink and CSP_M to inject faults and obtain failure expressions from a nominal model.

Finally, in Section 3.6 we present basic usage of Isabelle/HOL and Intelligible semi-automated reasoning (Isar), which were essential to carry out the proofs presented in this thesis.

3.1 Fault Tree Analysis and structure expressions

FTA was introduced in the Fault Tree Handbook [17] with Static Fault Trees. FTA is a deductive method that investigates what are the possible causes of an unwanted event. The method starts with the top-level event as the unwanted event and the combination of lower-level events that can cause it. Events are combined using gates, and each gate has a well defined semantics. It continues until basic (atomic) events are reached. An SFT represents, in a single view—very often considering faults outside of the boundaries of a system—, different states in which a particular failure is active in a system. The most traditional gates are AND and OR, which are equivalent to Boolean operators. These gates are also called coherent gates because they construct coherent trees (see Section 3.4 about the use of NOT gates). The Fault Tree Handbook shows other gates as, for example, the PAND gate, but the FTA with these gates is not well defined. SFT's gates and analysis are detailed in Subsection 3.1.1.

TFTs were created aiming at fully implementing the Fault Tree Handbook. The PAND gate was first defined for SFTs, but its analysis was left open in the handbook. The semantics (and analysis) of TFTs is defined in terms of a denotational semantics based on *sequence values* to express ordering of events, thus tackling PAND's order. We explain TFTs and the sequence values in Subsection 3.1.2.

With component and system design evolution, DFTs were created to tackle dynamic behaviour: fault-tolerance-related components (CSp), functional dependency (FDEP), and analysis of particular order of occurrence of faults (SEQ). SFT's gates are still present as DFT's gates. We explain them and DFT's analysis in Subsection 3.1.3.

The structure of an FT (or the structure of an MCS, explained further) is represented with a formula. The variables represent occurrences of basic events. Unary and binary relation symbols capture the semantics of gates. A formula with these characteristics is called *structure expression* or *structure function* (as the expression depends on the variables). The semantics of a structure expression is that the top-level event occurs if some combination of basic events occur.

The results obtained from the FTAs are shown in the Fault Tree Handbook. We summarize them as:

a) Qualitative

MCSs: Smallest combinations of components failures causing system failure. They are obtained from the reduction of structure expressions to a normal form. For example, in SFTs, structure expressions are reduced to disjunctive normal form (DNF). Each term in a reduced DNF is an MCS.

Importances: Qualitative rankings on contributions to system failure. A single fault causing a catastrophic failure is usually unacceptable. Ranking MCSs is the same as ordering them in ascending order of their size (smaller first).

b) Quantitative

Numerical probabilities: Probabilities of system and MCS' failures. A system failure probability is obtained by assigning probabilities to basic events and then calculating it accordingly to gates' semantics. MCS' failure probability is the calculation of the probability of the occurrence of *all* basic events of a specific MCS.

Importances: Quantitative rankings on contributions to system failure. Ranking MCSs is the same as ordering them in descending order of some unreliability formula (higher first). These formulas used to calculate importance vary. The most common are: (i) system unavailability, and (ii) system failure occurrence rate.

Sensitivity evaluation: Modifying characteristics of components and evaluate their impact. For a particular event in a tree, a higher and a lower failure probability value are assigned. If system's unavailability is not changed, then such an event is not important—the system is not sensitive to such an event.

As stated in [60], there are other uses of FTA. One of great importance is using it to minimize and optimize resources, which has been object of study in HiP-HOPS [61]. Through importance measures, FTA not only identifies what is important but also what is unimportant. This removes components without impacting the overall failure probability, which is related to the quantitative importance and sensitivity evaluation.

In important stages of critical systems, FTA plays an essential role. At least three dependability means can be achieved using FTs:

Removal. An FTA calculates if the probability of failure of a subsystem. If such a probability is higher than a certain maximum reference, such a subsystem should be removed or left to be incorporated in combination with a more reliable component.

Tolerance. An FTA indicates whether a single fault—or fewer combinations than expected—could lead to a catastrophic failure. In this case, a system should have replication, or stages of fault detection and error handling. Also, the probability of failure of the chosen fault tolerance method can be evaluated.

In Subsections 3.1.1 to 3.1.3 we briefly show FTs's symbology and means to analyse FTs. We will detail its structure expressions extraction because they are a common means to perform both qualitative and quantitative analysis.

3.1.1 Static Fault Trees

SFT's gates and structure expressions were used as basis for other kinds of tree, as in TFTs and DFTs. We explain their symbology and semantics in this section.

The Fault Tree Handbook shows traditional symbols for gates and events. Basic events are usually drawn as rectangle (for the text) and a circle below it, as shown in Figure 3, or as a circle with the text of the basic event, as shown in Figure 4. Top-level and intermediary events are drawn as a rectangle (for the text) and a gate below it, as shown in Figures 3 and 4. When an FT becomes too large, transfer in and out symbols can be used. They are usually drawn as triangles with a letter or a number. Figure 4 depicts traditional gates as specified in the Fault Tree Handbook, and Figure 3 shows an FT using the Fault Tree Analyser[1]—a free commercial tool. In this work, to keep a visual identity with other FTs, and to avoid symbols confusion, we use gates symbols as shown in Figure 5.

[1] <http://www.fault-tree-analysis-software.com>, accessed 2/feb/2016

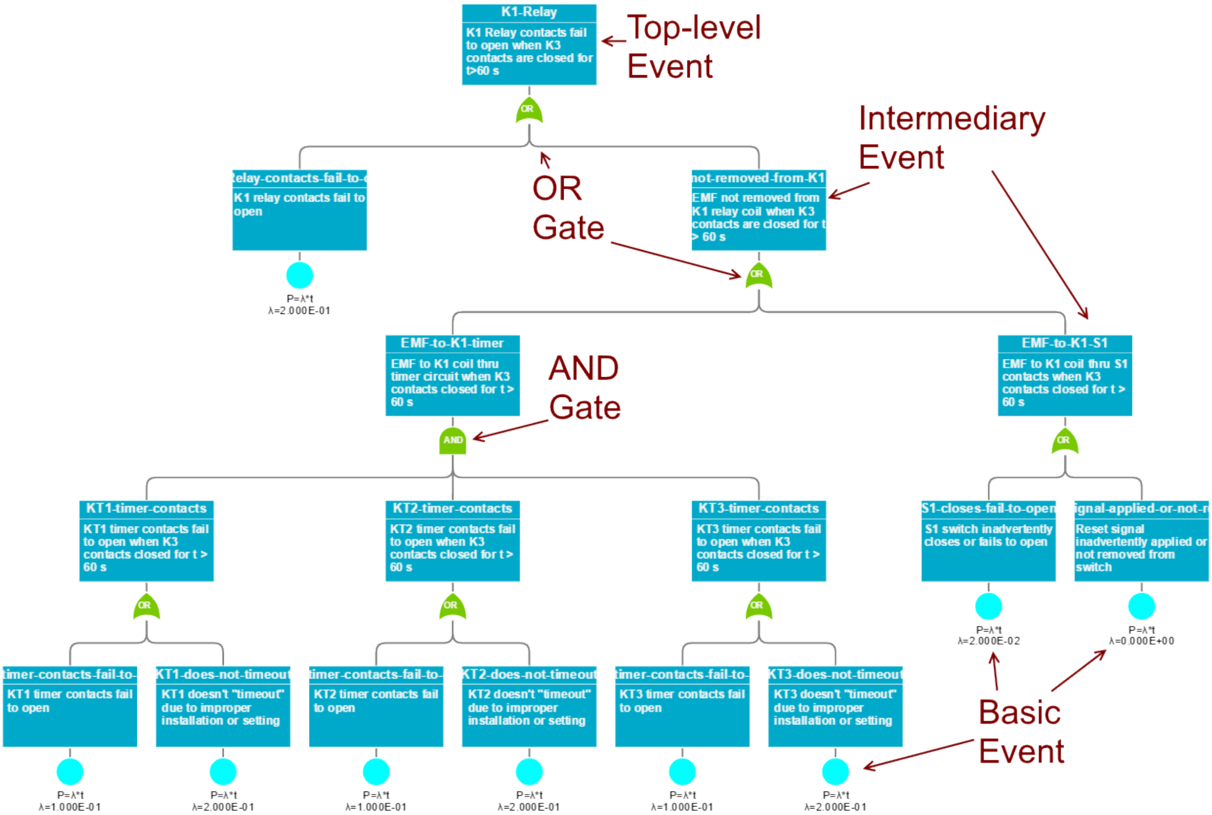


Figure 3 – SFT symbols using a free commercial tool

Structure expressions in FTA are defined in terms of set theory, using symbols for fault events occurrence. If a fault event symbol is in a set, then it means that this fault has occurred. A set is a combination of fault events that causes the occurrence of the top-level event of a tree. A structure expression of a tree is denoted by a set of sets of fault event combinations. The OR gate becomes the union operator between sets and the AND gate, the intersection. For example, if a system contains fault events a, b, and c, fault trees for this system contain at most all these three events. The occurrence of the fault event a is denoted by a set of sets A, which contains the following sets:

a) $\{a\}$: only a occurs;

b) $\{a, b\}$: a and b occur in any order;

c) $\{a, c\}$: a and c occur in any order;

d) $\{a, b, c\}$: all three events occur in any order.

All sets of A contain the fault event a. Similarly, the sets of sets B—that represents the occurrence of b—contains all sets that contain the fault event b (it includes the set $\{a, b, c\}$, for example).

The fault tree in Figure 6 contains only two events and the resulting structure expression for this FT is the expression $A \cap B$ (TOP), where A and B are the sets of sets that contain a and b, respectively. The resulting combinations for TOP are $\{a, b\}$ and

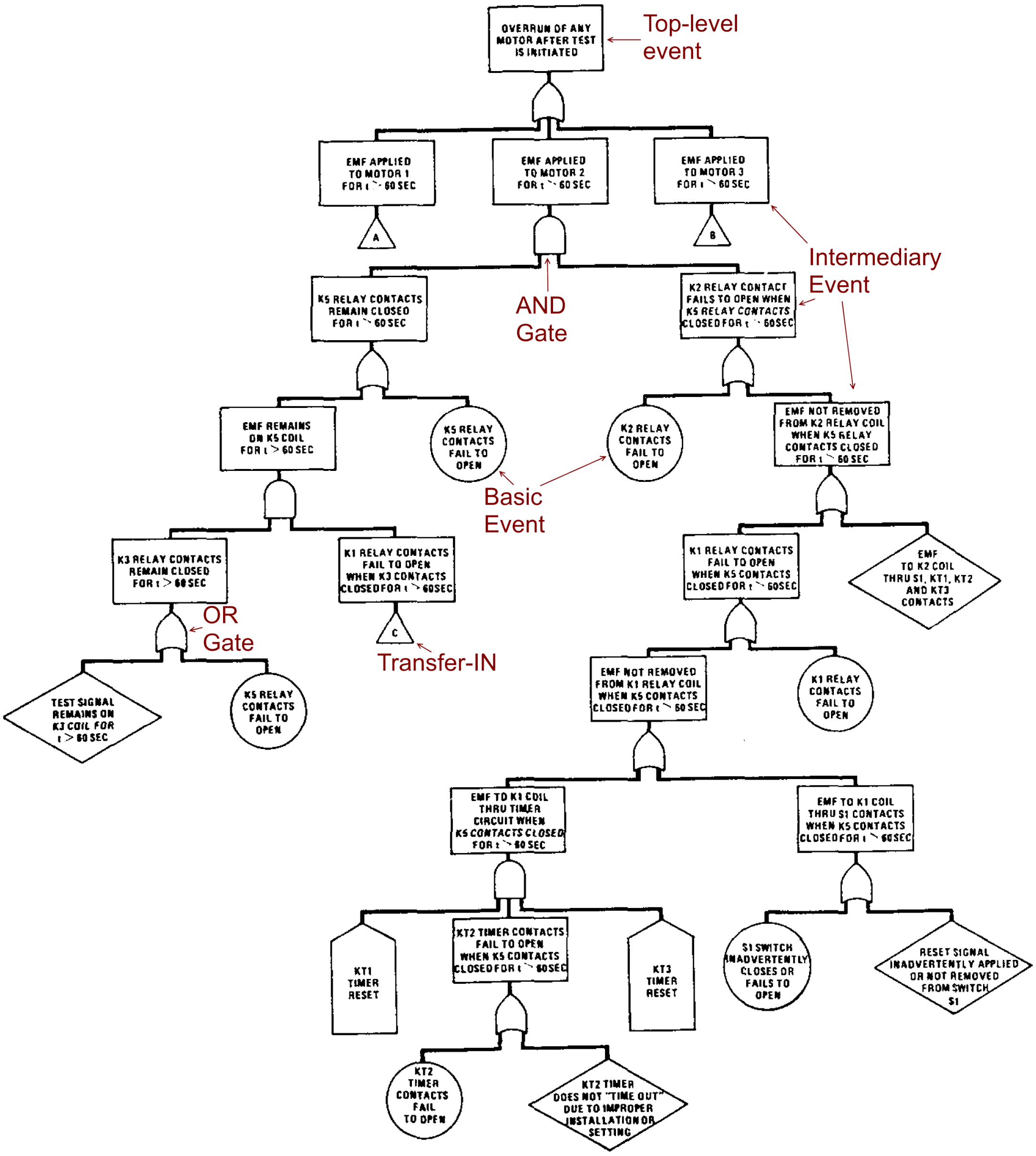
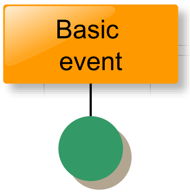


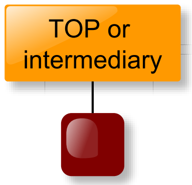
Figure 4 – SFT symbols as in the Fault Tree Handbook

$\{a, b, c\}$ (fault events a and b occur in all possibilities).

After obtaining structure expressions, the next step is to reduce the expressions to a canonical form to obtain the *MCSs*, which are the sets that contain the minimum and sufficient events to activate the top-level failure. Probabilistic analysis is then performed on these events to obtain the overall probability of occurrence of the top-level event. The Fault Tree Handbook shows an algorithm based on Shannon's method to reduce structure expressions to obtain minimal cut sets. The Boolean expression of the tree shown in Figure 6 is $TOP = A \wedge B$. A technique called BDD—which derives from Shannon's method—is explained in Subsection 3.2.2.



(a) Basic event

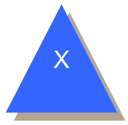


(b) Intermediary event

(c) AND gate

(d) OR gate

(e) NOT gate



(f) Transfer symbol

Figure 5 – SFT gates

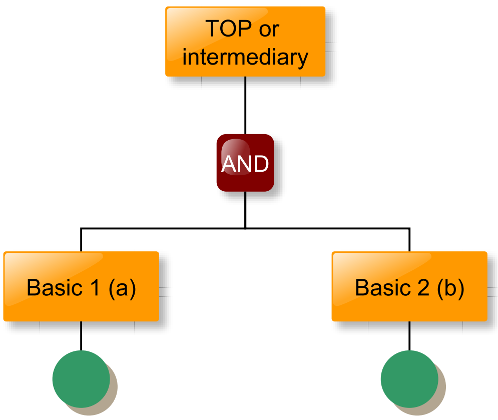


Figure 6 – Very simple example of a fault tree

3.1.2 Temporal Fault Trees

There are at least two versions of TFTs. One is described in [62] and use a more traditional style of temporal logic (a variation of linear temporal logic (LTL)). The other version is called Pandora and is the one we refer to in the following.

TFTs express ordering of events by directly focusing on ordering relationships rather than different states of a system. Basically they extend SFT's PAND gates, allowing analysis of FT with such gates. It is simpler to express than DFT, but lacks the fault-tolerance-related gate of DFTs (which we show in Subsection 3.1.3).

Structure expressions are also present in TFTs [19, 20, 31], through the Pandora methodology. These expressions use the SFT operators OR and AND, and three new operators related to events ordering: priority-AND (PAND), priority-OR (POR), and simultaneous-AND (SAND). The semantics of the PAND in TFTs is similar to the semantics of the PAND described in the Fault Tree Handbook. To avoid ambiguous

Table 1 – TTT of TFT's operators and sequence value numbers

A	B	AND	OR	PAND	POR	SAND
0	0	0	0	0	0	0
0	1	0	1	0	0	0
1	0	0	1	0	1	0
1	1	1	1	0	0	1
1	2	2	1	2	1	0
2	1	2	1	0	0	0

expressions, the semantics in TFTs is stated in terms of natural numbers, using a *sequence value* function. For every possible combination of events ordering, it assigns a sequence value to each fault event. For example, if event A occurs before event B, then the sequence value of A is lower than the sequence value of B, and one formula to express this is $A < B$.

An invariant on sequence values is that there are no gaps for assigned values. For example, if faults A and B occur at the same time and there are only these two events, then they should both be assigned value 1. On the other hand, if A occurs before B, then the assigned values are 1 and 2, respectively. Value zero means that the event is not active on the combination. Similar to Boolean's truth tables, the Pandora methodology defines TTTs. They represent formula values for every combination of events. Table 1 shows the TTT of all TFT operators accordingly to the semantics described in terms of a sequence value function S as follows:

$$S(A \wedge B) = \begin{cases} \max(S(A), S(B)) & \text{if } S(A) > 0 \wedge S(B) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1a}$$

$$S(A \vee B) = \begin{cases} \min(S(A), S(B)) & \text{if } S(A) > 0 \wedge S(B) > 0 \\ \max(S(A), S(B)), & \text{otherwise} \end{cases} \tag{3.1b}$$

$$S(A < B) = \begin{cases} S(B) & \text{if } S(A) > 0 \wedge S(B) > 0 \wedge S(A) < S(B) \\ 0 & \text{otherwise} \end{cases} \tag{3.1c}$$

$$S(A \mid B) = \begin{cases} S(A) & \text{if } S(A) < S(B) \vee S(B) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1d}$$

$$S(A \,\&\, B) = \begin{cases} S(A) & \text{if } S(A) > 0 \wedge S(B) > 0 \wedge S(A) = S(B) \\ 0 & \text{otherwise} \end{cases} \tag{3.1e}$$

Figure 7 shows TFT-specific symbols used in this work. To illustrate TFTs, for the formula $(A < C) \vee (A \wedge B)$, we show: (i) the TFT in Figure 8, and (ii) its corresponding TTT in Table 2 (the column '#' indicates the MCSeq number).

From structure expressions in order-sensitive FTs (TFT and DFT), MCSeqs are obtained. Several approaches represent MCSeq's ordering differently. For the best of our

(a) PAND gate **(b)** POR gate **(c)** SAND gate

Figure 7 – TFT-specific gates

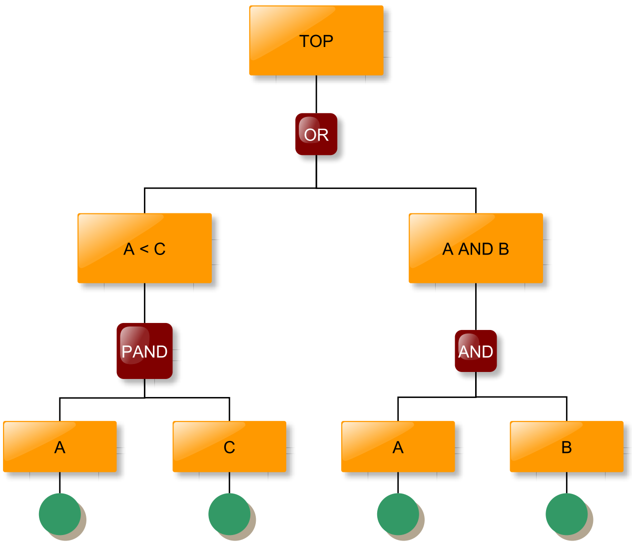


Figure 8 – TFT small example

knowledge they are introduced in the work [63] similarly as MCS, allowing set elements with arrows ("→") to represent order.

For TFTs, in the work [20] MCSeqs are represented as a DNF using AND and the temporal operators (PAND, POR, and SAND) as doublets (a single temporal relation)—which are the minimal terms—or prime implicants—in the DNF. In a doublet, the expression is a product (of AND) of temporal operators, and each temporal operator contains *exactly* two events. The conversion to doublets uses the temporal laws as shown in [20]. For example, the expression $(X \,\&\, Y) \mid Z$ is a temporal relation (POR) of a temporal relation (SAND). To extract MCSeqs it needs to be converted to $[X \,\&\, Y] \wedge [X \mid Z] \wedge [Y \mid Z]$ (the square brackets is the doublets notation and it is the direct application of the *Temporal Distributive Law* [20, p. 120]).

The normal form for TFT is similar to SFT: it is a DNF with temporal operators (PAND, POR, SAND) in the minimal terms. The reduction of TFT structure expressions is achieved using TDT. In a TDT, if all children of a tree node are true, then the node is also true. Conversely, if a node is true, then all its children are also true. An issue with TDTs is that they grow exponentially. Accordingly to the work [31], it is already infeasible to deal with seven fault events in TFTs. Although there is a solution, it is based

Table 2 – TTT of a simple example

#	A	B	C	$A < C$	$A \wedge B$	$(A < C) \vee (A \wedge B)$
01	0	0	0	0	0	**0**
02	0	0	1	0	0	**0**
03	0	1	0	0	0	**0**
04	0	1	1	0	0	**0**
05	0	1	2	0	0	**0**
06	0	2	1	0	0	**0**
07	1	0	0	0	0	**0**
08	1	0	1	0	0	**0**
09	1	0	2	2	0	**2**
10	1	1	0	0	1	**1**
11	1	1	1	0	1	**1**
12	1	1	2	2	1	**1**
13	1	2	1	0	2	**2**
14	1	2	2	2	2	**2**
15	1	2	3	3	2	**2**
16	1	3	2	2	3	**2**
17	2	0	1	0	0	**0**
18	2	1	0	0	2	**2**
19	2	1	1	0	2	**2**
20	2	1	2	0	2	**2**
21	2	1	3	3	2	**2**
22	2	2	1	0	2	**2**
23	2	3	1	0	3	**3**
24	3	1	2	0	3	**3**
25	3	2	1	0	3	**3**

on a mixed application of TDTs, modularisation of independent subtrees, and algebraic laws [19]. We show TDTs in Subsection 3.2.3. Some of these algebraic laws are:

$$(X < Y) \vee (X \,\&\, Y) \vee (Y < X) = X \wedge Y \qquad \text{Conjunctive Completion Law} \tag{3.2a}$$

$$(X \,|\, Y) \vee (X \,\&\, Y) \vee (Y \,|\, X) = X \vee Y \qquad \text{Disjunctive Completion Law} \tag{3.2b}$$

$$(X \,|\, Y) \vee (X \,\&\, Y) \vee (Y < X) = X \qquad \text{Reductive Completion Law 1st} \tag{3.2c}$$

$$(X \wedge Y) \vee (X \,|\, Y) = X \qquad \text{Reductive Completion Law 2nd} \tag{3.2d}$$

3.1.3 Dynamic Fault Trees

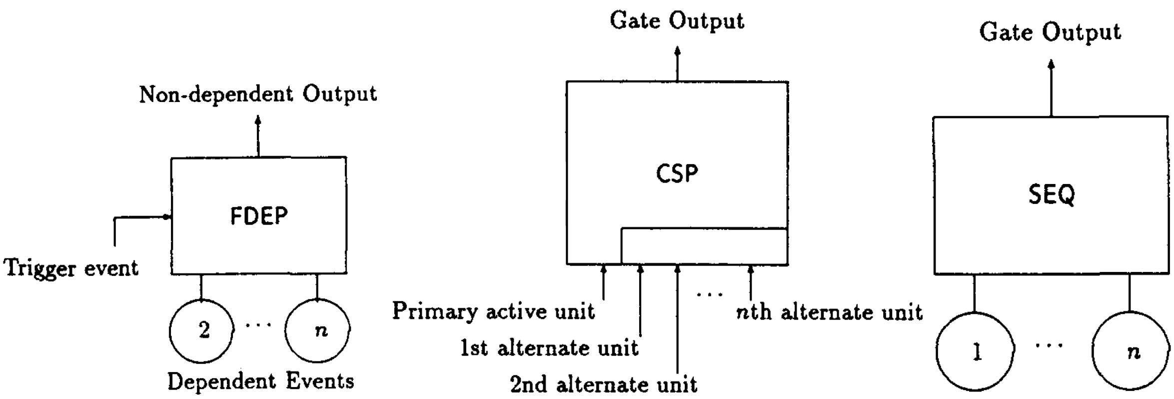
Dynamic Fault Trees were designed with the goal of analysing complex systems with dynamic redundancy management and complex fault and recovery mechanisms [1]. The idea was to create easy-to-use and less error-prone modelling tools than using DTMCs—or simply *Markov chains*—directly. So, since the very beginning, DFTs were intended to be evaluated using Markov chains. Figure 9 depicts the original gate symbols as shown in [1, 2]. In this work, we use gates symbols as depicted in Figure 10. The informal semantics of them are:

FDEP: When the trigger event occurs, the dependent events are forced to occur. Timing in this gate between trigger event and dependent events occurrences can be at the same time (like in TFT's SAND gate), or in a small amount of time, thus implying an order of occurrence, depending on the kind of dependency.

CSp: It is a specific gate to handle spare components. It is important to note that connected inputs are not components—they are fault events of connected components. If the i*th* input is already active (fault has occurred), then it is expected that the input $(i+1)th$ is not, following the specified order. The output becomes true after all connected inputs become true. A spare event can be connected to more than one CSp gate, representing the spare unit connection to one or more components.

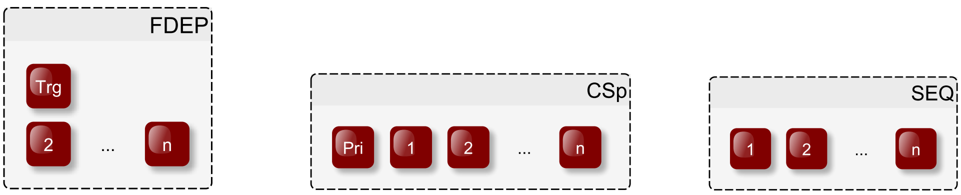
PAND: The same as in TFT: when the connected input events occur in the specified order, it outputs true.

SEQ: The connected events *shall* occur in the specified order. It is different from the PAND gate, because the latter *detects* the specified order. The usage of this gate is usually associated with FDEP.



(a) FDEP gate **(b)** CSp gate **(c)** SEQ gate

Figure 9 – DFTs's original gates symbols



(a) FDEP gate **(b)** CSp gate **(c)** SEQ gate

Figure 10 – DFTs's gates symbols

There are several means to analyse DFTs qualitative and quantitatively. The works reported in [3, 64, 21, 22] use structure expressions to perform both qualitative and quantitative analysis, and the work reported in [22] summarizes other approaches. We increment their summary (Table 3) and categorize them as:

a) Finding MCSeqs (qualitative analysis) is obtained by replacing DFT gates with SFT gates, using the text as its logical constraints. MCSs in the SFT are

Table 3 – DFT conversion to calculate probability of top-level event

Conversion	Calculation	Explained in
Automaton-like structure	CTMC	[33]
Bayesian network (BN) [65]	Inference algorithm (model-specific)	[34]
Stochastic well-formed net (SWN) [66] (a kind of coloured Petri-net (CPN) [67])	CTMC	[68]
SBDD (a modified version of BDD)	model-specific	[41, 42]

expanded using timing constraints from the texts into MCSeq. In this case, the behaviour of spare events cannot be correctly taken into account;

b) Quantitative analysis consists in converting a DFT to a well-defined formalism to calculate the probability of its top-level event. Table 3 shows the conversion, the calculation, and where the method is explained.

In [3, 64, 21] fault events occur in a specific time and are instantaneous (similar to detected faults), stated through a "date-of-occurrence" function. As the "date-of-occurrence" function is stated in continuous time, the probability of two events occurring at the same time is negligible. In fact, useful information is obtained from the possibilities of relation in time of the occurrence of the events. DFT gates' algebraic model is summarized in Table 4. Structure expressions are written with an algebra that has operators OR and AND, and three new operators to express events ordering: (i) non-inclusive-before (NIBefore), (ii) simultaneous (SIMLT), and (iii) inclusive-before (IBefore). The NIBefore and the SIMLT operators are similar to TFT's POR and SAND operators, respectively. The IBefore is a composition of NIBefore and SIMLT operators. Table 5 summarizes the date-of-occurrence function for all operators. An infinite value means the event never occurs.

MCSeqs are extracted from canonical form of structure expressions written in a DNF. Minimal terms are products of variables and NIBefore operators (the other operators can be written as combinations of NIBefore). The reduction of DFT structure expressions uses algebraic laws as, for example:

$$(a \lhd b) \vee (a \bigtriangleup b) \vee (b \lhd a) = a \vee b \tag{3.3a}$$

$$(a \wedge (b \lhd a)) \vee (a \bigtriangleup b) \vee (b \wedge (a \lhd b)) = a \wedge b \tag{3.3b}$$

$$(a \trianglelefteq b) \wedge (b \trianglelefteq a) = a \bigtriangleup b \tag{3.3c}$$

Figure 11 shows an example of DFT extracted from [22]. It is a cardiac assist system (HCAS), which is divided in four modules: trigger, CPU unit, motor section, and

Table 4 – Algebraic model of DFT gates with inputs A and B

Gate	Algebraic model of gate's output	Note
FDEP	$A_T = T \vee A$ and $B_T = T \vee B$	A_T and B_T replace A and B on the resulting expression
CSp	$(B_a \wedge (A \lhd B_a)) \vee (A \wedge (B_d \lhd A))$	A is the active input, and B is the spare. Subscripts a and d represent component's state—*active* and *dormant*, respectively, which are used on the failure distribution formulas
PAND	$B \wedge (A \trianglelefteq B)$	No distinction of active or dormant states.

Table 5 – Date-of-occurrence function for operators defined in [3]

Operator	Expression	Value if $\mathbf{d}(\mathbf{a}) < \mathbf{d}(\mathbf{b})$	Value if $\mathbf{d}(\mathbf{a}) = \mathbf{d}(\mathbf{b})$	Value if $\mathbf{d}(\mathbf{a}) > \mathbf{d}(\mathbf{b})$
OR	$d(a \vee b)$	$d(a)$	$d(a)$	$d(b)$
AND	$d(a \wedge b)$	$d(b)$	$d(a)$	$d(a)$
NIBefore	$d(a \lhd b)$	$d(a)$	$+\infty$	$+\infty$
SIMLT	$d(a \triangle b)$	$+\infty$	$d(a)$	$+\infty$
IBefore	$d(a \trianglelefteq b)$	$d(a)$	$d(a)$	$+\infty$

pumps. The trigger is divided in two components, CS and SS. The failure of any CS or SS, triggers a CPU unit failure. The primary CPU (P) has a warm[2] spare (B). The motor module fails if both M and MC fail. In order for the pumps unit to fail, all three pumps need to fail, and the left-hand side spare gate needs to fail before (or at the same time as) the right-hand side spare gate (PAND gate[3]). The top-level event structure expression is:

$$
\begin{aligned}
SYSTEM = & CS \vee SS \vee (M \wedge MC) \vee \\
& (P \wedge (B_d \lhd P)) \vee (B_a \wedge (P \lhd B_a)) \vee \\
& (BP_a \wedge (P2 \lhd P1) \wedge (P1 \lhd BP_a)) \vee (P2 \wedge (P1 \lhd BP_a) \wedge (BP_a \lhd P2))
\end{aligned} \tag{3.4}
$$

3.2 Structure expressions analysis

In this section we detail the non-state-based methods to analyse structure expressions. Another common approach to analyse an FT is to perform structure expression analysis based on algebraic laws. Boolean laws are well-known and are used for SFTs, temporal laws [20, 31] for TFTs, and the works reported in [3, 21] show laws for DFTs. An issue with algebraic laws is that, in some cases, the expression needs to be expanded

2 Warm spare gates only differ from CSp on the activation time.

3 Although the original example uses a PAND gate, accordingly to the informal description, a SEQ gate would fit better.

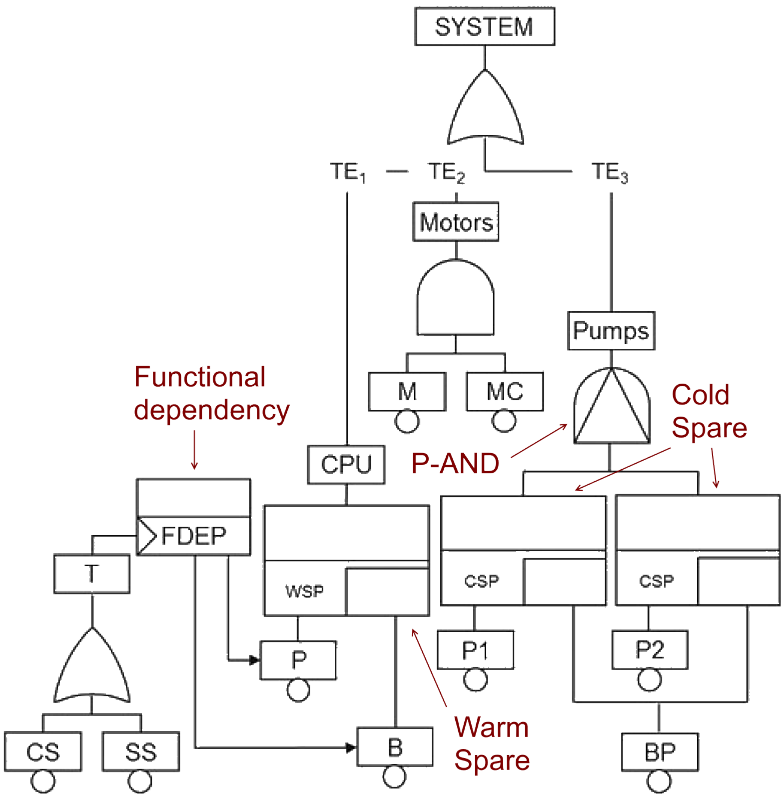


Figure 11 – DFT example

before it gets reduced, so reduction automation is not trivial without a theorem prover. For example, the following TFT's structure expression needs to be expanded [31] before it gets reduced:

$$
\begin{aligned}
& (X \wedge Y) \vee ((X < Y) \wedge Z) & \\
& (X < Y) \vee (X \,\&\, Y) \vee (Y < X) \vee ((X < Y) \wedge Z) & \text{by Eq. (3.2a)} \\
& (X < Y) \vee (X \,\&\, Y) \vee (Y < X) & \\
& X \wedge Y & \text{by Eq. (3.2a)}
\end{aligned}
$$

A denotational semantics to Boolean expressions—and consequently to SFT—is given by FBAs (Section 3.3).

There are several works with state-based analysis for FTs (SFT, TFT, and DFT). We show some of them in Subsection 3.2.1.

3.2.1 State-based and temporal logic analysis

The work reported in [69] shows a formal approach to analyse SFT using Interval Temporal Logic (ITL) [70]. Instead of tackling basic events ordering (as in PAND), it considers a causal relation over a gate, as for example, a relation of a basic event and a higher-level intermediary event.

For TFTs, the works reported in [71, 72] show an inverse solution. They map Finite State Machines (FSMs) to Pandora logic, then verify system properties. They show that such a mapping simplifies expressions reduction, thus improving performance on the analysis.

Although there is formal modelling for DFTs, they do not implement a direct modelling of the DFT itself. Instead, most of the works propose a formal modelling of a state-based approach. The work reported in [33] shows a formal model of Markov chains in the Z Notation (Z) [73] and each DFT element (basic events and gates). The analysis uses a quantifier on states of the resulting Markov chain automaton. The work reported in [74] shows a methodology to perform a modular analysis of DFTs based on BDD and Markov chain. As DFT extends SFT, it identifies subtrees that are purely SFT and uses BDD, otherwise. It performs Markov chain analysis. Still on the state-based approaches, the work reported in [75] maps DFTs to high-level Petri-net (HLPN) [76] to analyse false alarms.

In the following we show specific methods that are designed to reduce structure expressions. In essence, the methods are very similar. Structure expressions for SFTs can be reduced using BDDs (Subsection 3.2.2), TFTs can be reduced using TDTs (Subsection 3.2.3), MCSeqs of DFTs can be obtained using Zero-suppresed Binary Decision Diagram (ZBDD) [63] (Subsection 3.2.4), and the works reported in [41, 42] show the analysis of standby systems (CSp gates) using SBDDs (Subsection 3.2.5).

3.2.2 Binary Decision Diagrams

BDDs are directed acyclic graphs that represent a Boolean expression. They are still referred to as BDD, but the more spread version is the Reduced Ordered Binary Decision Diagram (ROBDD) [77], which is an optimisation. There are two ways to generate a BDD for an expression: (i) derive a diagram from the truth-table, or (ii) expand the paths based on Shannon's method (described in the Fault Tree Handbook).

To demonstrate the expressiveness of a BDD, Figure 12 shows a diagram for a truth table with three variables (Table 6). In a node, when a path is chosen, the variable of the node assumes the edge value. For example, the top-level node variable of Figure 12 is A. Following the right-hand side of the node, all leaf nodes correspond to the lines of the truth table that A has "0" values (the first four lines). The symbol nodes are replaced by the values assumed by a specific formula.

Following Shannon's method, we choose a variable and build the lower level BDD assuming the edge value for the chosen variable. In the remainder of the path, the variable's value is unchanged. For example, the expression $A \vee (\neg B \wedge C)$ has value "0" in the lines a and c, and value "1" in the other lines. By choosing the variable A first, then B and

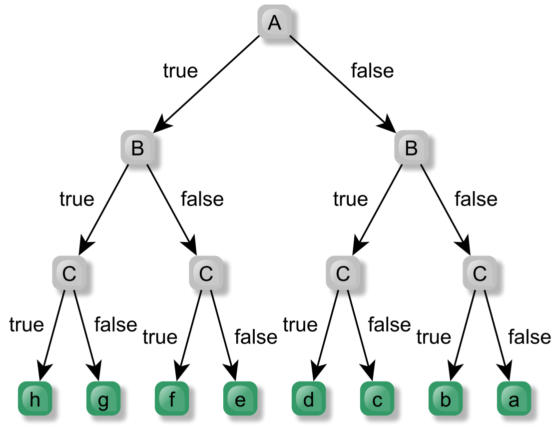


Figure 12 – A diagram for a truth table

Table 6 – Truth table for a formula outputs with three variables (A, B, and C)

A	B	C	Formula
0	0	0	a
0	0	1	b
0	1	0	c
0	1	1	d
1	0	0	e
1	0	1	f
1	1	0	g
1	1	1	h

C, the resulting BDD with the binary values nodes (called sink nodes "false" and "true") for this formula is depicted in Figure 13. Starting from the top-level node A, the formula expressed by the BDD is true if A assumes value true. If A is false, and B is false, the expression is only true if C is true.

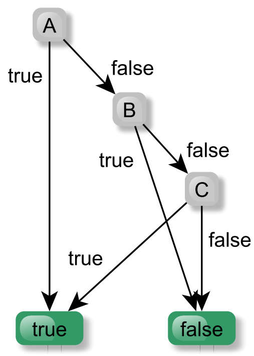


Figure 13 – A BDD for the expression $A \vee (\neg B \wedge C)$

Figure 13 is an ROBDD. To be considered an ROBDD, the BDD must meet the following constraints [77]:

a) the variables are assigned a constant ordering;

b) every path to sink nodes visit the input variables in ascending order;

c) each node represents a distinct logic function.

The size of an ROBDD, for a given expression, depends on the chosen variables ordering. The work reported in [78] shows initial findings on variable ordering, and the work reported in [79] shows heuristics to improve the performance for optimal order search.

For SFTs the evaluation of a BDD is the calculation of the probability of the paths ending in *true*. For example, the probability of the expression in Figure 13 is obtained from the formula: $\Pr(A \vee (\neg A \wedge \neg B \wedge C))$. Note that the formula in the probability calculation is different from the formula that originated the diagram.

3.2.3 Dependency tree

Dependency tree (TDT) [31] is a hierarchical acyclic graph of expressions that shows all possible cut sequences for any given set of events. It is a graphical view of a TTT. At the top of a TDT are the variables, that is, the single events that occur in an expression. On the lower levels are the increasingly complex expressions. Each node represents an MCSeq. Figure 14 shows a TDT with all nodes for variables X and Y.

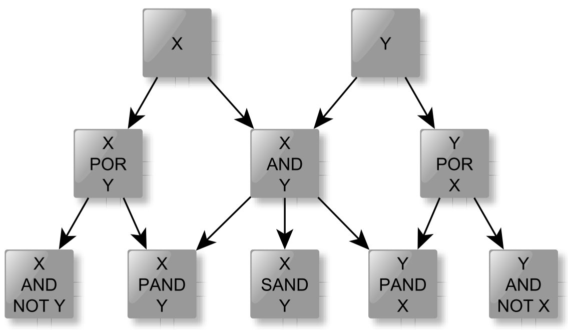


Figure 14 – TDT for variables X and Y

The reduction of a structure expression is given by the activation (true values) of nodes. If a node is active (true), then all child nodes are also active; the converse is also true: if all node's children are active, then it is also active. The reduced expression is given by the DNF created with the expressions of higher active level nodes. To reduce the formula $(X \wedge Y) \vee ((X < Y) \wedge Z)$, given on the beginning of this section, we create the TDT depicted in Figure 15. Nodes marked as "1" are those MCSeqs given directly by the formula. Nodes marked as "2" are child nodes of the "1"'s nodes, and so forth. The node of the expression $((X < Y) \wedge Z)$ is a grandchild of $X \wedge Y$ and thus it is not necessary. The final expression is obtained by the active higher level node, which is $X \wedge Y$.

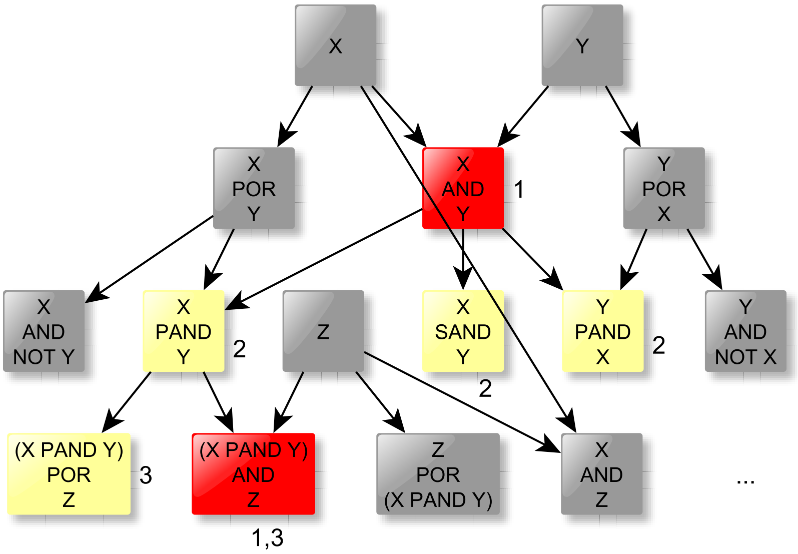


Figure 15 – TDT for the formula $(X \wedge Y) \vee ((X < Y) \wedge Z)$

3.2.4 Zero-suppresed Binary Decision Diagrams

The work reported in [63] proposes Zero-suppresed Binary Decision Diagram, which is a variant of BDD, and uses set manipulations (union, intersection, difference, and product) to obtain MCSeqs of DFTs.

To reduce a BDD to a ZBDD, the nodes that have the "true" ('1') path pointing to the "false" ('0') sink node are removed, and the parent node is connected directly to the "false" subgraph of the removed node. Figure 16 shows an example of ZBDD for the combination set $\{a, b\}$, as shown in [63]. The idea of the reduction is to remove irrelevant variables and nodes. The irrelevant variables are set to "false". The method obtains the MCSeqs by navigating the paths to sink node "true".

Although the work reported in [63] shows ZBDD, the final solution does not use them directly. Instead, it defines a hierarchical manipulation of DFT to obtain the MCSeqs when traversing the a DFT. The order-related operators in a DFT are replaced by a new event, which takes ordering into account. The idea is to transform the DFT into an SFT, in a very similar way as the one shown in [41]. Finally, the MCSeqs are obtained using set manipulation with elements that are basic events alone or order-related operators. These order-related operators are event-to-event only, so they cannot be combined with other sets.

The use of sets in [63] is very related to our ATF. We use sets of sequences to define the ATF, but keep the analysis with set operators. In ATF we do not create new events that represent an order-related operator. Our order-related operator has a set-based semantics that can be combined with other non-order-related (Boolean) operators.

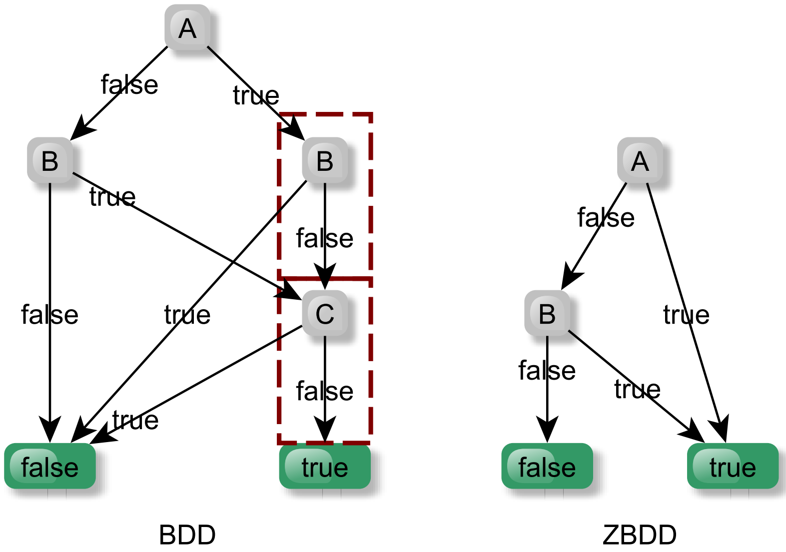


Figure 16 – ZBDD example of combination set $\{a, b\}$

3.2.5 Sequential Binary Decision Diagrams

SBDD is an extension of BDD to tackle ordering of events in DFTs for CSp and WSp gates. Ordering of events in CSp gates [42] is slightly different compared to WSp [41]. A backup system in CSp gets activated slower than in WSp, which implies that there are less failure possibilities in CSp, but its the readiness is lower than in WSp. SBDD adds a new node kind that contains a binary operation of fault events, which allows to express the ordering of events. One kind of operation expresses the slowness of the relation of the fault events of CSp, and another one expresses the readiness of the WSp. The latter semantics is similar to the semantics of PAND and IBefore (combined with AND) gates.

SBDD creation has two steps: (i) CSp or WSp DFT conversion, and (ii) SBDD model generation. In (i), it is a DFT-to-DFT conversion. CSp and WSp gates are converted to a new, but equivalent DFT without CSp and WSp gates, where the operations appear as basic events and are combined using other gates. In (ii), the SBDD model is created. The model may contain nodes that are contradictory as, for example, nodes that assumes that an event A is false and a binary operation that contains A is true. This step ends when all contradictions are removed. The evaluation is similar to BDD's: each path ending in true is a minimal term in the DNF that may contain one of the binary operations and individual events.

3.3 Free Boolean Algebras

Another means to analyse SFTs is to use an FBA to perform set-theoretical operations (intersection, difference, etc.) to reduce expressions. In this section we briefly present the FBA theory and its elements.

Instead of using an axiomatic definition of a Boolean algebra, we follow its set-theoretical definition, as shown in [80, pp. 254–258] and [16, pp. 8–11]. This definition is more elegant because it represents a Boolean algebra as an algebra of sets and does not rely on axioms (which can be misleading, case there is an unfounded axiom).

Definition 3.1 (Boolean Algebra)**.** *A Boolean algebra is defined as a triple $\langle B, \cap, - \rangle$, where B is a set with at least two elements, $\cap$ is the intersection (also called meet or* infimum*) and $-$ is the complement (also called negation).*

The other Boolean elements (union, $\perp$, and $\top$) are derived from the previous two operators:

$\cup$ is the union (also called join or *supremum*): $A \cup B = -(-A \cap -B)$

$\perp$ is the bottom (also called zero): $\perp = A \cap -A$

$\top$ is the top (also called unit): $\top = -\perp$

A Free Boolean Algebra is defined from a set E of generators. A generator can be represented as a proposition in statement calculus [80, p. 274]. For example, "valve A is stuck closed" and "motor X is malfunctioning" are valid statements. A Free Boolean Algebra is constructed from $\mathbb{P}(E)$, where $\mathbb{P}$ is the power set. Note that if E has n symbols, $\mathbb{P}(E)$ has 2^n elements, called *atoms* of a finite Boolean algebra. For the two statements above, the atoms are:

a) "Valve A is stuck closed" and "motor X is malfunctioning"

b) "Valve A is stuck closed" and "motor X is *not* malfunctioning"

c) "Valve A is *not* stuck closed" and "motor X is malfunctioning"

d) "Valve A is *not* stuck closed" and "motor X is *not* malfunctioning"

Such a Boolean algebra has 2^{2^n} formulas [16, p. 261]. For example, if $E = \{a, b\}$, then $\mathbb{P}(E) = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$, hence the Boolean algebra generated by E contains sixteen (2^{2^2}) formulas: $\{\}$, $\{\{\}\}$, $\{\{\}, \{a\}\}$, $\{\{\}, \{b\}\}, \ldots$, $\{\{a\}, \{a, b\}\}, \ldots$, $\{\{b\}, \{a, b\}\}, \ldots$, $\{\{\}, \{a\}, \{b\}, \{a, b\}\}$.

The Boolean algebra B can be inductively defined using some constructs.

Definition 3.2 (Inductive Free Boolean Algebra)**.** *Let s be a statement, then:*

$$\textbf{var}\, s = \{X | s \in X\} \implies \textbf{var}\, s \in B \qquad \textit{(variable)} \tag{3.5a}$$

$$X \in B \implies -X \in B \qquad \textit{(complement)} \tag{3.5b}$$

$$X \in B \wedge Y \in B \implies X \cap Y \in B \qquad \textit{(intersection)} \tag{3.5c}$$

The characterisation of a "free" Boolean algebra comes from that, for some valuation function a, some of the formulas evaluates to "1". Given a function $p : B \times \{0,1\} \to B$, such that:

$$p(i,j) = \begin{cases} i & j = 1 \\ -i & j = 0 \end{cases} \tag{3.6}$$

Lemma 3.1 (Free generators (valuation))**.** *Let F be a finite set, such that $F \subseteq E$, and $a : F \to \{0,1\}$, a necessary and sufficient condition for a set E of generators of a Boolean algebra B to be free is then:*

$$\bigwedge_{i \in F} p(i, a(i)) \neq 0 \tag{3.7}$$

Essentially, Lemma 3.1 states that there is no relation between generators, such as $a = -b$.

Lemma 3.2 (Free generators (algebraic))**.** *Let i and j be statements, such that $i, j \in E$, hence from Definition 3.2 and Lemma 3.1 it is necessary and sufficient that:*

$$\textbf{var}\, i = \textbf{var}\, j \iff i = j \tag{3.8a}$$

$$\textbf{var}\, i \neq -\textbf{var}\, j \tag{3.8b}$$

$$-\textbf{var}\, i \neq \textbf{var}\, j \tag{3.8c}$$

3.4 Using the NOT operator in Static Fault Trees

Although the Fault Tree Handbook introduces several gates, the vast majority of SFT analyses would fit in FTs with only AND and OR gates (coherent FTs). Qualitative analysis requires the reduction of the structure expression of FTs and, when NOT gates are present (non-coherent FTs), such a reduction can cause the interpretation of failure expression to be misled [9, 11, 10, 12, 13]. The work reported in [11] shows three funny examples of this kind of problem, and the works reported in [9, 11, 12] show how to solve it using BDDs. In the following we show: (i) the second example presented in [11], which highlights the problem when using NOT gates (Subsection 3.4.1), and (ii) the second example presented in [9], which defends the usefulness of NOT gates in a multitasking system (Subsection 3.4.2).

Negated events in a non-coherent analysis are in fact the working state of a component. The failure probability contribution of a negated basic event is close to 1. The problem with non-coherent FTs is that its analysis can cause impossible situations. The general formula to identify coherency is given in [9, 12] in terms of a structure function.

Definition 3.3 (FT Coherency)**.** *Let $\Phi(x) : B^n \to B^1$ be a binary function of a vector of binary variables, such that $x = [x_1, x_2, \ldots, x_n]$, representing the states of n system's components.*

A binary structure function $\Phi(x)$ is coherent if all the following hold:

a) $\Phi(x)$ *is monotonic (non-decreasing) in each variable;*

b) Each x_i is relevant, which means that $\Phi(x)[x_i/1] \neq \Phi(x)[x_i/0]$ for some vector x.

where $B^1 = \{0, 1\}$, $B^n = B^{n-1} \times B^1$, $x_i = 1$ implies that component i failed, and $\Phi(x) = 1$ implies the system failed. For $y = [y_1, y_2, \ldots, y_n]$, monotonicity of Φ means that for *all* i, $x_i \geq y_i$ ($y_i = 1 \implies x_i = 1$), and for *some* i, $x_i > y_i$ ($x_i = 1$ and $y_i = 0$). Variable replacement ($[a/b]$) is as usual: $x[x_i/a] = [x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n]$

3.4.1 Non-coherent fault tree misleads

In this section we illustrate—with the second example detailed in [11]—how non-coherent FT misleads.

A college student who wants to visit her mother in another city has two options: wake up early (x_3) and take a ride with a friend (x_1), or wake up late ($\neg x_3$) and take the metro (x_2). The top-event failure is "fail to visit mother" with expression $S = (x_1 \wedge x_3) \vee (x_2 \wedge \neg x_3)$. Its fault tree is depicted in Figure 17. It is clear that the structure function is non-coherent in x_3 accordingly to Definition 3.3: $\Phi(1, 1, x_3)[x_3/1] = \Phi(1, 1, x_3)[x_3/0]$.

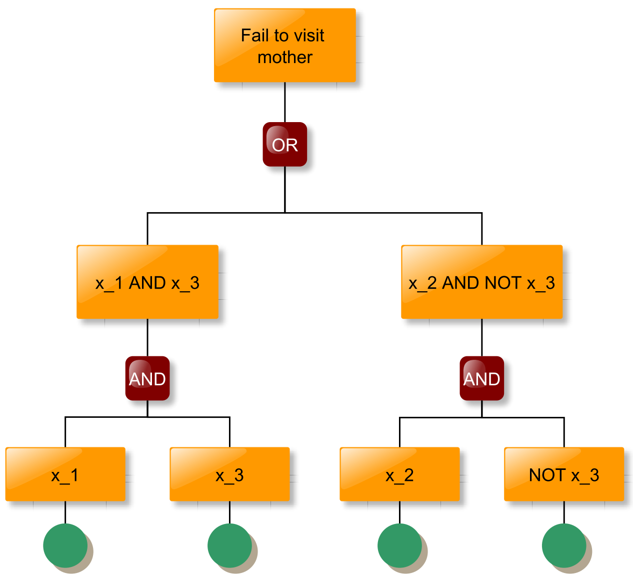


Figure 17 – Non-coherent FT college student's example

The problem with this tree is the interpretation of the qualitative results. One of the possibilities in this scenario is that the college student would take a ride AND take the metro ($x_1 \wedge x_2$). Quantitatively, the analysis of the probabilities shows that this result is not negligible, but its interpretation is impossible.

3.4.2 Usefulness of NOT gates in FTA

In this section we show the second example detailed in [9].

The gas detection system depicted in Figure 18 has two sensors D_1 and D_2 which are used to detect a leakage in a confined space. When a leakage is detected, these sensors send a signal to the logic control unit LU, which performs three tasks:

a) shuts-down the main system (process isolation) by de-energizing relay R_1;

b) informs the operator of the leakage by lamp and siren L;

c) deactivates all possible ignition sources, which is the interruption of power supply by de-energizing relay R_2.

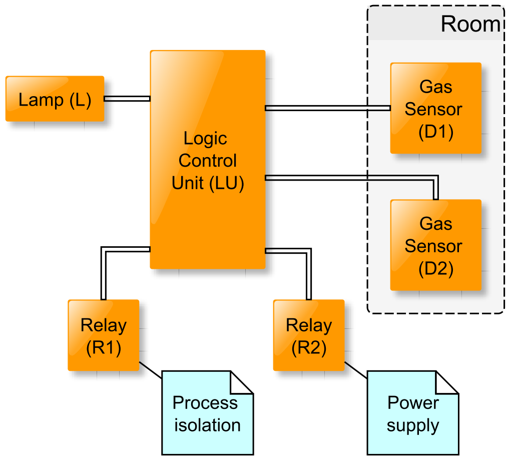


Figure 18 – Gas detection system

The system is in fail state if it does not perform one of these three tasks. The fault tree that represents this generic failure is depicted in Figure 19. G_1, G_2, and G_3 are subtrees that represents the three tasks “Operator not informed”, “Process shut-down fails”, and “Power supply not isolated”, respectively. All three tasks will fail if their respective main component fails (L, R_1, and R_2) or there is no signal from LU (LU fails or both D_1 and D_2 fail). The structure expressions for the subtrees are:

$$G_1 = L \lor LU \lor (D_1 \land D_2)$$
$$G_2 = R_1 \lor LU \lor (D_1 \land D_2)$$
$$G_3 = R_2 \lor LU \lor (D_1 \land D_2)$$

Analysing in more detail, there are different degrees of system failure. There are eight outcomes (given the three tasks) and the most critical one is when both process shut-down (G_2) and power supply isolation (G_3) fail keeping energized upon a leakage, and the operator is not informed (G_1), but the operator information system is working (lamp and siren are off, but they are operational). The coherent FT of this outcome is depicted

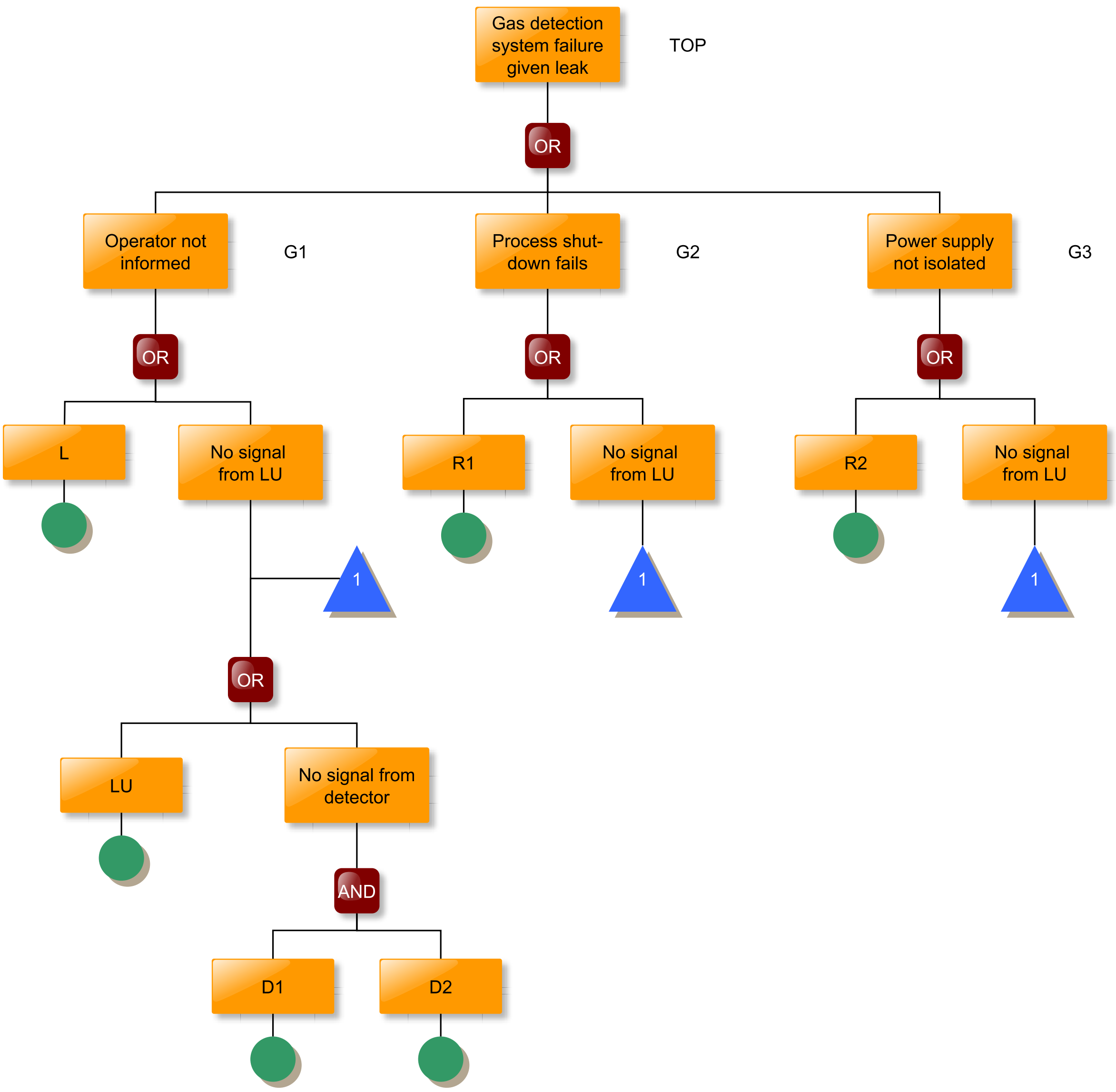


Figure 19 – FT for a generic failure in the gas detection system

in Figure 20. The minimal cut sets obtained from this will be: $\{R_1, R_2\}$, $\{D_1, D_2\}$, and $\{LU\}$.

Quantification of the coherent FT will overestimate the probability of the critical outcome unless the part of the system that is working (lamp and siren L, LU, and sensors D_1 and D_2) is taken into account. The non-coherent FT with the working part is shown in Figure 21.

If the operator *can* be informed, then cut sets $\{D_1, D_2\}$ and $\{LU\}$ could not have occurred (see Figure 19), thus the correct qualitative analysis should consider only cut set $\{R_1, R_2\}$. Reducing the expressions of the non-coherent FT (Figure 21), we obtain the structure expression: $\neg L \wedge \neg LU \wedge R_1 \wedge R_2 \wedge (\neg D_1 \vee \neg D_2)$. The approximation for this expression, removing the negated events, gives the cut set $\{R_1, R_2\}$, which gives a correct

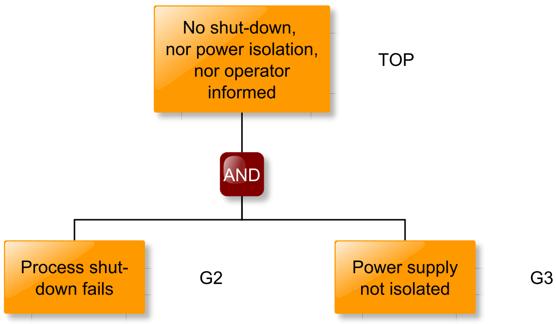


Figure 20 – *Coherent* FT for the most critical outcome of the gas detection system

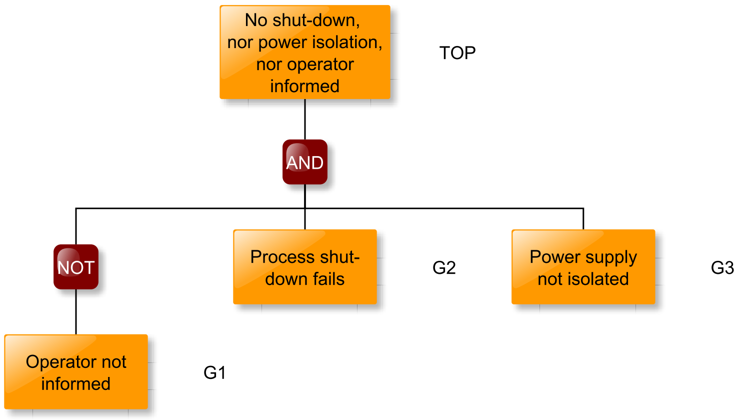


Figure 21 – *Non-coherent* FT for the most critical outcome of the gas detection system

quantitative analysis.

3.5 Systems' nominal model and faults injection

Control system modelling using Simulink block diagrams [81] is recommended in [28] and have been used by our industrial partner. It is a complementary tool of Matlab [82]. In fact, it works as a graphical interface to Matlab. A Simulink model has blocks and connections between these blocks, named signals. Each block has inputs and outputs and an internal behaviour expressed by its mathematical formula, which defines a function of the inputs for each output. There are many predefined blocks in the tool. It is also possible to create new blocks or use subsystems that encapsulate other blocks. A simulation adds extra parameters to a block diagram, like elapsed time and time between states. The elapsed time of a simulation is an abstraction for the quantity of possible simulation states and the time between states is related to the lowest common denominator of the sample time. Some components define different sample times, depending on their mode of operation. Usually, the value for this property is set to *auto*, allowing Simulink to choose a proper value automatically.

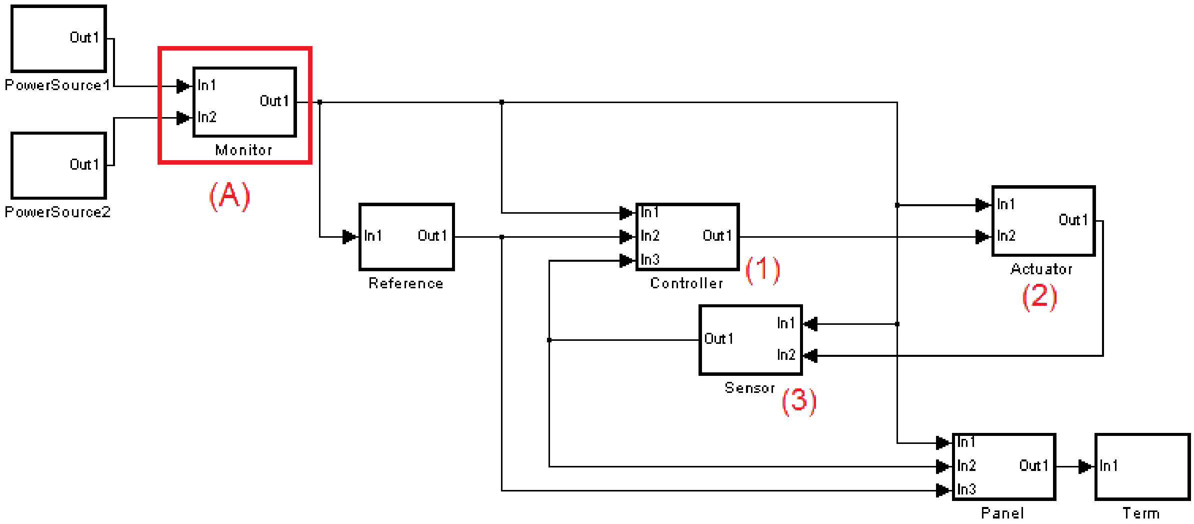


Figure 22 – Block diagram of the ACS provided by EMBRAER (nominal model)

Nowadays, control systems are usually composed of an electromechanical part and a processor. Figure 22 shows the components of a feedback system [83] which was provided by EMBRAER. In this system, the feedback behaviour is given by the *Controller* (1), *Actuator* (2) and *Sensor* (3). A command is received by the *Controller*, which sends a signal to the *Actuator* to start its movement. The *Sensor* detects the actual position of the *Actuator* and sends it back to the *Controller*, which adjusts the given command to achieve the desired position. This loop (feedback) continues until the desired position given by the original command is reached.

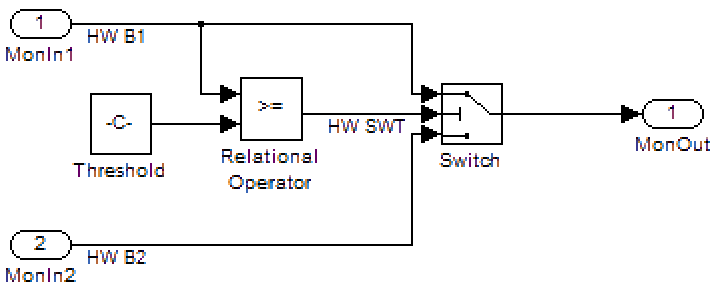


Figure 23 – Internal diagram of the monitor component (Figure 22 (A)).

Figure 23 shows the internal elements of the monitor component (Figure 22 (A)), which is used as a case study in Chapter 5 to illustrate our strategy. The outputs of the hardware elements are annotated with HW, which are the two power sources and an internal component of the monitor (switch command).

To perform a formal verification in a Simulink system model we use a model-checking tool, FDR. It is a refinement checker for formal models written in CSP_M. To verify a refinement, it takes two specifications: (i) a specification with more abstract properties, and (ii) an implementation with more concrete properties. If a refinement does not hold (the implementation fails to refine the specification), FDR shows counter-examples as traces of events. The CSP_M language is suitable to model concurrent behaviour and is very expressive to model systems' states. The work reported in [29] translates a Simulink model to the CSP_M language. The resulting CSP_M code (implementation) is then used to check if it meets functional requirements also encoded in CSP_M (specification).

In our previous work, reported in [26], we modified such a translation to perform fault injection using hardware annotations allowing a subsystem or part to "break" randomly. We designed a CSP_M process to act as an observer (specification), watching outputs of the nominal version and comparing to the outputs of the "breakable" version (with injected faults—the implementation) of the system. When the CSP_M process of the model and the observer are loaded into the FDR model-checker, counter-examples are generated for each output that differs from the nominal model, thus obtaining a *sequence* of injected faults combinations that leads to the unexpected output, which are indeed *fault traces*.

In what follows, injected faults and the top-level failure have generic names based on the names of the Simulink model blocks. It is out of the scope of [26] to define event names.

For the Simulink model shown in Figure 23, some representative fault traces are:

```
TRACE 1:
failure.Hardware.N04_RelationalOperator.1.EXP.B.true
failure.Hardware.N04_RelationalOperator.1.ACT.B.false
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION

TRACE 2:
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.true
failure.Hardware.N04_RelationalOperator.1.ACT.B.false
out.1.OMISSION

TRACE 3:
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION

TRACE 4:
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
out.1.OMISSION
```

```
TRACE 5:
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
out.1.OMISSION

TRACE 6:
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
out.1.OMISSION

TRACE 7:
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true

TRACE 8:
failure.Hardware.N04_MonIn2.1.EXP.I.5
failure.Hardware.N04_MonIn2.1.ACT.OMISSION
failure.Hardware.N04_MonIn1.1.EXP.I.5
failure.Hardware.N04_MonIn1.1.ACT.OMISSION
failure.Hardware.N04_RelationalOperator.1.EXP.B.false
failure.Hardware.N04_RelationalOperator.1.ACT.B.true
```

where `N04` is the subsystem name of the monitor in the Simulink diagram, `MonIn1` (first input of the monitor), `MonIn2` (second input of the monitor), and `RelationalOperator` (switcher controller) are the names of the hardware components in the Simulink diagram.

We only show eight counter-examples, but FDR generates a total of 64 counter-examples for this system. The other counter-examples are similar to the traces shown with different internal events.

To reuse HiP-HOPS, which is based on SFTs, we "remove" the ordering information of the traces to generate a failure expression. Each fault trace is abstracted as a conjunction (AND combination of the inner events, thus losing the ordering information), and the several conjunction-based fault events are combined using ORs (disjunctions). The result of the combination is a Boolean expression that represents the conditions that cause an undesirable output, the failure expression of the model. With the ATF proposed in this work we do not "remove" the ordering information, so we are able to use this information to generate or perform DFT and TFT analyses (TFTs have order-related operators, and it is shown in [3, 23, 21] that DFTs can be expressed by order-related operators).

If the failure expression is obtained for a whole system, it is indeed the structure expression of a fault tree for a general failure as the top-level event. Although it is possible to obtain the failure expression for a larger system, it may be impractical due to state-space explosion in CSP_M model analysis. Thus it should be used for components and subsystems or small systems following HiP-HOPS compositional structure. Using failure expression as subsystem annotations in [24], it is possible to obtain structure expressions for a larger system. It is worth noting that the goal of the work reported in [26] was to connect with HiP-HOPS, which is based on static fault trees. But we already knew that we had a richer fault modelling information than that presented in [26] because we abstracted traces (which already capture fault events ordering) to create propositions (any fault events order combination).

To show how these traces become failure expression, let us abbreviate fault names as:

```
A = failure.Hardware.N04_MonIn1.1
B = failure.Hardware.N04_MonIn2.1
S = failure.Hardware.N04_RelationalOperator
```

So, for each trace, we obtain an expression:

$$\begin{aligned}
\texttt{TRACE}\,1 &= S \wedge B \\
\texttt{TRACE}\,2 &= B \wedge S \\
\texttt{TRACE}\,3 &= A \wedge B \\
\texttt{TRACE}\,4 &= B \wedge A \\
\texttt{TRACE}\,5 &= A \wedge S \\
\texttt{TRACE}\,6 &= A \wedge S \wedge B \\
\texttt{TRACE}\,7 &= A \wedge B \wedge S \\
\texttt{TRACE}\,8 &= B \wedge A \wedge S
\end{aligned}$$

And we combine them as a single Boolean expression: $\texttt{TRACE}\,1 \vee \texttt{TRACE}\,2 \vee \texttt{TRACE}\,3 \vee \texttt{TRACE}\,4 \vee \texttt{TRACE}\,5 \vee \texttt{TRACE}\,6 \vee \texttt{TRACE}\,7 \vee \texttt{TRACE}\,8$, which by a traditional Boolean reduction strategy results in:

$$(A \wedge B) \vee (S \wedge (A \vee B))$$

The above expression is exactly the same failure expression provided by EMBRAER if we use the following association (Table 7):

$$\begin{aligned}
A &= \text{LowPower-In1} \\
B &= \text{LowPower-In2} \\
S &= \text{SwitchFailure}
\end{aligned}$$

Table 7 – Annotations table of the ACS provided by EMBRAER

Component	Deviation	Port	Annotation
PowerSource	LowPower	Out1	PowerSourceFailure
Monitor	LowPower	Out1	(SwitchFailure AND (LowPower-In1 OR LowPower-In2)) OR (LowPower-In1 AND LowPower-In2)
Reference	OmissionSignal	Out1	ReferenceDeviceFailure OR LowPower-In1

Note that when we combine each fault with AND gates, we lose the information about order[4]: $S \wedge B$ and $B \wedge S$ are equal, due to the commutative law of Boolean expressions.

Our strategy finds fault combinations S and B (in the sense of S occurring before B) as well as B and S (in the sense of B occurring before S) but abstracts this ordering information obtaining B and S, which is equivalent to S and B in Boolean Algebra. If A fails before S, the system fails because it should switch to B, but the switcher is in a faulty state. On the other hand, if S fails before A, the switcher fails because it inadvertently switched to B when A was still operational. When A fails, nothing changes and the output of the system is obtained from B.

We also employed the strategy proposed in the work [26] in another case study and obtained a weaker failure expression (that is, our expression considers more cases). The failure expression provided by the engineers of our industrial partner was stronger because they considered that one component has a very low probability of failure and removed it from the failure analysis. Although acceptable, it may cause incorrect analysis. Our strategy avoids this kind of issue by being completely systematic.

3.6 Isabelle/HOL

From the site[5] of the creators:

> Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. The main application is the formalization of mathematical proofs and in particular formal verification, which includes proving the correctness of computer hardware or software and proving properties of computer languages and protocols.

4 In our previous work we designed the observer to ignore order as well, by making similar traces—with different ordering—the same size. Here we modified the observer specification to make similar traces with different sizes.

5 Accessed 27/jan/2016: <https://isabelle.in.tum.de/overview.html>

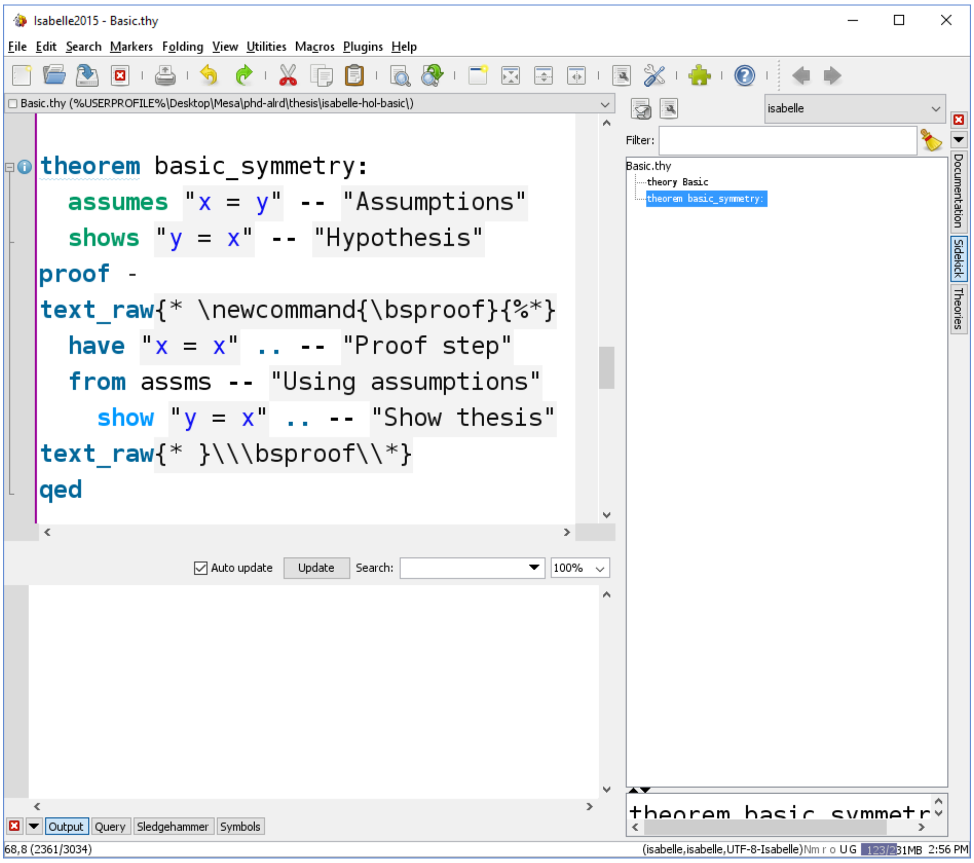


Figure 24 – Isabelle/HOL window, showing the basic symmetry theorem

Isabelle/HOL is the most widespread instance of Isabelle. HOL stands for higher-order logic. Isabelle/HOL provides a HOL proving environment ready to use, which includes: (co)datatypes, inductive definitions, recursive functions, locales, custom syntax definition, etc. Proofs can be written in both human[6] and machine-readable language based on Isar. The tool also includes the *sledgehammer*, a port to call external first-order provers to find proofs fully automatically. The user interface is based on jEdit[7], which provides a text editor, syntax parser, shortcuts, etc. (see Figure 24).

Theories on Isabelle/HOL are based in a few axioms. Isabelle/HOL Library's theories—which comes with the installer—and user's theories are based on these axioms. This design decision avoids inconsistencies and paradoxes (similar as it is in Z).

Besides the provided theories, its active community provides a comprehensive archive of formal proofs[8] (AFP). Each entry in this archive can be cited and usually contains an *abstract*, a document, and a theory file. For example, a Free Boolean Algebra theory is available in [84]. To use it, it is enough to download and put on the same directory of your own theory files.

6 By human we mean that anyone with mathematics and logic basic knowledge—it means that deep programming knowledge is not essential.

7 Accessed 27/jan/2016: <http://www.jedit.org/>

8 Accessed 27/jan/2016: <http://afp.sourceforge.net/>

Bellow we show an example and explain the overall syntax of the human and machine-readable language.

theorem *basic_symmetry:*
  assumes *"x = y"* — Assumptions
  shows *"y = x"* — Hypothesis
proof -
have *"x = x"* **..** — Proof step
  from *assms* — Using assumptions
    show *"y = x"* **..** — Show thesis
qed

Finally, Isabelle/HOL provides LaTeX syntax sugar and allow easy document preparation: this entire section was written in a theory file mixing Isabelle's and LaTeX's syntax). The above theorem can be written using Isabelle's quotation and anti-quotations. For example, we can write it using usual LaTeX theorem environment:

Theorem 3.1 (Basic symmetry)**.** *Assuming* x = y*, thus:*

$$y = x$$

Proof. **have** "x = x" **..** — Proof step

from assms — Using assumptions

show "y = x" **..** — Show thesis ☐

Otherwise specified, in the next sections we will omit proofs because they are all verified using Isabelle/HOL. The complete listing is in Appendix A.

Part II

Contributions

4 A free algebra to express structure expressions of ordered events

Recall from Sections 2.2 and 3.1 that fault events are independent on one another if the events are not susceptible to a common cause. The set-theoretical abstraction of structure expressions for SFTs [17, pp. VI-11] is very close to an FBA, where each generator in FBAs corresponds to a fault event symbol in fault trees. In FBAs, as generators are "free", they are independent on one another and Boolean formulas are written as a set of sets of possibilities, which are similar to the structure expressions of SFTs.

We showed in Section 3.1 that there is an omnipresence of order-based operators to analyse TFTs and DFTs. And that each approach describes a new algebra based on different representations of events ordering with similar theorems to reduce expressions to a canonical form.

From the need to tackle events ordering and from the ordering information we had from fault injection that we developed in [26], we defined a lists-based algebra, called Algebra of Temporal Faults (ATF), to express and analyse systems considering events ordering. We also provide a mapping from fault traces [26] (from CSP_M models) to this algebra. The order-specific operations are expressed with a new operator ($\rightarrow$) that we call exclusive-before (XBefore) ~~(or exclusive before)~~.

The set of sets for FBAs are the denotational semantics for Boolean algebras. We use the concept of generators to propose the ATF with a denotational semantics of a set of lists without repetition (distinct lists). The choice of lists is because this structure inherently associates a generator to an index, making implicit the representation of order. These lists are composed by non-repeated elements (distinct lists) because the events in fault trees are non-repairable, thus they do not occur more than once.

This list representation is different from the Sequence Number function used in [19, 20], but is related to the concept that there should be no gaps between consecutive events occurrence. It is different because order 0 (zero) in [19, 20] means non-occurrence. It may cause a discontinuity because 0 to 1 is different of 1 to 2. In FBAs the non-occurrence of an event is just the absence of the event. Thus we use the same representation of non-occurrence in ATF to avoid this discontinuity.

In the following we show the definitions and laws of our proposed ATF. To avoid repetition, let S, T and U be sets of distinct lists. A list xs is distinct if it has no repeated element. So, if x is in xs, then it has a unique associated index i and we denote it as $x = xs_i$. Furthermore, as we follow an FBA characterisation, we also need to show that the generators are independent.

The ATF form a free algebra, similarly to FBAs. *Infimum* and *Supremum* are defined as set intersection ($\cap$) and union ($\cup$) respectively. The order within the algebra is defined with set inclusion ($\subseteq$).

To distinguish the permutations that are not defined in FBA, we need a new operator. We give the definition of XBefore ($\rightarrow$) in terms of list concatenation, similar to the work reported in [85]:

$$S \rightarrow T = \{zs|\exists xs, ys \bullet (\mathbf{set}\ xs) \cap (\mathbf{set}\ ys) = \{\} \wedge xs \in S \wedge ys \in T \wedge zs = xs \mathbin{@} ys\} \quad (4.1)$$

where the **set** function returns the set of the elements of a list, and @ concatenates two lists.

In some cases it is more intuitive to use the XBefore definition in terms of lists slicing because it uses indexes explicitly. Lists slicing is the operation of taking or dropping elements, obtaining a sublist. In slicing, the starting index is inclusive, and the ending is exclusive. Thus the first index is 0 and the last index is the list length. For example, the list $xs_{[i..|xs|]}$ is equal to the xs list, where $|xs|$ is the list length. We use the following notation for list slicing:

$$xs_{[i..j]} = \text{starts at } i \text{ and ends at } j-1 \quad (4.2a)$$

$$xs_{[..j]} = xs_{[0..j]} \quad (4.2b)$$

$$xs_{[i..]} = xs_{[i..|xs|]} \quad (4.2c)$$

List slicing and concatenation are complementary: concatenating two consecutive slices results in the original list:

$$\forall i \bullet xs_{[..i]} \mathbin{@} xs_{[i..]} = xs \quad (4.3)$$

There is an equivalent definition of XBefore with concatenation using lists slicing:

$$S \rightarrow T = \left\{zs|\exists i \bullet zs_{[..i]} \in S \wedge zs_{[i..]} \in T\right\} \quad (4.4)$$

A variable in ATF is defined by one generator, and denotes its occurrence:

$$\mathbf{var}\ x = \{zs|x \in zs\} \quad (4.5)$$

The following expressions are sufficient to define the ATF in terms of an inductively defined set (**atf**):

$$\mathbf{var}\ x \in \mathbf{atf} \qquad \text{Variable} \quad (4.6a)$$

$$S \in \mathbf{atf} \implies -S \in \mathbf{atf} \qquad \text{Complement, Negation} \quad (4.6b)$$

$$S \in \mathbf{atf} \wedge T \in \mathbf{atf} \implies S \cap T \in \mathbf{atf} \qquad \text{Intersection, } \textit{Infimum} \quad (4.6c)$$

$$S \in \mathbf{atf} \wedge T \in \mathbf{atf} \implies S \rightarrow T \in \mathbf{atf} \qquad \text{XBefore} \quad (4.6d)$$

Following the definitions, the expressions below are also valid for **atf**:

$$\text{UNIV} \in \textbf{atf} \qquad \text{Universal set, True} \tag{4.6e}$$

$$\{\} \in \textbf{atf} \qquad \text{Empty set, False} \tag{4.6f}$$

$$S \in \textbf{atf} \wedge T \in \textbf{atf} \implies S \cup T \in \textbf{atf} \qquad \text{Union, } \textit{Supremum} \tag{4.6g}$$

The following expressions are valid for generators a and b and are sufficient to show that the generators are independent:

$$\textbf{var}\, a \subseteq \textbf{var}\, b \iff a = b \tag{4.7a}$$

$$\textbf{var}\, a = \textbf{var}\, b \iff a = b \tag{4.7b}$$

$$\textbf{var}\, a \nsubseteq -\textbf{var}\, b \tag{4.7c}$$

$$\textbf{var}\, a \neq -\textbf{var}\, b \tag{4.7d}$$

$$-\textbf{var}\, a \nsubseteq \textbf{var}\, b \tag{4.7e}$$

$$-\textbf{var}\, a \neq \textbf{var}\, b \tag{4.7f}$$

Expressions (4.6a) to (4.6g) and (4.7a) to (4.7f) implies that the ATF without the XBefore operator (4.1) forms a Boolean algebra based on sets of lists. And this is also equivalent to an FBA with the same generators.

In our previous work [85] we stated a relation of XBefore and *supremum*, provided the operands are variables (4.5). Now we generalise this relation in terms of abstract properties of the operands of the XBefore. We name these properties as *temporal properties*.

4.1 Temporal properties (*tempo*)

Temporal properties give a more abstract and less restrictive shape on the XBefore laws. These properties avoid the requirement that every operand of XBefore should be a variable (4.5).

The first temporal property is about disjoint split. If the first part of a list is in a given set, then every remainder part is not. So, if a generator is in the beginning of a list, it must not be at the ending (and vice-versa).

$$\textbf{tempo}_1\, S = \forall i, j, zs \bullet i \leq j \implies \neg\left(zs_{[..i]} \in S \wedge zs_{[j..]} \in S\right) \tag{4.8a}$$

$$\textbf{tempo}_2\, S = \forall i, zs \bullet zs \in S \iff zs_{[..i]} \in S \vee zs_{[i..]} \in S \tag{4.8b}$$

$$\textbf{tempo}_3\, S = \forall i, j, zs \bullet j < i \implies \left(zs_{[j..i]} \in S \iff zs_{[..i]} \in S \wedge zs_{[j..]} \in S\right) \tag{4.8c}$$

$$\textbf{tempo}_4\, S = \forall zs \bullet zs \in S \iff (\exists i \bullet zs_{[i..(i+1)]} \in S) \tag{4.8d}$$

The second temporal property is about belonging to one sublist in the beginning or in the end. If a generator is in a list, then it must be at the beginning or at the ending.

The third temporal property is about belonging to one sublist in the middle. If a generator belongs to a sublist between ~~i and j,~~ then it belongs to the sublist that starts at first position and ends in j and to the sublist that starts at i and ends at the last position (both sublists contain the sublist in the middle).

Finally, if a generator belongs to a list, then there is a sublist of size one that contains the generator.

Variables have all four temporal properties. For a generator x, the following is valid:

$$\mathbf{tempo}_1\,(\mathbf{var}\,x) \wedge \mathbf{tempo}_2\,(\mathbf{var}\,x) \wedge \mathbf{tempo}_3\,(\mathbf{var}\,x) \wedge \mathbf{tempo}_4\,(\mathbf{var}\,x) \tag{4.9}$$

In our previous work [85] we used set difference to specify the XBefore operator. Provided $\mathbf{tempo}_1\,S$ and $\mathbf{tempo}_1\,T$, XBefore in [85] is equivalent to (4.1):

$$S \to T = \{zs | \exists xs, ys \bullet xs \in S - T \wedge ys \in T - S \wedge \text{distinct}\,zs \wedge zs = xs\,@\,ys\} \tag{4.10}$$

Other expressions also meet one or more temporal properties:

$$\mathbf{tempo}_1\,S \wedge \mathbf{tempo}_1\,T \implies \mathbf{tempo}_1\,(S \cap T) \tag{4.11a}$$
$$\mathbf{tempo}_3\,S \wedge \mathbf{tempo}_3\,T \implies \mathbf{tempo}_3\,(S \cap T) \tag{4.11b}$$
$$\mathbf{tempo}_2\,S \wedge \mathbf{tempo}_2\,T \implies \mathbf{tempo}_2\,(S \cup T) \tag{4.11c}$$
$$\mathbf{tempo}_4\,S \wedge \mathbf{tempo}_4\,T \implies \mathbf{tempo}_4\,(S \cup T) \tag{4.11d}$$

4.2 XBefore laws

We now show some laws to be used in the algebraic reduction of ATF formulas. The laws follow from the definition of XBefore, from events independence, and from the temporal properties.

We use a normal form similar to the DNF of Boolean algebra. In DNF each sub-expression is a minimal cut set for SFT. In our normal form, also called DNF, we allow ANDs, NOTs, and XBefores to be in the sub-expressions. Each sub-expression is a set of minimal cut sequences for TFT and DFT. The following formulas are in DNF:

$$(A \cap -B) \cup ((A \to B) \cap C)$$
$$A \cup B$$
$$A \to B$$
$$A \cap B$$
$$A \to B \to C$$

The following formulas are *not* in DNF:

$$-(A \cup B)$$
$$A \cap (B \cup C)$$
$$A \rightarrow (B \cup C)$$
$$A \rightarrow (B \cap C)$$

But to transform the last two formulas into DNF, one can use Laws (4.15a), (4.15b), (4.15c) and (4.15d), for instance.

We define events independence ($\triangleleft\!\triangleright$) as the property that one operand does not imply the other. For example, we need to avoid that the operands of XBefore are **var** a and **var** $a \cup$ **var** b (it results in $\{\}$, see (4.13e)).

$$S \triangleleft\!\triangleright T = \forall i, zs \bullet \neg \left(zs_{[i..(i+1)]} \in S \wedge zs_{[i..(i+1)]} \in T \right) \tag{4.12}$$

The absence of occurrences ($\{\}$, the empty set of **atf**) is a "0" for the XBefore operator.

$$\{\} \rightarrow S = \{\} \qquad \text{left-false-absorb} \tag{4.13a}$$
$$S \rightarrow \{\} = \{\} \qquad \text{right-false-absorb} \tag{4.13b}$$
$$(S \rightarrow T) \cup S = S \qquad \text{left-union-absorb} \tag{4.13c}$$
$$(T \rightarrow S) \cup S = S \qquad \text{right-union-absorb} \tag{4.13d}$$
$$\mathbf{tempo}_1\, S \implies S \rightarrow S = \{\} \qquad \text{non-idempotent} \tag{4.13e}$$
$$\mathbf{tempo}_1\, S \wedge \mathbf{tempo}_1\, T \wedge \mathbf{tempo}_1\, U \implies S \rightarrow (T \rightarrow U) = (S \rightarrow T) \rightarrow U \qquad \text{associativity} \tag{4.13f}$$

The XBefore is absorbed by one of the operands: if one of the operands may happen alone, thus the order with any other operand is irrelevant. However, an event cannot come before itself, thus XBefore is not idempotent. The XBefore but is associative.

To allow formula reduction we need the relation of XBefore to the other Boolean operators. First we use the XBefore as operands of union and intersection.

$$\mathbf{tempo}_1\, S \wedge \mathbf{tempo}_1\, T \implies (S \rightarrow T) \cap (T \rightarrow S) = \{\} \qquad \text{inter-equiv-false} \tag{4.14a}$$
$$\mathbf{tempo}_{1-4}\, S \wedge \mathbf{tempo}_{1-4}\, T \wedge S \triangleleft\!\triangleright T \implies (S \rightarrow T) \cup (T \rightarrow S) = S \cap T \qquad \text{union-equiv-inter} \tag{4.14b}$$

As the XBefore is not symmetric, the intersection of symmetrical sets is empty. The union of the symmetric is a partition of the intersection of the operands.

In our previous work [85], we stated that S and T had to be variables. For example, of the form **var** s and **var** t. Now, each law requires that the operands satisfy some of the temporal properties, avoiding using variables explicitly.

Boolean operators are used as operands of the XBefore in the following laws.

$$(S \cup T) \to U = (S \to U) \cup (T \to U) \qquad \text{left-union-dist} \tag{4.15a}$$

$$S \to (T \cup U) = (S \to T) \cup (S \to U) \qquad \text{right-union-dist} \tag{4.15b}$$

$$\begin{aligned} &\mathbf{tempo}_{1-4}\, S \wedge \mathbf{tempo}_{1-4}\, T \wedge S \lhd\!\rhd T \implies \\ &\quad (S \cap T) \to U = (S \to T \to U) \cup \\ &\quad\quad (T \to S \to U) \qquad \text{left-inter-dist} \end{aligned} \tag{4.15c}$$

$$\begin{aligned} &\mathbf{tempo}_{1-4}\, T \wedge \mathbf{tempo}_{1-4}\, U \wedge T \lhd\!\rhd U \implies \\ &\quad S \to (T \cap U) = (S \to T \to U) \cup \\ &\quad\quad (S \to U \to T) \qquad \text{right-inter-dist} \end{aligned} \tag{4.15d}$$

$$\begin{aligned} \mathbf{tempo}_2\, S \implies S \cap (T \to U) = {} & ((S \cap T) \to U) \cup \\ & (T \to (S \cap U)) \qquad \text{unordered} \end{aligned} \tag{4.15e}$$

XBefore is distributive over union. On the other hand, the intersection is related to order. Thus it is not distributive with XBefore. Finally, the intersection of an event with an XBefore states that such an event can occur in any order within the events in the XBefore.

The law name, unordered, of (4.15e) is clearer if we expand (4.15e) with (4.15c) and (4.15d):

$$\begin{aligned} &\mathbf{tempo}_{1-4}\, S \wedge \mathbf{tempo}_{1-4}\, T \wedge \\ &\mathbf{tempo}_{1-4}\, U \wedge S \lhd\!\rhd T \wedge S \lhd\!\rhd U \implies \\ &\quad S \cap (T \to U) = (S \to T \to U) \cup \\ &\quad\quad (T \to S \to U) \cup \\ &\quad\quad (T \to U \to S) \qquad \text{expanded-unordered} \end{aligned} \tag{4.16}$$

4.3 Propositions

In this section we discuss the theorems and definitions the still need to be proved. We present them as propositions.

Soundness and completeness of the ATF is given in terms of the algebraic form and its denotational semantics (Subsection 4.3.1). The ActA is defined in terms of a logic that is solved by decision and some output value (Subsection 4.3.2).

4.3.1 Soundness and completeness of ATF

Given the semantics of a formula of ATF, there is always a set of sequences that represents exactly the formula. To guarantee the completeness we show that for every set of sequences there is a corresponding formula in ATF.

Proposition 4.1 (Soundness and completeness of ATF)**.** *Let F be the set of all formula in ATF, and SS be the set of all sets of sequences:*

$$\forall f \in F.\, \exists S \in SS.\, f = S \qquad \textit{Soundness} \tag{4.17a}$$

$$\forall S \in SS.\, \exists f \in F.\, f = S \qquad \textit{Completeness} \tag{4.17b}$$

The equality in the proposition is set-based, thus, in both cases, $f \subseteq S \wedge S \subseteq f$.

4.3.2 ActA concepts

The Activation Algebra (ActA) is used to model systems faults. When reasoning about faults, engineers analyse component by component, defining its outputs in the presence of each possible fault. The ActA is nothing more than this: the outputs of the components in the presence of (some combination of) faults. To ensure that all possibilities are considered (those that the engineer reasoned about them), the formula that results from the output conditions shall be a *tautology*. For example, if the engineer defines that a component produces as outputs: (i) A if F_1 occurs, and (ii) B if F_2 occurs, then there should be an output for condition $\neg F_1 \wedge \neg F_2$, and A and B must converge when both F_1 and F_2 occur. By convergence, an initial idea is that $A = B$ in this case. To connect components and to generate FTs, we ask questions to the ActA formulas: we define *predicates*.

A nominal value is required to handle the conditions that do not result in a failure. Recall from previous example, if F_1 and F_2 do not occur, then the system should be in a normal state, and its output is signalled as nominal with some nominal value. Nominal values are used to analyse value-based failures. In general, failure outputs do not have an associated value.

In some cases a degraded state can be an undesired state, as for example, if one wants to check the probability of operating in high-cost conditions. For these situations, the output values are signalled as degraded, but they have an associated value.

To connect components, instead of using a fixed condition like F_1 and F_2, we use a predicate. For example: if an omission is detected in the first input, the output of this component is A; the component outputs B if F_1 occurs, and it outputs its nominal value, otherwise.

To obtain a fault tree from ActA we define a predicate over a whole ActA formula of a system. For example: what are the conditions that generate and output *omission*?

The underlying conditions in ActA can be in Boolean algebra or in ATF. In any case, soundness and completeness in ActA is given in terms of the underlying algebra: given a formula in ActA, any predicate generates a valid expression in the underlying algebra, and there exists a formula and a predicate for any expression in the underlying algebra.

Proposition 4.2 (Soundness and completeness of ActA)**.** *Let F be the set of all formulas in ActA, G be the set of all formulas in its underlying algebra, and P a predicate over output values of ActA, then:*

$$\forall f \in F. \exists g \in G. P(f) \equiv g \qquad \textit{Soundness} \tag{4.18a}$$

$$\forall g \in G. \exists f \in F, P. P(f) \equiv g \qquad \textit{Completeness} \tag{4.18b}$$

5 Case study

EMBRAER provided us with the Simulink model of an Actuator Control System (depicted in Figure 22). The failure expression of this system (that is, for each of its constituent components) was also provided by EMBRAER (we show some of them in Table 7). In what follows we illustrate our strategy using the Monitor component.

A monitor component is a system commonly used for fault tolerance [86, 87]. Initially, the monitor connects the main input (power source on input port 1) with its output. It observes the value of this input port and compares it to a threshold. If the value is below the threshold, the monitor disconnects the output from the main input and connects to the secondary input. We present the Simulink model for this monitor in Figure 23.

Now we show two contributions: (i) using only Boolean operators, thus ignoring ordering, we can obtain the same results obtained in [26], and (ii) we represent each of the fault traces reported in [26] as a term in our proposed algebra of temporal faults. Similarly to the association of fault events of Table 7 in Section 3.5, we associate the fault events as:

$$
\begin{aligned}
a &= \text{LowPower-In1} & \qquad A &= \mathbf{var}\, a \\
b &= \text{LowPower-In2} & \qquad B &= \mathbf{var}\, b \\
s &= \text{SwitchFailure} & \qquad S &= \mathbf{var}\, s
\end{aligned}
$$

5.1 Structure expressions with Boolean operators

In this section we show that the same result reported in [26] in terms of static failure expression (or Boolean propositions) can be obtained with our Boolean operator without using XBefore. For each trace shown in Section 3.5, a mapping function[1] ($\overset{\leadsto}{\text{B}}$)

[1] In this work we do not show the mapping function from traces to ATF (and the mapping function with XBefore in Section 5.2). The mapping rules follow the traces: XBefore is obtained by the order of occurrence and the absence of an event is the complement ($-$).

generates the following sets of lists:

$$
\begin{array}{lll}
\texttt{TRACE 1:} & [s,b] \overset{\rightsquigarrow}{\text{B}} S \cap B \cap -A & \{[s,b],[b,s]\} \\
\texttt{TRACE 2:} & [b,s] \overset{\rightsquigarrow}{\text{B}} B \cap S \cap -A & \{[s,b],[b,s]\} \\
\texttt{TRACE 3:} & [a,b] \overset{\rightsquigarrow}{\text{B}} A \cap B \cap -S & \{[a,b],[b,a]\} \\
\texttt{TRACE 4:} & [b,a] \overset{\rightsquigarrow}{\text{B}} B \cap A \cap -S & \{[a,b],[b,a]\} \\
\texttt{TRACE 5:} & [a,s] \overset{\rightsquigarrow}{\text{B}} A \cap S \cap -B & \{[a,s],[s,a]\} \\
\texttt{TRACE 6:} & [a,s,b] \overset{\rightsquigarrow}{\text{B}} A \cap S \cap B & \{[a,b,s],[a,s,b],\ldots,[s,b,a]\} \\
\texttt{TRACE 7:} & [a,b,s] \overset{\rightsquigarrow}{\text{B}} A \cap B \cap S & \{[a,b,s],[a,s,b],\ldots,[s,b,a]\} \\
\texttt{TRACE 8:} & [b,a,s] \overset{\rightsquigarrow}{\text{B}} B \cap A \cap S & \{[a,b,s],[a,s,b],\ldots,[s,b,a]\}
\end{array}
$$

They represent the same faults shown in Section 3.5. Note that the negation in the formula is very simple to represent in ATF (and FBA) because it is just the absence of the generator.

Combining the above sets with unions (ORs), we obtain the following formula set:

$$\{[s,b],[b,s],[a,b],[b,a],[a,s],[s,a],[a,b,s],[a,s,b],\ldots,[s,b,a]\}$$

If we use Boolean expression reduction instead, it results in the following expression in ATF (and in FBA):

$$(A \cap B) \cup (S \cap (A \cup B))$$

which is equivalent to the set of sets above and is equivalent to EMBRAER failure expression shown in Table 7 (with AND gates as $\cap$ and OR gates as $\cup$). This shows that ATF can represent (static) failure expression as in our previous work [26].

5.2 Structure expressions with XBefore

Now, by using ATF with the XBefore operator and a mapping function ($\overset{\rightsquigarrow}{\text{XB}}$), we can capture each possible individual sequences as generated by the work [26]:

$$
\begin{array}{lll}
\texttt{TRACE 1:} & [s,b] \overset{\rightsquigarrow}{\text{XB}} (S \rightarrow B) \cap -A & \{[s,b]\} \\
\texttt{TRACE 2:} & [b,s] \overset{\rightsquigarrow}{\text{XB}} (B \rightarrow S) \cap -A & \{[b,s]\} \\
\texttt{TRACE 3:} & [a,b] \overset{\rightsquigarrow}{\text{XB}} (A \rightarrow B) \cap -S & \{[a,b]\} \\
\texttt{TRACE 4:} & [b,a] \overset{\rightsquigarrow}{\text{XB}} (B \rightarrow A) \cap -S & \{[b,a]\} \\
\texttt{TRACE 5:} & [a,s] \overset{\rightsquigarrow}{\text{XB}} (A \rightarrow S) \cap -B & \{[a,s]\} \\
\texttt{TRACE 6:} & [a,s,b] \overset{\rightsquigarrow}{\text{XB}} A \rightarrow S \rightarrow B & \{[a,s,b]\} \\
\texttt{TRACE 7:} & [a,b,s] \overset{\rightsquigarrow}{\text{XB}} A \rightarrow B \rightarrow S & \{[a,b,s]\} \\
\texttt{TRACE 8:} & [b,a,s] \overset{\rightsquigarrow}{\text{XB}} B \rightarrow A \rightarrow S & \{[b,a,s]\}
\end{array}
$$

Using ATF and combining each trace with ORs (unions), we obtain the following set:

$$M_L = \{[a, b], [b, a], [b, s], [s, b], [a, s], [a, b, s], [a, s, b], [s, a, b]\}$$

From the above traces, we also build an ATF expression by mapping each trace to an XBefore expression, composing all resulting XBefore expressions with ORs and reducing them using the XBefore laws (Section 4.2), resulting in an expression (M_A) that is equivalent to the above set of lists ($M_L \equiv M_A$). The failure expression of the monitor[2] is:

$$\begin{aligned}
M_A = {} & ((S \rightarrow B) \cap -A) \cup ((B \rightarrow S) \cap -A) \cup \\
& ((A \rightarrow B) \cap -S) \cup ((B \rightarrow A) \cap -S) \cup \\
& ((A \rightarrow S) \cap -B) \cup \\
& (A \rightarrow S \rightarrow B) \cup (A \rightarrow B \rightarrow S) \cup (B \rightarrow A \rightarrow S) \\
= {} & (B \cap S \cap -A) \cup && \text{by (4.14b)} \\
& (B \cap A \cap -S) \cup && \text{by (4.14b)} \\
& ((A \rightarrow S) \cap -B) \cup \\
& (A \rightarrow S \rightarrow B) \cup (A \rightarrow B \rightarrow S) \cup (B \rightarrow A \rightarrow S) \\
= {} & (B \cap S \cap -A) \cup \\
& (B \cap A \cap -S) \cup \\
& ((A \rightarrow S) \cap -B) \cup \\
& ((A \rightarrow S) \cap B) && \text{by (4.16)} \\
= {} & (B \cap S \cap -A) \cup (B \cap A \cap -S) \cup (A \rightarrow S) && \text{by absorption}
\end{aligned}$$

The semantics of the above expression is: (i) fault b (**var** b) occurs and fault a (**var** a) or fault s (**var** s) occurs (but not both a and s), or (ii) fault a occurs before fault s, which is more precise than the expression found without considering order of events.

[2] In the final formula, $(B \cap S \cap -A) \cup (A \cap B \cap -S)$ is equivalent to $(B \cap (S \oplus A))$. There is a typo in our previous work [85]. The expression was written with an OR ($\vee$) but it should an XOR ($\oplus$).

Part III

Final remarks

6 Conclusion

In this work we presented a foundational theory to support a more precise representation of fault events as compared to our previous strategy for injecting faults [26]. The failure expression is essential for system safety assessment because it is used as basic input for building fault trees [24, 29, 88]. Furthermore, we still connect the strategy presented in [89] with the works reported in [29] (functional analysis) and in [88, 24] (safety assessment) because our new algebra is at least a Boolean algebra.

The work reported in [20, 19, 31] tackles simultaneity with “nearly simultaneous” events [90]. But we consider instantaneous events, like the work reported in [22], because we assume that simultaneity is probabilistically impossible.

6.1 Status

In Figure 25 we show: (i) what was done in previous work and is used as input, (ii) what was done in the current work, (iii) what will be done in the next months (see Table 8 for tasks schedule), and (iv) what will be done in future work, after the thesis' defence. Many of the tasks shown in Table 8 already started. The Table shows the estimated execution of the tasks by year's quarters. The second quarter of 2016 is from April to June/2016, the third quarter is from July to September/2016, the fourth quarter, from October to December/2016, and we expected to defend the thesis by February/2017.

Table 8 – Tasks schedule

Task	2nd	3rd	4th	1st
Qualification	•			
Elaborate a theory for the ActA	•			
Elaborate a theory for the acceptance criteria	•			
Prepare a paper about ActA and acceptance criteria	•	•		
Submit paper about ActA and acceptance criteria		•		
Prove soundness and completeness theorems for the DNF of ATF		•	•	
Define the mapping rules from traces to ATF			•	
Demonstrate the relations of NOT and XBefore and other operators			•	
Define the conditions that cause non-coherent analysis with NOT			•	
Write the results in the thesis		•	•	
Prepare thesis' defence			•	•
Defence				•

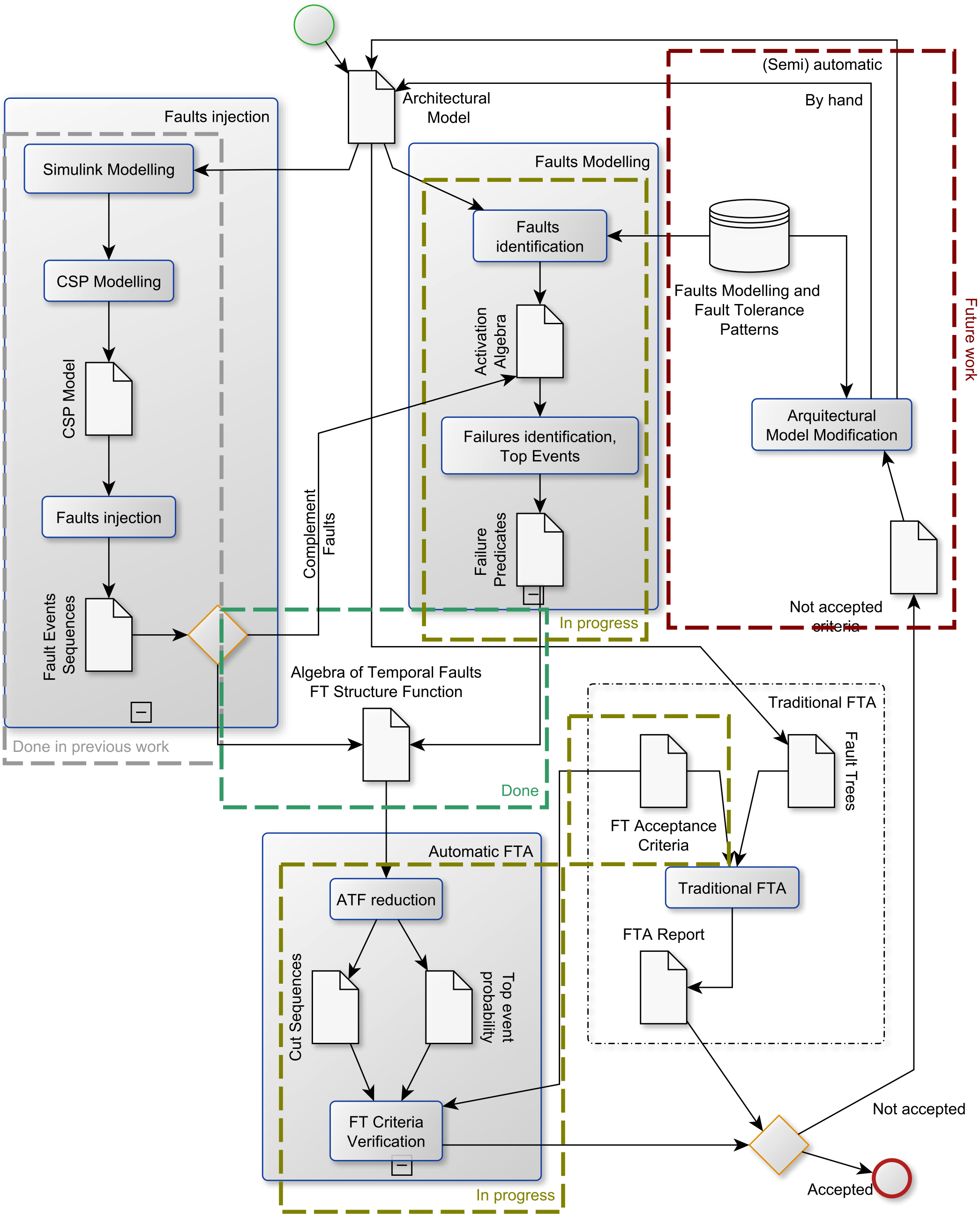


Figure 25 – Status of this thesis using the strategy overview (see Figure 1)

6.2 Next steps in this thesis

The next steps in this thesis are the conclusion of the work-in-progress of the "Faults Modelling" and "Automatic FTA" blocks in Figure 25. An initial version of the theory of the ActA is done. The case study shown in this thesis is also modelled in ActA. But we need to adapt the failure predicates to ATF. It may require a full refactoring of the theory of ActA.

We developed a small set of rules for the acceptance criteria verification, but we need to add more sophisticated rules, as for example, to consider phase and latency.

We showed the DNF for ATF, but we did not demonstrate that every formula can be converted into DNF. The laws shown in this work should be sufficient for this demonstration. Also, we did not show the mapping rules from traces to ATF (with Boolean operators only AND with XBefore). The mapping rules follow those for the traces: XBefore is obtained by the order of occurrence and the absence of an event is the complement $(-)$.

Although we do not use negation (NOT operator) with XBefore in our case study, it is part of ATF, so it could be used. As future work we will demonstrate the relations of NOT and XBefore, as we did for AND and OR. We will also define laws to avoid the conditions that cause non-coherent analysis [11]. The issue with negated events comes up when both an event and its negation appear on the same tree. One very restrictive solution to this issue is applying the *generators independence* laws (4.7d, 4.7f) on basic events of a tree, by actually considering a new event ne in place of the negation of another event e (for instance, $\mathbf{var}\, e$ and $\mathbf{var}\, ne = -\,\mathbf{var}\, e$). We look forward to obtain a less restrictive law.

6.3 Future work, out of the scope of this thesis

Boolean formulas reduction can be achieved by: (i) application of Boolean laws, (ii) BDD, or (iii) FBAs. We used Boolean and XBefore laws to reduce ATF formulas. The work reported in [41, 42] uses Sequential BDDs to reduce formulas with order-based operators. We plan to use similar concepts in a future work.

The work reported in [7] states that DTMCs (Markov chain) is more appropriate to represent several states than SFTs. Considering that DFTs were conceived as a visual representation of Markov chains, then we may say that DFTs can be used to represent several states. Thus they are suitable to propose the architectural model modifications as shown in Figures 1 and 25. The definition and the theory of “Faults Modelling and Fault Tolerance Patterns” and the automatic proposal of “Architectural Model Modifications” blocks are left as future work.

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Appendix

APPENDIX A – Formal proofs in Isabelle/HOL

In the following we list all theorems and proofs concerning the laws presented in Chapter 4. The complete set of verifiable theory files is available at <http://www.cin.ufpe.br/~alrd/phd/phd-alrd.zip> (password: 6Zvq$5Vyj). We list only those files created in our work. Each theorem, proof or corollary is followed by its own proof.

The theory about lists of distinct elements—or simply *distinct lists*—is available in [91] (we used the 2015 version that is available with Isabelle/HOL).

This Appendix is organized as follows: (i) Appendix A.1 presents the base lemmas and theorems for sliceable types; (ii) sublists (sliceable distinct lists) are shown in Appendix A.2; (iii) algebraic definitions and laws of the ATF are shown in Appendix A.3, and (iv) proofs using the denotational semantics of sets of distinct lists are shown in Appendix A.4.

A.1 Sliceable

In this section we present a class to express sub-structures for a data type, and laws over such a class. For example, for lists, *sliceable* defines operators and theorems to obtain sublists.

```
class sliceable =
  fixes slice :: "'a ⇒ nat ⇒ nat ⇒ 'a" ("(3_†_.._)"  [80,80,80] 80)
  fixes size :: "'a ⇒ nat" ("(1#_)" 65)
  fixes empty_inter :: "'a ⇒ 'a ⇒ bool"
  fixes disjoint :: "'a ⇒ bool"

  assumes slice_none: "x†0..(#x) = x"
  assumes empty_seq_inter [simp]:
    "disjoint x ⟹ c ≤ k ⟹ empty_inter (x†0..c) (x†k..(#x))"
  assumes size_slice: "size (x†i..j) = max 0 ((min j (size x))-i)"
  assumes slice_slice: "(x†i..j)†a..b = x†(i+a)..(min j (i+b))"
  assumes disjoint_slice_suc:
    "disjoint x ⟹ i≠j ⟹ i < (#x) ⟹ j < (#x) ⟹
    x†i..(Suc i) ≠ x†j..(Suc j)"
  assumes disjoint_slice[simp]: "disjoint x ⟹ disjoint (x†i..j)"
  assumes forall_slice_implies_eq: "(#x) = (#y) ∧ (∀i j. (x†i..j) =
```

```
    (y†i..j)) ⟷ (x = y)"

notation (latex output) slice  ("(3_ [_.._])" [80,80,80] 80)
```

Teste $x_{[i..j]}$

```
definition slice_right :: "'a::sliceable ⇒ nat ⇒ 'a" ("(2_†.._)" [80,80] 80)
where "slice_right x i = x†0..i"

notation ("latex") slice_right  ("(2_ [.._])" [80,80] 80)



definition slice_left :: "'a::sliceable ⇒ nat ⇒ 'a" ("(2_†_..)" [80,80] 80)
where "x†i.. = x†i..(# x)"

notation ("latex") slice_left  ("(2_ [_..])" [80,80] 80)
```

A.1.1 Disjoint elements and sliceable

```
lemma (in sliceable) slice_right_disjoint[simp]: "disjoint xs ⟹
  disjoint (slice_right xs i)"
unfolding slice_right_def
by simp
```

The notation for $x_{[..i]}$ is $x_{[..i]}$

```
lemma (in sliceable) slice_left_disjoint[simp]: "disjoint xs ⟹
  disjoint (xs†i..)"
unfolding slice_left_def
by simp
```

A.1.2 n-th element in a sliceable

```
abbreviation sliceable_nth :: "'a::sliceable ⇒ nat ⇒ 'a"
where
"sliceable_nth l i ≡ l†i..(Suc i)"
```

A.1.3 Theorems for sliceable

```
theorem (in sliceable) empty_seq_inter_eq [simp]:
  "disjoint x ⟹ empty_inter (x†..i) (x†i..)"
by (simp add: slice_right_def slice_left_def)
```

```
theorem (in sliceable) empty_seq_sliced_inter [simp]:
  "disjoint x ⟹ b ≤ i ⟹ j ≤ a ⟹ i ≤ j ⟹ a ≤ size x ⟹
    empty_inter (x†b..i) (x†j..a)"
proof-
  let ?l = "x†b..a"
  assume lt0: "i ≤ j"
  assume lt1: "j ≤ a"
  assume lt2: "b ≤ i"
  assume lt3: "a ≤ size x"
  assume lt4: "disjoint x"
  have blta: "b ≤ a" using lt0 lt1 lt2 by simp
  have ilta: "i ≤ a" using lt0 lt1 by simp
  hence 2: "empty_inter (?l†0..(i-b)) (?l†(j-b)..(#?l))"
    using lt0 lt4 disjoint_slice by simp
  hence "empty_inter ((x†b..a)†0..(i-b)) ((x†b..a)†(j-b)..(#?l))" by simp
  hence 3: "empty_inter (x†b..i) ((x†b..a)†(j-b)..(#(x†b..a)))" using ilta lt2
    by (simp add: slice_slice min_absorb2)
  hence 3: "empty_inter (x†b..i) (x†j..a)"
    using blta lt0 lt2 lt3
    by (auto simp add: size_slice slice_slice min_def)
  thus ?thesis by simp
qed

theorem distinct_slice_lte_inter_empty[simp]:
  "distinct l ⟹ i ≤ j ⟹
    set (take i (drop 0 l))
    ∩ set (take (length l-i) (drop i l)) = {}"
by (simp add: set_take_disj_set_drop_if_distinct )

lemma (in sliceable) size_slice_right_absorb: "(#(l†..i)) = min i (#l)"
by (simp add: slice_right_def sliceable_class.size_slice)

lemma (in sliceable) size_slice_left_absorb: "(#(l†i..)) = (#l)-i"
by (simp add: slice_left_def sliceable_class.size_slice)

corollary (in sliceable) slice_right_slice_left_absorb: "(l†..i)†j.. = l†j..i"
unfolding slice_left_def slice_right_def
by (metis (mono_tags, hide_lams) add.left_neutral add.right_neutral max_0L
  min.left_idem size_slice_right_absorb slice_right_def
  sliceable_class.size_slice sliceable_class.slice_none
  sliceable_class.slice_slice)
```

```
corollary (in sliceable) slice_right_slice_left_absorb_empty:
  "i ≤ j ⟹ (#((l†..i)†j..)) = 0"
by (simp add: size_slice_left_absorb size_slice_right_absorb)

corollary (in sliceable) slice_left_slice_right_absorb:
  "(l†i..)†..j = l†i..(i+j)"
unfolding slice_left_def slice_right_def
proof -
  have "(l†i..(#l))†0..j = (l†0..(#l))†i..(i + j)"
    by (simp add: sliceable_class.slice_slice)
  thus "(l†i..(#l))†0..j = l†i..(i + j)"
    by (simp add: sliceable_class.slice_none)
qed

corollary (in sliceable) slice_right_slice_right_absorb:
  "(l†..i)†..j = (l†..(min i j))"
unfolding slice_left_def slice_right_def
by (simp add: sliceable_class.slice_slice)

corollary (in sliceable) slice_left_slice_left_absorb:
  "(l†i..)†j.. = l†(i+j).."
unfolding slice_left_def slice_right_def
by (simp add: sliceable_class.slice_slice sliceable_class.size_slice
  min_absorb1)

corollary (in sliceable) slice_slice_right_absorb:
  "(l†i..j)†..b = l†i..(min j (i+b))"
unfolding slice_left_def slice_right_def
by (simp add: add.commute sliceable_class.slice_slice)

corollary (in sliceable) slice_slice_left_absorb:
  "(l†i..j)†a.. = l†(i+a)..j"
unfolding slice_left_def slice_right_def
by (metis (mono_tags, hide_lams) add.assoc diff_diff_left max_0L
  slice_left_def slice_left_slice_right_absorb slice_right_def
  slice_slice_right_absorb sliceable_class.size_slice
  sliceable_class.slice_none sliceable_class.slice_slice)

corollary (in sliceable) slice_left_slice_absorb:
  "(l†i..)†a..b = l†(i+a)..(i+b)"
unfolding slice_left_def slice_right_def
by (metis (mono_tags, lifting) slice_left_slice_right_absorb slice_right_def
```

```
  slice_right_slice_left_absorb slice_slice_left_absorb
  sliceable_class.slice_none)

corollary (in sliceable) slice_right_slice_absorb:
  "(l†..j)†a..b = l†a..(min j b)"
unfolding slice_left_def slice_right_def
by (simp add: sliceable_class.slice_slice)


lemmas (in sliceable) slice_slice_simps =
  slice_left_slice_left_absorb slice_left_slice_right_absorb
  slice_right_slice_left_absorb slice_right_slice_right_absorb slice_slice
  slice_slice_right_absorb slice_slice_left_absorb slice_left_slice_absorb
  slice_right_slice_absorb

lemmas (in sliceable) size_slice_defs =
  size_slice size_slice_left_absorb size_slice_right_absorb

lemma (in sliceable) slice_f_min_neutral:
  "(P (l†i..(min f k)) ∧ f ≤ k) ⟷ (P (l†i..f) ∧ f ≤ k)"
by linarith

lemma (in sliceable) slice_i_min_neutral:
  "(P (l†(min i k)..f) ∧ i ≤ k) ⟷ (P (l†i..f) ∧ i ≤ k)"
by linarith

lemma (in sliceable) slice_i_min_neutral_lt:
  "(P (l†(min k i)..f) ∧ i < k) ⟷ (P (l†i..f) ∧ i < k)"
by linarith

lemma (in sliceable) slice_foral_i_min_neutral:
  "(∀ i f . P (l†(min i k)..f) ∧ i ≤ k) ⟷ (∀ i f . P (l†i..f) ∧ i ≤ k)"
using not_less by auto

lemma (in sliceable) slice_f_max_neutral:
  "(P (l†i..(max f k)) ∧ f ≥ k) ⟷ (P (l†i..f) ∧ f ≥ k)"
by (metis max.orderE)

lemma (in sliceable) slice_i_max_neutral:
  "(P (l†(max i k)..f) ∧ i ≥ k) ⟷ (P (l†i..f) ∧ i ≥ k)"
by (metis max.orderE)
```

lemma *(***in** *sliceable) empty_slice[simp]: "i ≤ j ⟹ (#(l†j..i)) = 0"*
using *local.size_slice* **by** *auto*

corollary *(***in** *sliceable) forall_disjoint_slice_suc:*
"∀ i j . (disjoint x ∧ i≠j ∧ i < (#x) ∧ j < (#x)) ⟶
(x†i..(Suc i) ≠ x†j..(Suc j))"
by *(simp add: local.disjoint_slice_suc)*

A.2 Sliceable distinct lists

The following is the instantiation of the sliceable class for the dlist type.

instantiation *dlist :: (type) sliceable*
begin

definition
"l†i..f = Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l)))"

definition
"size l = length (list_of_dlist l)"

definition
"empty_inter l k =
((set (list_of_dlist l)) ∩ (set (list_of_dlist k)) = {})"

definition
"disjoint l = distinct (list_of_dlist l)"

lemma *list_of_dlist_slice :*
"list_of_dlist (l†i..f) = take (max 0 (f-i)) (drop i (list_of_dlist l))"
unfolding *slice_dlist_def*
by *simp*

lemma *Dlist_slice_inverse :*
"list_of_dlist (Dlist (take (max 0 (c-i)) (drop i (list_of_dlist x))))
= (take (max 0 (c-i)) (drop i (list_of_dlist x)))"
by *simp*

lemma *Dlist_empty_seq_inter: "c ≤ k ⟹*
(
set (take c (list_of_dlist x)) ∩

```
  set (drop k (list_of_dlist x))
  ) = {}"
by (simp add: set_take_disj_set_drop_if_distinct)

lemma Dlist_forall_slice_eq1:
  "(∀i f. (Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l1))) =
  Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l2))))) ⟹
  l1 = l2"
by (metis (mono_tags, hide_lams) Dlist_list_of_dlist
  Sliceable_dlist.list_of_dlist_slice drop_0 drop_take max_0L take_equalityI)

lemma Dlist_forall_slice_eq:
  "l1 = l2 ⟷
  (∀i f. (Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l1))) =
  Dlist (take (max 0 (f-i)) (drop i (list_of_dlist l2)))))"
using Dlist_forall_slice_eq1 by blast

lemma distinct_list_take_1_uniqueness:
  "distinct l ⟹ i≠j ⟹ i < length l ⟹ j < length l ⟹
    take 1 (drop i l) ≠ take 1 (drop j l)"
by (simp add: hd_drop_conv_nth nth_eq_iff_index_eq take_Suc)

lemmas list_of_dlist_simps = slice_left_def slice_right_def slice_dlist_def
  size_dlist_def disjoint_dlist_def empty_inter_dlist_def Dlist_slice_inverse

instance proof

  fix l::"'a dlist"
  show "l†0..(#l) = l" by (simp add: slice_dlist_def size_dlist_def
    list_of_dlist_inverse)
  next
  fix l::"'a dlist" and c::nat and k
  assume "c ≤ k"
  thus "empty_inter (l†0..c) (l†k..(#l))"
  by (simp add: size_dlist_def empty_inter_dlist_def
    set_take_disj_set_drop_if_distinct list_of_dlist_slice )
  next
  fix l::"'a dlist" and i and j and a and b
  show "size (l†i..j) = max 0 (min j (#l) - i)"
  proof (cases "j ≤ #l")
    case True
    assume "j ≤ #l"
```

```
  thus ?thesis
    by (metis (no_types, hide_lams) list_of_dlist_simps(7) size_dlist_def
      drop_take length_drop length_take list_of_dlist_simps(3) max_0L
      min.commute)
  next
  case False
  assume "¬ (j ≤ #l)"
  hence "j > #l" by simp
  thus ?thesis
    by (metis (no_types, lifting) list_of_dlist_simps(3)
      list_of_dlist_simps(7) size_dlist_def length_drop length_take max_0L
      min.commute min_diff)
qed
next
fix l::"'a dlist" and i and j and a and b
show "(l†i..j)†a..b = l†(i + a)..(min j (i + b))"

proof -
  have f1: "take b (take (max 0 (j - i)) (drop i (list_of_dlist l))) =
    drop i (take (min (i + b) j) (list_of_dlist l))"
    by (metis (no_types) diff_add_inverse drop_take max_0L take_take)
  have "∀n na. min (n::nat) na = min na n"
    by (metis min.commute)
  thus ?thesis
    using f1 by (metis (no_types) list_of_dlist_slice add.commute drop_drop
      drop_take max_0L slice_dlist_def)
qed
next
fix l::"'a dlist" and i and j
assume "disjoint l" "i≠j" "i < (#l)" "j < (#l)"
hence "take 1 (drop i (list_of_dlist l)) ≠
  take 1 (drop j (list_of_dlist l))"
  using distinct_list_take_1_uniqueness size_dlist_def by auto
hence "take (Suc i - i) (drop i (list_of_dlist l)) ≠
  take (Suc j - j) (drop j (list_of_dlist l))"
  by simp
hence "take (max 0 (Suc i - i)) (drop i (list_of_dlist l)) ≠
  take (max 0 (Suc j - j)) (drop j (list_of_dlist l))"
  by simp
thus "l†i..Suc i ≠ l†j..Suc j"
by (metis list_of_dlist_slice)
next
```

```
  fix l::"'a dlist" and i and j
  assume "disjoint l"
  thus "disjoint (l†i..j)"
    by (simp add: disjoint_dlist_def)
  next
  fix l1::"'a dlist" and l2::"'a dlist"
  show "(#l1) = (#l2) ∧ (∀i j. l1†i..j = l2†i..j) ⟷ (l1 = l2)"
    using Dlist_forall_slice_eq
    by (metis Sliceable_dlist.list_of_dlist_slice)
qed

end
```

A.2.1 Properties of sliceable distinct lists

In the following we present lemmas, corollaries and theorems about sliceable distinct lists.

```
abbreviation dlist_nth :: "'a dlist ⇒ nat ⇒ 'a"
where
"dlist_nth l i ≡ (list_of_dlist (sliceable_nth l i))!0"

theorem set_slice :
  "set (list_of_dlist l) =
    set (list_of_dlist (l†..i)) ∪ set (list_of_dlist (l†i..))"
unfolding slice_dlist_def slice_right_def slice_left_def size_dlist_def
apply (simp add: list_of_dlist_inject)
by (metis append_take_drop_id set_append)

theorem take_slice_right: "take n (list_of_dlist l) = list_of_dlist (l†..n)"
unfolding slice_right_def slice_dlist_def
by (metis Dlist_slice_inverse drop_0 max_0L minus_nat.diff_0)

theorem slice_right_cons: "distinct (x # xs) ⟹
  (Dlist (x # xs))†..(Suc n) = Dlist (x # (list_of_dlist ((Dlist xs)†..n)))"
unfolding slice_right_def slice_dlist_def
by (simp add: distinct_remdups_id)

theorem slice_append:
  "∀n. Dlist ((list_of_dlist (l†..n)) @ (list_of_dlist (l†n..))) = l"
unfolding size_dlist_def slice_left_def slice_right_def
by (simp add: list_of_dlist_inverse list_of_dlist_slice )
```

```
theorem slice_append_mid:
"∀ i s e. s ≤ i ∧ i ≤ e ⟶
  ((list_of_dlist (l†s..i)) @ (list_of_dlist (l†i..e))) =
    list_of_dlist (l†s..e)"
unfolding size_dlist_def slice_left_def slice_right_def list_of_dlist_slice
by (smt Nat.diff_add_assoc2 drop_drop le_add_diff_inverse
  le_add_diff_inverse2 max_0L take_add)

theorem slice_append_3:
"∀ i j. i ≤ j ⟶
  ((list_of_dlist (l†..i)) @
    (list_of_dlist (l†i..j)) @ (list_of_dlist (l†j..))) = list_of_dlist l"
unfolding size_dlist_def slice_left_def slice_right_def list_of_dlist_slice
by (metis append_assoc append_take_drop_id drop_0 le_add_diff_inverse
  length_drop max.cobounded2 max_0L minus_nat.diff_0 take_add take_all)

theorem distinct_slice_lte_inter_empty[simp]:
  "i ≤ j ⟹ set (list_of_dlist (l†..i)) ∩ set (list_of_dlist (l†j..)) = {}"
unfolding size_dlist_def slice_left_def slice_right_def
by (simp add: Dlist_empty_seq_inter list_of_dlist_slice )

corollary distinct_slice_inter_empty [simp]:
  "set (list_of_dlist (l†..i)) ∩ set (list_of_dlist (l†i..)) = {}"
by simp

corollary distinct_slice_lt_inter_empty [simp]:
  "i < j ⟹ set (list_of_dlist (l†..i)) ∩ set (list_of_dlist (l†j..)) = {}"
by simp

corollary distinct_slice_diff1:
  "set (list_of_dlist (l†..i)) - set (list_of_dlist (l†i..)) =
    set (list_of_dlist (l†..i))"
by (simp add: Diff_triv)

corollary distinct_slice_diff2:
  "set (list_of_dlist (l†i..)) - set (list_of_dlist (l†..i)) =
    set (list_of_dlist (l†i..))"
using distinct_slice_diff1 by fastforce
```

theorem *distinct_in_set_slice1_not_in_slice2:*
"i ≤ j ⟹
x ∈ set (list_of_dlist (l†..i)) ∧ x ∈ set (list_of_dlist (l†j..)) ⟹
False"
using *distinct_slice_lte_inter_empty* **by** *fastforce*

corollary *distinct_in_set_slice1_implies_not_in_slice2:*
"i ≤ j ⟹ x ∈ set (list_of_dlist (l†..i)) ⟹
x ∈ set (list_of_dlist (l†j..)) ⟹ False"
by *(meson distinct_in_set_slice1_not_in_slice2)*

lemma *exists_sublist_or_not_sublist [simp]: "∃ i. l†..i ∈ T ∨ l†i.. ∉ T"*
unfolding *slice_right_def slice_left_def*
by *auto*

lemma *forall_slice_left_implies_exists [simp]:*
"∀ i . l†i.. ∈ S ⟹ ∃ i . l†(Suc i).. ∈ S"
unfolding *slice_right_def slice_left_def*
by *(simp add: slice_dlist_def)*

lemma *forall_slice_right_implies_exists [simp]:*
"∀ i . l†..i ∈ S ⟹ ∃ i . l†..(i-1) ∈ S"
unfolding *slice_right_def slice_left_def*
by *auto*

lemma *take_Suc_Cons_hd_tl: "length l > 0 ⟹*
take (Suc n) l = hd l # (take n (tl l))"
apply *(induct l)*
by *auto*

corollary *take_Suc_Cons_hd_tl_singleton:*
"length l > 0 ⟹ take (Suc 0) l = [hd l]"
apply *(induct l)*
by *auto*

lemma *take_drop_suc: "i < length l ⟹ length l > 0 ⟹*
take (max 0 ((Suc i) - i)) (drop i l) = [l!i]"
by *(metis (no_types, lifting) Suc_diff_Suc Suc_eq_plus1_left add.commute*
append_eq_append_conv cancel_comm_monoid_add_class.diff_cancel
hd_drop_conv_nth lessI max_0L numeral_1_eq_Suc_0 numeral_One take_add

```
  take_hd_drop)

lemma slice_right_take:"l†..i = Dlist (take i (list_of_dlist l))"
unfolding slice_right_def slice_dlist_def
by auto

lemma slice_left_drop: "l†i.. = Dlist (drop i (list_of_dlist l))"
unfolding slice_left_def slice_dlist_def size_dlist_def
by auto

lemma take_one_singleton_hd: "l ≠ [] ⟹ take (Suc 0) l = [hd l]"
apply (induct l, simp)
by auto

lemma take_one_singleton_nth: "l ≠ [] ⟹ take (Suc 0) l = [l!0]"
apply (induct l, simp)
by auto

lemma take_one_drop_n_append_singleton_nth:
  "ys ≠ [] ⟹ take 1 (drop (length xs) (xs @ ys)) =
  [(xs @ ys)!(length xs)]"
by (induct xs, auto simp add: take_one_singleton_nth)

lemma append_length_nth_hd: "ys ≠ [] ⟹ [(xs @ ys)!(length xs)] = [hd ys]"
by (induct ys, auto)

lemma take_one_drop_n_singleton_nth: "l ≠ [] ⟹ n < length l ⟹
  take 1 (drop n l) = [l!n]"
proof-
  assume 0: "l ≠ []"
  assume 1: "n < length l"
  obtain xs where "xs = take n l" by simp
  obtain ys where "ys = drop n l" by simp
  have "take 1 (drop n l) = take 1 (drop (length xs) (xs @ ys))" using 0 1
    by (simp add: ‘ys = drop n l‘)
  also have "... = [(xs @ ys)!(length xs)]" using 0 1
    by (metis ‘ys = drop n l‘ drop_eq_Nil not_le
      take_one_drop_n_append_singleton_nth)
  also have "... = [l!(length xs)]"
    by (simp add: ‘xs = take n l‘ ‘ys = drop n l‘)
  finally show ?thesis using 0 1
    by (simp add: hd_drop_conv_nth take_one_singleton_hd)
```

qed

lemma *slice_singleton: "(list_of_dlist l) ≠ [] ⟹ i < (#l) ⟹
list_of_dlist (l†i..(Suc i)) = [(list_of_dlist l)!i]"*
by *(metis list_of_dlist_slice length_greater_0_conv size_dlist_def
take_drop_suc)*

lemma *slice_right_zero_eq_empty: "list_of_dlist (l†..0) = []"*
by *(simp add: slice_right_def slice_dlist_def)*

lemma *slice_left_size_eq_empty: "list_of_dlist (l†(#l)..) = []"*
by *(simp add: slice_left_def slice_dlist_def)*

lemma *slice_right_singleton_eq_element: "list_of_dlist l ≠ [] ⟹
list_of_dlist (l†..1) = [(list_of_dlist l)!0]"*
by *(metis One_nat_def take_one_singleton_nth take_slice_right)*

lemma *slice_left_singleton_eq_element: "list_of_dlist l ≠ [] ⟹
list_of_dlist (l†((#l)-1)..) = [(list_of_dlist l)!((#l)-1)]"*
by *(metis (no_types, lifting) Cons_nth_drop_Suc list_of_dlist_slice
Suc_diff_Suc Suc_leI diff_Suc_eq_diff_pred diff_less drop_0 drop_all
drop_take length_greater_0_conv max_0L minus_nat.diff_0 size_dlist_def
slice_left_def slice_none zero_less_one)*

lemma *dlist_empty_slice[simp]: "i ≤ j ⟹ (l†j..i) = Dlist []"*
by *(simp add: slice_dlist_def)*

lemma *dlist_append_extreme_left:
"i≤j ⟹ list_of_dlist (l†..j) =
(list_of_dlist (l†..i)) @ (list_of_dlist (l†i..j))"*
by *(metis list_of_dlist_slice le_add_diff_inverse max_0L take_add
take_slice_right)*

lemma *dlist_append_extreme_right:
"i≤j ⟹ list_of_dlist (l†i..) =
(list_of_dlist (l†i..j)) @ (list_of_dlist (l†j..))"*
unfolding *list_of_dlist_slice slice_left_def slice_right_def*
by *(metis append_take_drop_id drop_drop le_add_diff_inverse2 length_drop
max.cobounded2 max_0L size_dlist_def take_all)*

lemma *dlist_disjoint[simp]: "disjoint (l::'a dlist)"*
by *(simp add: disjoint_dlist_def)*

```
lemma dlist_member_suc_nth1:
  "x ∈ set (list_of_dlist(l†i..(Suc i))) ⟹ x = (list_of_dlist l)!i"
proof-
  assume 0: "x ∈ set (list_of_dlist (l†i..(Suc i)))"
  obtain rl where 1:"rl = list_of_dlist l" by blast
  hence "x ∈ set (take (max 0 (Suc i - i)) (drop i rl))"
    using 0 by (metis list_of_dlist_slice )
  hence "x ∈ set (take 1 (drop i rl))" by simp
  hence "x = rl!i"
    by (metis drop_Nil drop_all empty_iff list.inject list.set(1)
      list.set_cases not_less take_Nil take_one_drop_n_singleton_nth)
  thus ?thesis using 1 by simp
qed

lemma dlist_member_suc_nth2:
  "i < (#l) ⟹ x = (list_of_dlist l)!i ⟹
  x ∈ set (list_of_dlist (l†i..(Suc i)))"
unfolding size_dlist_def slice_dlist_def
by (metis Dlist_slice_inverse drop_Nil drop_eq_Nil leD length_greater_0_conv
  list.set_intros(1) take_drop_suc)

lemma dlist_member_suc_nth: "i < (#l) ⟹
  (x = (list_of_dlist l)!i) ⟷ (x ∈ set (list_of_dlist (l†i..(Suc i))))"
using dlist_member_suc_nth1 dlist_member_suc_nth2
by fastforce
```

A.3 Algebra of Temporal Faults

In the following we present the algebraic laws for the ATF.

A.3.1 Basic ATF operators and tempo1

```
class temporal_faults_algebra_basic = boolean_algebra +
  fixes xbefore :: "'a ⇒ 'a ⇒ 'a"
  fixes tempo1 :: "'a ⇒ bool"
  assumes xbefore_bot_1: "xbefore bot a = bot"
  assumes xbefore_bot_2: "xbefore a bot = bot"
  assumes xbefore_not_idempotent: "tempo1 a ⟹ xbefore a a = bot"
```

```
  assumes inf_tempo1: "⟦tempo1 a; tempo1 b⟧ ⟹ tempo1 (inf a b)"
  assumes xbefore_not_sym:
    "⟦tempo1 a; tempo1 b⟧ ⟹ (xbefore a b) ≤ -(xbefore b a)"
```

A.3.2 Definition of associativity of XBefore

```
class temporal_faults_algebra_assoc = temporal_faults_algebra_basic +
  assumes xbefore_assoc: "⟦tempo1 a; tempo1 b; tempo1 c⟧ ⟹
    xbefore (xbefore a b) c = xbefore a (xbefore b c)"
```

A.3.3 Equivalences in the ATF and properties

```
class temporal_faults_algebra_equivs = temporal_faults_algebra_assoc +
  fixes independent_events :: "'a ⇒ 'a ⇒ bool"
  fixes tempo2 :: "'a ⇒ bool"
  fixes tempo3 :: "'a ⇒ bool"
  fixes tempo4 :: "'a ⇒ bool"
  assumes xbefore_inf_equiv_bot:
    "⟦tempo1 a; tempo1 b⟧ ⟹ inf (xbefore a b) (xbefore b a) = bot"
  assumes xbefore_sup_equiv_inf:
    "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup (xbefore a b) (xbefore b a) = inf a b"
  assumes sup_tempo2: "⟦tempo2 a; tempo2 b⟧ ⟹ tempo2 (sup a b)"
  assumes inf_tempo3: "⟦tempo3 a; tempo3 b⟧ ⟹ tempo3 (inf a b)"
  assumes sup_tempo4: "⟦tempo4 a; tempo4 b⟧ ⟹ tempo4 (sup a b)"
```

A.3.4 XBefore transitivity

```
class temporal_faults_algebra_trans = temporal_faults_algebra_equivs +
  assumes xbefore_trans:
    "⟦tempo1 a; tempo1 b; tempo1 c⟧ ⟹ ⟦tempo2 a; tempo2 b; tempo2 c⟧ ⟹
    less_eq (inf (xbefore a b) (xbefore b c)) (xbefore a c)"
```

A.3.5 Mixed operators in ATF

```
class temporal_faults_algebra_mixed_ops = temporal_faults_algebra_trans +
  assumes xbefore_sup_1:
    "xbefore (sup a b) c = sup (xbefore a c) (xbefore b c)"
  assumes xbefore_sup_2:
    "xbefore a (sup b c) = sup (xbefore a b) (xbefore a c)"
  assumes xbefore_not: "
    independent_events a b ⟹
```

```
  ⟦tempo1 a; tempo1 b⟧ ⟹
  ⟦tempo2 a; tempo2 b⟧ ⟹
  ⟦tempo3 a; tempo3 b⟧ ⟹
  ⟦tempo4 a; tempo4 b⟧ ⟹
  - (xbefore a b) = sup (sup (- a) (- b)) (xbefore b a)"
 assumes inf_xbefore_equiv_sups_xbefore: "tempo2 a ⟹ inf a (xbefore b c) =
  sup (xbefore (inf a b) c) (xbefore b (inf a c))"

class temporal_faults_algebra = temporal_faults_algebra_mixed_ops
```

A.3.6 Theorems in the context of ATF

The following theorems are valid for ATF. They are valid for any instantiation of the ATF class as, for example, for the sets of distinct lists type.

```
context temporal_faults_algebra
begin

theorem xbefore_inf_1:
 "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
 ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
 xbefore (inf a b) c =
  sup (xbefore (xbefore a b) c) (xbefore (xbefore b a) c)"
proof-
 assume "independent_events a b" "tempo1 a" "tempo1 b"
 "tempo2 a" "tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
 hence "xbefore (inf a b) c = xbefore (sup (xbefore a b) (xbefore b a)) c"
  by (simp add: xbefore_sup_equiv_inf)
 thus ?thesis by (simp add: xbefore_sup_1)
qed

theorem xbefore_inf_2:
 "independent_events b c ⟹ ⟦tempo1 b; tempo1 c⟧ ⟹
 ⟦tempo2 b; tempo2 c⟧ ⟹ ⟦tempo3 b; tempo3 c⟧ ⟹ ⟦tempo4 b; tempo4 c⟧ ⟹
 xbefore a (inf b c) =
  sup (xbefore a (xbefore b c)) (xbefore a (xbefore c b))"
proof-
 assume "independent_events b c" "tempo1 b" "tempo1 c" "tempo2 b" "tempo2 c"
 "tempo3 b" "tempo3 c" "tempo4 b" "tempo4 c"
 hence "xbefore a (inf b c) = xbefore a (sup (xbefore b c) (xbefore c b))"
  by (simp add: xbefore_sup_equiv_inf)
 thus ?thesis by (simp add: xbefore_sup_2)
```

qed

lemma *xbefore_sup_absorb_1b:*
  "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup (xbefore b a) a = a"
by *(metis inf_le1 order_trans sup.absorb2 sup.cobounded2*
  xbefore_sup_equiv_inf)

lemma *xbefore_sup_absorb_2:*
  "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup a (xbefore a b) = a"
by *(metis dual_order.trans inf.cobounded1 sup.absorb1 sup.cobounded1*
  xbefore_sup_equiv_inf)

corollary *xbefore_sup_absorb_1:*
  "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup (xbefore a b) a = a"
proof-
  assume *0: "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a"*
  "tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
  hence *"sup a (xbefore a b) = sup (xbefore a b) a"*
    by *(simp add: sup.commute)*
  thus *?thesis* **using** *0* **by** *(simp add: xbefore_sup_absorb_2)*
qed

corollary *xbefore_sup_absorb_2b:*
  "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup a (xbefore b a) = a"
proof-
  assume *0: "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a"*
  "tempo2 b" "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
  hence *"sup a (xbefore b a) = sup (xbefore b a) a"*
    by *(simp add: sup.commute)*
  thus *?thesis* **using** *0* **by** *(simp add: xbefore_sup_absorb_1b)*
qed

```
corollary inf_xbefore_equiv_sups_xbefore_expanded:
  "independent_events a b ⟹ independent_events a c ⟹
  ⟦tempo1 a; tempo1 b; tempo1 c⟧ ⟹ ⟦tempo2 a; tempo2 b; tempo2 c⟧ ⟹
  ⟦tempo3 a; tempo3 b; tempo3 c⟧ ⟹ ⟦tempo4 a; tempo4 b; tempo4 c⟧ ⟹
    inf a (xbefore b c) =
    sup (sup (xbefore (xbefore a b) c)
      (xbefore (xbefore b a) c))
      (xbefore (xbefore b c) a)"
proof-
  assume "independent_events a b" "independent_events a c"
    "tempo1 a" "tempo1 b" "tempo1 c"
    "tempo2 a" "tempo2 b" "tempo2 c"
    "tempo3 a" "tempo3 b" "tempo3 c"
    "tempo4 a" "tempo4 b" "tempo4 c"
  hence "inf a (xbefore b c) =
    sup (xbefore (inf a b) c) (xbefore b (inf a c))"
    "xbefore (inf a b) c =
    sup (xbefore (xbefore a b) c) (xbefore (xbefore b a) c)"
    "xbefore b (inf a c) =
    sup (xbefore (xbefore b a) c) (xbefore (xbefore b c) a)"
    by (auto simp add: inf_xbefore_equiv_sups_xbefore xbefore_inf_1
      xbefore_inf_2 xbefore_assoc)
  thus ?thesis by (simp add: sup.assoc)
qed

lemma xbefore_sup_compl_inf_absorb1:
  "independent_events a b ⟹ ⟦tempo1 a; tempo1 b⟧ ⟹
    ⟦tempo2 a; tempo2 b⟧ ⟹ ⟦tempo3 a; tempo3 b⟧ ⟹ ⟦tempo4 a; tempo4 b⟧ ⟹
    sup (inf a (-b)) (xbefore a b) = inf a (- (xbefore b a))"
proof -
  assume a1: "independent_events a b"
  assume a2: "tempo1 a"
  assume a3: "tempo1 b"
  assume a4: "tempo2 a"
  assume a5: "tempo2 b"
  assume a6: "tempo3 a"
  assume a7: "tempo3 b"
  assume a8: "tempo4 a"
  assume a9: "tempo4 b"
  hence f10: "- xbefore a b = sup (sup (- a) (- b)) (xbefore b a)"
    using a8 a7 a6 a5 a4 a3 a2 a1 by (metis (no_types) local.xbefore_not)
  have f11: "∀a aa ab. inf (a::'a) (sup aa ab) = sup (inf a aa) (inf a ab)"
```

```
    using local.distrib_imp2 local.sup_inf_distrib1 by blast
  hence "sup (inf a (- b)) (inf a b) = a"
    by (metis (no_types) local.compl_sup_top local.inf_top_right)
  hence "sup (inf a (- b)) (xbefore a b) =
    inf a (sup (inf a (- b)) (- xbefore b a))"
    using f10 by (metis (no_types) local.compl_sup local.double_compl
      local.sup_inf_distrib1)
  hence f12: "sup (inf a (- b)) (xbefore a b) =
    sup (inf a (- xbefore b a)) (inf a (- b))"
    using f11 by (simp add: local.sup.commute)
  have f13: "sup (xbefore a b) (xbefore b a) = inf a b"
    using a9 a8 a7 a6 a5 a4 a3 a2 a1 by (metis (no_types)
      local.xbefore_sup_equiv_inf)
  have "sup a (xbefore b a) = a"
    using a9 a8 a7 a6 a5 a4 a3 a2 a1 by (meson xbefore_sup_absorb_2b)
  hence f14: "sup (inf a (- xbefore b a)) (xbefore b a) = a"
    using local.sup_inf_distrib2 by auto
  have "sup (xbefore a b) a = a"
    using a9 a8 a7 a6 a5 a4 a3 a2 a1 by (meson xbefore_sup_absorb_1)
  hence "sup (inf a (- xbefore b a)) (inf a b) = a"
    using f14 f13 by (metis (no_types) local.sup_left_commute)
  hence "sup (inf a (- xbefore b a)) (inf a b) =
    sup (inf a (- xbefore b a)) a"
    using local.sup.commute by auto
  hence "sup (inf a (- b)) (xbefore a b) =
    sup (inf a (- xbefore b a)) (inf (- b) (inf a b))"
    using f12 local.inf_commute local.sup_inf_distrib1 by auto
  thus ?thesis
    using local.inf_left_commute by auto
qed


end
```

A.4 Denotational semantics for ATF

In the following we present the denotation semantics for ATF in terms of sets of distinct lists.

A.4.1 Formula: distinct lists

The definition of a formula in the ATF is a set of sets of distinct lists (dlist).

```
typedef 'a formula = "UNIV::'a dlist set set" by simp
```

A.4.1.1 Formula as Boolean algebra

In the following we instantiate the formula as a Boolean algebra and prove that Boolean operators are valid.

```
instantiation formula :: (type) boolean_algebra
begin

definition
  "x ⊓ y = Abs_formula (Rep_formula x ∩ Rep_formula y)"

definition
  "x ⊔ y = Abs_formula (Rep_formula x ∪ Rep_formula y)"

definition
  "⊤ = Abs_formula UNIV"

definition
  "⊥ = Abs_formula {}"

definition
  "x ≤ y ⟷ Rep_formula x ⊆ Rep_formula y"

definition
  "x < y ⟷ Rep_formula x ⊂ Rep_formula y"

definition
  "- x = Abs_formula (- (Rep_formula x))"

definition
  "x - y = Abs_formula (Rep_formula x - Rep_formula y)"

lemma Rep_formula_inf:
  "Rep_formula (x ⊓ y) = Rep_formula x ∩ Rep_formula y"
unfolding inf_formula_def
by (simp add: Abs_formula_inverse Rep_formula)
```

```
lemma Rep_formula_sup:
  "Rep_formula (x ⊔ y) = Rep_formula x ∪ Rep_formula y"
unfolding sup_formula_def
by (simp add: Abs_formula_inverse Rep_formula)

lemma Rep_formula_top[simp]: "Rep_formula ⊤ = UNIV"
unfolding top_formula_def
by (simp add: Abs_formula_inverse)

lemma Rep_formula_bot[simp]: "Rep_formula ⊥ = {}"
unfolding bot_formula_def
by (simp add: Abs_formula_inverse)

lemma Rep_formula_compl: "Rep_formula (- x) = - Rep_formula x"
unfolding uminus_formula_def
by (simp add: Abs_formula_inverse Rep_formula)

lemma Rep_formula_diff:
  "Rep_formula (x - y) = Rep_formula x - Rep_formula y"
unfolding minus_formula_def
by (simp add: Abs_formula_inverse Rep_formula)

lemmas eq_formula_iff = Rep_formula_inject [symmetric]

lemmas Rep_formula_simps =
  less_eq_formula_def less_formula_def eq_formula_iff
  Rep_formula_sup Rep_formula_inf Rep_formula_top Rep_formula_bot
  Rep_formula_compl Rep_formula_diff

instance proof
qed (unfold Rep_formula_simps, auto)
end
```

A.4.1.2 Tempo properties

In this section we define the tempo properties.

Tempo1: disjoint split

```
definition dlist_tempo1 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo1 S ≡ ∀i j l. i ≤ j ⟶ ¬ ((S (l↑..i) ∧ S (l↑j..)))"
```

Tempo2: belonging iff

```
definition dlist_tempo2 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo2 S ≡ ∀i l. S l ⟷ (S (l†..i) ∨ S (l†i..))"

definition dlist_tempo3 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo3 S ≡ ∀i j l. j < i ⟶ (S (l†j..i) ⟷
  (S (l†..i) ∧ S (l†j..)))"

definition dlist_tempo4 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo4 S ≡ ∀ l. S l ⟷ (∃i. S (l†i..(Suc i)))"

definition dlist_tempo5 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo5 S ≡
  ∀ i j l. (i ≠ j ∧ i < (#l) ∧ j < (#l)) ⟶
    ¬(S (l†i..(Suc i)) ∧ S (l†j..(Suc j)))"

definition dlist_tempo6 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo6 S ≡ ∀l. (∀ i j. ¬ S (l†i..j)) ⟷ ¬ S l"

definition dlist_tempo7 :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo7 S ≡ ∀l. (∃ i j. i < j ∧ S (l†i..j)) ⟷ S l"

definition dlist_tempo :: "('a dlist ⇒ bool) ⇒ bool"
where
"dlist_tempo S ≡ dlist_tempo1 S ∧ dlist_tempo2 S ∧
  dlist_tempo3 S ∧ dlist_tempo5 S ∧ dlist_tempo4 S ∧ dlist_tempo6 S ∧
  dlist_tempo7 S"

lemmas tempo_defs = dlist_tempo_def dlist_tempo1_def dlist_tempo2_def
  dlist_tempo3_def dlist_tempo5_def dlist_tempo4_def dlist_tempo6_def
  dlist_tempo7_def

lemma dlist_tempo_1_no_gap:
  "dlist_tempo1 S ⟹ ∀i l. ¬ ((S (l†..i) ∧ S (l†i..)))"
unfolding dlist_tempo1_def
by auto
```

```
corollary dlist_tempo_1_no_gap_append:
  "dlist_tempo1 S ⟹
    ∀zs xs ys. list_of_dlist zs = list_of_dlist xs @ list_of_dlist ys ⟶
    ¬ ((S xs ∧ S ys))"
using dlist_tempo_1_no_gap
by (metis Dlist_list_of_dlist append_eq_conv_conj slice_left_drop
  take_slice_right)
```

A.4.1.3 Tempo properties for list member

We use the naming convention of variable, but in fact, a variable is equivalent to a list membership: *var a = {xs | a ∈ list_of_dlist xs}*.

```
lemma dlist_tempo1_member: "dlist_tempo1 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo1_def
by (meson distinct_in_set_slice1_not_in_slice2)

lemma dlist_tempo2_member: "dlist_tempo2 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo2_def
by (metis (no_types, lifting) Un_iff set_slice )

lemma dlist_tempo3_member: "dlist_tempo3 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo3_def
by (metis DiffD2 Un_iff distinct_slice_diff2 dlist_append_extreme_left
  dlist_append_extreme_right less_imp_le_nat set_append)

lemma dlist_tempo5_member: "dlist_tempo5 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo5_def
by (metis Dlist_list_of_dlist Suc_leI disjoint_dlist_def disjoint_slice_suc
  distinct_list_of_dlist dlist_empty_slice dlist_member_suc_nth1 empty_slice
  less_Suc_eq_0_disj not_less_eq slice_singleton)

lemma dlist_tempo4_member: "dlist_tempo4 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo4_def

by (metis dlist_member_suc_nth in_set_conv_nth in_set_dropD in_set_takeD
  list_of_dlist_Dlist set_remdups size_dlist_def slice_dlist_def)

lemma dlist_tempo6_member: "dlist_tempo6 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo6_def
by (metis append_Nil in_set_conv_decomp in_set_conv_nth in_set_dropD
  in_set_takeD length_pos_if_in_set list_of_dlist_slice take_drop_suc)
```

```
lemma dlist_tempo7_member: "dlist_tempo7 (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo7_def
by (metis Un_iff dlist_append_extreme_left dlist_member_suc_nth2
  in_set_conv_nth lessI less_imp_le_nat set_append set_slice size_dlist_def)

theorem dlist_tempo_member: "dlist_tempo (λxs. a ∈ set (list_of_dlist xs))"
unfolding dlist_tempo_def
by (simp add: dlist_tempo1_member dlist_tempo2_member dlist_tempo3_member
  dlist_tempo5_member dlist_tempo4_member dlist_tempo6_member
  dlist_tempo7_member)
```

A.4.1.4 Tempo properties for other operators

```
lemma dlist_tempo1_inf: "⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
  dlist_tempo1 (λzs. a zs ∧ b zs)"
unfolding dlist_tempo1_def
by simp

lemma dlist_tempo3_inf: "⟦dlist_tempo3 a; dlist_tempo3 b⟧ ⟹
  dlist_tempo3 (λzs. a zs ∧ b zs)"
unfolding dlist_tempo3_def
by auto

lemma dlist_tempo2_sup: "⟦dlist_tempo2 a; dlist_tempo2 b⟧ ⟹
  dlist_tempo2 (λzs. a zs ∨ b zs)"
unfolding dlist_tempo2_def
by auto

lemma dlist_tempo4_sup: "⟦dlist_tempo4 a; dlist_tempo4 b⟧ ⟹
  dlist_tempo4 (λzs. a zs ∨ b zs)"
unfolding dlist_tempo4_def
by blast
```

A.4.2 XBefore of distinct lists

```
definition dlist_xbefore :: "('a dlist ⇒ bool) ⇒ ('a dlist ⇒ bool) ⇒
  'a dlist ⇒ bool"
where
"dlist_xbefore a b xs ≡ ∃i. a (xs↑..i) ∧ b (xs↑i..)"
```

A.4.2.1 XBefore and temporal properties

lemma *dlist_tempo1_xbefore: "⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
dlist_tempo1 (dlist_xbefore a b)"*
unfolding *dlist_tempo1_def dlist_xbefore_def slice_slice_simps*
by *(smt le_add1 min.absorb2 min.coboundedl slice_right_slice_left_absorb
slice_right_slice_right_absorb)*

A.4.2.2 XBefore and appending

lemma *Rep_slice_append:
"list_of_dlist zs = (list_of_dlist (zs†..i)) @ (list_of_dlist (zs†i..))"*
by *(metis distinct_append distinct_list_of_dlist distinct_slice_inter_empty
list_of_dlist_Dlist remdups_id_iff_distinct slice_append)*

lemma *dlist_xbefore_append:
"dlist_xbefore a b zs ⟷
(∃xs ys. set (list_of_dlist xs) ∩ set (list_of_dlist ys) =
{} ∧ a xs ∧ b ys ∧
list_of_dlist zs = ((list_of_dlist xs) @ (list_of_dlist ys)))"*
unfolding *dlist_xbefore_def*
by *(metis Rep_slice_append append_Nil2 append_eq_conv_conj
distinct_slice_inter_empty dlist_xbefore_def drop_take max_0L
size_dlist_def slice_append slice_dlist_def slice_left_def slice_right_def
take_slice_right)*

A.4.2.3 XBefore, bot and idempotency

lemma *dlist_xbefore_bot_1: "dlist_xbefore (λxs. False) b zs = False"*
unfolding *dlist_xbefore_def*
by *simp*

corollary *dlistset_xbefore_bot_1:
"Collect (dlist_xbefore (λxs. False) b) = {}"*
by *(simp add: dlist_xbefore_bot_1)*

lemma *dlist_xbefore_bot_2: "dlist_xbefore a (λxs. False) zs = False"*
unfolding *dlist_xbefore_def*
by *simp*

lemma *dlistset_xbefore_bot_2:
"Collect (dlist_xbefore a (λxs. False)) = {}"*
by *(simp add: dlist_xbefore_bot_2)*

```
lemma dlist_xbefore_idem:
  "dlist_tempo1 a ⟹ dlist_xbefore a a zs = False"
unfolding dlist_xbefore_def dlist_tempo1_def
by blast

lemma dlistset_xbefore_idem:
  "dlist_tempo1 a ⟹ Collect (dlist_xbefore a a) = {}"
by (simp add: dlist_xbefore_idem)

lemma dlist_xbefore_implies_idem:
  "∀xs. b xs ⟶ a xs ⟹ dlist_tempo1 a ⟹ dlist_xbefore a b zs = False"
unfolding dlist_tempo1_def dlist_xbefore_def
by blast
```

A.4.2.4 XBefore associativity

```
theorem dlist_xbefore_assoc1:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹ dlist_tempo1 U ⟹
  (dlist_xbefore (dlist_xbefore S T) U zs) ⟷
    (dlist_xbefore S (dlist_xbefore T U) zs)"
unfolding dlist_xbefore_def slice_slice_simps dlist_tempo_def
apply auto
apply (metis diff_is_0_eq less_imp_le max_0L min_def not_le
  ordered_cancel_comm_monoid_diff_class.le_iff_add slice_dlist_def
  take_eq_Nil)
by (metis le_add1 min.absorb2)

corollary dlist_xbefore_assoc:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹ dlist_tempo1 U ⟹
  (dlist_xbefore (dlist_xbefore S T) U) =
    (dlist_xbefore S (dlist_xbefore T U))"
using dlist_xbefore_assoc1 by blast

corollary dlistset_xbefore_assoc:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹ dlist_tempo1 U ⟹
  Collect (dlist_xbefore (dlist_xbefore S T) U) =
    Collect (dlist_xbefore S (dlist_xbefore T U))"
by (simp add: dlist_xbefore_assoc)
```

A.4.2.5 XBefore equivalences

```
lemma dlist_tempo1_le_uniqueness:
```

```
  "dlist_tempo1 S ⟹ S (l†..i) ⟹ i ≤ j ⟹ ¬ S (l†j..)" and
  "dlist_tempo1 S ⟹ S (l†j..) ⟹ i ≤ j ⟹ ¬ S (l†..i)"
unfolding dlist_tempo1_def
by auto

lemma dlist_xbefore_not_sym:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹ dlist_xbefore S T xs ⟹
  dlist_xbefore T S xs ⟹ False"
by (metis dlist_xbefore_def le_cases dlist_tempo1_le_uniqueness)

corollary dlist_xbefore_and:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
    ((dlist_xbefore S T zs) ∧ (dlist_xbefore T S zs)) = False"
using dlist_xbefore_not_sym by blast


corollary dlistset_xbefore_and:
  "dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
  (Collect (dlist_xbefore S T)) ∩ (Collect (dlist_xbefore T S)) = {}"
using dlist_xbefore_and
by auto

lemma dlist_tempo2_left_absorb: "dlist_tempo2 S ⟹ S (l†i..) ⟹ S l"
unfolding dlist_tempo2_def
by auto

lemma dlist_tempo2_right_absorb: "dlist_tempo2 S ⟹ S (l†..i) ⟹ S l"
unfolding dlist_tempo2_def
by auto

lemma dlist_xbefore_implies_member1[simp]:
  "dlist_tempo2 S ⟹ dlist_xbefore S T l ⟹ S l"
by (meson dlist_xbefore_def dlist_tempo2_right_absorb)

lemma dlist_xbefore_implies_member2[simp]:
  "dlist_tempo2 T ⟹ dlist_xbefore S T l ⟹ T l"
by (meson dlist_xbefore_def dlist_tempo2_left_absorb)

lemma dlist_xbefore_or1:
  "dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
  dlist_xbefore S T l ∨ dlist_xbefore T S l ⟹ S l ∧ T l"
using dlist_xbefore_implies_member1 dlist_xbefore_implies_member2 by blast
```

```
definition dlist_independent_events ::
  "('a dlist ⇒ bool) ⇒ ('a dlist ⇒ bool) ⇒ bool"
where
"dlist_independent_events S T ≡
  (∀i l. ¬ (S (l†i..(Suc i)) ∧ T (l†i..(Suc i))))"


lemma "dlist_independent_events a b ⟹ ∀xs. b xs ⟶ a xs ⟹ False"
unfolding dlist_independent_events_def
sorry


lemma dlist_and_split9:
  "dlist_independent_events S T ⟹
    dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
    dlist_tempo3 S ⟹ dlist_tempo3 T ⟹
    dlist_tempo4 S ⟹ dlist_tempo4 T ⟹
  S l ∧ T l ⟷ (∃i j. i ≤ j ∧
    ((S (l†..i) ∧ T (l†j..)) ∨ (S (l†j..) ∧ T (l†..i))))"
unfolding dlist_independent_events_def
  dlist_tempo2_def dlist_tempo3_def dlist_tempo4_def
by (metis le_refl not_less not_less_eq_eq)

lemma dlist_tempo_equiv_xor:
  "dlist_tempo1 S ⟹ dlist_tempo2 S ⟹
  ∀l. S l ⟷ (∀i. (S (l†..i) ∧ ¬ S (l†i..)) ∨ (¬ S (l†..i) ∧ S (l†i..)))"
unfolding tempo_defs
by (meson order_refl)

corollary dlist_tempo_equiv_not_eq: "dlist_tempo1 S ⟹ dlist_tempo2 S ⟹
  ∀l. S l ⟷ (∀i. S (l†..i) ≠ S (l†i..))"
using dlist_tempo_equiv_xor
by auto

lemma dlists_xbefore_or2:
  "dlist_independent_events S T ⟹
  dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
  dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
  dlist_tempo3 S ⟹ dlist_tempo3 T ⟹
  dlist_tempo4 S ⟹ dlist_tempo4 T ⟹
```

```
  S l ∧ T l ⟹ dlist_xbefore S T l ∨ dlist_xbefore T S l"
unfolding dlist_xbefore_def dlist_tempo_def
by (metis dlist_and_split9 dlist_tempo_equiv_not_eq
  dlist_tempo1_le_uniqueness)

theorem dlist_xbefore_or_one_list:
  "dlist_independent_events S T ⟹
  dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
  dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
  dlist_tempo3 S ⟹ dlist_tempo3 T ⟹
  dlist_tempo4 S ⟹ dlist_tempo4 T ⟹
  dlist_xbefore S T l ∨ dlist_xbefore T S l ⟷ S l ∧ T l"
using dlist_xbefore_or1 dlists_xbefore_or2 dlist_tempo_def
by blast

corollary dlist_xbefore_or:
  "dlist_independent_events S T ⟹
  dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
  dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
  dlist_tempo3 S ⟹ dlist_tempo3 T ⟹
  dlist_tempo4 S ⟹ dlist_tempo4 T ⟹
  (λzs. (dlist_xbefore S T zs) ∨ (dlist_xbefore T S zs)) =
    (λzs. S zs ∧ T zs)"
using dlist_xbefore_or_one_list
by blast

corollary dlistset_xbefore_or:
  "dlist_independent_events S T ⟹
  dlist_tempo1 S ⟹ dlist_tempo1 T ⟹
  dlist_tempo2 S ⟹ dlist_tempo2 T ⟹
  dlist_tempo3 S ⟹ dlist_tempo3 T ⟹
  dlist_tempo4 S ⟹ dlist_tempo4 T ⟹
  (Collect (dlist_xbefore S T)) ∪ (Collect (dlist_xbefore T S)) =
    Collect S ∩ Collect T"
using dlist_xbefore_or
by (smt Collect_cong Collect_conj_eq Collect_disj_eq)
```

A.4.2.6 XBefore transitivity

```
theorem dlist_xbefore_trans: "
  ⟦dlist_tempo1 a; dlist_tempo1 b; dlist_tempo1 c⟧ ⟹
  ⟦dlist_tempo2 a; dlist_tempo2 b; dlist_tempo2 c⟧ ⟹
```

```
  dlist_xbefore a b zs ∧ dlist_xbefore b c zs ⟹
  dlist_xbefore a c zs"
using dlist_xbefore_not_sym
by (metis dlist_tempo2_def dlist_xbefore_def)

corollary dlistset_xbefore_trans: "
  ⟦dlist_tempo1 a; dlist_tempo1 b; dlist_tempo1 c⟧ ⟹
  ⟦dlist_tempo2 a; dlist_tempo2 b; dlist_tempo2 c⟧ ⟹
  (Collect (dlist_xbefore a b) ∩ Collect (dlist_xbefore b c)) ⊆
    Collect (dlist_xbefore a c)"
using dlist_xbefore_trans
by auto
```

A.4.2.7 Boolean operators mixed with XBefore

```
theorem mixed_dlist_xbefore_or1: "
  dlist_xbefore (λxs. a xs ∨ b xs) c zs =
  ((dlist_xbefore a c zs) ∨ (dlist_xbefore b c zs))"
unfolding dlist_xbefore_def by auto

corollary mixed_dlistset_xbefore_or1: "
  Collect (dlist_xbefore (λxs. a xs ∨ b xs) c) =
  Collect (dlist_xbefore a c) ∪ Collect (dlist_xbefore b c)"
proof-
  have "Collect (λzs. (dlist_xbefore a c zs) ∨ (dlist_xbefore b c zs)) =
    (Collect (dlist_xbefore a c) ∪ Collect (dlist_xbefore b c))"
    by (simp add: Collect_disj_eq)
  thus ?thesis using mixed_dlist_xbefore_or1 by blast
qed

theorem mixed_dlist_xbefore_or2: "
  dlist_xbefore a (λxs. b xs ∨ c xs) zs =
  ((dlist_xbefore a b zs) ∨ (dlist_xbefore a c zs))"
unfolding dlist_xbefore_def by auto

corollary mixed_dlistset_xbefore_or2: "
  Collect (dlist_xbefore a (λxs. b xs ∨ c xs)) =
  Collect (dlist_xbefore a b) ∪ Collect (dlist_xbefore a c)"
proof-
  have "Collect (λzs. (dlist_xbefore a b zs) ∨ (dlist_xbefore a c zs)) =
    Collect (dlist_xbefore a b) ∪ Collect (dlist_xbefore a c)"
    by (simp add: Collect_disj_eq)
```

```
  thus ?thesis using mixed_dlist_xbefore_or2 by blast
qed

lemma and_dlist_xbefore_equiv_or_dlist_xbefore:
  "dlist_tempo2 a ⟹
  (a zs ∧ dlist_xbefore b c zs) ⟷
    (dlist_xbefore (λ xs. a xs ∧ b xs) c zs ∨
      dlist_xbefore b (λxs. a xs ∧ c xs) zs)"
proof-
  assume "dlist_tempo2 a"
  hence 0: "∀i xs. a xs ⟷ (a (xs↑..i) ∨ a (xs↑i..))"
    using dlist_tempo2_def by auto
  have "a zs ∧ dlist_xbefore b c zs ⟷
    a zs ∧ (∃i. b (zs↑..i) ∧ c (zs↑i..))"
    by (auto simp add: dlist_xbefore_def)
  thus ?thesis using 0 by (auto simp add: dlist_xbefore_def)
qed

corollary and_dlistset_xbefore_equiv_or_dlistset_xbefore:
  "dlist_tempo2 a ⟹
  ((Collect a) ∩ (Collect (dlist_xbefore b c)))=
    (Collect (dlist_xbefore (λ xs. a xs ∧ b xs) c) ∪
      Collect (dlist_xbefore b (λxs. a xs ∧ c xs)))"
by (smt Collect_cong Collect_conj_eq Collect_disj_eq dlist_tempo2_def
  dlist_xbefore_def)

lemma dlist_xbefore_implies_not_sym_dlist_xbefore: "
  ⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
  dlist_xbefore a b zs ⟹ ¬ dlist_xbefore b a zs"
unfolding dlist_xbefore_def dlist_tempo1_def
by (meson nat_le_linear)

corollary dlistset_xbefore_implies_not_sym_dlistset_xbefore:
  "⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
  Collect (dlist_xbefore a b) ⊆ - Collect (dlist_xbefore b a)"
using dlist_xbefore_implies_not_sym_dlist_xbefore
by (metis (mono_tags, lifting) CollectD ComplI subsetI)

theorem mixed_not_dlist_xbefore: "dlist_independent_events a b ⟹
  ⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
  ⟦dlist_tempo2 a; dlist_tempo2 b⟧ ⟹
  ⟦dlist_tempo3 a; dlist_tempo3 b⟧ ⟹
```

```
  ⟦dlist_tempo4 a; dlist_tempo4 b⟧ ⟹
  (¬ (dlist_xbefore a b zs)) =
  ((¬ a zs) ∨ (¬ b zs) ∨ (dlist_xbefore b a zs))"
using dlist_xbefore_implies_not_sym_dlist_xbefore dlist_xbefore_or_one_list
by blast

corollary mixed_not_dlistset_xbefore: "dlist_independent_events a b ⟹
  ⟦dlist_tempo1 a; dlist_tempo1 b⟧ ⟹
  ⟦dlist_tempo2 a; dlist_tempo2 b⟧ ⟹
  ⟦dlist_tempo3 a; dlist_tempo3 b⟧ ⟹
  ⟦dlist_tempo4 a; dlist_tempo4 b⟧ ⟹
  (- Collect (dlist_xbefore a b)) =
  ((- Collect a) ∪ (- Collect b) ∪ Collect (dlist_xbefore b a))"
proof-
  assume 0: "dlist_independent_events a b" "dlist_tempo1 a" "dlist_tempo1 b"
  "dlist_tempo2 a" "dlist_tempo2 b" "dlist_tempo3 a" "dlist_tempo3 b"
  "dlist_tempo4 a" "dlist_tempo4 b"
  have "((- Collect a) ∪ (- Collect b) ∪ Collect (dlist_xbefore b a)) =
    ((Collect (λzs. ¬ a zs ∨ ¬ b zs)) ∪ Collect (dlist_xbefore b a))"
    by blast
  also have "... = (Collect (λzs. ¬ a zs ∨ ¬ b zs ∨ dlist_xbefore b a zs))"
    by blast
  hence "Collect (λzs. (¬ a zs) ∨ (¬ b zs) ∨ (dlist_xbefore b a zs)) =
    ((- Collect a) ∪ (- Collect b) ∪ Collect (dlist_xbefore b a))"
    "Collect (λzs. ¬ (dlist_xbefore a b zs)) =
      - Collect (dlist_xbefore a b)"
    by blast+
  thus ?thesis using 0 mixed_not_dlist_xbefore by blast
qed
```

A.4.3 Formulas as ATF

In the following we prove that a formula is a valid type instantation for all ATF classes.

A.4.3.1 Basic properties of ATF

```
instantiation formula :: (type) temporal_faults_algebra_basic
begin

definition
  "xbefore a b = Abs_formula { zs .
```

```
  dlist_xbefore (λxs. xs ∈ Rep_formula a) (λys. ys ∈ Rep_formula b) zs }"

definition
  "tempo1 a = dlist_tempo1 (λxs. xs ∈ Rep_formula a)"

lemma Rep_formula_xbefore_to_dlist_xbefore:
  "Rep_formula (xbefore a b) =
  Collect (dlist_xbefore (λx. x ∈ Rep_formula a) (λy. y ∈ Rep_formula b))"
unfolding dlist_xbefore_def xbefore_formula_def
by (simp add: Abs_formula_inverse)

lemma Rep_formula_xbefore_bot_1: "Rep_formula (xbefore bot a) =
  Rep_formula bot"
unfolding xbefore_formula_def
by (simp add: Abs_formula_inverse dlist_xbefore_bot_1)

lemma Rep_formula_xbefore_bot_2: "Rep_formula (xbefore a bot) =
  Rep_formula bot"
unfolding xbefore_formula_def
by (simp add: Abs_formula_inverse dlist_xbefore_bot_2)

lemma Rep_formula_xbefore_not_idempotent:
  "tempo1 a ⟹ Rep_formula (xbefore a a) = Rep_formula bot"
unfolding xbefore_formula_def tempo1_formula_def
by (simp add: Abs_formula_inverse dlist_xbefore_idem)

lemma Rep_formula_xbefore_not_sym:
  "⟦ tempo1 a; tempo1 b⟧ ⟹
    Rep_formula (xbefore a b) ⊆ Rep_formula (-xbefore b a)"
unfolding xbefore_formula_def tempo1_formula_def uminus_formula_def
by (simp add: Abs_formula_inverse
  dlistset_xbefore_implies_not_sym_dlistset_xbefore)

instance proof
  fix a::"'a formula"
  show "xbefore bot a = bot"
  unfolding eq_formula_iff Rep_formula_xbefore_bot_1 by auto
  next
  fix a::"'a formula"
  show "xbefore a bot = bot"
  unfolding eq_formula_iff Rep_formula_xbefore_bot_2 by auto
  next
```

```
  fix a::"'a formula"
  assume "tempo1 a"
  thus "xbefore a a = bot"
  unfolding eq_formula_iff
  using Rep_formula_xbefore_not_idempotent by auto
  next
  fix a::"'a formula" and b::"'a formula"
  assume "tempo1 a" "tempo1 b"
  thus "xbefore a b ≤ - xbefore b a"
  unfolding eq_formula_iff less_eq_formula_def
  using Rep_formula_xbefore_not_sym by simp
  fix a::"'a formula" and b::"'a formula"
  assume "tempo1 a" "tempo1 b"
  thus "tempo1 (inf a b)"
  unfolding tempo1_formula_def
  by (simp add: dlist_tempo1_inf Rep_formula_inf)
qed

end
```

A.4.3.2 Associativity of ATF

```
instantiation formula :: (type) temporal_faults_algebra_assoc
begin

instance proof
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo1 a" "tempo1 b" "tempo1 c"
  thus "xbefore (xbefore a b) c = xbefore a (xbefore b c)"
  unfolding xbefore_formula_def tempo1_formula_def
  by (simp add: Abs_formula_inverse dlist_xbefore_assoc)
qed

end
```

A.4.3.3 Equivalences in ATF

```
instantiation formula :: (type) temporal_faults_algebra_equivs
begin

definition
  "independent_events a b =
    dlist_independent_events
```

```
    (λxs. xs ∈ Rep_formula a) (λxs. xs ∈ Rep_formula b)"
```

definition

```
  "tempo2 a = dlist_tempo2 (λxs. xs ∈ Rep_formula a)"
```

definition

```
  "tempo3 a = dlist_tempo3 (λxs. xs ∈ Rep_formula a)"
```

definition

```
  "tempo4 a = dlist_tempo4 (λxs. xs ∈ Rep_formula a)"
```

instance proof

```
  fix a::"'a formula" and b::"'a formula"
  assume "tempo1 a" "tempo1 b"
  thus "inf (xbefore a b) (xbefore b a) = bot"
  unfolding xbefore_formula_def tempo1_formula_def bot_formula_def
    inf_formula_def
  by (simp add: dlistset_xbefore_and Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a" "tempo2 b"
    "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
  thus "sup (xbefore a b) (xbefore b a) = inf a b"
  unfolding xbefore_formula_def tempo1_formula_def tempo2_formula_def
    tempo3_formula_def tempo4_formula_def independent_events_formula_def
    sup_formula_def inf_formula_def
  by (simp add: dlistset_xbefore_or Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "tempo2 a" "tempo2 b"
  thus "tempo2 (sup a b)"
  unfolding tempo2_formula_def
  by (simp add: dlist_tempo2_sup Rep_formula_sup)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "tempo3 a" "tempo3 b"
  thus "tempo3 (inf a b)"
  unfolding tempo3_formula_def
  by (simp add: dlist_tempo3_inf Rep_formula_inf)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "tempo4 a" "tempo4 b"
```

```
  thus "tempo4 (sup a b)"
  unfolding tempo4_formula_def
  by (simp add: dlist_tempo4_sup Rep_formula_sup)
qed

end
```

A.4.3.4 Transitivity in ATF

```
instantiation formula :: (type) temporal_faults_algebra_trans
begin
instance proof
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo1 a" "tempo1 b" "tempo1 c" "tempo2 a" "tempo2 b" "tempo2 c"
  thus "inf (xbefore a b) (xbefore b c) ≤ xbefore a c"
  unfolding tempo1_formula_def tempo2_formula_def xbefore_formula_def
    less_eq_formula_def inf_formula_def
  by (simp add: dlistset_xbefore_trans Abs_formula_inverse)
qed
end
```

A.4.3.5 Mixed operators in ATF

```
instantiation formula :: (type) temporal_faults_algebra_mixed_ops
begin
instance proof
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  show "xbefore (sup a b) c = sup (xbefore a c) (xbefore b c)"
  unfolding xbefore_formula_def sup_formula_def
  by (simp add: mixed_dlistset_xbefore_or1 Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  show "xbefore a (sup b c) = sup (xbefore a b) (xbefore a c)"
  unfolding xbefore_formula_def sup_formula_def
  by (simp add: mixed_dlistset_xbefore_or2 Abs_formula_inverse)
  next
  fix a::"'a formula" and b::"'a formula"
  assume "independent_events a b" "tempo1 a" "tempo1 b" "tempo2 a" "tempo2 b"
    "tempo3 a" "tempo3 b" "tempo4 a" "tempo4 b"
  thus "(- xbefore a b) = (sup (sup (- a) (- b)) (xbefore b a))"
  by (simp add: Abs_formula_inverse xbefore_formula_def uminus_formula_def
    sup_formula_def independent_events_formula_def tempo1_formula_def
    tempo2_formula_def tempo3_formula_def tempo4_formula_def
```

```
    mixed_not_dlistset_xbefore)
  next
  fix a::"'a formula" and b::"'a formula" and c::"'a formula"
  assume "tempo2 a"
  thus "inf a (xbefore b c) =
    sup (xbefore (inf a b) c) (xbefore b (inf a c))"
  apply (auto simp add: xbefore_formula_def sup_formula_def inf_formula_def
    tempo2_formula_def Abs_formula_inverse)
  using and_dlistset_xbefore_equiv_or_dlistset_xbefore Abs_formula_inverse
  by fastforce
qed
end
```

A.4.4 Equivalence of the new definition of XBefore with the old one

```
definition old_dlist_xbefore
where
"old_dlist_xbefore S T zs ≡
  (∃ xs ys. S xs ∧ ¬ T xs ∧ T ys ∧ ¬ S ys ∧
    set (list_of_dlist xs) ∩ set (list_of_dlist ys) = {} ∧
    list_of_dlist zs = (list_of_dlist xs) @ (list_of_dlist ys))"

theorem old_dlist_xbefore_equals_new_xbefore:
  "⟦ dlist_tempo1 S; dlist_tempo1 T ⟧ ⟹
    dlist_xbefore S T zs = old_dlist_xbefore S T zs"
unfolding dlist_xbefore_append old_dlist_xbefore_def
using dlist_tempo_1_no_gap_append
by blast
```