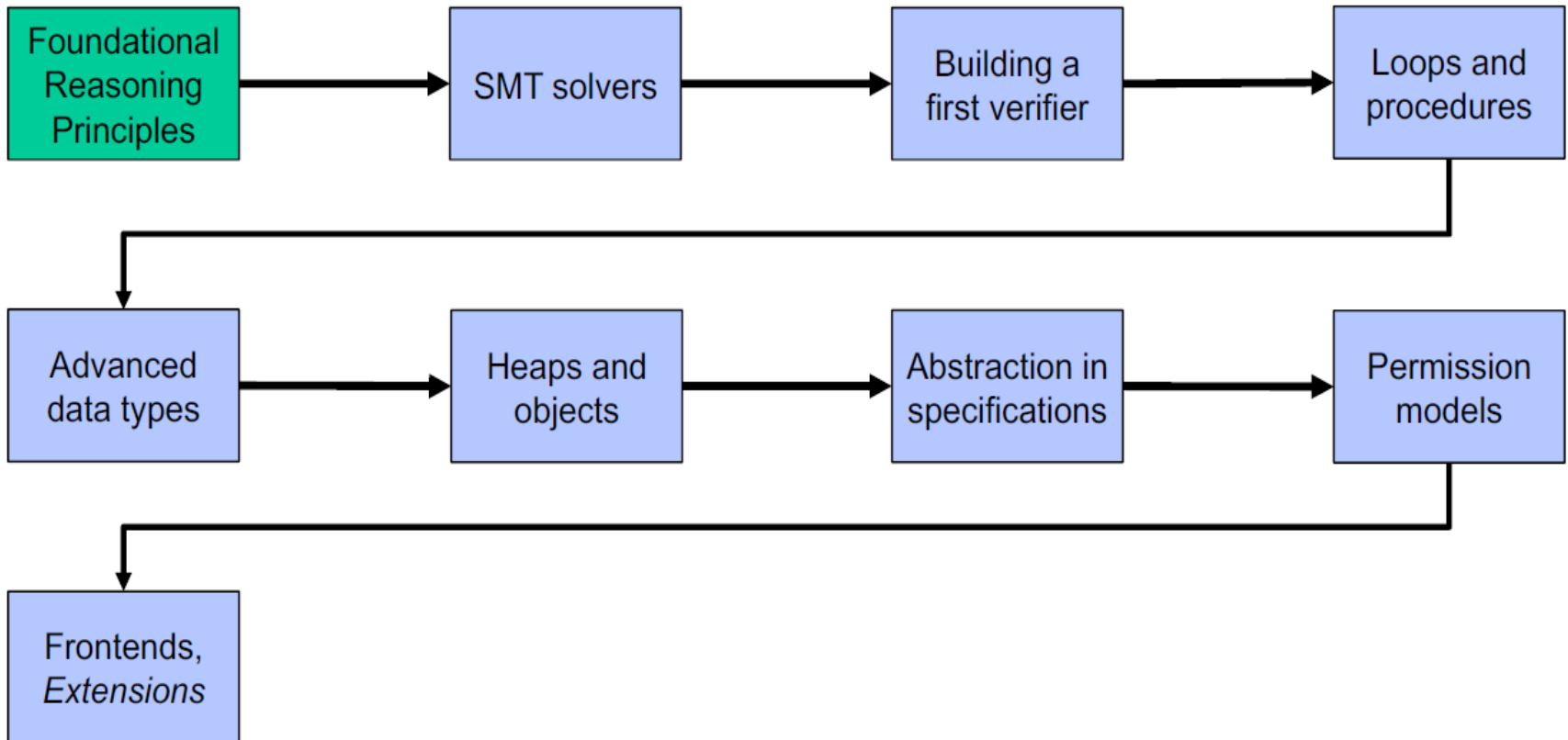


Methodologies for Software Processes

Lecture 2

Tentative course outline



Outline

1. Why do we need formal foundations?
2. Formalizing contracts
3. Reasoning about contracts
4. Epilogue

The Program Verification Task


Given a **program**

and a **specification spec**,

give a **proof**

that all program executions

comply with **spec**



```
fn abs(x:i32) -> i32 {  
  if x >= 0 {  
    return x  
  } else {  
    return -x  
  }  
}
```

spec: `abs(x)` returns `|x|`

Does every execution
comply with **spec**?

Verifying abs(x)

```
// Viper model of abs(x)
method abs(x: Int) returns (r: Int)
  ensures x >= 0 ==> r == x
  ensures x <= 0 ==> r == -x
{
  if (x < 0) { r := -x } else { r := x }
}
```



i32: 32-bit integers in two's complement!

i32::MIN is -2_147_483_648i32

i32::MAX is 2_147_483_647i32

abs(i32::MIN) == ???



```
fn abs(x:i32) -> i32 {
  if x >= 0 {
    return x
  } else {
    return -x
  }
}
```



spec: abs(x) returns |x|

Problem: Viper model does not capture the semantics of the Rust program

Refined verification of abs(x)

```
define i32MIN (-2147483648)
define i32MAX (2147483647)

method abs(x: Int) returns (r: Int)
  requires i32MIN <= x && x <= i32MAX
  ensures i32MIN <= r && r <= i32MAX
  ensures x >= 0 ==> r == x
  ensures x <= 0 ==> r == -x
{
  if (x < 0) { r := -x } else { r := x }
}
```



Refined specification for abs(x)

```
define i32MIN (-2147483648)
define i32MAX (2147483647)

method abs(x: Int) returns (r: Int)
  requires i32MIN <= x && x <= i32MAX
  requires x != i32MIN
  ensures i32MIN <= r && r <= i32MAX
  ensures x >= 0 ==> r == x
  ensures x <= 0 ==> r == -x
{
  if (x < 0) { r := -x } else { r := x }
}
```



Verification must be rooted in rigorous mathematics

Given a **program**

semantics

and a **specification spec**,

formal logic

give a **proof**

automated provers

that all program executions

comply with **spec**

program logics



Outline

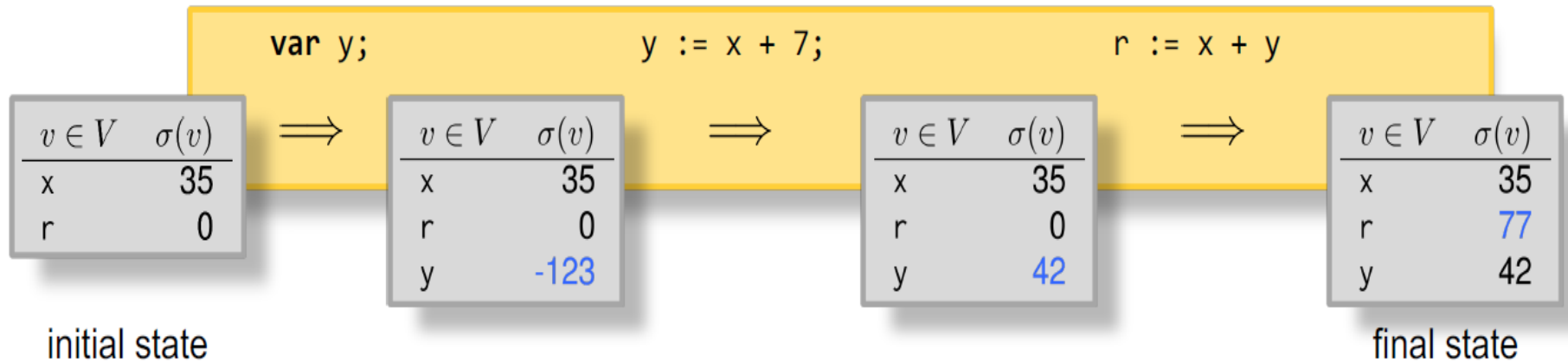
1. Why do we need formal foundations?
2. Formalizing contracts
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Making it formal

Program **states** assign **values** to **variables** in **Var**

$$\mathbf{States} = \{ \sigma: V \rightarrow \mathbf{Int} \mid V \subseteq \mathbf{Var} \text{ and } V \text{ finite} \}$$

Program **semantics** describes how *states* evolve during program execution



Making it formal

Predicates capture properties of program states

$$\mathbf{Pred} = \{ P \mid P: \mathbf{States} \rightarrow \mathbf{Bool} \}$$

Logical characterization

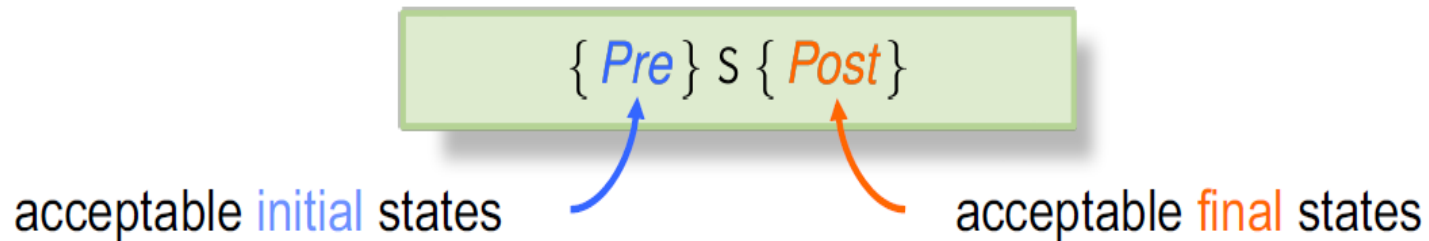
$$x \neq 0$$

Set characterization

$$P = \{ \sigma \in \mathbf{States} \mid \sigma(x) \neq 0 \}$$

Making it formal

Floyd-Hoare triples capture properties of (possibly infinitely many) executions



```
method foo(x: Int)
  returns (r: Int)
  requires x > 0
  ensures r > y
{
  var y: Int
  y := x + 7
  r := x + y
}
```

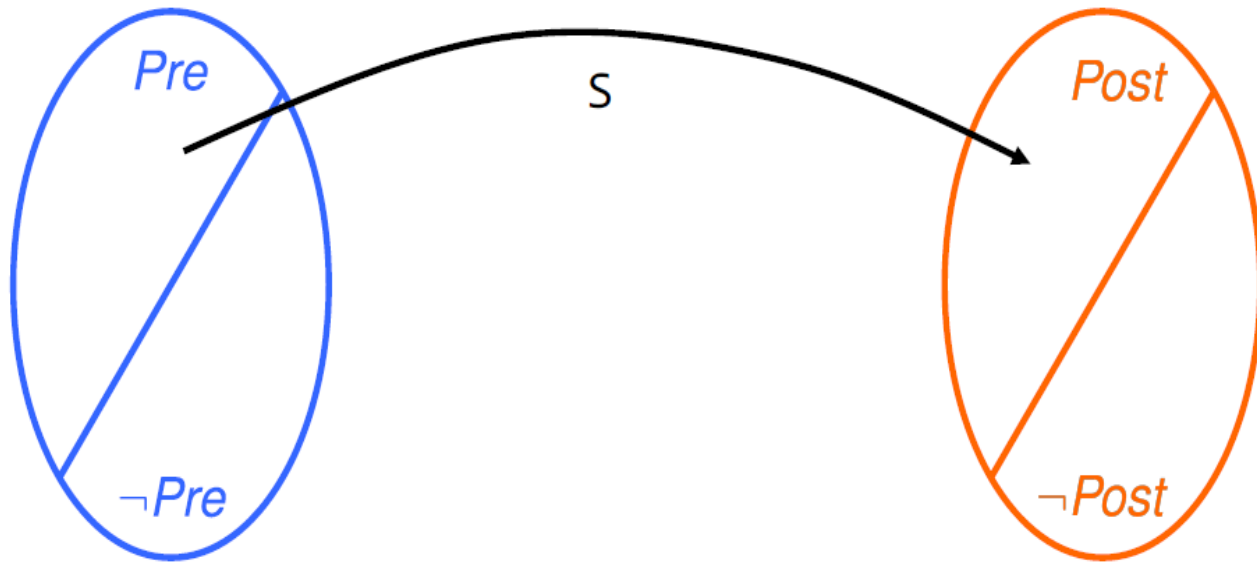


```
{ x > 0 }
var y;
y := x + 7;
r := x + y
{ r > y }
```

- Implicit I/O parameters
- Omit types (only **Int**)
- Moved pre- and postcondition

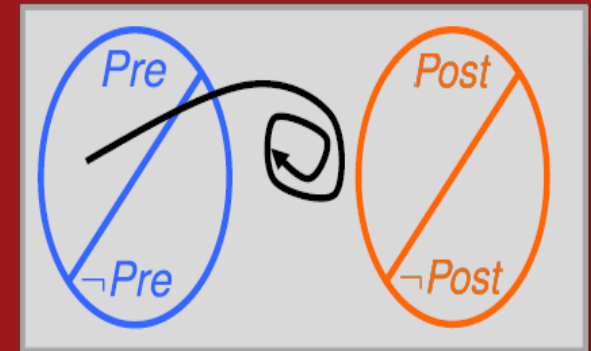
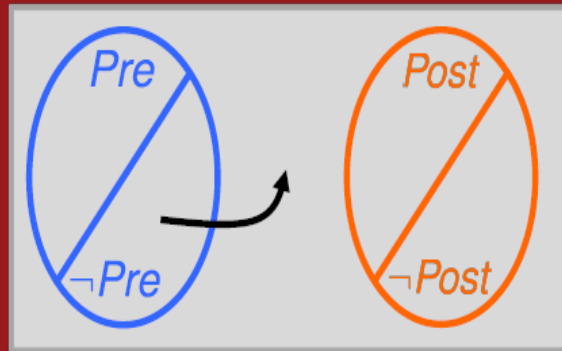
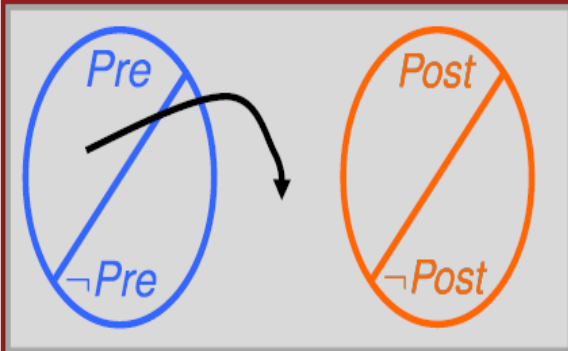
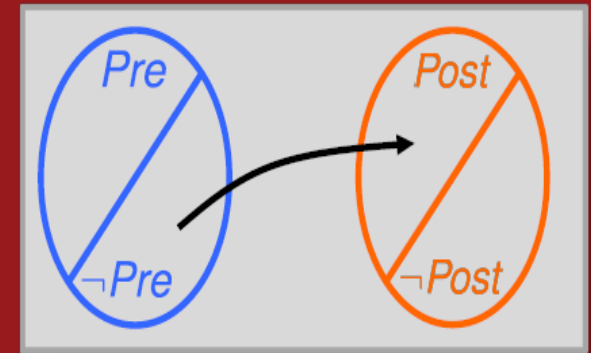
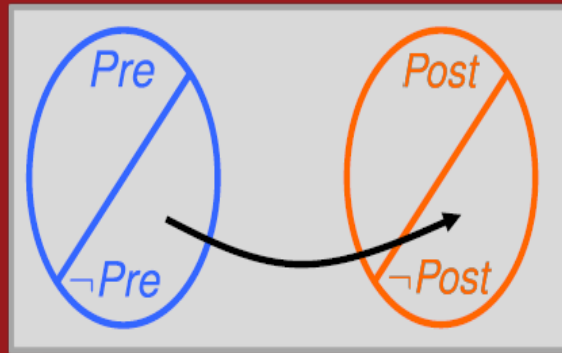
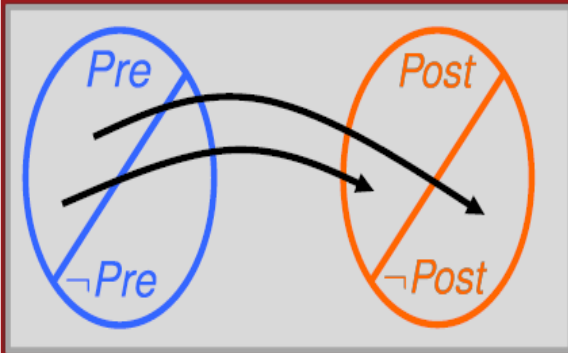
Making it formal

The triple $\{ \textit{Pre} \} S \{ \textit{Post} \}$ is **valid** if and only if
when program S is started in any state in \textit{Pre} ,
then S terminates in a state in \textit{Post} .



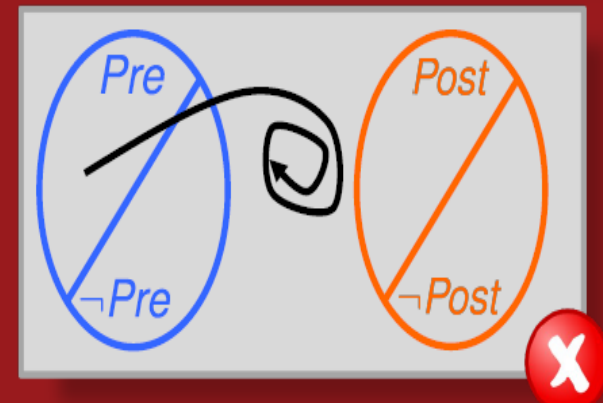
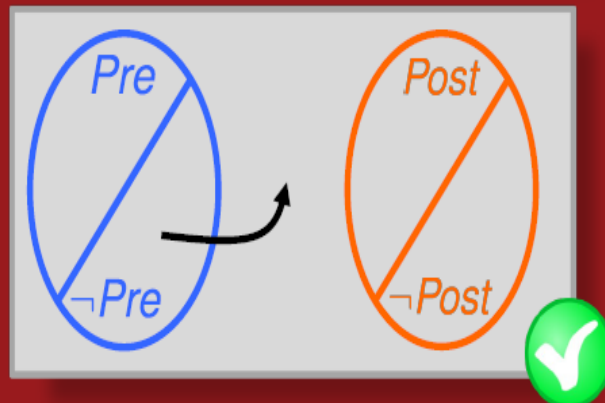
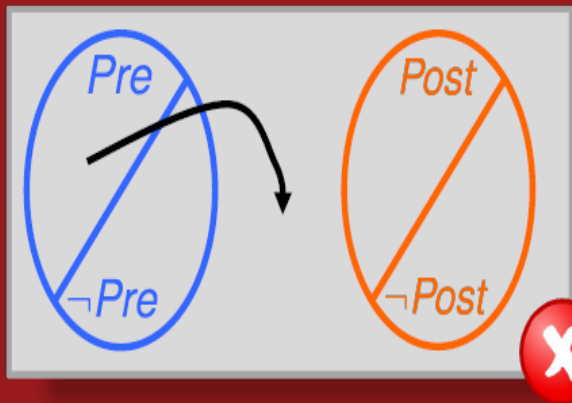
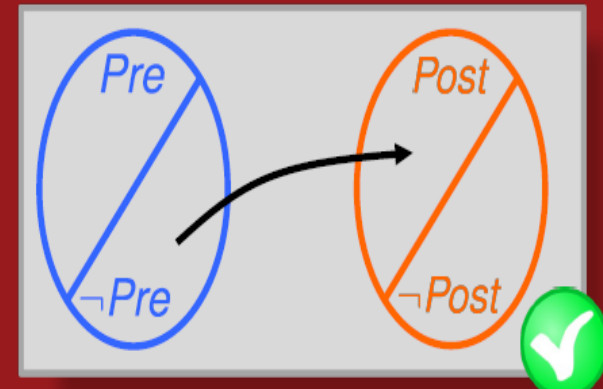
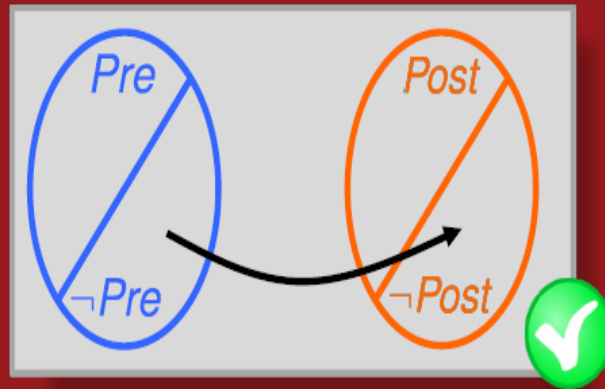
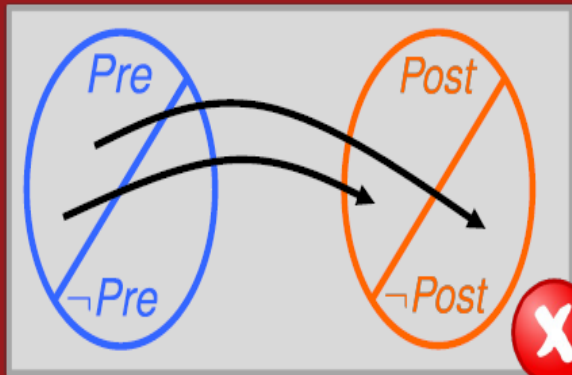
Which pictures correspond to valid Floyd-Hoare triples?

$\{ \text{Pre} \} S \{ \text{Post} \}$ is **valid** iff when program S is started in any state in Pre , then S terminates in a state in Post .



Solution

$\{ \textit{Pre} \} S \{ \textit{Post} \}$ is **valid** iff when program S is started in any state in \textit{Pre} , then S terminates in a state in \textit{Post} .

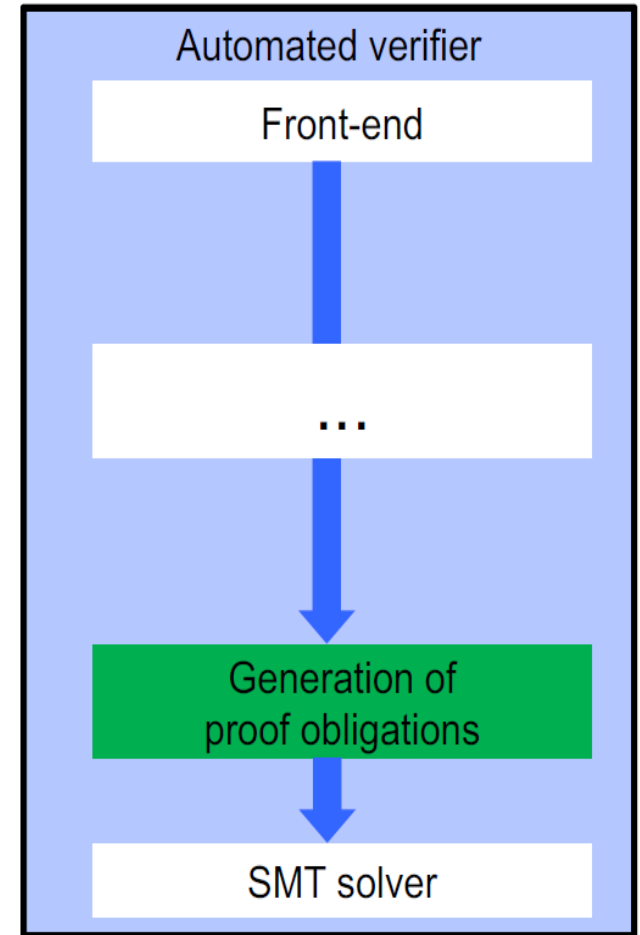


Outline

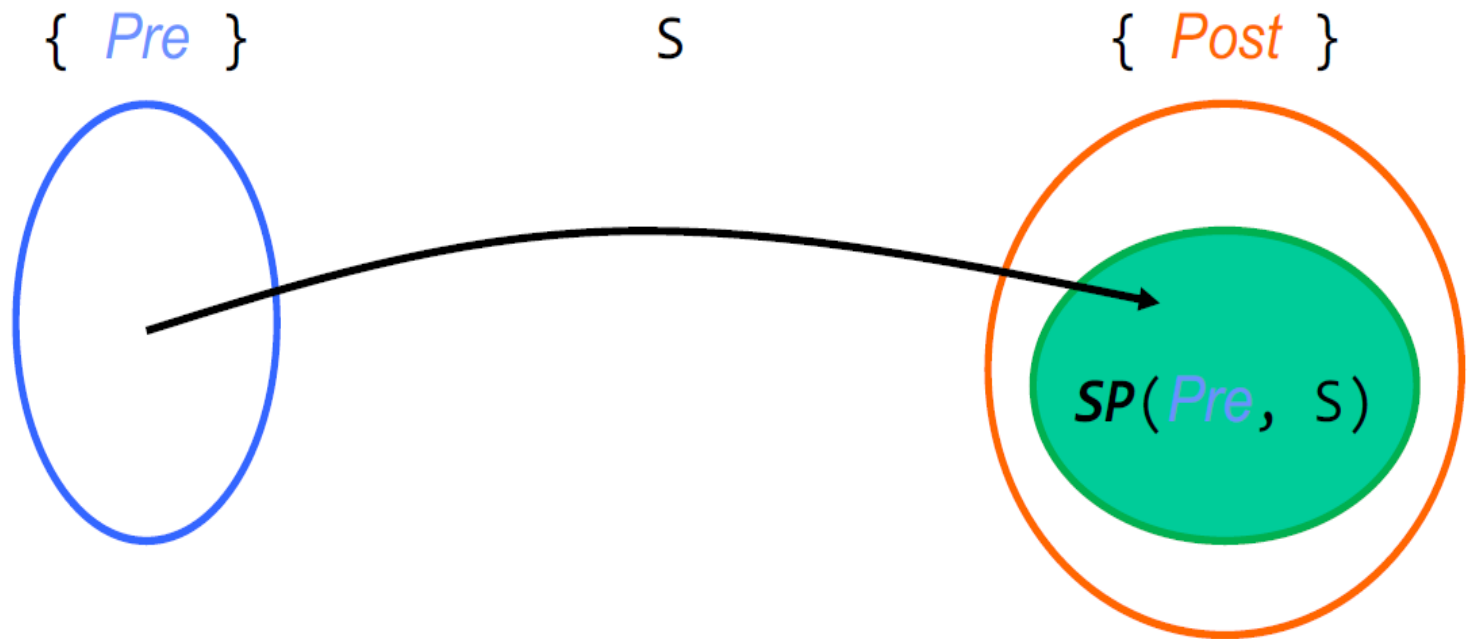
1. Why do we need formal foundations?
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4. Epilogue

How do we systematically prove a triple valid?

- Determine a **verification condition** VC
 - VC is a predicate
 - VC is **valid** iff it is true for *all* states
- **Soundness:** VC is valid \rightarrow triple is valid
- **Completeness:** triple is valid \rightarrow VC is valid
- **Predicate transformers** describe how *predicates* evolve during program execution



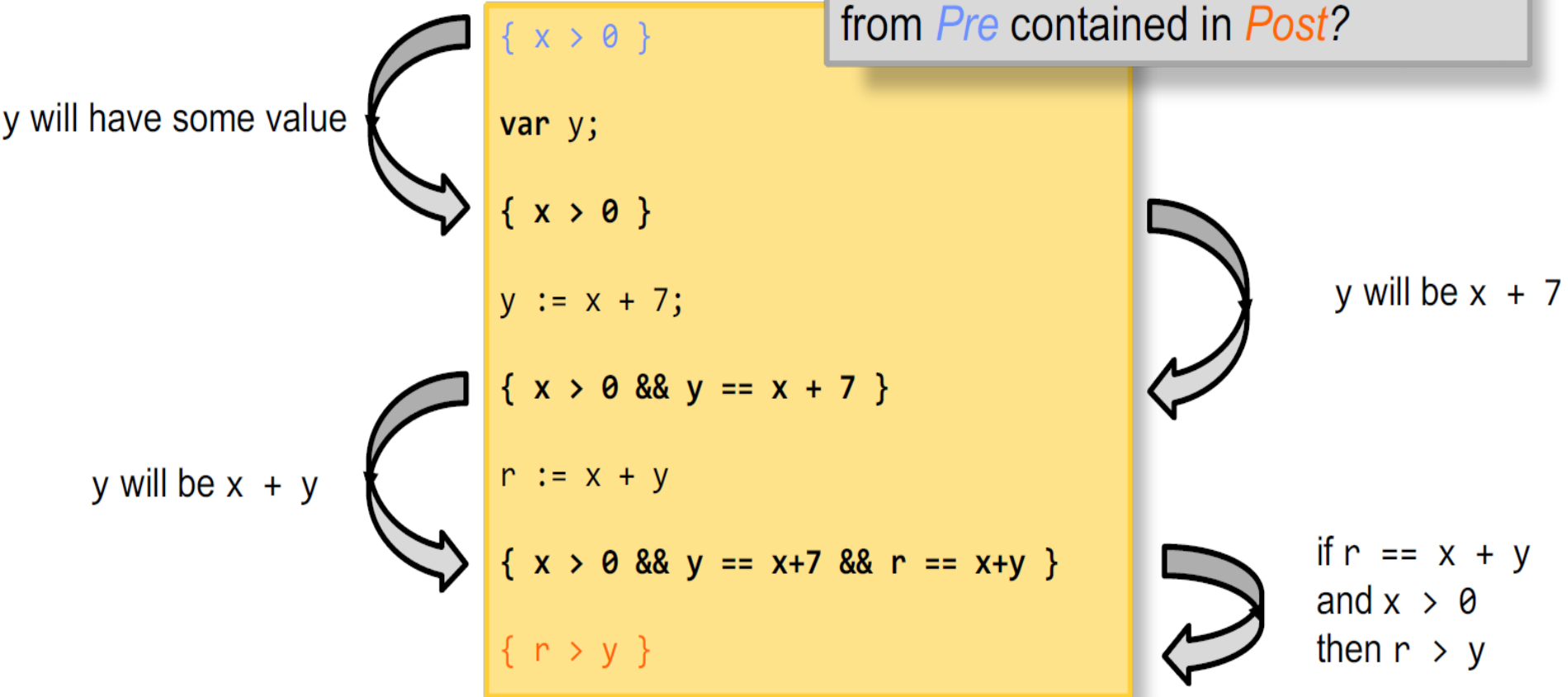
Forward Reasoning



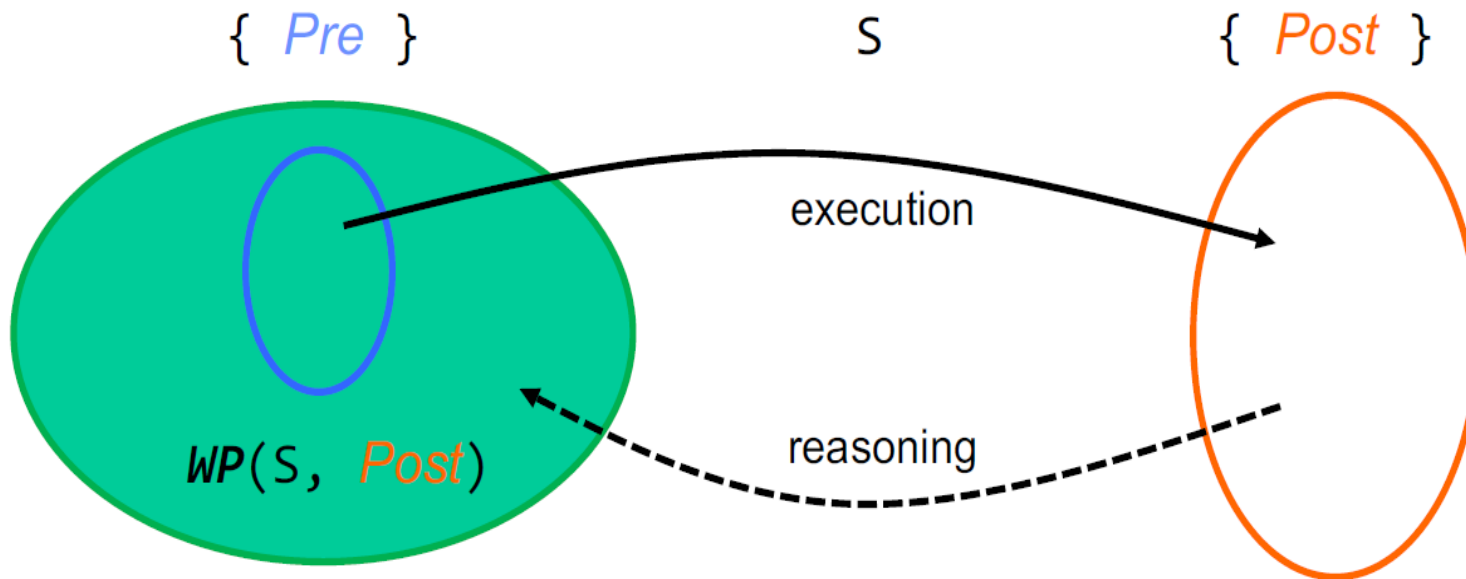
Forward VC: is the **strongest postcondition** $SP(Pre, S)$ (all final states that we can reach from Pre) of Pre and program S contained in $Post$?

Informal Forward Reasoning

Are all final states that we can reach from *Pre* contained in *Post*?

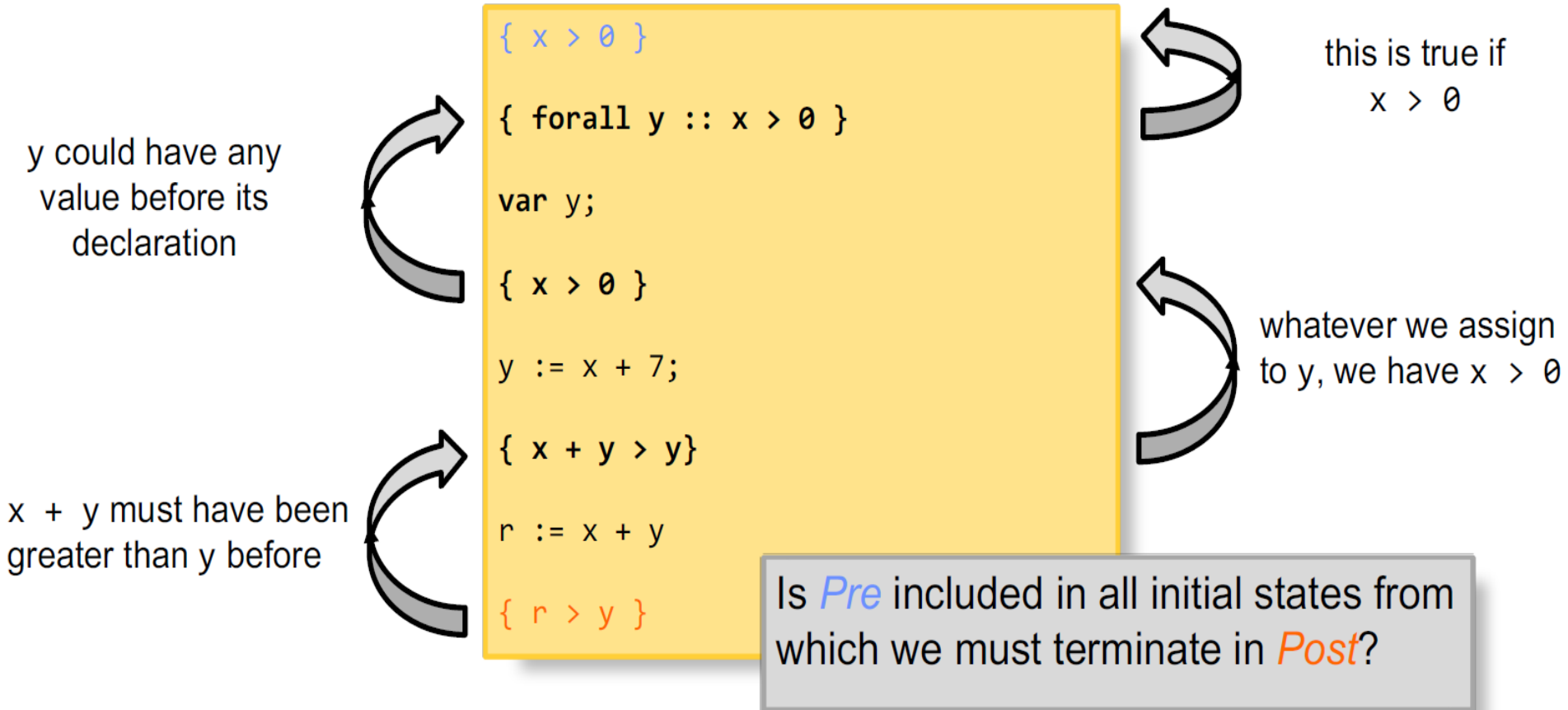


Backward Reasoning



Backward VC: is Pre included in the **weakest precondition** $WP(S, Post)$ (all initial states from which we must terminate in $Post$) of program S and $Post$?

Informal Backward Reasoning



PL0: a first programming language

x is a variable in **Var**

z is a constant in **Int**

Arithmetic expressions

a ::= x | z | a + a | a - a | a / a | a % a

Boolean expressions

b ::= true | false | a < a | a = a | b && b | b || b | !b | ...

Predicates (incomplete)

P, Q, R ::= b | P && P | P ==> P | **exists** x :: P | **forall** x :: P | ...

Statements in PL0

S ::= **var** x | x := a | S;S | S [] S | **assert** P | **assume** P

Local variable declarations: **var** x

```
{ true }  
var x;  
{ x >= 0 }
```



```
{ x == 0 }  
var x;  
{ x <= 0 }
```



```
{ x == 5 && y > x }  
var x;  
{ y > 5 }
```



Declares an uninitialized variable x that overshadows any existing x .

$SP(P, \text{var } x) ::= \text{exists } x :: P$

$WP(\text{var } x, Q) ::= \text{forall } x :: Q$

Assertions: `assert R`

```
{ x > 7 }
```

```
assert x > 5
```

```
{ x > 7 }
```



```
{ x > 5 }
```

```
assert x > 7
```

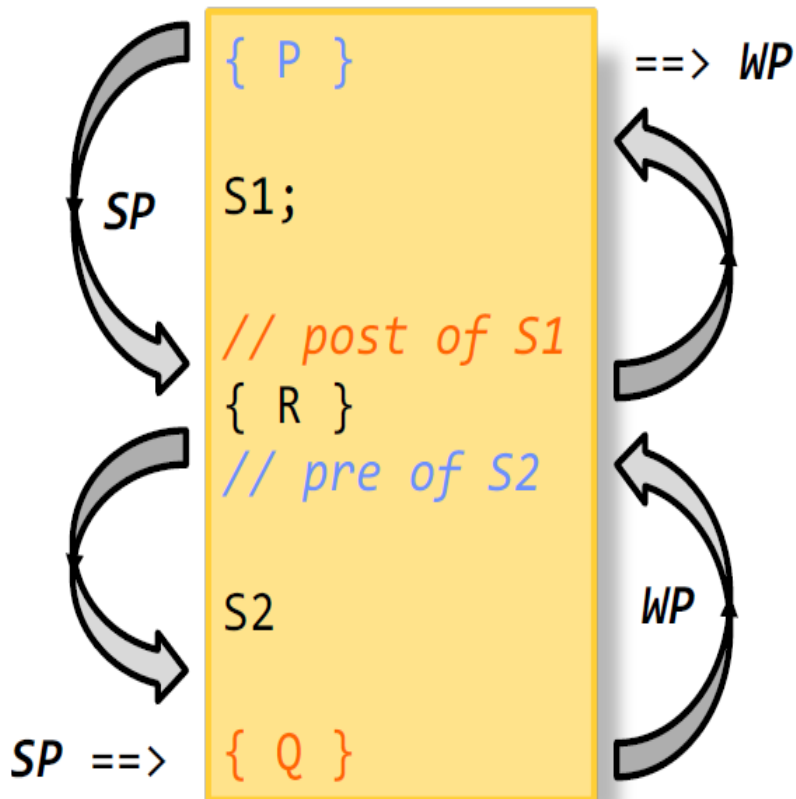
```
{ x > 7 }
```



Crashes if R does not hold in the current state; otherwise, *no effect*.

$$SP(P, \text{assert } R) ::= P \ \&\& \ R$$
$$WP(\text{assert } R, Q) ::= R \ \&\& \ Q$$

Sequential composition: $S1;S2$



First execute $S1$, then $S2$.

$$SP(P, S1;S2) ::= SP(SP(P, S1), S2)$$

$$WP(S1;S2, Q) ::= WP(S1, WP(S2, Q))$$

Nondeterministic choice: $S1 \ [] \ S2$

```
{ x == 7 }  
assert x > 0 [] assert y > 0  
{ x > 0 }
```



```
{ x > 0 && y > 0 }  
assert x*x > 0 [] assert x+y > 0  
{ x * x > 0 || x + y > 0 }
```



```
{ x > 0 && y > 0 }  
assert x > 0 [] assert y > 0  
{ x > 0 }
```



Executes *either* $S1$ or $S2$.

$$\begin{aligned} SP(P, S1 \ [] \ S2) \\ ::= SP(P, S1) \ || \ SP(P, S2) \end{aligned}$$
$$\begin{aligned} WP(S1 \ [] \ S2, Q) \\ ::= WP(S1, Q) \ \&\& \ WP(S2, Q) \end{aligned}$$

Assumptions: **assume** R

- Verification-specific statement
- *Not* executable
- Part of trusted code base

```
{ x == 0 }  
assume x > 0  
{ x > 0 }
```



```
{ x > 0 }  
assume x > 5  
{ x > 5 }
```



```
{ true }  
assume false  
{ false }
```



```
{ x == 1 }  
assume x > 0  
{ x == 2 }
```



Nothing happens if R holds in the current state; otherwise, *magic*.

$SP(P, \text{assume } R) ::= P \ \&\& \ R$

$WP(\text{assume } R, Q) ::= R \implies Q$

Assignment: $x := a$

$\{ y > 0 \}$

$x := 17 + y$

$\{ y > 0 \ \&\& \ x == 17 + y \}$



$\{ y < 23 \}$

$x := 23$

$\{ y < x \}$



$\{ x + 1 > 42 \}$

$x := x + 1$

$\{ x > 42 \}$



$\{ x > 42 \}$

$x := x + 1$

$\{ x > 42 \ \&\& \ x == x + 1 \}$



Assigns the value of a (evaluated in the initial state) to x in the final state.

$WP(x := a, Q) ::= Q[x / a]$

$E[x / F]$: E where every x is replaced by F

Assignment: $x := a$

```
{ y > 0 }  
x := 17 + y  
{ y > 0 && x == 17 + y }
```



```
{ y < 23 }  
x := 23  
{ y < x }
```



```
{ x + 1 > 42 }  
x := x + 1  
{ x > 42 }
```



```
{ x > 42 }  
x := x + 1  
{ x > 42 && x == x + 1 }
```



Assigns the value of a (evaluated in the initial state) to x in the final state.

$WP(x := a, Q) ::= Q[x / a]$

$E[x / F]$: E where every x is replaced by F

$SP(P, x := a) ::= P \ \&\& \ x == a$



Assignment: $x := a$

```
{ y > 0 }
```

```
x := 17 + y
```

```
{ y > 0 && x == 17 + y }
```



```
{ y < 23 }
```

```
x := 23
```

```
{ y < x }
```



```
{ x + 1 > 42 }
```

```
x := x + 1
```

```
{ x > 42 }
```



```
{ x > 42 }
```

```
x := x + 1
```

```
{ x > 42 && x == x + 1 }
```



Assigns the value of a (evaluated in the initial state) to x in the final state.

$WP(x := a, Q) ::= Q[x / a]$

$E[x / F]$: E where every x is replaced by F

$SP(P, x := a) ::= \text{exists } x_0 ::$
 $P[x / x_0] \ \&\& \ x == a[x / x_0]$

Proof annotations via overlapping Floyd-Hoare triples

upper implies lower

upper implies lower

valid triple

valid triple

valid triple

```
{ x > 0 }
```

```
{ 5 * x > 4 * x }
```

```
{ 2 * x + 3 * x > 4 * x }
```

```
y := 2 * x;
```

```
{ y + 3 * x > 4 * x }
```

```
z := 3 * x;
```

```
{ y + z > 4 * x }
```

```
r := y + z
```

```
{ r > 4 * x }
```

valid triple



Outline

1. Why do we need formal foundations?
2. Formalizing contracts
3. Reasoning about contracts
4. Epilogue

Strongest Post vs. Weakest Pre – Does it matter?

```
{ x == 0 } assert x < 0 { x == 1 }
```



```
x == 0 ==> WP(assert x < 0, x == 1)
=
x == 0 ==> (x < 0 && x == 1)

// not valid
```



Total correctness:

$Pre \Rightarrow WP(S, Post)$ valid
iff $\{ Pre \} S \{ Post \}$ valid
iff when program S is started in any state in Pre ,
then S terminates in a state in $Post$.

```
SP(x == 0, assert x < 0) ==> x == 1
=
(x == 0 && x < 0) ==> x == 1

// valid
```



Partial correctness:

$SP(Pre, S)$ valid
iff when program S is started in any state in Pre
and terminates without crashing,
then S terminates in a state in $Post$.

Wrap-up

```
method foo(...)  
  returns (...)  
  requires Pre  
  ensures Post  
{  
  S  
}
```



```
{ Pre }  
S  
{ Post }
```

Verification condition

(total correctness)

$Pre \implies WP(S, Post)$ valid?

Verification condition

(partial correctness)

$SP(S, Post) \implies Pre$ valid?

Operational Semantics

We will define the semantics of statements using four execution relations:

1. a relation \Longrightarrow_a for evaluating arithmetic expressions in a program state;
2. a relation \Longrightarrow_b for evaluating Boolean expressions in a program state;
3. a relation \Longrightarrow_p for evaluating predicates in a program state; and
4. a relation \Longrightarrow for performing one execution step from one program *configuration* (a statement coupled with a program state) to another configuration.

Evaluation of arithmetic expressions

For arithmetic expressions a in AExpr , we denote by $a, \sigma \Longrightarrow_a v$ that evaluating expression a in state σ yields integer value v .

Formally, the evaluation relation

$$\Longrightarrow_a \subseteq (\text{AExpr} \times \text{States}) \times \text{Int}$$

is given by the following inference rules:

$$\frac{}{v, \sigma \Longrightarrow_a v} \quad \frac{}{x, \sigma \Longrightarrow_a \sigma(x)}$$
$$\frac{\bullet \in \{+, -, *, /, \%\}}{a1, \sigma \Longrightarrow_a v1 \quad a2, \sigma \Longrightarrow_a v2 \quad v = v1 \bullet v2} \quad a1 \bullet a2, \sigma \Longrightarrow_a v$$

Notice that the last rule from above uses the standard arithmetic operators on integers $v1$ and $v2$ with one exception: we assume that all arithmetic operators are defined for every input, that is, they are total functions. If an operation is not well-defined, then it returns some value but we do not know which one. For example, evaluating $5 / 0$ yields such an unknown value. We will address how to spot and report such runtime errors later in the course.

Evaluation of Boolean expressions

For Boolean expressions b in BExpr , we denote by $b, \sigma \Rightarrow_b t$ that evaluating expression b in state σ yields truth value t .

Formally, the evaluation relation

$$\Rightarrow_b \subseteq (\text{BExpr} \times \text{States}) \times \text{Bool}$$

is defined by structural induction using the following inference rules:

$$\begin{array}{c}
 \frac{}{\text{true}, \sigma \Rightarrow_b \text{true}} \quad \frac{}{\text{false}, \sigma \Rightarrow_b \text{false}} \\
 \\
 \frac{a1, \sigma \Rightarrow_a v1 \quad a2, \sigma \Rightarrow_a v2 \quad t = (v1 \bowtie v2) \quad \bowtie \in \{=, !=, <=, >=, <, >\}}{a1 \bowtie a2, \sigma \Rightarrow_b t} \\
 \\
 \frac{b, \sigma \Rightarrow_b \text{false}}{!b, \sigma \Rightarrow_b \text{true}} \quad \frac{b, \sigma \Rightarrow_b \text{false}}{!b, \sigma \Rightarrow_b \text{true}} \\
 \\
 \frac{b1, \sigma \Rightarrow_b \text{false}}{b1 \ \&\& \ b2, \sigma \Rightarrow_b \text{false}} \quad \frac{b2, \sigma \Rightarrow_b \text{false}}{b1 \ \&\& \ b2, \sigma \Rightarrow_b \text{false}} \quad \frac{b1, \sigma \Rightarrow_b \text{true} \quad b2, \sigma \Rightarrow_b \text{true}}{b1 \ \&\& \ b2, \sigma \Rightarrow_b \text{true}} \\
 \\
 \frac{b1, \sigma \Rightarrow_b \text{true}}{b1 \ || \ b2, \sigma \Rightarrow_b \text{true}} \quad \frac{b2, \sigma \Rightarrow_b \text{true}}{b1 \ || \ b2, \sigma \Rightarrow_b \text{true}} \quad \frac{b1, \sigma \Rightarrow_b \text{false} \quad b2, \sigma \Rightarrow_b \text{false}}{b1 \ || \ b2, \sigma \Rightarrow_b \text{false}}
 \end{array}$$

In the above rules, we write $v1 \bowtie v2$ to denote the value obtained from performing a standard comparison \bowtie between two integer values.

Evaluation of predicates

For predicates P in Pred , we denote by $P, \sigma \Rightarrow_b \mathbf{t}$ that evaluating predicate P in state σ yields truth value \mathbf{t} .

Formally, the evaluation relation

$$\Rightarrow_P \subseteq (\text{Pred} \times \text{States}) \times \text{Bool}$$

is defined by structural induction using the following inference rules:

$$\frac{b, \sigma \Rightarrow_b \mathbf{t}}{b, \sigma \Rightarrow_P \mathbf{t}} \quad \frac{P, \sigma \Rightarrow_P \text{false}}{!P, \sigma \Rightarrow_P \text{true}} \quad \frac{P, \sigma \Rightarrow_P \text{false}}{!P, \sigma \Rightarrow_P \text{true}}$$

$$\frac{v \in \text{Int} \quad P, \sigma[x := v] \Rightarrow_P \text{true}}{\text{exists } x :: P, \sigma \Rightarrow_P \text{true}}$$

$$\frac{\dots \quad P, \sigma[x := -1] \Rightarrow_P \text{false} \quad P, \sigma[x := 0] \Rightarrow_P \text{false} \quad P, \sigma[x := 1] \Rightarrow_P \text{false}}{\text{exists } x :: P, \sigma \Rightarrow_P \text{false}}$$

$$\frac{v \in \text{Int} \quad P, \sigma[x := v] \Rightarrow_P \text{false}}{\text{forall } x :: P, \sigma \Rightarrow_P \text{false}}$$

$$\frac{\dots \quad P, \sigma[x := -1] \Rightarrow_P \text{true} \quad P, \sigma[x := 0] \Rightarrow_P \text{true} \quad P, \sigma[x := 1] \Rightarrow_P \text{true}}{\text{forall } x :: P, \sigma \Rightarrow_P \text{true}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{false}}{P1 \ \&\& \ P2, \sigma \Rightarrow_P \text{false}} \quad \frac{P2, \sigma \Rightarrow_P \text{false}}{P1 \ \&\& \ P2, \sigma \Rightarrow_P \text{false}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{true} \quad P2, \sigma \Rightarrow_P \text{true}}{P1 \ \&\& \ P2, \sigma \Rightarrow_P \text{true}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{true}}{P1 \ || \ P2, \sigma \Rightarrow_P \text{true}} \quad \frac{P2, \sigma \Rightarrow_P \text{true}}{P1 \ || \ P2, \sigma \Rightarrow_P \text{true}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{false} \quad P2, \sigma \Rightarrow_P \text{false}}{P1 \ || \ P2, \sigma \Rightarrow_P \text{false}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{true} \quad P2, \sigma \Rightarrow_P \text{true}}{P1 \ ==> \ P2, \sigma \Rightarrow_P \text{true}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{true} \quad P2, \sigma \Rightarrow_P \text{false}}{P1 \ ==> \ P2, \sigma \Rightarrow_P \text{false}}$$

$$\frac{P1, \sigma \Rightarrow_P \text{false}}{P1 \ ==> \ P2, \sigma \Rightarrow_P \text{true}}$$

Program Configurations

A *program configuration* is either a pair (S, σ) consisting of a statement S and a program state σ , or a pair (\mathbf{done}, σ) indicating termination with final state σ , or **error**, which indicates the occurrence of a runtime error, or **magic** indicating that we ended up in a magical place where anything is possible (i.e., we can verify any property, even wrong ones). Formally, the set **Conf** of program configurations is given by

$$\mathbf{Conf} ::= (\mathbf{Stmt} \times \mathbf{States}) \cup (\{\mathbf{done}\} \times \mathbf{States}) \cup \{\mathbf{error}, \mathbf{magic}\}.$$

Execution relation for statements

For **PL0 statements** S in **Stmt**, we denote by $S, \sigma \Longrightarrow C$ that executing one step starting in configuration (S, σ) results in configuration C . Formally, the execution relation

$$\Longrightarrow \subseteq \mathbf{Conf} \times \mathbf{Conf}$$

is defined by structural induction using the following inference rules:

$$\frac{v \in \mathbf{Int}}{\mathbf{var} \ x, \sigma \Longrightarrow \mathbf{done}, \sigma[x := v]}$$
$$\frac{a, \sigma \Longrightarrow_a v}{x := a, \sigma \Longrightarrow \mathbf{done}, \sigma[x := v]}$$

$\frac{P, \sigma \Rightarrow_P \text{true}}{\text{assert } P, \sigma \Rightarrow \text{done}, \sigma}$	$\frac{P, \sigma \Rightarrow_P \text{false}}{\text{assert } P, \sigma \Rightarrow \text{error}}$
$\frac{P, \sigma \Rightarrow_P \text{true}}{\text{assume } P, \sigma \Rightarrow \text{done}, \sigma}$	$\frac{P, \sigma \Rightarrow_P \text{false}}{\text{assume } P, \sigma \Rightarrow \text{magic}}$
$\frac{S1, \sigma \Rightarrow S1', \sigma'}{S1; S2, \sigma \Rightarrow S1'; S2, \sigma'}$	$\frac{S1, \sigma \Rightarrow \text{done}, \sigma'}{S1; S2, \sigma \Rightarrow S2, \sigma'}$
$\frac{S1, \sigma \Rightarrow \text{error}}{S1; S2, \sigma \Rightarrow \text{error}}$	$\frac{S1, \sigma \Rightarrow \text{magic}}{S1; S2, \sigma \Rightarrow \text{magic}}$
$\frac{}{S1 \text{ [] } S2, \sigma \Rightarrow S1, \sigma}$	$\frac{}{S1 \text{ [] } S2, \sigma \Rightarrow S2, \sigma}$

Notice that we can always perform at least one step of the execution relation until we reach a final configuration of the form `error`, `magic`, or `(done, _)`.

Floyd-Hoare triples

Extended execution relation

Towards a formal definition of Floyd-Hoare triples, we first extend the execution relation \Longrightarrow to perform exactly $n \geq 0$ (as opposed to exactly one) execution steps:

- $C \Longrightarrow^0 C$, and
- $C \Longrightarrow^{n+1} C'$ if $C \Longrightarrow C''$ and $C'' \Longrightarrow^n C'$.

We write $C \Longrightarrow^* C'$ if there exists some $n \geq 0$ such that $C \Longrightarrow^n C'$. Furthermore, we write $C \not\Longrightarrow^n$ if there exists *no* configuration C' such that $C \Longrightarrow^n C'$.

Formal Definition

A **Floyd-Hoare triple** $\{P\} S \{Q\}$ consists of a precondition P in Pred , a program statement S , and a postcondition Q in Pred .

Intuitively, a triple is valid if every execution of program S that starts in a state satisfying the precondition P will not cause a runtime error (for example, by failing an assertion) and will eventually terminate in a state satisfying the postcondition Q .

Towards a formal definition, recall from the definition of our language that $\text{dom}(\sigma)$ denotes the variables for which the state σ is defined. Moreover, let $\text{Var}(P)$ denote all free variables that appear in predicate P . Then $\text{Var}(P) \subseteq \text{dom}(\sigma)$ means that state σ assigns values at least to all variables in predicate P .

The Floyd-Hoare triple $\{P\} S \{Q\}$ is *valid* (for total correctness) if and only if for every state σ with $\text{Var}(P) \subseteq \text{dom}(\sigma)$, if $P, \sigma \Rightarrow_P \text{true}$, then

1. (safety) there exists no $n \geq 0$ such that $S, \sigma \Rightarrow^n \text{error}$;
 2. (termination) there exists $n \geq 0$ such that for every $m \geq n$, $S, \sigma \not\Rightarrow^m$; and
 3. (partial correctness) if $S, \sigma \Rightarrow^* (\text{done}, \sigma')$ and $\text{Var}(Q) \subseteq \text{dom}(\sigma')$, then $Q, \sigma' \Rightarrow_P \text{true}$.
-

Here, the first condition (safety) ensures that we cannot encounter a runtime error (for example an assertion failure) when starting execution in a state that satisfies the precondition. The second condition ensures termination, that is, we cannot perform execution steps ad infinitum. The third condition makes sure that we satisfy the postcondition whenever we manage to terminate without a runtime error.

We say that a triple is valid for *partial correctness*, whenever the third (but not necessarily the first or second) property holds.

In principle, we can directly apply the above definition of validity together with our operational semantics to check whether a triple is valid. However, in most cases this is not feasible, since we would have to consider all possible program executions.

Instead, we use verification conditions to reason about the validity of a triple in a symbolic way, that is by reasoning about predicates, without invoking the operational program semantics. In the lecture, we considered two such approaches for constructing verification conditions: weakest preconditions and strongest postconditions.

Weakest preconditions

Weakest preconditions generate a verification condition for checking the validity of Floyd-Hoare triples by starting at the postcondition and moving backward through the program.

Definition

The *weakest precondition* of a program statement s and a postcondition q is the predicate WP describing the largest possible set of states such that the Floyd-Hoare triple

$$\{ WP \} s \{ Q \}$$

is valid for total correctness. In other words, whenever $\{ P \} s \{ Q \}$ is valid, then P implies WP .

Intuition

The weakest precondition is a predicate, which describes all possible initial states (think: inputs) on which we can run program S such that it terminates without runtime errors in final states (think: outputs) specified by the postcondition Q .

When determining the weakest precondition of a program S and a postcondition Q , we thus aim to modify Q in a way that reverts changes made to the state by S . The result is a predicate that is true for a given initial state whenever running S on that initial state leads to a final state satisfying Q again. Thus, one can think about the weakest precondition as the predicate that anticipates the truth value of postcondition Q but *before* (that is, evaluated in initial states) rather than *after* (that is, evaluated in final states) running program S .

Computation

We typically do not need to apply the above (semantic) definition, since we can compute weakest preconditions (syntactically) on the program structure using the following recursive definition:

S	WP(S, Q)
=====	
var x	forall x :: Q
assert R	R && Q
assume R	R ==> Q
x := a	Q[x / a]
S1; S2	WP(S1, WP(S2, Q))
S1 [] S2	WP(S1, Q) && WP(S2, Q)

Here, $Q[x / a]$ is the predicate Q in which every (free) appearance of x has been replaced by a . A formal definition is given [below](#).

We briefly go over the above rules for computing weakest preconditions.

Variable declarations

The variable declaration `var x` declares `x` as an uninitialized (integer-valued) variable named `x`. All information about the value of some other variables named `x` that might have existed before are forgotten. This is reflected by the operational rule

$$\frac{v \in \text{Int}}{\text{var } x, \sigma \Longrightarrow \text{done}, \sigma[x := v]}$$

which nondeterministically assigns any integer `v` to `x`.

Now, assume that postcondition `Q` holds for the final state $\sigma[x := v]$. What are the possible initial states such that executing `var x` could lead to such a state?

In fact, such states must coincide with $\sigma[x := v]$ for all variables except `x`, since `var x` does not change any other variable. Moreover, the value of `x` does not matter at all, since `x` will be set to an arbitrary integer by executing `var x`.

Hence, we can (but, due to nondeterminism do not have to) reach $\sigma[x := v]$ from any state $\sigma[x := u]$ for all initial values `u` of `x`.

As a predicate, we can use a universal quantifier over `x` to express that the value of `x` does not matter. This yields `WP(var x, Q) ::= forall x :: Q`.

Assignments

The assignment `x := a` evaluates expression `a` in the current (initial) program state and assigns the resulting value to variable `x`. This is reflected by the operational rule

$$\frac{\mathbf{a}, \sigma \Longrightarrow_a \mathbf{v}}{\mathbf{x} := \mathbf{a}, \sigma \Longrightarrow \mathbf{done}, \sigma[\mathbf{x} := \mathbf{v}]}$$

Similarly to the variable declaration, only variable x is potentially changed by the assignment. However, the original value of x now does matter, since we may need it to evaluate expression a . For example, the assignment $x := x + 1$ depends on the value of x in the initial state.

Now, assume that postcondition Q holds for the final state $\sigma[x := v]$. What are the possible initial states such that executing $x := a$ leads to such a state?

Intuitively, we would like to undo the change made by the assignment. For example, if $x \geq 17$ holds after the assignment $x := x + 1$, then $x \geq 16$ should hold before the assignment. One way to do this is to observe that the expression $x + 1$ is evaluated in initial states, just like the weakest precondition. We can thus use it to refer to the value v . To undo the assignment, we then replace every occurrence of x by $x+1$ and obtain $x + 1 \geq 17$.

The effect of such a replacement is that the effect of setting x to the value of $x+1$ now appears on both sides of every constraint involving x . For the arithmetic operations in our language, they will thus cancel each other out. More concretely, $x + 1 \geq 17$ is equivalent to $x \geq 16$, our desired precondition.

Formally, the weakest precondition of assignments is defined as $WP(x:=a, Q) ::= Q[x/a]$, that is we formally substitute x by a in Q .

The following lemma justifies the above intuition, that is, we can substitute x by a to "undo" the effect of executing the assignment $x := a$:

Substitution Lemma Consider a predicate Q , an arithmetic expression a , and a state σ . Moreover, assume that a evaluates to value v for state σ , that is, $a, \sigma \rightarrow v$.

Then, $\sigma[x := v] \models Q$ if and only if $\sigma \models Q[x / a]$.

Assertions

The assertion `assert R` checks whether the predicate `R` holds in the current (initial) state and causes a runtime error if this is not the case; otherwise, it has no effect. This is reflected by two operational rules, one for the case that `R` holds and one for the case that it does not hold:

$$\frac{P, \sigma \Longrightarrow_P \text{true}}{\text{assert } P, \sigma \Longrightarrow \text{done}, \sigma} \qquad \frac{P, \sigma \Longrightarrow_P \text{false}}{\text{assert } P, \sigma \Longrightarrow \text{error}}$$

Now, assume that postcondition `Q` holds for the final state σ . We do not consider the configuration `error` because reason about total correctness; a triple is not valid if we can reach `error` from an initial state that satisfies the precondition.

What are the possible initial states such that executing `assert R` leads to σ ? We first observe that the assertion does not change σ : if `Q` holds after execution of `assert R`, then it also holds before its execution. We then reach the final state σ if and only if initial state σ additionally satisfies the predicate `R`.

Hence, the weakest precondition of assertions is defined as $\text{WP}(\text{assert } R, Q) ::= R \ \&\& \ Q$.

Assumptions

The assumption `assume R` is a somewhat strange statement because we cannot execute it on an actual machine; it is a verification-specific statement, similar to making an assumption in a mathematical proof. Intuitively, `assume R` checks whether the predicate `R` holds in the current (initial) state. Similar to an assertion, there is no effect if the predicate holds. However, in contrast to an assertion, we reach a "magical state" in which everything goes if the predicate does not hold. This corresponds to the case that we have made a wrong assumption in a proof: from a wrong assumption we can conclude whatever we want, so the result becomes worthless. This is reflected by two operational rules, one for the case that `R` holds and one for the case that it does not hold:

$$\frac{P, \sigma \Rightarrow_P \text{true}}{\text{assume } P, \sigma \Rightarrow \text{done}, \sigma} \qquad \frac{P, \sigma \Rightarrow_P \text{false}}{\text{assume } P, \sigma \Rightarrow \text{magic}}$$

Now, assume that postcondition `Q` holds for the final state σ . As for assertions, we do not consider the configuration `magic`; executions that move to `magic` cannot invalidate a triple anyway.

What are the possible initial states such that executing `assume R` leads to σ ? We first observe that the assertion does not change σ : if `Q` holds after execution of `assume R`, then it also holds before its execution. There are now two cases:

1. σ satisfies `R` and postcondition `Q`, or
2. σ does not satisfy `R` and we thus do not care about the result due to a wrong assumption; the weakest precondition should be `true` in this case, since everything follows from a wrong assumption.

Formally, this can be expressed as the predicate `(R && Q) || !R` or, equivalently, `WP(assume R) ::= R ==> Q`.

Sequential composition

The sequential composition $S1;S2$ first executes program $S1$ and then program $S2$. This is reflected by four operational rules

$$\begin{array}{c} \frac{S1, \sigma \Longrightarrow S1', \sigma'}{S1;S2, \sigma \Longrightarrow S1';S2, \sigma'} \qquad \frac{S1, \sigma \Longrightarrow \text{done}, \sigma'}{S1;S2, \sigma \Longrightarrow S2, \sigma'} \\ \frac{S1, \sigma \Longrightarrow \text{error}}{S1;S2, \sigma \Longrightarrow \text{error}} \qquad \frac{S1, \sigma \Longrightarrow \text{magic}}{S1;S2, \sigma \Longrightarrow \text{magic}} \end{array}$$

Here, the first rule keeps executing $S1$; termination of $S1$ is handled by the second rule, which moves on with executing $S2$. The last two rules propagate the case that we already encountered a runtime error or a wrong assumption.

What are the possible initial states such that executing $S1;S2$ leads to final states that satisfy postcondition Q ?

By the inductive definition of weakest preconditions, we can assume that we already know how to construct weakest preconditions for $S1$ and $S2$, namely $WP(S2, Q)$ and $WP(S1, WP(S2, Q))$. The set of all initial states on which we can run $S2$ to end up in a state that satisfies Q is given by $WP(S2, Q)$. We can use this predicate as the postcondition of program $S1$. By the same argument, $WP(S1, WP(S2, Q))$ is the set of all initial states on which we can run $S1$ to end up in a state that satisfies $WP(S2, Q)$.

Putting both arguments together, $WP(S1, WP(S2, Q))$ is the set of all initial states such that running $S1$ followed by $S2$ leads us to states that satisfy Q . Hence, the weakest precondition of sequential composition is defined as $WP(S1;S2, Q) ::= WP(S1, WP(S2, Q))$.

Nondeterministic choice

The nondeterministic choice $S1 \sqcup S2$ executes either program $S1$ or program $S2$; we do not know which one. This is reflected by two operational rules, one for running $S1$ and one for running $S2$:

$$\frac{}{S1 \sqcup S2, \sigma \Longrightarrow S1, \sigma} \qquad \frac{}{S1 \sqcup S2, \sigma \Longrightarrow S2, \sigma}$$

What are the possible initial states such that executing $S1 \sqcup S2$ leads to final states that satisfy postcondition Q ? By the inductive definition of weakest preconditions, we can assume that we already know how to construct weakest preconditions for $S1$ and $S2$, namely $WP(S1, Q)$ and $WP(S2, Q)$. Since we do not know whether $S1$ or $S2$ is executed, we can only guarantee that we will certainly reach a state that satisfies Q if we start in a state from which it does not matter whether we run $S1$ or $S2$. Our desired initial states must thus satisfy both $WP(S1, Q)$ and $WP(S2, Q)$.

Hence, the weakest precondition of nondeterministic choice is defined as $WP(S1 \sqcup S2, Q) ::= WP(S1, Q) \ \&\& \ WP(S2, Q)$.

Variable substitution

$F[x/a]$ denotes the *substitution* of every (free) occurrence of variable x by expression a in F . It is defined recursively as follows:

F	$F[x/a]$
=====	=====
true	true
false	false
x	a
y	y (where $x \neq y$)
v	v
$a1 <> a2$	$a1[x/a] <> a2[x/a]$ for $<>$ in { $=$, \neq , \leq , \geq , $<$, $>$, $+$, $*$, $-$, $/$, $\%$ }
$Q1 \ \&\& \ Q2$	$Q1[x/a] \ \&\& \ Q2[x/a]$
$Q1 \ \ Q2$	$Q1[x/a] \ \ Q2[x/a]$
$\neg Q1$	$\neg(Q1[x/a])$
$Q1 \implies Q2$	$Q1[x/a] \implies Q2[x/a]$
$\text{exists } y :: Q1$	$\text{exists } y :: Q1[x/a]$
$\text{forall } y :: Q1$	$\text{forall } y :: Q1[x/a]$

Strongest postconditions

Strongest postconditions generate a verification condition for checking the validity of Floyd-Hoare triples by starting at the precondition and moving forward through the program, just like an ordinary program execution.

Definition

The *strongest postcondition* of a program statement S and a precondition P is the predicate SP describing the smallest possible set of states such that the Floyd-Hoare triple

$$\{ P \} S \{ SP \}$$

is valid for *partial* correctness. In other words, whenever $\{ P \} S \{ Q \}$ is valid for partial correctness, then SP implies Q .

Notice that strongest postcondition on their own only make sense for partial correctness. For example, how should we choose SP such that $\{ \text{true} \} \text{assert false} \{ SP \}$ is valid? There is no SP such that the triple is valid for total correctness. For partial correctness, we have $SP = \text{false}$ and, in fact, every postcondition leads to a triple that is valid for partial correctness.

Intuition

The strongest postcondition of a program S and a precondition P is the predicate that consists of all final states in which program S terminates when started on states that satisfy P . It thus corresponds to symbolically executing the program, that is, we run the program by modifying sets of states - described by predicates - instead of single states.

When determining strongest postconditions, we thus aim to modify the precondition such that it captures the changes made to the program states by the executed program.

Computation

Similar to weakest preconditions, we can compute strongest postconditions on the program structure using the following recursive definition:

S	SP(P, S)
=====	
var x	exists x :: P
assert R	P && R
assume R	P && R
x := a	exists x0 :: P[x / x0] && x == a[x / x0]
S1; S2	SP(SP(P, S1), S2)
S1 [] S2	SP(P, S1) SP(P, S2)

Here, $P[x / x_0]$ is the predicate P in which every (free) appearance of x has been substituted by x_0 .

As for [weakest preconditions](#), we briefly go over the above rules for computing strongest postconditions.

Variable declarations

The variable declaration `var x` declares `x` as an uninitialized (integer-valued) variable named `x`. All information about the value of some other variables named `x` that might have existed before are forgotten. This is reflected by the operational rule

$$\frac{v \in \text{Int}}{\text{var } x, \sigma \Longrightarrow \text{done}, \sigma[x := v]}$$

which nondeterministically assigns any integer `v` to `x`.

Now, assume that precondition `P` holds for the initial state σ . What are the final states that we can reach by running `var x` on σ ? The state $\sigma[x := v]$ for some (nondeterministically chosen) integer v . Hence, the strongest postcondition is `sp(P, var x) ::= exists x :: P`.

Assignments

The assignment $x := a$ evaluates expression a in the current (initial) program state and assigns the resulting value to variable x . This is reflected by the operational rule

$$\frac{a, \sigma \Rightarrow_a v}{x := a, \sigma \Rightarrow \text{done}, \sigma[x := v]}$$

Similarly to the variable declaration, only variable x is potentially changed by the assignment.

Now, assume that precondition P holds for the initial state σ . What are the possible final states that we can reach by executing $x := a$ on σ ?

A first guess might be to define $SP(P, x := a) ::= P \ \&\& \ x == a$, that is, P should still hold and after the assignment x equals a . However, the following triples are *not* valid:

- $\{ x == 4 \} x := 7 \{ x == 4 \ \&\& \ x == 7 \}$
- $\{ \text{true} \} x := x + 1 \{ \text{true} \ \&\& \ x == x + 1 \}$

Hence, the above proposal is not necessarily correct if P or a depend on x . The underlying issue is that both P and a are originally evaluated in initial states but the strongest postconditions talks about final states. The variable x is the only variable that might have a different value before and after executing the assignment.

To define the correct strongest postcondition, we introduce an auxiliary variable, say x_0 , that represents the original value of x when evaluated in some initial state. The predicate $P[x / x_0]$ then means that precondition P was true in some initial state; similarly, $a[x/x_0]$ refers to the evaluation of a in the same initial state. Put together, we then use the original proposal but state that P and a were true in some initial state.

Hence, the strongest postcondition of assignments is defined as $SP(P, x:=a) ::= \text{exists } x_0 :: P[x / x_0] \ \&\& \ x == a[x / x_0]$.

As an example, let consider precondition $x == 4$ and the assignment $x := x + 7$:

```
sp(x == 4, x := x + 7)
= (by definition)
exists x0 :: x0 == 4 && x == x0 + 7
= (simplification)
x == 11
```

Indeed, x will be 11 after running $x := x + 7$ on a state where x is initially 4.

Assumptions

The assumption `assume R` is a somewhat strange statement because we cannot execute it on an actual machine; it is a verification-specific statement, similar to making an assumption in a mathematical proof. Intuitively, `assume R` checks whether the predicate `R` holds in the current (initial) state; there is no effect if the predicate holds; we reach a "magical state" in which everything goes if the predicate does not hold. This is reflected by two operational rules, one for the case that `R` holds and one for the case that it does not hold:

$$\frac{P, \sigma \implies p \text{ true}}{\text{assume } P, \sigma \implies \text{done}, \sigma} \qquad \frac{P, \sigma \implies p \text{ false}}{\text{assume } P, \sigma \implies \text{magic}}$$

Now, assume that precondition `p` holds for the initial state σ . We do not consider the configuration `magic`; executions that move to `magic` cannot [invalidate a triple](#) anyway.

What are the possible final states that we can reach by executing `assume R` on σ ?

Since we do not have to consider executions that move to `magic`, we know that σ satisfies `R`. Moreover, since the assumption does not change σ , `P` also holds in the final state.

Hence, the strongest postcondition of assumptions is defined as `SP(P, assume R) ::= P && R`.

Assertions

The assertion `assert R` checks whether the predicate `R` holds in the current (initial) state and causes a runtime error if this is not the case; otherwise, it has no effect. Since we reason about partial correctness with strongest postconditions, it does not matter whether we reach an error configuration or a magical configuration.

The strongest postcondition of the assertion `assert R` thus does not differ from the strongest postcondition of the assumption `assume R`. That is, $SP(P, \text{assert } R) ::= P \ \&\& \ R$.

Sequential composition

The sequential composition $S1;S2$ first executes program $S1$ and then program $S2$. This is reflected by four operational rules

$$\begin{array}{c} \frac{S1, \sigma \Longrightarrow S1', \sigma'}{S1;S2, \sigma \Longrightarrow S1';S2, \sigma'} \qquad \frac{S1, \sigma \Longrightarrow \text{done}, \sigma'}{S1;S2, \sigma \Longrightarrow S2, \sigma'} \\ \frac{S1, \sigma \Longrightarrow \text{error}}{S1;S2, \sigma \Longrightarrow \text{error}} \qquad \frac{S1, \sigma \Longrightarrow \text{magic}}{S1;S2, \sigma \Longrightarrow \text{magic}} \end{array}$$

Here, the first rule keeps executing $S1$; termination of $S1$ is handled by the second rule, which moves on with executing $S2$. The last two rules propagate the case that we already encountered a runtime error or a wrong assumption.

What are the possible final states that we can reach when executing $S1;S2$ on some initial state that satisfies precondition P ?

By the inductive definition of strongest postconditions, we can assume that we already know how to construct strongest postconditions for $S1$ and $S2$, namely $SP(P, S1)$ and $SP(SP(P, S1), S2)$.

$SP(P, S1)$ is then the set of all final states we can reach after running $S1$ on a state satisfying P . We then use this predicate as the precondition of $S2$. By the same argument, $SP(SP(P, S1), S2)$ is then the set of final states we can reach after running $S2$ on a state satisfying $SP(P, S1)$.

Putting both arguments together, $SP(SP(P, S1), S2)$ is the set of all final states reached after running $S1$ followed by $S2$ on initial states that satisfy P .

Hence, the strongest postcondition of sequential composition is defined as $SP(P, S1;S2) ::= SP(SP(P, S1), S2)$.

Nondeterministic choice

The nondeterministic choice $S1 \ [] \ S2$ executes either program $S1$ or program $S2$; we do not know which one. This is reflected by two operational rules, one for running $S1$ and one for running $S2$:

$$\frac{}{S1 \ [] \ S2, \sigma \Longrightarrow S1, \sigma} \quad \frac{}{S1 \ [] \ S2, \sigma \Longrightarrow S2, \sigma}$$

If we run $S1 \ [] \ S2$ on an initial states that satisfy precondition P , then we end up in the states reached by executing $S1$ on P or in the states reached by executing $S2$ on P , respectively.

Hence, the strongest postcondition of nondeterministic choice is defined as $SP(P, S1 \ [] \ S2) ::= SP(P, S1) \ || \ SP(P, S2)$.

Summary

- Program verification means proving that a program meets its specification.
- How automated program verification compares, on a high level, to other techniques that aim to increase our confidence in the correctness of software.
- What is the look and feel of automated verifiers that follow the principle of *verification like compilation*, including *modular reasoning with contracts*, *incremental* construction of verified code, and the notion of a *trusted codebase*.
- Program verification relies on rigorous mathematical definitions. We formalize the verification problem using *Floyd-Hoare triples*. There are different notions of validity or correctness for such triples; the strongest one is *total correctness*.
- To check whether a Floyd-Hoare triple is valid, we attempt to construct a *verification condition*, a logical predicate, which we can then check for validity, that is, it must be true for all possible program states.
- There are two systematic approaches for constructing verification conditions: start at the precondition and construct a suitable predicate by moving forward through the program or start at the postcondition and move backward through the program. The resulting *predicate transformers* compute the *strongest postcondition* and the *weakest precondition*, respectively.