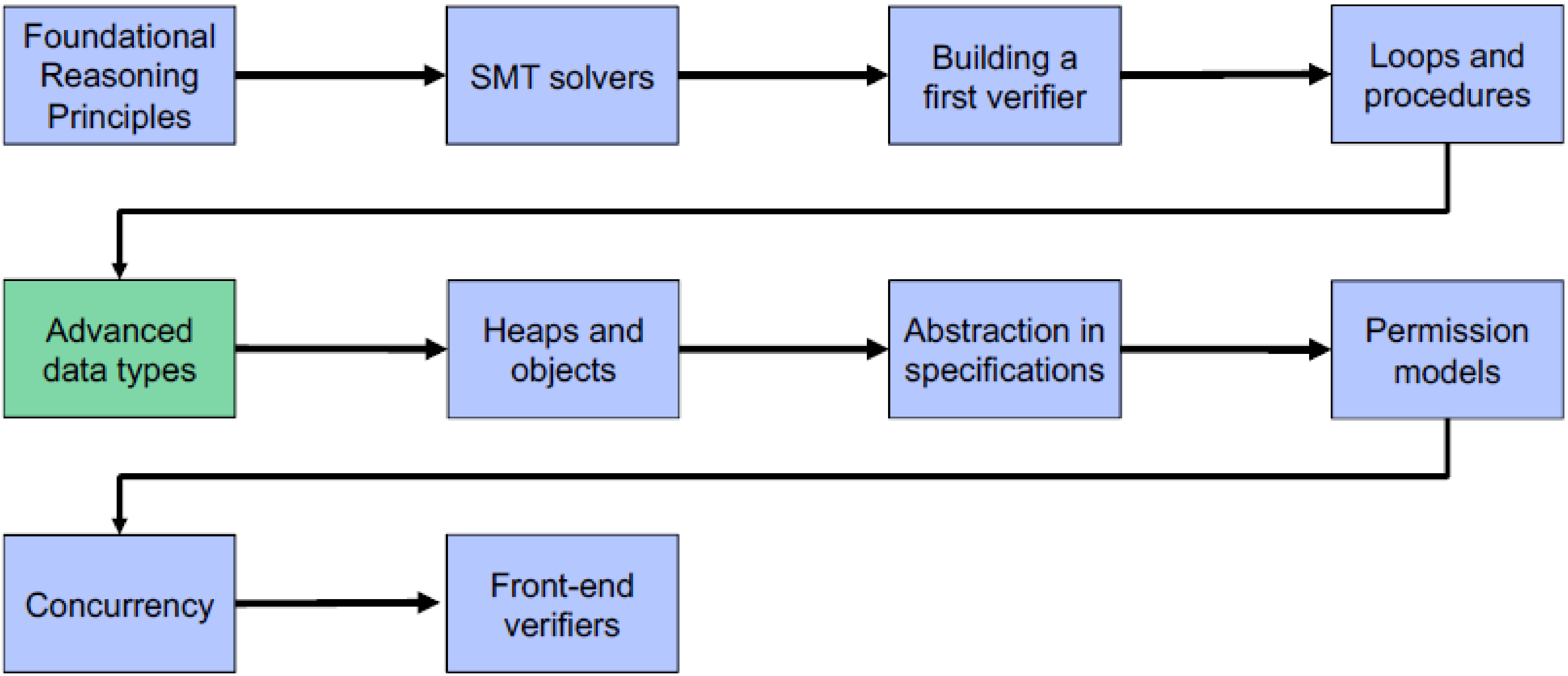
Methodologies for Software Processes

Lecture 7

ADVANCED DATATYPES

Tentative course outline



Outline

- Mathematical data types
- User-defined functions
- Function encoding

Mathematical data types

- Our language so far supports only three types

```
Types
T ::= Bool | Int | Rational
```

- Many functional languages feature mathematical data types
  - lists, tuples, sets, trees, etc.

- Subset of abstract data types (ADTs)
  - What are values of a type?
  - What are operations on data of a type?
  - immutable, no side-effects
  - ➔ *"programming & specification vocabulary"*

```
domain Set {
  function empty(): Set
  function add(s: Set, x: Int): Set
  function contains(s: Set, x: Int): Bool
  function union(s: Set, t: Set): Set
  function is_empty(s: Set): Bool
}
```

- Mathematical data types are for specifying imperative code ➔ module 8
  - *"Array sort leaves the multiset of elements unchanged"*
  - *"All implementations of Java's List interface store a sequence of elements"*

Common mathematical data types (PL4)

- We extend our language to support commonly-used data types
- The built-in data types
  - are generic
  - represent immutable, mathematical values
  - represent finite collections
  - are available in Viper
- We use Viper's expression syntax
  - See tutorial for other data types
  - https://viper.ethz.ch/tutorial

Types

```
T ::= Bool | Int | Rational | Set[T]
    | Seq[T] | Multiset[T] | Map[T, T]
```

Expressions

```
e ::= ...                          as before
    | Set[T]()                     empty set
    | Set(ē)                       set literal
    | e union e
    | e intersection e
    | e setminus e
    | e subset e
    | e in e                       membership
    | |e|                          cardinality
```

Example

```
method collect(s: Seq[Int]) returns (res: Set[Int])
  ensures forall j: Int :: 0 <= j && j < |s| ==> s[j] in res
  ensures forall x: Int :: x in res ==> x in s
{
  res := Set[Int]()
  var i: Int := 0
  while (i < |s|)
    invariant 0 <= i && i <= |s|
    invariant forall j: Int :: 0 <= j && j < i ==> s[j] in res
    invariant forall x: Int :: x in res ==> x in s
  {
    res := res union Set(s[i])
    i := i + 1
  }
}
```

Set operations

Sequence operations

Custom data types (PL3)

Declarations

```
D ::= ...                                          as before

    | domain <name> {                        define new type
        (function <name>(x:T): T)*           define function
        (axiom <name> { P })*                   define axiom
      }
```

Types

```
T ::= Bool | Int | Rational
        | <name>      defined types
```

Expressions

```
e ::= ...                  as before
       | <name>(ē)   function call
```

```
domain Point {
  function cons(x: Int, y: Int): Point
  function first(p: Point): Int
  function second(p: Point): Int

  axiom destruct_over_construct {
      forall x: Int, y: Int ::
        first(cons(x,y)) == x && second(cons(x,y)) == y
   }
}
```

- Every domain declares a new type and associated functions
- Corresponds to a axiomatizing a new theory

Example: binary trees with values at leafs

```
// Java-like code
interface Tree {
 Tree leaf(int value);
 Tree node(Tree left, Tree right);

 bool is_leaf();
 Tree left();
 Tree right();
 int value();
}
```

```
var t: Tree := node(
  node(leaf(3), leaf(17)),
  leaf(22)
)
assert !is_leaf(t)
var t2: Tree := right(left(t))
assert value(t2) == 17
```

```
domain Tree {

  function leaf(value: Int): Tree
  function node(left: Tree, right: Tree): Tree

  function is_leaf(t: Tree): Bool
  function value(t: Tree): Int
  function left(t: Tree): Tree
  function right(t: Tree): Tree

  axiom value_over_leaf {
    forall x:Int :: value(leaf(x)) == x
  }

  axiom right_over_node {
    forall l:Tree, r:Tree :: right(node(l, r)) == r
  }

  // ... (see 02-tree.vpr)

}
```

Example: binary trees with values at leafs

```
// Java-like code
interface Tree {
 Tree leaf(int value);
 Tree node(Tree left, Tree right);

 bool is_leaf();
 Tree left();
 Tree right();
 int value();
}
```

```
var t: Tree := node(
  node(leaf(3), leaf(17)),
  leaf(22)
)
assert !is_leaf(t)
var t2: Tree := right(left(t))
assert value(t2) == 17
```

constructors

```
domain Tree {
  function leaf(value: Int): Tree
  function node(left: Tree, right: Tree): Tree

  function is_leaf(t: Tree): Bool

           value(t: Tree): Int
           left(t: Tree): Tree
           right(t: Tree): Tree

  axiom value_over_leaf {
    forall x:Int :: value(leaf(x)) == x
  }

  axiom right_over_node {
    forall l:Tree, r:Tree :: right(node(l, r)) == r
  }

  // ... (see 02-tree.vpr)
}
```

Example: binary trees with values at leafs

```
// Java-like code
interface Tree {
 Tree leaf(int value);
 Tree node(Tree left, Tree right);

 bool is_leaf();
 Tree left();
 Tree right();
 int value();
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```
var t: Tree := node(
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  leaf(22)
)
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var t2: Tree := right(left(t))
assert value(t2) == 17
```

```
domain Tree {

  function leaf(value: Int): Tree
  function node(left: Tree, right: Tree): Tree

  function is_leaf(t: Tree): Bool
  function value(t: Tree): Int
           left(t: Tree): Tree
           right(t: Tree): Tree

           ue_over_leaf {
           x:Int :: value(leaf(x)) == x

  axiom right_over_node {
    forall l:Tree, r:Tree :: right(node(l, r)) == r
  }

  // ... (see 02-tree.vpr)
}
```

discriminators

Example: binary trees with values at leafs

```
// Java-like code
interface Tree {
 Tree leaf(int value);
 Tree node(Tree left, Tree right);

 bool is_leaf();
 Tree left();
 Tree right();
 int value();
}
```

```
var t: Tree := node(
  node(leaf(3), leaf(17)
  leaf(22)
)
assert !is_leaf(t)
var t2: Tree := right(left(t))
assert value(t2) == 17
```

destructors

```
domain Tree {

  function leaf(value: Int): Tree
  function node(left: Tree, right: Tree): Tree

  function is_leaf(t: Tree): Bool
  function value(t: Tree): Int
  function left(t: Tree): Tree
  function right(t: Tree): Tree

  axiom value_over_leaf {
          x:Int :: value(leaf(x)) == x

          ght_over_node {
          l:Tree, r:Tree :: right(node(l, r)) == r
  }

  // ... (see 02-tree.vpr)
}
```

Example: binary trees with values at leafs

```
// Java-like code
interface Tree {
 Tree leaf(int value)
 Tree node(Tree left,

 bool is_leaf();
 Tree left();
 Tree right();
 int value();
}
```

```
var t: Tree := node(
  node(leaf(3), leaf(17)),
  leaf(22)
)
assert !is_leaf(t)
var t2: Tree := right(left(t))
assert value(t2) == 17
```

Axioms

- Discriminators over constructors
- All trees are built from constructors
- Destructors over constructors

```
                     Tree
                    right: Tree): Tree

                     Bool
                     nt
                     ee
  function right(t: Tree): Tree

  axiom value_over_leaf {
    forall x:Int :: value(leaf(x)) == x
  }

  axiom right_over_node {
    forall l:Tree, r:Tree :: right(node(l, r)) == r
  }

  // ... (see 02-tree.vpr)
}
```

Encoding of custom data types

- We encode custom data types into SMT by axiomatizing them
  - new type ➔ uninterpreted sort
  - new operation ➔ uninterpreted function
  - new axiom ➔ assert axiom (add to BP)

Background Predicate:
conjunction of all axioms

Verification condition:

`BP ==> P ==> WP(S, Q)` valid

```
domain Set {
  function empty(): Set
  function card(s: Set): Int
  // ...

  axiom card_empty { card(empty()) == 0 }
  // ...
}
```

Conceptually, data types are encoded to PL0 as `assume BP`; the SMT language also needs declarations which are *not* in PL0.

```
(declare-sort Set)

(declare-const empty Set)
(declare-fun card (Set) Int)
; ...

(assert (= (card empty) 0)) ; axiom
; ...
```

Pragmatically, we can enrich PL0 by a statement for SMT declarations or "inline SMT code"

Encoding of built-in data types

- Built-in data types define domains with carefully crafted axioms and more convenient syntax
- Encoding: PL4 ➔ PL3

- Generics can be handled via monomorphization: generate a separate axiomatization for every instance of a generic type `T` that is used in a given program

Types

```
T ::= ...
    | Set[T]
```

Expressions

`e ::= ...`	as before
`| Set[T]()`	empty set
`| |e|`	cardinality

`T ::= Int`

```
domain IntSet {
  function empty(): IntSet
  function card(s: IntSet): Int
  // ...

  axiom card_empty { card(empty()) == 0 }
  // ...
}
```

Outline

- Mathematical data types
- User-defined functions
- Function encoding

Writing stronger specifications

- The built-in types and operators allow one to specify many interesting properties
- However, there are many methods whose behavior cannot be specified (easily)
- It is often useful to define additional mathematical vocabulary to specify the intended behavior

→ Axiomatizations have a fixed pattern

→ Use functional programs

```
method fac(n: Int) returns (res: Int)
  requires 0 <= n
  ensures  res == facDef(n)
{
  res := 1
  var i: Int := 1
  while(i <= n) {
    res := res * i
    i := i + 1
  }
}
```

```
domain X {
 function facDef(n: Int): Int
 axiom {
   forall n: Int ::
     (n <= 1 ==> facDef(n) == 1) &&
     (n > 1 ==> facDef(n)
                == n * facDef(n-1))
}}
```

User-defined functions (PL5)

- Functions abstract over expressions
  - can appear in specifications
  - can be recursive
  - can be uninterpreted (no definition)

- Model of mathematical functions
  - no side-effects
  - must always terminate (*not checked by Viper!*)
  - deterministic
  - well-defined for every input (total)

```
function facDef(n: Int): Int
{
  n <= 1 ? 1 : n * facDef(n-1)
}
```

Declarations

```
D ::= ...
    | function <name>(x̄: T̄): T
        (requires P)*
        (ensures Q)*
        ({ e })?
```

Expressions

```
e ::= ... | <name>(ē)
```

Reasoning about function calls

- Functions generally do not require a specification
  - Postconditions are typically equal the function definition
- We reason about calls by using the function definition

```
function facDef(n: Int): Int
{
  n <= 1 ? 1 : n * facDef(n-1)
}
```

- In contrast to methods, reasoning about function calls is not modular

```
x := facDef(1)
assert x == 1
```

- Non-modularity has drawbacks
  - All callers need to be re-verified when a function definition changes
  - But mathematical vocabulary is typically more stable

Partial functions

- Many operations are inherently partial functions
  - Meaningful only on a subset of the possible arguments
  - Example: division by zero
- Option 1: construct artificially total functions
  - Often leads to awkward function definitions
  - May cause misleading error messages
- Option 2: equip functions with preconditions
  - Needs to be checked for every function call
  - Also called “well-definedness conditions”
  - Supported by Viper

```
function facDef(n: Int): Int
{ n <= 1 ? 1 : n * facDef(n-1) }
```

```
x := facDef(-1)
```

```
function facDef(n: Int): Int
  requires 0 <= n
{ n <= 1 ? 1 : n * facDef(n-1) }
```

```
x := facDef(-1)
```

Function postconditions

- Since reasoning about function calls uses the function definition, functions typically do not have postconditions

- But postconditions are permitted
  - Use keyword `result` to refer to the returned value

```
function facDef(n: Int): Int
  requires 0 <= n
  ensures  1 <= result
{ n <= 1 ? 1 : n * facDef(n-1) }
```

- When reasoning about function calls, Viper uses the function definition and the postcondition

```
function f(): Bool
  ensures  false
{ f() }
```

- Postcondition is verified against function definition
  - Assumed for recursive calls
  - Dangerous when functions do not terminate!

```
x := f()
assert false
```

Use cases for function postconditions

- Abstract functions
  - Shortcut for axiomatizing certain functions
  - In the absence of a function definition, calls are verified using only the postcondition

```
function sqrt(n: Int): Int
  requires 0 <= n
  ensures  0 <= result
  ensures  result * result <= n &&
           n < (result+1) * (result+1)
```

```
c := sqrt(a*a + b*b)
assert a*a + b*b - c*c < 2*c + 1
```

Use cases for function postconditions

```
function facDef(n: Int): Int
  requires 0 <= n
  ensures  1 <= result
{ n <= 1 ? 1 : n * facDef(n-1) }
```

```
assume 0 <= y
x := facDef(y)
assert 1 <= x // fails without post
```

- Automating induction proofs
  - SMT solvers are generally not able to prove properties about recursive functions using induction
  - By declaring a function postcondition, we provide the necessary induction hypothesis
  - Also works with methods ➔ lemmas

```
function facDef(n: Int): Int
  requires 0 <= n
  ensures  1 <= result
{
    n <= 1
      ? 1
      : n * facDef(n-1)
}
```

```
Induction hypothesis:
for all m < n, 1 <= facDef(m)
```

```
Induction base:
facDef(0) >= 1, facDef(1) >= 1
```

```
Induction step: for n > 1,
facDef(n)
=  n * facDef(n-1)
>= facDef(n-1)          (n > 1)
>= 1                   (by I.H.)
```

Outline

- Mathematical data types
- User-defined functions
- Function encoding

Simplified encoding of functions

- User-defined functions are encoded into the background predicate as an uninterpreted function and a definitional axiom

```
function f(x: T): TT {
  E
}
```

```
function f(x: T): TT

axiom forall x: T :: f(x) == E
```

- The axiom above is simplified; it omits
  - pre- and postconditions
  - checks that partial expressions are well-defined

Simplified encoding with pre- and postconditions

- Function pre- and postconditions are added to the definitional axiom

```
function f(x: T): TT
  requires P
  ensures  Q
{ E }
```

```
function f(x: T): TT

axiom {
  forall x: T ::
    P ==> f(x) == E && Q[result/f(x)]
}
```

- Sound, but recursive functions may lead to non-termination ➔ next module

- Note that postconditions are encoded in the axiom
  - An inconsistent postcondition can compromise soundness, even if the function is never called!

```
function f(): Bool
  ensures  false
{ f() }
```

~~x := f()~~

```
assert false
```

Well-definedness conditions for partial expressions

- New proof obligation: all expressions are well-defined
  - Example: no division by zero
  - User-defined functions are are called with arguments that satisfy their preconditions
- Well-definedness condition `DEF: Expr → Pred`
  - `DEF(e)` holds in state σ iff expression e can be evaluated in σ

Short-circuit evaluation

Expression e	*DEF*(e)
0, 1, -3, false, … (constants)	true
e1 + e2, e1 < e2, e1 && e2, …	*DEF*(e1) && *DEF*(2)
e1 / e2	*DEF*(e1) && *DEF*(e2) && e2 != 0
foo(e)	*DEF*(e) && “precondition of foo”
e1 ==> e2	*DEF*(e1) && (e1 ==> *DEF*(e2))

Encoding partial expressions

- Every statement first asserts well-definedness of its expressions

```
x := f(y) + 2 * z / n
```

precondition of `f(y)`

```
assert DEF(f(y)) && n != 0
x := f(y) + 2 * z / n
```

```
assume x/y > 1
```

```
assert y != 0
assume x/y > 1
```

- Alternative: redefine ***WP***

```
WP(x := e, Q)      ::= DEF(e) && Q[x / e]
WP(assert P, Q)    ::= DEF(P) && P && Q
WP(assume P, Q)    ::= DEF(P) && P ==> Q
                       ...
```

Wrap-up

- Writing specifications often requires a suitable mathematical vocabulary
  - added via a background predicate BP that axiomatizes uninterpreted sorts and functions
  - Verification condition: `BP ==> P ==> WP(S, Q)`

```
method collect(s: Seq[Int])
  returns (res: Set[Int])
  ensures forall j: Int ::
    0 <= j && j < |s| ==> s[j] in res
{ ... }
```

- Viper's background predicate collects axioms from multiple features
  - Built-in types and their operations
  - User-defined functions
  - Custom axiomatizations via domains

```
function f(n: Int): Int
{ n <= 1 ? 1 : n * f(n-1) }
```

```
domain Set {
  function empty(): Set
  function union(s: Set, t: Set): Set
  // ...
}
```