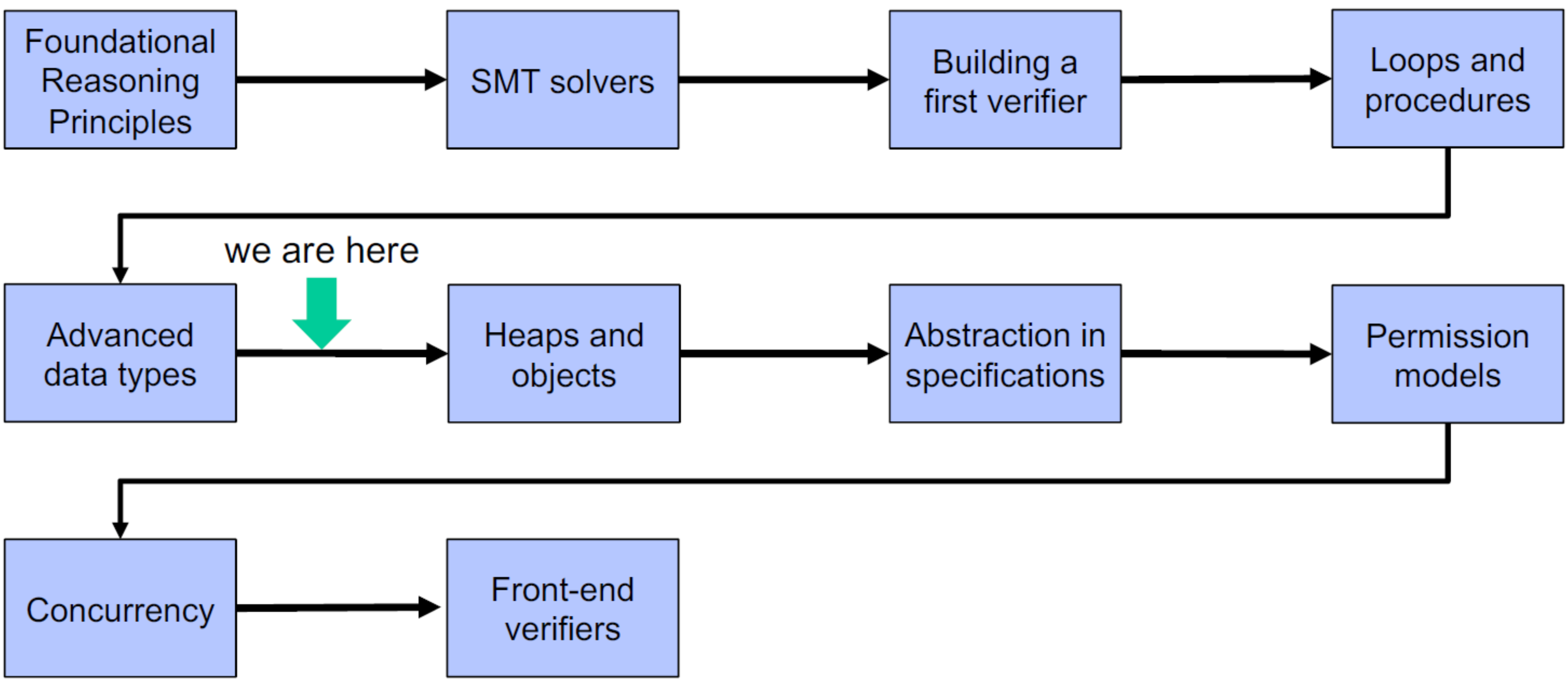
Methodologies for Software Processes

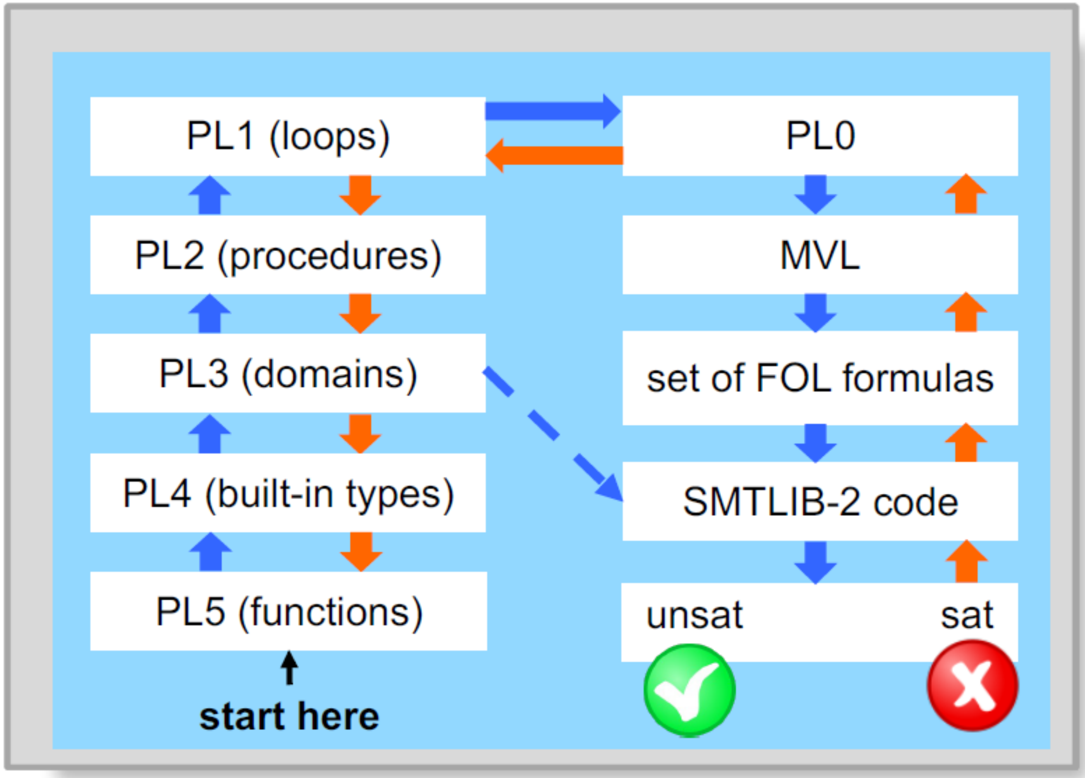
Lecture 8

VERIFICATION TACTICS

Tentative course outline



The language PL5

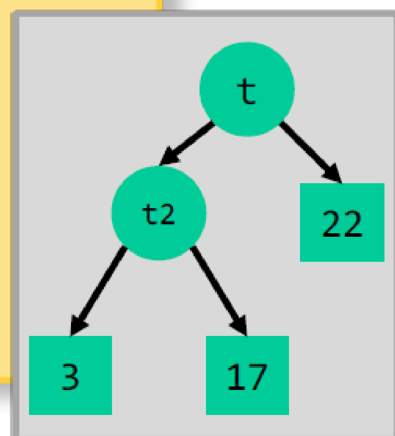


Example – summing values in a binary tree

```
method client() {
    var t: Tree := node(
      node(leaf(3), leaf(17)),
      leaf(22)
    )
    assert sum(t) == 42
}

function sum(t: Tree): Int

domain Tree {
  // ...
}
```



	Approach
1	implement function
2	abstract function with postcondition
3	definitional axiom using Tree functions
4	manually written definitional axioms
5	implement function + assertion in client

- All approaches are logically equivalent
- Most theories with quantifiers are undecidable

➔ To effectively use automated verifiers, we need to understand how tools deal with quantifiers

Outline: verification tactics

- Excursion: quantifiers
- Lemmas & proofs
- Hands-on program verification

Universal quantifier instantiation

- **Our problem:** Is the **FO** formula `F` unsatisfiable?
  - equivalent: is !F satisfiable?

- To prove `forall x :: G` `unsat`, we can try out all possible candidate values until we find *one value* v such that `G[x/v]` becomes `unsat`

- **Issue 1:** How do we choose *good* candidates?
  - most values may be irrelevant for our VCs

- **Issue 2:** When do we give up trying more values?
  - Logics with quantifiers are often undecidable
  - Better to quickly report that we cannot verify a problem than trying out values indefinitely

Verification condition:

```
BP && !WP(S, true)                          unsat
```

```
     forall x :: G                          unsat
iff  !(forall x :: G)                       sat
iff  exists x :: !G                         sat
iff  for some value v, !G[x/v]              sat
iff  for some value v, G[x/v]               unsat
```

```
forall x :: G
<==>
G[x/v1] && ... && G[x/vn] &&
        && forall x :: G
```

Universal quantifier instantiation – approaches

- Due to undecidability, all approaches are incomplete
  - May return `unknown` or not terminate

- Model-based quantifier instantiation (MBQI)
  - Focuses on proving satisfiability
  - Possibly returns `unknown` instead of `unsat`
  - ➔ Not well-suited for our verification problem

Verification condition:
`BP && !`***`WP`***`(S, true) unsat`

- Heuristic quantifier instantiation with E-matching
  - Focuses on proving unsatisfiability
  - Possibly returns `unknown` instead of `sat`
  - We may not get a counterexample if verification fails
  - ➔ Most common approach used by verification tools

Heuristic quantifier instantiation

- Main idea: try out a subset V of all values
  - return unsat if `G[x/v]` is unsat for some v in V
  - return unknown if `G[x/v]` is sat for all v in V

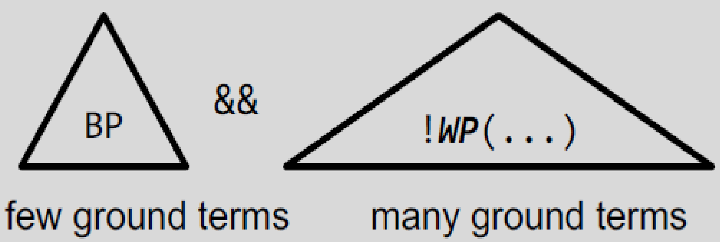
```
forall x :: x in V ==> G unsat

iff for some v in V, G[x/v]  unsat

implies forall x :: G unsat
```

- Hypothesis: V should contain...
  - all ground terms
    - terms without quantifier-bound variables
    - “expressions used in program or specification”
    - E.g., `0, 1+2, x+2` (where x is a free variable)
  - function applications to ground terms
    - “unfolding of function calls”
    - E.g., `fib(fib(1))`

Structure of our VCs



Heuristic quantifier instantiation loop (for one quantifier)

Input: FO formula *F* `&& forall x :: G`

Output: `unsat` or `unknown`

```
h(0) == 1 &&
forall x :: h(x) == 1+h(x-1) && h(x) < 3
```

Algorithm:

```
F(0) :=  F
```

```
F(0) := h(0) == 1
```

```
for i = 0, 1, 2, ...

    (Choose) pick a ground term t in F(i)
    or G such that F(i) does not contain a
    conjunct equal to G[x/t]; return
    unknown if no such t exists

    (Instantiate) F(i+1) := G[x/t] && F(i)

    (Check) If F(i+1) is unsat, then
            return unsat
```

```
i = 0: choose t = 1
G[x/t] = 1 + h(0) && h(1) < 3
Instantiate: F(1) := G[x/t] && F(0)
Check: F(1) is sat ➔ continue
```

```
i = 1: choose t = 1 + h(0) = 2
G[x/t] = h(2) == 1+h(2-1) && h(2) < 3
Instantiate: F(2) := G[x/t] && F(1)
= h(2) == 1+h(2-1) && h(2) < 3
  && h(1) == 1 + h(0) && h(1) < 3
  && h(0) == 1
F(2) unsat ➔ return unsat
```

E-matching

- Problems with heuristic
  - Formulas may have exponentially many ground terms
  - Function applications admit infinitely many ground terms
    - ➔ Let user determine relevant ground terms

Predicates

```
P ::= ... | forall x:T :: { p } P
```

- A pattern (or trigger) is a term p such that
  - p contains all bound variables in the scope of the quantifier
  - p contains at least one non-constant uninterpreted function
  - p contains at most constant interpreted function

```
x == f(7) && g(x) == 3 &&
  forall y: Int ::
    { g(f(y)) } g(f(y)) > 5
```

- Consider only ground terms `t` that e-match pattern p, that is, we can find some ground term `t'` provably equal to p`[x / t]`

How can we instantiate the above?

E-matching

- Problems with heuristic
  - Formulas may have exponentially many ground terms
  - Function applications admit infinitely many ground terms
    - ➔ Let user determine relevant ground terms

Predicates

```
P ::= ... | forall x:T :: { p } P
```

- A pattern (or trigger) is a term p such that
  - p contains all bound variables in the scope of the quantifier
  - p contains at least one non-constant uninterpreted function
  - p contains at most constant interpreted function

```
x == f(7) && g(x) == 3 &&
  forall y: Int ::
    { g(f(y)) } g(f(y)) > 5
```

- Consider only ground terms `t` that e-match pattern p, that is, we can find some ground term `t'` provably equal to `p[x / t]`

`p[x/t] = g(f(7)) = 3` ← `t'` appears above
➔ instantiate `g(f(7)) > 5`

How can we instantiate the above?

Example

```
f(0) != f(1) && g(1) == 0 && f(0) == 1
&& forall x: Int ::{ p } f(x) == f(g(x))
```

- Patterns are typically terms in the quantified formulas body, e.g. `p = g(x)`
  - e-match 1: `g(1) == 0` and `0` is a ground term
  - instantiating `f(1) == f(g(1))` makes the whole formula unsatisfiable ➔ return `unsat`
- Too restrictive patterns may often yield unknown, e.g. `p = g(g(x))`
  - No e-matching possible ➔ return `unknown`
- Too permissive patterns may lead to matching loops, e.g. `p = f(x)`
  - e-match `0`, instantiate `f(0) == f(g(0))`
  - e-match `g(0)`, instantiate `f(g(0)) == f(g(g(0)))`, e-match `g(g(0)) ...`

Reasoning about recursive functions

➔ `07-factorial.vpr`

- Problem: Recursive functions can always be unfolded to instantiate new ground terms

- There is no natural condition for stopping the unfolding, even if the recursive function terminates

- Consequences:
  - Recursive functions lead to matching loops
  - SMT solver may never terminate

- Solution: limit the unfolding depth

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
var n: Int; assert fac(n) != 0
```

```
function fac(x: Int): Int

axiom forall x: Int ::
 fac(x) == (x <= 1 ? 1 : x * fac(x-1))
```

```
fac(0) == 1 && fac(n) != 0
```

```
fac(1)==1 && fac(0)==1 && fac(n)!=0
```

```
fac(fac(1)) == 1 && fac(1) == 1 && ...
```

```
fac(fac(fac(1))) == 1 && ...
```

Limited functions

→ 08-factorial.vpr

- Goal: encode recursive functions such that they can be unfolded only a limited number of times

```
function fac(x: Int): Int
function fac0(x: Int): Int

axiom forall x: Int ::
 (x <= 1 ==> fac(x) == 1) &&
 (x > 1 ==> fac(x) == x * fac0(x-1))
```

- Idea: to stop unfolding, call a different function without a definitional axiom

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
var n: Int; assert fac(n) != 0
```

```
fac(0) == 1 && fac(n) != 0
```

```
fac(1)==1 && fac(0)==1 && fac(n)!=0
```

```
fac(fac(1)) == fac(1) * fac0(fac(1)-1)
&& fac(1)==1 && fac(0)==1 && fac(n)!=0
```

Since `fac0` is not constrained by axioms, the SMT solver can choose a function for `fac0` such that the formula becomes `unsat`

Improving limited functions

→ `09-factorial.vpr`

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings
- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
→ fac(1) == 1 && fac(2) == 2 * fac(2-1)
→ fac(1) == 1 && fac(2) == 2 * 1
→ fac(1) == 1 && fac(2) == 2
```

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```

✔

```
assert fac(2) == 2 // two unfoldings
```

✘

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```

✘

Improving limited functions

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings
- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
➔ fac(1) == 1 && fac(2) == 2 * fac(2-1)
➔ fac(1) == 1 && fac(2) == 2 * 1
➔ fac(1) == 1 && fac(2) == 2
```

- Idea: axiomatize `fac(x) == fac0(x)`
  - Problem: may reintroduce matching loop

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```

```
assert fac(2) == 2 // two unfoldings
```

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```

```
axiom {
  forall x: Int ::
    fac(x) == fac0(x)
}
```

Improving limited functions

→ `10-factorial.vpr`

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings
- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
→ fac(1) == 1 && fac(2) == 2 * fac(2-1)
→ fac(1) == 1 && fac(2) == 2 * 1
→ fac(1) == 1 && fac(2) == 2
```

- Idea: axiomatize `fac(x) == fac0(x)`
  - Problem: may reintroduce matching loop
  - Solution: trigger axiom only for function `fac(x)`

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```

✓

```
assert fac(2) == 2 // two unfoldings
```

✗

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```

✓

```
axiom {
  forall x: Int :: { fac(x) }
    fac(x) == fac0(x)
}
```

Working with limited functions

➔ `11-factorial.vpr`

- To work around unfolding limits, it is often sufficient to mention a ground term that is required for the proof

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(2) == 2
```

```
var n: Int := fac(1)
// ground term fac(1) is available
assert fac(2) == 2
```

Existential quantifier instantiation

➔ `12-exists.vpr`

- **Our problem:** Is the **FO** formula `F` unsatisfiable?
  - equivalent: is !F satisfiable?

- To prove **`exists`** `x :: G` `unsat`, we have to show that `G[x/v]` becomes `unsat` for all values v

- When aiming to prove unsatisfiability, SMT solvers often struggle with existentials

```
assert exists x: Int :: x == 0
```

- Try to avoid existential quantifiers in specifications

- If needed, manually instantiate or introduce existential quantifiers ➔ user-defined **lemmas**

Verification condition:

```
BP && !WP(S, true)                unsat
```

```
     exists x :: G                unsat
iff  !(exists x :: G)               sat
iff  forall x :: !G                 sat
iff  for all values v, !G[x/v]      sat
iff  for all values v, G[x/v]     unsat
```

Outline

- Excursion: quantifiers
- Lemmas & proofs
- Hands-on program verification

Lemmas – in mathematics

- A lemma consists of
  - a premise determining whether the lemma can be used
  - a conclusion stating what property is guaranteed
  - a proof checking that the conclusion indeed always follows from the premise

- To apply a lemma, we check its premises and, if yes, can use its conclusion

- Lemmas are “subroutines of a larger proof”

Lemma 1.
Premise: `n >= 0`
Conclusion: `fac(n) > 0`
Proof: by induction on n.

```
Theorem. For all x > 0,
fac(x) + fac(x) + fac(x) > 2.

Proof.
Let x > 0. Then, since x >= 0,
Lemma 1 yields fac(x) > 0.
Hence,
fac(x) + fac(x) + fac(x) > 2.
```

Why do we need lemmas for program verification?

```
function fac(x: Int): Int {
  x <= 1 ? 1 : x * fac(x-1)
}

method client(x: Int)
  returns (y: Int)
  requires x > 0
  ensures y > 2
{
    var z: Int := fac(x)

    y := z + z + z
}
```

➔ `13-factorial-positive.vpr`

- SMT solver does not notice that `fac(x) > 0`
  - No automatic proofs by induction
  - Could be added as postcondition to `fac(x)`

- We may not want to add all needed properties as axioms (of functions)
  - Some properties might be specialized and are only useful in very specific cases
  - Many axioms might slow down proof generation

Lemmas – as ghost methods

```
method lemma(<arguments>)
  requires Premise
  ensures Conclusion
{
    Proof
}
```

```
lemma(x,y)
assert Conclusion(x,y)
```

- Lemmas are ghost methods
  - They may not affect program execution
  - They can be removed from production code
- Method body represents a correctness proof
  - Abstract methods are trusted (unproven)
- By invoking a lemma, we learn its postcondition only for the supplied the arguments

Using a lemma in Viper

➔ `14-factorial-lemma.vpr`

```
function fac(x: Int): Int {
  x <= 1 ? 1 : x * fac(x-1)
}

method client(x: Int)
  returns (y: Int)
  requires x > 0
  ensures y > 2
{
    var z: Int := fac(x)
    lemma_fac_pos(x)
    y := z + z + z
}
```

we now know `fac(x) > 0`

we do *not* know `fac(z) > 0`

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
```

- By invoking a lemma, we learn its postcondition only for the supplied the arguments
- To use a lemma, we just call the method
  - Checks that premise holds for supplied arguments
  - Guarantees that conclusion holds afterward

Proving lemmas by implementing ghost methods

Statement	Meaning in proofs
`x := e`	Name an expression
`assert P`	Make a correct statement (possibly to introduce ground terms)
`assume P`	Make a (possibly wrong) assumption
`if (b) {S1} else {S2}`	Case distinction on b
method call	Invoke another lemma
recursive method call (for proofs by induction)	Invoke the induction hypothesis given by the lemma's contract

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
  // decreases n // variant
{
    var v: Int := n; assert v >= 0
    // proof by induction on n
    if (n == 0) { // base case
      assert fac(0) > 0
    } else { // induction step
        assert n-1 >= 0
        // invoke I.H.
        assert n-1 < v
        lemma_fac_pos(n-1)
        assert fac(n-1) > 0
    }
}
```

➔ 15-lemma-proof.vpr

Why we need termination proofs for lemmas

Claim: for all integers n, the following triple is valid:

```
{ n >= 0 }
lemma_fac_pos(n)
{ fac(n) > 0 }
```

Induction hypothesis (I.H.): for all m < n, the triple

```
{ m >= 0 }
lemma_fac_pos(m)
{ fac(m) > 0 }
```

is valid.

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
{

    if (n == 0) {
```

induction base: claim holds for `n == 0`

```
    } else {
```

illegal use of **I.H.**: argument is not < n

```
        lemma_fac_pos(n)
    }
}
```

unsound proof!

➔ `15-lemma-proof.vpr`

Why we need termination proofs for lemmas

Claim: for all integers n, the following triple is valid:

```
{ n >= 0 }
lemma_fac_pos(n)
{ fac(n) > 0 }
```

Induction hypothesis (I.H.): for all m < n, the triple

```
{ m >= 0 }
lemma_fac_pos(m)
{ fac(m) > 0 }
```

is valid.

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
{
  var v: Int := n; assert v >= 0
    if (n == 0) {
```

induction base: claim holds for n `== 0`

```
    } else {
```

illegal use of **I.H.**: argument is not < n

```
        lemma_fac_pos(n)      assert n < v
    }
}
```

- For terminating proof programs, calls always decrease some variant

➔ only valid proofs verify

variant does not decrease

➔ unsound proof fails

➔ `15-lemma-proof.vpr`

Lemmas for existential quantifiers

➔ 18-exists-lemmas.vpr

- Sometimes the solver may be unable to deal with parts of a predicate
  - Existential quantifiers, very complex predicates
- We can "hide" such predicates in a boolean function with no definitional axiom
  - The solver has nothing to unfold
  - Function calls are kept around
- We can then introduce abstract lemmas to
  - *unfold* the predicate: make its definition visible, possibly with concrete values for existentials
  - *fold* the predicate: store its definition behind a call, possibly abstracting concrete values

```
function divides(x:Int, y: Int): Bool
  requires x >= 0 && y >= 0
  //ensures result == exists z:Int ::
  //              z >= 0 && x * z == y
```

```
method divides_unfold(x: Int, y: Int)
  returns (z: Int)
  requires x > 0 && y > 0
  requires divides(x, y)
  ensures z >= 0 && x * z == y
```

```
method divides_fold(x: Int,
                    y: Int, z: Int)
  requires x > 0 && y > 0
  requires z >= 0 && x * z == y
  ensures divides(x, y)
```