Homework no. 3

A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is considered sparse if the size of the matrix $\boldsymbol{n}$ is *‚large'* and it has *'few'* elements $\boldsymbol{a_{ij} \neq 0}$. For this homework we shall consider two types of sparse matrices. The first type are the matrices that have at least 20 non-zero elements on each row. The second type are the tridiagonal matrices, for which the nonzero coefficients are placed on the main diagonal and on 2 secondary diagonal, one on the lower triangular part of the matrix (diagonal $\boldsymbol{p}$) and the other on the upper triangular part of matrix $\boldsymbol{A}$ (diagonal $\boldsymbol{q}$). If matrix $\boldsymbol{A}$ is tridiagonal, the coefficients are defined by:

$$a_{ij} = \begin{cases} a_i & \textbf{daca } i = j \\ b_i & \textbf{daca } j - i = q \\ c_j & \textbf{daca } i - j = p \\ 0 & \textbf{in rest} \end{cases}$$

In vectors $\boldsymbol{a} \in \mathbb{R}^{n}, \boldsymbol{b} \in \mathbb{R}^{n-q+1}, \boldsymbol{c} \in \mathbb{R}^{n-p+1}$ are stored the elemets of the three diagonals with nonzero entries for matrix $\boldsymbol{A}$. For this homework we consider tridiagonal matrices with $\boldsymbol{p=q=2}$. Such a matrix has the following form:

$$A = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-1} & a_n \end{pmatrix}$$

In files a.txt, b.txt, aplusb.txt, aorib.txt posted on the lab's web page are stored 4 sparse matrices. The matrix from file b.txt is a tridiagonal matrix, for which we store the following elements:

- ***n*** data size, ***p, q*** – indices where the secondary nonzero diagonals begin
- ***n*** values for the main diagonal of the matrix (vector ***a***), followed by ***n-1 (n-p+1)*** values of vector ***b*** and then the ***n-1 (n-q-1)*** elements of vector ***c***. The three vectors are separated by empty lines.

The matrices that have atmost 20 nonzero elements are stored in the file in the following way:

- ***n*** data size,
- $a_{ij} \neq 0$, ***i, j*** **–** the value of a non-zero element from the sparse matrix $A \in \mathbb{R}^{n\times n}$, followed by the row and column indices of the respective non-zero element (the position in the matrix),
- one assumes that the nonzero coefficients of the matrix are placed randomly in the file (they are not ordered by row or column indices, or otherwise)

Using these files, read the size of the matrix and generate the necessary data structures for economically storing the sparse matrix (the special way in which a sparse matrix is stored, is described below).

Let $A, B \in \mathbb{R}^{n\times n}$ be two sparse matrices (read from file a.txt and b.txt, respectively), the matrix ***B*** is a tridiagonal matrix. Using the special, economical way of storing sparse matrices, compute:

- ***A+B*** sum of the two matrices,
- ***A*B*** matrix multiplication result.

Verify that the sum/product of the matrices from files a.txt and b.txt is the matrix from file aplusb.txt/aorib.txt. Two elements placed on the same position (that have the same row and column indices) ***(i, j)*** are considered numerically equal if $|c_{ij}-d_{ij}| < \varepsilon$. For the result matrices one uses the general storing scheme.

Remarks: 1) Don't use classically stored matrices for solving the above problems and don't use a function ***val(i,j)*** that computes the value of a_{ij} the corresponding element in the matrix, for all pairs ***(i,j)***.

2) The sum of matrices from files a.txt and b.txt is a matrix with at most 23 non-zero elements on each line. For the multiplication of sparse matrices, the sparsity of the result (the proportion of zero elements in the matrix) cannot be specified in advance.

3) If in the homework's attached files, there are multiple entries with the same line and column indices:

val_1 , i, j

…

val_2 , i, j

…

val_k , i, j

this situation has the following meaning:

$$a_{ij} = val_1 + val_2 + \cdots + val_k \,.$$

Bonus (30pt): Write a function that computes the product of two tridiagonal matrices. There are no restrictions on the values of p and q, the values for matrix A are (p^A, q^A) and (p^B, q^B) for matrix B.

Sparse matrix storing method

A sparse vector is a vector that has 'few' non-zero elements. Such a vector can be stored economically in a data structure that will keep only the non-zero values and the position in the vector of the respective value:

$$\{(val \neq 0, i);\ x_i = val\}\,.$$

A sparse matrix can be economically stored as a vector of sparsely stored vectors: each line of the matrix is stored in a sparse vector structure (for Homework 4 the diagonal element should be the last memorized element of the line).

Examples

Matrix:

$$A = \begin{pmatrix} \mathbf{102.5} & \mathbf{0.0} & \mathbf{2.5} & \mathbf{0.0} & \mathbf{0.0} \\ \mathbf{3.5} & \mathbf{104.88} & \mathbf{1.05} & \mathbf{0.0} & \mathbf{0.33} \\ \mathbf{0.0} & \mathbf{0.0} & \mathbf{100.0} & \mathbf{0.0} & \mathbf{0.0} \\ \mathbf{0.0} & \mathbf{1.3} & \mathbf{0.0} & \mathbf{101.3} & \mathbf{0.0} \\ \mathbf{0.73} & \mathbf{0.0} & \mathbf{0.0} & \mathbf{1.5} & \mathbf{102.23} \end{pmatrix}$$

can be sparsely stored in the following way:

{ { (2.5, 3) , (102.5, 1)},
{ (3.5, 1), (0.33, 5),(1.05, 3), (104.88, 2)},
{ (100.0, 3},
{ (1.3, 2) , (101.3, 4)},
{ (1.5, 4), (0.73, 1), (102.23, 5)} }.

Tridiagonal matrix

$$A = \begin{pmatrix} \mathbf{102.5} & \mathbf{2.5} & \mathbf{0.0} & \mathbf{0.0} & \mathbf{0.0} \\ \mathbf{3.5} & \mathbf{104.88} & \mathbf{1.05} & \mathbf{0.0} & \mathbf{0.0} \\ \mathbf{0.0} & \mathbf{1.3} & \mathbf{100.0} & \mathbf{0.33} & \mathbf{0.0} \\ \mathbf{0.0} & \mathbf{0.0} & \mathbf{0.73} & \mathbf{101.3} & \mathbf{0.0} \\ \mathbf{0.0} & \mathbf{0.0} & \mathbf{0.0} & \mathbf{1.5} & \mathbf{102.23} \end{pmatrix}$$

$$a = \begin{pmatrix} \mathbf{102.5} \\ \mathbf{104.88} \\ \mathbf{100.0} \\ \mathbf{101.3} \\ \mathbf{102.23} \end{pmatrix}, \quad b = \begin{pmatrix} \mathbf{2.5} \\ \mathbf{1.05} \\ \mathbf{0.33} \\ \mathbf{0.0} \end{pmatrix}, \quad c = \begin{pmatrix} \mathbf{3.5} \\ \mathbf{1.3} \\ \mathbf{0.73} \\ \mathbf{1.5} \end{pmatrix}, \quad p = 2, \quad q = 2$$