Homework no. 7

Let P be a polynomial of degree n with real coefficients:

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_kx^{n-k} + \cdots + a_{n-1}x + a_n \ , \quad a_0 \neq 0$$

Compute the interval $[-R, R]$ where all the real roots of the polynomial P can be found. Implement Olver's method for approximating the real roots of a polynomial. In all computations use Horner's algorithm for calculating the value of the polynomial in a point. Approximate as many as possible real roots of the polynomial P with Olver's method, starting from distinct initial points x_0. Display the results on screen and also write them in a file. Write in the file only the distinct roots (2 real values v_1 and v_2 are considered distinct if $|v_1 - v_2| > \epsilon$).

Bonus 20 pt.: Eliminate the root of multiplicity > 1 computing the greatest common divisorof polynomials P and P', $Q = \text{g.c.d.}(P, P')$ and simplify polynomial P, $P = P/Q$. Implement the algorithm for computing the greatest common divisor for two polynomials (don't use a library function that computes the g.c.d. of two polynomials).

Olver's Method for Approximating Real Roots of Polynomials

Let P be a polynomial of degree n:

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \ , \quad (a_0 \neq 0) \tag{1}$$

All the real roots of the polynomial P are in the interval $[-R,\ R]$ where R is given by:

$$R = \frac{|a_0| + A}{|a_0|} \quad , \quad A = \max\{|a_i| \ ; \ i = \overline{1,n}\} \tag{2}$$

For approximating a real root x^*, $x^* \in [-R,\ R]$, of the polynomial P defined by (1), one computes a sequence of real numbers, $\{x_k\}$, which converges to the root x^*, $x_k \longrightarrow x^*$ for $k \to \infty$.

In order to build the sequence $\{x_k\}$, one needs the first element x_0, then, the other elements are computed in the following way (x_{k+1} is computed using x_k):

$$\begin{aligned} x_{k+1} &= x_k - \frac{P(x_k)}{P'(x_k)} - \frac{1}{2}c_k \ , \ k = 0, 1, \ldots \ , \ x_0 - \text{given} \\ x_{k+1} &= x_k - \Delta \ x_k \ \big(\Delta \ x_k = \frac{P(x_k)}{P'(x_k)} + \frac{1}{2}c_k\big) \qquad (3) \\ c_k &= \frac{[P(x_k)]^2 P''(x_k)}{[P'(x_k)]^3} \end{aligned}$$

We denoted by P' and P'' the first and the second derivative of polynomial P, respectively.

Important remark: The way the first element of the sequence, x_0, is selected, can influence the convergence of the sequence x_k to x^* (or the divergence). Usually, a selection of the initial iteration x_0 in the neighbourhood of a root x^*, guarantees the convergence $x_k \longrightarrow x^*$ for $k \to \infty$.

Not all the elements of the sequence $\{x_k\}$ must be memorized, in order to obtain an approximation for the root we only need the 'last' computed value x_{k_0}. A value $x_{k_0} \approx x^*$ approximates a root (thus, is the 'last' computed element of the sequence) when the difference between two successive elements of the sequence is sufficiently small, i.e.:

$$|x_{k_0} - x_{k_0-1}| < \epsilon$$

where ϵ is the precision with which we want to approximate the root x^*.

A possible approximation scheme for Olver's method for approximating the root x^*, is the following:

Olver's Method

choose randomly x ; $k = 1$;
(for the convergence of the sequence $\{x_k\}$ is preferable to choose x_0 in the neighbourhood of a root)

do
{
$\star$ if $(|P'(x_k)| \leq \epsilon\ )$ EXIT;
(try restarting the algorithm, changing x_0)
$\star$ compute $\Delta\, x$ using formula (3) ;
$\star$ $x = x - \Delta\, x$;
$\star$ k=k+1;
}
while $(|\Delta\, x| \geq \epsilon$ and $k \leq k_{\max}$ and $|\Delta\, x| \leq 10^8)$
if ($|\Delta\, x| < \epsilon$) $x_k \approx x^*$;
else *divergence* ; (try new values for x_0)

Horner's method for computing $P(v)$

Let P be a polynomial of degree p:

$$P(x) = c_0 x^p + c_1 x^{p-1} + \cdots + c_{p-1} x + c_p \ , \quad (c_0 \neq 0)$$

We can write polynomial P also as:

$$P(x) = ((\cdots(((c_0 x + c_1)x + c_2)x + c_3)x + \cdots)x + c_{p-1})x + c_p$$

Taking into account this grouping of the coefficients, one obtains an efficient way to compute the value of polynomial P in any point $v \in \mathbf{R}$, this procedure is knowns as *Horner's method*:

$$\begin{array}{rcl} d_0 & = & c_0 \, , \\ d_i & = & c_i + d_{i-1} v \ , \quad i = \overline{1,p} \end{array} \tag{4}$$

In the above sequence:

$$P(v) := d_p$$

the rest of the computed elements of the sequence $d_i, i = 1, ..., p-1$, are the coefficients of the quotient polynomial Q, obtained in the division:

$$\begin{array}{rcl} P(x) & = & (x - v)Q(x) + r \ , \\ Q(x) & = & d_0 x^{p-1} + d_1 x^{p-2} \cdots + d_{p-2} x + d_{p-1} \ , \\ r & = & d_p = P(v). \end{array}$$

Computing $P(v)$ (d_p) with formula (4) can be performed using only one real variable $d \in \mathbf{R}$ instead of using a vector $d \in \mathbf{R}^p$.

Examples

$$
\begin{aligned}
P(x) &= (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6 \ , \\
& \quad a_0 = 1.0\,, \quad a_1 = -6.0\,, \quad a_2 = 11.0\,, \quad a_3 = -6. \\
P(x) &= (x-\frac{2}{3})(x-\frac{1}{7})(x+1)(x-\frac{3}{2}) \\
&= \frac{1}{42}(42x^4 - 55x^3 - 42x^2 + 49x - 6) \\
& \quad a_0 = 42.0\,, \quad a_1 = -55.0\,, \quad a_2 = -42.0\,, \quad a_3 = 49.0\,, \quad a_4 = -6.0. \\
P(x) &= (x-1)(x-\frac{1}{2})(x-3)(x-\frac{1}{4}) \\
&= \frac{1}{8}(8x^4 - 38x^3 + 49x^2 - 22x + 3) \\
& \quad a_0 = 8.0\,, \quad a_1 = -38.0\,, \quad a_2 = 49.0\,, \quad a_3 = -22.0\,, \quad a_4 = 3.0. \\
P(x) &= (x-1)^2(x-2)^2 \\
&= x^4 - 6x^3 + 13x^2 - 12x + 4 \\
& \quad a_0 = 1.0\,, \quad a_1 = -6.0\,, \quad a_2 = 13.0\,, \quad a_3 = -12.0\,, \quad a_4 = 4.0.
\end{aligned}
$$

Compute the complex value of polynomial P with a complex argument

Consider the complex number $z = c + id$. We want to compute the complex number (its real and imaginary parts, C and D):

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = C + iD$$

The real polynomial of degree 2 with the complex roots z and $\bar{z} = c - id$ is:

$$T(x) = x^2 + px + q = (x - z)(x - \bar{z}) \quad \text{with} \quad p = -2c \ , \ q = c^2 + d^2$$

Consider the division of polynomial P by polynomial T, where Q is the quotient polynomial and R is the remainder polynomial, i.e.:

$$\begin{aligned} P(x) &= T(x)Q(x) + R(x) \\ Q(x) &= b_0 x^{n-2} + b_1 x^{n-3} + \cdots + b_{n-3}x + b_{n-2} \ , \\ R(x) &= r_0 x + r_1 \ . \end{aligned} \tag{5}$$

The real coefficients $\{b_i\}$ and $\{r_i\}$ of the quotient and remainder polynomials Q and R can be computed by identifying the coefficients for x^k from the left hand-side with those from the right hand-side in relation (5). We deduce the following recurrence formulae:

$$\begin{aligned} b_0 &= a_0 \quad , \quad b_1 = a_1 - p\, b_0 \\ b_i &= a_i - p\, b_{i-1} - q\, b_{i-2} \quad , \quad i = 2, \ldots n \\ r_0 &= b_{n-1} \quad , \quad r_1 = b_n + p\, b_{n-1}. \end{aligned} \tag{6}$$

If one uses the relation (5) with $x = z$, one obtains:

$$P(z) = T(z)\, Q(z) + R(z) = R(z) = r_0\, z + r_1 = (r_0\, c + r_1) + i\, r_0\, d.$$

Thus, one gets:

$$\begin{aligned} C &= r_0\, c + r_1 = b_{n-1}\, c + b_n + p\, b_{n-1} \\ D &= r_0\, d = b_{n-1}\, d. \end{aligned} \tag{7}$$