Homework no. 6

Given $(n+1)$ distinct points, x_0, x_1, …, x_n ($x_i \in \mathbf{R}\ \forall i$, $x_i \neq x_j$, $\ i \neq j$) and the $(n+1)$ values of an unknown function$f$ at these points, $y_0 = f(x_0)$, $y_1 = f(x_1)$, …, $y_n = f(x_n)$:

x	x_0	x_1	$\cdots$	x_n
f	y_0	y_1	$\cdots$	y_n

approximate the value of function f in $\bar{x}$, $f(\bar{x})$, for a given $\bar{x}$, a value which is not in the above table, $\bar{x} \neq x_i$, $i = 0, \ldots, n$ using:

- polynomial approximation computed with the least squares method. For computing the value of least squares polynomial in $\bar{x}$, use Horner's algorithm; display $P_m(\bar{x})$, $|P_m(\bar{x}) - f(\bar{x})|$ and $\displaystyle\sum_{i=0}^{n} |P_m(x_i) - y_i|$. For m introduce values smaller than 6.
- using the quadratic spline function of class C^1. In this case, consider also known about function f, the value of the first derivative in x_0, $d_a = f'(x_0 = a)$. Display $S_f(\bar{x})$ şi $|S_f(\bar{x}) - f(\bar{x})|$.

For solving the linear systems involved in solving the above interpolation problems use the library employed in **Homework 2**.

Generate the interpolation nodes $\{x_i, i = 0, ..., n\}$ in the following way: x_0 şi x_n are introduced from the keyboard or from a file such that $x_0 < x_n$, and x_i are randomly generated such that $x_i \in (x_0, x_n)$ and $x_{i-1} < x_i$; the values $\{y_i, i = 0, ..., n\}$ are computed using a function f declared in your program (examples of how to choose the nodes x_0, x_n and the funcion $f(x)$ are at the end of this document), $y_i = f(x_i), i = 0, ..., n$;

Bonus (10 pt): Draw the graphs of function f and of the approximative computed functions P_m and T_n.

Least Squares Interpolation

Let $a = x_0 < x_1 < \cdots < x_n = b$. Given $\bar{x} \in [a, b]$ approximate $f(\bar{x})$ knowing that the $n+1$ values y_i of function f in the interpolation nodes.

One computes a polynomial of degree m:

$$P_m(x) = P_m(x; a_0, a_1, \dots, a_m) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{m} a_k\, x^k$$

the coefficients $\{a_i; i = \overline{0, m}\}$ being the solution to the optimization problem:

$$\min \left\{ \sum_{r=0}^{n} \left| P_m(x_r; a_0, a_1, \dots, a_m) - y_r \right|^2 ;\ a_0, a_1, \dots, a_m \in \mathbb{R} \right\}$$

Solving this problem leads to solving the linear system:

$$Ba = f$$

$$B = (b_{ij})_{i,j=0,\dots,m} \in \mathbb{R}^{(m+1)\times(m+1)} \quad f = (f_i)_{i=0,\dots,m} \in \mathbb{R}^{m+1}$$

$$\sum_{j=0}^{m} \Big(\sum_{k=0}^{n} x_k^{i+j} \Big) a_j = \sum_{k=0}^{n} y_k x_k^i \quad , \quad i = 0, \dots, m$$

This linear system can be solved with the same numerical library that was used for *Homework 2*.

The value of function f in $\bar{x}$ is approximated by the value of the polynomial P_m in $\bar{x}$:

$$f(\bar{x}) \approx P_m(\bar{x}; a_0, a_1, \dots, a_m)$$

For computing the value of polynomial $P_m(\bar{x})$ use Horner's method described below.

Horner's method for computing $P(v)$

Let P be a polynomial of degree p:

$$P(x) = c_0 x^p + c_1 x^{p-1} + \cdots + c_{p-1} x + c_p \ , \quad (c_0 \neq 0)$$

We can write polynomial P also as:

$$P(x) = ((\cdots(((c_0 x + c_1)x + c_2)x + c_3)x + \cdots)x + c_{p-1})x + c_p$$

Taking into account this grouping of the coefficients, one obtains an efficient way to compute the value of polynomial P in any point $v \in \mathbf{R}$, this procedure is knowns as *Horner's method*:

$$\begin{aligned} d_0 &= c_0 \,, \\ d_i &= c_i + d_{i-1} v \ , \quad i = \overline{1, p} \end{aligned} \tag{1}$$

In the above sequence:

$$P(v) := d_p$$

the rest of the computed elements of the sequence $d_i, i = 1, ..., p-1$, are the coefficients of the quotient polynomial Q, obtained in the division:

$$\begin{array}{rcl} P(x) & = & (x-v)Q(x) + r \ , \\ Q(x) & = & d_0 x^{p-1} + d_1 x^{p-2} \cdots + d_{p-2}x + d_{p-1} \ , \\ r & = & d_p = P(v). \end{array}$$

Computing $P(v)$ (d_p) with formula (1) can be performed using only one real variable $d \in \mathbf{R}$ instead of using a vector $d \in \mathbf{R}^p$.

Quadratic *spline* functions of class C^1

Let $a = x_0 < x_1 < \cdots < x_n = b$. Given $\bar{x} \in [a, b]$, approximate the value of $f(\bar{x})$ knowing beside the $n+1$ pairs $(x_i, y_i = f(x_i))$ the value d_a of the first derivative of function f in a:

$$f'(a) = d_a.$$

One looks for a function S_f of class C^1 on $[a, b]$ such that:

$$S_f(x) = a_i x^2 + b_i x^2 + c_i \ , \quad \text{for } x \in [x_i, x_{i+1}], \ i = \overline{0, n-1}$$

$$S_f(x_i) = y_i \ , \ i = \overline{0, n} \ , \quad S_f'(a) = d_a (d_a = f'(a))$$

The function S_f that satisfies the above stated properties has the following form:

$$S_f(x) = \frac{(A_{i+1} - A_i)}{2h_i}(x - x_i)^2 + A_i(x - x_i) + y_i \quad , \text{ for } x \in [x_i, x_{i+1}], \ i = \overline{0, n-1}$$

where

$$\begin{array}{rcl} h_i & = & x_{i+1} - x_i \quad , \quad i = 0, \ldots, n-1 \\ A_{i+1} & = & -A_i + \dfrac{2(y_{i+1} - y_i)}{h_i} \quad , \quad i = 0, \ldots, n-1 \\ A_0 & = & d_a \end{array}$$

The value of function f in $\bar{x}$, $f(\bar{x})$, is approximated by the value $S_f(\bar{x})$ ($\bar{x}$ must be a value in the interval $[a, b]$). One searches an index i_0 such that $\bar{x} \in [x_{i_0}, x_{i_0+1}]$. We have:

$$f(\bar{x}) \approx S_f(\bar{x}) = \frac{(A_{i_0+1} - A_{i_0})}{2h_{i_0}}(x - x_{i_0})^2 + A_{i_0}(x - x_{i_0}) + y_{i_0}.$$

Input data - examples

$$x_0 = a = 1 \quad , \quad x_n = b = 5 \quad , \quad f(x) = x^2 - 12x + 30$$

$$x_0 = a = 0 \quad , \quad x_n = b = 1.5 \quad , \quad f(x) = \sin(x) - \cos(x)$$

$$x_0 = a = 0 \quad , \quad x_n = b = 2 \quad , \quad f(x) = 2x^3 - 3x + 15$$