Homework no. 8

Consider $F : \mathbf{R} \longrightarrow \mathbf{R}$ a real function. Approximate a (local or global) minimum point of function F using Dehghan-Hajarian [1] method. Verify that the obtained solution is a minimum point by checking the sign of the second derivative. Compare the solutions obtained using the two different methods for approximating the first derivative of function F (see relations (4), (5)) by displaying the number of iterations necessary to obtain the approximative solution for the same precision $\epsilon > 0$.

Minimization of functions of one variable

Consider $F : \mathbf{R} \longrightarrow \mathbf{R}$ a real, twice differentiable function, $F \in C^2(\mathbf{R})$, for which one wants to approximate the solution x^* of the minimization problem:

$$\min\{F(x); x \in V\} \quad \longleftrightarrow \quad F(x^*) \leq F(x) \quad \forall x \in V \tag{1}$$

where $V = \mathbf{R}$ (x^* is a global minimum point) or $V = [\bar{x} - r, \bar{x} + r]$ (local minimum point). A *critical point* for function F, is a point $\tilde{x}$ which is the root of the first derivative of F:

$$F'(\tilde{x}) = 0. \tag{2}$$

For the twice differentiable real functions, the minimum points of function F are among the critical points of F. A critical point is a minimum point if:

$$F''(x^*) > 0.$$

We shall search the minimum points of F among the solutions of the nonlinear equation (2). Below is described the secant method for approximating the roots of a nonlinear equation:

$$g(x) = 0 \qquad (g(x) = F'(x)).$$

[1] Dehghan, M., Hajarian, M. (2010). Some derivative free quadratic and cubic convergence iterative formulas for solving nonlinear equations. Computational and Applied Mathematics, 29(1), 19-30.

Dehghan-Hajarian's Method

The root x^* is approximated by computing a sequence $\{x_k\}$ which, if certain conditions are fulfilled, converges to x^*. The convergence depends on the selection of the first two values of the sequence.

The $k+1$ element of the sequence, x_{k+1}, is computed using the previous element, x_k, in the following way:

$$x_{k+1} = x_k - \frac{g(x_k)[g(z) - g(x_k)]}{g(x_k + g(x_k)) - g(x_k)} = x_k - \Delta x_k \ , \qquad k = 0, 1, \dots \ ,$$

$$x_0 - \ \text{dat},$$

$$z = x_k + \frac{[g(x_k)]^2}{g(x_k + g(x_k)) - g(x_k)} \quad , \quad \Delta x_k = \frac{g(x_k)[g(z) - g(x_k)]}{g(x_k + g(x_k)) - g(x_k)} . \tag{3}$$

The convergence of the sequence depends on the selection of the first element of the sequence, x_0.

Important remark: The choice of the initial data, x_0, influences the convergence (or divergence) of the sequence x_k to x^*. Usually, a selection of the initial iterations in the neighborhood of a root x^*, guarantees the convergence $x_k \longrightarrow x^*$ for $k \to \infty$.

Not all the elements of the sequence $\{x_k\}$ must be memorized, in order to obtain an approximation for the root we only need the 'last' computed value x_{k_0}. A value $x_{k_0} \approx x^*$ approximates a root (thus, is the 'last' computed element of the sequence) when the difference between two successive elements of the sequence is sufficiently small, i.e.:

$$|x_{k_0} - x_{k_0 - 1}| < \epsilon$$

where ϵ is the precision with which we want to approximate the root x^*.

Dehghan-Hajarian's Method

```
choose randomly x_0 ;
//(for the convergence of the sequence {x_k} is preferable
choosing x_0 in the neighbourhood of a root)
x = x_0 ;
k = 0 ;
do
  {
    - if ( |g(x + g(x)) - g(x)| <= ϵ) return x;
    - compute z and Δx with formula (3) ;
    - x = x - Δx ;;
    - k = k + 1;
  }
while (|Δx| >= ϵ and k <= k_max and |Δx| <= 10^8)
if ( |Δx| < ϵ ) return x; // x_k ≈ x^* ;
else "divergence" ; //(try changing the first element x_0)
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In order to compute/approximate the value of the first derivative of function F, employing only the values of F, one can use one of the following approximation formulae:

$$F'(x) \approx G_i(x,h) \quad , \quad i = 1,2$$

where

$$G_1(x,h) = \frac{3F(x) - 4F(x-h) + F(x-2h)}{2h} \tag{4}$$

$$G_2(x,h) = \frac{-F(x+2h) + 8F(x+h) - 8F(x-h) + F(x-2h)}{12h} \tag{5}$$

with $h = 10^{-5}$ or 10^{-6} (h can be considered an input parameter). For verifying that the critical point approximated with Dehghan-Hajarian's method is a minimum point, the following relation must be fulfilled:

$$F''(x^*) > 0.$$

The following formula provides an approximation of the second derivative of F'':

$$F''(x) \approx \frac{-F(x+2h) + 16F(x+h) - 30F(x) + 16F(x-h) - F(x-2h)}{12h^2}$$

Examples

$$\begin{aligned}
F(x) &= \frac{1}{3}x^3 - 2x^2 + 2x + 3 \ , & x^* = 2 + \sqrt{2} \approx 3.41421356237 \\
F(x) &= x^2 + \sin(x) \ , & x^* \approx -0.4501836112948 \\
F(x) &= x^4 - 6x^3 + 13x^2 - 12x + 4 \ , & x^* \in \{1, 2\}
\end{aligned}$$