

The background of the slide features a black and white aerial photograph of the University of Regina campus. In the foreground, there's a large, multi-story stone building with classical architectural details. Behind it, a river flows through a park area with many trees. Further back, the city of Regina's skyline is visible, featuring several modern office buildings and residential areas.

UNIVERSITY OF REGINA

CS310-002

DISCRETE

COMPUTATIONAL

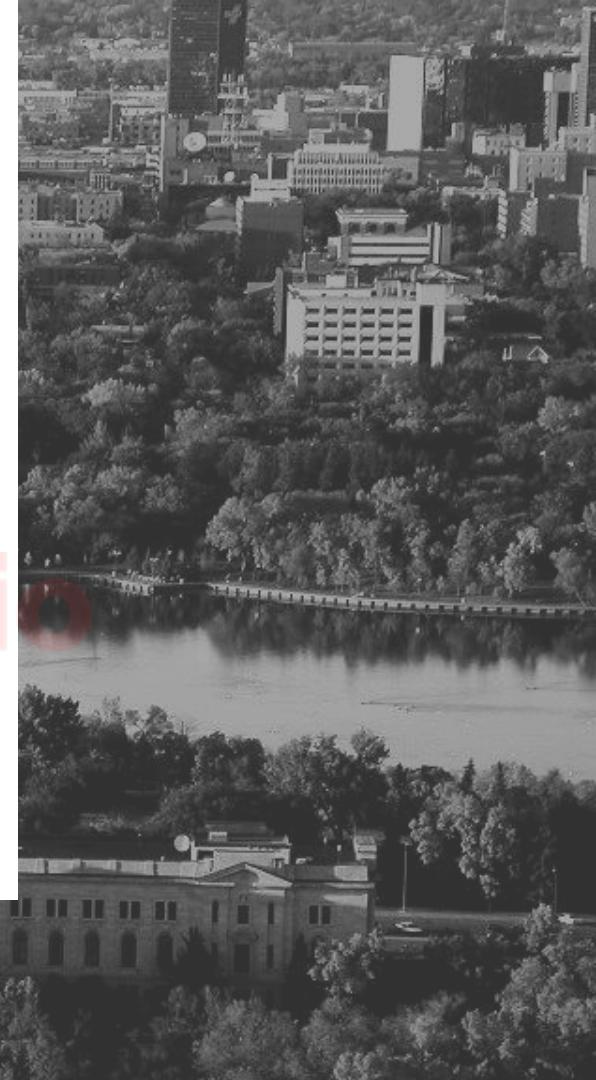
STRUCTURES

andreeds.github.io

ANDRÉ E. DOS SANTOS

dossantos@cs.uregina.ca

andreeds.github.io



CS310-002
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ADVANCED COUNTING TECHNIQUES

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ANDRÉ E. DOS SANTOS
dossantos@cs.uregina.ca
andreeds.github.io



Review

A recursive definition of a sequence specifies

- Initial conditions
- Recurrence relation

Example:

$$a_0 = 0 \text{ and } a_1 = 3$$

Initial conditions

$$a_n = 2a_{n-1} - a_{n-2}$$

Recurrence relation

$$a_n = 3n$$

Solution

Review

Linear recurrences: Each term of a sequence is a linear function of earlier terms in the sequence.

Example:

$$a_0 = 1 \quad a_1 = 6 \quad a_2 = 10$$

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$$

$$\begin{aligned} a_3 &= a_0 + 2a_1 + 3a_2 \\ &= 1 + 2(6) + 3(10) = 43 \end{aligned}$$

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Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

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**linear homogeneous recurrence relation
of degree k with constant coefficients**

is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

- a_n is expressed in terms of the previous k terms of the sequence, so its degree is k .
- This recurrence includes k initial conditions

$$a_0 = C_0$$

$$a_1 = C_1$$

...

$$a_k = C_k$$

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linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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The right-hand side is a sum of previous terms of the sequence each multiplied by function of n

linear **homogeneous** recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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No terms occur that are not multiples of the a_j s

linear homogeneous recurrence relation
of degree k with **constant coefficients**

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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The coefficients of the terms of the sequence are all constants, rather than functions that depend on n

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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The degree is k because a_n is expressed in terms of the previous k terms of the sequence

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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The degree is k because a_n is expressed in terms of the previous k terms of the sequence

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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Examples:

★ $P_n = (1.11)P_{n-1}$

★ $f_n = f_{n-1} + f_{n-2}$

★ $a_n = a_{n-5}$

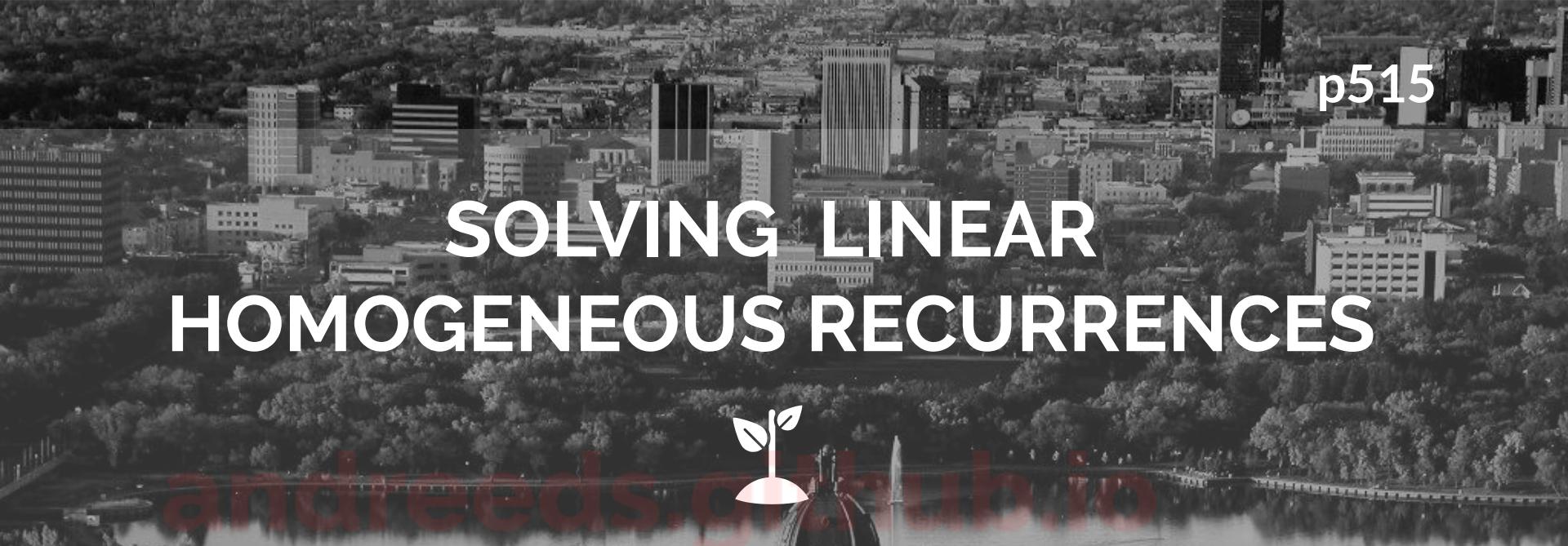
linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

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NOT Examples:

- ★ $a_n = a_{n-1} + a^2_{n-2}$
- ★ $H_n = 2H_{n-1} + 1$
- ★ $B_n = nB_{n-1}$

A black and white aerial photograph of a city skyline, likely Boise, Idaho, featuring several skyscrapers and a large park area with a fountain in the foreground.

p515

SOLVING LINEAR HOMOGENEOUS RECURRENCES



andrews.cs.utb.edu

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Proposition

- Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.
 - ◆ α is any constant.

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Proposition

It follows from **Proposition**, if we find some solutions to a linear homogeneous recurrence, then **any linear combination** of them will also be a solution to the linear homogeneous recurrence.

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Solving linear homogeneous recurrences

- Recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

- Try to find a solution of form r^n

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \dots - c_k r^{n-k} = 0$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0 \quad \text{dividing both sides by } r^{n-k}$$

This equation is called the **characteristic equation**.

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Example

The *Fibonacci* recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

Its characteristic equation is

$$r^2 - r - 1 = 0$$

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Example

What is the characteristic equation of $F_n = 4F_{n-1} - 4F_{n-2}$?

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Proposition

- r is a solution of $r^k - c_1r^{k-1} + c_2r^{k-2} + \dots + c_k = 0$ if and only if r^n is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$.

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Example

consider the characteristic equation $r^2 - 4r + 4 = 0$.

$$r^2 - 4r + 4 = (r - 2)^2 = 0$$

So, $r = 2$.

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So, 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$.

$$2^n = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2}$$

Theorem

- Consider the characteristic equation

$$r^k - c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k = 0$$

and the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

- Assume r_1, r_2, \dots and r_m all satisfy the equation.

- Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be any constants.

- So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_m r_m^n$ satisfies the recurrence.

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Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

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Example

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

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Example

What is the solution of the recurrence relation

$$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$

with $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$?

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REVIEW QUESTIONS

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$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.