UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

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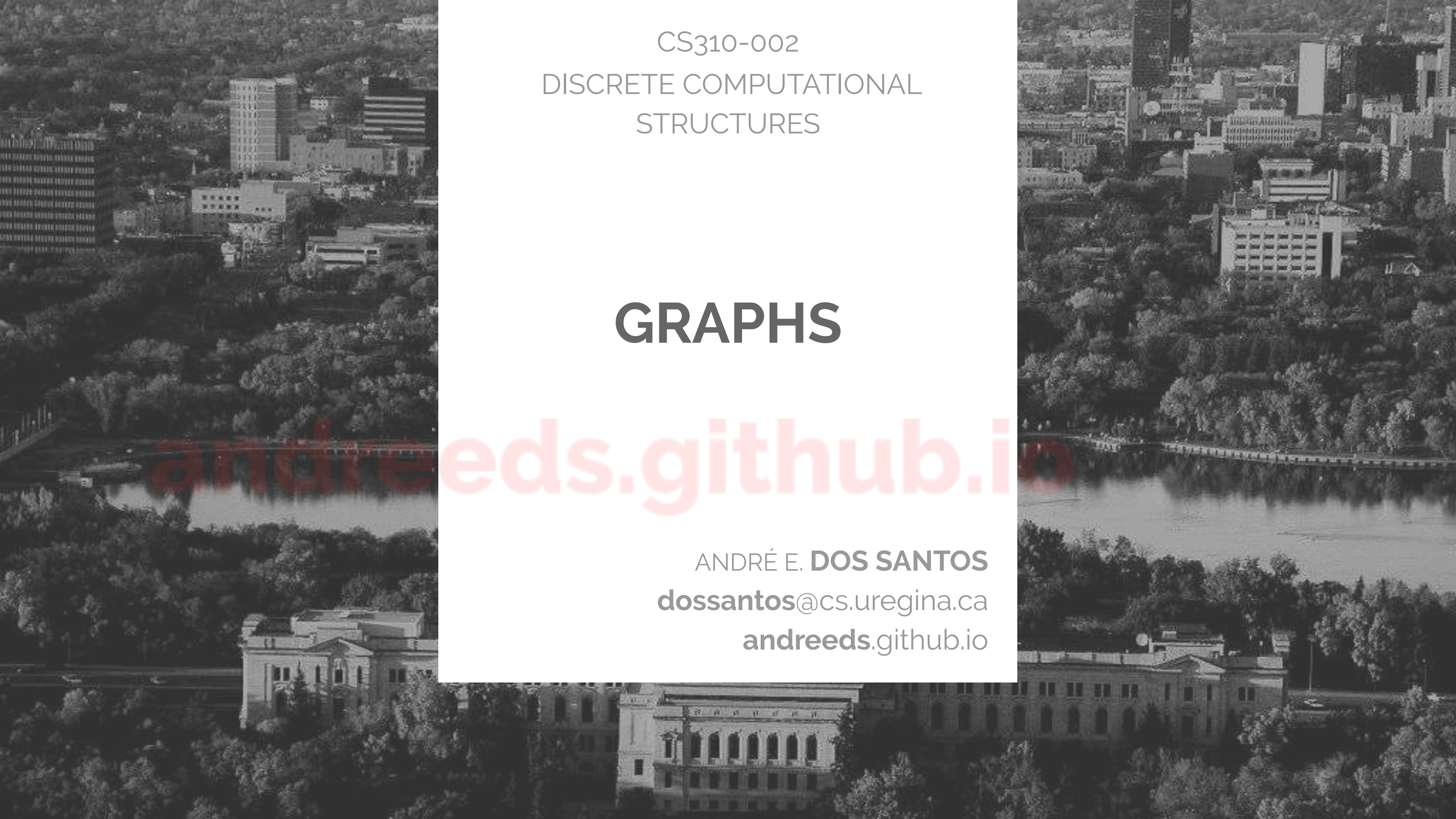
ANDRÉ E. DOS SANTOS
dossantos@cs.uregina.ca
andreeds.github.io



CS310-002

DISCRETE COMPUTATIONAL STRUCTURES

GRAPHS

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ANDRÉ E. **DOS SANTOS**
dossantos@cs.uregina.ca
andreeds.github.io

EULER AND HAMILTON PATHS

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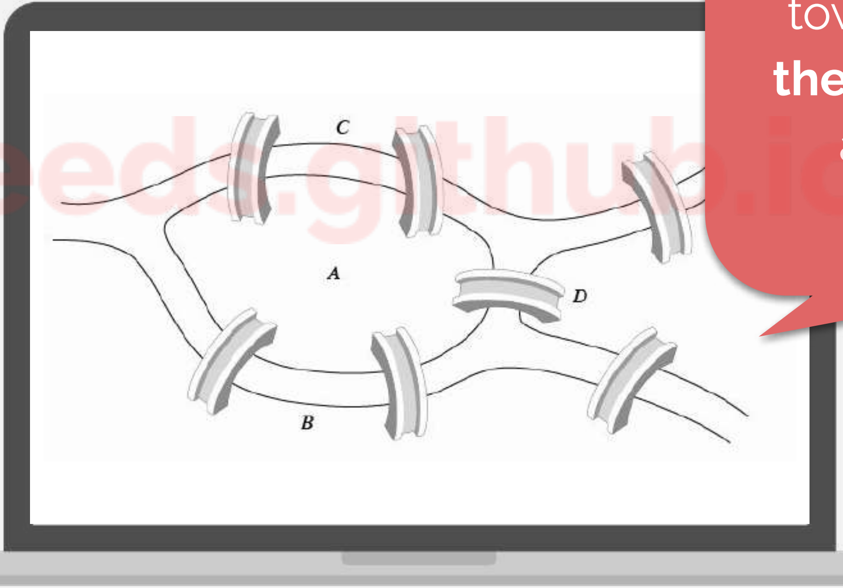
THE SEVEN BRIDGES OF KÖNIGSBERG



Is it possible to start at some location in the town, travel across **all the bridges only once**, and **return** to the **starting point**?

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Definition

An ***Euler circuit*** in a graph G is a **simple circuit** containing **every edge** of G.
An ***Euler path*** in G is a simple path containing **every edge** of G.

Example

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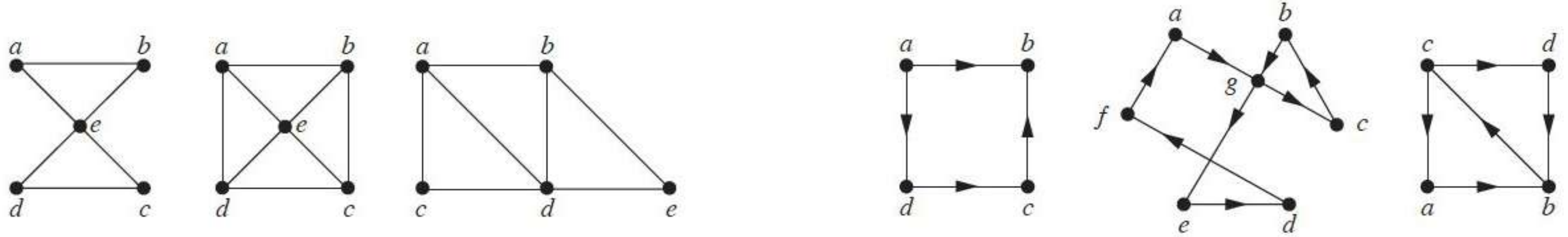
Which of the undirected graphs and directed graphs have an Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3 H_1 H_2 H_3

Definition

An ***Euler graph*** is a graph that has an ***Euler circuit***.

Examples

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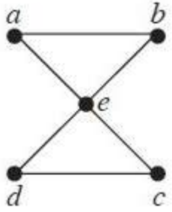
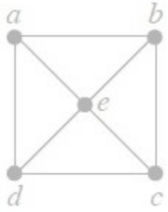
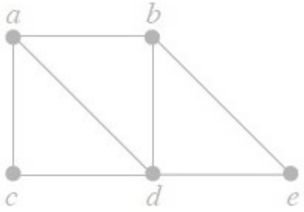


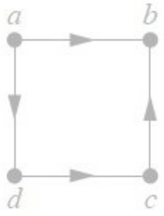
G_1



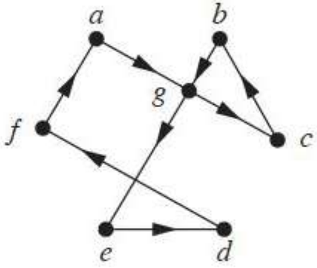
G_2



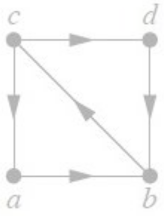
G_3



H_1



H_2



H_3

Theorem

A **connected multigraph** with at least two vertices has an ***Euler circuit*** if and only if **every vertex has even degree**.

Examples

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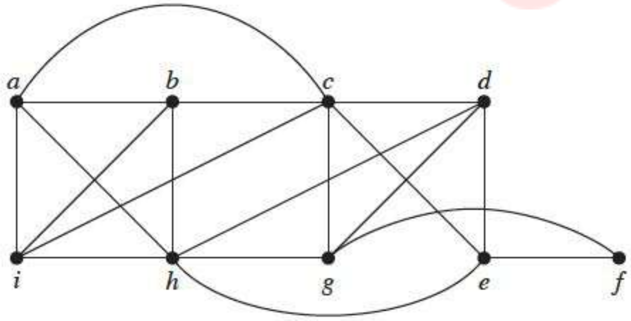
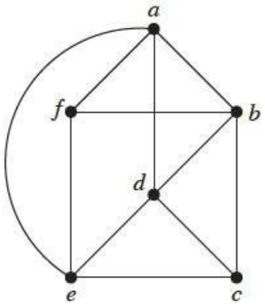
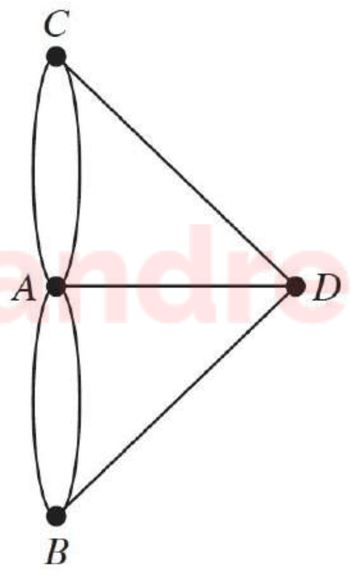
Which of the connected multigraph have an Euler circuit?

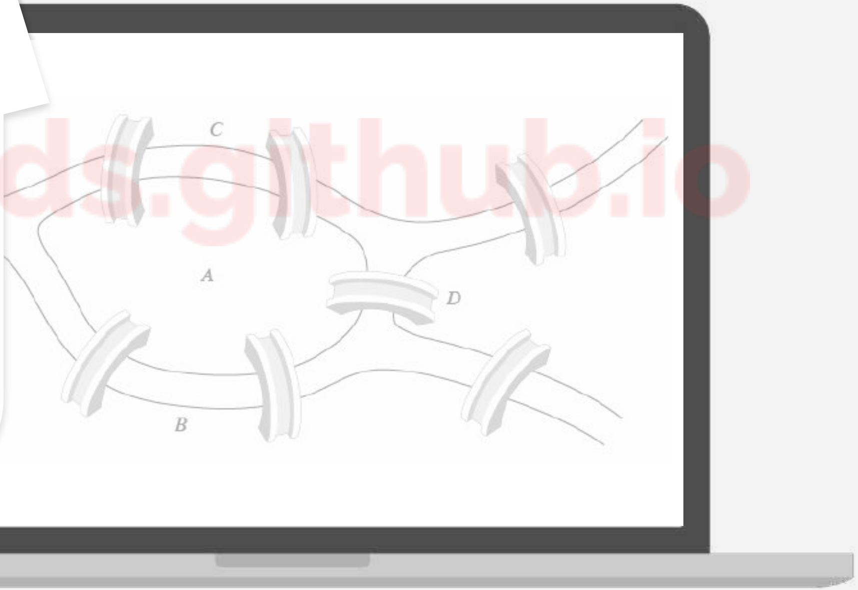




MULTIGRAPH MODEL OF THE TOWN OF KÖNIGSBERG

p696





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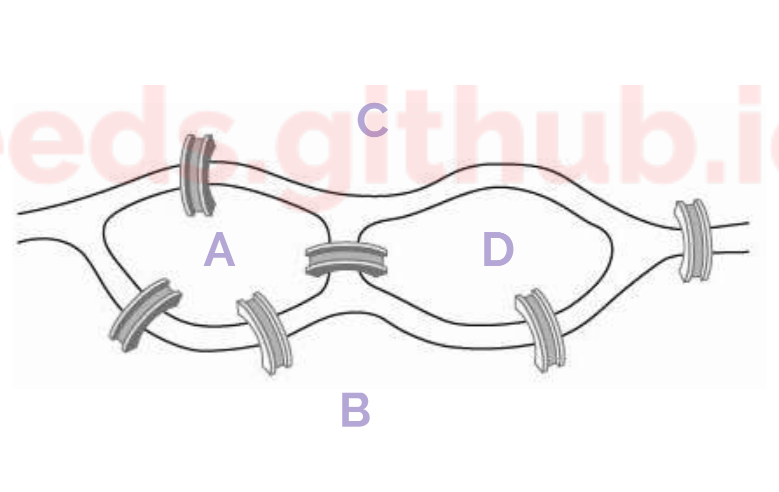
MULTIGRAPH MODEL OF THE TOWN OF NOWHERE



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Theorem

A **connected multigraph** has an ***Euler path*** **but not an *Euler circuit*** if and only if it has **exactly two vertices of odd degree**.

Example

Which graphs have an Euler path?

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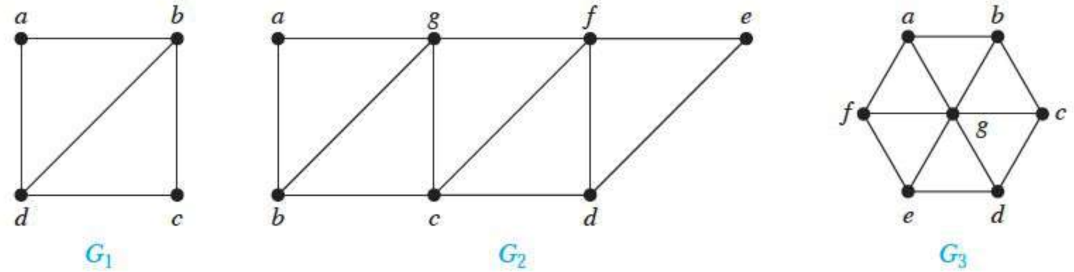

Theorem

A ***strongly connected**** **directed multigraph** has an ***Euler circuit*** if and only if ***in*-degree(v) = *out*-degree(v)** for **every vertex** v.

Example

Which graphs have an *Euler circuit*?

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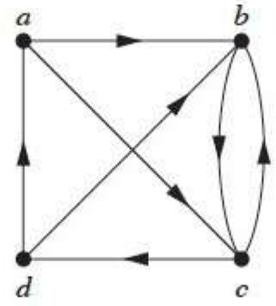

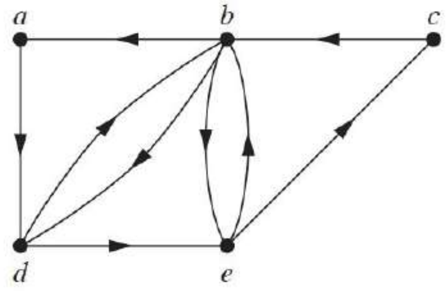

A directed graph is **strongly connected** if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph.

CHINESE POSTMAN PROBLEM

Find a minimum length circuit that traverses each edge at least once

p698

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Applications

Garbage collection,
street sweeping,
snow plowing,
school buses routing,
etc.

Definition

A ***Hamilton circuit*** in a graph G is a **circuit** that passes through **every vertex** of G **exactly once**.
A ***Hamilton path*** in G is a path that passes through **every vertex** of G **exactly once**.

Example

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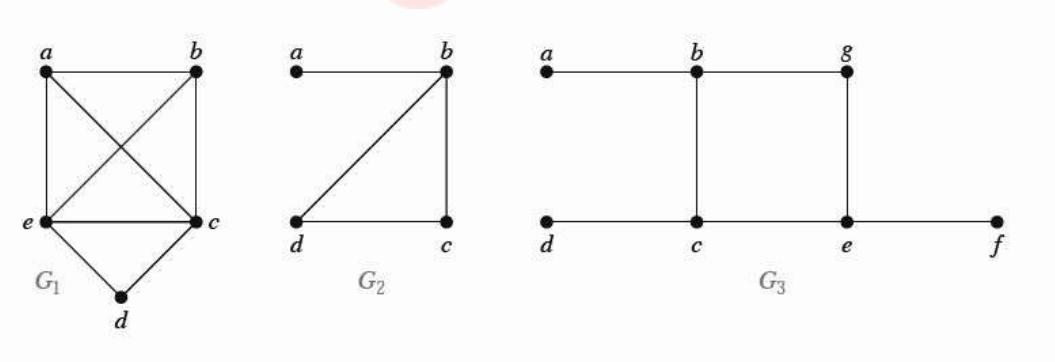
Which of the simple graphs have a Hamilton circuit or, if not, a Hamilton path?



HAMILTON PATHS

Definition

A ***Hamilton graph*** is a graph that has a ***Hamilton circuit***.

Examples

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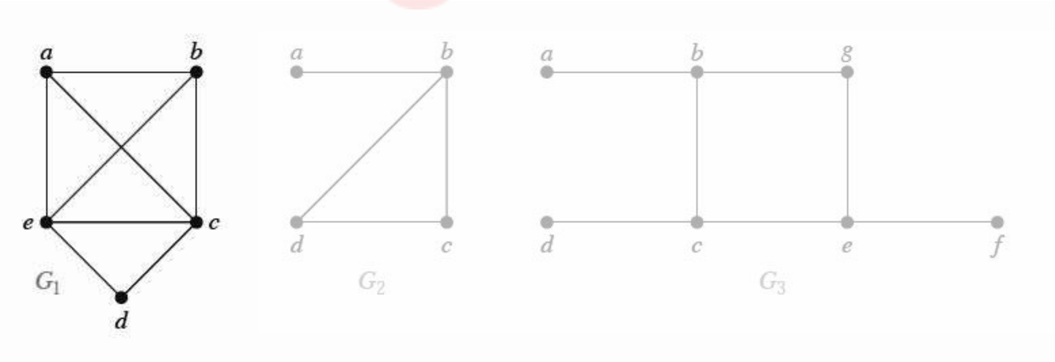

Dirac's Theorem

If G is a simple graph with n vertices ($n \geq 3$) such that the degree of **every vertex** in G is **at least** $\boldsymbol{n/2}$, then G has a ***Hamilton circuit***.

sufficient condition

Example

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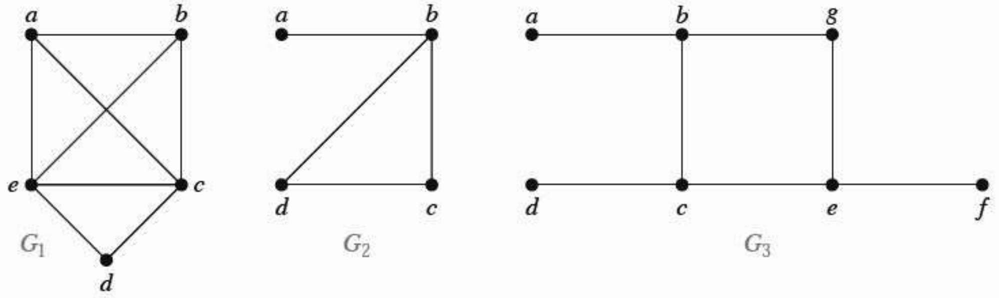

Ore's Theorem

If G is a simple graph with n vertices ($n \geq 3$) such that the $\boldsymbol{deg(u) + deg(v) \geq n}$ for **every pair of nonadjacent vertices u and v** in G, then G has a ***Hamilton circuit***.

sufficient condition

Example

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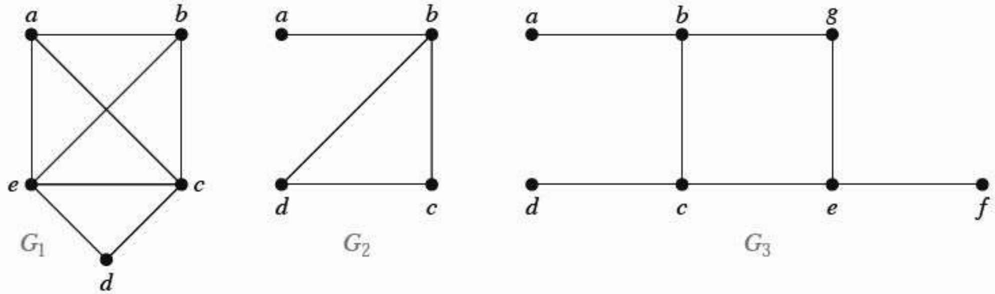

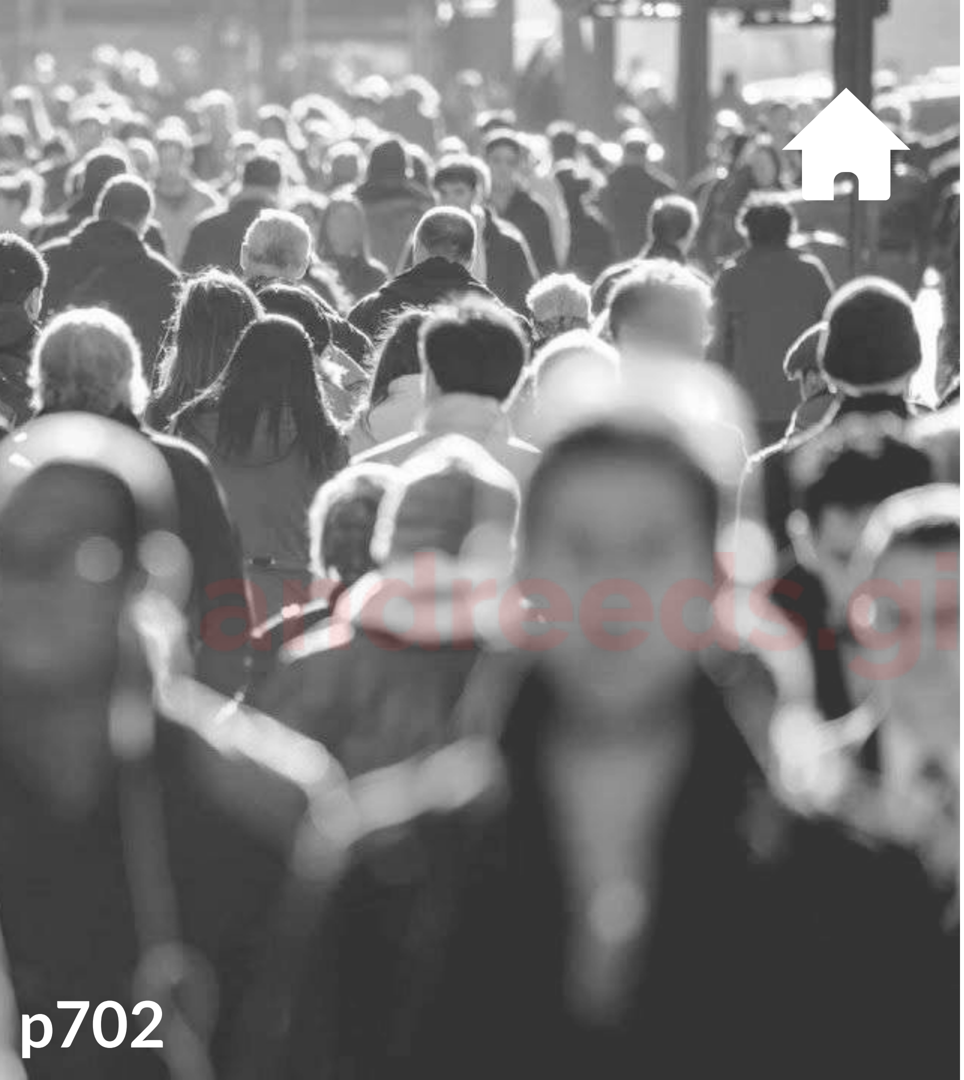

TRAVELING SALESMAN PROBLEM

Find a path through a weighted graph which starts and ends at the same vertex, includes every other vertex exactly once, and minimizes the total weight of edges.

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Applications

School buses routing,
the scheduling of a machine to drill holes in a circuit board,
3D printing,
etc.

~~FUN FACTS~~

JUST SOME PROPERTIES

If two edges incident on a vertex are required in the construction of a *Hamilton circuit*, then all the others can be deleted without changing the *Hamiltonicity* of the graph.

If a vertex v has **degree two**, then both its **incident edges must lie on a Hamilton circuit**, if there is one

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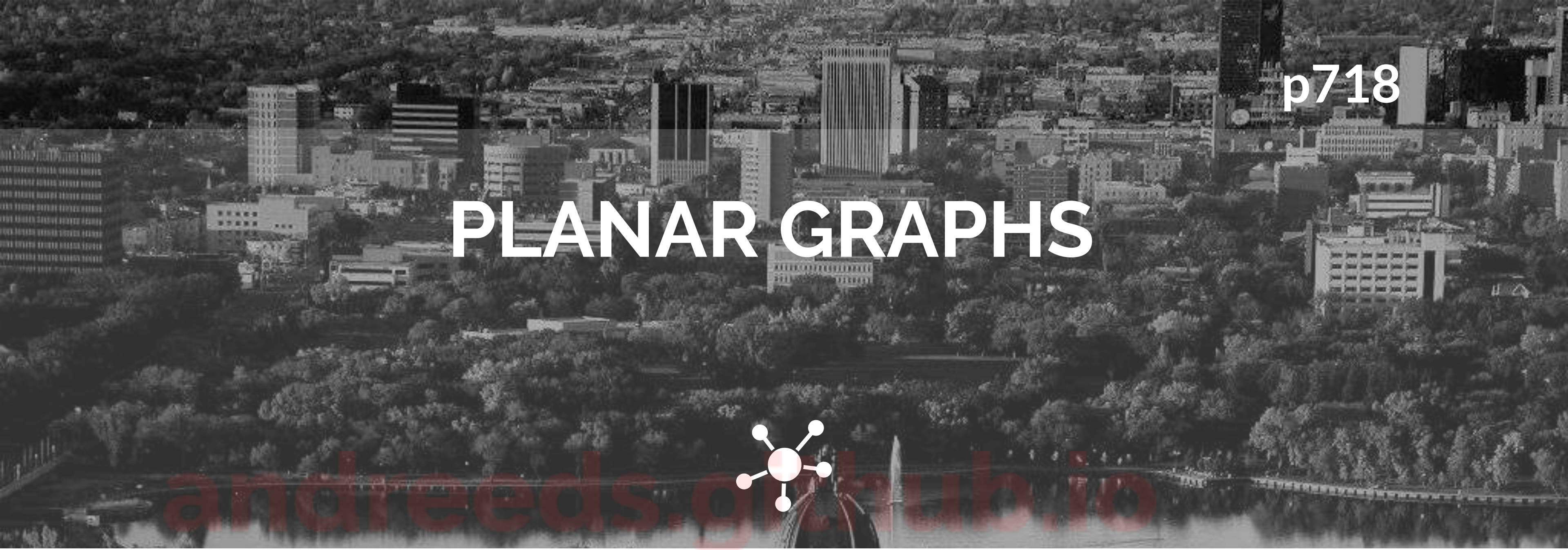
A *Hamilton graph* has no articulation points

If $G(V,E)$ has a ***Hamilton circuit***, then, for **every** nonempty **subset** S of V, the number of **connected components in $G - S$ is less than or equal to $|S|$**

PLANAR GRAPHS



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Definition

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

Example

Which one is planar: $\mathbf{Q_3}$, $\mathbf{K_4}$, $\mathbf{K_{3,3}}$?

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Definition

Let G be a connected planar simple graph with e edges and v vertices. Let $\boldsymbol{r}$ **be the number of regions** in a planar representation of G. Then $\boldsymbol{v - e + r = 2}$.

Example

Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

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Corollary

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $\boldsymbol{e \leq 3v - 6}$.

Corollary

If G is a connected planar simple graph, then G has a **vertex of degree not exceeding 5**.

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Example

Show that $\mathbf{K_5}$ is nonplanar.

Corollary

If a connected planar simple graph has e edges and v vertices $(v \geq 3)$ and **no circuits of length 3**, then $\boldsymbol{e \leq 2v - 4}$.

Example

Show that $\mathbf{K}_{3,3}$ is nonplanar.

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Definition

If a graph is planar, so will be any graph obtained by **removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$**. Such an operation is called an **elementary subdivision**.

Example

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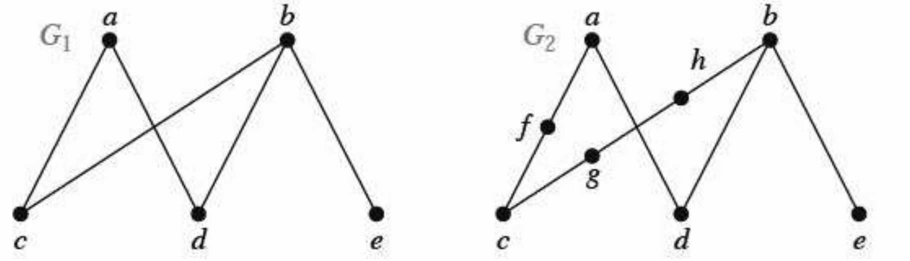
How can G_2 be obtained from G_1? How can G_1 be obtained from G_1?



Definition

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be **obtained from the same graph by a sequence of elementary subdivisions**.

Example

Show that the graphs G_1, G_2, and G_3 are all homeomorphic.

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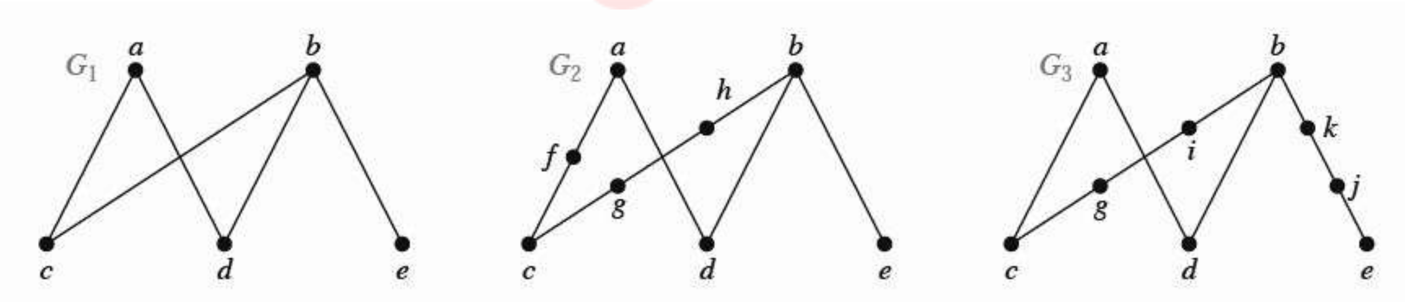
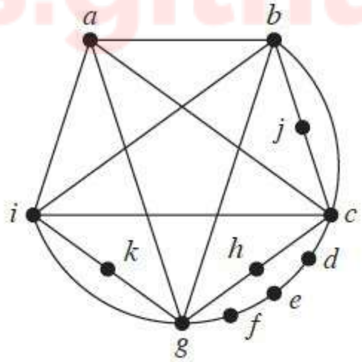


Kuratowski Theorem

A graph is **nonplanar** if and only if it contains a subgraph **homeomorphic** to $K_{3,3}$ or K_5.

Example

Is the following graph planar?



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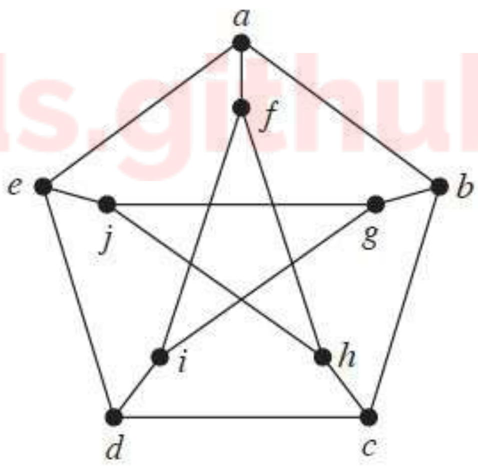
Example

Is the Petersen graph planar?



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REVIEW QUESTIONS Pt. 4
GRAPHS

Determine whether the given graph has an Euler circuit

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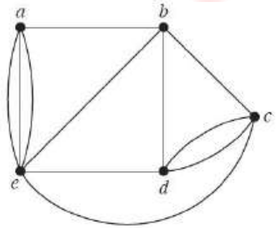
5.



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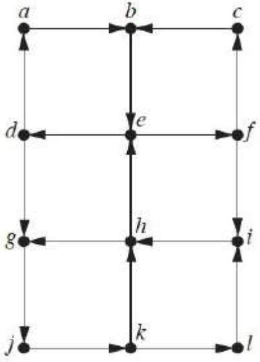

REVIEW QUESTIONS Pt. 4
GRAPHS

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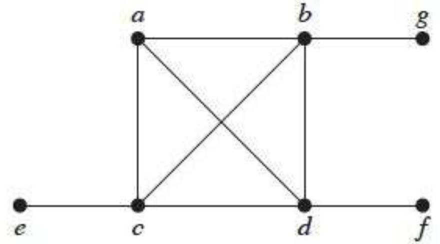
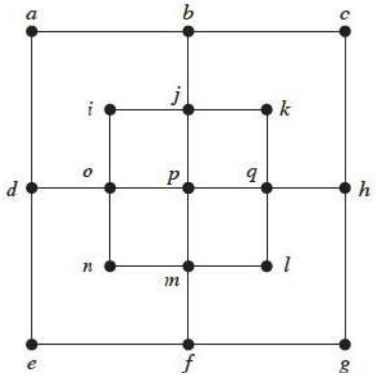
Determine whether the given graph has a Hamilton circuit

33.



34.



REVIEW QUESTIONS Pt. 4

GRAPHS

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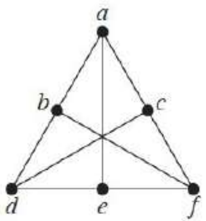
Determine whether the given graph is planar. If so, draw it so that no edges cross.

5.



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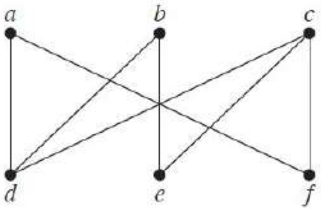
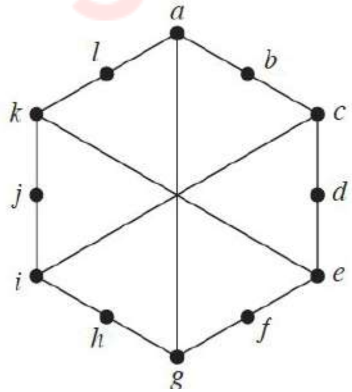


REVIEW QUESTIONS Pt. 4

GRAPHS

determine whether the given graph is homeomorphic to $K_{3,3}$.



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