UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

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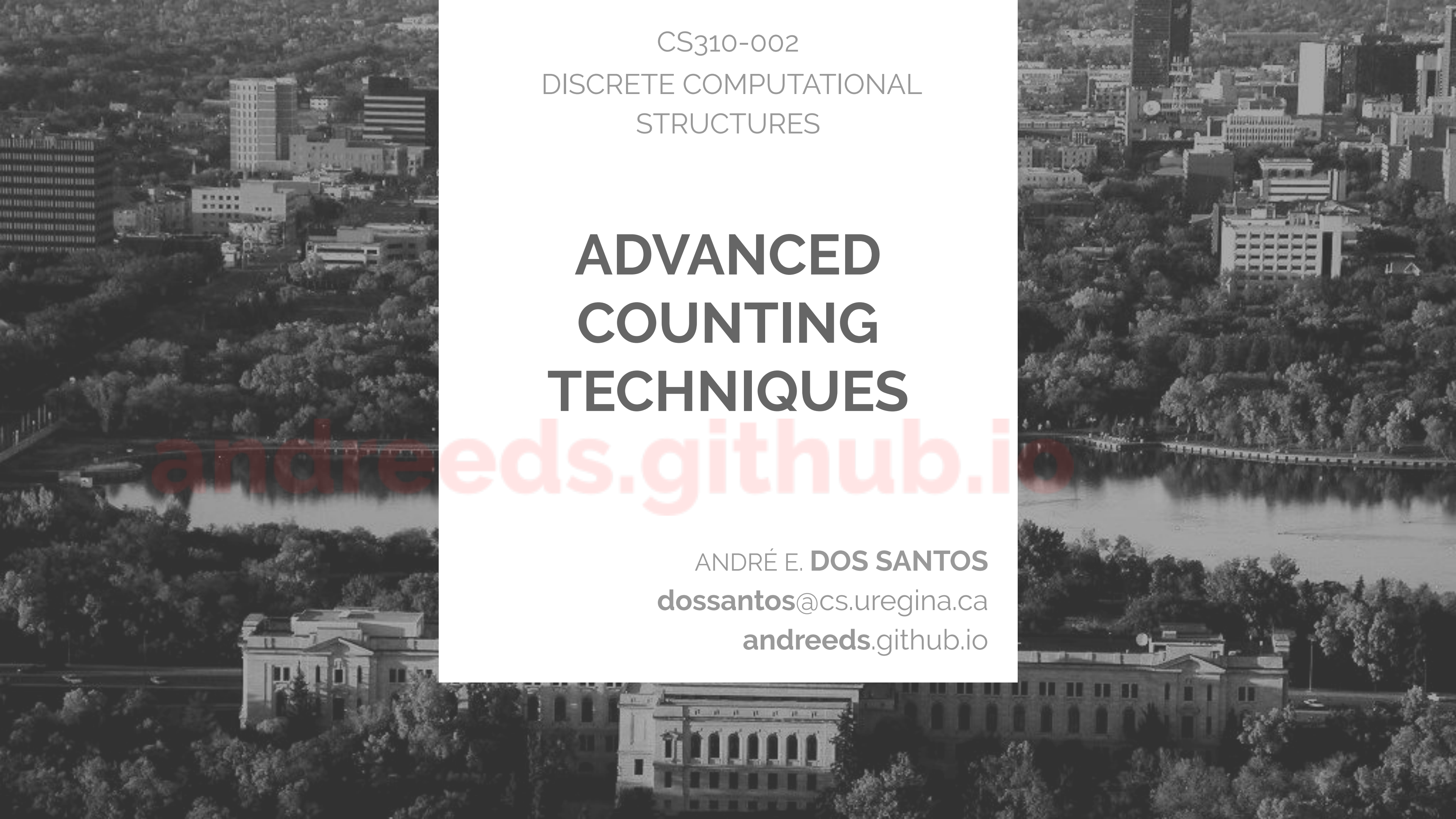

CS310-002

DISCRETE COMPUTATIONAL STRUCTURES

ADVANCED COUNTING TECHNIQUES

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Review solving linear homogeneous recurrence relations

1. **Check** it is linear, homogeneous, and with constant coefficients.
2. Identify the **degree** k.
3. Find the **characteristic equation**
4. Find the **roots** of the equation in 3.
5. The solution will be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$
6. Substitute the **initial conditions** to find the α's value

Review solving linear homogeneous recurrence relations of degree 2

1. **Check** it is linear, homogeneous, and with constant coefficients.
2. The **degree** is 2. (Duh!)
3. Find the **characteristic equation**
4. Find the **roots** of the equation in 3.
5. The solution will be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
6. Substitute the **initial conditions** to find the α's value

Theorem

- Suppose
- $r^k - c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k = 0$ has t **distinct** roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively,
- So,

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n^1 + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^{\,n} + (\alpha_{2,0} + \alpha_{2,1} n^1 + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^{\,n} + \ldots + (\alpha_{t,0} + \alpha_{t,1} n^1 + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^{\,n}$$

satisfies the recurrence for $n = 0, 1, 2, \ldots$,

where $\alpha_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$.

Solving linear homogeneous recurrence relations of degree 2 with only one root

1. **Check** it is linear, homogeneous, and with constant coefficients.
2. The **degree** is 2.
3. Find the **characteristic equation**
4. Find the **root** of the equation in 3.
5. The solution will be of the form $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$
6. Substitute the **initial conditions** to find the α's value

Example

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Example

What is the solution of the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$?

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SOLVING LINEAR NON-HOMOGENEOUS RECURRENCES

linear nonhomogeneous recurrence relation with constant coefficients

is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

→ $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ is called the **associated homogeneous recurrence relation**.

linear nonhomogeneous recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + \boxed{F(n),}$$

function depending only on n

linear nonhomogeneous recurrence relation with constant coefficients

$$a_n = \boxed{c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}} + F(n),$$

associated homogeneous recurrence relation

linear nonhomogeneous recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Examples:

- ★ $a_n = 2a_{n-1} + 1$
- ★ $a_n = a_{n-1} + a_{n-2} + n$
- ★ $a_n = a_{n-1} + 2^n$

Theorem

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

Note that there is **no general method** for finding a particular solution that works for every function $F(n)$.

Example

What is the solution of the recurrence relation

$$a_n = 3a_{n-1} + 2n$$

with $a_1 = 3$?

REVIEW QUESTIONS

ADVANCED COUNTING TECHNIQUES

What is the solution of the recurrence relation

$$a_n = 8a_{n-2} - 16a_{n-4}$$

for $n \geq 4$, with $a_0 = 1$, $a_1 = 4$, $a_2 = 28$, and $a_3 = 32$?

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$