UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

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DISCRETE COMPUTATIONAL STRUCTURES

THE FOUNDATIONS

RULES OF INFERENCE & PROOFS

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$$\begin{array}{ll} & \text{premise}_1 \\ & \text{premise}_2 \\ & \ldots \\ & \text{premise}_n \\ \hline \therefore & \textbf{conclusion} \end{array}$$

argument

An argument is **valid** if the

truth of all its premises implies that the conclusion is true

RULES OF INFERENCE

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \rightarrow p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

Rules of Inference

Example:

"Randy works hard" and **"If Randy works hard, then he is a dull boy"** imply the conclusion **"If Randy is a dull boy, then he will not get the job."**

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d is "Randy is a dull boy"
j is "Randy will get the job."

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2. $\mathbf{w} \rightarrow \mathbf{d}$
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modus ponens (1 and 2)

4. $\mathbf{d}$

modus ponens (3 and 4)

5. $\neg\mathbf{j}$

RULES OF INFERENCE for Quantified Statements

Rule of Inference	*Name*
$\dfrac{\forall x\, P(x)}{\therefore\; P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\; \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore\; P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\; \exists x\, P(x)}$	Existential generalization

Combining Rules of Inference for Propositions and Quantified Statements

Example:

Assume that

1. **"For all positive integers n, if n is greater than 4, then n^2 is less than 2^n"** is **true**.

Show that $\mathbf{100^2 < 2^{100}}$.

Let **P(n)** denote **"n > 4"**
and **Q(n)** denote **"$n^2 < 2^n$"**

(1) can be represented as

2. $\forall n(P(n) \rightarrow Q(n))$

with domain = $\mathbb{N}+$

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(2) is true

P(100) is true
because 100 > 4

by universal modus ponens

Q(100) is true

$\therefore 100^2 < 2^{100}$

Methods of Proving Theorems

I. **Direct Proofs**
II. **Proof by Contraposition**
III. **Proofs by Contradiction**

Methods of Proving Theorems

$$p \rightarrow q$$

METHODS	HOW	WHY
I. **Direct Proofs**	assume **p** is true	show **p** → **q** is true
II. **Proof by Contraposition**	assume **¬q** is true	show **¬q** →**¬p** is true
III. **Proofs by Contradiction**	assume **¬p** is true	show **¬p** → **q** is false

Example:

Give a **direct proof** of the theorem **"If n is an odd integer, then n^2 is odd."**

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Give a **direct proof** of the theorem **"If n is an odd integer, then n^2 is odd."**

1. **$\forall n(P(n) \rightarrow Q(n))$**,
   where **P(n)** is **"n is an odd integer"**
   and **Q(n)** is **"n^2 is odd."**

2. Assume **P(n)** is **true**

3. By definition, an odd integer
   $n = 2k + 1$,
   where k is some integer.

4. $$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

5. $\therefore$ **n^2** is an **odd integer** $\square$

Example: Prove that **if n is an integer and $3n + 2$ is odd, then n is odd.**

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By direct proof

1. if **3n + 2** is an odd integer, then
   3n + 2 = 2k + 1
   for some integer **k.**

2. **3n + 1 = 2k**

3. **?**

Example: Prove that **if n is an integer and 3n + 2 is odd, then n is odd.**

By contraposition

Example: Prove that **if n is an integer and 3n + 2 is odd, then n is odd.**

By contraposition

1. **$\forall n(P(n) \rightarrow Q(n))$**,
   where
   P(n) is **"3n + 2 is an odd integer"**
   and **Q(n)** is **"n is odd."**

2. Assume **$\neg Q(n)$**
   then **n** is **even**

3. By definition, an even integer
   n = 2k,
   where **k** is some integer.

4. **3n + 2** **= 3(2k) + 2**
   = 6k + 2
   = 2(3k + 1)

5. **"3n + 2 is an even integer"**
   $\neg P(n)$ □

   (because **$\neg q \rightarrow \neg p \equiv p \rightarrow q$**)

Example: Give a proof by **contradiction** of the theorem **“If 3n + 2 is odd, then n is odd.”**

Example: Give a proof by **contradiction** of the theorem **"If 3n + 2 is odd, then n is odd."**

1. $\forall n(P(n) \rightarrow Q(n))$,
   where
   P(n) is **"3n + 2 is an odd integer"**
   and **Q(n)** is **"n is odd."**

2. Assume **P(n)** is **true**
   and **Q(n)** is **false**
   then **n** is **even**

3. By definition, **n = 2k,**
   where **k** is some integer.

4. $3n + 2 = 3(2k) + 2$
   $= 6k + 2$
   $= 2(3k + 1)$

5. **"3k + 2** is **even"**
   `¬P(n)

6. A **contradiction** to **(2)** □

A Constructive Existence Proof

Example:

> Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

A Constructive Existence Proof

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Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

After considerable computation (such as a computer search) we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3 \qquad \square$$

A Nonconstructive Existence Proof

Example:

Show that there exist **irrational** numbers **x** and **y** such that x^y is **rational**.

A Nonconstructive Existence Proof

Example:

Show that there exist **irrational** numbers **x** and **y** such that x^y is **rational**.

1. $\sqrt{2}$ is **irrational**
2. Consider $\sqrt{2}^{\sqrt{2}}$
3. Either one of the following is **true:**

A. If $\sqrt{2}^{\sqrt{2}}$ is **rational:**

$x = y = \sqrt{2}$

$x^y = \sqrt{2}^{\sqrt{2}}$ $\square$

B. if $\sqrt{2}^{\sqrt{2}}$ is **irrational:**

$x = \sqrt{2}$

$y = \sqrt{2}^{\sqrt{2}}$

$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$ $\square$