UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

ANDRÉ E. **DOS SANTOS**
dossantos@cs.uregina.ca
andreeds.github.io

CS310-002

DISCRETE COMPUTATIONAL STRUCTURES

RELATIONS

ANDRÉ E. **DOS SANTOS**

dossantos@cs.uregina.ca

andreeds.github.io

RELATIONS AND THEIR PROPERTIES

RELATIONS

Let A and B be sets.
A **binary relation** from A to B is a subset of $A \times B$.

A **relation** on the set A is a relation from A to A, which is a subset of $A \times A$.

Example:

➔ Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

➔ How many relations are there on a set with n elements?

A relation R on a set A is called **reflexive** if $(x, x) \in R$ for every element $x \in A$.

Example

Which of the following relations on the set of integers are reflexive?

$R_1 = \{(a, b) \mid a \leq b\}$,
$R_2 = \{(a, b) \mid a > b\}$,
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) \mid a = b - 1\}$,
$R_5 = \{(a, b) \mid a + b \leq 6\}$.

Example

Is the "divides" relation on the set of integers reflexive?

A relation R on a set A is called **symmetric** if

$$(\forall x, y \in A)[(x, y) \in R \leftrightarrow (y, x) \in R\ ].$$

A relation R on a set A is called **antisymmetric** if

$$(\forall x, y \in A)\ [\ ((x, y) \in R) \wedge ((y, x) \in R) \rightarrow (y = x)\ ].$$

Example

Which of the following relations on $\{1, 2, 3, 4\}$ are symmetric and which are antisymmetric?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 4)\}$,
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$,
$R_4 = \{(1, 1), (2, 2), (3, 3), (3, 1), (3, 4), (4, 4)\}$.

Example

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$R_5 = \{(a, b) \mid a + b \leq 6\}$.

Example

Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

A relation R on a set A is called **transitive** if

$$(\forall x, y, z \in A)[((x, y) \in R) \land ((y, z) \in R) \to (x, z) \in R].$$

Example

Which of the following relations on $\{1, 2, 3, 4\}$ are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 4)\}$,
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$,
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Example

Which of the following relations on the set of integers are transitive?

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$R_5 = \{(a, b) \mid a + b \leq 6\}$.

Example

Is the "divides" relation on the set of positive integers transitive?

EQUIVALENCE RELATIONS

A relation on a set A is called an **equivalence relation** if it is **reflexive**, **symmetric**, and **transitive**.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are *equivalent elements* with respect to a particular equivalence relation.

Example

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Let R be an equivalence relation on a set A. The *set of all elements* that are related to an element a of A is called the **equivalence class** of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

So,

$$[a]_R = \{ s \mid (a, s) \in R\}.$$

When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

Example

Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. It follows that R is an equivalence relation. What are the equivalence classes with respect to the equivalence relation R?

Theorem

Let $\boldsymbol{R}$ be an equivalence relation on a set $\boldsymbol{A}$. These statements for elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\boldsymbol{A}$ are equivalent:

I. $\boldsymbol{aRb}$
II. $[\boldsymbol{a}] = [\boldsymbol{b}]$, and
III. $[\boldsymbol{a}] \cap [\boldsymbol{b}] \neq \emptyset$.

A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Example

Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S.

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example

List the ordered pairs in the equivalence relation $\boldsymbol{R}$ produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

REVIEW QUESTIONS
RELATIONS

For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.


p618
PARTIAL ORDERINGS
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A relation R on a set S is called a **partial ordering** or **partial order** if it is **reflexive**, **antisymmetric**, and **transitive**.

A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R). Members of S are called elements of the poset.

Example

- Is the "greater than or equal" relation ($\geq$) a partial ordering on the set of integers?
- Is the "divides" relation a partial ordering on the set of positive integers?
- Is the relation $\subseteq$ a partial ordering on the power set of a set S?

The elements a and b of a poset $(S, \preccurlyeq)$ are called **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called **incomparable**.

Example

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, then S is called a **totally ordered** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or **linear order**.

A totally ordered set is also called a **chain**.

Example

Is the poset $(\mathbf{Z}^+, |\,)$ totally ordered?

Is the poset $(\mathbf{Z}^+, \leq)$ totally ordered?

$(S, \preccurlyeq)$ is a **well-ordered set** if it is a poset such that $\preccurlyeq$ is a *total ordering* and every nonempty subset of S has a least element.

Example

Is the poset $(\mathbf{Z}^+, |\,)$ well-ordered?

Is the poset $(\mathbf{Z}^+, \leq)$ well-ordered?

The Principle of Well-ordered Induction

Suppose that S is a well-ordered set. Then $\mathbf{P}(x)$ is true for all $x \in S$, if

Induction Step: For every $y \in S$, if $\mathbf{P}(x)$ is true for all $x \in S$ with $x \prec y$, then $\mathbf{P}(y)$ is true.

Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the **lexicographic order** $\preccurlyeq$ on $A_1 \times A_2$ is defined by $(a_1, a_2) \prec (b_1, b_2)$ if

I. $a_1 \prec_1 b_1$, or if
II. both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

Example

Determine whether $(3, 5) \prec (4, 8)$, $(3, 8) \prec (4, 5)$, and $(4, 9) \prec (4, 11)$.

RELATIONS

A relation R on a set S is **quasi-ordering** on S if R is reflexive and transitive.

Example

Let R be the relation on the set of all functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that $(f, g) \in R$ if and only if f is $O(g)$. Is R a quasi-ordering?

A subset of a poset such that every two elements of this subset are comparable is called a **chain**.

A subset of a poset such that every two elements of this subset are incomparable is called an **antichain**.

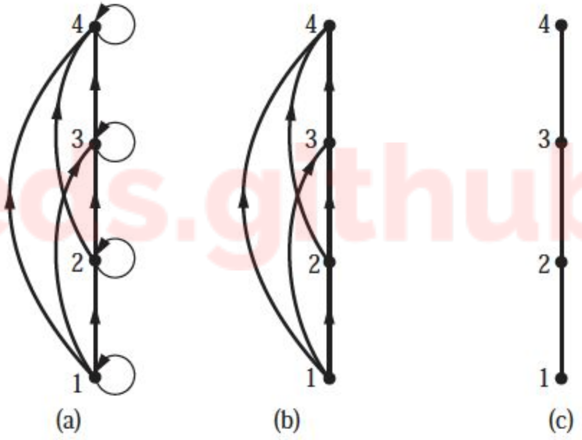
Example

Find all chains and antichains in the poset $(\{1, 2, 3, 10, 12\}, \mid)$.

HASSE DIAGRAMS

p339

Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.



a is a **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$

a is a **minimal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$

Example

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?

a is the **greatest element** in the poset $(S, \preccurlyeq)$ if $a \in S$ and $b \preccurlyeq a$ for all $b \in S$.

a is the **least element** in the poset $(S, \preccurlyeq)$ if $a \in S$ and $a \preccurlyeq b$ for all $b \in S$.

Example

➔ Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ is the greatest element, and which is the least element?

➔ Which elements of the poset $(P(S), \subseteq)$ is the greatest element, and which is the least element?

➔ Which elements of the poset $(Z^{+}, \mid)$ is the greatest element, and which is the least element?

u is an **upper bound** of a subset A in the poset $(S, \preccurlyeq)$
if $u \in S$ and $a \preccurlyeq u$ for all $a \in A$.

l is an **lower bound** of a subset A in the poset $(S, \preccurlyeq)$
if $l \in S$ and $l \preccurlyeq a$ for all $a \in A$.

x is the **least upper bound** of a subset A in the poset $(S, \preccurlyeq)$
if $x \in S$ and $a \preccurlyeq x$ for all $a \in A$ and $x \preccurlyeq z$ for all $z \in A$.

x is the **greatest lower bound** of a subset A in the poset $(S, \preccurlyeq)$
if $x \in S$ and $x \preccurlyeq a$ for all $a \in A$ and $z \preccurlyeq x$ for all $z \in A$.

Example

Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(Z^+, |)$.

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Examples

Is the poset $(\{1, 2, 3, 4, 5\}, |)$ a lattice?
Is the poset $(\{1, 2, 4, 8, 16\}, |)$ a lattice?
Is the poset $(P(S), \subseteq)$ a lattice?
Is the poset $(Z^+, |)$ a lattice?

REVIEW QUESTIONS
RELATIONS

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{\{2\}, \{4\}\}$.

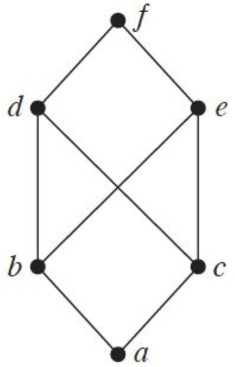
f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

g) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.

h) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists.

REVIEW QUESTIONS
RELATIONS

Is this poset represented by a Hasse diagrams a lattice?



List at least 2 problems that relations can help solve?