UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

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CS310-002

DISCRETE COMPUTATIONAL STRUCTURES

GRAPHS

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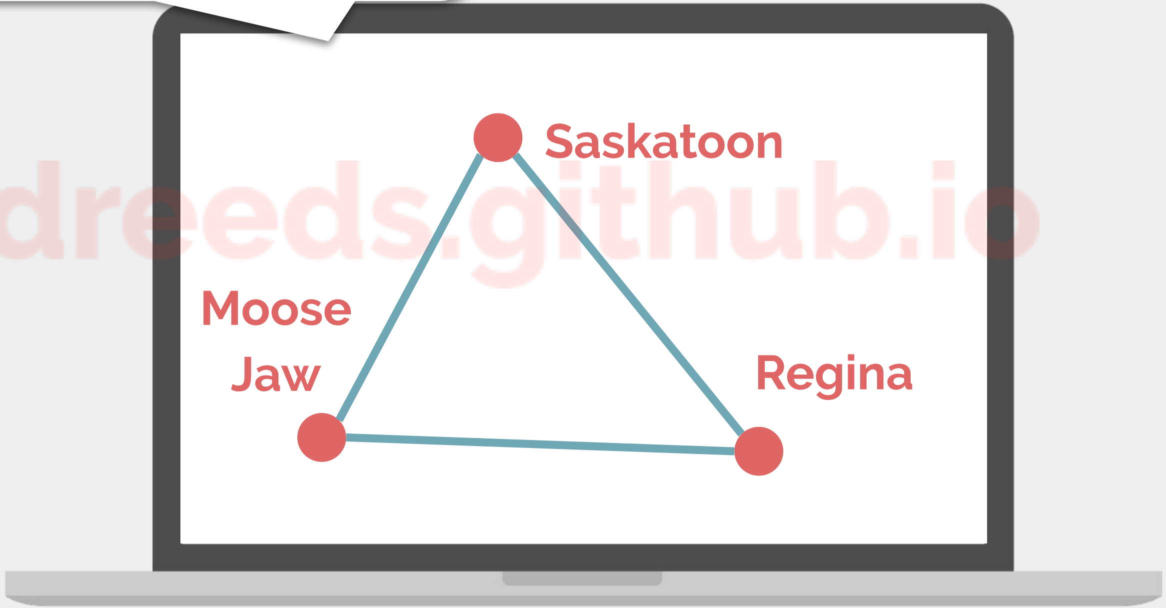
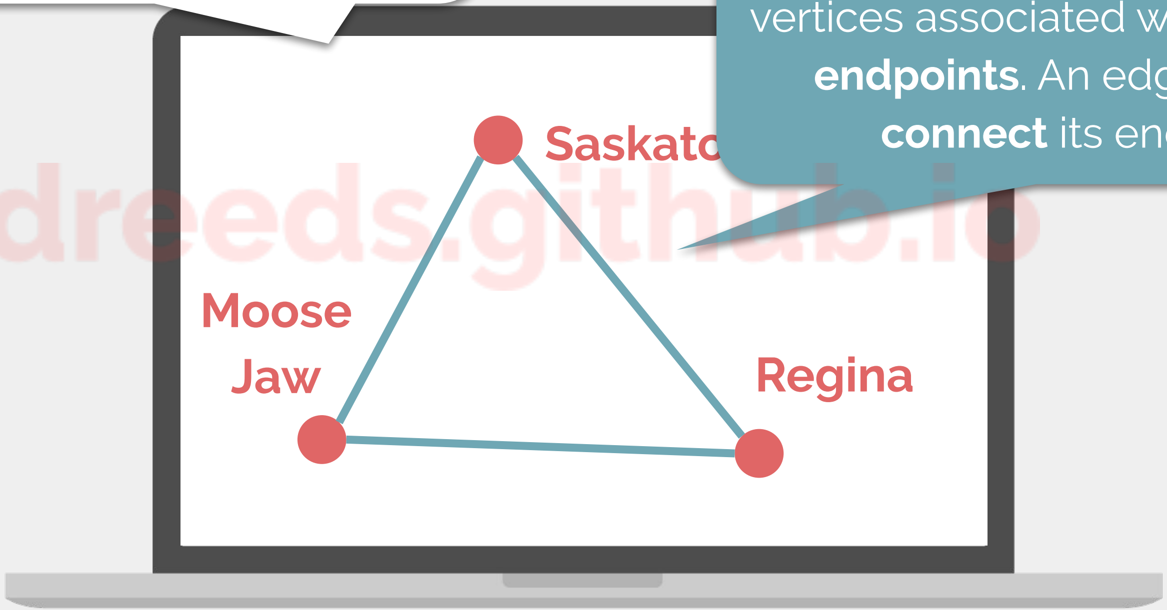
CONNECTIVITY

A **graph** $G = (V, E)$ consists of V, a nonempty set of **vertices**, and E, a set of **edges**.

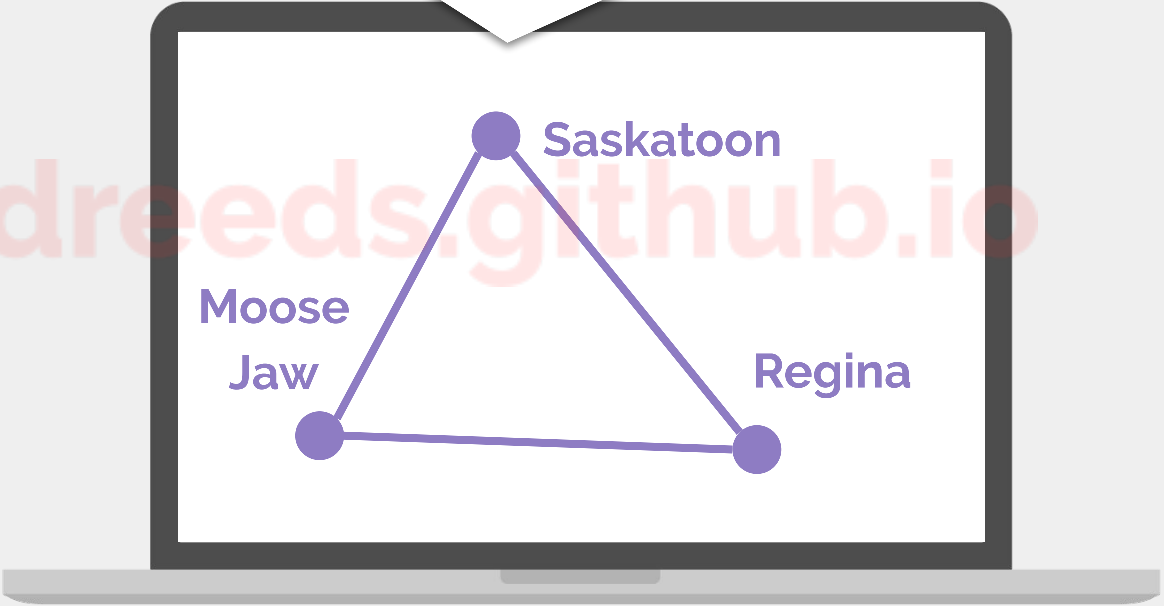


A **graph** $G = (V, E)$ consists of V, a nonempty set of **vertices**, and E, a set of **edges**.

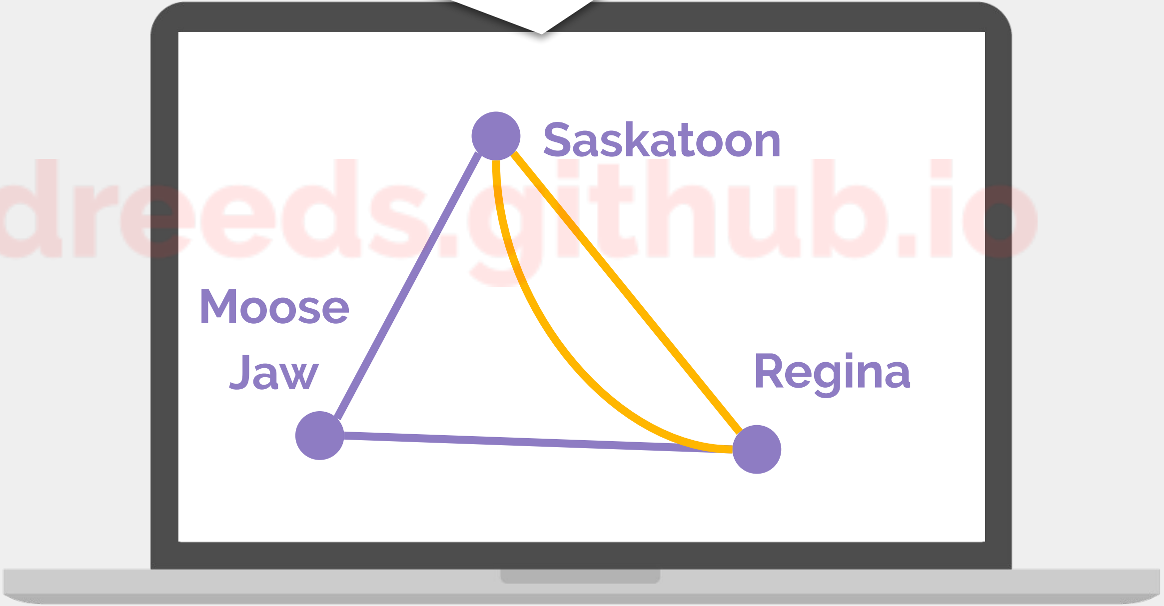
Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.



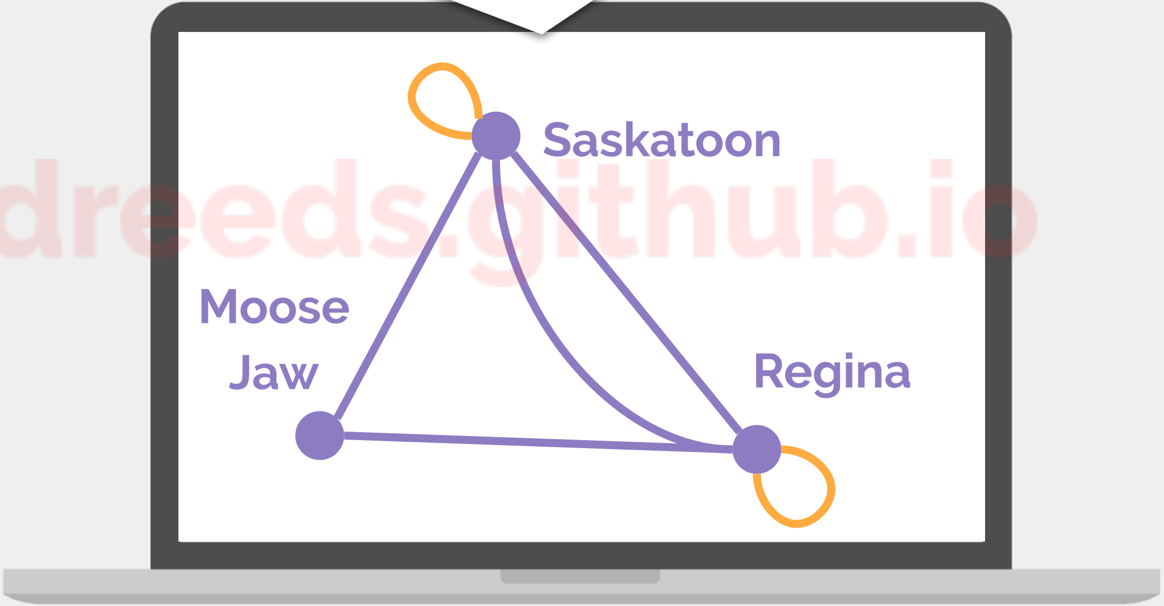
A **simple graph** is a graph in which each edge connects two distinct vertices and where **no two edges connect the same pair of vertices.**



A **multigraph** is a graph in which each edge connects two distinct vertices. Two edges connect the same pair of vertices are called **multiple edges**.



An edge is called a **loop** if it connects a vertex to itself. A **pseudograph** is a graph which may contain loops and multiple edges.



A **directed graph** (or **digraph**) $D = (V, E)$ consists of a set of vertices V and a set of *directed edges* (or *arcs*) E.

Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

A directed graph is called a **simple directed graph** if it has no loops and has no multiple directed edges.

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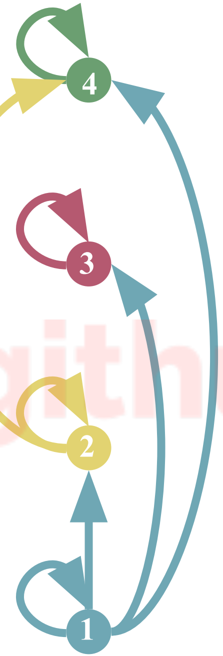
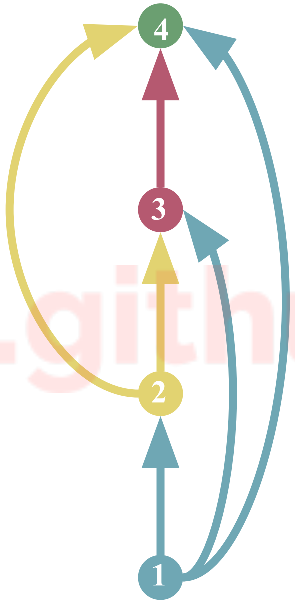
directed graph



poset ({1, 2, 3, 4}, |)

simple directed graph



poset $(\{1, 2, 3, 4\}, <)$

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TYPE	DIRECTED EDGES	MULTIPLE EDGES	LOOPS
	NO	NO	NO
	NO	YES	NO
	NO	YES	YES
	YES	NO	NO
	YES	YES	YES

TYPE	DIRECTED EDGES	MULTIPLE EDGES	LOOPS
simple graph	NO	NO	NO
multigraph	NO	YES	NO
pseudograph	NO	YES	YES
simple directed graph	YES	NO	NO
directed multigraph	YES	YES	YES

APPLICATIONS

SOCIAL NETWORKS

Friendship Graphs, Influence Graphs, Collaboration Graphs, Call Graphs, …

INFORMATION NETWORKS

Web Graphs, Citation Graphs, …

SOFTWARE DESIGN APPLICATIONS

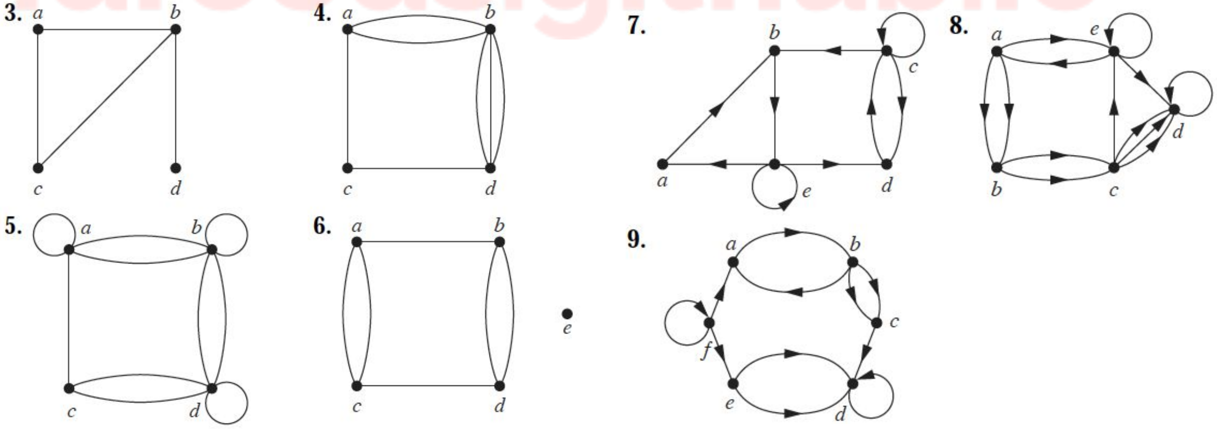
Module Dependency Graphs, Precedence Graphs and Concurrent Processing, …

TRANSPORTATION NETWORKS

Airline Routes, Road Networks, …

REVIEW QUESTIONS Pt. 1
GRAPHS

Identify the type of the following graphs



REVIEW QUESTIONS Pt. 1
GRAPHS

For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are prerequisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?

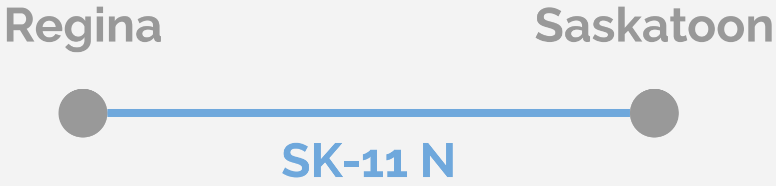
Construct the directed graph $G = (V, E)$, where
V = **{MATH110, MATH102, CS110, CS115, CS210, CS310}**
and E = **{ (MATH102, CS110), (MATH110, CS110), (CS110, CS115), (MATH110, CS115), (MATH110, CS210), (CS115, CS210), (MATH110, CS310), (CS210, CS310) }**.

GRAPH TERMINOLOGY AND SPECIAL TYPES OF GRAPHS

NEIGHBORS

Definition

Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.



NEIGHBORHOOD

Definition

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

Definition

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Example

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

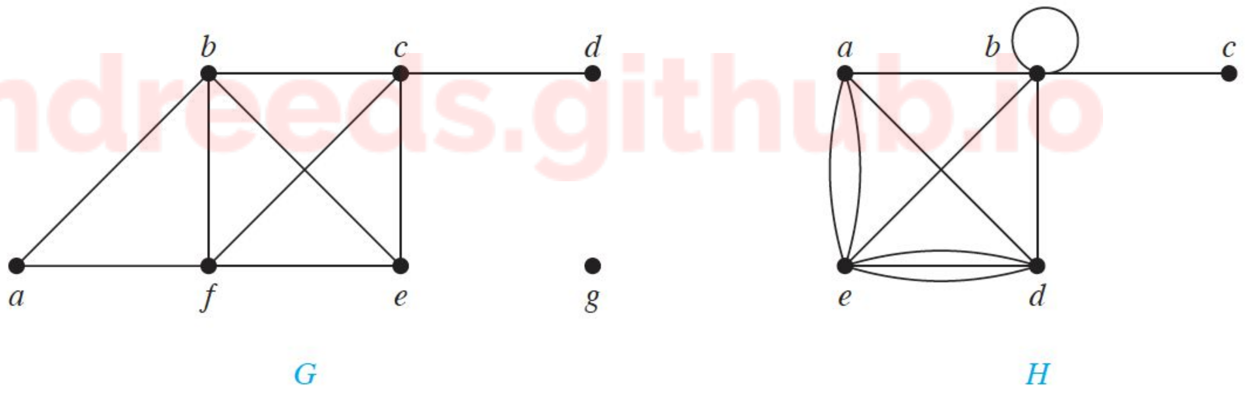


FIGURE 1 **The Undirected Graphs G and H.**

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

Example

How many edges are there in a graph with 10 vertices each of degree six?

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

Theorem

An undirected graph has an even number of vertices of odd degree.

Example

How many vertices of odd degree exist in the graphs G and H shown in Figure 1?

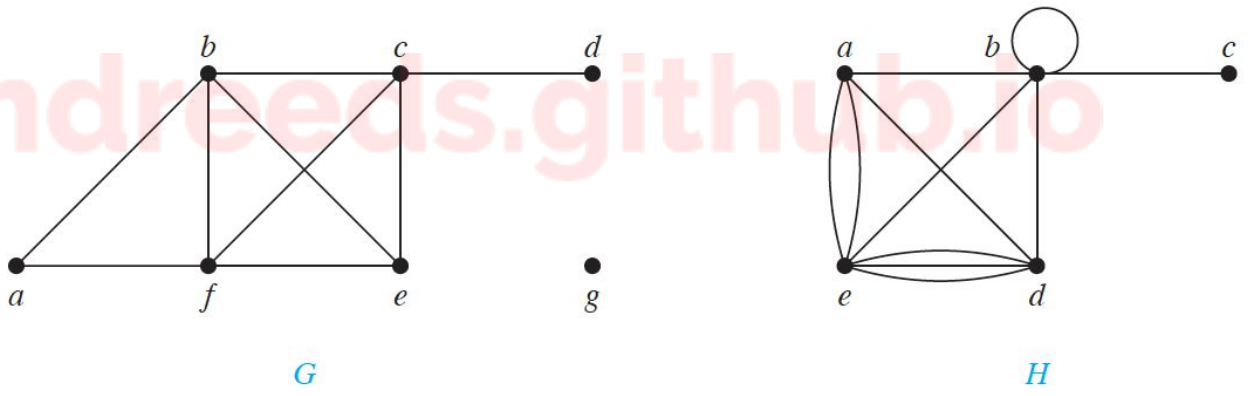


FIGURE 1 **The Undirected Graphs *G* and *H*.**

Definition

When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

Definition

In a graph with directed edges the *in-degree of a vertex* v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of* v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Example

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.

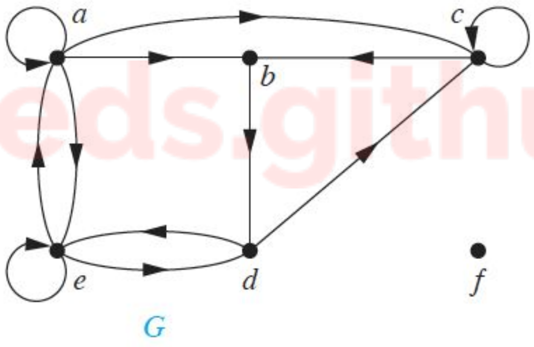


FIGURE 2 **The Directed Graph G.**

Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Example

How many edges are there in the graph of Figure 2?

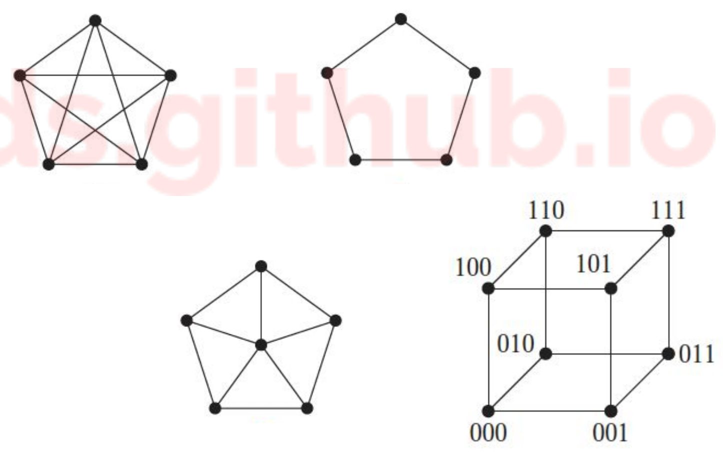
SOME SPECIAL SIMPLE GRAPHS

Complete Graphs

Cycles

Wheels

n-Cubes



SOME SPECIAL SIMPLE GRAPHS

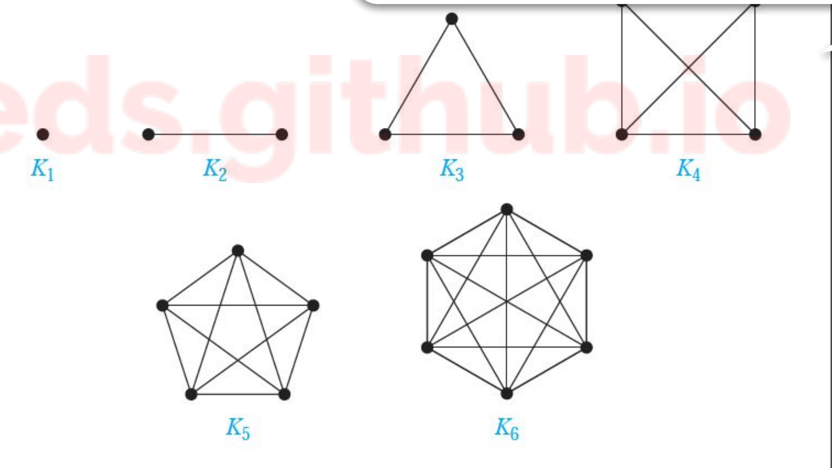
Complete Graphs

Cycles

Wheels

n-Cubes

K_n ($|V(K_n)| = n$, and $|E(K_n)| = n(n-1)/2$)



SOME SPECIAL SIMPLE GRAPHS

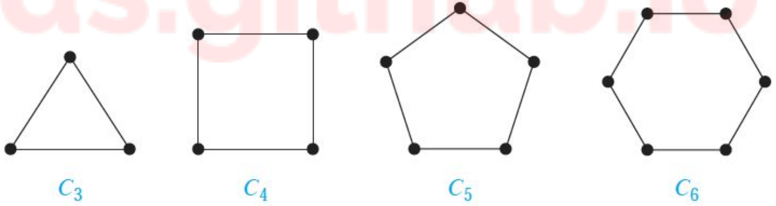
Complete Graphs

Cycles

Wheels

n-Cubes

$C_n, n \geq 3$ ($|V(C_n)| = n$, and $|E(C_n)| = n$)



SOME SPECIAL SIMPLE GRAPHS

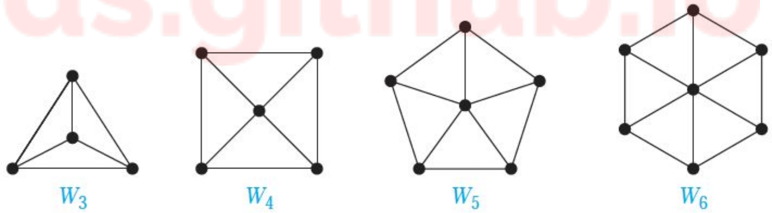
Complete Graphs

Cycles

Wheels

n-Cubes

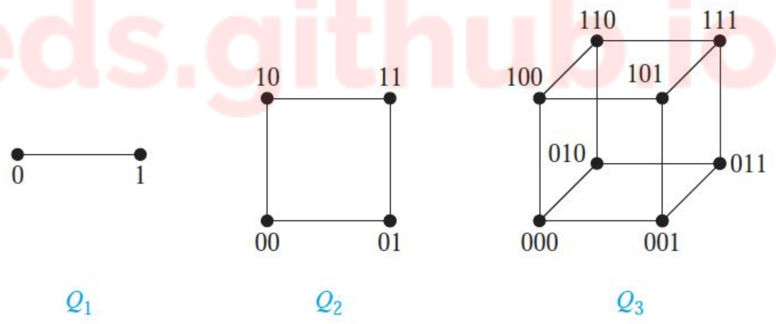
$W_n, n \geq 3$ ($|V(W_n)| = n + 1$, and $|E(W_n)| = 2n$)



SOME SPECIAL SIMPLE GRAPHS

- Complete Graphs
- Cycles
- Wheels
- ***n*-Cubes**

Q_n ($|V(Q_n)| = 2^n$, and $|E(Q_n)| = n2^{n-1}$)



Definition

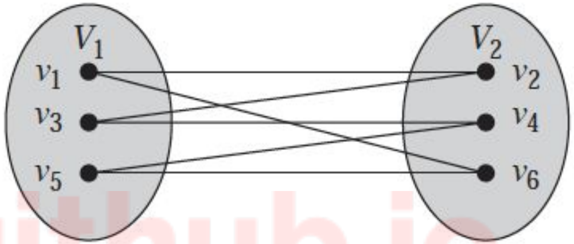
A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

Example

Is C_6 bipartite? Is K_3 bipartite?

Examples

C_6



K_3

Example

Are the graphs G and H displayed in Figure 8 bipartite?

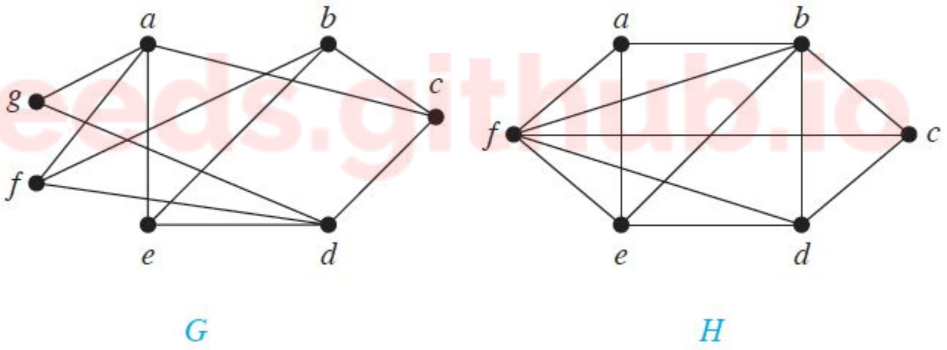


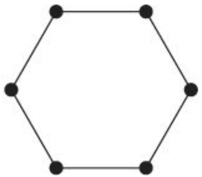
FIGURE 8 **The Undirected Graphs G and H.**

BIPARTITE GRAPHS

Theorem

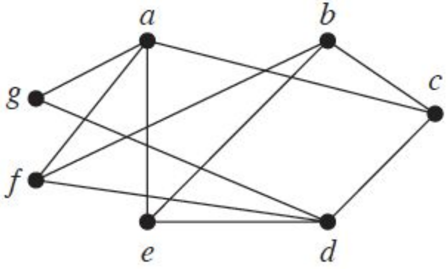
A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Examples

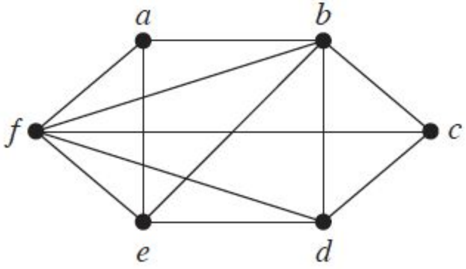


C_6

K_3



G



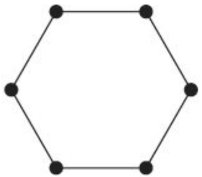
H

BIPARTITE GRAPHS

Theorem

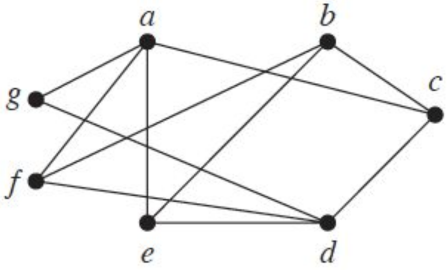
A simple graph is bipartite if and only if it contains no odd cycle.

Examples

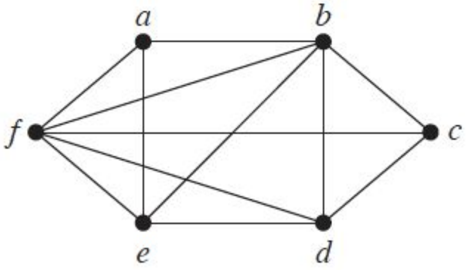


C_6

K_3



G



H

definition

A **matching** M in a simple graph $G = (V, E)$ is a subset of E such that no two edges of M are incident with the same vertex of G.

Examples

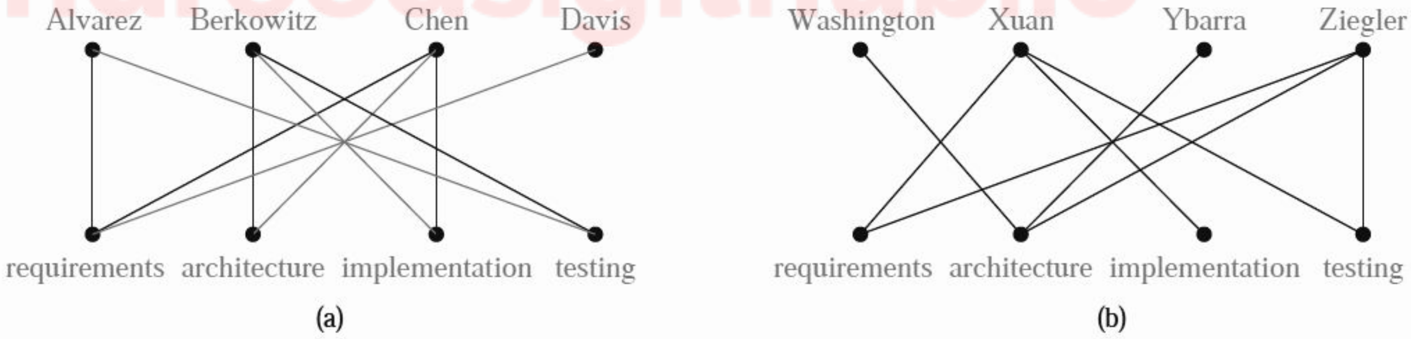


FIGURE 10 Modeling the Jobs for Which Employees Have Been Trained.

definition

A **maximum matching** in a simple graph G is a matching in G with the largest number of edges.

Examples

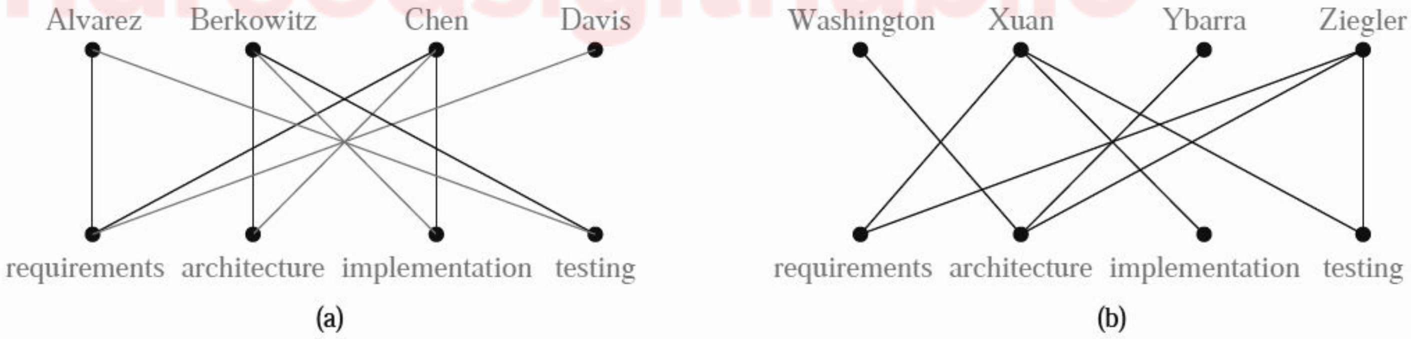


FIGURE 10 Modeling the Jobs for Which Employees Have Been Trained.

definition

A matching in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching from V_1 to V_2** if every vertex in V_1 is the endpoint of an edge in the matching.

Examples

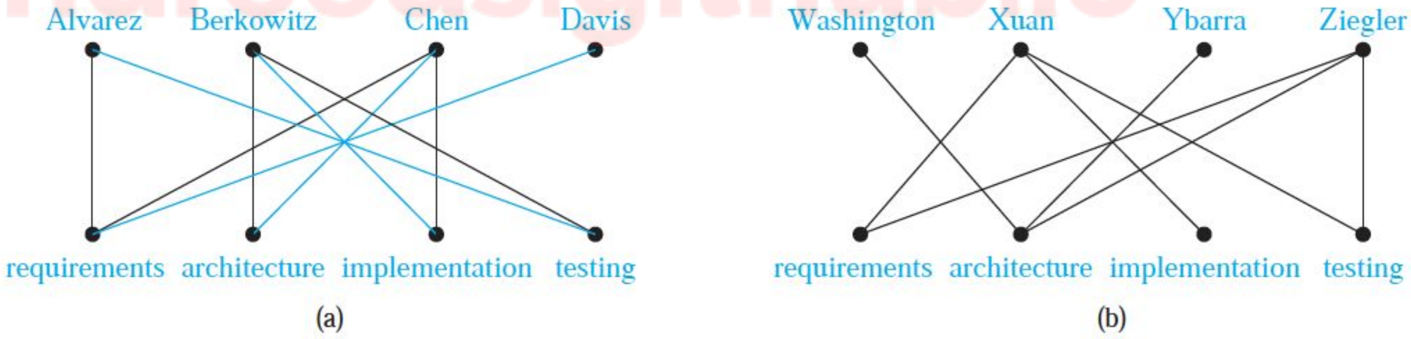


FIGURE 10 **Modeling the Jobs for Which Employees Have Been Trained.**

HALL'S MARRIAGE THEOREM The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1. $N(X)$ denotes the neighbourhood of the set X.

Examples

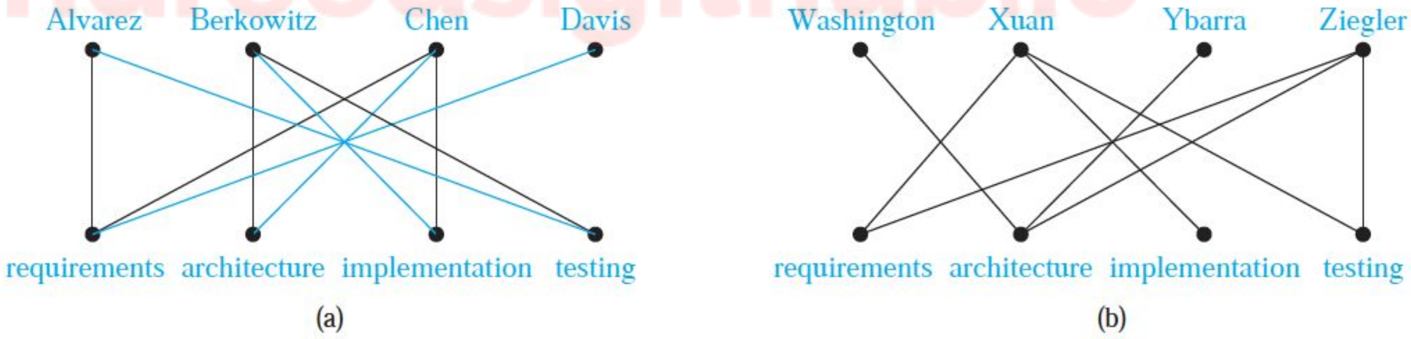


FIGURE 10 **Modeling the Jobs for Which Employees Have Been Trained.**

SUBGRAPHS

Definition

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

Definition

Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

Example

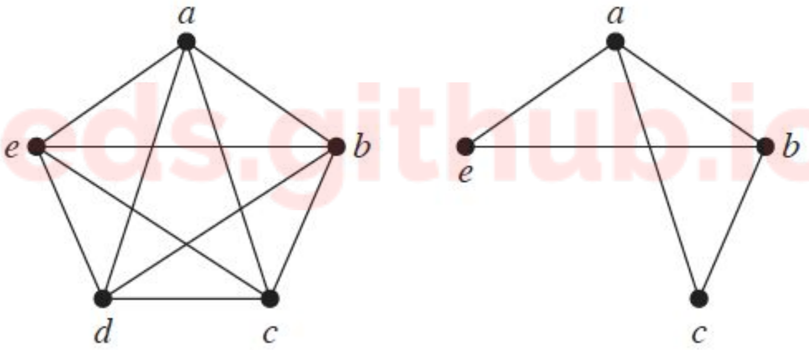


FIGURE 15 A Subgraph of K_5.

REMOVING OR ADDING EDGES OF A GRAPH

Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph $G - e = (V, E - \{e\})$.

We can also add an edge e to G to produce a new larger graph when this edge connects two vertices already in G. The resulting graph $G + e = (V, E \cup \{e\})$.

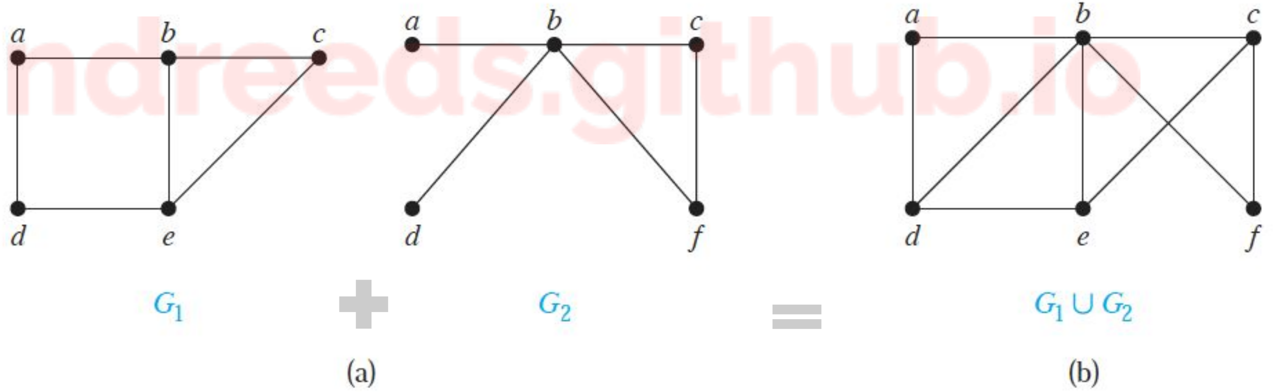
REMOVING VERTICES FROM A GRAPH

Given a graph $G = (V, E)$ and a vertex $v \in V$, we can produce a subgraph of G by removing the vertex v. The resulting subgraph $G - v = (V - \{v\}, E')$, where E' is the set of edges of G not incident to v.

Given a graph $G = (V, E)$ and a subset of vertices $V' \subset V$, we can produce a subgraph of G by removing all vertices of V'. The resulting subgraph $G - V' = (V - V', E')$, where E' is the set of edges of G not incident to a vertex in V'.

Definition

The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



REVIEW QUESTIONS Pt. 2
GRAPHS

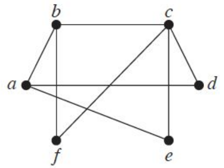
Can a simple graph exist with 15 vertices each of degree five?

For which values of n are these graphs bipartite?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

Determine whether these graphs are bipartite:

23.



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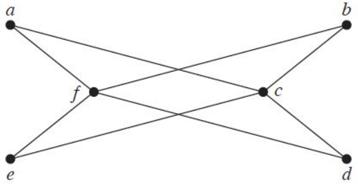

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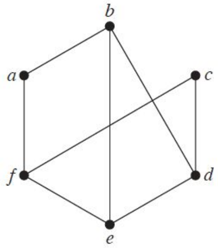

REVIEW QUESTIONS Pt. 2
GRAPHS

Find the union of the given pair of simple graphs.
(Assume edges with the same endpoints are the same.)

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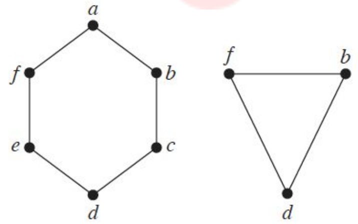

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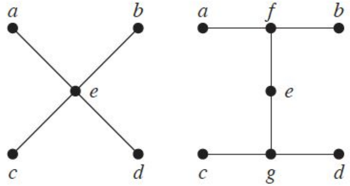

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