UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

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DISCRETE COMPUTATIONAL STRUCTURES

DISCRETE PROBABILITY
INTRODUCTION AND PROBABILITY THEORY

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AN INTRODUCTION TO
DISCRETE PROBABILITY
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DISCRETE PROBABILITY

An **experiment** is a process that produces *one* of a given *set of possible outcomes*.

The **sample space** of the experiment is the set of possible outcomes.

An **event** is a subset of a sample space.

The **event space** is the *power set* of the sample space.

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Theorem

The **probability** of an *event* E, which is a subset of a finite *sample space* S of equally likely outcomes, is

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$$p(E) = \frac{|E|}{|S|}$$

Examples:

- ★ What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?
- ★ A base-10 numeral is randomly chosen from the range 0, 1, 2, …, 99. What is the probability that the numeral contains no 6's or 8's?
- ★ A base-10 numeral is randomly chosen from the range0, 1, 2, …, 99. What is the probability that the numeral contains one 6 and no 8's?

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Examples (consider a deck of 52 standard cards):

- ★ What is the probability that a hand of 5 cards in poker contains 2 cards of 1 kind (numeral)?
- ★ What is the probability that a hand of 5 cards in poker contains 4 cards of 1 kind (numeral)?
- ★ What is the probability that a hand of 5 cards in poker contains full house?
- ★ What is the probability that a hand of 5 cards in poker contains 3 of one kind but no full house?

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Examples:

★ What is the probability that the numbers 16, 8, 18, 38, and 26 are drawn in that order from a bin containing 50 balls labeled with the numbers 1, 2, ..., 50 if

- the ball selected is not returned to the bin before the next ball is selected and
- the ball selected is returned to the bin before the next ball is selected?

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Theorem

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complement event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

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Example:

★ A coin is tossed 10 times. Find the probability of at least one tail.

Theorem

Let E_1 and E_2 be events in a sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

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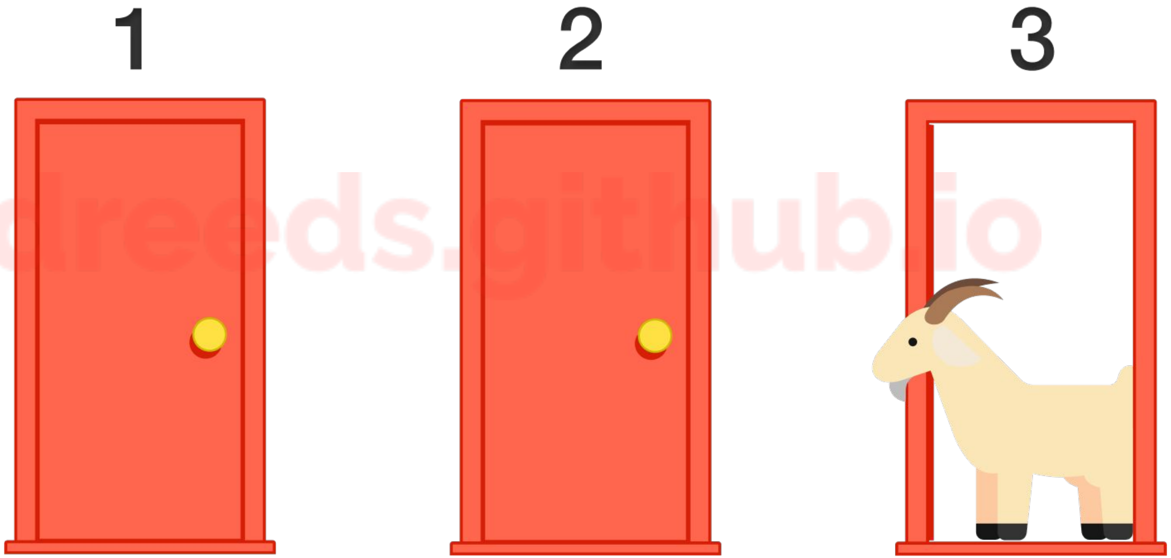
Example:

★ An integer is chosen from the interval [1, …, 100]. Find the probability that it is divisible either by 2 or by 7.

★

THE MONTY HALL THREE-DOOR PUZZLE



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PROBABILITY THEORY

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DISCRETE PROBABILITY

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome s such that:

i. for each $s \in S$

$$0 \leq p(s) \leq 1$$

ii. $$\sum_{s \in S} p(s) = 1.$$

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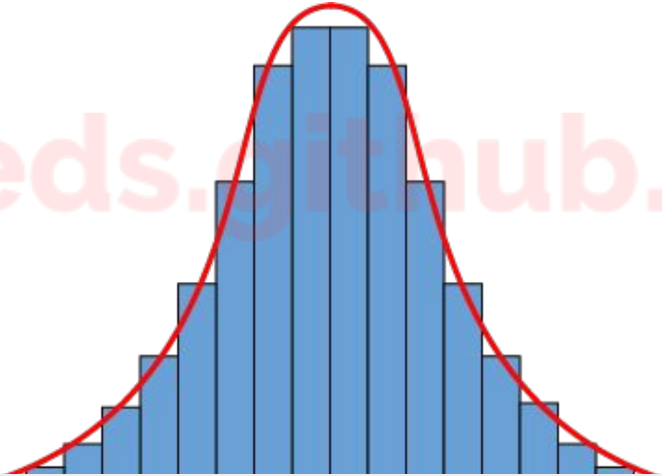
DISCRETE PROBABILITY

The function p from the set of all outcomes of the sample space S is called a **probability distribution**.



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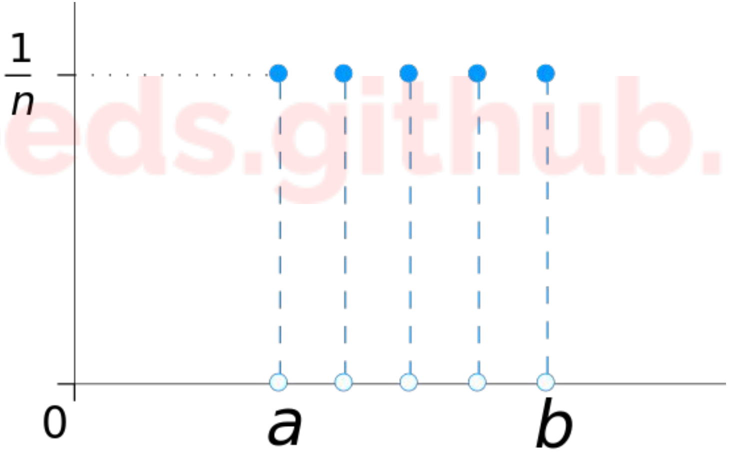
Suppose that S is a set with n elements.
The **uniform distribution** assign the probability $1/n$ to each element of S.



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Theorem

If $E_1, E_2, \ldots$ is a sequence of pairwise *disjoint* events in a sample space S, then

$$p\left(\bigcup_i E_i\right) = \sum_i p(E_i).$$

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Let E and F be events with $p(F) > 0$.
The **conditional probability** of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

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Examples:

★ Two fair dice are rolled. Given that at least one is a 4, what is the conditional probability that both are 4s?

:

Events $\boldsymbol{E}$ and $\boldsymbol{F}$ are **independent** *if and only if*

$$p(E \cap F) = p(E)p(F).$$

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Examples:

- ★ Are the events $\boldsymbol{E}$, that a family with three dogos has puppers of both sexes, and $\boldsymbol{F}$, that this family has at most one good boi, **independent**?
- ★ Are the events $\boldsymbol{E}$, that a family with three doggos has puppers of both sexes, and $\boldsymbol{F}$, that this family has at most two good bois, **independent**?

The events $E_1, E_2, \ldots, E_n$ are **pairwise independent** *if and only if*

$$p(E_i \cap E_j) = p(E_i)p(E_j)$$

for all pairs of integers i and j with $1 \leq i < j \leq n$.

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The events $E_1, E_2, \ldots, E_n$ are **mutual independent** *if and only if*

$$p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$$

whenever i_j, $j = 1;,2, \ldots, m$, are integers with $1 \leq i_1 < i_2 \cdots i_m \leq n$ and $m \geq 2$.

The Birthday Problem

What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday **month** is greater than 1/2?

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What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday **day** is greater than 1/2?

DISCRETE PROBABILITY

A **random variable** is a *function* from the sample space of an experiment to the *set of real numbers*.

That is, **a random variable assigns a real number to each possible outcome**.

The **distribution** *of a random variable* X on a sample space S is the *set of pairs*

$$(x, p(X = x)) \text{ for all } x \in X(S),$$

where $p(X = x)$ is the probability that X takes the value x.

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The **set of pairs** in this distribution is determined by all $p(X = x)$ for $x \in X(S)$,

Example:

> Suppose that a coin is flipped 3 times. Let $H(t)$ be the random variable that equals the number of heads that appear when t is the outcome. Compute the distribution of H on the sample space.

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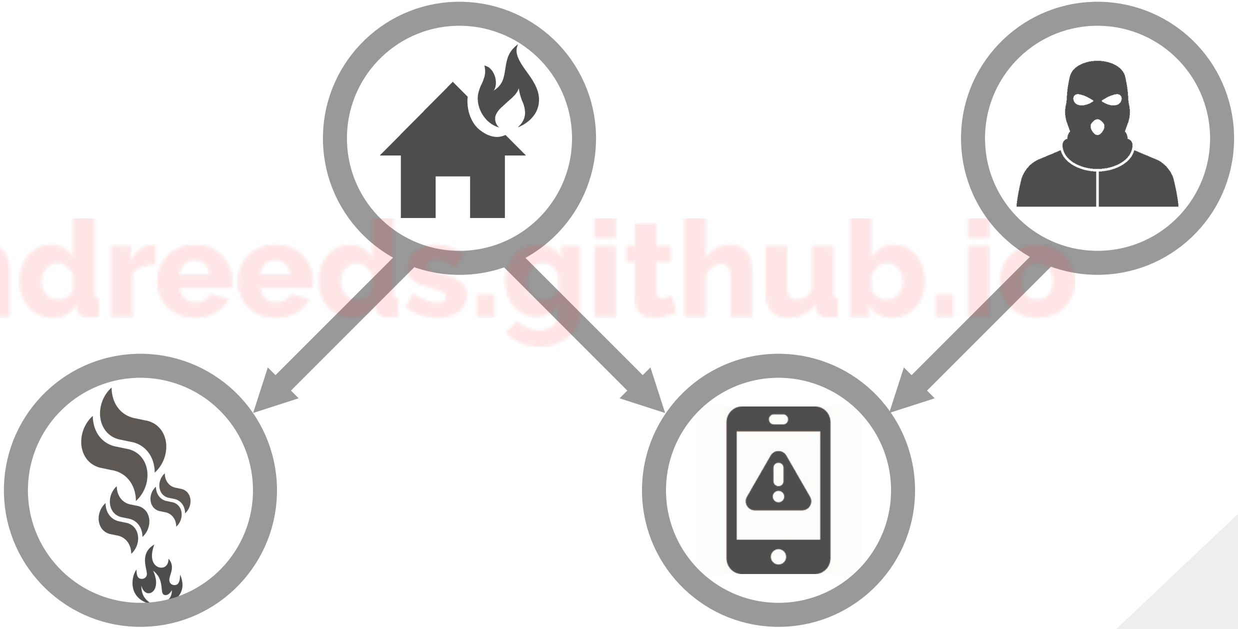
PGM EXAMPLE



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THE PROBABILISTIC METHOD

If the probability that an element chosen at random from a S *does not have a particular property* is less than 1, there exists an element in S with this property.

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- to create non-constructive proofs
- can show that objects with certain properties exist
- $p(E) = 1 - p(\bar{E})$

WORTH CHECKING OUT

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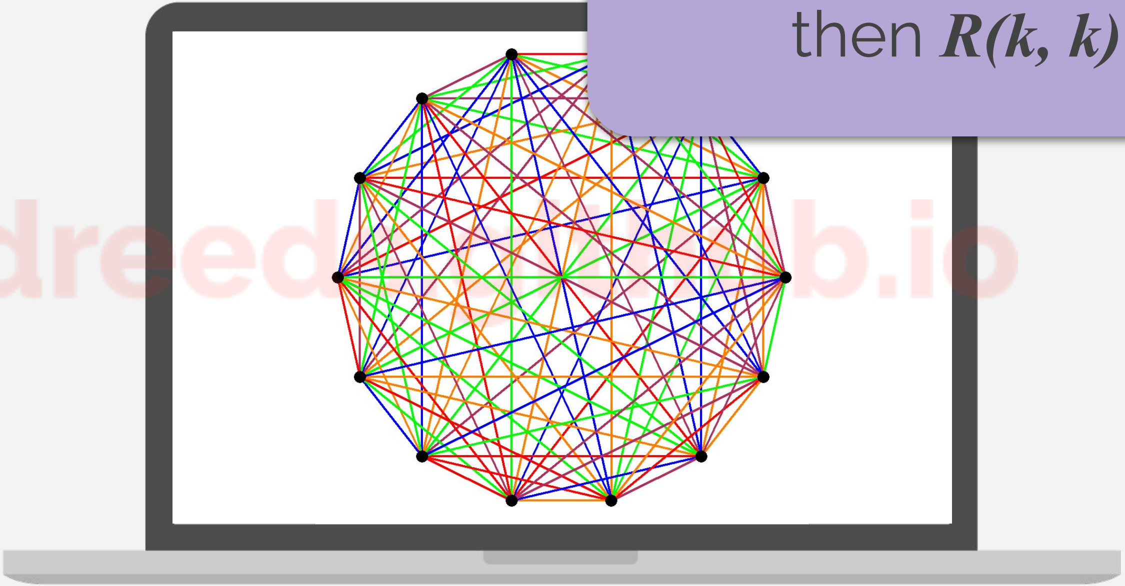
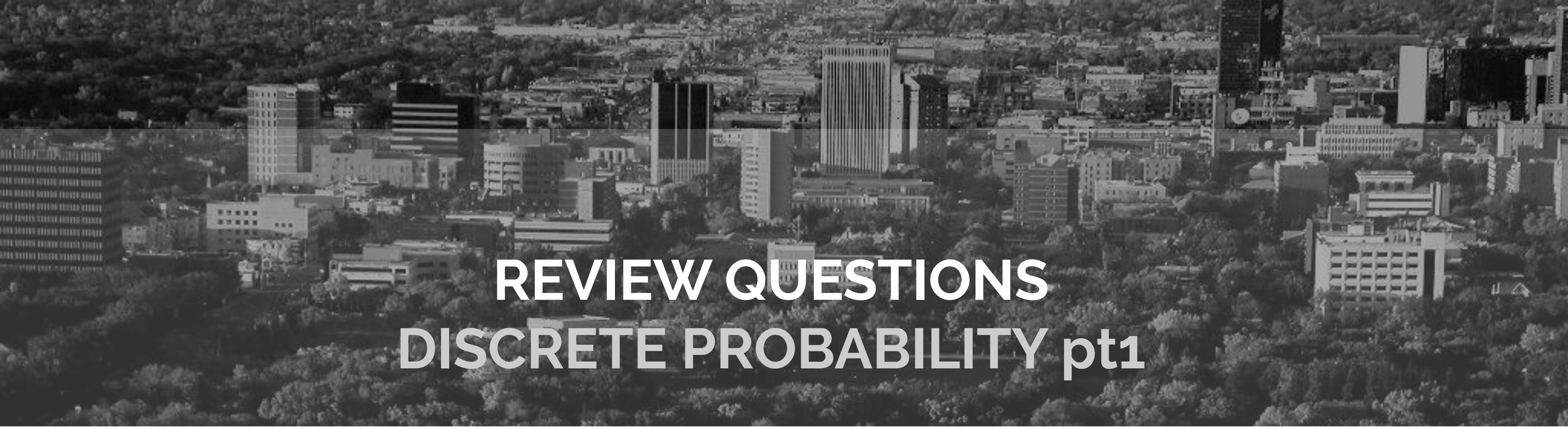
If k is an integer with $k \geq 2$,
then $R(k, k) \geq 2^{k/2}$





REVIEW QUESTIONS

DISCRETE PROBABILITY pt1

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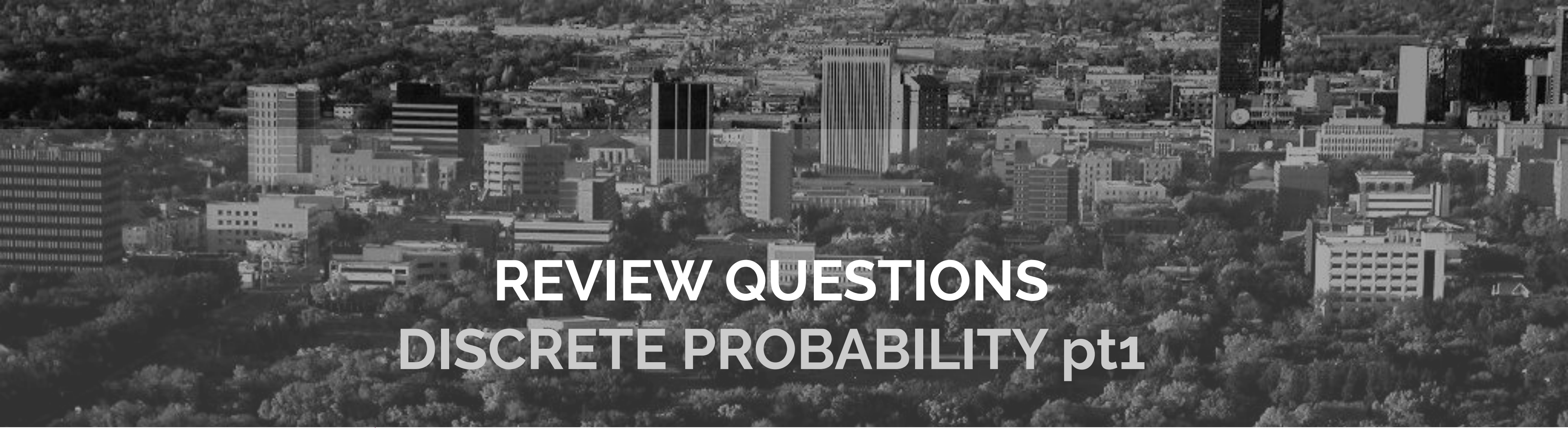
What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)

Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.



REVIEW QUESTIONS
DISCRETE PROBABILITY pt1

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Write out the conditions required for three events E_1, E_2, and E_3 to be mutually independent.

Let E_1, E_2, and E_3 be the events that the first flip comes up heads, that the third flip comes up heads, and that exactly one of the first flip and third flip come up heads, respectively, when a fair coin is flipped three times. Are E_1, E_2, and E_3 pairwise independent? Are they mutually independent?

What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)