UNIVERSITY OF REGINA

CS310-002
DISCRETE COMPUTATIONAL STRUCTURES

ANDRÉ E. **DOS SANTOS**
dossantos@cs.uregina.ca
andreeds.github.io

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ADVANCED COUNTING TECHNIQUES

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Review

A recursive definition of a sequence specifies

- ➔ **Initial conditions**
- ➔ **Recurrence relation**

Example:

$a_0 = 0$ and $a_1 = 3$	**Initial conditions**
$a_n = 2a_{n-1} - a_{n-2}$	**Recurrence relation**
$a_n = 3n$	**Solution**

Review

Linear recurrences: Each term of a sequence is a linear function of earlier terms in the sequence.

Example:

$a_0 = 1 \qquad a_1 = 6 \qquad a_2 = 10$

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$$

$$a_3 = a_0 + 2a_1 + 3a_2$$

$$= 1 + 2(6) + 3(10) = 43$$

Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

linear homogeneous recurrence relation of degree k with constant coefficients

is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- ➔ a_n is expressed in terms of the previous k terms of the sequence, so its degree is k.
- ➔ This recurrence includes k initial conditions

$a_0 = C_0$ $a_1 = C_1$ … $a_k = C_k$

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

The right-hand side is a sum of previous terms of the sequence each multiplied by function of n

linear **homogeneous** recurrence relation
of degree k with constant coefficients

$$a_n = \boxed{c_1 a_{n-1}} + \boxed{c_2 a_{n-2}} + \cdots + \boxed{c_k a_{n-k}}$$

No terms occur that are not multiples of the a_js

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = \boxed{c_1} a_{n-1} + \boxed{c_2} a_{n-2} + \cdots + \boxed{c_k} a_{n-k}$$

The coefficients of the terms of the sequence are all constants, rather than functions that depend on n

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{\boxed{n-k}}$$

The degree is k because a_n is expressed in terms of the previous k terms of the sequence

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{\boxed{n-k}}$$

The degree is k because a_n is expressed in terms of the previous k terms of the sequence

linear homogeneous recurrence relation of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Examples:

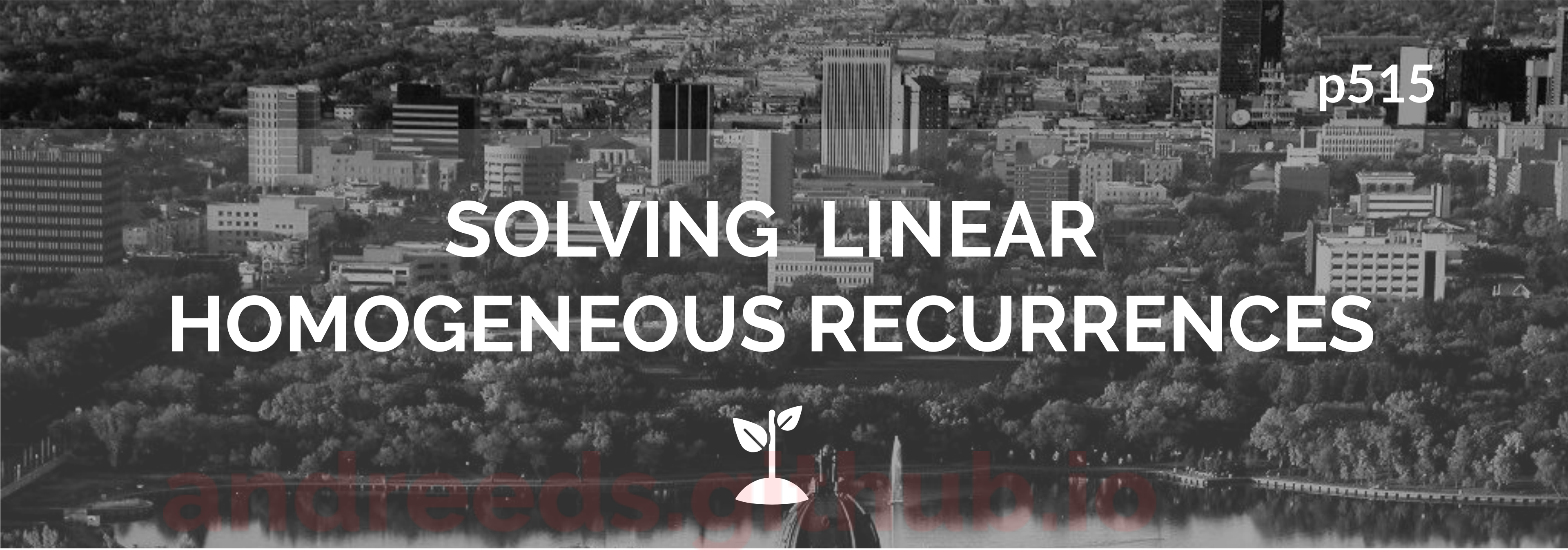
- ★ $P_n = (1.11) P_{n-1}$
- ★ $f_n = f_{n-1} + f_{n-2}$
- ★ $a_n = a_{n-5}$

linear homogeneous recurrence relation
of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

NOT Examples:

- ★ $a_n = a_{n-1} + a^2_{n-2}$
- ★ $H_n = 2H_{n-1} + 1$
- ★ $B_n = nB_{n-1}$



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SOLVING LINEAR HOMOGENEOUS RECURRENCES

Proposition

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.
  - α is any constant.

Proposition

It follows from **Proposition**, if we find some solutions to a linear homogeneous recurrence, then **any linear combination** of them will also be a solution to the linear homogeneous recurrence.

Solving linear homogeneous recurrences

- Recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

- Try to find a solution of form rn

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \ldots - c_k r^{n-k} = 0$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

dividing both sides by r^{n-k}

This equation is called the **characteristic equation**.

Example

The *Fibonacci* recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

Its characteristic equation is

$$r^2 - r - 1 = 0$$

Example

What is the characteristic equation of $F_n = 4F_{n-1} - 4F_{n-2}$?

Proposition

➔ r is a solution of $r^k - c_1r^{k-1} + c_2r^{k-2} + \ldots + c_k = 0$ if and only if r^n is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$.

Example

consider the characteristic equation $r^2 - 4r + 4 = 0$.

$$r^2 - 4r + 4 = (r - 2)^2 = 0$$

So, $r = 2$.

So, 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$.

$$2^n = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2}$$

Theorem

- Consider the characteristic equation

$$r^k - c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k = 0$$

and the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

- Assume $r_1, r_2, \ldots$ and r_m all satisfy the equation.
- Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any constants.
- So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$ satisfies the recurrence.

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Example

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

Example

What is the solution of the recurrence relation

$$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$

with $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$?

REVIEW QUESTIONS

ADVANCED COUNTING TECHNIQUES

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.