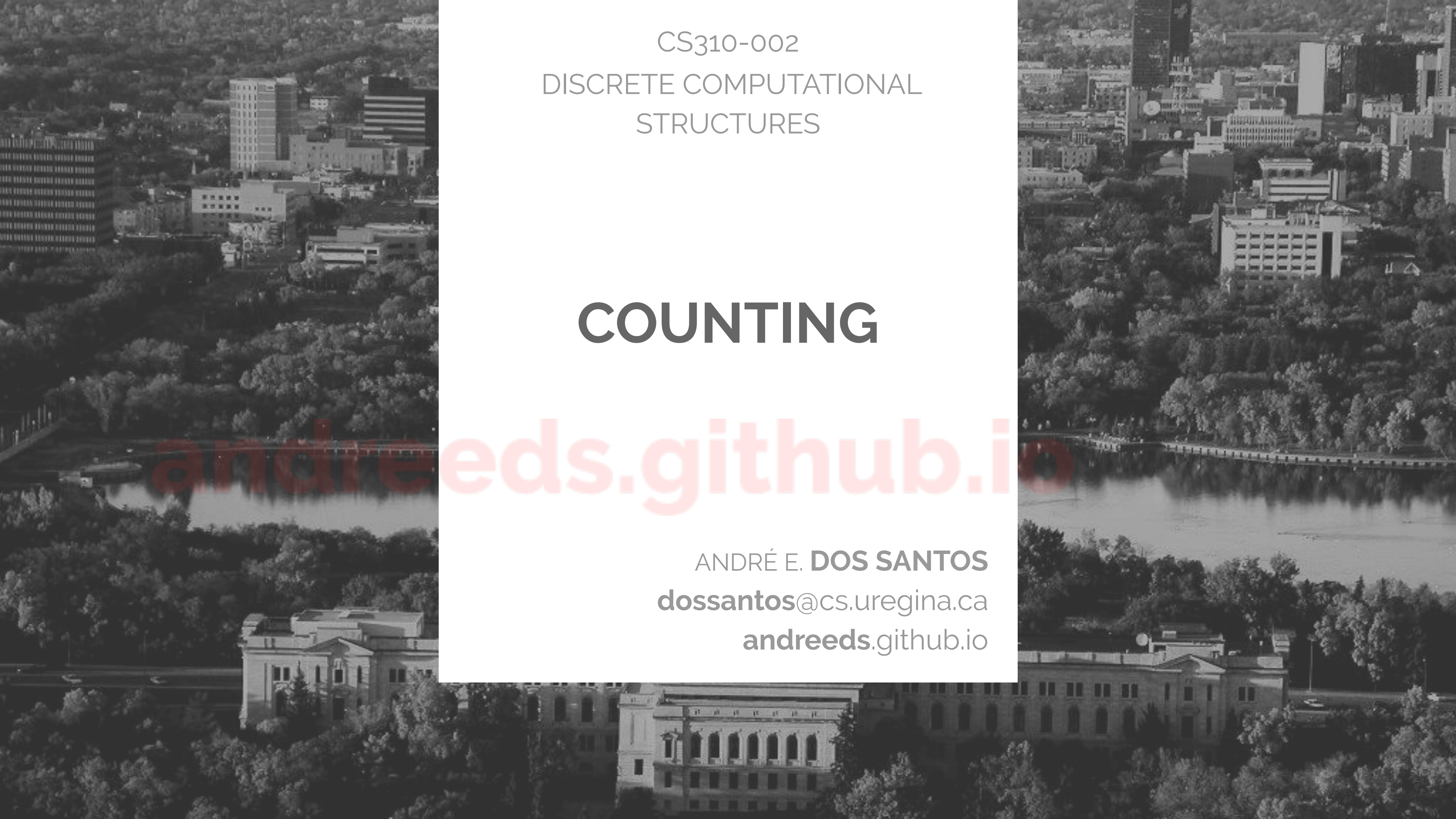
UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

ANDRÉ E. **DOS SANTOS**
dossantos@cs.uregina.ca
andreeds.github.io



CS310-002

DISCRETE COMPUTATIONAL STRUCTURES

COUNTING

ANDRÉ E. **DOS SANTOS**

dossantos@cs.uregina.ca

andreeds.github.io

THE BASICS OF COUNTING

The Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are $\mathbf{n_1}$ ways to do the rst task and $\mathbf{n_2}$ ways to do the second task after the rst task has been done, then there are $\mathbf{n_1 n_2}$ ways to do the procedure.

Examples:

★ The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

★ How many functions are there from a set with **m** elements to one with **n** elements?

★ How many *one-to-one* functions are there from a set with **m** elements to one with **n** elements?

★ Given a set **S** of **n** elements, how many different subsets of **S** are there?

The Sum Rule

If a rst task can be done in $\mathbf{n_1}$ ways and a second task can be done in $\mathbf{n_2}$ ways, and if these tasks *cannot be done at the same time*, then there are $\mathbf{n_1 + n_2}$ ways to do one of these tasks

Example:

★ A student can choose a project from one of three lists. The three lists contain 6, 8 and 10 projects respectively. How many possible projects are there to choose from?

The Subtraction Rule

If a task can be done in *either* $\mathbf{n_1}$ ways or $\mathbf{n_2}$ ways, then the number of ways to do the task is $\mathbf{n_1 + n_2}$ *minus* the number of ways to do the task that are common to the two different ways

Example:

- ★ A computer company receives 300 applications from computer graduates for a job planning a line of new Web servers. Suppose that 180 of these applicants majored in computer science, 120 majored in math, and 50 majored both in computer science and in math. How many of these applicants majored neither in computer science nor in math?

The Division Rule

There are **n/d** ways to do a task if it can be done using a *procedure that can be carried out in* **n** *ways*, and for every way **w**, exactly **d** of the **n** ways correspond to way **w**

Example:

★ How many different ways are there to seat five people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Tree Diagrams

A **tree diagram** can be used to count a set. In a tree diagram, each branch represent each possible choice

Example:

★ Count the subsets of {2, 5, 11, 17, 23} whose sum is at most 29 p399

THE PIGEONHOLE PRINCIPLE

The Pigeonhole Principle

If **k + 1** or more objects are placed into **k** boxes, then there is at least one box containing two or more of the objects

Corollary

A function **f** from a set with **k + 1** or more elements to a set with **k** elements is *not one-to-one*

Examples:

- ★ Among any group of 367 people, there must be at least two with the same birthday.
- ★ Among any group of **$n \geq 8$** people, there must be two of them who were born on the same day of the week.
- ★ In your drawer, you have 10 pairs of different socks, all loosely mixed together. In total darkness, you pack your suitcase. How many socks must you pack to be sure that you have at least one pair?

The Generalized Pigeonhole Principle

If **N** objects are placed into **k** boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects

Examples:

- ★ Among any group of 100 people, there must be at least $\lceil 100/12 \rceil = 9$ who were born in the same month.
- ★ How many cards must be selected from a standard deck of 52 cards to guarantee that at least four cards of the same suit are chosen?
- ★ How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

Applications of the Pigeonhole Principle

★ Show that among any **$n + 1$** positive integers not exceeding **$2n$** there must be an integer that divides one of the other integers.

Hint

Write each of the **$n+1$** integers as a power of **2** times an odd integer, that is, let $a_i = 2^{k_i} b_i$ for $i = 1, 2, \ldots, n + 1$, where k_i is a nonnegative integer and b_i is odd.

Applications of the Pigeonhole Principle

★ During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Hint

Let $\mathbf{a_j}$ be the number of games played on or before the **j**-th day of the month.

Ramsey Theory

★ The *Ramsey number* **R(m, n)** is the minimum number of guests **R(m, n)** that must be invited to a party so that at least **m** know each other or at least **n** do not know each other.

p399

PERMUTATIONS AND COMBINATIONS

PERMUTATIONS

A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an ***r**-permutation*.

Theorem

If **n** is a positive integer and **r** is an integer with **1 ≤ r ≤ n,** then the number of ***r**-permutations* of a set with **n** distinct elements is

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

PERMUTATIONS

$$P(n, r) = \frac{n!}{(n-r)!}$$

if **n** and **r** are integers with **0 ≤ r ≤ n**.

Examples:

★ Suppose there are 7 flags, all of different colors. How many different signals can be formed by running three flags to the top of a flagpole?

★ How many ways are there to select a third-prize winner, a second-prize winner, and a first-prize winner from 100 different people who have entered a contest?

★ How many ways are there to select three third-prize winners from 100 different people who have entered a contest?

COMBINATIONS

An ***r**-combination* of elements of a set is an unordered selection of **r** elements from the set.

COMBINATIONS

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

The number of ***r****-combinations* of a set with **n** *distinct* elements, where **n** is a positive integer and **r** is an integer with

0 ≤ r ≤ n

COMBINATIONS

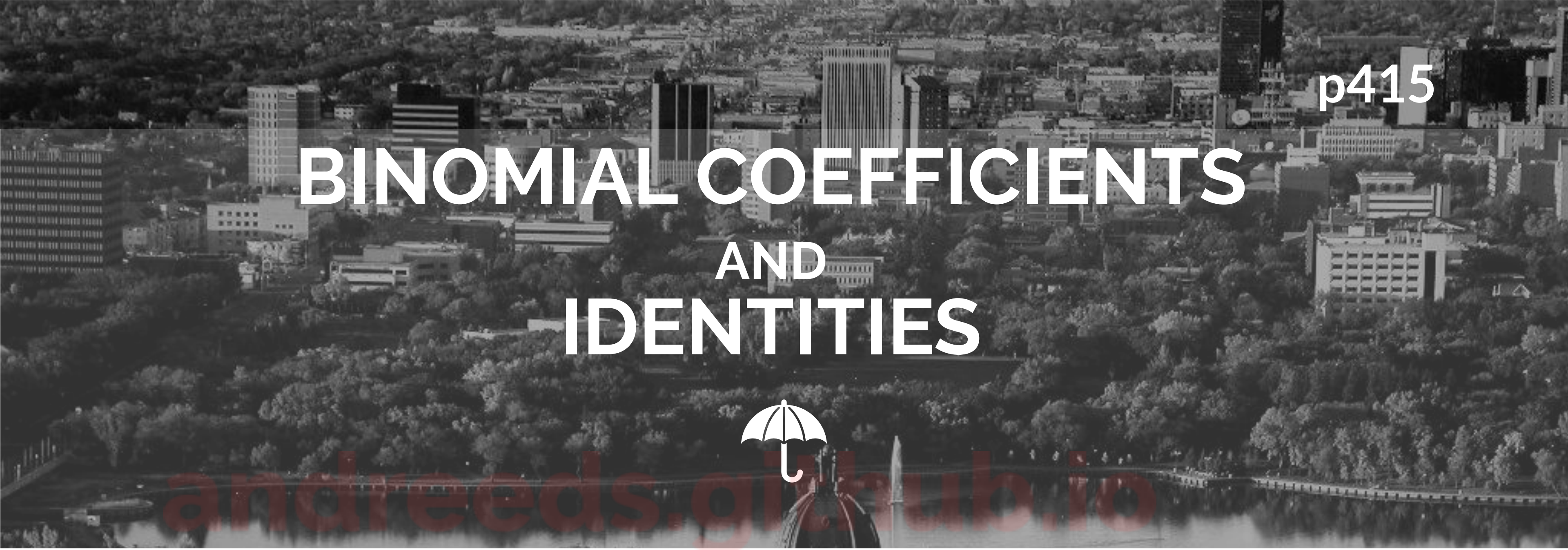
Theorem

Let **n** and **r** be nonnegative integers with **r ≤ n**. Then

$$C(n, r) = C(n, n-r)$$

Examples:

★ How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?



BINOMIAL COEFFICIENTS AND IDENTITIES

THE BINOMIAL THEOREM

The number of ***r**-combinations* from a set with **n** elements, i.e., **C(n, r)**, is often denoted by $\binom{n}{r}$.

THE BINOMIAL THEOREM

C(n, r) is also called a **binomial coecient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as **$(x + y)^n$**.

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Examples:

- ★ What is the expansion of $(x + y)^4$?
- ★ What is the coecient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?
- ★ What is the coecient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

APPLICATIONS OF THE BINOMIAL THEOREM

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

APPLICATIONS OF THE BINOMIAL THEOREM

Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

APPLICATIONS OF THE BINOMIAL THEOREM

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

PASCAL'S IDENTITY

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

PASCAL'S IDENTITY

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\cdots
\end{array}
$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

OTHER IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

OTHER IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

then follows that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

If n is a nonnegative integer

OTHER IDENTITIES INVOLVING BINOMIAL COEFFICIENTS

Theorem

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

REVIEW QUESTIONS
COUNTING

State the generalized pigeonhole principle.

Explain how the generalized pigeonhole principle can be used to show that among any 91 integers, there are at least ten that end with the same digit.

How many ways are there to select six students from a class of 25 to serve on a committee?

How many ways are there to select six students from a class of 25 to hold six different executive positions on a committee?

Find the coefficient of $x^{100}y^{101}$ in the expansion of $(2x + 5y)^{201}$.