UNIVERSITY OF REGINA

CS310-002 DISCRETE COMPUTATIONAL STRUCTURES

ANDRÉ E. **DOS SANTOS**
dossantos@cs.uregina.ca
andreeds.github.io

CS310-002

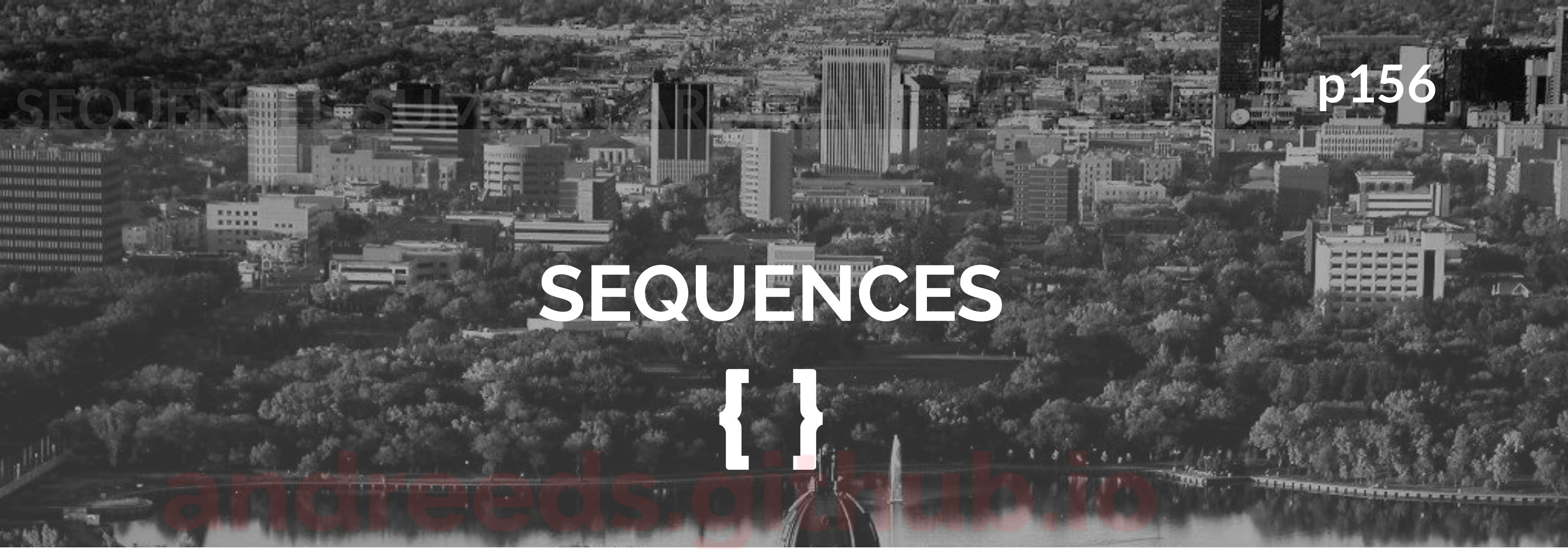
DISCRETE COMPUTATIONAL STRUCTURES

BASIC STRUCTURES
SEQUENCES, SUMS, AND CARDINALITY OF SETS

ANDRÉ E. **DOS SANTOS**

dossantos@cs.uregina.ca

andreeds.github.io



SEQUENCES

{ }

Examples:

1, -1, 1, -1, 1, -1, ...
2, 10, 50, 250, 1250, ...
6, 2 ,2/3, 2/9, 2/27, ...

A **sequence** is a *function* from a *subset of the set of the integers** to a set **S**

Examples:

8, 14, 32, 86, 248, ...

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...

* usually either the set **{0, 1, 2, ...}** or the set **{1, 2, 3, ...}**

A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* **a** and the *common ratio* **r** are real numbers

Examples:

1, -1, 1, -1, 1, -1, ...

2, 10, 50, 250, 1250, ...

6, 2 ,2/3, 2/9, 2/27, ...

An **arithmetic progression** is a sequence of the form

a, a + d, a + 2d, ..., a + nd, ...

where the *initial term* **a** and the *common difference* **d** are real numbers

Examples:

-1, 3, 7, 11, ...
7, 4, 1, -2, ...

inferring a rule: Given some initial terms $a_0, a_1, ..., a_k$ of a *sequence*, construct a rule that is consistent with those initial terms

A **recursion** for a_n is a function whose arguments are earlier terms in the sequence

A **closed form** for a_n is a formula whose argument is the subscript n

Example geometric progression:

Find a *recursion function* and a *closed form formula*
to produce a sequence with the first 5 terms **2, 10, 50, 250, 1250**

Example geometric progression:

Find a *recursion function* and a *closed form formula*
to produce a sequence with the first 5 terms **2, 10, 50, 250, 1250**

recursion function

$\{a_n\}$, where $a_n = 5a_{n-1}$, with $n = 1, 2, \ldots, 4$ and $a_0 = 2$

closed form formula

$\{a_n\}$, where $a_n = 2 * 5^n$, with $n = 0, 1, \ldots, 4$

inferring a rule of a geometric progression

$a = a_0$ and $r = a_n / a_{n-1}$

Example:

Find a *recursion function* and a *closed form formula*
to produce a sequence with the first 5 terms **6, 2 , 2/3, 2/9, 2/27**

Example arithmetic progression:

Find a *recursion function* and a *closed form formula*
to produce a sequence with the first 10 terms **2, 5, 8, 11, 14, 17, 20, 23, 26, 29**

Example arithmetic progression:

Find a *recursion function* and a *closed form formula*
to produce a sequence with the first 10 terms **2, 5, 8, 11, 14, 17, 20, 23, 26, 29**

recursion function

$\{a_n\}$, where $a_n = 3+a_{n-1}$, with $n = 1, 2, \ldots, 9$ and $a_0 = 2$

closed form formula

$\{a_n\}$, where $a_n = 2 + 3n$, with $n = 0, 1, \ldots, 9$

inferring a rule of a arithmetic progression

$$a + d(n-1)$$

Example:

Find a *recursion function* to produce a sequence with the first 12 terms **0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89**

SUMMATIONS

Σ

Examples:

$1 + 2 + 3 + \ldots + n$

$1^2 + 2^2 + 3^2 + \ldots + n^2$

$$a_m + a_{m+1} + \cdots + a_n$$

$$\sum_{j=m}^{n} a_j \,, \qquad \textstyle\sum_{j=m}^{n} a_j \,, \qquad \text{or} \qquad \sum_{m \le j \le n} a_j$$

What is the value of $\sum_{j=1}^{5} j^2$?

What is the value of $\sum_{k=4}^{8}(-1)^k$?

Find $\sum_{k=101}^{200} k^2$.

Find $\sum_{i=1}^{5} \sum_{j=2}^{3} ij$.

Find $\sum_{k \in \{1,3,5\}} 100k$.

Find a Σ-notation for $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128$.

Find a Σ-notation for $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10$.

Fill in the blanks: $\sum_{k=1}^{7} 2^k = \sum_{k=[\ \ ]}^{10} 2^{[\ \ ]}$.

arithmetic progression

Theorem:

If a and d are real numbers, then

$$\sum_{i=0}^{n}(a+id)=(n+1)a+\frac{n(n+1)d}{2}$$

geometric progression

Theorem:

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1}, & \text{if } r \neq 1 \\ (n+1)a, & \text{if } r = 1 \end{cases}$$

SUMMATION FORMULAE

p166

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$

SEQUENCES, SUMS, & CARDINALITY OF SETS

Find $\sum_{k=101}^{200} k^2$.

Find $\sum_{i=1}^{5} \sum_{j=2}^{3} ij$.

Find $\sum_{k\in\{1,3,5\}} 100k$.

Find a Σ-notation for $1+2+4+8+16+32+64+128$.

Find a Σ-notation for $1-2+3-4+5-6+7-8+9-10$.

Fill in the blanks: $\sum_{k=1}^{7} 2^k = \sum_{k=[\ \ ]}^{10} 2^{[\ \ ]}$.

Find $\sum_{k=0}^{n} (\frac{1}{2})^k$.

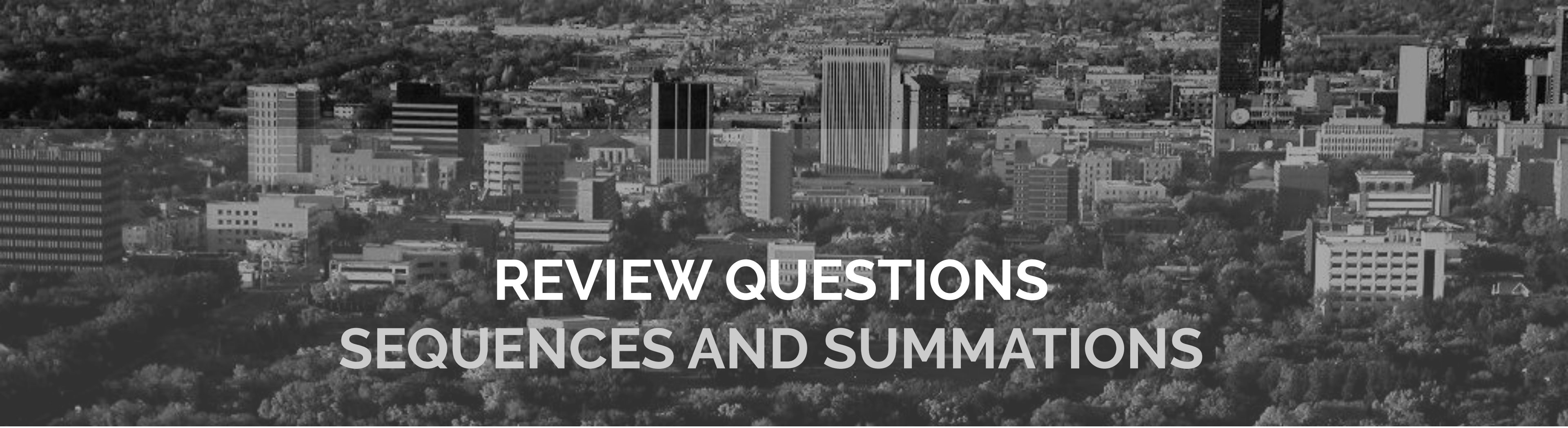
Find $\sum_{k=0}^{n}(\frac{1}{2})^k$.

Let x be a real number with $|x| < 1$. Find $\sum_{k=0}^{\infty} x^k$.

SUMMATION FORMULAE

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Find $\sum_{k=0}^{n}(\frac{1}{2})^k$.



REVIEW QUESTIONS
SEQUENCES AND SUMMATIONS

Conjecture a formula for the terms of the sequence that begins 8, 14, 32, 86, 248 and find the next three terms of your sequence.

Suppose that $a_n = a_{n-1} - 5$ for $n = 1, 2, \ldots$. Find a formula for a_n.

What is the sum of the terms of the geometric progression $a + ar + \cdots + ar^n$ when $r \neq 1$?

CARDINALITY OF SETS

$| \quad |$

The sets **A** and **B** have the same **cardinality** if and only if there is a *one-to-one* correspondence from **A** to **B**.
When **A** and **B** have the same cardinality, we write **|A| = |B|**

If there is a *one-to-one* function from **A** to **B**, the cardinality of **A** is less than or the same as the cardinality of **B** and we write **|A| ≤ |B|**. Moreover, when **|A| ≤ |B|** and **A** and **B** have different cardinality, we say that the cardinality of **A** is less than the cardinality of **B** and we write **|A| < |B|**

A set that is either

(i) *finite* or

(ii) has the same *cardinality as the set of positive integers*

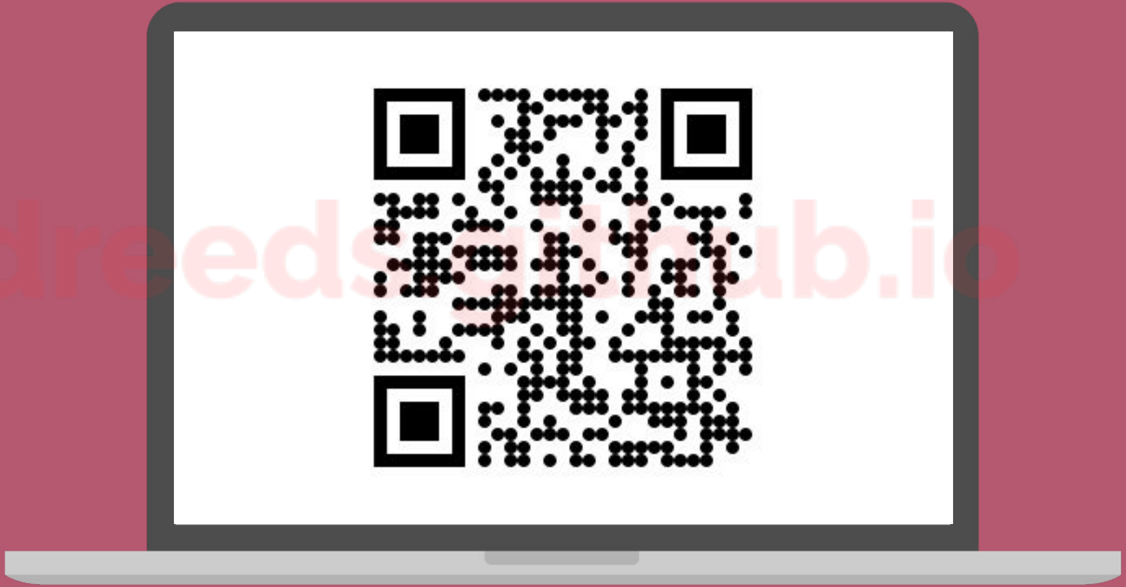
is called **countable**.

When an **infinite** set **S** is countable, we denote the cardinality of **S by** $\aleph_0$, and we write **|S|** = $\aleph_0$

A set that is not countable is called **uncountable**

HOW TO COMPARE INFINITIES

https://www.youtube.com/watch?v=_3PwEXk67Wg



Example:

Assume **A = {0, 2, 4, 6, … }** set of even numbers. Is it countable?

Theorem

If **A** and **B** are countable sets, then **A** ∪ **B** is also countable.

Schroder-Bernstein Theorem

If **A** and **B** are sets with **|A| ≤ |B|** and **|B| ≤ |A|**, then **|A| = |B|**. In other words, if there are *one-to-one functions* **f** from **A** to **B** and **g** from **B** to **A**, then there is a *one-to-one correspondence* between **A** and **B**.

Show that **|(0, 1)| = |(0, 1]|**

Theorem

The set of real numbers is uncountable.

Proof idea

Cantor diagonalization argument (1879).

Applications

To prove the **Halting Problem**, to show that some languages are not *Turing-recognizable*, to show some functions are not computable, etc..

SOME INFINITIES ARE BIGGER THAN OTHER INFINITIES

DIAGONALIZATION

https://www.youtube.com/watch?v=dEOBDIyz0BU

We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function.

If a function is not computable we say it is **uncomputable**.

Theorem

There are functions from the set of positive integers to **{0, 1}** that are uncomputable.

IMPOSSIBLE PROGRAMS
THE HALTING PROBLEM

https://www.youtube.com/watch?v=wGLQiHXHWNk

REVIEW QUESTIONS
CARDINALITY OF SETS

- Show that the set of odd integers is countable.
- Give an example of an uncountable set.